Ekman Layers of Rotating Fluids with Vanishing Viscosity between Two Infinite Parallel Plates

Van-Sang Ngo

Laboratoire de Mathématiques Raphaël Salem, UMR 6085 CNRS,
Université de Rouen, France
van-sang.ngo@univ-rouen.fr

Abstract. In this paper, we prove the convergence of weak solutions of fast rotating fluids between two infinite parallel plates towards the two-dimensional limiting system. We also put in evidence the existence of Ekman boundary layers when Dirichlet boundary conditions are imposed on the domain.

Keywords: Navier-Stokes, Rotating fluids, Ekman layer.

1 Introduction

In this paper, we consider a simplified model of geophysical fluids, that is the system of fast rotating, incompressible, homogeneous fluids between two parallel plates, with Dirichlet boundary conditions, as in [16], [18] or [8]

$$\begin{cases} \partial_t u^\varepsilon - \nu_h(\varepsilon)\Delta_h u^\varepsilon - \beta\varepsilon\partial_3^2 u^\varepsilon + u^\varepsilon\cdot\nabla u^\varepsilon + \dfrac{e_3\wedge u^\varepsilon}{\varepsilon} = -\nabla p^\varepsilon & \text{in } \mathbb{R}_+\times\Omega_h\times[0,1] \\ \operatorname{div} u^\varepsilon = 0 & \text{in } \mathbb{R}_+\times\Omega_h\times[0,1] \\ u^\varepsilon{}_{|t=0} = u_0^\varepsilon, & \text{in } \Omega_h\times[0,1]. \end{cases} \tag{1}$$

Here, the fluid rotates in the domain $\Omega_h\times[0,1]$, between two "horizontal plates" $\Omega_h\times\{0\}$ and $\Omega_h\times\{1\}$, where Ω_h is a subdomain of $\mathbb{R}^2$. We are interested in the case where Rossby number ε goes to zero and we suppose that $\nu_h(\varepsilon)$ also goes to zero with ε. We emphasize that all along this paper, we always use the index "h" to refer to the horizontal terms and horizontal variables, and the index "v" or "3" to the vertical ones.

The Coriolis force $\varepsilon^{-1}e_3\wedge u^\varepsilon$ has a very important impact on the behaviors of fast rotating fluids (corresponding to a small Rossby number ε). Indeed, if we suppose that $\overline{u}$ is the formal limit of u^ε when ε goes to zero, we can prove that $\overline{u}$ does not depend on the third space variable x_3. Since the Coriolis force becomes very large as ε becomes small, the "only theorical way" to balance that force is to use the pressure force term $-\nabla p^\varepsilon$. This means that there exists a function φ such that $e_3\wedge\overline{u} = \nabla\varphi$, or in a equivalent way

$$\begin{pmatrix} -\overline{u}^2 \\ \overline{u}^1 \\ 0 \end{pmatrix} = \begin{pmatrix} \partial_1\varphi \\ \partial_2\varphi \\ \partial_3\varphi \end{pmatrix}.$$

N.T. Nguyen, T. Van Do, and H.A. Le Thi (Eds.): *ICCSAMA 2013*, SCI 479, pp. 187–207.
DOI: 10.1007/978-3-319-00293-4_15

Thus, φ (and so $\overline{u}^1$ and $\overline{u}^2$) does not depend on x_3. The incompressibility of the fluid implies that

$$\partial_3\overline{u}^3 = -\partial_1\overline{u}^1 - \partial_2\overline{u}^2 = -\partial_1\partial_2\varphi + \partial_1\partial_2\varphi = 0,$$

which means the third component $\overline{u}^3$ does not depend on x_3 neither. This independence was justified in the experiment of G.I. Taylor (see [10]), drops of dye injected into a rapidly rotating, homogeneous fluid, within a few rotations, formed perfectly vertical sheets of dyed fluid, known as *Taylor curtains*. In large-scale atmospheric and oceanic flows, the Rossby number is often observed to be very small, and the fluid motions also have a tendency towards columnar behaviors (*Taylor columns*). For example, currents in the western North Atlantic have been observed to extend vertically over several thousands meters without significant change in amplitude and direction ([23]).

The columnar behaviors of the solution of the system (1), in the case where $\nu(\varepsilon) > 0$ is fixed and where the domain has no boundary ($\mathbb{T}^3$ or $\mathbb{R}^3$), were studied by many authors. In the case of periodic domains, Babin, Mahalov and Nicolaenko [1]-[2], Embid and Majda [11], Gallagher [13] and Grenier [15] proved that the weak (and strong) solutions of the system (1) converge to the solution of the limiting system, which is a two-dimensional Navier-Stokes system with three components. In the case of $\mathbb{R}^3$, Chemin, Desjardins, Gallagher and Grenier proved in [6] and [7] that if the initial data are in $\mathbf{L}^2(\mathbb{R}^3)$ then the limiting system is zero. If the initial data are of the form

$$u_0 = \overline{u}_0 + v_0, \tag{2}$$

where

$$\overline{u}_0 = \left(\overline{u}_0^1(x_1,x_2), \overline{u}_0^2(x_1,x_2), \overline{u}_0^3(x_1,x_2)\right)$$

is a divergence-free vector field, independent of x_3 and

$$v_0 = \left(v_0^1(x_1,x_2,x_3), v_0^2(x_1,x_2,x_3), v_0^3(x_1,x_2,x_3)\right),$$

the limiting system is also proved to be a two-dimensional Navier-Stokes system with three components. The case where $\nu_h(\varepsilon) \to 0$ as $\varepsilon \to 0$ and the domain is $\mathbb{R}^3$ was studied by the author of this paper in [19] and [20]. We also refer to [14] in which, Gallagher and Saint-Raymond proved the convergence of the weak solutions of the system (1) to the solution of the two-dimensional limiting system in the more general case where the axis of rotation is not fixed to be e_3.

Things are very different in the case where the domain is $\Omega_h \times [0,1]$ with Dirichlet boundary conditions. Indeed, when the rotation goes to infinity, the *Taylor columns* are only formed in the interior of the domain. Near the boundary, Ekman boundary layers exist. The behaviors of the fluid become very complex and the friction slows the fluid down in a way that the velocity is zero on the boundary. In the works of Grenier and Masmoudi [16] ($\Omega_h = \mathbb{T}^2$) and Chemin *et al.* [8] ($\Omega_h = \mathbb{R}^2$), it was proved that, in the limiting system, we obtain an additional damping term of the form $\sqrt{2\beta}\overline{u}$. This phenomenon is well known in fluid mechanics as the Ekman pumping.

Since the viscosity is positive in all three directions ($\nu_h = \nu_h(\varepsilon) > 0$ and $\nu_v = \beta\varepsilon > 0$), the system (1) possesses a weak Leray solution

$$u^\varepsilon \in \mathbf{L}^\infty(\mathbb{R}_+, \mathbf{L}^2(\Omega_h \times [0,1])) \cap \mathbf{L}^2(\mathbb{R}_+, \dot{\mathbf{H}}^1(\Omega_h \times [0,1])).$$

In the case where $\nu_h > 0$ is fixed and where the initial data are well prepared, *i.e.*

$$\lim_{\varepsilon\to 0} u_0^\varepsilon = \overline{u}_0 = (\overline{u}_0^1(x_1,x_2), \overline{u}_0^2(x_1,x_2), 0) \quad \text{in } \mathbf{L}^2(\mathbb{R}_h^2 \times [0,1])$$

and $\overline{u}_0$ is a divergence-free two-dimensional vector field in $\mathbf{H}^\sigma(\mathbb{R}_h^2)$, $\sigma > 2$, it was proved by Grenier and Masmoudi in [16] ($\Omega_h = \mathbb{T}_h^2$) and by Chemin *et al.* in [8] ($\Omega_h = \mathbb{R}_h^2$) that, when ε goes to zero, u^ε converges to the solution of the following limiting system in $\mathbf{L}^\infty(\mathbb{R}_+, \mathbf{L}^2(\mathbb{R}^3))$

$$\begin{cases} \partial_t \overline{u}^h - \nu_h \Delta_h \overline{u}^h + \overline{u}^h \cdot \nabla_h \overline{u}^h + \sqrt{2\beta}\, \overline{u}^h = -\nabla_h \overline{p} \\ \partial_t \overline{u}^3 - \nu_h \Delta_h \overline{u}^3 + \overline{u}^h \cdot \nabla_h \overline{u}^3 + \sqrt{2\beta}\, \overline{u}^3 = 0 \\ \operatorname{div}_h \overline{u}^h = 0 \\ \partial_3 \overline{u} = 0 \\ \overline{u}_{|t=0} = \overline{u}_0. \end{cases} \tag{3}$$

The case of ill-prepared data, where $\overline{u}_0 = (\overline{u}_0^1(x_1,x_2), \overline{u}_0^2(x_1,x_2), \overline{u}_0^3(x_1,x_2))$ has all the three components different from 0, was studied in [8].

In this paper, we consider the system (1) in the case where $\Omega_h = \mathbb{R}^2$, where $\nu_h(\varepsilon) \to 0$ as $\varepsilon \to 0$ and where the data are well prepared. The limiting system is the following

$$\begin{cases} \partial_t \overline{u}^h + \overline{u}^h \cdot \nabla_h \overline{u}^h + \sqrt{2\beta}\, \overline{u}^h = -\nabla_h \overline{p} \\ \partial_t \overline{u}^3 + \overline{u}^h \cdot \nabla_h \overline{u}^3 + \sqrt{2\beta}\, \overline{u}^3 = 0 \\ \operatorname{div}_h \overline{u}^h = 0 \\ \partial_3 \overline{u} = 0 \\ \overline{u}_{|t=0} = \overline{u}_0. \end{cases} \tag{4}$$

We want to remark that in this case where the data are well prepared, as $\overline{u}_0^3 = 0$, the third component $\overline{u}^3 = 0$ for any $t > 0$. In [16] and [8], it was proved that, in the case where $\nu_h \to 0$ as $\varepsilon \to 0$, the weak solutions of the system (1) converge to the solution of the limiting system (4), but the convergence is only local with respect to the time variable. In this paper, we show the exponential decay of the solution of the system (4) in appropriate Sobolev norms, and we improve the result of [16] and [8]. More precisely, we prove the uniform convergence (with respect to the time variable) of (1) towards (4).

Theorem 1. *Suppose that*

$$\lim_{\varepsilon\to 0} \nu_h(\varepsilon) = 0 \quad \textit{and} \quad \lim_{\varepsilon\to 0} \frac{\varepsilon^{\frac{1}{2}}}{\nu_h(\varepsilon)} = 0.$$

Let $u_0^\varepsilon \in \mathbf{L}^2(\mathbb{R}_h^2 \times [0,1])$ *be a family of initial data such that*

$$\lim_{\varepsilon \to 0} u_0^\varepsilon = \overline{u}_0 = (\overline{u}_0^1(x_1,x_2), \overline{u}_0^2(x_1,x_2), 0) \quad in \quad \mathbf{L}^2(\mathbb{R}_h^2 \times [0,1]),$$

where $\overline{u}_0$ *is a divergence-free two-dimensional vector field in* $\mathbf{H}^\sigma(\mathbb{R}_h^2)$, $\sigma > 2$. *Let* $\overline{u}$ *be the solution of the limiting system* (4) *with initial data* $\overline{u}_0$ *and, for each* $\varepsilon > 0$, *let* u^ε *be a weak solution of* (1) *with initial data* u_0^ε. *Then*

$$\lim_{\varepsilon \to 0} \|u^\varepsilon - \overline{u}\|_{\mathbf{L}^\infty\left(\mathbb{R}_+, \mathbf{L}^2(\mathbb{R}_h^2 \times [0,1])\right)} = 0.$$

2 Preliminaries

In this section, we briefly recall the properties of dyadic decompositions in the Fourier space and give some elements of the Littlewood-Paley theory. Using dyadic decompositions, we redefine some classical function spaces, which will be used in this paper. In what follows, we always denote by (c_q) (respectively (d_q)) a square-summable (respectively summable) sequence, with $\sum_q c_q^2 = 1$ (respectively $\sum_q d_q = 1$), of positive numbers (which can depend on several parameters). We also remarque that, in order to simplify the notations, we use the bold character $\mathbf{X}$ to indicate the space of vector fields, each component of which belongs to the space X.

We recall that $\mathcal{F}$ and $\mathcal{F}^{-1}$ are the Fourier transform and its inverse, and that we also write $\widehat{u} = \mathcal{F}u$. For any $d \in \mathbb{N}^*$ and $0 < r < R$, we denote $B_d(0,R) = \left\{\xi \in \mathbb{R}^d \mid |\xi| \le R\right\}$, and $C_d(r,R) = \left\{\xi \in \mathbb{R}^d \mid r \le |\xi| \le R\right\}$. The following Bernstein lemma gives important properties of a distribution u when its Fourier transform is well localized. We refer the reader to [5] for the proof of this lemma.

Lemma 2. *Let* $k \in \mathbb{N}$, $d \in \mathbb{N}^*$ *and* $r_1, r_2 \in \mathbb{R}$ *satisfy* $0 < r_1 < r_2$. *There exists a constant* $C > 0$ *such that, for any* $a, b \in \mathbb{R}$, $1 \le a \le b \le +\infty$, *for any* $\lambda > 0$ *and for any* $u \in L^a(\mathbb{R}^d)$, *we have*

$$supp\ (\widehat{u}) \subset B_d(0, r_1\lambda) \quad \Longrightarrow \quad \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^b} \le C^k \lambda^{k+d\left(\frac{1}{a}-\frac{1}{b}\right)} \|u\|_{L^a}, \tag{5}$$

and

$$supp\ (\widehat{u}) \subset C_d(r_1\lambda, r_2\lambda) \quad \Longrightarrow \quad C^{-k}\lambda^k \|u\|_{L^a} \le \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^a} \le C^k \lambda^k \|u\|_{L^a}. \tag{6}$$

Let ψ be an even smooth function in $C_0^\infty(\mathbb{R})$, whose support is contained in the ball $B_1(0, \frac{4}{3})$, such that ψ is equal to 1 on a neighborhood of the ball $B_1(0, \frac{3}{4})$. Let

$$\varphi(z) = \psi\left(\frac{z}{2}\right) - \psi(z).$$

Then, the support of φ is contained in the ring $C_1(\frac{3}{4}, \frac{8}{3})$, and φ is identically equal to 1 on the ring $C_1(\frac{4}{3}, \frac{3}{2})$. The functions ψ and φ allow us to define a dyadic partition of $\mathbb{R}^d$, $d \in \mathbb{N}^*$, as follows

$$\forall z \in \mathbb{R}, \quad \psi(z) + \sum_{j \in \mathbb{N}} \varphi(2^{-j} z) = 1.$$

Moreover, this decomposition is almost orthogonal, in the sense that, if $|j - j'| \geq 2$, then

$$\text{supp } \varphi(2^{-j}(\cdot)) \cap \text{supp } \varphi(2^{-j'}(\cdot)) = \emptyset.$$

We introduce the following dyadic frequency cut-off operators. We refer to [3] and [5] for more details.

Definition 3. *For any $d \in \mathbb{N}^*$ and for any tempered distribution $u \in \mathcal{S}'(\mathbb{R}^d)$, we set*

$$\begin{aligned}
&\Delta_q u = \mathcal{F}^{-1}\left(\varphi(2^{-q}|\xi|)\widehat{u}(\xi)\right), && \forall q \in \mathbb{N},\\
&\Delta_{-1} u = \mathcal{F}^{-1}\left(\psi(|\xi|)\widehat{u}(\xi)\right), &&\\
&\Delta_q u = 0, && \forall q \leq -2,\\
&S_q u = \sum_{q' \leq q-1} \Delta_{q'} u, && \forall q \geq 1.
\end{aligned}$$

Using the properties of ψ and φ, one can prove that for any tempered distribution $u \in \mathcal{S}'(\mathbb{R}^d)$, we have

$$u = \sum_{q \geq -1} \Delta_q u \qquad \text{in} \qquad \mathcal{S}'(\mathbb{R}^d),$$

and the (isotropic) nonhomogeneous Sobolev spaces $H^s(\mathbb{R}^d)$, with $s \in \mathbb{R}$, can be characterized as follows

Proposition 4. *Let $d \in \mathbb{N}^*$, $s \in \mathbb{R}$ and $u \in H^s(\mathbb{R}^d)$. Then,*

$$\|u\|_{H^s} := \left(\int_{\mathbb{R}^d} (1 + |\xi|^2)^s |\widehat{u}(\xi)|^2 \, d\xi\right)^{\frac{1}{2}} \sim \left(\sum_{q \geq -1} 2^{2qs} \|\Delta_q u\|_{L^2}^2\right)^{\frac{1}{2}}$$

Moreover, there exists a square-summable sequence of positive numbers $\{c_q(u)\}$ with $\sum_q c_q(u)^2 = 1$, such that

$$\|\Delta_q u\|_{L^2} \leq c_q(u) 2^{-qs} \|u\|_{H^s}.$$

The decomposition into dyadic blocks also gives a very simple characterization of Hölder spaces.

Definition 5. *Let $d \in \mathbb{N}^*$ and $r \in \mathbb{R}_+ \setminus \mathbb{N}$.*

1. *If $r \in]0,1[$, we denote $C^r(\mathbb{R}^d)$ the set of bounded functions $u : \mathbb{R}^d \to \mathbb{R}$ such that there exists $C > 0$ satisfying*

$$\forall\, (x,y) \in \mathbb{R}^d \times \mathbb{R}^d, \qquad |u(x) - u(y)| \leq C\,|x-y|^r .$$

2. *If $r > 1$ is not an integer, we denote $C^r(\mathbb{R}^d)$ the set of $[r]$ times differentiable functions u such that $\partial^\alpha u \in C^{r-[r]}(\mathbb{R}^d)$, for any $\alpha \in \mathbb{N}^d$, $|\alpha| \leq [r]$, where $[r]$ is the largest integer smaller than r.*

One can prove that the set $C^r(\mathbb{R}^d)$, endowed with the norm

$$\|u\|_{C^r} := \sum_{|\alpha|\leq [r]} \left(\|\partial^\alpha u\|_{L^\infty} + \sup_{x\neq y} \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|}{|x-y|^{r-[r]}} \right)$$

is a Banach space. Moreover, we have the following result, the proof of which can be found in [5].

Proposition 6. *There exists a constant $C > 0$ such that, for any $r \in \mathbb{R}_+ \setminus \mathbb{N}$ and for any $u \in C^r(\mathbb{R}^d)$, we have*

$$\sup_q 2^{qr} \|\Delta_q u\|_{L^\infty} \leq \frac{C^{r+1}}{[r]!} \|u\|_{C^r} .$$

Conversely, if the sequence $\left(2^{qr} \|\Delta_q u\|_{L^\infty}\right)_{q\geq -1}$ is bounded, then

$$\|u\|_{C^r} \leq C^{r+1} \left(\frac{1}{r-[r]} + \frac{1}{[r]+1-r} \right) \sup_q 2^{qr} \|\Delta_q u\|_{L^\infty} .$$

Finally, we need the following results (for a proof, see [21]). Let $[.,.]$ denote the usual commutator.

Lemma 7. *Let $d \in \mathbb{N}^*$. There exists a constant $C > 0$ such that, for any tempered distributions u, v in $\mathcal{S}'(\mathbb{R}^d)$, we have*

$$\|[\Delta_q, u]\, v\|_{L^2} = \|\Delta_q(uv) - u\Delta_q v\|_{L^2} \leq C 2^{-q} \|\nabla u\|_{L^\infty} \|v\|_{L^2} .$$

3 Estimates for the Limiting System

In this section, we give useful auxiliary results concerning the 2D limiting system (4). Throughout this paper, for any vector field $\overline{u} = (\overline{u}^1, \overline{u}^2, \overline{u}^3)$ independent of the vertical variable x_3, we denote by $\overline{w}$ the associated horizontal vorticity, $\overline{w} = \partial_1 \overline{u}^2 - \partial_2 \overline{u}^1$. For the sake of the simplicity, let $\gamma = \sqrt{2\beta}$. The first result of this section is the following lemma

Lemma 8. *Let* $\overline{u}_0 = (\overline{u}_0^1(x_1,x_2), \overline{u}_0^2(x_1,x_2), \overline{u}_0^3(x_1,x_2)) \in \mathbf{L}^2(\mathbb{R}_h^2)$ *be a divergence-free vector field, the horizontal vorticity of which*

$$\overline{w}_0 = \partial_1 \overline{u}_0^2 - \partial_2 \overline{u}_0^1 \in L^2(\mathbb{R}_h^2) \cap L^\infty(\mathbb{R}_h^2).$$

Then, the system (4)*, with initial data* $\overline{u}_0$*, has a unique, global solution*

$$\overline{u} \in \mathbf{C}(\mathbb{R}_+, \mathbf{L}^2(\mathbb{R}_h^2)) \cap \mathbf{L}^\infty(\mathbb{R}_+, \mathbf{L}^2(\mathbb{R}_h^2)).$$

Moreover,

(i) There exists a constant $C > 0$ *such that, for any* $p \geq 2$ *and for any* $t > 0$*, we have*

$$\left\|\nabla_h \overline{u}^h(t)\right\|_{\mathbf{L}^p(\mathbb{R}_h^2)} \leq CMp\, e^{-\gamma t}, \tag{7}$$

$$\left\|\overline{u}^h(t)\right\|_{\mathbf{L}^p(\mathbb{R}_h^2)} \leq CM e^{-\gamma t}, \tag{8}$$

where

$$M = \max\left\{\left\|\overline{u}_0^h\right\|_{\mathbf{L}^2(\mathbb{R}_h^2)}, \left\|\overline{w}_0\right\|_{L^2(\mathbb{R}_h^2)}, \left\|\overline{w}_0\right\|_{L^\infty(\mathbb{R}_h^2)}\right\}.$$

(ii) For any $p \geq 2$*, if* $\overline{u}_0^3 \in \mathbf{L}^p(\mathbb{R}_h^2)$*, then,*

$$\left\|\overline{u}^3(t)\right\|_{L^p(\mathbb{R}_h^2)} \leq \left\|\overline{u}_0^3\right\|_{L^p(\mathbb{R}_h^2)} e^{-\gamma t}. \tag{9}$$

To prove of Lemma 8, we remark that in (4), the first two components of $\overline{u}$ verify a two-dimensional Euler system with damping term. Then, according to the Yudovitch theorem [25] (see also [5]), this system has a unique solution $\overline{u}^h \in \mathbf{C}\left(\mathbb{R}_+, \mathbf{L}^2(\mathbb{R}_h^2)\right) \cap \mathbf{L}^\infty\left(\mathbb{R}_+, \mathbf{L}^2(\mathbb{R}_h^2)\right)$ such that the horizontal vorticity $\overline{w} \in \mathbf{L}^\infty(\mathbb{R}_+, \mathbf{L}^2(\mathbb{R}_h^2)) \cap \mathbf{L}^\infty(\mathbb{R}_+, \mathbf{L}^\infty(\mathbb{R}_h^2))$. Since the third component $\overline{u}^3$ satisfies a linear transport-type equation, then we can deduce the existence and uniqueness of the solution $\overline{u}$ of the limiting system (4). Then, Inequalities (7)-(9) can be deduced from classical $\mathbf{L}^p$ estimates for Euler equations and transport equations. □

Next, we need the following Brezis-Gallouet type inequality. For a proof of Lemma 9 below, see [5] or [20]. We also refer to ([4]).

Lemma 9. *Let* $r > 1$*. Under the hypotheses of Lemma 8 and the additional hypothesis that* $\overline{w}_0 \in H^r(\mathbb{R}_h^2)$*, there exists a positive constant* C_r *such that*

$$\left\|\nabla_h \overline{u}^h\right\|_{\mathbf{L}^\infty(\mathbb{R}_h^2)} \leq C_r \left\|\overline{w}\right\|_{L^\infty(\mathbb{R}_h^2)} \ln\left(e + \frac{\|\overline{w}\|_{H^r(\mathbb{R}_h^2)}}{\|\overline{w}\|_{L^\infty(\mathbb{R}_h^2)}}\right). \tag{10}$$

Now, we can give a $\mathbf{L}^\infty$-estimate of $\nabla_h \overline{u}^h$ in the following lemma.

Lemma 10. *Under the hypotheses of Lemma 9, there exist positive constants* C_1*,* C_2*, depending on* γ*,* $\|\overline{w}_0\|_{H^r(\mathbb{R}^2)}$*, such that*

$$\|\overline{w}(t)\|_{H^r(\mathbb{R}_h^2)} \leq C_1 e^{-\gamma t}, \tag{11}$$

and

$$\left\|\nabla_h \overline{u}^h(t)\right\|_{\mathbf{L}^\infty(\mathbb{R}_h^2)} \leq C_2 e^{-\gamma t}. \tag{12}$$

Proof of Lemma 10

First of all, there exist a constant $C > 0$ and a summable sequence of positive numbers $(d_q)_{q\geq -1}$ such that (see [20])

$$\left|\left\langle \Delta_q^h(\overline{u}^h \cdot \nabla_h \overline{w}) \mid \Delta_q^h \overline{w}\right\rangle\right| \leq C d_q 2^{-2qr} \left(\left\|\nabla_h \overline{u}^h\right\|_{\mathbf{L}^\infty} + \|\overline{w}\|_{L^\infty}\right) \|\overline{w}\|_{H^r}. \quad (13)$$

Then, for any $r > 1$, we get the following energy estimate in Sobolev H^r-norm:

$$\frac{1}{2}\frac{d}{dt}\|\overline{w}(t)\|_{H^r}^2 + \gamma \|\overline{w}(t)\|_{H^r}^2 \leq C\left(\|\overline{w}(t)\|_{L^\infty} + \left\|\nabla_h \overline{u}^h(t)\right\|_{\mathbf{L}^\infty}\right)\|\overline{w}(t)\|_{H^r}^2. \quad (14)$$

Taking into account Estimate (10), we rewrite (14) as follows

$$\frac{d}{dt}\|\overline{w}(t)\|_{H^r} + \gamma \|\overline{w}(t)\|_{H^r} \leq C \|\overline{w}\|_{L^\infty}\left(1 + \ln\left(e + \frac{\|\overline{w}\|_{H^r}}{\|\overline{w}\|_{L^\infty}}\right)\right)\|\overline{w}(t)\|_{H^r}. \quad (15)$$

Since $\overline{w}$ is solution of a linear transport equation, it is easy to prove that $\|\overline{w}(t)\|_{L^p} \leq CM\mathrm{e}^{-\gamma t}$, where C is a positive constant and

$$M = \max\left\{\left\|\overline{u}_0^h\right\|_{\mathbf{L}^2(\mathbb{R}_h^2)}, \|\overline{w}_0\|_{L^2(\mathbb{R}_h^2)}, \|\overline{w}_0\|_{L^\infty(\mathbb{R}_h^2)}\right\}.$$

Since C and M do not depend on p, we have

$$\|\overline{w}(t)\|_{L^\infty} \leq CM\mathrm{e}^{-\gamma t}.$$

Therefore, considering $y(t) = \|\overline{w}(t)\|_{H^r}\mathrm{e}^{\gamma t}$, we can deduce from (15) that

$$\frac{d}{dt}y(t) \leq CM\mathrm{e}^{-\gamma t}y(t)\left[1 + \ln\left(e + y(t)\right)\right]. \quad (16)$$

Integrating (16) with respect to t, we obtain the existence of $C_1 > 0$ such that

$$\|\overline{w}(t)\|_{H^r} \leq C_1\mathrm{e}^{-\gamma t}.$$

Combining the above estimate with (10) and using the fact that $x\ln\left(e + \frac{\alpha}{x}\right)$ is an increasing function, we obtain the existence of a positive constant C_2, depending on γ and $\|\overline{w}_0\|_{H^r}$, such that

$$\left\|\nabla_h \overline{u}^h(t)\right\|_{\mathbf{L}^\infty} \leq C_2\mathrm{e}^{-\gamma t}. \qquad \square$$

In what follows, we wish to prove an estimate similar to (12) for the third component $\overline{u}^3$ of the solution $\overline{u}$ of the system (4).

Lemma 11. *Let $2 < r < 3$ and $\overline{u}(t,x)$ be a solution of (4), with initial data $\overline{u}_0$ in $\mathbf{H}^r(\mathbb{R}_h^2)$. Then, there exist a positive constant C_3, depending on γ and $\|\overline{u}_0\|_{\mathbf{H}^r(\mathbb{R}_h^2)}$ such that, for any $t \geq 0$,*

$$\left\|\overline{u}^3(t)\right\|_{H^r(\mathbb{R}_h^2)} \leq C_3 e^{-\gamma t}. \quad (17)$$

Proof of Lemma 11

Differentiating two times the equation verified by $\overline{u}^3$, for any $i, j \in \{1, 2\}$, we have

$$\partial_t\partial_i\partial_j\overline{u}^3 + \gamma\partial_i\partial_j\overline{u}^3 + (\partial_i\partial_j\overline{u}^h)\cdot\nabla_h\overline{u}^3 + (\partial_i\overline{u}^h)\cdot\nabla_h\partial_j\overline{u}^3 \\ + (\partial_j\overline{u}^h)\cdot\nabla_h\partial_i\overline{u}^3 + \overline{u}^h\cdot\nabla_h\partial_i\partial_j\overline{u}^3 = 0.$$

Taking the H^{r-2} scalar product of the above equation with $\partial_i\partial_j\overline{u}^3$, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}&\left\|\partial_i\partial_j\overline{u}^3\right\|_{H^{r-2}}^2 + \gamma\left\|\partial_i\partial_j\overline{u}^3\right\|_{H^{r-2}}^2 \\ &\le \left|\left\langle(\partial_i\partial_j\overline{u}^h)\cdot\nabla_h\overline{u}^3 \mid \partial_i\partial_j\overline{u}^3\right\rangle_{H^{r-2}}\right| + \left|\left\langle(\partial_i\overline{u}^h)\cdot\nabla_h\partial_j\overline{u}^3 \mid \partial_i\partial_j\overline{u}^3\right\rangle_{H^{r-2}}\right| \\ &+ \left|\left\langle(\partial_j\overline{u}^h)\cdot\nabla_h\partial_i\overline{u}^3 \mid \partial_i\partial_j\overline{u}^3\right\rangle_{H^{r-2}}\right| + \left|\left\langle\overline{u}^h\cdot\nabla_h\partial_i\partial_j\overline{u}^3 \mid \partial_i\partial_j\overline{u}^3\right\rangle_{H^{r-2}}\right|.\end{aligned} \tag{18}$$

The divergence-free property allow us to write

$$\begin{aligned}(\partial_i\partial_j\overline{u}^h)\cdot\nabla_h\overline{u}^3 &= \partial_i\left((\partial_j\overline{u}^h)\cdot\nabla_h\overline{u}^3\right) - (\partial_j\overline{u}^h)\cdot\nabla_h\partial_i\overline{u}^3 \\ &= \partial_i\left((\partial_j\overline{u}^h)\cdot\nabla_h\overline{u}^3\right) - \mathrm{div}_h\left(\partial_i\overline{u}^3\partial_j\overline{u}^h\right).\end{aligned}$$

Then, using the Cauchy-Schwarz inequality, classical estimates for Sobolev spaces (see [[5], Theorem 2.4.1]) and the Sobolev embedding $H^{r-1}(\mathbb{R}_h^2) \hookrightarrow L^\infty(\mathbb{R}_h^2)$, we obtain

$$\begin{aligned}&\left|\left\langle(\partial_i\partial_j\overline{u}^h)\cdot\nabla_h\overline{u}^3 \mid \partial_i\partial_j\overline{u}^3\right\rangle_{H^{r-2}}\right| \\ &\quad\le \left\|(\partial_j\overline{u}^h)\cdot\nabla_h\overline{u}^3\right\|_{H^{r-1}}\left\|\partial_i\partial_j\overline{u}^3\right\|_{H^{r-2}} + \left\|\partial_i\overline{u}^3\partial_j\overline{u}^h\right\|_{\mathbf{H}^{r-1}}\left\|\partial_i\partial_j\overline{u}^3\right\|_{H^{r-2}} \\ &\quad\le \left(\left\|\partial_j\overline{u}^h\right\|_{\mathbf{L}^\infty}\left\|\overline{u}^3\right\|_{H^r} + \left\|\nabla_h\overline{u}^3\right\|_{\mathbf{L}^\infty}\left\|\partial_j\overline{u}^h\right\|_{\mathbf{H}^{r-1}}\right)\left\|\overline{u}^3\right\|_{H^r} \\ &\qquad\qquad + \left(\left\|\partial_i\overline{u}^3\right\|_{L^\infty}\left\|\partial_j\overline{u}^h\right\|_{\mathbf{H}^{r-1}} + \left\|\partial_j\overline{u}^h\right\|_{\mathbf{L}^\infty}\left\|\overline{u}^3\right\|_{H^r}\right)\left\|\overline{u}^3\right\|_{H^r} \\ &\quad\le C\left\|\overline{w}\right\|_{H^{r-1}}\left\|\overline{u}^3\right\|_{H^r}^2.\end{aligned}$$

The same arguments imply

$$\begin{aligned}&\left|\left\langle(\partial_i\overline{u}^h)\cdot\nabla_h\partial_j\overline{u}^3 \mid \partial_i\partial_j\overline{u}^3\right\rangle_{H^{r-2}}\right| \\ &\quad\le \left\|\mathrm{div}_h\left(\partial_j\overline{u}^3\partial_i\overline{u}^h\right) - \partial_j\overline{u}^3\partial_i(\mathrm{div}_h\overline{u}^h)\right\|_{H^{r-2}}\left\|\partial_i\partial_j\overline{u}^3\right\|_{H^{r-2}} \\ &\quad\le \left(\left\|\partial_j\overline{u}^3\right\|_{L^\infty}\left\|\partial_i\overline{u}^h\right\|_{\mathbf{H}^{r-1}} + \left\|\partial_i\overline{u}^h\right\|_{\mathbf{L}^\infty}\left\|\overline{u}^3\right\|_{H^r}\right)\left\|\overline{u}^3\right\|_{H^r} \\ &\quad\le C\left\|\overline{w}\right\|_{H^{r-1}}\left\|\overline{u}^3\right\|_{H^r}^2,\end{aligned}$$

and likewise,

$$\left|\left\langle(\partial_j\overline{u}^h)\cdot\nabla_h\partial_i\overline{u}^3 \mid \partial_i\partial_j\overline{u}^3\right\rangle_{H^{r-2}}\right| \le C\left\|\overline{w}\right\|_{H^{r-1}}\left\|\overline{u}^3\right\|_{H^r}^2.$$

For the last term of (18), since $2 < r < 3$, a slightly different version of Estimate (13) yields

$$\left|\left\langle\overline{u}^h\cdot\nabla_h\partial_i\partial_j\overline{u}^3 \mid \partial_i\partial_j\overline{u}^3\right\rangle_{H^{r-2}}\right| \le C\left\|\nabla_h\overline{u}^h\right\|_{\mathbf{L}^\infty}\left\|\partial_i\partial_j\overline{u}^3\right\|_{H^{r-2}}^2 \le C\left\|\nabla_h\overline{u}^h\right\|_{\mathbf{L}^\infty}\left\|\overline{u}^3\right\|_{H^r}^2.$$

Multiplying (18) by $\mathrm{e}^{\gamma t}$, then integrating the obtained equation with respect to time and using Lemma 10, we get

$$\begin{aligned}\left\|\overline{u}^3(t)\right\|_{H^r}\mathrm{e}^{\gamma t} &\le \left\|\overline{u}_0^3\right\|_{H^r} + C\int_0^t \left(\left\|\nabla_h\overline{u}^h(\tau)\right\|_{\mathbf{L}^\infty} + \left\|\overline{w}(\tau)\right\|_{H^{r-1}}\right)\left\|\overline{u}^3(\tau)\right\|_{H^r}\mathrm{e}^{\gamma\tau}d\tau \\ &\le \left\|\overline{u}_0^3\right\|_{H^r} + C(C_1+C_2)\int_0^t \left(\left\|\overline{u}^3(\tau)\right\|_{H^r}\mathrm{e}^{\gamma\tau}\right)\mathrm{e}^{-\gamma\tau}d\tau.\end{aligned}$$

Thus, the Gronwall lemma allow us to obtain (17). □

In the next paragraphs, we will not directly compare the system (1) with the limiting system (4) because of technical difficulties. Instead of (4), we consider the following system

$$\begin{cases}\partial_t\overline{u}^{\varepsilon,h} - \nu_h(\varepsilon)\Delta_h\overline{u}^{\varepsilon,h} + \gamma\overline{u}^{\varepsilon,h} + \overline{u}^{\varepsilon,h}\cdot\nabla_h\overline{u}^{\varepsilon,h} = -\nabla\overline{p}^\varepsilon \\ \partial_t\overline{u}^{\varepsilon,3} - \nu_h(\varepsilon)\Delta_h\overline{u}^{\varepsilon,3} + \gamma\overline{u}^{\varepsilon,3} + \overline{u}^{\varepsilon,h}\cdot\nabla_h\overline{u}^{\varepsilon,3} = 0 \\ \mathrm{div}_h\,\overline{u}^{\varepsilon,h} = 0 \\ \partial_3\overline{u}^\varepsilon = 0 \\ \overline{u}^\varepsilon{}_{|t=0} = \overline{u}_0\end{cases} \tag{19}$$

with $\lim_{\varepsilon\to 0}\nu_h(\varepsilon) = 0$.

Proposition 12. *Like the system* (4)*, the system* (19) *has a unique, global solution*

$$\overline{u}^\varepsilon \in \mathbf{C}\left(\mathbb{R}_+, \mathbf{L}^2(\mathbb{R}_h^2)\right) \cap \mathbf{L}^\infty\left(\mathbb{R}_+, \mathbf{L}^2(\mathbb{R}_h^2)\right) \cap \mathbf{L}^2\left(\mathbb{R}_+, \dot{\mathbf{H}}^1(\mathbb{R}_h^2)\right),$$

which also satisfies Lemmas 8, 9, 10 and 11.

In the following lemma, we will prove the convergence of $\overline{u}^\varepsilon$ towards $\overline{u}$ when ε goes to zero.

Lemma 13. *Suppose that $\nu_h(\varepsilon)$ converges to 0 when ε goes to 0 and that $\overline{u}_0 \in \mathbf{H}^\sigma(\mathbb{R}_h^2)$, $\sigma > 2$. Then, $\overline{u}^\varepsilon$ converges towards the solution $\overline{u}$ of* (4) *in $\mathbf{L}^\infty(\mathbb{R}_+, \mathbf{L}^2(\mathbb{R}_h^2))$, as ε goes to 0.*

Proof of Lemma 13

Using the previously proved results of this section, for any $t > 0$, we have

$$\left\|\overline{u}(t)\right\|_{\mathbf{L}^2(\mathbb{R}_h^2)} \le M\mathrm{e}^{-\gamma t} \quad \text{and} \quad \left\|\overline{u}^\varepsilon(t)\right\|_{\mathbf{L}^2(\mathbb{R}_h^2)} \le M\mathrm{e}^{-\gamma t}. \tag{20}$$

Thus, for fixed $\mu > 0$, there exists $T_\mu > 0$ such that, for any $t \ge T_\mu$,

$$\left\|\overline{u}^\varepsilon(t)\right\|_{\mathbf{L}^2(\mathbb{R}_h^2)} + \left\|\overline{u}(t)\right\|_{\mathbf{L}^2(\mathbb{R}_h^2)} \le \frac{\mu}{2}.$$

On the interval $[0, T_\mu]$, let $v^\varepsilon = \overline{u}^\varepsilon - \overline{u}$. Then, v^ε is a solution of the following system

$$\begin{cases} \partial_t v^{\varepsilon,h} - \nu_h(\varepsilon)\Delta_h v^{\varepsilon,h} + \gamma v^{\varepsilon,h} + \overline{u}^{\varepsilon,h}\cdot\nabla_h v^{\varepsilon,h} + v^{\varepsilon,h}\cdot\nabla_h \overline{u}^h = \nu_h \Delta_h \overline{u}^h - \nabla\tilde{p}, \\ \partial_t v^{\varepsilon,3} - \nu_h(\varepsilon)\Delta_h v^{\varepsilon,3} + \gamma v^{\varepsilon,3} + \overline{u}^{\varepsilon,h}\cdot\nabla_h v^{\varepsilon,3} + v^{\varepsilon,h}\cdot\nabla_h \overline{u}^3 = \nu_h \Delta_h \overline{u}^3, \\ \mathrm{div}_h v^{\varepsilon,h} = 0, \\ \partial_3 v^\varepsilon = 0, \\ v^\varepsilon_{|t=0} = 0. \end{cases}$$

Taking the $\mathbf{L}^2$-scalar product of the first two equations of the above system with $v^{\varepsilon,h}$ and $v^{\varepsilon,3}$ respectively, we get

$$\frac{1}{2}\frac{d}{dt}\|v^\varepsilon\|^2_{\mathbf{L}^2} + \nu_h(\varepsilon)\|\nabla_h v^\varepsilon\|^2_{\mathbf{L}^2} + \gamma\|v^\varepsilon\|^2_{\mathbf{L}^2} \leq \nu_h(\varepsilon)\|\nabla_h\overline{u}\|_{\mathbf{L}^2}\|\nabla_h v^\varepsilon\|_{\mathbf{L}^2} + \left|\left\langle v^{\varepsilon,h}\cdot\nabla_h\overline{u}|v^\varepsilon\right\rangle\right|.$$

Hence,

$$\frac{1}{2}\frac{d}{dt}\|v^\varepsilon\|^2_{\mathbf{L}^2} + \gamma\|v^\varepsilon\|^2_{\mathbf{L}^2} \leq \nu_h(\varepsilon)\|\nabla_h\overline{u}\|^2_{\mathbf{L}^2} + \|\nabla_h\overline{u}\|_{\mathbf{L}^\infty}\|v^\varepsilon\|^2_{\mathbf{L}^2}.$$

Integrating the obtained inequality, we come to

$$\|v^\varepsilon(t)\|^2_{\mathbf{L}^2} \leq \frac{\nu_h(\varepsilon)}{\gamma}\|\nabla_h\overline{u}\|^2_{\mathbf{L}^\infty([0,T_\mu],\mathbf{L}^2)} + \int_0^t \|\nabla_h\overline{u}(\tau)\|_{\mathbf{L}^\infty} \mathrm{e}^{-\gamma(t-\tau)}\|v^\varepsilon(\tau)\|^2_{\mathbf{L}^2}\,d\tau.$$

Then, the Gronwall Lemma proves that, for any $0 < t < T_\mu$,

$$\|v^\varepsilon(t)\|^2_{\mathbf{L}^2} \leq C\nu_h(\varepsilon)M^2 T_\mu \exp\left\{\int_0^t \|\nabla_h\overline{u}(\tau)\|_{\mathbf{L}^\infty}\,d\tau\right\}.$$

Combining with (20), this above estimate implies that

$$\lim_{\varepsilon\to 0}\|\overline{u}^\varepsilon - \overline{u}\|_{\mathbf{L}^\infty(\mathbb{R}_+,\mathbf{L}^2(\mathbb{R}^2_h))} = 0. \qquad \square$$

4 Ekman Boundary Layers

As mentioned in the introduction, when ε goes to 0, the fluid has the tendency to have a two-dimensional behavior. In the interior part of the domain, far from the boundary, the fluid moves in vertical columns, according to the Taylor-Proudman theorem. Near the boundary, the Taylor columns are destroyed and thin boundary layers are formed. The movements of the fluid inside the layers are very complex and the friction stops the fluid on the boundary. The goal of this paragraph is to briefly recall the mathematical construction of these boundary layers. More precisely, we will “correct” the solution of the limiting system (4) (which is a divergence-free vector field, independent of x_3) by adding a “boundary layer term” $\mathcal{B}$ such that $\overline{u}+\mathcal{B}$ is a divergence-free vector field which vanishes on the boundary.

In order to construct such boundary layers, a typical approach consists in looking for the approximate solutions of the system (1) in the following form (the Ansatz):

$$\begin{aligned} u^\varepsilon_{app} &= u_{0,int} + u_{0,BL} + \varepsilon u_{1,int} + \varepsilon u_{1,BL} + \dots \\ p^\varepsilon_{app} &= \frac{1}{\varepsilon} p_{-1,int} + \frac{1}{\varepsilon} p_{-1,BL} + p_{0,int} + p_{0,BL} + \dots, \end{aligned} \tag{21}$$

where the terms with the index "int" stand for the "interior" part, which is smooth functions of (x_h, x_3) and the index "BL" refers to the boundary layer part, which is smooth functions of the form

$$(x_h, x_3) \to F_0(t, x_h, \frac{x_3}{\delta}) + F_1(t, x_h, \frac{1-x_3}{\delta}),$$

where $F_0(x_h, \zeta)$ and $F_1(x_h, \zeta)$ rapidly decrease in ζ at infinity. The quantity $\delta > 0$, which goes to zero as ε goes to zero, represents the size of the boundary layers. It is proved that δ is of the same order as ε (see [16], [18], [8] and [9]). In this paper, we simply choose $\delta = \varepsilon$.

Let $E = 2\beta\varepsilon^2$ be the Ekman number and $\overline{u}$ be the solution of the limiting system (4). We recall that the third component $\overline{u}^3 = 0$ and we pose $\mathrm{curl}(\overline{u}) = \partial_1 \overline{u}^2 - \partial_2 \overline{u}^1$. In [16], [18] and [8], by studying carefully the Ansatz (21), the authors proved that we can write the boundary layer part in the following form

$$\mathcal{B} = \mathcal{B}^1 + \mathcal{B}^2 + \mathcal{B}^3 + \mathcal{B}^4,$$

where $\mathcal{B}^i$, $i \in \{1, 2, 3, 4\}$, are defined as follows.
1. The term $\mathcal{B}^1$ is defined by

$$\mathcal{B}^1 = \begin{pmatrix} \tilde{w}_1 + \breve{w}_1 \\ \tilde{w}_2 + \breve{w}_2 \\ \sqrt{\frac{E}{2}}\, \mathrm{curl}(\overline{u})\, G(x_3) \end{pmatrix}$$

where

$$\begin{aligned} \tilde{w}_1 &= -\mathrm{e}^{-\frac{x_3}{\sqrt{E}}} \left(\overline{u}^1 \cos\left(\frac{x_3}{\sqrt{E}}\right) + \overline{u}^2 \sin\left(\frac{x_3}{\sqrt{E}}\right)\right), \\ \tilde{w}_2 &= -\mathrm{e}^{-\frac{x_3}{\sqrt{E}}} \left(\overline{u}^2 \cos\left(\frac{x_3}{\sqrt{E}}\right) - \overline{u}^1 \sin\left(\frac{x_3}{\sqrt{E}}\right)\right), \\ \breve{w}_1 &= -\mathrm{e}^{-\frac{1-x_3}{\sqrt{E}}} \left(\overline{u}^1 \cos\left(\frac{1-x_3}{\sqrt{E}}\right) + \overline{u}^2 \sin\left(\frac{1-x_3}{\sqrt{E}}\right)\right), \\ \breve{w}_2 &= -\mathrm{e}^{-\frac{1-x_3}{\sqrt{E}}} \left(\overline{u}^2 \cos\left(\frac{1-x_3}{\sqrt{E}}\right) - \overline{u}^1 \sin\left(\frac{1-x_3}{\sqrt{E}}\right)\right), \\ G(x_3) &= -\mathrm{e}^{-\frac{x_3}{\sqrt{E}}} \sin\left(\frac{x_3}{\sqrt{E}} + \frac{\pi}{4}\right) + \mathrm{e}^{-\frac{1-x_3}{\sqrt{E}}} \sin\left(\frac{1-x_3}{\sqrt{E}} + \frac{\pi}{4}\right). \end{aligned}$$

2. The terms $\mathcal{B}^2$ and $\mathcal{B}^3$ are defined by

$$\mathcal{B}^2 = \begin{pmatrix} \sqrt{E}\,\overline{u}^2 \\ -\sqrt{E}\,\overline{u}^1 \\ \sqrt{E}\,\mathrm{curl}(\overline{u})\ \left(\frac{1}{2} - x_3\right) \end{pmatrix}$$

$$\mathcal{B}^3 = \mathrm{e}^{-\frac{1}{\sqrt{E}}} \cos\left(\frac{1}{\sqrt{E}}\right) \begin{pmatrix} \overline{u}^1 \\ \overline{u}^2 \\ 0 \end{pmatrix}.$$

3. Finally,

$$\mathcal{B}^4 = f(x_3) \begin{pmatrix} \overline{u}^2 \\ -\overline{u}^1 \\ 0 \end{pmatrix} + g(x_3) \begin{pmatrix} 0 \\ 0 \\ \mathrm{curl}(\overline{u}) \end{pmatrix},$$

where

$$\begin{aligned} f(x_3) &= a\left(\mathrm{e}^{-\frac{x_3}{\sqrt{E}}} + \mathrm{e}^{-\frac{1-x_3}{\sqrt{E}}}\right) + b, \\ g(x_3) &= -\sqrt{\frac{E}{2}}\, \mathrm{e}^{-\frac{1}{\sqrt{E}}} \sin\left(\frac{1}{\sqrt{E}} + \frac{\pi}{4}\right) - \int_0^{x_3} f(s)ds, \end{aligned}$$

and where (a, b) is the solution of the linear system

$$\begin{cases} a\left(1 + \mathrm{e}^{-\frac{1}{\sqrt{E}}}\right) + b = -\sqrt{E} + \mathrm{e}^{-\frac{1}{\sqrt{E}}} \sin\left(\frac{1}{\sqrt{E}}\right) \\ 2a\sqrt{E}\left(1 - \mathrm{e}^{-\frac{1}{\sqrt{E}}}\right) + b = \sqrt{2E}\mathrm{e}^{-\frac{1}{\sqrt{E}}} \sin\left(\frac{1}{\sqrt{E}} + \frac{\pi}{4}\right). \end{cases} \tag{22}$$

We remark that the determinant of the system (22) is

$$D = 1 + \mathrm{e}^{-\frac{1}{\sqrt{E}}} - 2\sqrt{E}\left(1 - \mathrm{e}^{-\frac{1}{\sqrt{E}}}\right).$$

Thus, for $\varepsilon > 0$ small enough, we have $D > \frac{1}{2}$ and (22) always has the following solution

$$a = \frac{J_E - K_E}{D} \qquad \text{and} \qquad b = \frac{K_E\left(1 + \mathrm{e}^{-\frac{1}{\sqrt{E}}}\right) - 2J_E\sqrt{E}\left(1 - \mathrm{e}^{-\frac{1}{\sqrt{E}}}\right)}{D},$$

where

$$J_E = -\sqrt{E} + \mathrm{e}^{-\frac{1}{\sqrt{E}}} \sin\left(\frac{1}{\sqrt{E}}\right),$$

$$K_E = \sqrt{2E}\mathrm{e}^{-\frac{1}{\sqrt{E}}} \sin\left(\frac{1}{\sqrt{E}} + \frac{\pi}{4}\right).$$

It is easy to prove that when $\varepsilon > 0$ is small enough, then

$$|a| < 4(\beta + \sqrt{\beta})\varepsilon \quad \text{and} \quad |b| < 32\beta\varepsilon^2.$$

With the previously defined boundary layer term $\mathcal{B}$, we can verify that

$$\mathrm{div}\,(\overline{u}+\mathcal{B})=0 \qquad \text{and} \qquad (\overline{u}+\mathcal{B})_{|\{x_3=0\}}=(\overline{u}+\mathcal{B})_{|\{x_3=1\}}=0.$$

Now, let

$$B_0(x_3)=\begin{bmatrix} -\mathrm{e}^{-\frac{x_3}{\sqrt{E}}}\cos\frac{x_3}{\sqrt{E}}-\mathrm{e}^{-\frac{1-x_3}{\sqrt{E}}}\cos\frac{1-x_3}{\sqrt{E}} & -\mathrm{e}^{-\frac{x_3}{\sqrt{E}}}\sin\frac{x_3}{\sqrt{E}}-\mathrm{e}^{-\frac{1-x_3}{\sqrt{E}}}\sin\frac{1-x_3}{\sqrt{E}} \\ \mathrm{e}^{-\frac{x_3}{\sqrt{E}}}\sin\frac{x_3}{\sqrt{E}}+\mathrm{e}^{-\frac{1-x_3}{\sqrt{E}}}\sin\frac{1-x_3}{\sqrt{E}} & -\mathrm{e}^{-\frac{x_3}{\sqrt{E}}}\cos\frac{x_3}{\sqrt{E}}-\mathrm{e}^{-\frac{1-x_3}{\sqrt{E}}}\cos\frac{1-x_3}{\sqrt{E}} \end{bmatrix}$$

Then, we can write $\mathcal{B}$ in the following form

$$\mathcal{B}=\mathcal{M}(x_3)A(t,x_1,x_2),$$

where

$$A(t,x_1,x_2)={}^t\big(\overline{u}^1,\overline{u}^2,\mathrm{curl}(\overline{u})\big) \qquad \text{and} \qquad \mathcal{M}(x_3)=\begin{bmatrix} M(x_3) & 0 \\ 0 & m(x_3)\end{bmatrix}$$

with $M(x_3)$ and $m(x_3)$ defined by

$$\begin{aligned} M(x_3)&=B_0(x_3)+\left(\sqrt{E}+f(x_3)\right)\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}+\mathrm{e}^{-\frac{1}{\sqrt{E}}}\cos\frac{1}{\sqrt{E}}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\\ m(x_3)&=\sqrt{\frac{E}{2}}\,G(x_3)+\sqrt{E}\left(\frac{1}{2}-x_3\right)+g(x_3). \end{aligned}$$

We can also prove the existence of a constant $C>0$ such that, for any $p\geq 1$, we have

$$\begin{cases} \|\mathcal{M}(\cdot)\|_{\mathbf{L}^p_{x_3}}\leq C\varepsilon^{\frac{1}{p}}, \quad \|\mathcal{M}(\cdot)\|_{\mathbf{L}^\infty_{x_3}}\leq C, \quad \|\mathcal{M}'(\cdot)\|_{\mathbf{L}^p_{x_3}}\leq C\varepsilon^{\frac{1}{p}-1},\\ \|m(\cdot)\|_{L^\infty_{x_3}}\leq C\varepsilon, \quad \|m(\cdot)\|_{L^p_{x_3}}\leq C\varepsilon \\ \sup\limits_{x_3\in[0,\frac{1}{2}]}\left|x_3^2M'(x_3)\right|\leq C\varepsilon \quad \text{and} \quad \sup\limits_{x_3\in[\frac{1}{2},1]}\left|(1-x_3)^2M'(x_3)\right|\leq C\varepsilon. \end{cases} \tag{23}$$

5 Convergence to the Limiting System

In this paragraph, we provide a priori estimates needed and a sketch the proof of Theorem 14. These a priori estimates can be justified by a classical approximation by smooth fonctions (see for instance [9]). For any $\varepsilon>0$, we consider the following 2D damped Navier-Stokes system with three components:

$$\begin{cases} \partial_t\,\overline{u}^{\varepsilon,h}-\nu_h(\varepsilon)\Delta_h\,\overline{u}^{\varepsilon,h}+\sqrt{2\beta}\,\overline{u}^{\varepsilon,h}+\overline{u}^{\varepsilon,h}\cdot\nabla_h\,\overline{u}^{\varepsilon,h}=-\nabla_h\,\overline{p}^{\varepsilon}\\ \partial_t\,\overline{u}^{\varepsilon,3}-\nu_h(\varepsilon)\Delta_h\,\overline{u}^{\varepsilon,3}+\sqrt{2\beta}\,\overline{u}^{\varepsilon,3}+\overline{u}^{\varepsilon,h}\cdot\nabla_h\,\overline{u}^{\varepsilon,3}=0\\ \mathrm{div}_h\overline{u}^{\varepsilon,h}=0\\ \partial_3\overline{u}^{\varepsilon}=0\\ \overline{u}^{\varepsilon}{}_{|t=0}=\overline{u}_0. \end{cases} \tag{24}$$

Then, Lemma 13 implies that Theorem 1 is a corollary of the following theorem

Theorem 14. *Suppose that*

$$\lim_{\varepsilon\to 0} \nu_h(\varepsilon) = 0 \quad \textit{and} \quad \lim_{\varepsilon\to 0} \frac{\varepsilon^{\frac{1}{2}}}{\nu_h(\varepsilon)} = 0.$$

Let $u_0^\varepsilon \in \mathbf{L}^2(\Omega)$ *be a family of initial data such that*

$$\lim_{\varepsilon\to 0} u_0^\varepsilon = \overline{u}_0 = (\overline{u}_0^1(x_1,x_2), \overline{u}_0^2(x_1,x_2), 0) \quad \textit{in} \quad \mathbf{L}^2(\mathbb{R}_h^2 \times [0,1]),$$

where $\overline{u}_0$ *is a divergence-free two-dimensional vector field in* $\mathbf{H}^\sigma(\mathbb{R}_h^2)$, $\sigma > 2$. *Let* $\overline{u}^\varepsilon$ *be the solution of the system* (24) *with initial data* $\overline{u}_0$ *and for each* $\varepsilon > 0$, *let* u^ε *be a weak solution of* (1) *with initial data* u_0^ε. *Then*

$$\lim_{\varepsilon\to 0} \|u^\varepsilon - \overline{u}^\varepsilon\|_{\mathbf{L}^\infty\left(\mathbb{R}_+, \mathbf{L}^2(\mathbb{R}_h^2\times[0,1])\right)} = 0.$$

Proof of Theorem 14

We first remark that we can construct the boundary layers term $\mathcal{B}^\varepsilon$ for the system (24) in the same way as we did to construct $\mathcal{B}$, with $\overline{u}$ being replaced by $\overline{u}^\varepsilon$. It is easy to prove that $\mathcal{B}^\varepsilon$ is small, *i.e.*, $\mathcal{B}^\varepsilon$ goes to 0 in $\mathbf{L}^\infty\left(\mathbb{R}_+, \mathbf{L}^2(\mathbb{R}_h^2 \times [0,1])\right)$ as ε goes to 0. Then, our goal is to prove that $v^\varepsilon = u^\varepsilon - \overline{u}^\varepsilon - \mathcal{B}^\varepsilon$ converge to 0 in $\mathbf{L}^\infty\left(\mathbb{R}_+, \mathbf{L}^2(\mathbb{R}_h^2 \times [0,1])\right)$ as ε goes to 0.

We recall that a two-dimensional divergence-free vector field (independant of x_3) belongs to the kernel of the operator $\mathbb{P}(e_3 \wedge \cdot)$, where $\mathbb{P}$ is the Leray projection of $\mathbf{L}^2(\mathbb{R}^3)$ onto the subspace of divergence-free vector fields. As a consequence, $e_3 \wedge \overline{u}^\varepsilon$ is a gradient term. Replacing u^ε by $v^\varepsilon + \overline{u}^\varepsilon + \mathcal{B}^\varepsilon$ in the system (1), we deduce that v^ε satisfied the following equation

$$\begin{aligned} \partial_t v^\varepsilon - \nu_h(\varepsilon)\Delta_h v^\varepsilon - \beta\varepsilon\partial_3^2 v^\varepsilon + L_1 + u^\varepsilon\cdot\nabla v^\varepsilon + \mathcal{B}^\varepsilon\cdot\nabla\mathcal{B}^\varepsilon \\ + \mathcal{B}^\varepsilon\cdot\nabla\overline{u}^\varepsilon + v^\varepsilon\cdot\nabla\mathcal{B}^\varepsilon + v^\varepsilon\cdot\nabla\overline{u}^\varepsilon - L_2 + \frac{e_3\wedge v^\varepsilon}{\varepsilon} = -\nabla\widetilde{p}^\varepsilon, \end{aligned} \quad (25)$$

where

$$\begin{aligned} L_1 &= \partial_t\mathcal{B}^\varepsilon - \nu_h(\varepsilon)\Delta_h\mathcal{B}^\varepsilon + \overline{u}^\varepsilon\cdot\nabla\mathcal{B}^\varepsilon \\ L_2 &= \beta\varepsilon\partial_3^2\mathcal{B}^\varepsilon - \frac{e_3\wedge\mathcal{B}^\varepsilon}{\varepsilon} + \sqrt{2\beta}\,\overline{u}^\varepsilon. \end{aligned}$$

Taking the $\mathbf{L}^2$ scalar product of (25) with v^ε, then integrating by parts the obtained equation and taking into account the fact that v^ε satisfies the Dirichlet boundary condition, we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|v^\varepsilon\|_{\mathbf{L}^2}^2 + \nu_h(\varepsilon)\|\nabla_h v^\varepsilon\|_{\mathbf{L}^2}^2 + \beta\varepsilon\|\partial_3 v^\varepsilon\|_{\mathbf{L}^2}^2 \\ = -\langle L_1, v^\varepsilon\rangle - \langle u^\varepsilon\cdot\nabla v^\varepsilon, v^\varepsilon\rangle - \langle\mathcal{B}^\varepsilon\cdot\nabla\mathcal{B}^\varepsilon, v^\varepsilon\rangle - \langle\mathcal{B}^\varepsilon\cdot\nabla\overline{u}^\varepsilon, v^\varepsilon\rangle \\ - \langle v^\varepsilon\cdot\nabla\mathcal{B}^\varepsilon, v^\varepsilon\rangle - \langle v^\varepsilon\cdot\nabla\overline{u}^\varepsilon, v^\varepsilon\rangle + \langle L_2, v^\varepsilon\rangle. \end{aligned} \quad (26)$$

In what follows, we will seperately estimate the seven terms on the right-hand side of Inequation (26). Using the same notations as in [16] and [18], we denote B_1, B_2 and b (V_1, V_2 and v respectively) the three components of $\mathcal{B}^\varepsilon$ (v^ε respectively) and we write $B = (B_1, B_2)$ et $V = (V_1, V_2)$.

1. Applying the operator curl to the first two equations of the system (24) (we recall that in this paper, curl only acts on the horizontal components and we already defined $\mathrm{curl}(\overline{u}) = \partial_1\overline{u}^2 - \partial_2\overline{u}^1$), we obtain

$$\partial_t(\mathrm{curl}\,\overline{u}^\varepsilon) - \nu_h(\varepsilon)\Delta_h(\mathrm{curl}\,\overline{u}^\varepsilon) + \sqrt{2\beta}\,(\mathrm{curl}\,\overline{u}^\varepsilon) + \overline{u}^\varepsilon\cdot\nabla\mathrm{curl}\,(\overline{u}^\varepsilon) = 0.$$

We recall that $A(t,x_1,x_2) = {}^t(\overline{u}_1^\varepsilon, \overline{u}_2^\varepsilon, \mathrm{curl}\,(\overline{u}^\varepsilon))$. So combining the above equation with the first two equations of (24), we deduce that

$$\partial_t A - \nu_h(\varepsilon)\Delta_h A + \sqrt{2\beta}\,A + \overline{u}^\varepsilon\cdot\nabla A = -(\nabla_h\overline{p}^\varepsilon, 0).$$

Since $\overline{u}^{\varepsilon,3} = 0$, div $v^\varepsilon = 0$, $\partial_3\overline{p} = 0$ and $\mathcal{B}^\varepsilon = \mathcal{M}(x_3)A(t,x_1,x_2)$, we can write

$$\begin{aligned}|\langle L_1, v^\varepsilon\rangle| &= |\langle\mathcal{M}(x_3)\left(\partial_t A - \nu_h(\varepsilon)\Delta_h A + \overline{u}^\varepsilon\cdot\nabla A\right), v^\varepsilon\rangle|\\ &\le \sqrt{2\beta}\,\|\mathcal{M}(\cdot)\|_{\mathbf{L}^2_{x_3}}\|A\|_{\mathbf{L}^2_{x_h}}\|v^\varepsilon\|_{\mathbf{L}^2}.\end{aligned}$$

Then, Estimate (23), Lemma 8 and Young's inequality imply

$$|\langle L_1, v^\varepsilon\rangle| \le C(\overline{u}_0)\,\varepsilon^{\frac12}\mathrm{e}^{-t\sqrt{2\beta}}\left(1 + \|v^\varepsilon\|^2_{\mathbf{L}^2}\right). \tag{27}$$

2. For the second term, using the divergence-free property of u^ε, we simply have

$$\langle u^\varepsilon\cdot\nabla v^\varepsilon, v^\varepsilon\rangle = 0. \tag{28}$$

3. We decompose the third term into two parts:

$$\langle\mathcal{B}^\varepsilon\cdot\nabla\mathcal{B}^\varepsilon, v^\varepsilon\rangle = \langle B\cdot\nabla_h\mathcal{B}^\varepsilon, v^\varepsilon\rangle + \langle b\partial_3\mathcal{B}^\varepsilon, v^\varepsilon\rangle.$$

Using an integration by parts, the "horizontal" part can be bounded as follows

$$|\langle B\cdot\nabla_h\mathcal{B}^\varepsilon, v^\varepsilon\rangle| \le |\langle(\mathrm{div}_h B)\mathcal{B}^\varepsilon, v^\varepsilon\rangle| + |\langle B\otimes\mathcal{B}^\varepsilon, \nabla_h v^\varepsilon\rangle| = |\langle B\otimes\mathcal{B}^\varepsilon, \nabla_h v^\varepsilon\rangle|.$$

Hence, Hölder's inequality, Estimates (23), Lemma 8 and Young's inequality yield

$$\begin{aligned}|\langle B\cdot\nabla_h\mathcal{B}^\varepsilon, v^\varepsilon\rangle| &\le \|B\|_{\mathbf{L}^4}\|\mathcal{B}^\varepsilon\|_{\mathbf{L}^4}\|\nabla_h v^\varepsilon\|_{\mathbf{L}^2} \\ &\le \|\mathcal{M}(\cdot)\|^2_{\mathbf{L}^4_{x_3}}\left(\|\overline{u}^\varepsilon\|^2_{\mathbf{L}^4_{x_h}} + \|\nabla_h\overline{u}^\varepsilon\|^2_{\mathbf{L}^4_{x_h}}\right)\|\nabla_h v^\varepsilon\|_{\mathbf{L}^2} \\ &\le C(\overline{u}_0)\,\varepsilon^{\frac12}\nu_h(\varepsilon)^{-1}\mathrm{e}^{-t\sqrt{2\beta}} + \frac{\nu_h(\varepsilon)}{16}\|\nabla_h v^\varepsilon\|^2_{\mathbf{L}^2}\end{aligned} \tag{29}$$

Likewise, we have the following estimate for the vertical part:

$$\begin{aligned}|\langle b\,\partial_3\mathcal{B}^\varepsilon, v^\varepsilon\rangle| &\le \|b\|_{L^\infty}\|\partial_3\mathcal{B}^\varepsilon\|_{\mathbf{L}^2}\|v^\varepsilon\|_{\mathbf{L}^2} \\ &\le C(\overline{u}_0)\,\mathrm{e}^{-t\sqrt{2\beta}}\|m(\cdot)\|_{L^\infty_{x_3}}\|\mathcal{M}'(\cdot)\|_{\mathbf{L}^2_{x_3}}\|v^\varepsilon\|_{\mathbf{L}^2} \\ &\le C(\overline{u}_0)\,\varepsilon^{\frac12}\mathrm{e}^{-t\sqrt{2\beta}}\left(1 + \|v^\varepsilon\|^2_{\mathbf{L}^2}\right).\end{aligned} \tag{30}$$

4. For the fourth term, taking into account the fact that $\overline{u}^\varepsilon$ is independent of x_3, Estimates (23) and Lemma 8 imply

$$\begin{aligned}|\langle \mathcal{B}^\varepsilon \cdot \nabla \overline{u}^\varepsilon, v^\varepsilon\rangle| &\le \|\mathcal{B}^\varepsilon\|_{\mathbf{L}^4_{x_h}\mathbf{L}^2_{x_3}} \|\nabla_h \overline{u}^\varepsilon\|_{\mathbf{L}^4_{x_h}} \|v^\varepsilon\|_{\mathbf{L}^2} \\ &\le \|\mathcal{M}(\cdot)\|_{\mathbf{L}^2_{x_3}} \left(\|\overline{u}^\varepsilon\|^2_{\mathbf{L}^4} + \|\nabla_h \overline{u}^\varepsilon\|^2_{\mathbf{L}^4}\right) \|v^\varepsilon\|_{\mathbf{L}^2} \\ &\le C(\overline{u}_0)\, \varepsilon^{\frac{1}{2}} \mathrm{e}^{-t\sqrt{2\beta}} \left(1 + \|v^\varepsilon\|^2_{\mathbf{L}^2}\right).\end{aligned} \tag{31}$$

5. The fifth term is the most difficult to treat. First, we decompose this term as follows

$$\langle v^\varepsilon \cdot \nabla \mathcal{B}^\varepsilon, v^\varepsilon\rangle = \langle V \cdot \nabla_h B, V\rangle + \langle V \cdot \nabla_h b, v\rangle + \langle v\, \partial_3 b, v\rangle + \langle v\, \partial_3 B, V\rangle .$$

For the first term on the right-hand side, Hölder inequality implies that

$$|\langle V \cdot \nabla_h B, V\rangle| \le C\, \|V\|_{\mathbf{L}^2} \|\nabla_h B\|_{\mathbf{L}^\infty} \|V\|_{\mathbf{L}^2} \le C\, \|\mathcal{M}(\cdot)\|_{\mathbf{L}^\infty_{x_3}} \|\nabla_h \overline{u}^\varepsilon\|_{\mathbf{L}^\infty_{x_h}} \|v^\varepsilon\|^2_{\mathbf{L}^2} .$$

Then, using Estimates (23) and Lemma 8, we obtain

$$|\langle V \cdot \nabla_h B, V\rangle| \le C(\overline{u}_0)\, \mathrm{e}^{-t\sqrt{2\beta}} \|v^\varepsilon\|^2_{\mathbf{L}^2} . \tag{32}$$

Next, by integrating by parts and using Hölder's inequality, we deduce that

$$|\langle V \cdot \nabla_h b, v\rangle| \le C\, \|\nabla_h v^\varepsilon\|_{\mathbf{L}^2} \|b\|_{\mathbf{L}^\infty} \|v^\varepsilon\|_{\mathbf{L}^2} .$$

So, Estimates (23), Lemmas 8 and 10 and Young's inequality imply

$$\begin{aligned}|\langle V \cdot \nabla_h b, v\rangle| &\le C\, \|m(\cdot)\|_{\mathbf{L}^\infty_{x_3}} \|\mathrm{curl}\, \overline{u}^\varepsilon\|_{\mathbf{L}^\infty_{x_h}} \|\nabla_h v^\varepsilon\|_{\mathbf{L}^2} \|v^\varepsilon\|_{\mathbf{L}^2} \\ &\le C(\overline{u}_0)\, \varepsilon^2 \nu_h(\varepsilon)^{-1} \mathrm{e}^{-t\sqrt{2\beta}} \|v^\varepsilon\|^2_{\mathbf{L}^2} + \frac{\nu_h(\varepsilon)}{16} \|\nabla_h v^\varepsilon\|^2_{\mathbf{L}^2} .\end{aligned} \tag{33}$$

Performing an integration by parts, we can control the third term in the same way as the second one:

$$\begin{aligned}|\langle v\, \partial_3 b, v\rangle| &= 2\, |\langle bv, \partial_3 v\rangle| = 2\, |\langle bv, \mathrm{div}\,_h V\rangle| \\ &\le C(\overline{u}_0)\, \varepsilon^2 \nu_h(\varepsilon)^{-1} \mathrm{e}^{-t\sqrt{2\beta}} \|v^\varepsilon\|^2_{\mathbf{L}^2} + \frac{\nu_h(\varepsilon)}{16} \|\nabla_h v^\varepsilon\|^2_{\mathbf{L}^2} .\end{aligned} \tag{34}$$

In order to estimate the last term of the right-hand side, we decompose it into two parts, the first part corresponding to the boundary layer near $\{x_3 = 0\}$ and the other corresponding to the one near $\{x_3 = 1\}$:

$$\langle v\, \partial_3 B, V\rangle = \int_{\mathbb{R}^2_h \times \left[0, \frac{1}{2}\right]} (v\, \partial_3 B) \cdot V dx + \int_{\mathbb{R}^2_h \times \left[\frac{1}{2}, 1\right]} (v\, \partial_3 B) \cdot V dx$$

For the fisrt part, since v^ε vanishes on $\{x_3 = 0\}$, using Hölder's inequality and Hardy-Littlewood inequality, we get

$$\begin{aligned}\left|\int_{\mathbb{R}^2_h \times \left[0, \frac{1}{2}\right]} (v\, \partial_3 B) \cdot V dx\right| &\le \sup_{x_3 \in \left[0, \frac{1}{2}\right]} \left|x_3^2 M'(x_3)\right| \|\overline{u}^\varepsilon\|_{\mathbf{L}^\infty_{x_h}} \left\|\frac{v}{x_3}\right\|_{L^2} \left\|\frac{V}{x_3}\right\|_{\mathbf{L}^2} \\ &\le \sup_{x_3 \in \left[0, \frac{1}{2}\right]} \left|x_3^2 M'(x_3)\right| \|\overline{u}^\varepsilon\|_{\mathbf{L}^\infty_{x_h}} \|\partial_3 v\|_{L^2} \|\partial_3 V\|_{\mathbf{L}^2} .\end{aligned}$$

We recall that $\partial_3 v = -\mathrm{div}_h V$. Then, Lemmas 8 and 10, Estimates (23) and Young's inequality imply

$$\left| \int_{\mathbb{R}^2_h \times [0,\frac{1}{2}]} (v\,\partial_3 B)\cdot V\,dx \right| \leq C(\overline{u}_0)\,\varepsilon\,\mathrm{e}^{-t\sqrt{2\beta}}\,\|\mathrm{div}_h v\|_{\mathbf{L}^2}\,\|\partial_3 V\|_{\mathbf{L}^2} \tag{35}$$

$$\leq C(\overline{u}_0)\,\varepsilon\,\|\nabla_h v^\varepsilon\|^2_{\mathbf{L}^2} + \frac{\beta\varepsilon}{4}\,\|\partial_3 v^\varepsilon\|^2_{\mathbf{L}^2}\,.$$

For the second part concerning the boundary layer near $\{x_3 = 1\}$, since $v^\varepsilon = (V, v)$ vanishes on $\{x_3 = 1\}$, Hardy-Littlewood inequality implies that

$$I_v = \int_{\mathbb{R}^2_h} \left(\int_{\frac{1}{2}}^1 \left| \frac{v(x_h, x_3)}{1 - x_3} \right|^2 dx_3 \right) dx_h = \int_{\mathbb{R}^2_h} \left(\int_0^{\frac{1}{2}} \left| \frac{v(x_h, 1 - x_3)}{x_3} \right|^2 dx_3 \right) dx_h$$

$$\leq C \int_{\mathbb{R}^2_h} \left(\int_0^{\frac{1}{2}} |\partial_3 v(x_h, 1 - x_3)|^2\, dx_3 \right) dx_h$$

$$\leq C\,\|\partial_3 v\|^2_{L^2} = C\,\|\mathrm{div}_h V\|^2_{L^2}\,.$$

Likewise,

$$I_V = \int_{\mathbb{R}^2_h} \left(\int_{\frac{1}{2}}^1 \left| \frac{V(x_h, x_3)}{1 - x_3} \right|^2 dx_3 \right) dx_h \leq C\,\|\partial_3 V\|^2_{L^2}\,.$$

Thus, using Hölder inequality, we get

$$\left| \int_{\mathbb{R}^2_h \times [\frac{1}{2},1]} (v\,\partial_3 B)\cdot V\,dx \right| \leq \sup_{x_3 \in [\frac{1}{2},1]} \left| (1 - x_3)^2 M'(x_3) \right| \|\overline{u}^\varepsilon\|_{\mathbf{L}^\infty_{x_h}} \sqrt{I_v}\sqrt{I_V} \tag{36}$$

$$\leq C(\overline{u}_0)\,\varepsilon\,\mathrm{e}^{-t\sqrt{2\beta}}\,\|\mathrm{div}_h v\|_{\mathbf{L}^2}\,\|\partial_3 V\|_{\mathbf{L}^2}$$

$$\leq C(\overline{u}_0)\,\varepsilon\,\|\nabla_h v^\varepsilon\|^2_{\mathbf{L}^2} + \frac{\beta\varepsilon}{4}\,\|\partial_3 v^\varepsilon\|^2_{\mathbf{L}^2}\,.$$

6. The sixth term on the right-hand side of (26) can be treated using Hölder inequality and Lemma 10. We have

$$|\langle v^\varepsilon \cdot \nabla \overline{u}^\varepsilon, v^\varepsilon \rangle| \leq C\,\|\nabla_h \overline{u}^\varepsilon\|_{\mathbf{L}^\infty(\mathbb{R}^2_h)}\,\|v^\varepsilon\|^2_{\mathbf{L}^2} \leq C(\overline{u}_0)\,\mathrm{e}^{-t\sqrt{2\beta}}\,\|v^\varepsilon\|^2_{\mathbf{L}^2} \tag{37}$$

7. We will evaluate the seventh term as in [16] or [18]. We have

$$\langle L_2, v^\varepsilon \rangle = \left\langle \beta\varepsilon\partial_3^2 \mathcal{B}^\varepsilon, v^\varepsilon \right\rangle - \left\langle \frac{e_3 \wedge \mathcal{B}^\varepsilon}{\varepsilon}, v^\varepsilon \right\rangle + \left\langle \sqrt{2\beta}\,\overline{u}^\varepsilon, v^\varepsilon \right\rangle.$$

We recall that

$$\mathcal{B}^\varepsilon = \mathcal{B}^{\varepsilon,1} + \mathcal{B}^{\varepsilon,2} + \mathcal{B}^{\varepsilon,3} + \mathcal{B}^{\varepsilon,4},$$

and for any $i \in \{1,2,3,4\}$, we set $\mathcal{B}^{\varepsilon,i} = (B^i, b^i)$, where B^i and b^i denote the horizontal and vertical components of $\mathcal{B}^{\varepsilon,i}$ respectively. Then, the following identities are immediate

$$\partial_3^2 B^3 = 0,$$

$$\beta\varepsilon\partial_3^2 B^1 - \frac{e_3 \wedge \mathcal{B}^1}{\varepsilon} = 0,$$

$$\beta\varepsilon\partial_3^2 B^2 - \frac{e_3 \wedge \mathcal{B}^2}{\varepsilon} + \sqrt{2\beta}\,\overline{u}^\varepsilon = 0.$$

For the remaining terms, we have

$$\begin{aligned} \beta\varepsilon\left|\left\langle\partial_3^2 b, v\right\rangle\right| &\leq \beta\varepsilon\left\|\partial_3 b\right\|_{\mathbf{L}^2}\left\|\partial_3 v\right\| \\ &\leq \beta\varepsilon\left\|\nabla_h B\right\|_{\mathbf{L}^2}\left\|\nabla_h v^\varepsilon\right\|_{\mathbf{L}^2} \\ &\leq \beta\varepsilon\left\|M(x_3)\right\|_{\mathbf{L}^2_{x_3}}\left\|\nabla_h\overline{u}^\varepsilon\right\|_{\mathbf{L}^2_{x_h}}\left\|\nabla_h v^\varepsilon\right\|_{\mathbf{L}^2} \\ &\leq C(\overline{u}_0)\,\varepsilon^3\nu_h(\varepsilon)^{-1}\mathrm{e}^{-t\sqrt{2\beta}} + \frac{\nu_h(\varepsilon)}{16}\left\|\nabla_h v^\varepsilon\right\|^2_{\mathbf{L}^2}; \end{aligned} \tag{38}$$

$$\left|\left\langle\frac{e_3 \wedge \mathcal{B}^3}{\varepsilon}, V\right\rangle\right| \leq C\varepsilon^{-1}\mathrm{e}^{-\frac{1}{\varepsilon}}\left\|\overline{u}^\varepsilon\right\|_{\mathbf{L}^2}\left\|v^\varepsilon\right\|_{\mathbf{L}^2} \leq C(\overline{u}_0)\,\varepsilon\,\mathrm{e}^{-t\sqrt{2\beta}}\left(1 + \left\|v^\varepsilon\right\|^2_{\mathbf{L}^2}\right). \tag{39}$$

We recall that

$$B^4 = f(x_3)\begin{pmatrix}\overline{u}^{\varepsilon,2} \\ -\overline{u}^{\varepsilon,1}\end{pmatrix},$$

where

$$f(x_3) \;=\; a\left(\mathrm{e}^{-\frac{x_3}{\sqrt{E}}} + \mathrm{e}^{-\frac{1-x_3}{\sqrt{E}}}\right) + b,$$

and where $E = 2\beta\varepsilon^2$ is the Ekman number. We also recall that, if $\varepsilon > 0$ is small enough, we have

$$|a| < 4(\beta + \sqrt{\beta})\varepsilon \quad \text{and} \quad |b| < 32\beta\varepsilon^2.$$

Then,

$$\begin{aligned} \beta\varepsilon\left|\left\langle\partial_3^2 B^4, V\right\rangle\right| &= \beta\varepsilon\left|\left\langle f''(x_3)\overline{u}^\varepsilon, V\right\rangle\right| \\ &\leq C\varepsilon^{\frac{1}{2}}\left\|\overline{u}^\varepsilon\right\|_{\mathbf{L}^2}\left\|v^\varepsilon\right\|_{\mathbf{L}^2} \\ &\leq C(\overline{u}_0)\,\varepsilon^{\frac{1}{2}}\mathrm{e}^{-t\sqrt{2\beta}}\left(1 + \left\|v^\varepsilon\right\|^2_{\mathbf{L}^2}\right). \end{aligned} \tag{40}$$

Finally, we have

$$\begin{aligned} \left|\left\langle\frac{e_3 \wedge \mathcal{B}^4}{\varepsilon}, V\right\rangle\right| &\leq C\left[\left(\int_0^1\left|\mathrm{e}^{-\frac{x_3}{\sqrt{E}}} + \mathrm{e}^{-\frac{1-x_3}{\sqrt{E}}}\right|^2 dx_3\right)^{\frac{1}{2}} + \beta\varepsilon\right]\left\|\overline{u}^\varepsilon\right\|_{\mathbf{L}^2}\left\|V\right\|_{\mathbf{L}^2} \\ &\leq C(\overline{u}_0)\,\varepsilon^{\frac{1}{2}}\mathrm{e}^{-t\sqrt{2\beta}}\left(1 + \left\|v^\varepsilon\right\|^2_{\mathbf{L}^2}\right). \end{aligned} \tag{41}$$

End of the proof: Summing all the inequalities from (27) to (41), we deduce from (26) that

$$\frac{d}{dt}\left\|v^\varepsilon\right\|_{\mathbf{L}^2}^2+\nu_h(\varepsilon)\left\|\nabla_h v^\varepsilon\right\|_{\mathbf{L}^2}^2+\beta\varepsilon\left\|\partial_3 v^\varepsilon\right\|_{\mathbf{L}^2}^2 \\ \leq C(\overline{u}_0)\,\varepsilon^{\frac{1}{2}}\nu_h(\varepsilon)^{-1}\mathrm{e}^{-t\sqrt{2\beta}}+C(\overline{u}_0)\,\mathrm{e}^{-t\sqrt{2\beta}}\left\|v^\varepsilon\right\|_{\mathbf{L}^2}^2+C(\overline{u}_0)\,\varepsilon\left\|\nabla_h v^\varepsilon\right\|_{\mathbf{L}^2}^2.$$

Since

$$\lim_{\varepsilon\to 0}\frac{\varepsilon^{\frac{1}{2}}}{\nu_h(\varepsilon)}=0,$$

there exists $\varepsilon_0=\varepsilon_0(\overline{u}_0)\in]0,1[$ such that, for any $0<\varepsilon<\varepsilon_0$, we have $C(\overline{u}_0)\,\varepsilon<\nu_h(\varepsilon)$. Therefore, for any $\varepsilon\in]0,\varepsilon_0[$ small enough, by integrating the above inequality with respect to the time variable, we get

$$\left\|v^\varepsilon(t)\right\|_{\mathbf{L}^2}^2\leq\left\|v^\varepsilon(0)\right\|_{\mathbf{L}^2}^2+\overline{C}(\overline{u}_0)\varepsilon^{\frac{1}{2}}\nu_h(\varepsilon)^{-1}+C(\overline{u}_0)\int_0^t\mathrm{e}^{-s\sqrt{2\beta}}\left\|v^\varepsilon(s)\right\|_{\mathbf{L}^2}^2 ds. \quad (42)$$

We recall that $v^\varepsilon=u^\varepsilon-\overline{u}^\varepsilon-\mathcal{B}^\varepsilon$. Thus,

$$\left\|v^\varepsilon(0)\right\|_{\mathbf{L}^2}^2\leq\left\|u^\varepsilon(0)-\overline{u}^\varepsilon(0)\right\|_{\mathbf{L}^2}^2+\left\|\mathcal{B}^\varepsilon(0)\right\|_{\mathbf{L}^2}^2 \\ \leq\left\|u_0^\varepsilon-\overline{u}_0\right\|_{\mathbf{L}^2}^2+\left\|\mathcal{M}(\cdot)\right\|_{\mathbf{L}^2_{x_3}}^2\left\|\overline{u}_0\right\|_{\mathbf{L}^2}^2\leq\left\|u_0^\varepsilon-\overline{u}_0\right\|_{\mathbf{L}^2}^2+C\varepsilon^{\frac{1}{2}}\left\|\overline{u}_0\right\|_{\mathbf{L}^2}^2.$$

According to Gronwall lemma, it follows from (42) that

$$\left\|v^\varepsilon(t)\right\|_{\mathbf{L}^2}^2\leq\left(\left\|u_0^\varepsilon-\overline{u}_0\right\|_{\mathbf{L}^2}^2+C\varepsilon^{\frac{1}{2}}\left\|\overline{u}_0\right\|_{\mathbf{L}^2}^2+\overline{C}(\overline{u}_0)\varepsilon^{\frac{1}{2}}\nu_h(\varepsilon)^{-1}\right)\exp\left\{\frac{\overline{C}(\overline{u}_0)}{\sqrt{2\beta}}\right\}.$$

Using the hypotheses that

$$\lim_{\varepsilon\to 0}\frac{\varepsilon^{\frac{1}{2}}}{\nu_h(\varepsilon)}=0 \quad\text{and}\quad \lim_{\varepsilon\to 0}\left\|u_0^\varepsilon-\overline{u}_0\right\|_{\mathbf{L}^2}=0,$$

we obtain

$$\lim_{\varepsilon\to 0}\left\|v^\varepsilon\right\|_{\mathbf{L}^\infty(\mathbb{R}_+,\mathbf{L}^2)}=0,$$

and Theorem 14 is proved. □

References

1. Babin, A., Mahalov, A., Nicolaenko, B.: Global Splitting, Integrability and Regularity of 3D Euler and Navier-Stokes Equations for Uniformly Rotating Fluids. European Journal of Mechanics 15, 291–300 (1996)
2. Babin, A., Mahalov, A., Nicolaenko, B.: Global regularity of 3D rotating Navier-Stokes equations for resonant domains. Indiana University Mathematics Journal 48, 1133–1176 (1999)
3. Bony, J.-M.: Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires. Annales de l'École Normale Supérieure 14, 209–246 (1981)

4. Brezis, H., Gallouet, T.: Nonlinear Schrödinger evolution equations. Nonlinear Analysis 4(4), 677–681 (1980)
5. Chemin, J.-Y.: Fluides parfaits incompressibles. Astérisque 230 (1995)
6. Chemin, J.-Y., Desjardins, B., Gallagher, I., Grenier, E.: Anisotropy and dispersion in rotating fluids. Nonlinear Partial Differential Equations and their Application, Collège de France Seminar, Studies in Mathematics and its Applications 31, 171–191 (2002)
7. Chemin, J.-Y., Desjardins, B., Gallagher, I., Grenier, E.: Fluids with anisotropic viscosity, Special issue for R. Temam's 60th birthday. M2AN. Mathematical Modelling and Numerical Analysis 34(2), 315–335 (2000)
8. Chemin, J.-Y., Desjardins, B., Gallagher, I., Grenier, E.: Ekman boundary layers in rotating fluids. ESAIM Controle Optimal et Calcul des Variations, A tribute to J.-L. Lions 8, 441–466 (2002)
9. Chemin, J.-Y., Desjardins, B., Gallagher, I., Grenier, E.: Mathematical Geophysics: An introduction to rotating fluids and to the Navier-Stokes equations. Oxford University Press (2006)
10. Cushman-Roisin, B.: Introduction to geophysical fluid dynamics. Prentice-Hall (1994)
11. Embid, E., Majda, A.: Averaging over fast gravity waves for geophysical flows with arbitrary potential vorticity. Communications in Partial Differential Equations 21, 619–658 (1996)
12. Fujita, H., Kato, T.: On the Navier-Stokes initial value problem I. Archiv for Rational Mechanic Analysis 16, 269–315 (1964)
13. Gallagher, I.: Applications of Schochet's Methods to Parabolic Equation. Journal de Mathématiques Pures et Appliquées 77, 989–1054 (1998)
14. Gallagher, I., Saint-Raymond, L.: Weak convergence results for inhomogeneous rotating fluid equations. Journal d'Analyse Mathématique 99, 1–34 (2006)
15. Grenier, E.: Oscillatory perturbations of the Navier-Stokes equations. Journal de Mathématiques Pures et Appliquées 76, 477–498 (1997)
16. Grenier, E., Masmoudi, N.: Ekman layers of rotating fluid, the case of well prepared initial data. Communications in Partial Differential Equations 22(5-6), 953–975 (1997)
17. Leray, J.: Essai sur le mouvement d'un liquide visqueux emplissant l'espace. Acta Matematica 63, 193–248 (1933)
18. Masmoudi, N.: Ekman layers of rotating fluids: the case of general initial data. Communications on Pure and Applied Mathematics 53(4), 432–483 (2000)
19. Ngo, V.-S.: Rotating Fluids with small viscosity. International Mathematics Research Notices IMRN (10), 1860–1890 (2009)
20. Ngo, V.-S.: Rotating fluids with small viscosity - The case of ill-prepared data (submitted)
21. Paicu, M.: Equation anisotrope de Navier-Stokes dans des espaces critiques. Revista Matemática Iberoamericana 21(1), 179–235 (2005)
22. Pedlosky, J.: Geophysical fluid dynamics. Springer (1979)
23. Schmitz, W.J.: Weakly depth-dependent segments of the North Atlantic circulation. Journal of Marine Research 38, 111–133
24. Schochet, S.: Fast singular limits of hyperbolic PDEs. Journal of Differential Equations 114, 476–512 (1994)
25. Yudovitch, V.: Non stationnary flows of an ideal incompressible fluid. Zh. Vych. Math. 3, 1032–1066 (1963)