Prediction of Hidden Oscillations Existence in Nonlinear Dynamical Systems: Analytics and Simulation

Gennady A. Leonov and Nikolay V. Kuznetsov

Abstract. From a computational point of view, in nonlinear dynamical systems, attractors can be regarded as self-excited and hidden attractors. Self-excited attractors can be localized numerically by a standard computational procedure, in which after a transient process a trajectory, starting from a point of unstable manifold in a neighborhood of equilibrium, reaches a state of oscillation, therefore one can easily identify it. In contrast, for a hidden attractor, a basin of attraction does not intersect neighborhoods of equilibria. While classical attractors are self-excited, attractors can therefore be obtained numerically by the standard computational procedure, for localization of hidden attractors it is necessary to develop special procedures, since there are no similar transient processes leading to such attractors. This keynote lecture is devoted to affective analytical-numerical methods for localization of hidden oscillations in nonlinear dynamical systems and their application to well known fundamental problems and applied models.

1 Introduction

An oscillation in dynamical system can be easily localized numerically if initial conditions from its open neighborhood lead to long-time behavior that approaches the oscillation. Such oscillation (or a set of oscillations) is called an attractor, and its attracting set is called the basin of attraction [49, 27]. Thus, from a computational

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point of view in applied problems of nonlinear analysis of dynamical models, it is essential to regard attractors [44, 45, 40, 32] as *self-excited* and *hidden attractors* depending on simplicity of finding its basin of attraction in the phase space.

For a self-excited attractors its basin of attraction is connected with an unstable equilibrium: self-excited attractors can be localized numerically by *standard computational procedure*, in which after a transient process a trajectory, started from a point of unstable manifold in a neighborhood of equilibrium, reaches a state of oscillation therefore one can easily identify it. In contrast, for a *hidden attractor, its basin of attraction does not intersect with small neighborhoods of equilibria*.

For numerical localization of hidden attractors it is necessary to develop special analytical-numerical procedures, since there are no similar transient processes leading to such attractors from neighborhoods of equilibria. Remark, that one cannot guarantee finding of an attractor (especially for multidimensional systems) by the integration of trajectories with random initial conditions since basin of attraction can be very small.

2 Self-excited Attractor Localization

In the first half of the last century during the initial period of the development of the theory of nonlinear oscillations [55, 19, 3, 53], a main attention was given to analysis and synthesis of oscillating systems, for which the problem of the existence of oscillations can be solved with relative ease.

These investigations were encouraged by the applied research of periodic oscillations in mechanics, electronics, chemistry, biology and so on (see, e.g., [54]) The structure of many applied systems (see, e.g., Duffing [13], van der Pol [51], Tricomi [56], Beluosov-Zhabotinsky [4] systems) was such that the existence of oscillations was "almost obvious" since the oscillations were excited from unstable equilibria (*self-excited oscillation*). This allowed one, following Poincare's advice "*to construct the curves defined by differential equations*" [50], to visualize periodic oscillations by standard computational procedure.

Then, in the middle of 20th century, it was found numerically the existence of chaotic oscillations [57, 47], which were also excited from an unstable equilibrium and could be computed by the standard computational procedure. Nowadays there is enormous number of publications devoted to the computation and analysis of self-exited chaotic oscillations (see, e.g., [52, 11, 9] and others).

In Fig. 1 numerical localization of classical self-exited oscillation are shown: van der Pol oscillator [51], one of the modification of Belousov-Zhabotinsky reaction [4], two prey and one predator model [14].

In Fig. 2 examples of visualization of classical self-excited chaotic attractors are presented: Lorenz system [47], Rössler system [52], "double-scroll" attractor in Chua's circuit [5].

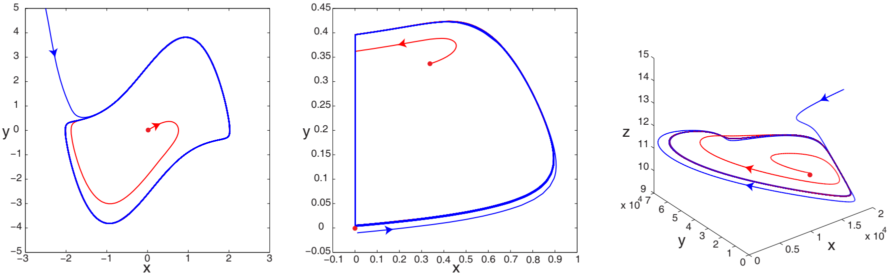

Fig. 1 Standard computation of classical self-excited periodic oscillations

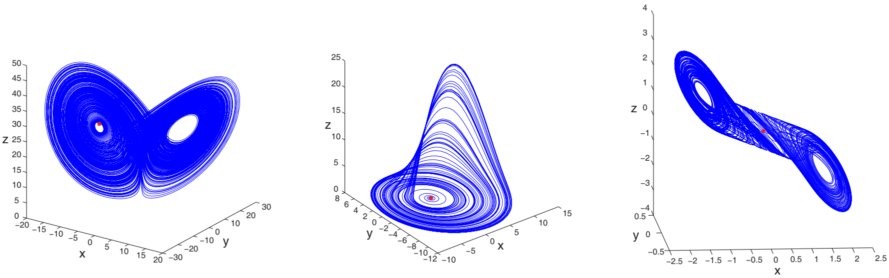

Fig. 2 Standard computation of classical self-excited chaotic attractors

3 Hidden Attractor Localization

While classical attractors are self-exited attractors therefore can be obtained numerically by the standard computational procedure, for localization of hidden attractors it is necessary to develop special procedures, since there are no similar transient processes leading to such attractors. At first, the problem of investigating hidden oscillations arose in the second part of Hilbert's 16th problem (1900) on the number and possible dispositions of limit cycles in two-dimensional polynomial systems (see, e.g., [46] and authors' works [34, 23, 42, 36, 43, 21, 32]). The the problem is still far from being resolved even for a simple class of quadratic systems.

Later, the problem of analyzing hidden oscillations arose from engineering problems in automatic control. In 50s of the last century in M.Kapranov's works [17] on stability of phase locked-loops (PLL) systems, widely used nowadays in telecommunications and computer architectures, the qualitative behavior of systems was studied and the estimate of stability domain was obtained. In these investigations Kapranov assumed that in PLL systems there were self-excited oscillations only. However, in 1961, N.Gubar' [32] revealed a gap in Kapranov's work and showed analytically the possibility of the existence of hidden oscillations in two-dimensional system of phase locked-loop: thus from the computational point of view the system considered was globally stable (all the trajectories tend to equilibria), but, in fact, there was a bounded domain of attraction only.

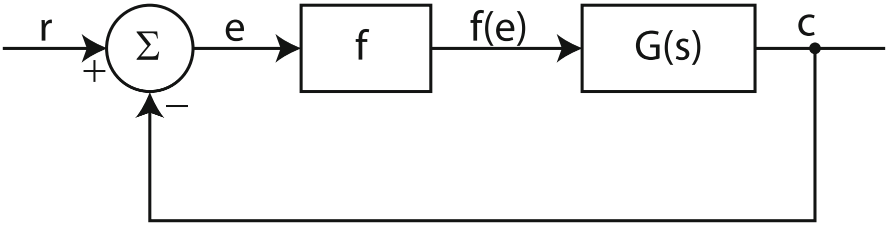

Fig. 3 Nonlinear control system. $G(s)$ is a linear transfer function, $f(e)$ is a single-valued, continuous, and differentiable function [16].

In 1957 year R.E. Kalman stated the following [16]: "*If $f(e)$ in Fig. 1* [see Fig. 3] *is replaced by constants K corresponding to all possible values of $f'(e)$, and it is found that the closed-loop system is stable for all such K, then it intuitively clear that the system must be monostable; i.e., all transient solutions will converge to a unique, stable critical point.*" Kalman's conjecture is a strengthening of Aizerman's conjecture [2], in which in place of condition on the derivative of nonlinearity it is required that the nonlinearity itself belongs to linear sector.

In the last century the investigations of widely known Aizerman's, and Kalman's conjectures on absolute stability have led to the finding of hidden oscillations in automatic control systems with a unique stable stationary point and with a nonlinearity, which belongs to the sector of linear stability (see, e.g., [31, 30, 6, 37, 7, 24, 38, 41, 32]).

The generalization of Kalman's conjecture to multidimensional nonlinearity is known as Markus-Yamabe conjecture [48] (which is also proved to be false [12]).

At the end of the last century the difficulties the difficulties of numerical analysis of hidden oscillations arose in simulations of aircraft's control systems (anti-windup) and drilling systems which caused crashes [26, 29, 8, 18, 33, 32].

Further investigations on hidden oscillations were greatly encouraged by the present authors' discovery, in 2009-2010 (for the first time), of *chaotic hidden attractor* in Chua's circuits (simple electronic circuit with nonlinear feedback) [25, 44, 7].

Until recently, only self excited attractors have been found in Chua circuits. Note that L. Chua himself, in analyzing various cases of attractors existence in Chua's circuit [10], does not admit the existence of hidden attractor (discovered later) in his circuits. Now, it is shown that Chua's circuit and its various modifications [44, 45, 20] can exhibit hidden chaotic attractors (see, Fig.4, b), Fig.5) with positive largest Lyapunov exponent[22, 35][1].

[1] While there are known effects of the largest Lyapunov exponent (LE) sign inversion ([35, 22]) for nonregular time-varying linearizations, computation of Lyapunov exponents for linearization of nonlinear autonomous system along non stationary trajectories is widely used for investigation of chaos, where positiveness of the largest Lyapunov exponent is often considered as indication of chaotic behavior in considered nonlinear system.

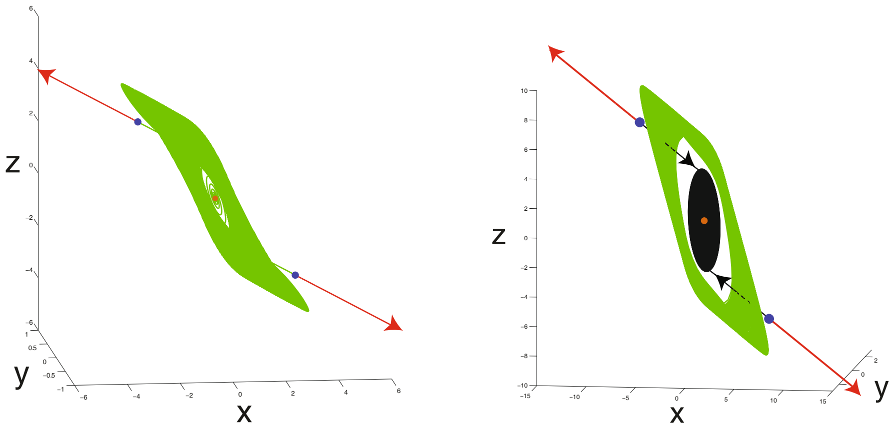

Fig. 4 a) Self-excited and b) Hidden Chua attractor with similar shapes

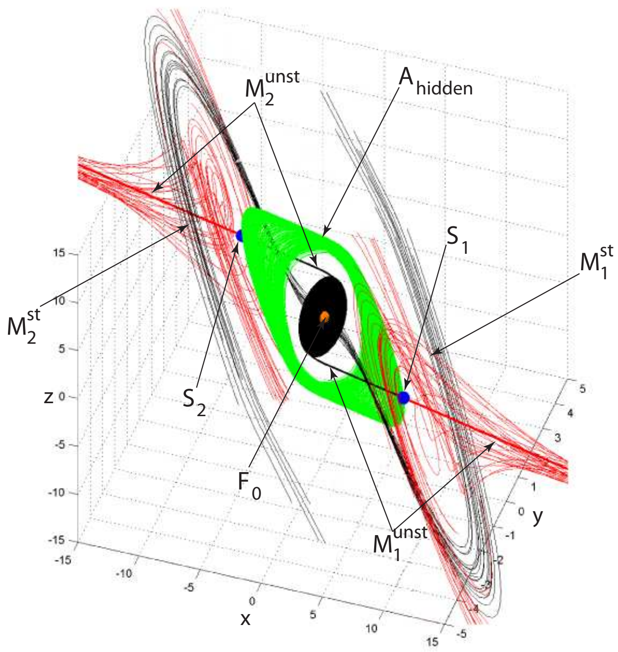

Fig. 5 Hidden chaotic attractor (green domain) in classical Chua circuit: locally stable zero equilibrium F_0 attracts trajectories (black) from stable manifolds $M^{st}_{1,2}$ of two saddle points $S_{1,2}$; trajectories (red) from unstable manifolds $M^{unst}_{1,2}$ tend to infinity

4 Conclusion

Since one cannot guarantee revealing complex oscillations regime by linear analysis and standard simulation, rigorous nonlinear analysis and special numerical methods should be used for investigation of nonlinear dynamical systems.

It was found [28, 38, 24, 39, 41, 32] that the effective methods for the numerical localization of hidden attractors in multidimensional dynamical systems are the methods based on special modifications of *describing function method*[2] and *numerical continuation*: it is constructed a sequence of similar systems such that for the first (starting) system the initial data for numerical computation of possible oscillating solution (starting oscillation) can be obtained analytically and then the transformation of this starting oscillation when passing from one system to another is followed numerically.

Also some recent examples of hidden attractors can be found in [63, 60, 59, 58, 61, 62, 1, 15].

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[2] In engineering practice for the analysis of the existence of periodic solutions it is widely used classical harmonic linearization and describing function method (DFM). However these classical method is not strictly mathematical reasonable and can lead to untrue results (e.g., DFM proves validity of Aizerman's and Kalman's conjectures on absolute system if control system, while counterexample with hidden oscillation are well known).

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