A Recognition Algorithm and Some Optimization Problems on Weakly Quasi-Threshold Graphs

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Abstract Graph theory provides algorithms and tools to handle models for important applications in medicine, such as drug design, diagnosis, validation of graph-theoretical methods for pattern identification in public health datasets. In this chapter we characterize weakly quasi-threshold graphs using the weakly decomposition, determine: density and stability number for weakly quasi-threshold graphs.

1 Introduction

The well-known class of cographs is recursively defined by using the graph operations of 'union' and 'join' [1]. Bapat et al. [2], introduced a proper subclass of cographs, namely the class of weakly quasi-threshold graphs, by restricting the join operation. The class of cographs coincides with the class of graphs having no induced P_4 [3]. Trivially-perfect graphs, also known as quasi-threshold graphs, are characterized as the subclass of cographs having no induced C_4, that is, such graphs are $\{P_4, C_4\}$-free graphs, and are recognized in linear time [4, 5]. Another subclass of cographs are the $\{P_4, C_4, 2K_2\}$-free graphs known as threshold graphs, for which there are several linear-time recognition algorithms [4, 5]. Every threshold graph is trivially-perfect but the converse is not true.

When searching for recognition algorithms, frequently appears a type of partition for the set of vertices in three classes A, B, C, which we call a *weakly decomposition*, such that: A induces a connected subgraph, C is totally adjacent to B, while C and A are totally nonadjacent.

The structure of the chapter is the following. In Sect. 2 we present the notations to be used, in Sect. 3 we give the notion of weakly decomposition and in Sect. 4

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we give a recognition algorithm and determine the clique number, the stability number on weakly quasi-threshold graphs.

2 General Notations

Throughout this chapter, $G = (V, E)$ is a connected, finite and undirected graph, without loops and multiple edges [6], having $V = V(G)$ as the vertex set and $E = E(G)$ as the set of edges. $\overline{G}$ is the complement of G. If $U \subseteq V$, by $G(U)$ we denote the subgraph of G induced by U. By $G - X$ we mean the subgraph $G(V - X)$, whenever $X \subseteq V$, but we simply write $G - v$, when $X = \{v\}$. If $e = xy$ is an edge of a graph G, then x and y are adjacent, while x and e are incident, as are y and e. If $xy \in E$, we also use $x \sim y$, and $x \nsim y$ whenever x, y are not adjacent in G. A vertex $z \in V$ distinguishes the non-adjacent vertices $x, y \in V$ if $zx \in E$ and $zy \notin E$. If $A, B \subset V$ are disjoint and $ab \in E$ for every $a \in A$ and $b \in B$, we say that A, B are *totally adjacent* and we denote by $A \sim B$, while by $A \nsim B$ we mean that no edge of G joins some vertex of A to a vertex from B and, in this case, we say that A and B are *non-adjacent*.

The *neighbourhood* of the vertex $v \in V$ is the set $N_G(v) = \{u \in V : uv \in E\}$, while $N_G[v] = N_G(v) \cup \{v\}$; we simply write $N(v)$ and $N[v]$, when G appears clearly from the context. The neighbourhood of the vertex v in the complement of G will be denoted by $\overline{N}(v)$.

The neighbourhood of $S \subset V$ is the set $N(S) = \cup_{v \in S} N(v) - S$ and $N[S] = S \cup N(S)$. A *clique* is a subset Q of V with the property that $G(Q)$ is complete. The *clique number* or *density* of G, denoted by $\omega(G)$, is the size of the maximum clique. A clique cover is a partition of the vertices set such that each part is a clique. $\theta(G)$ is the size of a smallest possible clique cover of G; it is called the *clique cover number* of G. A stable set is a subset X of vertices where every two vertices are not adjacent. $\alpha(G)$ is the number of vertices is a stable set o maximum cardinality; it is called the *stability number* of G. $\chi(G) = \omega(\overline{G})$ and it is called *chromatic number*.

By P_n, C_n, K_n we mean a chordless path on $n \geq 3$ vertices, a chordless cycle on $n \geq 3$ vertices, and a complete graph on $n \geq 1$ vertices, respectively.

A graph is called *cograph* if it does not contain P_4 as an induced subgraph.

Let $\mathscr{F}$ denote a family of graphs. A graph G is called *$\mathscr{F}$-free* if none of its subgraphs is in F.

3 Preliminary Results

3.1 Weakly Decomposition

At first, we recall the notions of weakly component and weakly decomposition.

Definition 1 [7–9] A set $A \subset V(G)$ is called a weakly set of the graph G if $N_G(A) \neq V(G) - A$ and $G(A)$ is connected. If A is a weakly set, maximal with respect to set inclusion, then $G(A)$ is called a weakly component. For simplicity, the weakly component $G(A)$ will be denoted with A.

Definition 2 [7–9] Let $G = (V, E)$ be a connected and non-complete graph. If A is a weakly set, then the partition $\{A, N(A), V - A \cup N(A)\}$ is called a weakly decomposition of G with respect to A.

Below we remind a characterization of the weakly decomposition of a graph. The name of "*weakly component*" is justified by the following result.

Theorem 1 [8–10] *Every connected and non-complete graph* $G = (V, E)$ *admits a weakly component* A *such that* $G(V - A) = G(N(A)) + G(\overline{N}(A))$.

Theorem 2 [8, 9] *Let* $G = (V, E)$ *be a connected and non-complete graph and* $A \subset V$. *Then* A *is a weakly component of* G *if and only if* $G(A)$ *is connected and* $N(A) \sim \overline{N}(A)$.

The next result, that follows from Theorem 1, ensures the existence of a weakly decomposition in a connected and non-complete graph.

Corollary 1 *If* $G = (V, E)$ *is a connected and non-complete graph, then* V *admits a weakly decomposition* (A, B, C), *such that* $G(A)$ *is a weakly component and* $G(V - A) = G(B) + G(C)$.

Theorem 2 provides an $O(n + m)$ algorithm for building a weakly decomposition for a non-complete and connected graph.

Algorithm for the weakly decomposition of a graph ([10])
Input: A connected graph with at least two nonadjacent vertices, $G = (V, E)$.
Output: A partition $V = (A, N, R)$ such that $G(A)$ is connected, $N = N(A)$, $A \not\sim R = \overline{N}(A)$.
begin
    A := any set of vertices such that
    $A \cup N(A) \neq V$
    $N := N(A)$
    $R := V - A \cup N(A)$
    while ($\exists n \in N$, $\exists r \in R$ such that $nr \notin E$) *do*
        begin
            $A := A \cup \{n\}$
            $N := (N - \{n\}) \cup (N(n) \cap R)$
            $R := R - (N(n) \cap R)$
        end
end

In [7] we give:

Let $G = (V, E)$ *be a connected graph with at least two nonadjacent vertices and (A,N,R) a weakly decomposition, with A the weakly component. G is a* P_4-free graph if and only if:

(1) $A \sim N \sim R$;
(2) *G(A), G(N), G(R)* are P_4*-free graph.*

3.2 Weakly Quasi-Threshold Graphs

In this subsection we remind some results on weakly quasi-threshold graphs.

A *cograph* which is C_4-free is called a *quasi-threshold* graph.

In [2] we study the class of weakly quasi-threshold graphs that are obtained from a vertex by recursively applying the operations (1) adding a new isolated vertex, (2) adding a new vertex and making it adjacent to all old vertices, (3) disjoint union of two old graphs, and (4) adding a new vertex an making it adjacent to all neighbours of an old vertex.

Let $G = (V, E)$ be a graph. Define a relation on V [2] as follows: Let $u, v \in V$. Then $u \equiv v$ if $N(u) = N(v)$. We observe that $\equiv$ is an equivalence relation and the equivalence classes are stable sets in G.

Let G be a graph with $Q_1, \ldots, Q_k$ as the equivalence classes under the relation $\equiv$. For each $i = 1, \ldots, k$ choose a vertex $u_i \in Q_i$. We call the subgraph $\widetilde{G}$ of G induced by $u_1, \ldots, u_k$ as a subgraph of representatives of G.

Let G be a graph. Then G is weakly quasi-threshold [2] if an only if a subgraph of representatives is quasi-threshold.

Let $G = (V, E)$ be a connected graph. Then the following are equivalent [2]:

(1) G is a weakly quasi-threshold
(2) G ia a P_4-free and there is no induced $C_4 = [v_1, v_2, v_3, v_4]$ with $N(v_1) \neq N(v_3)$ and $N(v_2) \neq N(v_4)$.

A graph G is weakly quasi-threshold [11] if and only if G does not contain any P_4 or $co-(2P_3)$ as induced subgraphs.

4 New Results on Threshold Graphs

4.1 Characterization of a Weakly Quasi-Threshold Graph Using the Weakly Decomposition

In this paragraph we give a new characterization of weakly quasi-threshold graphs using the weakly decomposition.

Theorem 3 *Let* $G = (V, E)$ *be a connected graph with at least two nonadjacent vertices and (A, N, R) a weakly decomposition, with A the weakly component. G is a weakly quasi-threshold graph if and only if:*

(1) $A \sim N \sim R$;
(2) $G(N)$ is $\overline{P}_3$-*free graph*;
(3) $G(A \cup N)$, $G(N \cup R)$ *are weakly quasi-threshold graphs.*

Proof Let $G = (V, E)$ be a connected, uncomplete graph and (A, N, R) a weakly decomposition of G, with $G(A)$ as the weakly component.

At first, we assume that G is weakly quasi-threshold. Then G is P_4-free. So, $A \sim N \sim R$. Because G is weakly quasi-threshold graph it follows that $G(A \cup N)$, $G(N \cup R)$ are weakly quasi-threshold graphs. We suppose that $G(N)$ contain $\overline{P}_3 = (\{a, b, c\}, \{ac\})$ as induced subgraph. Because $G(A)$ is connected $\exists x, y \in A$ such that $xy \in E$. Because $A \nsim R$, $\forall z \in R$, $G(\{x, y, z\}) \simeq \overline{P}_3$. Because $N \sim A \cup R$, $G(\{a, y, a, b, c, z\}) \simeq co - (2P_3)$, in contradicting with G is weakly quasi-threshold graph.

Conversely, we suppose that (1), (2) and (3) hold. From (3), $G(A)$, $G(N)$, $G(R)$ are P_4-free. Because (1) hold, G is P_4-free. $G(A)$, $G(N)$, $G(R)$ are $\{co - (2P_3)\}$-free because (3) hold. $G(A \cup R)$ is $\{co - (2P_3)\}$-free because $A \nsim R$ and $\{co - (2P_3)\}$ is connected. We suppose that G contain $H = \{co - (2P_3)\}$ as induced subgraph such that $V(H) \cap A \neq \emptyset$, $V(H) \cap N \neq \emptyset$ and $V(H) \cap R \neq \emptyset$. Because (1) hold, $N \sim (A \cup R)$. The unique $S \subset V$ totally adjacent with $V(H) - S$, ($S \sim V(H) - S$), is S with $S = V(\overline{P}_3)$. Then $G(N)$ contain $\overline{P}_3$ as induced subgraph, contradicting (2). So, G is $\{co - (2P_3)\}$-free. So, G is weakly quasi- threshold graph.

4.2 Determination of Clique Number and Stability Number for a Weakly Quasi-Threshold Graph

In this paragraph we determine the stability number and the clique number for weakly quasi-threshold graphs.

Proposition 1 *If $G = (V, E)$ is a connected graph with at least two nonadjacent vertices and (A, N, R) a weakly decomposition with A the weakly component then*

$$\alpha(G) = max\{\alpha(G(A)) + \alpha(G(\overline{N}(A))), \alpha(G(A \cup N(A)))\}.$$

Proof Indeed, every stable set of maximum cardinality either intersects $\overline{N}(A)$ and in this case the cardinal is $\alpha(G(A)) + \alpha(G(\overline{N}(A)))$ or it does not intersect $\overline{N}(A)$ and has the cardinal $\alpha(G(A \cup N(A)))$.

Theorem 4 *Let $G = (V, E)$ be connected with at least two non-adjacent vertices and (A, N, R) a weakly decomposition with A the weakly component. If G is a weakly quasi-threshold graph then*

$$\alpha(G) = \alpha(G(A)) + max\{\alpha(G(N)), \alpha(G(R))\}$$

and

$$\omega(G) = \omega(G(N)) + max\{\omega(G(A)), \omega(G(R))\}.$$

Proof Because $A \sim N$, from Proposition 1, it follows that

$$\alpha(G) = \alpha(G(A)) + max\{\alpha(G(N)), \alpha(G(R))\}.$$

Because $A \sim N \sim R$, it follows that

$$\omega(G) = \omega(G(N)) + max\{\omega(G(A)), \omega(G(R))\}.$$

5 Conclusions and Future Work

In this chapter we characterize weakly quasi-threshold graphs using the weakly decomposition, determine: density and stability number for weakly quasi-threshold graphs. Our future work concerns we give some applications of weakly quasi-threshold graphs including the medicine. Also we will explore the connection of weakly quasi-threshold graphs with the intelligent systems.

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