An Elementary Proof that the First Hitting Time of an Open Set by a Jump Process is a Stopping Time

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Abstract We give a short and elementary proof that the first hitting time of an open set by the jump process of a càdlàg adapted process is a stopping time.

1 Introduction

For a stochastic process X and a subset B of the real numbers, the random variable $T = \inf\{t \geq 0 | X_t \in B\}$ is called the first hitting time of B by X. A classical result in the general theory of processes is the début theorem, which has as a corollary that under the usual conditions, the first hitting time of a Borel set for a progressively measurable process is a stopping time, see [3], Sect. III.44 for a proof of this theorem, or [1] and [2] for a recent simpler proof. For many purposes, however, the general début theorem is not needed, and weaker results may suffice, where elementary methods may be used to obtain the results. For example, it is elementary to show that the first hitting time of an open set by a càdlàg adapted process is a stopping time, see [4], Theorem I.3. Using somewhat more advanced, yet relatively elementary methods, Lemma II.75.1 of [5] shows that the first hitting time of a compact set by a càdlàg adapted process is a stopping time.

These elementary proofs show stopping time properties for the first hitting times of a càdlàg adapted process X. However, the jump process ΔX in general has paths with neither left limits nor right limits, and so the previous elementary results do not apply. In this note, we give a short and elementary proof that the first hitting time of

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an open set by ΔX is a stopping time when the filtration is right-continuous and X is càdlàg adapted. This result may be used to give an elementary proof that the jumps of a càdlàg adapted process are covered by the graphs of a countable sequence of stopping times.

2 A Stopping Time Result

Assume given a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t), P)$ such that the filtration $(\mathscr{F}_t)_{t \geq 0}$ is right-continuous in the sense that $\mathscr{F}_t = \cap_{s>t} \mathscr{F}_s$ for all $t \geq 0$. We use the convention that $X_{0-} = X_0$, so that there is no jump at the timepoint zero.

Theorem 1. *Let X be a càdlàg adapted process, and let U be an open set in $\mathbb{R}$. Define $T = \inf\{t \geq 0 | \Delta X_t \in U\}$. Then T is a stopping time.*

As X has càdlàg, ΔX is zero everywhere except for on a countable set, and so T is identically zero if U contains zero. In this case, T is trivially a stopping time. Thus, it suffices to prove the result in the case where U does not contain zero. Therefore, assume that U is an open set not containing zero. As the filtration is right-continuous, an elementary argument yields that to show the stopping time property of T, it suffices to show $(T < t) \in \mathscr{F}_t$ for $t > 0$, see Theorem I.1 of [4].

To this end, fix $t > 0$ and note that

$$(T < t) = (\exists\, s \in (0, \infty) : s < t \text{ and } X_s - X_{s-} \in U)\ . \tag{1}$$

Let $F_m = \{x \in \mathbb{R} | \ \forall\ y \in U^c : |x - y| \geq 1/m\}$, F_m is an intersection of closed sets and therefore itself closed. Clearly, $(F_m)_{m \geq 1}$ is increasing, and since U is open, $U = \cup_{m=1}^{\infty} F_m$. Also, $F_m \subseteq F_{m+1}^{\circ}$, where F_{m+1}° denotes the interior of F_{m+1}. Let Θ_k be the subset of $\mathbb{Q}^2$ defined by $\Theta_k = \{(p, q) \in \mathbb{Q}^2 | 0 < p < q < t, |p - q| \leq \frac{1}{k}\}$. We will prove the result by showing that

$$\begin{aligned}&(\exists\, s \in (0, \infty) : s < t \text{ and } X_s - X_{s-} \in U)\\ &= \cup_{m=1}^{\infty} \cup_{n=1}^{\infty} \cap_{k=n}^{\infty} \cup_{(p,q) \in \Theta_k} (X_q - X_p \in F_m)\ .\end{aligned} \tag{2}$$

To obtain this, first consider the inclusion towards the right. Assume that there is $0 < s < t$ such that $X_s - X_{s-} \in U$. Take m such that $X_s - X_{s-} \in F_m$. As $F_m \subseteq F_{m+1}^{\circ}$, we then have $X_s - X_{s-} \in F_{m+1}^{\circ}$ as well. As F_{m+1}° is open and as X is càdlàg, it holds that there is $\varepsilon > 0$ such that whenever $p, q \geq 0$ with $p \in (s - \varepsilon, s)$ and $q \in (s, s + \varepsilon)$, $X_q - X_p \in F_{m+1}^{\circ}$. Take $n \in \mathbb{N}$ such that $1/2n < \varepsilon$. We now claim that for $k \geq n$, there is $(p, q) \in \Theta_k$ such that $X_q - X_p \in F_{m+1}$. To prove this, let $k \geq n$ be given. By the density properties of $\mathbb{Q}_+$ in $\mathbb{R}_+$, there are elements

$p, q \in \mathbb{Q}$ with $p, q \in (0, t)$ such that $p \in (s - 1/2k, s)$ and $q \in (s, s + 1/2k)$. In particular, then $0 < p < q < t$ and $|p - q| \leq |p - s| + |s - q| \leq 1/k$, so $(p, q) \in \Theta_k$. As $1/2k \leq 1/2n < \varepsilon$, we have $p \in (s - \varepsilon, s)$ and $q \in (s, s + \varepsilon)$, and so $X_q - X_p \in F^\circ_{m+1} \subseteq F_{m+1}$. This proves the inclusion towards the right.

Now consider the inclusion towards the left. Assume that there is $m \geq 1$ and $n \geq 1$ such that for all $k \geq n$, there exists $(p, q) \in \Theta_k$ with $X_q - X_p \in F_m$. We may use this to obtain sequences $(p_k)_{k \geq n}$ and $(q_k)_{k \geq n}$ with the properties that $p_k, q_k \in \mathbb{Q}, 0 < p_k < q_k < t, |p_k - q_k| \leq \frac{1}{k}$ and $X_{q_k} - X_{p_k} \in F_m$. Putting $p_k = p_n$ and $q_k = q_n$ for $k < n$, we then find that the sequences $(p_k)_{k \geq 1}$ and $(q_k)_{k \geq 1}$ satisfy $p_k, q_k \in \mathbb{Q}, 0 < p_k < q_k < t$, $\lim_k |p_k - q_k| = 0$ and $X_{q_k} - X_{p_k} \in F_m$. As all sequences of real numbers contain a monotone subsequence, we may by taking two consecutive subsequences and renaming our sequences obtain the existence of two monotone sequences (p_k) and (q_k) in $\mathbb{Q}$ with $0 < p_k < q_k < t$, $\lim_k |p_k - q_k| = 0$ and $X_{q_k} - X_{p_k} \in F_m$. As bounded monotone sequences are convergent, both (p_k) are (q_k) are then convergent, and as $\lim_k |p_k - q_k| = 0$, the limit $s \geq 0$ is the same for both sequences.

We wish to argue that $s > 0$, that $X_{s-} = \lim_k X_{p_k}$ and that $X_s = \lim_k X_{q_k}$. To this end, recall that U does not contain zero, and so as $F_m \subseteq U$, F_m does not contain zero either. Also note that as both (p_k) and (q_k) are monotone, the limits $\lim_k X_{p_k}$ and $\lim_k X_{q_k}$ exist and are either equal to X_s or X_{s-}. As $X_{q_k} - X_{p_k} \in F_m$ and F_m is closed and does not contain zero, $\lim_k X_{q_k} - \lim_k X_{p_k} = \lim_k X_{q_k} - X_{p_k} \neq 0$. From this, we can immediately conclude that $s > 0$, as if $s = 0$, we would obtain that both $\lim_k X_{q_k}$ and $\lim_k X_{p_k}$ were equal to X_s, yielding $\lim_k X_{q_k} - \lim_k X_{p_k} = 0$, a contradiction. Also, we cannot have that both limits are X_s or that both limits are X_{s-}, and so only two cases are possible, namely that $X_s = \lim_k X_{q_k}$ and $X_{s-} = \lim_k X_{p_k}$ or that $X_s = \lim_k X_{p_k}$ and $X_{s-} = \lim_k X_{q_k}$. We wish to argue that the former holds. If $X_s = X_{s-}$, this is trivially the case. Assume that $X_s \neq X_{s-}$ and that $X_s = \lim_k X_{p_k}$ and $X_{s-} = \lim_k X_{q_k}$. If $q_k \geq s$ from a point onwards or $p_k < s$ from a point onwards, we obtain $X_s = X_{s-}$, a contradiction. Therefore, $q_k < s$ infinitely often and $p_k \geq s$ infinitely often. By monotonicity, $q_k < s$ and $p_k \geq s$ from a point onwards, a contradiction with $p_k < q_k$. We conclude $X_s = \lim_k X_{q_k}$ and $X_{s-} = \lim_k X_{p_k}$, as desired.

In particular, $X_s - X_{s-} = \lim_k X_{q_k} - X_{p_k}$. As $X_{q_k} - X_{p_k} \in F_m$ and F_m is closed, we obtain $X_s - X_{s-} \in F_m \subseteq U$. Next, note that if $s = t$, we have $p_k, q_k < s$ for all k, yielding that both sequences must be increasing and $X_s = \lim X_{q_k} = X_{s-}$, a contradiction with the fact that $X_s - X_{s-} \neq 0$ as $X_s - X_{s-} \in U$. Thus, $0 < s < t$. This proves the existence of $s \in (0, \infty)$ with $s < t$ such that $X_s - X_{s-} \in U$, and so proves the inclusion towards the right.

We have now shown (2). Now, as X_s is $\mathscr{F}_t$ measurable for all $0 \leq s \leq t$, it holds that the set $\cup_{m=1}^\infty \cup_{n=1}^\infty \cap_{k=n}^\infty \cup_{(p,q) \in \Theta_k} (X_q - X_p \in F_m)$ is $\mathscr{F}_t$ measurable as well. We conclude that $(T < t) \in \mathscr{F}_t$ and so T is a stopping time.

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