8 On a Compliant Mechanism Design Methodology Using the Synthesis with Compliance Approach for Coupled and Uncoupled Systems

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Abstract. Compliant mechanisms are defined as those that gain some or all of their mobility from the flexibility of their members. Suitable use of pseudo-rigid-body models for compliant segments, and state-of-the-art knowledge of rigid-body mechanism synthesis types, greatly simplifies the design of compliant mechanisms. Starting with a pseudo-rigid-body four-bar mechanism, with one to four torsional springs located at the revolute joints to represent mechanism characteristic compliance, a simple, heuristic approach is provided to develop various compliant mechanism types. The *synthesis with compliance* method is used for three, four and five precision positions, with consideration of one to four torsional springs, to develop design tables for standard mechanism synthesis types. These tables reflect the mechanism compliance by specification of either energy or torque. The approach, while providing credible solutions, experiences some limitations.

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The method is not yet robust, and research is continuing to further improve it. Examples are presented to demonstrate the use of weakly or strongly coupled sets of kinematic and energy/torque equations, as well as different compliant mechanism types in obtaining solutions.

1 Nomenclature

$\mathbf{Z}_n$	Vector notation of link n
R_n	Magnitude of $\mathbf{Z}_n$
θ_n	Angle of $\mathbf{Z}_n$, measured ccw from x-axis
P_j	j^{th} precision point
δ_j	Vector from first to j^{th} precision point
ϕ_j	Rotation of the input link from first to j^{th} position
γ_j	Rotation of the coupler from first to j^{th} position
ψ_j	Rotation of the output link from first to j^{th} position
E_j	Energy of the mechanism at j^{th} precision position
k_i	Spring constant of the i^{th} torsional spring
β_{ij}	j^{th} angular position of the i^{th} torsional spring
T_j	Torque specified at j^{th} precision position
h_{ij}	First-order kinematic coefficient of the i^{th} link at j^{th} position
K_θ	Characteristic stiffness coefficient
γ	Characteristic radius factor
Θ	Pseudo-rigid-body angle

2 Introduction

Methods developed in recent times for synthesizing compliant mechanisms comprise numerical synthesis [1, 2], a systematic application of structural optimization [3-6], graphical synthesis [7], loop closure [8, 9], and homotopy technique [10]. In using the pseudo-rigid-body model concept, loop closure offers invaluable benefits, such as use of existing knowledge base in rigid-body mechanism synthesis [11], generation of multiple solutions, and expediency of solution with accuracy. This paper describes a methodology to synthesize compliant mechanisms using the pseudo-rigid-body model concept and the loop-closure technique, while taking into account the mechanism's non-prescribed energy-free state.

The pseudo-rigid-body model concept [12, 13], which readily accommodates large deflection of flexible members, naturally bridges rigid-body synthesis to compliant mechanism design, providing the greatest benefit of all. For each flexible member (segment), a derived equivalent pseudo-rigid-body model predicts its deflection path and force-deflection characteristics. These segments are modeled by two or more rigid links attached at pin joints. A torsional spring, located at a pin joint, is used to model the force-deflection relationships of a compliant segment accurately.

A large-deflection cantilevered compliant (fixed-free) beam of length l and its equivalent pseudo-rigid-body model are shown in Figs. 1(a) and 1(b), respectively. It is assumed that the nearly circular path of the beam end can be modeled by two rigid links joined at a "characteristic" pivot along the beam [14]. A torsional spring at the pivot represents the beam's resistance to deflection. The stiffness coefficient K_θ is related to the torsional spring constant, k, of the beam. The location of this characteristic pivot is measured from the beam end as a fraction of the beam's length, γl, where γ is the characteristic radius factor [15, 16]. This distance γl is also known as the characteristic radius.

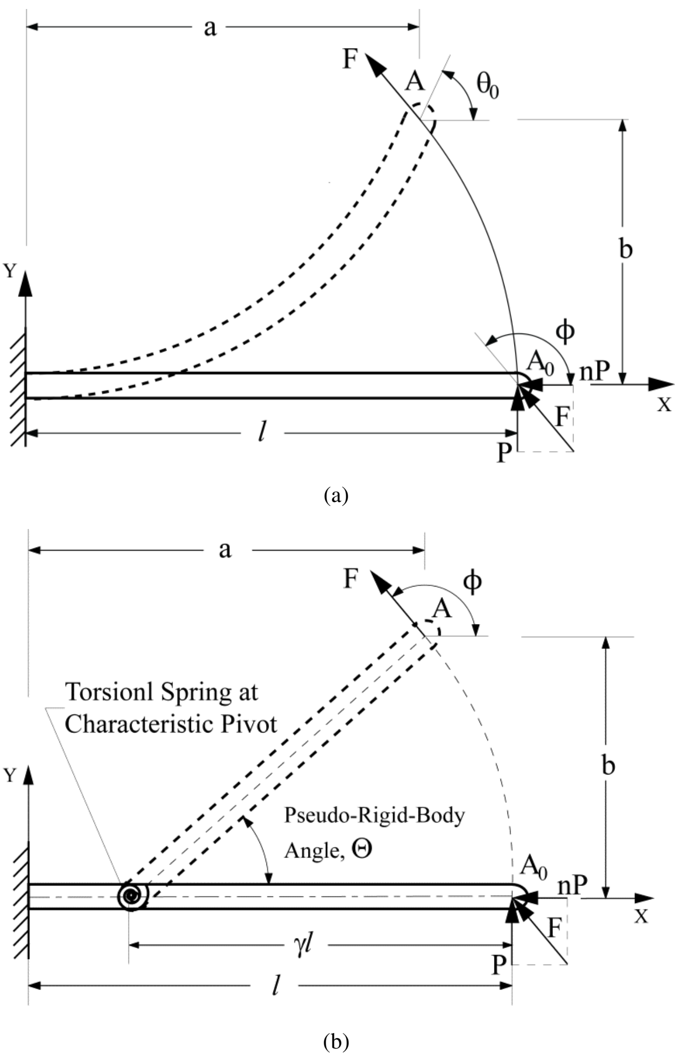

Fig. 1 (a) A cantilever beam with a force at the beam end, and (b) its pseudo-rigid-body model

The average value of the characteristic radius factor is found to be 0.85 [15], and may be used as a preliminary estimate in problem solving. The angle by which the characteristic radius is rotated is referred to as the pseudo-rigid-body angle, Θ. This concept, along with existing rigid-body mechanism theories for function, path, and motion generation, and path generation with prescribed timing [11], can be used advantageously to synthesize compliant mechanisms.

A loop-closure technique was developed to synthesize compliant mechanisms by combining loop-closure equations with energy/torque relations, which reflect the mechanism compliance [9]. This technique, termed as *synthesis with compliance*, thus relates the energy storage characteristics of compliant segments to the kinematic mobility of the mechanism. Therefore, for these two sets of equations, there are two sets of unknowns: 1) the kinematic variables, consisting of link lengths and angles of the pseudo-rigid-body model, corresponding to some select positions known as precision positions; and 2) the energy variables, composed of the undeflected spring position β_{i0}, related to the initial pseudo-rigid-body angle Θ_{n0}, and the spring stiffness k_i, related to the characteristic stiffness coefficient K_Θ for each compliant segment (Fig. 2).

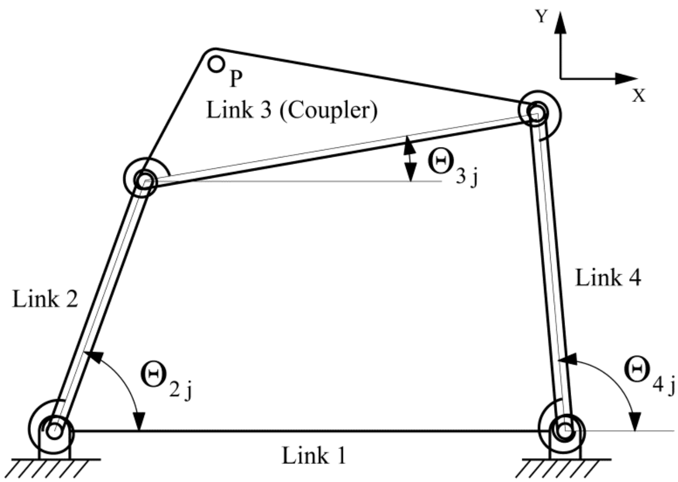

Fig. 2 Schematic of a pseudo-rigid-body four-bar mechanism with torsional springs

Using the above reduction technique, a pseudo-rigid-body kinematic chain with discrete compliances at the characteristic pivots may be obtained. With this in mind, the basic kinematic four-bar chain is selected, with its revolute joints representing the aforementioned characteristic pivots, and the springs the segment compliances.

Depending on the compliances (or the springs) introduced *synthesis with compliance* yields a set of weakly coupled or strongly coupled equations [15]. In weakly coupled set of equations, the kinematic equations are solved independently of energy/torque equations, whereas in the latter case, both the kinematic and energy/torque equations are simultaneously solved for all the unknowns. The possible compliant mechanism configurations, with fully compliant, small-length flexural pivot [15], and rigid segments, are heuristically derived [17] and improved upon in Fig. 3. In using equivalent pseudo-rigid-body model representations for various compliant segment types [15], and assuming four torsional springs in the rigid-body four-bar mechanism (Fig. 2), three possible compliant mechanisms (Figs. 3A-C) may be conceptualized. Similarly, five mechanism types result (Figs. 3D-H) from use of three springs, eight (Figs. 3I-P) from two springs, and two (Figs. 3Q, R) from a single spring, giving a total of 18 possible configuration types for solution. It is from these types that we shall draw upon for later examples.

3 Synthesis with Compliance for Energy Specifications

3.1 Kinematic Equations

In Function generation [11], the vector loop closure $\mathbf{Z}_2 \rightarrow \mathbf{Z}_3 \rightarrow \mathbf{Z}_4 \rightarrow \mathbf{Z}_{4j} \rightarrow \mathbf{Z}_{3j} \rightarrow \mathbf{Z}_{2j}$ (Fig. 4) gives the following equation:

$$\mathbf{Z}_2(1 - e^{i\phi_j}) + \mathbf{Z}_3(1 - e^{i\gamma_j}) + \mathbf{Z}_4(e^{i\psi_j} - 1) = 0 \tag{1}$$

where, j is the mechanism position.

For path generation, motion generation (rigid-body guidance), and path generation with prescribed timing [11], the loop-closure equations (2) and (3) are obtained for loops $\mathbf{Z}_2 \rightarrow \mathbf{Z}_5 \rightarrow \delta_j \rightarrow \mathbf{Z}_{5j} \rightarrow \mathbf{Z}_{2j}$ and $\mathbf{Z}_4 \rightarrow \mathbf{Z}_6 \rightarrow \delta_j \rightarrow \mathbf{Z}_{4j} \rightarrow \mathbf{Z}_{6j}$ (Fig. 5), formed by dyads $\mathbf{Z}_2 \rightarrow \mathbf{Z}_5$ and $\mathbf{Z}_4 \rightarrow \mathbf{Z}_6$, respectively.

$$\mathbf{Z}_2(e^{i\phi_j} - 1) + \mathbf{Z}_5(e^{i\gamma_j} - 1) = \delta_j \tag{2}$$

$$\mathbf{Z}_4(e^{i\psi_j} - 1) + \mathbf{Z}_6(e^{i\gamma_j} - 1) = \delta_j \tag{3}$$

3.2 Energy Equations

The stored energy of the compliant mechanism in the j^{th} precision position is estimated [8, 18] by the potential energy stored in the torsional springs of the pseudo-rigid-body model (Fig. 2) as:

$$E_j = \frac{1}{2}\sum_{i=1}^{m} k_i\left(\beta_{ij} - \beta_{i0}\right)^2; \qquad 1 \le m \le 4 \tag{4}$$

where, k_i is the spring constant, β_{ij} the j^{th} angular position of the i^{th} torsional spring, β_{i0} the angular position of the i^{th} spring in its undeflected position, and m the number of torsional springs.

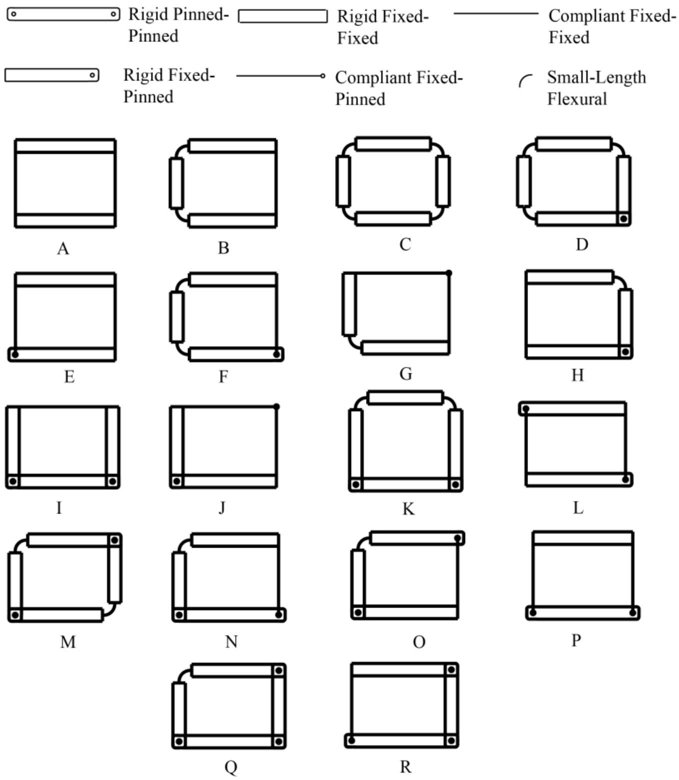

Fig. 3 Schematic representation of compliant mechanism types from a pseudo-rigid-body four-bar mechanism

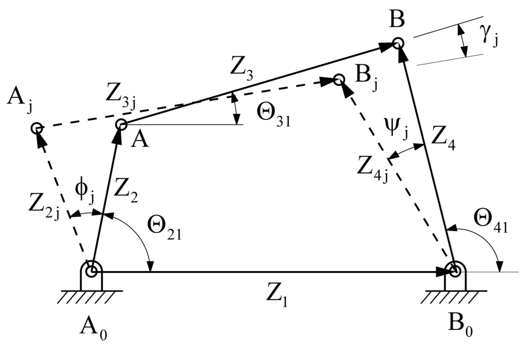

Fig. 4 Vector schematic of a four-bar function generation mechanism in its 1st and jth positions

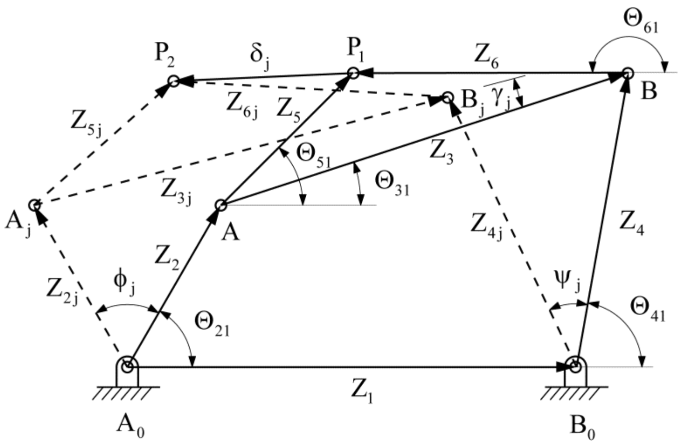

Fig. 5 Vector schematic of a four-bar mechanism showing vector dyads in the 1st and jth positions

The angle β_{ij} is related to the pseudo-rigid-body mechanism angles, Θ, [7, 8] as follows:

$$\beta_{1j} = \Theta_{2j} \tag{5a}$$

$$\beta_{2j} = 180° - (\Theta_{2j} - \Theta_{3j}) \tag{5b}$$

$$\beta_{3j} = \Theta_{4j} - \Theta_{3j} \tag{5c}$$

$$\beta_{4j} = \Theta_{4j} \tag{5d}$$

where, Θ_{nj} represents the angles of the n^{th} link at the j^{th} position. From equations (4) and (5), therefore, the mechanism potential energy in the j^{th} position is given as:

$$E_j = \frac{1}{2}\left\{\begin{array}{l} k_1\left(\Theta_{2j} - \Theta_{20}\right)^2 \\ + k_2\left[\left(\Theta_{3j} - \Theta_{30}\right) - \left(\Theta_{2j} - \Theta_{20}\right)\right]^2 \\ + k_3\left[\left(\Theta_{4j} - \Theta_{40}\right) - \left(\Theta_{3j} - \Theta_{30}\right)\right]^2 \\ + k_4\left(\Theta_{4j} - \Theta_{40}\right)^2 \end{array}\right\} \tag{6}$$

where, Θ_{n0} represents the angular position of the n^{th} link in the initial energy/torque free spring state.

Based on equations (1)-(3) and (6), Tables 1-4 outline the number of equations, unknowns, and free choices for a given number of torsional springs (m), for function, path, and motion generation, and path generation with prescribed timing, respectively. These tables encapsulate a methodology for the synthesis of an appropriate pseudo-rigid-body four-bar mechanism for the given criteria. As an example, in Table 1, for a single torsional spring (m=1) and the three-precision-position case, there are 7 equations and 10 unknowns, and hence 3 free choices, theoretically yielding solutions in the order of $(\infty)^3$ [11]. In the last column of the table, the notations s.c. and w.c. signify a strongly and weakly coupled system, respectively.

For a function generation five-precision-position synthesis, and m=1, the system is over-constrained with more equations than unknowns, and hence is excluded from Table 1. In Tables 3 and 4, the cases of five-precision-position synthesis for m=1 & 2 are not included for a similar reason. The numbers in brackets in Tables 2-4 refer to additional equations or unknowns arising from the case wherein torque (instead of energy) is specified, as explained below.

Table 1 Design choices based on number of torsional springs for *function generation* synthesis with compliance

Number of Torsional Springs	Number of Equations	Number of Unknowns		Number of Free Choices
		Three Precision Positions		
1	7	$Z_2, Z_3, Z_4, \gamma_2, \gamma_3, k_1, \beta_{10}$	(10)	3 (s.c.‡)
2	9†	$Z_2, Z_3, Z_4, \gamma_2, \gamma_3, k_1, k_2, \Theta_{20}, \Theta_{30}, \Theta_{40}$	(13)	4 (w.c.‡)
3	9†	”+ k_3	(14)	5 (w.c.)
4	9†	” + k_4	(15)	6 (w.c.)
		Four Precision Positions		
1	10	$Z_2, Z_3, Z_4, \gamma_2, \gamma_3, \gamma_4, k_1, \beta_{10}$	(11)	1 (s.c.)
2	14†	$Z_1, Z_2, Z_3, Z_4, \gamma_2, \gamma_3, \gamma_4, k_1, k_2, \Theta_{20}, \Theta_{30}, \Theta_{40}$	(16)	2 (s.c.)
3	12	$Z_2, Z_3, Z_4, \gamma_2, \gamma_3, \gamma_4, k_1, k_2, k_3, \Theta_{20}, \Theta_{30}, \Theta_{40}$	(15)	3 (w.c.)
4	12	” + k_4	(16)	4 (w.c.)
		Five Precision Positions		
2	17†	$Z_1, Z_2, Z_3, Z_4, \gamma_2, \gamma_3, \gamma_4, \gamma_5, k_1, k_2, \Theta_{20}, \Theta_{30}, \Theta_{40}$	(17)	0 (s.c.)
3	17†	$Z_1, Z_2, Z_3, Z_4, \gamma_2, \gamma_3, \gamma_4, \gamma_5, k_1, k_2, k_3, \Theta_{20}, \Theta_{30}, \Theta_{40}$	(18)	1 (s.c.)
4	15	” + k_4	(17)	2 (w.c.)

† Equation (14) gives two more scalar equations; ‡s.c.≡ strongly coupled system; w.c.≡ weakly coupled system.

4 Synthesis with Compliance for Torque Specifications

4.1 Kinematic Equations

These equations remain the same as in the case of "Energy Specifications" above.

4.2 Torque Equations

The general torque equation [8, 18] is given by

$$T_{2j} = \sum_{i=1}^{m} k_i \left(\beta_{ij} - \beta_{i0}\right) \frac{d\beta_{ij}}{dS}; \qquad 1 \le m \le 4 \tag{7}$$

where, S represents the input variable for the mechanism, and all other variables are as defined in equation (4). If Θ_2 is the input, then $d\beta_{ij}/dS$ may be expressed as:

$$\left(\frac{d\beta_1}{d\Theta_2}\right)_j = 1 \tag{8a}$$

$$\left(\frac{d\beta_2}{d\Theta_2}\right)_j = \left(\frac{d\Theta_3}{d\Theta_2}\right)_j - 1 = h_{3j} - 1 \tag{8b}$$

$$\left(\frac{d\beta_3}{d\Theta_2}\right)_j = \left(\frac{d\Theta_4}{d\Theta_2}\right)_j - \left(\frac{d\Theta_3}{d\Theta_2}\right)_j = h_{4j} - h_{3j} \tag{8c}$$

$$\left(\frac{d\beta_4}{d\Theta_2}\right)_j = \left(\frac{d\Theta_4}{d\Theta_2}\right)_j = h_{4j} \tag{8d}$$

where, h_{ij} represents the first-order kinematic coefficient of the i^{th} link at the j^{th} position, and is defined [19] as follows:

$$h_{3j} = \frac{R_2 \sin(\Theta_{4j} - \Theta_{2j})}{R_3 \sin(\Theta_{3j} - \Theta_{4j})} \tag{9}$$

$$h_{4j} = \frac{R_2 \sin(\Theta_{3j} - \Theta_{2j})}{R_4 \sin(\Theta_{3j} - \Theta_{4j})} \tag{10}$$

Substituting the values of β_{ij} in equation (7),

$$
\begin{aligned}
T_{2j} = {} & k_1\left(\Theta_{2j} - \Theta_{20}\right) \\
& + k_2\left[\left(\Theta_{3j} - \Theta_{30}\right) - \left(\Theta_{2j} - \Theta_{20}\right)\right]\left(h_{3j} - 1\right) \\
& + k_3\left[\left(\Theta_{4j} - \Theta_{40}\right) - \left(\Theta_{3j} - \Theta_{30}\right)\right]\left(h_{4j} - h_{3j}\right) \\
& + k_4\left(\Theta_{4j} - \Theta_{40}\right)h_{4j}
\end{aligned}
\tag{11}
$$

and expanding with the help of equations (9) and (10), we have

$$
\begin{aligned}
T_{2j} = {} & k_1\left(\Theta_{2j} - \Theta_{20}\right) + k_2\begin{bmatrix}\left(\Theta_{3j} - \Theta_{30}\right) - \\ \left(\Theta_{2j} - \Theta_{20}\right)\end{bmatrix}\left(\frac{R_2 \sin(\Theta_{4j} - \Theta_{2j})}{R_3 \sin(\Theta_{3j} - \Theta_{4j})} - 1\right) \\
& + k_3\begin{bmatrix}\left(\Theta_{4j} - \Theta_{40}\right) - \\ \left(\Theta_{3j} - \Theta_{30}\right)\end{bmatrix}\left(\frac{R_2 \sin(\Theta_{3j} - \Theta_{2j})}{R_4 \sin(\Theta_{3j} - \Theta_{4j})} - \frac{R_2 \sin(\Theta_{4j} - \Theta_{2j})}{R_3 \sin(\Theta_{3j} - \Theta_{4j})}\right) \\
& + k_4\left(\Theta_{4j} - \Theta_{40}\right)\frac{R_2 \sin(\Theta_{3j} - \Theta_{2j})}{R_4 \sin(\Theta_{3j} - \Theta_{4j})}
\end{aligned}
\tag{12}
$$

This equation, involving first-order kinematic coefficients, requires Θ_{3j} (yet an unknown), where j represent the j^{th} precision position of the mechanism. When j> 1, Θ_{3j} is given by $\Theta_{31} + \gamma_j$, where γ_j is the coupler position relative to the first precision position. Hence, if Θ_{31} is determined, then Θ_{3j} may be calculated. In function generation, this may either be a free choice, or be solved for explicitly from the kinematic equations. In all other synthesis methods, requiring the use of dyads, Θ_{31} is not readily available. However, it is easily obtained from the following equation (Fig. 4):

$$
\mathbf{Z}_3 - \mathbf{Z}_5 + \mathbf{Z}_6 = 0 \tag{13}
$$

Accordingly, for a strongly coupled, torque specification case, except for function generation synthesis, the number of unknowns is increased by consideration of Θ_{31} and R_3, as indicated in Tables 2-4 within brackets [21].

Table 2 Design choices based on number of torsional springs for *path generation* synthesis with compliance

Number of Torsional Springs	Number of Equations	Number of Unknowns		Number of Free Choices
		Three Precision Positions		
1	11 [+2*]	$Z_2, Z_4, Z_5, Z_6, \phi_2, \phi_3, \gamma_2, \gamma_3, \psi_2, \psi_3, k_1, \beta_{10}$	(16)[+2**]	5 (s.c.‡)
2	13	$Z_2, Z_4, Z_5, Z_6, \phi_2, \phi_3, \gamma_2, \gamma_3, \psi_2, \psi_3, k_1, k_2, \Theta_{20}, \Theta_{30}, \Theta_{40}$	(19)	6(w.c.‡)
3	13	” + k_3	(20)	7(w.c.)
4	13	” + k_4	(21)	8(w.c.)
		Four Precision Positions		
1	16 [+2*]	$Z_2, Z_4, Z_5, Z_6, \phi_2, \phi_3, \phi_4, \gamma_2, \gamma_3, \gamma_4, \psi_2, \psi_3, \psi_4, k_1, \beta_{10}$	(19)[+2**]	3(s.c.)
2	22†	$Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, \phi_2, \phi_3, \phi_4, \gamma_2, \gamma_3, \gamma_4, \psi_2, \psi_3, \psi_4, k_1, k_2, \Theta_{20}, \Theta_{30}, \Theta_{40}$	(26)	4 (s.c.)
3	18	$Z_2, Z_4, Z_5, Z_6, \phi_2, \phi_3, \phi_4, \gamma_2, \gamma_3, \gamma_4, \psi_2, \psi_3, \psi_4, k_1, k_2, k_3, \Theta_{20}, \Theta_{30}, \Theta_{40}$	(23)	5 (w.c.)
4	18	” + k_4	(24)	6 (w.c.)
		Five Precision Positions		
1	21 [+2*]	$Z_2, Z_4, Z_5, Z_6, \phi_2, \phi_3, \phi_4, \phi_5, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \psi_2, \psi_3, \psi_4, \psi_5, k_1, \beta_{10}$	(22)[+2**]	1 (s.c.)
2	27†	$Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, \phi_2, \phi_3, \phi_4, \phi_5, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \psi_2, \psi_3, \psi_4, \psi_5, k_1, k_2, \Theta_{20}, \Theta_{30}, \Theta_{40}$	(29)	2 (s.c.)
3	27†	” + k_3	(30)	3 (s.c.)
4	23	$Z_2, Z_4, Z_5, Z_6, \phi_2, \phi_3, \phi_4, \phi_5, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \psi_2, \psi_3, \psi_4, \psi_5, k_1, k_2, k_3, k_4, \Theta_{20}, \Theta_{30}, \Theta_{40}$	(27)	4 (w.c.)

∗Equation (13) contributes two more scalar equations. ∗∗$\mathbf{Z}_3$ introduces two additional unknowns. † Equation (14) gives two additional scalar equations.
‡s.c.≡ strongly coupled system; w.c.≡ weakly coupled system

5 General Synthesis Case with a Non-prescribed Energy-Free State

Let us consider a general case, where the energy-free position of the compliant mechanism is different from the prescribed positions. Currently, for this case, in a pseudo-rigid-body four-bar mechanism with more than one torsional spring, the deflection-free state of one spring does not govern the deflection-free states of the remaining springs. However, in a monolithic (one-piece) compliant mechanism, the energy-free state of one flexural segment implies that all other compliant segments are also in their energy-free states corresponding to that position. Thus, to model the compliant mechanism in an optimal way, all deflection-free angular positions of the torsional springs in a pseudo-rigid-body four-bar mechanism should be related to one another, as should be expected from the energy-free position of the compliant mechanism.

Without using additional equations, the torsional springs in their undeflected states are not yet constrained, and even though the resulting mechanism solution may be a valid pseudo-rigid-body four-bar mechanism with independent springs, it is not an acceptable one-piece compliant mechanism solution. These additional constraints will need to relate the deflection-free state angles (β_{i0}) in the energy/torque equations to one another and to the link angles of the pseudo-rigid-body four-bar mechanism.

At the energy-free position of the mechanism or its zeroth position, equation (5) relates β_{i0} of the ith torsional spring to the pseudo-rigid-body angles. Additionally, as Θ_{n0} are part of the designed pseudo-rigid-body four-bar mechanism, they need to satisfy the four-bar loop-closure equation in the energy-free state. Hence, the vector equation shown below will need to be enforced:

$$\mathbf{Z}_{20} + \mathbf{Z}_{30} = \mathbf{Z}_1 + \mathbf{Z}_{40} \tag{14}$$

where, the subscript ‘0’ represents the energy-free position. This provides additional constraints with no further unknowns appended to the system.

This additional equation (14) would suffice to satisfactorily synthesize a weakly coupled system. In a strongly coupled system, however, few more equations need to be included to ensure a satisfactory solution of the system. For a strongly coupled function generation synthesis case, with equations (1), (6) or (12), and (14) included in the system, $\mathbf{Z}_1$ is additionally an unknown. To accommodate this, the first precision position four-bar loop-closure equation, i.e.

$$\mathbf{Z}_2 + \mathbf{Z}_3 = \mathbf{Z}_1 + \mathbf{Z}_4 \tag{15}$$

is used, resulting in two more scalar equations added to the system. For the remaining three cases of strongly coupled system synthesis, $\mathbf{Z}_1$ and $\mathbf{Z}_3$ become additional unknowns. In order to solve them, the coupler loop-closure equation (13) is used in addition to equation (15). Consequently, the system accumulates four more scalar equations. The above discussion is applicable only for a pseudo-rigid-body four-bar mechanism with two or more torsional springs.

In a pseudo-rigid-body four-bar mechanism with a single spring (m = 1), a torsional spring deflection-free angle, β_{i0}, that identifies the energy-free state of the mechanism, does not impose any conditions on the other pseudo-rigid-body links that are without torsional springs. Hence, in this case, no additional constraints are required.

Table 3 Design choices based on number of torsional springs for *motion generation* synthesis with compliance

Number of Torsional Springs	Number of Equations	Number of Unknowns	Number of Free Choices
		Three Precision Positions	
1	11 [+2*]	$Z_2, Z_4, Z_5, Z_6, \phi_2, \phi_3, \psi_2, \psi_3, k_1, \beta_{10}$ (14)[+2**]	3 (s.c.‡)
2	13	$Z_2, Z_4, Z_5, Z_6, \phi_2, \phi_3, \psi_2, \psi_3, k_1, k_2, \Theta_{20}, \Theta_{30}, \Theta_{40}$ (17)	4 (w.c.‡)
3	13	” + k_3 (18)	5 (w.c.)
4	13	” + k_4 (19)	6 (w.c.)
		Four Precision Positions	
1	16 [+2*]	$Z_2, Z_4, Z_5, Z_6, \phi_2, \phi_3, \phi_4, \psi_2, \psi_3, \psi_4, k_1, \beta_{10}$ (16)[+2**]	0 (s.c.)
2	22†	$Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, \phi_2, \phi_3, \phi_4, \psi_2, \psi_3, \psi_4, k_1, k_2, \Theta_{20}, \Theta_{30}, \Theta_{40}$ (23)	1 (s.c.)
3	18	$Z_2, Z_4, Z_5, Z_6, \phi_2, \phi_3, \phi_4, \psi_2, \psi_3, \psi_4, k_1, k_2, k_3, \Theta_{20}, \Theta_{30}, \Theta_{40}$ (20)	2 (w.c.)
4	18	” + k_4 (21)	3 (w.c.)
		Five Precision Positions	
4	23	$Z_2, Z_4, Z_5, Z_6, \phi_2, \phi_3, \phi_4, \phi_5, \psi_2, \psi_3, \psi_4, \psi_5, k_1, k_2, k_3, k_4, \Theta_{20}, \Theta_{30}, \Theta_{40}$ (23)	0 (w.c.)

∗Equation (13) contributes two more scalar equations. ∗∗$\mathbf{Z}_3$ introduces two additional unknowns. † Equation (14) gives two additional scalar equations.
‡s.c.≡ strongly coupled system; w.c.≡ weakly coupled system.

Accordingly, Tables 1-4 have been updated to reflect these changes in the required number of equations, unknowns, and free choices, for different number of torsional springs and various precision position requirements [21]. Example 1 shows the application of this technique in designing a compliant mechanism with one fixed-fixed compliant segment, where three precision positions and the corresponding energies are specified.

As mentioned earlier, the above discussion assumes that the energy-free position of the compliant mechanism is different from the prescribed positions. If the energy-free position of the mechanism happens to be one of the prescribed positions, a reduced system of equations can be used to synthesize a compliant mechanism and is shown in example 2.

Table 4 Design choices based on number of torsional springs for *path generation with prescribed timing* synthesis with compliance

Number of Torsional Springs	Number of Equations	Number of Unknowns		Number of Free Choices
Three Precision Positions				
1	11 [+2*]	$Z_2, Z_4, Z_5, Z_6, \phi_2, \phi_3, \gamma_2, \gamma_3, k_1, \beta_{10}$	(14) [+2**]	3 (s.c.‡)
2	13	$Z_2, Z_4, Z_5, Z_6, \phi_2, \phi_3, \gamma_2, \gamma_3, k_1, k_2, \Theta_{20}, \Theta_{30}, \Theta_{40}$	(17)	4 (w.c.‡)
3	13	” + k_3	(18)	5 (w.c.)
4	13	” + k_4	(19)	6 (w.c.)
Four Precision Positions				
1	16 [+2*]	$Z_2, Z_4, Z_5, Z_6, \phi_2, \phi_3, \phi_4, \gamma_2, \gamma_3, \gamma_4, k_1, \beta_{10}$	(16)[+2**]	0 (s.c.)
2	22†	$Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, \phi_2, \phi_3, \phi_4, \gamma_2, \gamma_3, \gamma_4, k_1, k_2, \Theta_{20}, \Theta_{30}, \Theta_{40}$	(23)	1 (s.c.)
3	18	$Z_2, Z_4, Z_5, Z_6, \phi_2, \phi_3, \phi_4, \gamma_2, \gamma_3, \gamma_4, k_1, k_2, k_3, \Theta_{20}, \Theta_{30}, \Theta_{40}$,	(20)	2 (w.c.)
4	18	” + k_4	(21)	3 (w.c.)
Five Precision Positions				
4	23	$Z_2, Z_4, Z_5, Z_6, \phi_2, \phi_3, \phi_4, \phi_5, \gamma_2, \gamma_3, \gamma_4, \gamma_5, k_1, k_2, k_3, k_4, \Theta_{20}, \Theta_{30}, \Theta_{40}$	(23)	0 (w.c.)

*Equation (13) contributes two more scalar equations. **$\mathbf{Z}_3$ introduces two additional unknowns.
† Equation (14) gives two additional scalar equations.
‡s.c.≡ strongly coupled system; w.c.≡ weakly coupled system.

6 Examples

6.1 Example 1

It is desired to design a compliant mechanism for three-precision-position path generation with prescribed timing, with energy specified at the precision positions as follows:

δ_2 = -3 + 0.5i ; δ_3 = -5 + 0.25i; ϕ_2 = 20°; ϕ_3 = 35°; E_1 = 6.3 in-lb; E_2 = 28 in-lb; E_3 = 51.6 in-lb

Assuming two torsional springs are used in the pseudo-rigid-body four-bar mechanism, Table 4 shows there are 13 equations, 17 unknowns and 4 free choices, resulting in a weakly coupled system. Hence, the kinematic and energy variables can be solved for independently. A compliant mechanism configuration with one fixed-fixed segment, as shown in Figure 3(I), is chosen for synthesis. Four free choices are expended on R_2, R_4, θ_{21}, and θ_{41}. Using equations (2), (3), (6) and (14), the following solution is obtained:

$\mathbf{Z}_1 = 2.875 + 3.019i$; $\mathbf{Z}_2 = 0.479 + 5.479i$
$\mathbf{Z}_3 = 5.355 + 3.884i$; $\mathbf{Z}_4 = 2.958 + 6.344i$
$\mathbf{Z}_5 = 4.652 + 6.397i$; $\mathbf{Z}_6 = -0.703 + 2.537i$
$\gamma_2 = 9.286°$; $\gamma_3 = 14.613°$
$\psi_2 = 22.064°$; $\psi_3 = 36.74°$
$k_3 = 78.27$ in-lb/rad; $k_4 = 78.27$ in-lb/rad
$\Theta_{20} = 70°$ $\Theta_{30} = 24.152°$
$\Theta_{40} = 43.921°$

The length of the fixed-fixed compliant segment is determined using the equation:

$$\gamma l = |\mathbf{Z}| \tag{16}$$

where, γ is the characteristic radius factor, l the length of the compliant segment, and $|\mathbf{Z}|$ the magnitude of pseudo-rigid-body link length. The moment of inertia is obtained using the equation:

$$k = 2\gamma K_{\Theta} \frac{EI}{l} \tag{17}$$

where k is the torsional spring stiffness, K_{Θ} the stiffness coefficient, with an average value of 2.65, E the modulus of elasticity, and I the moment of inertia. Considering a rectangular cross section of width, w, and thickness, t, using Polypropylene, a thermoplastic material, and assuming the width, w to be 0.5 in., the value of the thickness, t, is obtained as 0.258 in. The resulting compliant mechanism is shown in Fig. 6.

The synthesis results obtained using the pseudo-rigid-body model (PRBM) are compared with the finite element analysis (FEA) software ABAQUS®. The coupler curve is obtained using the PRBM and precision position locations from both PRBM and FEA are shown plotted on the coupler curve in Fig. 7.

6.2 Example 2

A fully-compliant mechanism is to be designed for three-precision-position path generation with prescribed timing and energy specifications:

$\delta_2 = -3 + 0.5i$; $\delta_3 = -5 + 0.25i$; $\phi_2 = 20°$; $\phi_3 = 35°$; $E_1 = 0$ in-lb; $E_2 = 15$ in-lb; $E_3 = 44.8$ in-lb.

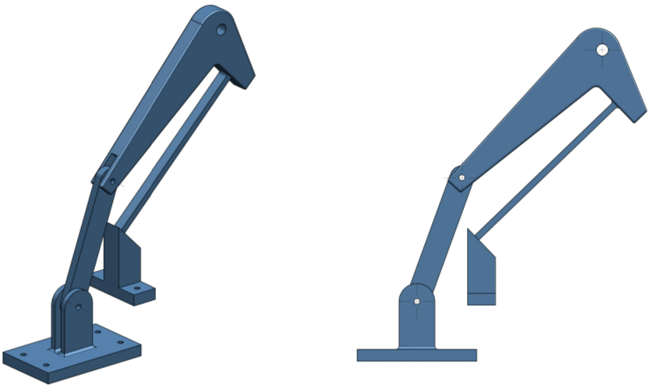

Fig. 6 Solid model of the compliant mechanism (with one fixed-fixed segment) designed in example 1

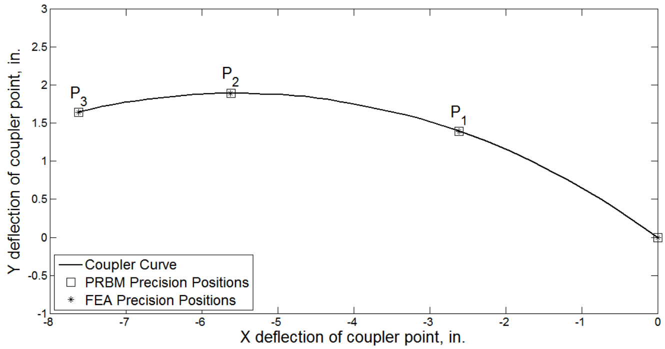

Fig. 7 Coupler curve of the mechanism (with precision positions) for example 1

In this example, the energy-free state of the mechanism is assumed to be the first precision position, and therefore the reduced system of kinematic and energy equations is used. Assuming four torsional springs are used in the pseudo-rigid-body four-bar mechanism, Table 4 shows that the resulting system is weakly coupled. A compliant mechanism configuration with two fixed-fixed compliant segments, as shown in Figure 3(A), is chosen for synthesis, and results in 10 equations, 14 unknowns, and hence, 4 free choices. Selecting R_2, Θ_{21}, R_4, and Θ_{41} as free choices, the following solution is obtained:

$\mathbf{Z}_1 = 2.876 + 3.019i$; $\mathbf{Z}_2 = 0.4794 + 5.479i$
$\mathbf{Z}_3 = 5.355 + 3.885i$; $\mathbf{Z}_4 = 2.958 + 4.110i$
$\mathbf{Z}_5 = 4.652 + 6.422i$; $\mathbf{Z}_6 = -0.703 + 2.537i$
$\gamma_2 = 9.286°$; $\gamma_3 = 14.613°$
$\psi_2 = 22.064°$; $\psi_3 = 36.74°$
$k_1 = 80.88$ in-lb/rad; $k_2 = 80.88$ in-lb/rad
$k_3 = 87.73$ in-lb/rad; $k_4 = 87.73$ in-lb/rad

The lengths and moment of inertias of the two compliant segments are determined using equations (16) and (17), respectively. The lengths of the input and output compliant segments are obtained as 6.4705 in. and 8.2350 in., respectively. The width, w, is assumed to be 0.5 in. for both segments, resulting in the thickness, t, as 0.2406 in. and 0.2679 in., for input and output segments, respectively. The resulting compliant mechanism is shown in Fig. 8.

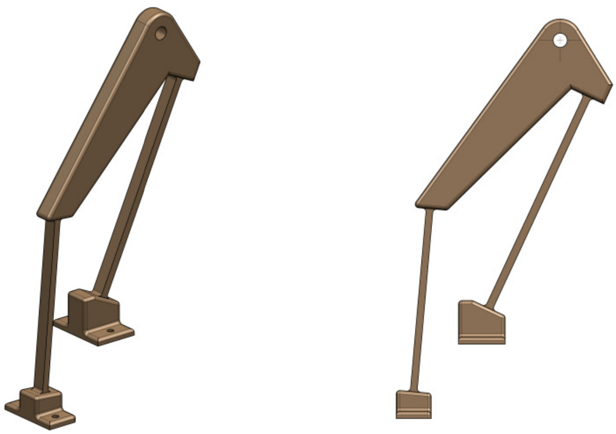

Fig. 8 Solid model of the compliant mechanism (with two fixed-fixed segments) designed in example 2

Fig. 9 shows the resulting coupler curve for this mechanism obtained using the PRBM. The precision position locations are shown plotted as obtained using the PRBM and FEA.

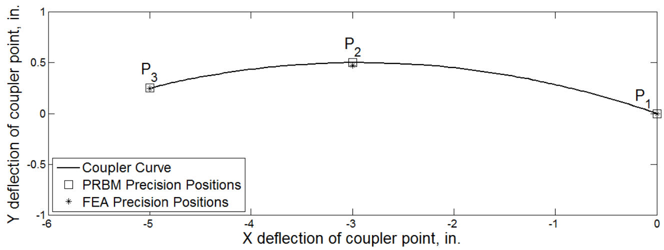

Fig. 9 Coupler curve of the mechanism (with precision positions) for example 2

6.3 Example 3

A compliant mechanism with small-length flexural pivots (SLFPs) is to be designed for three-precision-position path generation with specified energy:

$\delta_2 = -3 + 0.5i$; $\delta_3 = -5 + 0.2i$; $E_1 = 11.75$ in-lb; $E_2 = 37.25$ in-lb; $E_3 = 60$ in-lb.

Assuming four torsional springs are used in the pseudo-rigid-body four-bar mechanism; from Table 2, it can be seen that the resulting system is weakly coupled. A compliant mechanism configuration with four small-length flexural pivots, as shown in Figure 3(C), is chosen for synthesis, resulting in 13 equations, 21 unknowns, and hence, 8 free choices. Selecting R_2, Θ_{21}, R_4, Θ_{41}, γ_2, γ_2, Θ_{20}, and Θ_{30} as free choices, the following solution is obtained:

$\mathbf{Z}_1 = 3.479 - 1.537i$; $\mathbf{Z}_2 = 3 + 5.196i$
$\mathbf{Z}_3 = 5.841 - 2.233i$; $\mathbf{Z}_4 = 5.362 + 4.499i$
$\mathbf{Z}_5 = -1.464 + 3.809i$; $\mathbf{Z}_6 = -7.305 + 6.042i$
$\phi_2 = 24.473°$; $\phi_3 = 40.214°$
$\psi_2 = 22.635°$; $\psi_3 = 36.136°$
$\Theta_{40} = 8.7545°$
$k_1 = 30.688$ in-lb/rad; $k_2 = 29.0019$ in-lb/rad
$k_3 = 19.975$ in-lb/rad; $k_4 = 35.1406$ in-lb/rad

The length of SLFP is assumed to be 5% of the longer rigid segment. Thus, for the input and output pseudo-rigid-body link, the lengths of the corresponding SLFP are obtained as 0.2857 in. and 0.3333 in., respectively. The moment of inertia is calculated using the equation:

$$k = \frac{EI}{l} \tag{18}$$

Using the moment of inertias obtained, and assuming the width, w, to be 2.0 in. for all SLFPs, the thickness, t, is obtained for the four SLFPs, from bottom left to bottom right, as 0.06402 in., 0.06287 in., 0.05845 in. and 0.07056 in., respectively.

The resulting compliant mechanism is shown in Fig. 10, and Fig. 11 shows the coupler curve obtained using the PRBM for the mechanism, as well as the precision position locations.

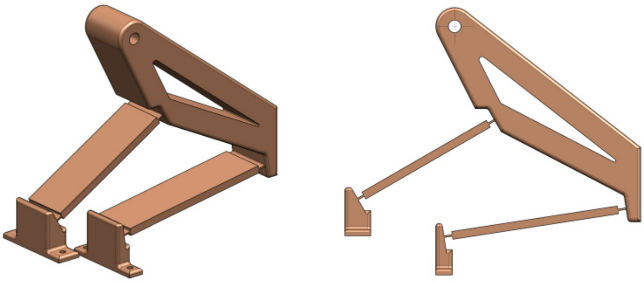

Fig. 10 Solid model of the compliant mechanism (with four small-length flexural pivots) designed in example 3

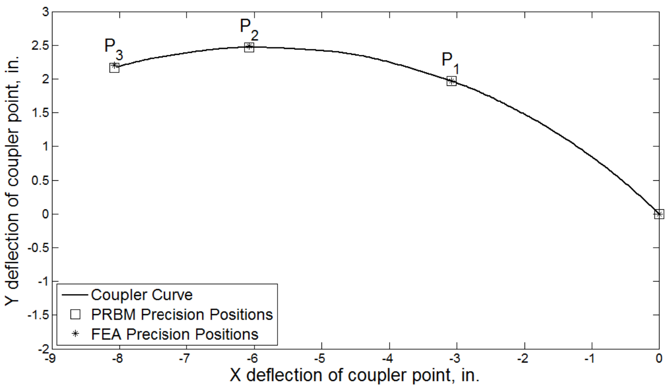

Fig. 11 Coupler curve of the mechanism (with precision positions) for example 3

7 Some Limitations Experienced in the Use of the Synthesis with Compliance Method

Though the *synthesis with compliance* technique is very useful for compliant mechanism design with energy/torque specifications, it suffers from some limitations in its current form, and are mentioned below:

This method solves the sets of kinematic and energy/torque equations as either a strongly coupled or a weakly coupled system depending on the number of unknowns shared between them. Generally, solving these coupled nonlinear kinematic and energy/torque equations presents increased complexity.

In a weakly coupled system, the kinematic and energy/torque equations are solved separately, and the kinematic configuration is solved for before solving the energy/torque equations. As a result, the latter system of equations frequently yields negative solutions for spring stiffness values. Although good solution were obtained, following a cumbersome process of iterations.

The number of variables involved in the sets of kinematic and energy equations are typically greater than number of equations available. In order to solve the equations, the user is required to assign reasonable values for the free choices and initial estimates. This process is highly cumbersome, and no guidelines currently exist to alleviate the situation.

Due to the nonlinearity of the sets of kinematic and energy/torque equations, the solutions obtained are rather sensitive to the values assigned for the free choices and initial estimates. Even the slightest changes in their values result in dramatic changes in the outcomes, which are frequently unrealistic.

In order to overcome the aforementioned limitations associated with the current state of the *synthesis with compliance* method, a more robust approach is currently being researched.

8 Conclusions

This paper is based on the use of existing concepts of equivalent pseudo-rigid-body models for compliant segments, and rigid-body mechanism synthesis for function, path and motion generation, and path generation with prescribed timing. Starting with a pseudo-rigid-body four-bar mechanism, with use of one to four torsional springs at the revolute joints to represent segment compliance, a heuristic approach is employed to develop a variety of compliant mechanism types. The *synthesis with compliance* technique has been used for a variable number of springs, for three, four and five precision positions of the mechanism. Exhaustive design tables have been systematically developed which enumerate the number of equations, unknowns and free choices for the above-mentioned synthesis types. The tables appropriately reflect these differences which result from the specification of either energy or torque. Examples have been presented to demonstrate the use of the *synthesis with compliance* method using different compliant segment types in obtaining the solutions. The results obtained are favorably compared with finite element analysis. Some insight is provided as to the limitations encountered in the method presented. Currently, the development of a more robust design methodology is underway, and will be reported in the near future.

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