6 Mobility Analysis and Type Synthesis with Screw Theory: From Rigid Body Linkages to Compliant Mechanisms

Hai-Jun Su, Lifeng Zhou, and Ying Zhang

Abstract. Mobility analysis is one of fundamental problems in kinematics and an important tool in type synthesis of linkages. In this paper, we will review screw theory as a mathematical tool for mobility analysis of overconstrained linkages and compliant mechanisms. Established by Ball in late 1800, screw theory has become one of the fundamental theories for characterizing instantaneous kinematics of spatial movements. In mid to late 1960, Waldron was one of the first modern kinematicians who systematically developed screw theory and its applications to the constraint analysis and synthesis of overconstrained linkages. Due to the screw theory, several overconstrained spatial linkages have been invented and designed, including the well known Waldron six-bar loop overconstrained linkage. In recent years, mobility analysis has been extended to compliant mechanisms which achieve motion through deflection of flexure joints. By the concept of relative compliance/stiffness, we can also define mobility of compliant mechanisms similar to their rigid body counterparts. This paper will summarize some recent work on applying screw theory to mobility analysis and synthesis of compliant mechanisms.

1 Introduction

Waldron (1966) [1] defined "mobility" or number of degrees of freedom of a mechanism as "the number of transformation parameters of joints of the mechanism which are required to determine the position of every point of every member with respect to a coordinate frame fixed to one of the members." Numerous authors have attempted to come up a general formula for calculating the mobility of general mechanisms. The most popular moblity formula is probably Kutzbach-Gruebler criterion, written as

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$$M = d(n - j - 1) + \sum_{i=1}^{j} f_i \tag{1}$$

where M is the mobility, n is the number of links, j is the number of joints, $d = 6$ for the general spatial case and $d = 3$ for planar and spherical cases, and f_i is the connectivity of the ith joint.

A screw is the geometric entity that underlies the foundation of statics and instantaneous (first-order) kinematics. Ball [2] was the first to establish a systematical formulation for screw theory. In the era of modern kinematics, a number of authors [3, 4, 5, 6] have contributed the development of screw theory and its application to analysis and design of spatial linkages. The two fundamental concepts in screw theory are "*twist*" representing a general helical motion of a rigid body about an instantaneous axis in space, and "*wrench*" representing a system of force and moment acting on a rigid body. These two concepts are often called duality [7] in kinematics and statics.

Waldron was probably one of the first modern kinematicians who applied screw theory for mobility analysis and synthesis of spatial linkages in 1960. In particular, Waldron systematically investigated and invented overconstrained linkages [1, 8]. This includes the well known Waldron six-bar overconstrained linkage [9]. Since then, screw theory has been applied to various research topics ranging from robotics [6, 10], mobility analysis [11], assembly analysis [12, 13] and topology synthesis [14] of parallel mechanisms.

In recent years, compliant mechanisms [15, 16] have received increasingly attention from the community due to their applications to precision machinery, aerospace and space structures and so on. Compliant mechanisms gain their mobility at least partially from deformation of their flexible members. Compared with their rigid body counterparts, compliant mechanisms or flexures have many advantages, such as high precision and a simplified manufacturing and assembly process due to integration of joints with rigid links. However the design and analysis of compliant mechanisms is complex due to the nonlinearity of deformation of the flexible members.

Similar to rigid body mechanisms, one important task in design of compliant mechanisms is so called "type synthesis" whose goal is to find one or more compliant mechanisms for achieving a prescribed motion pattern. As an important task of type synthesis, mobility analysis is to characterize the motion pattern for a particular compliant mechanism. Recently screw theory has been applied to mobility analysis [17, 18, 19] and type synthesis [20, 21] of compliant mechanisms. The basic principle is to first characterize freedom and constraints of flexure elements using twists and wrenches in screw theory under the assumption of ideal geometries of compliant mechanisms. For instance, a circular notch is considered as an idealized rotational joint, hence can be characterized as a pure rotational twist. Then we consider a compliant mechanism as a system of rigid bodies interconnected by these flexure elements. By applying kinematic transformation of screws, we can analyze and synthesize mobility of compliant mechanisms in a similar manner of rigid body mechanisms. In this paper, we will summarize some recent advances in this area.

2 Screw Theory Overview

In this section, we first review basic concepts of screw theory as a background preparation for the following sections.

In screw theory, an instantaneous screw motion is represented by a twist $\hat{T}$. And a constraint or forbidden motion is represented by a wrench $\hat{W}$. Both twist $\hat{T}$ and wrench $\hat{W}$ are 6 by 1 column vectors, written as

$$\hat{T} = \begin{Bmatrix} \boldsymbol{\Omega} \\ \mathbf{V} \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{\Omega} \\ \mathbf{c} \times \boldsymbol{\Omega} + p\boldsymbol{\Omega} \end{Bmatrix}, \tag{2}$$

$$\hat{W} = \begin{Bmatrix} \mathbf{F} \\ \mathbf{M} \end{Bmatrix} = \begin{Bmatrix} \mathbf{F} \\ \mathbf{c} \times \mathbf{F} + q\mathbf{F} \end{Bmatrix}, \tag{3}$$

where p and q are called pitches of twist and wrenches. And $\hat{T}$ and $\hat{W}$ satisfy the so called reciprocal condition:

$$\hat{T} \circ \hat{W} = \boldsymbol{\Omega} \cdot \mathbf{M} + \mathbf{V} \cdot \mathbf{F} = 0. \tag{4}$$

A general rotational or translational freedom respectively corresponds to a twist with zero or infinite pitch, written as

$$\hat{T}_R = \begin{Bmatrix} \boldsymbol{\Omega} \\ \mathbf{c} \times \boldsymbol{\Omega} \end{Bmatrix}, \quad \hat{T}_P = \begin{Bmatrix} \mathbf{0} \\ \mathbf{V} \end{Bmatrix}. \tag{5}$$

Similarly a general rotational or translation constraint removes a rotation or translation along a particular direction. They respectively correspond to a wrench with infinite or zero pitch, written as

$$\hat{W}_R = \begin{Bmatrix} \mathbf{0} \\ \mathbf{M} \end{Bmatrix}, \quad \hat{W}_P = \begin{Bmatrix} \mathbf{F} \\ \mathbf{c} \times \mathbf{F} \end{Bmatrix} \tag{6}$$

For convenience, we define six principal twists as the rotation and translations about all the three coordinate axes, written as

$$\begin{aligned} &\hat{R}_x = \left(1\ 0\ 0\ 0\ 0\ 0\right)^T \quad \hat{R}_y = \left(0\ 1\ 0\ 0\ 0\ 0\right)^T \quad \hat{R}_z = \left(0\ 0\ 1\ 0\ 0\ 0\right)^T \\ &\hat{P}_x = \left(0\ 0\ 0\ 1\ 0\ 0\right)^T \quad \hat{P}_y = \left(0\ 0\ 0\ 0\ 1\ 0\right)^T \quad \hat{P}_z = \left(0\ 0\ 0\ 0\ 0\ 1\right)^T \end{aligned} \tag{7}$$

Similarly, we define six principal wrenches as the rotational and translational constraint about all the three coordinate axes, written as

$$\begin{aligned} &\hat{F}_x = \left(1\ 0\ 0\ 0\ 0\ 0\right)^T \quad \hat{F}_y = \left(0\ 1\ 0\ 0\ 0\ 0\right)^T \quad \hat{F}_z = \left(0\ 0\ 1\ 0\ 0\ 0\right)^T \\ &\hat{M}_x = \left(0\ 0\ 0\ 1\ 0\ 0\right)^T \quad \hat{M}_y = \left(0\ 0\ 0\ 0\ 1\ 0\right)^T \quad \hat{M}_z = \left(0\ 0\ 0\ 0\ 0\ 1\right)^T \end{aligned} \tag{8}$$

The coordinate transformation of a twist or wrench is calculated as

$$\hat{T}' = [Ad]\hat{T}, \quad \hat{W}' = [Ad]\hat{W}, \tag{9}$$

where $\hat{T}, \hat{W}$ and $\hat{T}', \hat{W}'$ correspond to the twist and the wrench before and after the transformation. And $[Ad]$ is the so-called 6×6 adjoint matrix, written as

$$[Ad] = \begin{bmatrix} R & 0 \\ DR & R \end{bmatrix} \tag{10}$$

where $[R]$ is a 3 by 3 rotation matrix and $[D]$ is the 3 by 3 skew-symmetric matrix defined by the translational vector $\mathbf{d} = (d_x, d_y, d_z)^T$. They have the form

$$[R] = \begin{bmatrix} \mathbf{x}\ \mathbf{y}\ \mathbf{z} \end{bmatrix}, \quad [D] = \begin{bmatrix} 0 & -d_z & d_y \\ d_z & 0 & -d_x \\ -d_y & d_x & 0 \end{bmatrix}$$

3 Mobility Analysis of Overconstrained Linkages

It is well known that the Kutzbach-Gruebler formula applies to mechanisms with general dimensions and may fail for overconstrained mechanisms which gain some extra mobility due to their special dimensions. Many modern kinematicians have already contributed to generalize this formula to include various kinds of overconstrained mechanisms. For instance, in 1966, Waldron [1] proposed a formula for single loop linkages, written as

$$M = (N + n) - (m + n - k) \tag{11}$$

where N is the number of degrees of freedom of the serial chain by breaking the single loop linkage at any joint, n is the connectivity of the closing joint, and $m + n - k$ is the order of the equivalent screw system of the loop linkage.

3.1 Waldron Six-Bar Linkages

Waldron six-bar linkage also called hybrid six-bar linkage [9] is a kind of overconstrained linkage that has one mobility. It is formed by two Bennett four-bar linkages [22] sequentially connected. It is well known that a Bennett four-bar linkage is a spatial overconstrained linkage with one degree of freedom. And during the movement of the Waldron six-bar linkage, the two Bennett linkages will keep their geometrical constrains without change.

As shown in Fig. 1, the geometry of a Waldron six-bar linkage is described as the following. Let us denote the eight axes of the two Bennett linkages as $z_1 - z_8$ with the first four axes belong to the first Bennett linkage and the last four belong to the second Bennett linkage. When constructing the six-bar linkage with two four-bars linkages, z_1 coincides with z_5 while links 1, 8 are replaced by a single link 9 and links 4, 5 are replaced by link 10. As a result, links 2, 3, 10, 6, 7 and 9 are connected by six joints $z_2, z_3, z_4, z_6, z_7, z_8$ to form a single loop six-bar linkage.

The relative position of the links and joints is described by using Denavit and Hartenberg parameters. These parameters are presented as $\alpha_i, a_i, d_i, \theta_i$. α_i is the twist

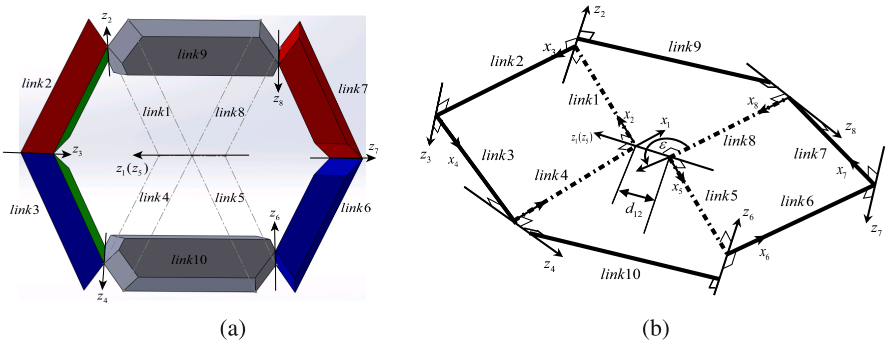


Fig. 1 The Waldron six-bar linakge is formed by two Bennett four-bar linkages

angle between the axes of z_i and z_{i+1}, a_i is the distance between z_i and z_{i+1}, d_i is the offset between links i and $i+1$ along z_i, θ_i is the angle from x_i to x_{i+1} measured about z_i. The values of those parameters are showed in Table 1. d_{12} is the distance of the two Bennett linkage along z_1, and ε is the angle between x_1 and x_8. And here in order to establish the position relationship of the two Bennett linkages clearly, we substituted the D-H parameters of z_4 to z_1 and z_1 to z_5 for z_4 to z_5. According to the geometric constraint of Bennett linkage, we have $\sin\alpha_1/a_1 = \sin\alpha_2/a_2$ and $\sin\alpha_{11}/a_{11} = \sin\alpha_{22}/a_{22}$.

Table 1 D-H parameters of the Waldron six bar linkage

i	joint i-joint j	α_i	a_i	d_i	θ_i
1	$z_1 - z_2$	α_1	a_1	0	θ_1
2	$z_2 - z_3$	α_2	a_2	0	θ_2
3	$z_3 - z_4$	α_1	a_1	0	$2\pi - \theta_1$
4	$z_4 - z_1$	α_2	a_2	0	$2\pi - \theta_2$
5	$z_1 - z_5$	0	0	d_{12}	ε
6	$z_5 - z_6$	α_{11}	a_{11}	0	θ_{11}
7	$z_6 - z_7$	α_{22}	a_{22}	0	θ_{22}
8	$z_7 - z_8$	α_{11}	a_{11}	0	$2\pi - \theta_{11}$
9	$z_8 - z_5$	α_{22}	a_{22}	0	$2\pi - \theta_{22}$

3.2 Mobility Analysis of Waldron Six-Bar Linkages

In this section, we show how to use screw theory to calculate the mobility of Waldron hybrid six-bar linkage.

Since the relative position between the first and second Bennett linkages is decided by d_{12} and ε, we only need to analyze one of two Bennett linkages and the second one could be calculated by using transformation of coordinates. Based on the Denavit and Hartenberg parameters, the transformation between the adjacent joints can be obtained easily by

$$T_i = X[\alpha_i, a_i] Z[\theta_i, d_i] \tag{12}$$

where $Z[\cdot]$ and $X[\cdot]$ represents the screw displacement along z and x axis respectively. The transformation from join z_i to the first joint can be calculated by

$${}^1_iT = T_1 T_2 \cdots T_{i-1}, \quad i = 2,3,4 \tag{13}$$

The transform matrix ${}^1_iT$ could be written as

$${}^1_iT = \begin{bmatrix} {}^1_iR & {}^1_i\mathbf{d} \\ 0 & 1 \end{bmatrix} \tag{14}$$

For the screw of z_1, we can choose it as $\$_1 = (0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0)^T$.Then the other axis vector could be calculated as $\mathbf{s}_i = {}^1_iR(0,0,1)^T$. An arbitrary point on z_i could be chosen as $\mathbf{r}_i = {}^1_i\mathbf{d}$. Then the screw of z_i will be $\$_i = (\mathbf{s}_i; \mathbf{r}_i \times \mathbf{s}_i)$.

To obtain the four screws $(\$_1^{'}, \$_2^{'}, \$_3^{'}, \$_4^{'})$ of the second Bennett linkage, we only need to replace the $\alpha_1, \alpha_2, a_1, a_2, \theta_1, \theta_2$ in $(\$_1, \$_2, \$_3, \$_4)$ by $\alpha_{11}, \alpha_{22}, a_{11}, a_{22}, \theta_{11}, \theta_{22}$ of the second Bennett linkage in the coordinate of joint 5 which are the D-H parameters of the second Bennett linkage given in Table 1. And the screws of the second Bennett linkage in the coordinate frame of joint 1 can be calculated as

$$\$_{i+4} = [Ad]\$_i^{'}, \quad i = 1,2,3,4. \tag{15}$$

where $[Ad]$ is the six by six adjoint transformation matrix of screws by substituting the following matrices

$$[R] = \begin{bmatrix} \cos(\varepsilon) & -\sin(\varepsilon) & 0 \\ \sin(\varepsilon) & \cos(\varepsilon) & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad [D] = \begin{bmatrix} 0 & -d_{12} & 0 \\ d_{12} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

into formula (10).

Finally the screw system of the hybrid six-bar linkage is obtained as $\$ = (\$_2, \$_3, \$_4, \$_6, \$_7, \$_8)$. The order N of the loop linkage is calculated as the rank of the 6 by 6 matrix formed by these six screws. By using Mathematica program, it is easy to figure out that this order is five. Since the six-bar linkage is connected by six revolute joints, the mobility of its serial chain will be 5 and the connectivity of the closing joint will be 1. Therefore, the mobility of the six bar linkage is calculated as $M = 6 - 5 = 1$ using Waldron mobility formula (11).

4 Mobility Analysis and Type Synthesis of Compliant Mechanisms

Inspired by the above work on rigid body mechanisms, we recently extended this work to mobility analysis and type synthesis of compliant mechanisms. Compliant mechanisms can be considered as a collection of relative rigid members (links) connected with flexible members (flexure joints). Compliant mechanisms gain at least part of their mobility from deformation of flexible members.

4.1 Mobility and Compliance

The mobility of a compliant mechanism is a subtle concept as virtually any material deforms more or less, hence results in movement. As we know, compliance C is defined as the ratio of movement over loading exerted for any specific direction determined by a screw $\hat{T}$. There are two kinds of compliance: rotational and translational, which have the unit of rad/Nm and $1/N$ respectively. For any member of a compliant mechanism, there are three rotational compliances and three translational compliances along the axes of coordinate system attached to that member, denoted by C_{Rx}, C_{Ry}, C_{Rz}, C_{Tx}, C_{Ty}, C_{Tz}. To compare rotational compliance with a translational one, we multiple the rotational compliances by a chosen constant l, i.e.

$$C_{tx} = C_{Rx}l, \quad C_{ty} = C_{Ry}l, \quad C_{tz} = C_{Rz}l. \tag{16}$$

The constant l can be chosen as the overall dimension of the member of interest, typically the motion stage of a compliant mechanism. Compliances C_{tx}, C_{ty}, C_{tz} represents the translation of the tip of a bar with length l that is attached to the motion stage of the mechanism when a tangent force is applied at the tip.

Now we redefine compliances of a member of a compliant mechanism as

$$C_1 = C_{tx}, \quad C_2 = C_{ty}, \quad C_3 = C_{tz}, \quad C_4 = C_{Tx}, \quad C_5 = C_{Ty}, \quad C_6 = C_{Tz}. \tag{17}$$

To define the mobility of a compliant mechanism, we introduce the concept of "compliance ratio" which is essentially the ratio of the compliance of the mechanism in a particular direction over the maximum compliance in all directions, i.e.

$$CR_i = \frac{C_i}{\max(C_i)}, \quad i = 1, \ldots 6 \tag{18}$$

The range of CR_i is between 0 and 1. If CR_i is below a specified small threshold, e.g. 0.01, we consider the mechanism has no mobility in that direction. Note this represents that the movement of the mechanism in the direction $\hat{T}_i$ is two order smaller than that in the direction with the maximum compliance when the same force is exerted. And the mobility of a compliant mechanism is counted as the number of mobility in three rotational and three translational directions.

4.2 Commonly Used Flexure Primitives

Here we first study the mobility of a list of flexure primitives commonly used in compliant mechanisms. A flexure primitive is defined as an "atomic" flexure mechanism that consists of only one flexure element and zero intermediate body. They cannot be further divided into substructures. In this section we first categorize commonly used flexure primitives and derive their freedom and constraint spaces. Then we will discuss a general synthesis methodology for constructing serial and parallel kinematic chains of these flexure primitives.

According to the mobility or the rank of their twist system, we can categorize the most commonly used flexure primitives. For instance, notch hinges, short beams and split tubes have one rotational degree of freedom. A spherical notch or short wire/rod has three rotational degrees of freedom. A thin beam or blade flexure, rotational symmetric cylinder or a disc coupling has two rotational and one translational mobility. And a long wire or corner blade has three rotational and two translational mobility. These flexure primitives and their freedom space and twist and wrench matrices are summarized in Table 2.

These primitives are basic building blocks for constructing more complex flexure systems. In what follows, we show how to build more complex mechanisms with these flexure primitives using a serial, parallel or hybrid structure.

4.3 Serial Flexure Chains

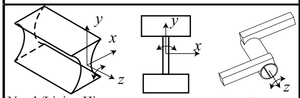
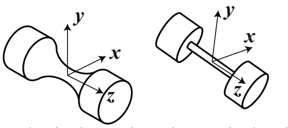
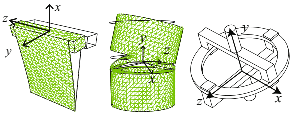
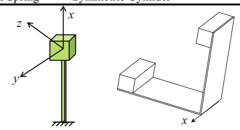
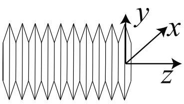
A serial flexure mechanism is formed by connecting a functional body to a fixed reference body through a serial chain of flexure elements that are joined with intermediate bodies. Let us denote the motion space of the jth flexure element in a serial flexure mechanism by a twist matrix $[T_j]$. The motion space of the rigid body constrained by this flexure system is the superimposition of the motion of individual elements. Mathematically the motion space of a serial chain of m flexures is given by the range space of the following matrix

$$[T] = [Ad_1T_1 \quad Ad_2T_2 \quad \cdots \quad Ad_mT_m]. \tag{19}$$

which is the column-wise combination of each $[T_j]$ after an appropriate coordinate transformation $[Ad_j]$. The column rank of $[T]$ gives the mobility of the functional body, denoted by $f = rank(T)$. Since it is not uncommon that the column vectors of $[T_j]$ are dependent, the mobility f is typically less than or equal to the total number of columns of $[T]$. By column reducing the matrix $[T]$, we can obtain a basis of the motion space of the flexure system. And the complementary constraint space is obtained by the standard screw algebra, denoted by a 6 by $6-f$ wrench matrix $[W]$.

Figure 2(a) shows a serial chain of two identical blade flexures. Blade flexure 2 is perpendicular to blade 1. We place the stage and its local coordinate system at the end of the second blade. The twist matrix for both blade flexures is $[T_b]$, already given in Table 2. The coordinate transformation from blade 1 to functional body is a pure translation along y axis for l units,

Table 2 The motion and constraint spaces of commonly used flexure primitives

Flexure	Freedom Symbol	$[T]$	$[W]$
 Notch/Living Hinge, Short Beam, Split Tube	R	$[\hat{R}_z]$	$[\hat{F}_x\ \hat{F}_y\ \hat{F}_z\ \hat{M}_x\ \hat{M}_y]$
 Spherical Notch, Short Wire/Rod	S=3R	$[\hat{R}_x\ \hat{R}_y\ \hat{R}_z]$	$[\hat{F}_x\ \hat{F}_y\ \hat{F}_z]$
 Blade/Sheet/ Leaf Spring, Rotational Symmetric Cylinder, Disc Coupling	B=2R-P	$[\hat{R}_x\ \hat{R}_z\ \hat{P}_y]$	$[\hat{F}_x\ \hat{F}_z\ \hat{M}_y]$
 long wire/rod, corner blade	W=3R-2P	$[\hat{R}_x\ \hat{R}_y\ \hat{R}_z\ \hat{P}_y\ \hat{P}_z]$	$[\hat{F}_x]$
 Bellow Spring	B_s=2R-3P	$[\hat{R}_x\ \hat{R}_y\ \hat{P}_x\ \hat{P}_y\ \hat{P}_z]$	$[\hat{M}_z]$

$$R_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0 & 0 & l \\ 0 & 0 & 0 \\ -l & 0 & 0 \end{bmatrix}.$$

And the transformation from the blade 2 to the functional body is pure rotation about z axis for $-\pi/2$,

$$R_2 = [Z(-\frac{\pi}{2})] = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where $[Z(\alpha)]$ represents the rotation about z axis for an angle of α.

The twist matrix of a serial chain of two blades is obtained by substituting them into (19),

$$[T_{bb}] = [Ad_1T_b \quad Ad_2T_b] = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ l & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -l & 0 & 0 & 0 \end{bmatrix}$$

$$\triangleq \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -l & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{20}$$

where the last step is a column-wise reduction process. Obviously $f = rank(T_{bb}) = 5$ as the elements of the last column are all zeros. Therefore, the flexure system provides a mobility of five degrees-of-freedom.

The corresponding reciprocal wrench matrix is

$$[W_{bb}] = [0, \quad 0, \quad 1; \quad l, \quad 0, \quad 0]^T, \tag{21}$$

which represents a constraint along a line parallel to z axis at the point **r** shown as the blue line in Fig. 2(a).

4.4 Parallel Flexure Chains

A parallel flexure mechanism is formed by connecting a functional body to a reference body through two or more flexure elements in parallel. Let us denote the

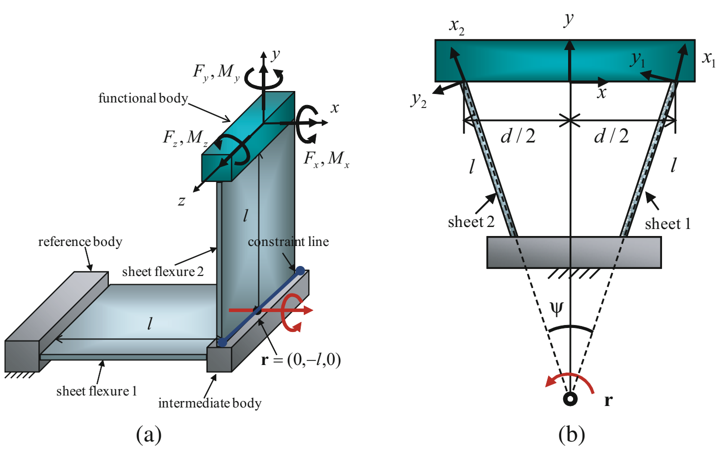


Fig. 2 (a) A serial chain compliant mechanism formed by two perpendicular blade flexures, (b) A parallel chain flexure mechanism formed by two parallel ideal blade flexures

constraint space of the jth flexure element by a wrench matrix $[W_j]$. The constraint space of the functional body is the superimposition of the constraint space of each element. Mathematically the constraint space of a parallel flexure mechanism with m flexures is given by the following wrench matrix

$$[W] = [Ad_1 W_1 \quad Ad_2 W_2 \quad \cdots \quad Ad_m W_m]. \tag{22}$$

Again matrices $[Ad_j]$ are coordinate transformation of jth flexures.

The column rank of $[W]$ gives the degree-of-constraint of the functional body, denoted by $c = rank(W)$. Similar to the case of serial chains, c is typically less than or equal to the total number of columns of $[W]$ as some column vectors are dependent. By column reducing the matrix $[W]$, we can obtain a basis of the constraint space of the flexure system. And the complementary motion space is obtained by the standard screw algebra, denoted by a 6 by $6-c$ twist matrix $[T]$.

Figure 2(b) shows a trapezoidal leaf-type flexure pivot that is formed by two identical blade flexures assembled symmetrically at an angle of ψ and a distance of d. The coordinate transformations for blade 1 and 2 are respectively

$$\begin{aligned} R_1 &= [Z(\frac{\pi-\psi}{2})], \quad \mathbf{d}_1 = (\frac{d}{2}, 0, 0), \\ R_2 &= [Z(\frac{\pi+\psi}{2})], \quad \mathbf{d}_2 = (-\frac{d}{2}, 0, 0). \end{aligned}$$

Substituting the above formula into (22) and applying a column-wise reduction, we obtain the following wrench matrix,

$$\begin{aligned} [W_t] &= [Ad_1 W_b \quad Ad_2 W_b] \\ &= \begin{bmatrix} 0 & \sin(\frac{\psi}{2}) & 0 & 0 & -\sin(\frac{\psi}{2}) & 0 \\ 0 & \cos(\frac{\psi}{2}) & 0 & 0 & \cos(\frac{\psi}{2}) & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\cos(\frac{\psi}{2}) & 0 & 0 & -\cos(\frac{\psi}{2}) \\ -\frac{d}{2} & 0 & \sin(\frac{\psi}{2}) & \frac{d}{2} & 0 & -\sin(\frac{\psi}{2}) \\ 0 & \frac{1}{2}d\cos(\frac{\psi}{2}) & 0 & 0 & -\frac{1}{2}d\cos(\frac{\psi}{2}) & 0 \end{bmatrix} \\ &\triangleq \begin{bmatrix} \sin\left(\frac{\psi}{2}\right) & 0\,0\,0\,0 \\ 0 & 1\,0\,0\,0 \\ 0 & 0\,1\,0\,0 \\ 0 & 0\,0\,1\,0 \\ 0 & 0\,0\,0\,1 \\ \frac{1}{2}d\cos\left(\frac{\psi}{2}\right) & 0\,0\,0\,0 \end{bmatrix} = \begin{bmatrix} \\ \\ \\ 0 \\ \\ \\ \end{bmatrix}, \end{aligned} \tag{23}$$

where W_b is the reciprocal wrench of T_b. Again the last step is obtained by a column-wise reduction. The corresponding reciprocal twist matrix of $[W_t]$ is

$$[T_t] = \begin{bmatrix} 0 \\ 0 \\ \sin\left(\frac{\psi}{2}\right) \\ -\frac{1}{2}d\cos\left(\frac{\psi}{2}\right) \\ 0 \\ 0 \end{bmatrix}, \tag{24}$$

which represents the rotation about the intersection line of the blades shown as $\mathbf{r} = (0, -d\cot(\psi/2)/2, 0)$ in Fig. 2(b).

4.5 Design of Freedom Elements

By using serial or parallel chains of flexure primitives as shown in Table 2, we have synthesized a catalogue of freedom and constraint elements which provide translational or rotational freedom or constraints. For convenience, we list all the possible freedom elements with one rotational (R) and translational (P) DOF motion, i.e. R-joints and P-joints, in Figs. 3 and 4.

These freedom elements are basic building blocks to construct hybrid structures of flexure mechanisms. For instance, if we would like to design a parallel structure with three rotations, we just need to use three translational constraint elements to remove all translations. As shown in Fig 5(a), We first design a serial chain of two blade flexures (denote as B-B) that functions as a single translational constraint. By combining three serial chains of B-B, we obtain a design. The functional body **A** can rotate about its center relative to the base body **B**, while its translations are constrained.

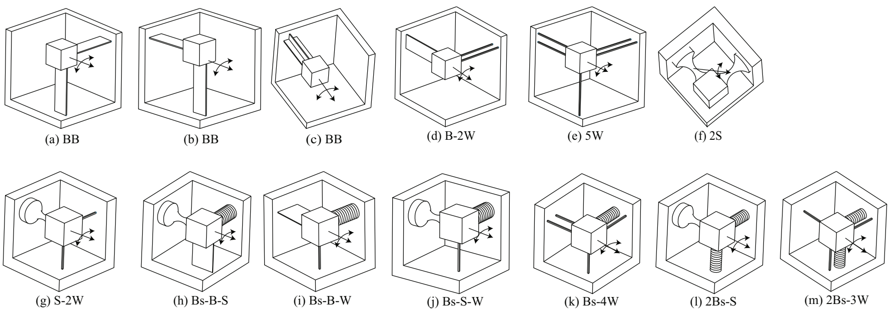


Fig. 3 Various designs of R-joints with flexure primitives: B, W, S and B_s. The double arrow arcs represent the rotation allowed by flexure R-joints. The box represents the functional body.

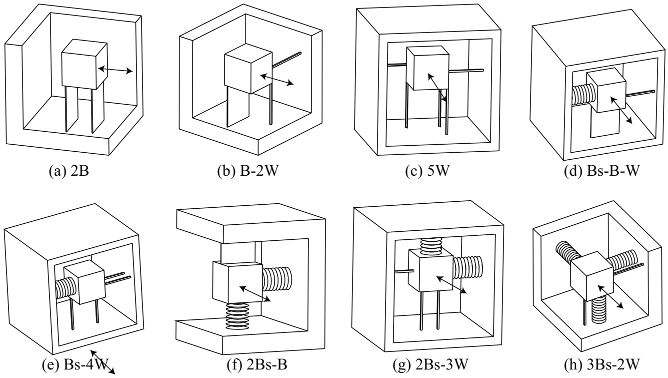


Fig. 4 Various designs of P-joints with parallel structures of flexure primitives: B, W, B_S. The arrowed lines indicate the direction of translation. The box represents the functional body.

4.6 Synthesis of Hybrid Structures

We can further build more complex flexure mechanisms with hybrid structures of flexure primitives together with the freedom and constraint elements synthesized in the previous sections. Here a hybrid structure is a structure with both serial and parallel connections.

Figure 5(a) shows a compliant parallel platform mechanism that has three rotational degrees of freedom. Each limb is a serial chain of two blade flexures. The functional body A can rotate about its center relative to the base body B while its translations are constrained.

As another example, we would like to design a parallel structure with three translational degrees of freedom. We just need to use three rotational constraint elements to remove all rotations. If we choose the BB design in for all three rotational

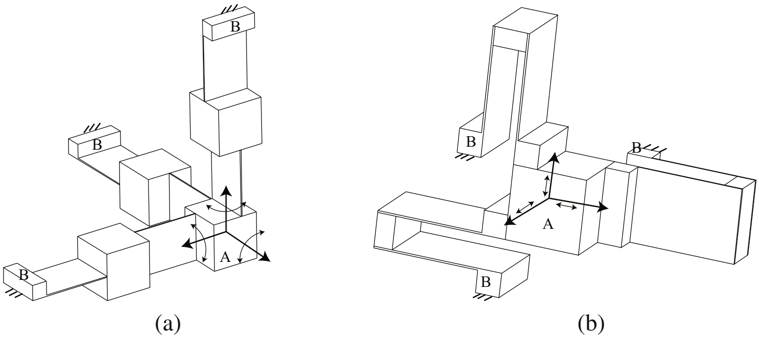


Fig. 5 (a) A parallel platform with three rotational degrees of freedom and (b) a parallel compliant platform with three translational degrees of freedom. The body B are fixed. The body A is the functional body.

constraints, we obtain the design shown in Fig 5(b). The functional body A can translate in all directions while its rotations are constrained.

5 Conclusions

In this paper, we first reviewed screw theory and its applications to mobility analysis and synthesis of rigid body linkages. In particular, we highlighted contributions of Waldron to mobility formula for general loop linkages with screw theory and synthesis of overconstrained linkages in 1960. As an example, we studied the mobility of the Waldron hybrid six-bar linkages using screw theory. Inspired by these work, we then reviewed some recent advances in applying screw theory to mobility analysis of compliant mechanisms. This screw theory based mobility formula is the foundation of mobility analysis and type synthesis of compliant mechanisms. We presented a screw theory representation of freedom and constraint spaces of commonly used flexure primitives, synthesis of R and P joints with flexure primitives and synthesis of hybrid structures such as compliant parallel platform mechanisms.

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