Chapter 5
Non-supersymmetric Extremal Black Holes: First-Order Flows and Stabilisation Equations

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We review the results of [1, 2] on reducing the second-order equations of motion for stationary extremal black holes in four-dimensional $N = 2$ supergravity to first-order flow equations and further to non-differential stabilisation equations.

5.1 Introduction

Supergravity theories, being extensions of general relativity, admit black hole solutions. Finding them, as indeed any type of solutions in any theory, can be greatly simplified by a judicious exploitation of symmetries. One example, which would be valid also in Einstein's relativity, concerns space-time symmetries: for solutions translationally invariant in time, we can take the metric to be stationary, i.e. to have

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a timelike Killing vector. In the simplest case, without rotation or NUT charge, the ansatz for the line element can be chosen to be static (invariant under time reversal) and spherically symmetric.

In supergravity, we can in addition take advantage of supersymmetry, as long as the solution is also, at least partly, supersymmetric (or 'BPS'), i.e. when it is invariant under some of the supersymmetry transformations (a supersymmetric variation of the fields vanishes). If we look for classical, purely bosonic solutions (the expectation value of anticommuting fields must be zero, in other words fermions have no classical limit), the only non-trivial conditions (Killing spinor equations) come from the supersymmetry transformations of the fermionic fields, as the result of the supersymmetry transformations acting on bosons is fermionic and this by our assumption is automatically zero. Since the Killing spinor equations are of first order in derivatives of the fields, they are usually easier to solve than the second-order equations of motion.

Finally, we can make use of other internal symmetries of the theory, especially its invariance under duality transformations. We shall return to this point later, but already now let us mention that it may be helpful if the existence of these symmetries can be reflected in the description of the theory itself.

5.2 Special Geometry

In this review we restrict our attention to the bosonic sector of $N = 2$ ungauged supergravity in four spacetime dimensions [3, 4], which is one of the most widely encountered settings in the study of supergravity black holes, for it is sufficiently complicated to have physically interesting solutions, yet simple enough to be tractable. We consider only vector multiplets, because hypermultiplets do not couple to them and can be taken as constants in a self-consistent solution.

The remaining part of the action contains the familiar Einstein–Hilbert term for gravity, kinetic terms for n_v neutral complex scalars and $n_\mathrm{v} + 1$ abelian gauge fields (where n_v is the number of vector multiplets; the extra gauge field is the graviphoton of the gravity multiplet):

$$I_{4\mathrm{D}} = \frac{1}{16\pi} \int \Big(R \star 1 - 2\, g_{a\bar{b}}(z, \bar{z})\, \mathrm{d}z^a \wedge \star\, \mathrm{d}\bar{z}^{\bar{b}} - \mathrm{Im}\, \mathcal{N}_{IJ}(z, \bar{z})\, F^I \wedge \star\, F^J - \mathrm{Re}\, \mathcal{N}_{IJ}(z, \bar{z})\, F^I \wedge F^J \Big). \tag{5.1}$$

Importantly for our subsequent discussion, both the scalar manifold metric $g_{a\bar{b}}$ and the gauge kinetic matrix $\mathcal{N}$ depend on the scalars. This dependence, however, is not totally arbitrary. The scalar manifold is special Kähler, which means that its metric follows from the (real) Kähler potential K

$$g_{a\bar{b}} = \partial_a \partial_{\bar{b}} K(z, \bar{z}), \tag{5.2}$$

and that the Kähler potential itself can be determined by specifying a prepotential F (not to be confused with the field strengths $F^I = \mathrm{d}A^I$).[1] The prepotential is usually displayed in homogeneous (projective) coordinates $X^I(z)$, rather than the affine coordinates $z^a = X^a(z)/X^0(z)$ corresponding to physical scalars, and we shall take it to be cubic (so-called 'very special geometry'):

$$F = -\frac{1}{6} D_{abc} \frac{X^a(z) X^b(z) X^c(z)}{X^0(z)}. \tag{5.3}$$

The Kähler potential can be calculated as the symplectic invariant

$$K = -\ln\left[\mathrm{i}\begin{pmatrix} X^I(z) & F_I(z)\end{pmatrix}\begin{pmatrix} 0 & -\mathbb{I} \\ \mathbb{I} & 0\end{pmatrix}\overline{\begin{pmatrix} X^I(z) \\ F_I(z)\end{pmatrix}}\right] \tag{5.4}$$

$$=: -\ln\Big(\mathrm{i}\langle \Omega_{\mathrm{hol}}(z), \overline{\Omega_{\mathrm{hol}}(z)}\rangle\Big), \tag{5.5}$$

where $F_I = \partial_I F = \partial F/\partial X^I(z)$ and $\Omega_{\mathrm{hol}}(z)$ is the holomorphic section of special geometry. (In practice it will be more convenient to use the covariantly holomorphic section $\Omega = \mathrm{e}^{K/2}\Omega_{\mathrm{hol}}$.) The detailed form of the matrix $\mathcal{N}$ will not be needed in our considerations, but it too can be obtained from the prepotential.

5.3 Static Supersymmetric Black Holes

The static, spherically symmetric case is the simplest to analyse, because all spacetime-dependent quantities become functions of the radial coordinate τ only. For the line element one can then take [6]

$$\mathrm{d}s^2 = -\mathrm{e}^{2U(\tau)}\mathrm{d}t^2 + \mathrm{e}^{-2U(\tau)}\left(\frac{c^4}{\sinh^4(c\tau)}\mathrm{d}\tau^2 + \frac{c^2}{\sinh^2(c\tau)}\mathrm{d}\Omega^2_{(2)}\right), \tag{5.6}$$

where c is a non-extremality parameter and $\mathrm{d}\Omega^2_{(2)}$ is the metric of a unit two-sphere. (The horizon is located at $\tau \to \pm\infty$, depending on conventions, and $\tau \to 0$ corresponds to spatial infinity, where we shall also require $U = 0$ for asymptotic flatness.) The vector fields can be expressed in terms of the magnetic and electric charges

$$\Gamma := \begin{pmatrix} p^I \\ q_I \end{pmatrix} = \frac{1}{4\pi}\int_{S^2}\begin{pmatrix} F^I \\ G_I\end{pmatrix}, \tag{5.7}$$

[1] Strictly speaking, the existence of a prepotential is not guaranteed in every symplectic frame, but a frame where a prepotential exists can always be found by a symplectic transformation [5].

where $G_I = \operatorname{Im}\mathcal{N}_{IJ} \star F^J + \operatorname{Re}\mathcal{N}_{IJ} F^J$ is the dual field strength. Upon substitution one obtains [7] the action[2]

$$I_{\text{eff}} = -\frac{1}{4\pi}\int \mathrm{d}t \int \mathrm{d}\tau \left(\dot{U}^2 + g_{a\bar{b}}\dot{z}^a\dot{\bar{z}}^{\bar{b}} + \mathrm{e}^{2U} V_{\text{bh}}\right) \tag{5.8}$$

with an effective 'black hole potential'

$$V_{\text{bh}} = -\tfrac{1}{2}\Gamma^{\mathrm{T}}\mathcal{M}(\mathcal{N})\Gamma, \tag{5.9}$$

which is a function of the charges and (through the gauge couplings) of the scalars (again, the detailed form of the matrix $\mathcal{M}$ is immaterial for our purposes). This action reproduces the original equations of motion, except for one component of the Einstein equations, which is referred to as the Hamiltonian constraint:

$$\dot{U}^2 + g_{a\bar{b}}\dot{z}^a\dot{\bar{z}}^{\bar{b}} - \mathrm{e}^{2U} V_{\text{bh}} = c^2. \tag{5.10}$$

The black hole potential can be expressed in terms of the central charge[3] $Z = \langle \Gamma, \Omega\rangle$ as

$$V_{\text{bh}} = |Z|^2 + 4g^{a\bar{b}}\partial_a|Z|\partial_{\bar{b}}|Z|. \tag{5.11}$$

This, together with the Hamiltonian constraint and the identification of the mass with the asymptotic value of the tt component of the metric differentiated with respect to τ (which, by asymptotic flatness, equals $\dot{U}$ here) implies that solutions that saturate the supersymmetric BPS bound

$$M = |Z(z_\infty, \Gamma)| \tag{5.12}$$

must be extremal: $c = 0$. In that case the line element simplifies to

$$\mathrm{d}s^2 = -\mathrm{e}^{2U(\tau)}\mathrm{d}t^2 + \mathrm{e}^{-2U(\tau)}\left(\frac{1}{\tau^4}\mathrm{d}\tau^2 + \frac{1}{\tau^2}\mathrm{d}\Omega^2_{(2)}\right) \tag{5.13}$$

with $\tau = \pm 1/|\mathbf{x}|$, so the terms in parentheses represent the flat metric on $\mathbb{R}^3$.

Formula (5.11) also makes it possible to rewrite the effective Lagrangian (up to a total derivative term, which can be neglected) as a sum of squares:

$$\mathcal{L}_{\text{eff}} \propto \left(\dot{U} + \mathrm{e}^U|Z|\right)^2 + \left\|\dot{z}^a + 2\mathrm{e}^U g^{a\bar{b}}\partial_{\bar{b}}|Z|\right\|^2. \tag{5.14}$$

[2] We adopt the plus sign for the potential term, as is customary in the black hole potential literature.

[3] More precisely, Z is a function of the radial coordinate. It coincides with the central charge of the supersymmetry algebra at spatial infinity.

Since $\delta(\ldots)^2 = 2(\ldots)\delta(\ldots)$, the action attains a stationary value when the two brackets vanish separately, which immediately leads to first-order gradient flow equations that by construction satisfy the (second-order) equations of motion:

$$\dot{U} = -\mathrm{e}^U |Z|, \tag{5.15}$$

$$\dot{z}^a = -2\mathrm{e}^U g^{a\bar{b}} \partial_{\bar{b}} |Z|. \tag{5.16}$$

The flow generated by these equations terminates at $\dot{z}^a = 0$ in the scalar manifold and on the horizon in spacetime. This means that the critical points of $|Z(z, \bar{z}, \Gamma)|$ determine the horizon values of the scalars in terms of the charges (attractor mechanism).

The conditions for critical points can be brought to the form known as the attractor equations [8]

$$2\mathrm{Im}(\bar{Z}\Omega) = \Gamma. \tag{5.17}$$

When the scalars are constant in spacetime (so-called 'doubly extremal black holes'), the attractor values are taken everywhere. The non-constant solutions are given in terms of harmonic functions by the similar stabilisation equations[4] [9]

$$2\mathrm{Im}(\Omega_{\mathrm{hol}}) = \mathcal{H}, \qquad \mathcal{H} = \Gamma\tau + h, \tag{5.18}$$

where h is the vector of constants related to the asymptotic values of the scalars.

5.4 Beyond the Static, Supersymmetric Case

For supersymmetric black holes, as we indicated in the introduction, the fact that the equations of motion can be reduced to first-order equations is not surprising. More unexpectedly, it turns out [10] that non-supersymmetric and even non-extremal black holes can enjoy a like description. In the non-supersymmetric extremal case this becomes evident [11] when the black hole potential remains invariant under rotations of the charge vector by a matrix S, $\Gamma \mapsto \tilde{\Gamma} = S\Gamma$:

$$S^{\mathrm{T}}\mathcal{M}S = \mathcal{M} \quad \Longrightarrow \quad V_{\mathrm{bh}} = W^2 + 4g^{a\bar{b}}\partial_a W \partial_{\bar{b}} W, \quad W = |\langle\tilde{\Gamma}, \Omega\rangle|. \tag{5.19}$$

Flow equations based on the superpotential[5] W (or 'fake central charge', as it is built from the 'fake charges' $\tilde{\Gamma}$) are entirely analogous to (5.15), (5.16), except that the respective attractor equations may be underdetermined and may not fix the horizon

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[4] Some authors interchange the meaning of 'stabilisation equations' and 'attractor equations' relative to our nomenclature or use the term 'generalised stabilisation equations' for those involving harmonic functions. Occasionally the name 'stabilisation equations' is given to the relation, implied by Eq. (5.18), between the real and imaginary parts of the symplectic section.

[5] In some papers called the prepotential.

values of scalars completely in terms of the charges (this is due to flat directions of the effective potential; the entropy, being related to the stationary value of the potential, is therefore still independent of the asymptotic values of the scalars [12]). In the non-extremal case [13] one seeks an expansion of the whole potential term (including the warp factor) into squares of partial derivatives of a (generalised) superpotential Y:

$$e^{2U} V_{bh} = (\partial_U Y)^2 + 4g^{a\bar{b}}\partial_a Y \partial_{\bar{b}} Y - c^2. \tag{5.20}$$

This again leads to gradient flow equations, but not to the attractor mechanism, because unlike Z or W, Y depends also on the metric function U.

What about extremal, but not necessarily static and spherically symmetric black holes (for instance, rotating stationary solutions or multicentre configurations)? Could one still reduce the equations of motion to first-order equations and integrate them to non-differential stabilisation equations? To answer these questions we proceed in a similar manner to the static case, but using a different formalism, originally devised by Denef [14] for supersymmetric solutions.

Let us first reinterpret or redefine the quantities that we have already encountered from the geometrical perspective, when the supergravity considered is viewed as a low-energy approximation of type IIB string theory compactified on a Calabi–Yau three-fold M_{CY}. Specifically, the number of vector multiplets is equal to one of the Hodge numbers, $n_v = h^{2,1}$, the holomorphic symplectic section Ω_{hol} can be identified with the unique up to rescaling (3, 0)-form that characterises the compactification manifold and the symplectic product becomes the intersection product, for instance

$$e^{-K} = i \int_{M_{CY}} \Omega_{hol} \wedge \bar{\Omega}_{hol}. \tag{5.21}$$

In the canonical basis $\{\alpha_I, \beta^I\}$ of the third integral cohomology $H^3(M_{CY}, \mathbb{Z})$:

$$\Omega_{hol} = X^I(z)\alpha_I - F_I(z)\beta^I, \qquad \Gamma = p^I\alpha_I - q_I\beta^I \tag{5.22}$$

and the electromagnetic fields can be seen as the components of the IIB five-form

$$\mathcal{F} = F^I \otimes \alpha_I - G_I \otimes \beta^I, \tag{5.23}$$

which should be self-dual in 10 dimensions. Decomposing this condition into 4 + 6 dimensions as $\star_{10}\mathcal{F} = (\star \otimes \diamond)\mathcal{F} = \mathcal{F}$ we can relate the Hodge operator on the internal manifold to matrix $\mathcal{M}$ appearing in the black hole potential:

$$\diamond \begin{pmatrix} \beta^I \\ \alpha_I \end{pmatrix} = \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix} \mathcal{M} \begin{pmatrix} \beta^I \\ \alpha_I \end{pmatrix}. \tag{5.24}$$

For the spacetime line element we take [15]

$$ds^2 = -e^{2U(\mathbf{x})}\big(dt + \omega(\mathbf{x})\big)^2 + e^{-2U(\mathbf{x})}\delta_{ij}dx^i dx^j, \quad \omega(\mathbf{x}) = \omega_i(\mathbf{x})dx^i \tag{5.25}$$

and this time we trade the manifest Lorentz invariance of the action for the duality invariance by writing the Lagrangian in terms of the spatial components of field strengths $\boldsymbol{\mathcal{F}}$ (boldface signals here and below the spatial part) and in terms of the product

$$(\boldsymbol{\mathcal{B}}, \boldsymbol{\mathcal{C}}) = \frac{e^{2U}}{1-w^2}\int_{M_{\text{CY}}} \boldsymbol{\mathcal{B}} \wedge \big[\star_0(\diamond\boldsymbol{\mathcal{C}}) - \star_0(w \wedge \diamond\boldsymbol{\mathcal{C}})\, w + \star_0(w \wedge \star_0\boldsymbol{\mathcal{C}})\big], \tag{5.26}$$

defined for any $\boldsymbol{\mathcal{B}}, \boldsymbol{\mathcal{C}} \in \Omega^2(\mathbb{R}^3) \otimes H^3(M_{\text{CY}})$, where $\star_0$ is the Hodge dual with respect to the flat metric δ_{ij} in space and $\Omega^2(\mathbb{R}^3)$ is the set of two-forms on space. The result is

$$\begin{aligned} I_{4\text{D eff}} = -\frac{1}{16\pi}\int dt \int_{\mathbb{R}^3} \Big[& 2\mathbf{d}U \wedge \star_0 \mathbf{d}U - \tfrac{1}{2}e^{4U}\mathbf{d}\omega \wedge \star_0\mathbf{d}\omega \\ &+ 2g_{a\bar b}\,\mathbf{d}z^a \wedge \star_0\mathbf{d}\bar z^{\bar b} + (\boldsymbol{\mathcal{F}}, \boldsymbol{\mathcal{F}})\Big]. \end{aligned} \tag{5.27}$$

Analogously to the derivation in the previous section, we need to rewrite the effective Lagrangian in a form that yields first-order equations. The suitable pairing between the scalars and the gauge fields is afforded by the combination

$$\boldsymbol{\mathcal{G}} = \boldsymbol{\mathcal{F}} - 2\text{Im}\,\star_0\mathbf{D}(e^{-U}e^{-i\alpha}\Omega) + 2\text{Re}\,\mathbf{D}(e^{U}e^{-i\alpha}\Omega\,\omega), \tag{5.28}$$

where

$$\mathbf{D} = \mathbf{d} + i(\mathbf{Q} + \mathbf{d}\alpha + \tfrac{1}{2}e^{2U}\star_0\mathbf{d}\omega), \qquad \mathbf{Q} = \text{Im}(\partial_a K\,\mathbf{d}z^a) \tag{5.29}$$

and $\alpha(\mathbf{x})$ is at this stage an arbitrary function. The Lagrangian becomes

$$\begin{aligned} \mathcal{L}_{\text{eff}} = (\boldsymbol{\mathcal{G}}, \boldsymbol{\mathcal{G}}) &- 4\,(\mathbf{Q} + \mathbf{d}\alpha + \tfrac{1}{2}e^{2U}\star_0\mathbf{d}\omega) \wedge \text{Im}\langle\boldsymbol{\mathcal{G}}, e^{U}e^{-i\alpha}\Omega\rangle \\ &+ \mathbf{d}\Big[2w \wedge (\mathbf{Q} + \mathbf{d}\alpha) + 4\text{Re}\langle\boldsymbol{\mathcal{F}}, e^{U}e^{-i\alpha}\Omega\rangle\Big]. \end{aligned} \tag{5.30}$$

Neglecting the total derivative term and requiring that the variations of the remaining two terms vanish separately, directly leads to the first-order equations

$$\boldsymbol{\mathcal{F}} - 2\text{Im}\,\star_0\mathbf{D}(e^{-U}e^{-i\alpha}\Omega) + 2\text{Re}\,\mathbf{D}(e^{U}e^{-i\alpha}\Omega\,\omega) = 0, \tag{5.31}$$

$$\mathbf{Q} + \mathbf{d}\alpha + \tfrac{1}{2}e^{2U}\star_0\mathbf{d}\omega = 0. \tag{5.32}$$

The second relation implies $\mathbf{D} = \mathbf{d}$, so taking the differential of the first yields the Laplace equation

$$2\,\mathbf{d}\star_0\mathbf{d}(e^{-U}e^{-i\alpha}\Omega) = 0 \tag{5.33}$$

on account of the closure of the field strength ($\mathbf{d}\mathcal{F} = 0$). The resulting supersymmetric stabilisation equations have the same form [16] as those for static, spherically black holes (in the gauge $K = 2U$, $\alpha = 0$), but now the harmonic functions can be multicentred, with the constituents located at $\mathbf{x}_n$:

$$2\mathrm{Im}(\mathrm{e}^{-U}\mathrm{e}^{-\mathrm{i}\alpha}\Omega) = \mathcal{H}, \qquad \mathcal{H} = \sum_{n=1}^{N} \Gamma_n \tau_n + h_n, \tag{5.34}$$

where $\tau_n = \pm 1/|\mathbf{x} - \mathbf{x}_n|$. The poles of harmonic functions must be the physical charges, as mandated by Eq. (5.31), and the constants h_n dictate the asymptotic values of the scalars. Once the stabilisation equations have been solved, all unknowns, in particular the metric warp factor and the one-form ω, can be calculated.

A non-supersymmetric generalisation [1, 2] inspired by the superpotential approach can be achieved through the replacement of $\mathcal{F}$ by a fake field strength $\tilde{\mathcal{F}}$ that reproduces the same gauge term: $(\tilde{\mathcal{F}}, \tilde{\mathcal{F}}) = (\mathcal{F}, \mathcal{F})$. If these field strengths are related by a constant matrix, the above derivation can be repeated without any other adjustments, but the relevant multicentre configurations, with centres at arbitrary distances from each other, have vanishing angular momentum and purely imaginary scalars z^a.

Allowing the relation between the fake and actual field strengths to be arbitrary causes a number of complications, due to the fact that $\tilde{\mathcal{F}}$ may not be closed. The condition of reproducing the original gauge term in the Lagrangian has to be relaxed by adding a possible three-form deviation $\boldsymbol{\Xi}$

$$(\tilde{\mathcal{F}}, \tilde{\mathcal{F}}) = (\mathcal{F}, \mathcal{F}) + \boldsymbol{\Xi}, \tag{5.35}$$

and a new term $\boldsymbol{\eta}$ appears in the rewriting:

$$\begin{aligned} \mathcal{L} = (\tilde{\mathcal{G}}, \tilde{\mathcal{G}}) - 4\,(\mathbf{Q} + \mathbf{d}\alpha + \boldsymbol{\eta} + \tfrac{1}{2}\mathrm{e}^{2U}\star_{\mathbf{0}}\mathbf{d}\omega) \wedge \mathrm{Im}\langle\tilde{\mathcal{G}}, \mathrm{e}^{U}\mathrm{e}^{-\mathrm{i}\alpha}\Omega\rangle \\ + \mathbf{d}\,[\,2w \wedge (\mathbf{Q} + \mathbf{d}\alpha) + 4\mathrm{Re}\langle\tilde{\mathcal{F}}, \mathrm{e}^{U}\mathrm{e}^{-\mathrm{i}\alpha}\Omega\rangle\,], \end{aligned} \tag{5.36}$$

where

$$\tilde{\mathcal{G}} = \tilde{\mathcal{F}} - 2\mathrm{Im}\star_{\mathbf{0}}\mathbf{D}(\mathrm{e}^{-U}\mathrm{e}^{-\mathrm{i}\alpha}\Omega) + 2\mathrm{Re}\,\mathbf{D}(\mathrm{e}^{U}\mathrm{e}^{-\mathrm{i}\alpha}\Omega\omega). \tag{5.37}$$

The two new quantities $\boldsymbol{\Xi}$ and $\boldsymbol{\eta}$ are constrained by consistency requirements to satisfy:

$$\boldsymbol{\Xi} = -2\mathbf{d}\boldsymbol{\eta} \wedge w, \tag{5.38}$$

$$\boldsymbol{\eta} \wedge \mathrm{Im}\langle\tilde{\mathcal{G}}, \mathrm{e}^{U}\mathrm{e}^{-\mathrm{i}\alpha}\Omega\rangle = \langle\mathbf{d}\tilde{\mathcal{F}}, \mathrm{Re}(\mathrm{e}^{U}\mathrm{e}^{-\mathrm{i}\alpha}\Omega)\rangle + \tfrac{1}{4}\boldsymbol{\Xi}. \tag{5.39}$$

Evidently, the general stationary extremal case can be reduced to first-order equations $\tilde{\mathcal{G}} = 0$ and $\mathbf{D} = \mathbf{d} - \mathrm{i}\boldsymbol{\eta}$, but the description is much more involved than before: even after elimination of $\boldsymbol{\Xi}$ we are left with an additional unknown object $\boldsymbol{\eta}$, which has to obey a complicated equation. Crucially, the combination $\mathrm{e}^{-U}\mathrm{e}^{-\mathrm{i}\alpha}\Omega$ is generically no longer determined by Laplace's equation, thus the solutions will no longer be given in terms of pure harmonic functions.

5.5 An Ansatz for Stabilisation Equations

Although in the non-supersymmetric case we are unable to integrate the equations of motion directly to non-differential stabilisation equations, we can still try to find a suitable ansatz [2]. To that end it is useful to look at the known 'almost-BPS' [17] seed solution [18] for single-centred under-rotating extremal black holes in four-dimensional theories with cubic prepotentials[6]:

$$2\mathrm{Im}\,\hat{\Omega} = \tilde{\mathcal{H}} + \tilde{\mathcal{R}}, \qquad \boldsymbol{\mathcal{F}} = \star_0 \mathbf{d}\mathcal{H} - 2\,\mathbf{d}(\mathrm{e}^{2U}\mathrm{Re}\,\hat{\Omega}\,\omega), \tag{5.40}$$

where $\hat{\Omega} = \mathrm{e}^{-U}\mathrm{e}^{-\mathrm{i}\alpha}\Omega$ and

$$\mathcal{H} = (H^0,\ 0;\ 0,\ H_a), \quad \tilde{\mathcal{H}} = (-H^0,\ 0;\ 0,\ H_a), \quad \tilde{\mathcal{R}} = \left(0,\ 0;\ \frac{M}{H^0},\ 0\right). \tag{5.41}$$

The harmonic functions are:

$$H^0 = h^0 + p^0\tau, \qquad H_a = h_a + q_a\tau, \qquad M = b + J\tau^2\cos\theta, \tag{5.42}$$

where J denotes the angular momentum.

Besides the familiar harmonic part, in this case $\tilde{\mathcal{H}}$, the stabilisation equations involve ratios of harmonic functions, $\tilde{\mathcal{R}}$. As one can verify, the new object that we had to introduce to compensate for the possible non-closure of $\tilde{\boldsymbol{\mathcal{F}}}$ does not vanish if $M \neq 0$:

$$\boldsymbol{\eta} = \mathrm{e}^{2U}\langle \mathbf{d}\tilde{\mathcal{R}}, \tilde{\mathcal{H}}\rangle = -\mathrm{e}^{2U} H^0\,\mathbf{d}\left(\frac{M}{H^0}\right). \tag{5.43}$$

Interestingly, the anharmonic part of the stabilisation equations persists even without the angular momentum, as long as the constant b responsible for the non-trivial flow of the axions $\mathrm{Re}\,z^a$ is non-zero.

Under duality transformations $\tilde{\mathcal{H}}$ and $\tilde{\mathcal{R}}$ will change: in particular the relation between $\tilde{\mathcal{H}}$ and $\mathcal{H}$ will be more complicated than merely a switch of sign. The structure of the ansatz for the stabilisation equations stays nonetheless the same. Since the starting configuration is a seed solution, we postulate that in general the right-hand side of the stabilisation equations for extremal black holes is given by a sum of harmonic functions and ratios of harmonic functions. Whereas the anharmonic part may vanish, as it did in the supersymmetric case, the harmonic part must be always present, to ensure the correct near-horizon behaviour.

[6] The notation has been slightly changed in comparison with [2].

5.6 Conclusions

As we have seen, it is possible to derive first-order equations not only for supersymmetric black holes. In the stationary extremal case this can be accomplished by a merger of the superpotential approach with Denef's duality-covariant formalism. The essential step in the rewriting is a suitable pairing between the scalar degrees of freedom and the gauge fields.

Unfortunately, the procedure applied to stationary non-supersymmetric extremal black holes does not offer nearly as much simplification as in the BPS case. This is partly due to the fact that the superpotential approach itself requires a new quantity (originally: the superpotential, here: the fake field strength or the vector of fake charges), whose relation to the physical parameters needs to be determined separately, and partly due to the anharmonicity of the stabilisation equations, reflecting the non-closure of the fake field strength. Nevertheless, the combinations of the variables entering the stabilisation equations are universal, in the sense that when one uses them as the new degrees of freedom, the equations of motion expressed in terms of them take the same form for all black hole solutions, irrespective of supersymmetry or extremality [19, 20].

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