19 Electro-Elastic Model for Dielectric Elastomers

Rocco Vertechy, Giovanni Berselli, and Vincenzo Parenti Castelli

Abstract. A continuum model is described for the study of the electro-elastic finite deformations of dielectric elastomers. The model: i) derives directly from a global energy balance; ii) does not require the postulation of any force or stress-tensor of electrical origin; iii) only requires the knowledge of permittivity and shear moduli of the considered material; and iv) is presented in Lagrangian form which is suited for the implementation in multi-physic finite element simulation environments.

1 Introduction

Dielectric Elastomers (DEs) are incompressible solids which exhibit non-linear elastic finite deformations and linear strain-independent dielectric properties. DEs can deform in response to an applied electric field, and can alter the electric field and/or potential in response to an undergone deformation. Thanks to this Electro-Mechanical (EM) coupling, DEs are currently being investigated as transduction materials for solid-state actuators, sensors and energy harvesters that are lightweight, energy-efficient, shock-resistant and low-cost [1].

Recently, a number of continuum EM models have been proposed that can be used for the accurate analysis and optimization of DE-based devices. An extensive review is provided in [2], which also presents a continuum thermo-EM model for general isotropic modified-entropic hyperelastic dielectrics.

Rocco Vertechy
Scuola Superiore Sant'Anna, Pisa, Italy
e-mail: r.vertechy@sssup.it

Giovanni Berselli
University of Modena and Reggio Emilia, Italy
e-mail: giovanni.berselli@unimore.it

Vincenzo Parenti Castelli
University of Bologna, Italy
e-mail: vincenzo.parenti@unibo.it

V. Kumar et al. (Eds.): *Adv. in Mech., Rob. & Des. Educ. & Res.*, MMS 14, pp. 245–250.
DOI: 10.1007/978-3-319-00398-6_19

This paper describes a reduced continuum finite-deformation EM model which is suited for the analysis and finite element simulation of DEs. The validity of the considered model has already been tested in an number practical case studies [3].

Section 2 provides the statement of the problem; Section 3 defines the total EM energy of a general system comprising elastic dielectrics and conductors; Section 4 derives the balance equations, boundary conditions and constitutive relations for the considered general EM system, Section 5 specifies the constitutive relations holding for typical DE materials.

2 Problem Definition

Referring to Fig. 1, consider a closed and electrically isolated EM system $\mathcal{B}$, which comprises dielectric and conducting bodies (electrodes) that move and deform in free space under the action of externally applied loads of electro-mechanical origin. For every motion and deformation of $\mathcal{B}$: 1) no mass can enter or leave the boundary of $\mathcal{B}$; 2) energy can cross the boundary of $\mathcal{B}$ in the form of electrical and mechanical work; 3) no interaction occurs between the electrical charges that lie within $\mathcal{B}$ and those outside (i.e. the boundary of $\mathcal{B}$ is either electrically shielded from its exterior or has an infinite extent).

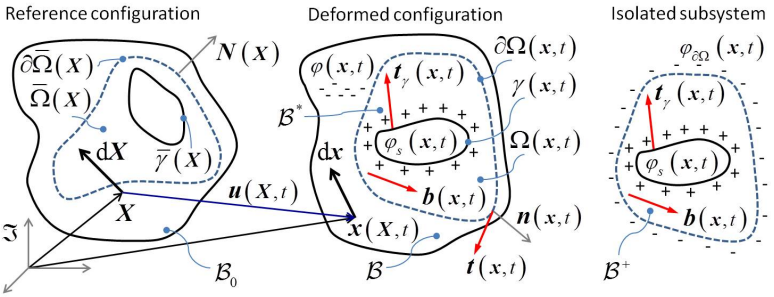

Fig. 1 EM system: reference and actual configuration, and isolated subsystem

Regarding kinematics, define with $\Im$ a fixed frame with respect to which the motions and deformations of $\mathcal{B}$ are measured (with $\mathcal{B}$ specifically indicating the current deformed configuration), and identify with $\mathcal{B}_0$ the reference (stress-free) configuration. For any arbitrary time instant $t \geq 0$, consider a general material point P of $\mathcal{B}$ and indicate with $\boldsymbol{X}$ and $\boldsymbol{x}(\boldsymbol{X},t)$ (where $\boldsymbol{x}(\boldsymbol{X}, 0) \equiv \boldsymbol{X}$) the position vectors expressing the location occupied by P when the EM system is in $\mathcal{B}_0$ and $\mathcal{B}$ respectively. Then, with reference to Fig. 1, the following definitions hold

$$\boldsymbol{u}\left(\boldsymbol{X},t\right)=\boldsymbol{x}\left(\boldsymbol{X},t\right)-\boldsymbol{X}\ ,\ \mathbf{F}\left(\boldsymbol{X},t\right)=\partial\boldsymbol{x}/\partial\boldsymbol{X}=\mathrm{GRAD}\left(\boldsymbol{x}\right)\ ,\ J\left(\boldsymbol{X},t\right)=\det\mathbf{F}\ , \tag{1}$$

where: $\boldsymbol{u}$ is the displacement field; $\mathbf{F}$ is the deformation gradient; J is the Jacobian determinant.

Regarding EM system loadings, the physical space contained within $\mathcal{B}$ features: a distribution of electric charges (namely free and injected electrons or ions), with densities $\varphi(\boldsymbol{x},t)$ and $\varphi_\gamma(\boldsymbol{x},t)$, respectively defined per unit of deformed volume dv of $\mathcal{B}$ and per unit of deformed area ds of the physical surface $\gamma(t)$ (for instance a conducting electrode); a distribution of matter with mass density, $\rho(\boldsymbol{x},t)$, defined per unit volume dv. The same physical space is also subjected to: purely mechanical loads represented by a body force field (for instance the gravity field), $\boldsymbol{b}(\boldsymbol{x},t)$, defined per unit volume dv; and a traction vector, $\boldsymbol{t}_\gamma(\boldsymbol{x},t)$, defined per unit area ds of $\gamma(t)$ (for instance a body boundary).

Beside the displacement field, the other variables that complete the description of the state of $\mathcal{B}$ are the electric potential $\phi(\boldsymbol{x},t)$, the electric displacement vector $\boldsymbol{D}(\boldsymbol{x},t)$, and the electric field $\boldsymbol{E}(\boldsymbol{x},t)$ such that

$$\boldsymbol{E}=-\partial\phi/\partial\boldsymbol{x}=-\mathrm{grad}\phi\ . \tag{2}$$

3 Conservation of Energy

Consider an arbitrary but closed subsystem of $\mathcal{B}$, hereafter called $\mathcal{B}^*$, which (for every time instant $t > 0$) is identified by the volume $\Omega(t)$ and bounded by the closed surface $\partial\Omega(t)$ with unit normal $\boldsymbol{n}(\boldsymbol{x},t)$. In Fig. 1, one of these possible subsystems is indicated with a blue dash-dotted line.

Irrespective of the specific response of the substances contained therein, the evolution of $\mathcal{B}^*$ is governed by a balance of EM energy. Differently from $\mathcal{B}$, $\mathcal{B}^*$ is not electrically isolated, and thus interactions may exist between the electrical charges that lie within $\Omega(t)$ and those outside. According to potential theory [2], and as represented on the right side of Fig. 1, $\mathcal{B}^*$ is equivalent to an identical electrically isolated subsystem $\mathcal{B}^+$ which has the boundary $\partial\Omega(t)$ covered by a single layer of charges with surface density

$$\varphi_{\partial\Omega}=-\boldsymbol{D}\cdot\boldsymbol{n}\ . \tag{3}$$

Thus, the conservation of total EM energy for the arbitrary subsystem $\mathcal{B}^*$ reads as

$$\mathrm{d}\left(\mathcal{K}+\mathcal{W}\right)/\mathrm{d}t=\mathcal{P}_{me}+\mathcal{P}_{el}\ , \tag{4}$$

where $\mathcal{P}_{me}(t)$ and $\mathcal{P}_{el}(t)$ are the external mechanical and electrical powers entering in $\mathcal{B}^+$ (that is in $\mathcal{B}^*$) from the outside of its boundary $\partial\Omega(t)$, namely

$$\mathcal{P}_{me} = \int_{\Omega(t)-\gamma(t)} \boldsymbol{b}\cdot\dot{\boldsymbol{u}}\,\mathrm{d}v + \int_{\gamma(t)} \boldsymbol{t}_{\gamma}\cdot\dot{\boldsymbol{u}}\,\mathrm{d}s + \int_{\partial\Omega(t)-\gamma(t)} \boldsymbol{t}\cdot\dot{\boldsymbol{u}}\,\mathrm{d}s\,, \tag{5.1}$$

$$\mathcal{P}_{el} = \int_{\Omega(t)-\gamma(t)} \phi\,\mathrm{d}\left(\varphi\mathrm{d}v\right)/\mathrm{d}t + \int_{\gamma(t)} \phi\,\mathrm{d}\left(\varphi_{\gamma}\mathrm{d}s\right)/\mathrm{d}t + \int_{\partial\Omega(t)-\gamma(t)} \phi\,\mathrm{d}\left(\varphi_{\partial\Omega}\mathrm{d}s\right)/\mathrm{d}t\,, \tag{5.2}$$

with $\dot{\boldsymbol{u}}(\boldsymbol{x},t)$ being the velocity field ($\dot{\boldsymbol{u}} = d\boldsymbol{u}/dt$); whereas $\mathcal{K}(t)$ and $\mathcal{W}(t)$ are the kinetic and potential energies associated to the physical space contained in $\mathcal{B}^{+}$

$$\mathcal{K}(t) = \int_{\Omega(t)-\gamma(t)} 0.5\rho\dot{\boldsymbol{u}}^{2}\mathrm{d}v\,,\ \mathcal{W} = \int_{\Omega(t)-\gamma(t)} \left(\rho\Psi + \boldsymbol{E}\cdot\boldsymbol{D}\right)\mathrm{d}v\,, \tag{5.3}$$

with $\Psi(\mathbf{F},\boldsymbol{E})$ being the energy density (per unit volume dv) of deformation and polarization of a given material. Note that Ψ does not include the energy required to build the electrostatic field in $\mathcal{B}^{+}$ (this is accounted by the term $\boldsymbol{E}\cdot\boldsymbol{D}$).

Equation (4), together with equations (5), represents the conservation of total EM energy of the arbitrary subsystem $\mathcal{B}^{*}$, expressed in global form and referred to the deformed configuration $\mathcal{B}$ of the overall EM system. For solids, it is generally more convenient to express the balance equations with respect to the reference configuration $\mathcal{B}_0$ (the so called Lagrangian description). This makes it possible to evaluate the considered integrals and to perform all time-derivatives with respect to fixed spatial domains; namely the arbitrary volume $\bar{\Omega} = \Omega(0)$ with boundary $\partial\bar{\Omega} = \partial\Omega(0)$, and the physical surface $\bar{\gamma} = \gamma(0)$.

Introducing from Eq. (1) the volume ratio relationship, dv = JdV, and the Nanson's formula, $\boldsymbol{n}$ds = $\boldsymbol{N}$dS (with dV and dS indicating the infinitesimal undeformed volume and surface of $\mathcal{B}_0$, and $\boldsymbol{N}$ being the unit normal to dS), and defining the Lagrangian electric field $\bar{\boldsymbol{E}}$ from Eqs. (1) and (2) as

$$\bar{\boldsymbol{E}} = -\partial\phi/\partial\boldsymbol{X} = -\mathrm{GRAD}\phi = \mathbf{F}^{T}\boldsymbol{E}\,, \tag{6}$$

equations (5) can then be rewritten as

$$\mathcal{P}_{me} = \int_{\bar{\Omega}-\bar{\gamma}} \bar{\boldsymbol{b}}\cdot\dot{\boldsymbol{u}}\,\mathrm{d}V + \int_{\bar{\gamma}} \bar{\boldsymbol{t}}_{\gamma}\cdot\dot{\boldsymbol{u}}\,\mathrm{d}S + \int_{\partial\bar{\Omega}-\bar{\gamma}} \bar{\boldsymbol{t}}\cdot\dot{\boldsymbol{u}}\,\mathrm{d}S\,, \tag{7.1}$$

$$\mathcal{P}_{el} = \int_{\bar{\Omega}-\bar{\gamma}} \phi\,\mathrm{d}\left(\bar{\varphi}\mathrm{d}V\right)/\mathrm{d}t + \int_{\bar{\gamma}} \phi\,\mathrm{d}\left(\bar{\varphi}_{\gamma}\mathrm{d}S\right)/\mathrm{d}t - \int_{\partial\bar{\Omega}-\bar{\gamma}} \phi\,\mathrm{d}\left(\bar{\boldsymbol{D}}\cdot\boldsymbol{N}\mathrm{d}S\right)/\mathrm{d}t\,, \tag{7.2}$$

$$\mathcal{K} = \int_{\bar{\Omega}-\bar{\gamma}} 0.5\bar{\rho}\dot{\boldsymbol{u}}^{2}\mathrm{d}V\,,\ \mathcal{W} = \int_{\bar{\Omega}-\bar{\gamma}} \left(\bar{\rho}\Psi + \bar{\boldsymbol{E}}\cdot\bar{\boldsymbol{D}}\right)\mathrm{d}V\,, \tag{7.3}$$

where $\bar{\boldsymbol{b}}$, $\bar{\boldsymbol{t}}$, $\bar{\boldsymbol{t}}_{\gamma}$, $\bar{\varphi}$, $\bar{\varphi}_{\gamma}$, $\bar{\rho}$ and $\bar{\boldsymbol{D}}$ are the Lagrangian variables

$$\bar{\boldsymbol{b}} = J\boldsymbol{b}\,,\ \bar{\boldsymbol{t}} = \boldsymbol{t}\frac{\mathrm{d}s}{\mathrm{d}S}\,,\ \bar{\boldsymbol{t}}_\gamma = \boldsymbol{t}_\gamma\frac{\mathrm{d}s}{\mathrm{d}S}\,,\ \bar{\varphi} = J\varphi\,,\ \bar{\varphi}_\gamma = \varphi_\gamma\frac{\mathrm{d}s}{\mathrm{d}S}\,,\ \bar{\rho} = J\rho\,,\ \bar{\boldsymbol{D}} = J\mathbf{F}^{-1}\boldsymbol{D}\,. \tag{8}$$

In obtaining Eq. (7.2), Eq. (3) has been used.

Resorting to the Gauss's divergence theorem along with Eqs. (1) and (6), the conservation of total EM energy of $\mathcal{B}^*$ in Lagrangian description follows as

$$\begin{aligned}&\int_{\bar{\Omega}-\bar{\gamma}}\left[\left(\bar{\rho}\ddot{\boldsymbol{u}} - \mathrm{DIV}\left(\bar{\rho}\left(\partial\Psi/\partial\mathbf{F}\right)^T\right) - \bar{\boldsymbol{b}}\right)\cdot\dot{\boldsymbol{u}} + \left(\bar{\boldsymbol{D}} + \bar{\rho}\,\partial\Psi/\partial\bar{\boldsymbol{E}}\right)\cdot\dot{\bar{\boldsymbol{E}}}\right]\mathrm{d}V + \\ &+\int_{\partial\bar{\Omega}-\bar{\gamma}}\left[\left(\bar{\rho}\left(\partial\Psi/\partial\mathbf{F}\right)^T\cdot\boldsymbol{N}\right) - \bar{\boldsymbol{t}}\right]\cdot\dot{\boldsymbol{u}}\,dS - \int_{\bar{\gamma}}\left(\left[\!\left[\bar{\rho}\left(\partial\Psi/\partial\mathbf{F}\right)^T\right]\!\right]\cdot\boldsymbol{N} + \bar{\boldsymbol{t}}_\gamma\right)\cdot\dot{\boldsymbol{u}}\,dS + \\ &-\int_{\bar{\Omega}-\bar{\gamma}}\phi\frac{\mathrm{d}}{\mathrm{d}t}\left(\bar{\varphi} - \mathrm{DIV}\left(\bar{\boldsymbol{D}}\right)\right)\mathrm{d}V - \int_{\bar{\gamma}}\phi\frac{\mathrm{d}}{\mathrm{d}t}\left(\bar{\varphi}_\gamma - \left[\!\left[\bar{\boldsymbol{D}}\right]\!\right]\cdot\boldsymbol{N}\right)\mathrm{d}S = 0\end{aligned} \tag{9}$$

4 Balance Equations and Constitutive Relations

Equation (9) holds for any arbitrary volume $\bar{\Omega}$ (with boundary $\partial\bar{\Omega}$) and for any general EM process. Thus, satisfaction of Eq. (9) requires:

$$\bar{\rho}\ddot{\boldsymbol{u}} = \mathrm{DIV}\left(\mathbf{P}\right) + \bar{\boldsymbol{b}} \ \text{ on } \bar{\Omega}-\bar{\gamma}\,, \text{ and } \bar{\boldsymbol{t}}_\gamma = -\left[\!\left[\mathbf{P}\right]\!\right]\cdot\boldsymbol{N} \ \text{ on } \bar{\gamma}\,, \tag{10}$$

$$\mathrm{DIV}\left(\bar{\boldsymbol{D}}\right) = \bar{\varphi} \ \text{ on } \bar{\Omega}-\bar{\gamma}\,, \text{ and } \left[\!\left[\bar{\boldsymbol{D}}\right]\!\right]\cdot\boldsymbol{N} = \bar{\varphi}_\gamma \ \text{ on } \bar{\gamma}\,, \tag{11}$$

$$\mathbf{P}^T = \bar{\rho}\,\partial\Psi/\partial\mathbf{F} \ \text{ with } \bar{\boldsymbol{t}} = \mathbf{P}\cdot\boldsymbol{N}\,, \tag{12}$$

$$\bar{\boldsymbol{D}} = -\bar{\rho}\,\partial\Psi/\partial\bar{\boldsymbol{E}}\,. \tag{13}$$

For the considered EM system, Eqs. (10) and (12) represent the Lagrangian form of the balance of linear momentum (with the second relation of Eq. (12) being the stress theorem holding in the reference configuration), whereas Eqs. (11) and (13) are the Lagrangian form of the electrostatic equations.

5 Constitutive Relations for Dielectric Elastomers

Equations (10)-(13) hold for any conservative elastic dielectric body that admits an energy density function of deformation and polarization. Particular problem solutions require specific definitions of $\Psi(\mathbf{F},\boldsymbol{E})$. A possible form for DEs is

$$\Psi = \Psi_{MR} + \Psi_{es} + \Psi_{vol}\,, \tag{14.1}$$

$$\Psi_{MR} = \frac{c_1}{\bar{\rho}}\left[\text{trace}\left(\mathbf{F}\mathbf{F}^T\right)-3\right]+\frac{c_2}{\bar{\rho}}\left[\left(\text{trace}\left(\mathbf{F}\mathbf{F}^T\right)\right)^2-\text{trace}\left(\left(\mathbf{F}\mathbf{F}^T\right)^2\right)-3\right], \tag{14.2}$$

$$\Psi_{es} = -0.5\varepsilon \boldsymbol{E}^2/\rho = -0.5\varepsilon J\bar{\boldsymbol{E}}\cdot\left(\mathbf{F}^{-1}\mathbf{F}^{-T}\bar{\boldsymbol{E}}\right)/\bar{\rho}\,,\ \Psi_{vol} = -p\left(J-1\right)/\bar{\rho}\,. \tag{14.3}$$

where Ψ_{MR} is the Mooney-Rivlin strain-energy function for hyperelastic materials (only dependent on the DE shear moduli c_1 and c_2), Ψ_{es} is a purely electrostatic energy function (only dependent on the DE permittivity ε), and Ψ_{vol} is a constraining term introduced to enforce the incompressibility condition (J = 1, with p being a Lagrange multiplier identifiable as a hydrostatic pressure).

With this energy density function, the constitutive relations (12) and (13), which complete the EM model for DEs together with Eqs. (10) and (11), read as

$$\begin{aligned}\mathbf{P}^T = 2\Big[c_1\mathbf{1}+c_2\left(\text{trace}\left(\mathbf{F}\mathbf{F}^T\right)\mathbf{1}-\mathbf{F}\mathbf{F}^T\right)\Big]\mathbf{F}-p\mathbf{F}^{-T}+\\ +\varepsilon\Big[\left(\mathbf{F}^{-T}\bar{\boldsymbol{E}}\right)\otimes\left(\mathbf{F}^{-T}\bar{\boldsymbol{E}}\right)-0.5\bar{\boldsymbol{E}}\cdot\left(\mathbf{F}^{-1}\mathbf{F}^{-T}\bar{\boldsymbol{E}}\right)\mathbf{1}\Big]\mathbf{F}^{-T}\,,\end{aligned} \tag{15}$$

$$\bar{\boldsymbol{D}} = \varepsilon\mathbf{F}^{-1}\mathbf{F}^{-T}\bar{\boldsymbol{E}}\,. \tag{16}$$

6 Conclusions

An electro-mechanical finite-deformation model for dielectric elastomers has been presented. The model is fully coupled and features the balance equations of linear momentum and of electrostatics, associated boundary conditions, and constitutive relations only dependent on three parameters. The model is expressed in a Lagrangian formulation which enables its direct use in finite element codes.

Acknowledgments. R.V. acknowledge the financial support from the EC, in the framework of the project "PolyWEC - New mechanisms and concepts for exploiting electroactive Polymers for Wave Energy Conversion" (FP7-ENERGY.2012.10.2.1, grant: 309139).

References

1. Carpi, F., De Rossi, D., Kornbluh, R., Pelrine, P., Sommer-Larsen, P.: Dielectric Elastomers as Electromechanical Transducers: Fundamentals, Materials, Devices, Models and Applications of an Emerging Electroactive Polymer Technology. Elsevier, Oxford (2008)
2. Vertechy, R., Berselli, G., Parenti Castelli, V., Bergamasco, M.: Continuum thermo-electro-mechanical model for electrostrictive elastomers. Journal of Intelligent Material Systems and Structures 24(6), 761–778 (2013), doi:10.1177/1045389X12455855
3. Vertechy, R., Frisoli, A., Bergamasco, M., Carpi, F., Frediani, G., De Rossi, D.: Modeling and experimental validation of buckling dielectric elastomer actuators. Smart Materials and Structures 21(9), 94005 (2012), doi:10.1088/0964-1726/21/9/094005