Chapter 4
Non-extremal Black-Hole Solutions of $\mathcal{N} = 2,\ d = 4,\ 5$ Supergravity

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4.1 Introduction

Black holes have been intensely studied in the framework of string theory for the last 20 years. They are described by classical solutions of the supergravity theories that describe effectively the low-energy dynamics of different string compactifications. Being solutions of theories with local supersymmetry one can distinguish among them the particular class of those that preserve some unbroken supersymmetries (called supersymmetric or, less precisely, BPS).

The special properties enjoyed by these black-hole solutions makes them very interesting (they are the ones for which the entropy was first computed by counting their microstates [1] and they are among those for which there is an attractor mechanism at work [2–5]) and easier to construct. For instance, it is known how to construct, systematically, all the black-hole solutions of any theory of ungauged $\mathcal{N} = 2,\ d = 4$ [6–9][1] and $d = 5$ [11, 12] supergravity coupled to any number vector supermultiplets and some general results are also known for higher-$\mathcal{N}$, $d = 4$ supergravities [13]. Some supersymmetric black-hole solutions of non-Abelian gauged $\mathcal{N} = 2,\ d = 4$ supergravity are also known in fully analytic form [14–16].

In spite of their interest, non-extremal black-hole solutions of these theories are much less known. Here we are going to review recent progress in the construction of non-extremal black holes and branes, particularly in ungauged theories of $\mathcal{N} = 2$, $d = 4$ and $d = 5$ supergravity coupled to vector supermultiplets [17–25].

This progress is based, first of all, in the use of the FGK formalism [26], conveniently generalized in [23] to arbitrary spacetime dimension $d \geq 4$ and worldvolume dimension $p \geq 0$. Usually, only the results of [26] concerning extremal black holes

[1] See also [10] for a review.

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and attractors are used, but the formalism provides a setting which simplifies the task of finding explicit solutions. We review this formalism in Sect. 4.2.

A second ingredient is the general ansatz for single, static, non-extremal black holes of $\mathcal{N} = 2, d = 4, 5$ supergravity presented in [19, 20]. This ansatz, which we review in Sect. 4.3, can be understood as a deformation of the general supersymmetric solution of [6–9] in which the harmonic functions (traditionally denoted by H^i) are replaced by linear combinations of hyperbolic sines and cosines, but the physical fields have the same form in terms of those functions as they had in terms of the H^i.

Several arguments in support of the generality of this ansatz were given in [19], but the main assumption that the functional form of the physical fields in terms of the functions H^i can be given stronger foundations. In [17, 18] for the $\mathcal{N} = 2,\ d = 5$ case and in [21, 22] for the $\mathcal{N} = 2,\ d = 4$ case, it was shown that the H^i can be used as dynamical variables in the reduced action of the FGK formalism. The change of variables from the physical fields to the H^i assumes the same functional dependence of the former on the later both for extremal (supersymmetric and non supersymmetric) and non-extremal black holes, proving the assumption. This allows the use of these variables in more complex settings, such as rotating black holes or black-holes in gauged supergravities, as had been observed before.

The use of these variables in combination with the FGK formalism (a combination that we call H-FGK formalism) simplifies considerably the task of finding general, explicit extremal and non-extremal solutions and also general results about families of solutions (see [24] and [25] for the 5- and 4-dimensional cases, respectively), as we will review in Sect. 4.4.

4.2 FGK Formalism

In this Section we will review the generalization presented in [23] to arbitrary spacetime dimension $d \geq 4$ and worldvolume dimension $p \geq 0$ of the formalism introduced in [26] for 4-dimensional black holes. The generalization to $d \geq 4$ is necessary to study the black holes of $\mathcal{N} = 2,\ d = 5$ theories and the generalization to black p-branes will allow us to study the black string solutions of those theories. The black holes of $\mathcal{N} = 2,\ d = 4$ can be obtained by direct dimensional reduction of the 5-dimensional black holes and double dimensional reduction of the black strings, hence the interest in these objects.

4.2.1 Derivation of the Effective Action

The main ingredients of the FGK formalism are a generic action which can describe the relevant bosonic sectors of most (or all) the ungauged supergravities and a generic metric and coordinate choice which can describe the exterior of all the single, static, black p-brane solutions of those theories. The generic action is then reduced using as reduction ansatz the generic metric, which has only one undetermined function that will remain a variable of the dimensionally reduced equations of motion. The staticity of the ansatz leaves us with only one parameter on which the physical fields (metric function plus scalar fields) can depend in the dimensionally reduced equations of motion. it is, then, possible to find an effective 1-dimensional (mechanical) action for the remaining variables (which has to be supplemented by a constraint) from which one can derive the equations of motion and general results concerning the black p-brane solutions of those theories.

The generic action that we propose is

$$\mathcal{I}[g, A^{\Lambda}_{(p+1)}, \phi^i] = \int d^d x \sqrt{|g|} \Big\{ R + \mathcal{G}_{ij}(\phi)\partial_\mu \phi^i \partial^\mu \phi^j + 4\tfrac{(-1)^p}{(p+2)!} I_{\Lambda\Sigma}(\phi) F^{\Lambda}_{(p+2)} \cdot F^{\Sigma}_{(p+2)} \Big\}, \tag{4.1}$$

where the scalar fields ϕ^i parametrize a non-linear σ-model with metric $\mathcal{G}_{ij}(\phi)$, $I_{\Lambda\Sigma}(\phi)$ is a scalar-dependent, negative-definite (kinetic) matrix that describes the coupling of the scalars to the $(p+1)$-forms $A^{\Lambda}_{(p+1)}$ to which the p-branes couple electrically,

$$F^{\Lambda}_{(p+2)\,\mu_1\cdots\mu_{p+2}} = (p+2)\partial_{[\mu_1|} A^{\Lambda}_{(p+1)\,|\mu_2\cdots\mu_{p+2}]}, \tag{4.2}$$

are their field strengths and we have used the notation

$$F^{\Lambda}_{(p+2)} \cdot F^{\Sigma}_{(p+2)} \equiv F^{\Lambda}_{(p+2)\,\mu_1\cdots\mu_{p+2}} F^{\Sigma}_{(p+2)}{}^{\mu_1\cdots\mu_{p+2}}. \tag{4.3}$$

We define, as usual, the worldvolume dimension of the dual brane $\tilde{p} \equiv d-p-4$. In general $p \neq \tilde{p}$ and neither the dual $\tilde{p}$-brane can couple to the $(p+1)$-forms $A^{\Lambda}_{(p+1)}$ nor the electric p-branes can couple to the dual $(\tilde{p}+1)$-forms $A_{\Lambda\,(\tilde{p}+1)}$. Thus, the above model is generically sufficient.

However, there are particular cases in which the above model is too simple: when $p = \tilde{p} = (d-4)/2$ one should consider additional terms in the action of the form

$$+4\xi^2 \tfrac{(-1)^p}{(p+2)!} R_{\Lambda\Sigma}(\phi) F^{\Lambda}_{(p+2)} \cdot \star F^{\Sigma}_{(p+2)}. \tag{4.4}$$

Here $R_{\Lambda\Sigma}(\phi)$ is a scalar-dependent matrix such that

$$R_{\Lambda\Sigma} = -\xi^2 R_{\Sigma\Lambda}, \tag{4.5}$$

where

$$\xi^2 \equiv \star^2 = -(-1)^{d/2} = (-1)^{p+1}, \tag{4.6}$$

and $\star$ is the operator that relates $(p+2)$-form field strengths to their $(\tilde{p}+2)$-form Hodge duals. In these cases our ansatz must take into account that the same brane can also be magnetically charged i.e. they can be dyonic.

There is yet another particular case: when $d = 4n+2$ the dyonic branes can also be self- or anti-self-dual because the $(p+2)$-form field strengths can also be self- or anti-self-duals. When this is the case, the electric and magnetic charges must be equal up to a sign that depends on the self- or anti-self-duality and on the conventions used.

The second ingredient mentioned at the beginning of this section is a generic ansatz for the metric of charged, static, flat[2], black p-brane in $d = p + \tilde{p} + 4$ dimensions, with a transverse radial coordinate ρ chosen in such a way that the event horizon is at $\rho \to \infty$.

We have arrived at this ansatz[3] by studying (see the Appendix in Ref. [23]) the metrics of well-known families of p-brane solutions, such as those originally found in Ref. [28] and reproduced in Ref. [29] whose conventions and notations we follow here.

The ansatz depends on two independent functions of the radial (in the $(\tilde{p}+3)$-dimensional transverse space) coordinate ρ $\tilde{U}(\rho)$ and $W(\rho)$ to be found by solving the equations of motion and a "background" metric in the $(\tilde{p}+3)$-dimensional transverse space $\gamma_{(\tilde{p}+3)\,\underline{mn}}$ which has the fixed form

$$\gamma_{(\tilde{p}+3)\,\underline{mn}}dx^m dx^n = \left(\frac{\omega/2}{\sinh\left(\frac{\omega}{2}\rho\right)}\right)^{\frac{2}{\tilde{p}+1}} \left[\left(\frac{\omega/2}{\sinh\left(\frac{\omega}{2}\rho\right)}\right)^2 \frac{d\rho^2}{(\tilde{p}+1)^2} + d\Omega^2_{(\tilde{p}+2)}\right], \tag{4.7}$$

where, $d\Omega^2_{(\tilde{p}+2)}$ is the metric of the round $(\tilde{p}+2)$-sphere of unit radius and ω is the *non-extremality parameter*, denoted by r_0 in the 4-dimensional case considered in Refs. [19, 26].[4] Furthermore, the worldvolume of the p-brane is parametrized by the time coordinate t and the p spacelike coordinates $(y^1, \cdots, y^p)$ that we denote collectively by $\mathbf{y}_{(p)}$.

With all these elements, the generic metric takes the form

$$ds^2_{(d)} = e^{\frac{2}{p+1}\tilde{U}}\left[W^{\frac{p}{p+1}}dt^2 - W^{-\frac{1}{p+1}}d\mathbf{y}^2_{(p)}\right] - e^{-\frac{2}{\tilde{p}+1}\tilde{U}}\gamma_{(\tilde{p}+3)\,\underline{mn}}dx^m dx^n. \tag{4.8}$$

Some comments are in order. First, observe that this metric reduces in the $p = 0$ case to the metrics used in $d = 4$ and arbitrary d-dimensional black holes in Refs. [20, 26] respectively (W disappears and $\tilde{U}$ is just U in the notation used in those references). Secondly, observe that, for general values of p, we have two independent functions $\tilde{U}$ and W instead of just one, as in the black-hole case which should be recovered after dimensional reduction. the presence of W cannot be "gauged away": while it is possible to redefine $\tilde{U}$ and the transverse metric $\gamma_{(\tilde{p}+3)\,\underline{mn}}$ so as to totally absorb W in some components of the metric, it is not possible to do it simultaneously in all of them.

[2] By this we mean that the metric of the spatial part of its worldvolume is Euclidean. As we are going to see, the metric of the full worldvolume is not flat.

[3] This metric has also been derived from the equations of motion in Refs. [27].

[4] In higher dimensions ω is not a length, hence the change in notation.

Although the presence of one additional independent function is somewhat unexpected, is should be clear that there is nothing wrong with using it as long as we perform the reduction substituting the ansatz for the metric directly in the equations of motion. The reduced equations of motion will then tell us whether we have two independent functions or just one and what is the relation between them in the second case. We will also use normalization and regularity conditions to further constrain these functions.

The ansatz for $(p+1)$-form potentials $A^{\Lambda}_{(p+1)}$ for electrically-charged p-branes is

$$A^{\Lambda}_{(p+1)\,ty_1\cdots y_p} = \psi^{\Lambda}(\rho), \tag{4.9}$$

(all the other components vanish).

In the particular case $p = \tilde{p} = (d-4)/2$, in which the branes can also be magnetically charged with respect to the dual (*magnetic*) $(p+1)$-form potentials that we are going to denote by $A_{(p+1)\,\Lambda}$, we have to start by giving a proper definition of these dual potentials. The starting point are the equations of motion of the *electric* $(p+1)$-form potentials which are the only ones that appear in the original action Eq. (4.1). As mentioned before, in this particular case the action has to be supplemented by the term in Eq. (4.4). Taking all this into account the equations of motion can be written as

$$dG_{(p+2)\,\Lambda} = 0, \tag{4.10}$$

where the $(p+2)$-form $G_{(p+2)\,\Lambda}$ (*magnetic field strengths*) is defined by

$$G_{(p+2)\,\Lambda} \equiv R_{\Lambda\Sigma}F^{\Sigma}_{(p+2)} + I_{\Lambda\Sigma} \star F^{\Sigma}_{(p+2)}. \tag{4.11}$$

As it is well known, these differential equations imply the local existence of the magnetic $(p+1)$-form potentials $A_{(p+1)\,\Lambda}$ satisfying

$$G_{(p+2)\,\Lambda} = dA_{(p+1)\,\Lambda}. \tag{4.12}$$

Then, in this particular cases, we also make the following ansatz for the magnetic potentials

$$A_{(p+1)\,\Lambda\,ty_1\cdots y_p} = \chi_{\Lambda}(\rho). \tag{4.13}$$

This ansatz implies that some of the spatial components of the electric potentials $A^{\Lambda}_{(p+1)}$ do not vanish and, actually, have complicated dependencies on the angular coordinates of the transverse $(p+2)$ sphere. The magnetic potentials codify very efficiently these complicated dependencies and their use (the relevant spatial components of the electric potentials can be expressed quite easily in terms of the time component of the magnetic ones in the static case that we are considering) simplifies the reduction of the equations of motion.

It is convenient to arrange all the electric and magnetic $(p+2)$-form field strengths and electrostatic and magnetostatic potentials into single vectors whose components are labeled by $M, N, \ldots$

$$\left(\mathcal{F}^M\right) \equiv \begin{pmatrix} F^\Lambda \\ G_\Lambda \end{pmatrix}, \qquad \left(\Psi^M\right) \equiv \begin{pmatrix} \psi^\Lambda \\ \chi_\Lambda \end{pmatrix}. \tag{4.14}$$

In terms of the vector of field strengths so the Bianchi identities and Maxwell equations can be written as

$$d\mathcal{F}^M = 0, \tag{4.15}$$

which is covariant under linear transformations

$$\begin{pmatrix} F' \\ G' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} F \\ G \end{pmatrix}, \tag{4.16}$$

where $A,\ B,\ C,\ D$ are constant matrices, but not all of them are consistent with the definition of the magnetic field strengths in terms of the electric ones Eq. (4.12). This definition must be preserved by the linear transformations if they are going to be symmetries of the equations of motion of the theory and this requires that the scalar-dependent matrices R, I transform according to

$$N' = (C + DN)\,(A + BN)^{-1}\,, \tag{4.17}$$

where we have defined the matrix N by

$$N \equiv R + \xi I. \tag{4.18}$$

In $d = 4$ $\xi = i$ and $N_{\Lambda\Sigma} \equiv \mathcal{N}_{\Lambda\Sigma}$ is the complex *period matrix*.

On the other hand, using the above-defined vectors, the contribution of the $(p+1)$-form potentials to the energy-momentum tensor can be written in the compact form

$$\Omega_{MN} \star \mathcal{F}^M{}_{\mu\alpha_1\cdots\alpha_{p+1}} \mathcal{F}^N{}_\nu{}^{\alpha_1\cdots\alpha_{p+1}}, \tag{4.19}$$

where

$$(\Omega_{MN}) \equiv \begin{pmatrix} 0 & \mathbb{I} \\ \xi^2\mathbb{I} & 0 \end{pmatrix}, \tag{4.20}$$

plays the rôle of a metric that we will use to raise and lower M, N indices. This implies that, in order to preserve the Einstein equations, the linear transformations of the n electric and n magnetic field strengths must be restricted to the group $\mathrm{O}(n, n)$ when $\xi^2 = +1$ and to the group $\mathrm{Sp}(2n+2, \mathbb{R})$ when $\xi^2 = -1$ (in particular, for $d = 4$ dimensions).

There is an alternative expression for this contribution to the energy-momentum tensor which turns out to be very useful when we perform the reduction of the Einstein equations with the above ansatz:

$$\mathcal{M}_{MN}(N)\mathcal{F}^M{}_{\mu\alpha_1\cdots\alpha_{p+1}} \mathcal{F}^N{}_\nu{}^{\alpha_1\cdots\alpha_{p+1}}, \tag{4.21}$$

where the symmetric matrix $\mathcal{M}_{MN}(N)$ is given by

$$
\begin{aligned}
(\mathcal{M}_{MN}(N)) &\equiv \begin{pmatrix} I - \xi^2 R I^{-1} R & \xi^2 R I^{-1} \\ \\ -I^{-1} R & I^{-1} \end{pmatrix}, \\
\left(\mathcal{M}^{MN}(N)\right) &= \begin{pmatrix} I^{-1} & -\xi^2 I^{-1} R \\ \\ R I^{-1} & I - \xi^2 R I^{-1} R \end{pmatrix} = (\mathcal{M}_{NP}(N))^{-1}.
\end{aligned}
\tag{4.22}
$$

These formulae are only relevant in the particular case $p = \tilde{p} = (d-4)/2$. However, we can use them in any dimension including the additional terms (matrix $R_{\Lambda\Sigma}$, magnetic charges p^Λ etc.) in the understanding that they vanish whenever the condition is not satisfied (and $R_{\Lambda\Sigma} = p^\Lambda = 0$).

The last piece of our ansatz is the assumption that the scalar fields ϕ^i only depend on the radial coordinate ρ.

Plugging this ansatz into the equations of motion derived from the action Eq. (4.1) supplemented by the term in Eq. (4.4) we get five sets of equations for $\tilde{U}$, W, the potentials Ψ^M and the scalars ϕ^i. We consider first the following two equations:

$$\frac{d^2 \ln W}{d\rho^2} = 0, \tag{4.23}$$

$$\frac{d}{d\rho}\left[e^{-2\tilde{U}}\, \mathcal{M}_{MN}\, \dot{\Psi}^N\right] = 0. \tag{4.24}$$

(overdots denoting derivatives w.r.t. ρ) that can be integrated immediately. The result is, normalizing $W(0) = 1$ at spatial infinity, introducing the integration constants γ and $\mathcal{Q}_M$, and the normalization constant α

$$W = e^{\gamma\rho}, \tag{4.25}$$

$$\dot{\Psi}^M = \alpha e^{2\tilde{U}} \mathcal{M}^{MN} \mathcal{Q}_N. \tag{4.26}$$

The constants $\mathcal{Q}_M$ are, up to the global normalization represented by the constant α, just the electric and magnetic charges of the dyonic p-brane with respect to the $(p+1)$-form potentials

$$\mathcal{Q}_M \sim \int_{S^{\tilde{p}+2}} \star \mathcal{M}_{MN} \mathcal{F}^N, \quad (\mathcal{Q}^M) \equiv \begin{pmatrix} p^\Lambda \\ q_\Lambda \end{pmatrix}, \quad \mathcal{Q}_M \equiv \Omega_{MN} \mathcal{Q}^N. \tag{4.27}$$

With $W = e^{\gamma\rho}$ the metric ansatz takes the form

$$ds^2_{(d)} = e^{\frac{2}{p+1}\tilde{U}}\left[e^{\frac{p}{p+1}\gamma\rho} dt^2 - e^{-\frac{1}{p+1}\gamma\rho} d\mathbf{y}^2_{(p)}\right] - e^{-\frac{2}{\tilde{p}+1}\tilde{U}} \gamma_{(\tilde{p}+3)\,\underline{mn}} dx^m dx^n, \tag{4.28}$$

and only depends on one undetermined function, $\tilde{U}$, as expected. It, however, depends on two constants, ω and γ which are, a priori, independent. We only expect one constant in the metric (since we should be able to reduce it to a black-hole's) and, actually, we can eliminate one of them by requiring the regularity of the black brane's horizon.

Let us study the near-horizon limit of the above metric. In this limit, the angular part of the transverse metric behaves as

$$\sim e^{\frac{1}{\tilde{p}+1}\omega\rho}(-\omega)^{\frac{2}{\tilde{p}+1}}d\Omega^2_{(\tilde{p}+2)}, \tag{4.29}$$

which means in that black p-branes with regular horizons $\tilde{U}$ must behave as

$$\tilde{U} \sim C + \frac{\omega}{2}\rho, \tag{4.30}$$

for the angular part of the complete metric to be regular in that limit. Defining the entropy density by unit (world-) volume $\tilde{S}$ by

$$\tilde{S} \equiv \frac{A_{\mathrm{h}\,(\tilde{p}+2)}}{\omega_{(\tilde{p}+2)}}, \tag{4.31}$$

where $A_{\mathrm{h}\,(\tilde{p}+2)}$ is the volume of the $(\tilde{p}+2)$-dimensional constant worldvolume sections of the horizon and $\omega_{(\tilde{p}+2)}$ is the volume of the round $(\tilde{p}+2)$-sphere of unit radius[5] we find that the above behavior of $\tilde{U}$ leads to the entropy density

$$\tilde{S} = \left(-e^{-C}\omega\right)^{\frac{\tilde{p}+2}{\tilde{p}+1}}, \quad \Rightarrow \quad e^{C} = -\omega\tilde{S}^{-\frac{\tilde{p}+1}{\tilde{p}+2}}. \tag{4.32}$$

Then, in order for the worldvolume metric to be regular in this limit, $\tilde{U}$ and W must behave as[6]

$$e^{\tilde{U}} \sim (-\omega)\tilde{S}^{-\frac{\tilde{p}+1}{\tilde{p}+2}}e^{\frac{\omega}{2}\rho}, \quad W \sim e^{\omega\rho}, \tag{4.33}$$

where have chosen arbitrarily a normalization constant. Since we have just seen that $W = e^{\gamma\rho}$, we conclude that in black branes with regular horizons $\omega = \gamma$ and the general metric for regular p-branes is, therefore, given by

$$ds^2_{(d)} = e^{\frac{2}{\tilde{p}+1}\tilde{U}}\left[e^{\frac{p}{p+1}\omega\rho}dt^2 - e^{-\frac{1}{p+1}\omega\rho}d\mathbf{y}^2_{(p)}\right] - e^{-\frac{2}{\tilde{p}+1}\tilde{U}}\gamma_{(\tilde{p}+3)\,\underline{mn}}dx^m dx^n. \tag{4.34}$$

We can now consider the near-horizon limit of the time-radial part of general metric and find that it can always (for $\omega \neq 0$) can be brought into the Rindler-like form

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[5] Not to be mistaken for the non-extremality parameter ω.

[6] This is true for $\omega \neq 0$. The near-horizon behavior for $\omega = 0$ is given by Eq. (4.46).

$$\sim e^{\frac{2}{p+1}C} \exp\left(-\frac{(\tilde{p}+1)e^{Cc}}{(-\omega)^{\frac{1}{\tilde{p}+1}}}\varrho\right)\Big[dt^2 - d\varrho^2\Big] = e^{-\frac{4\pi}{\beta}\rho}\Big[dt^2 - d\varrho^2\Big], \tag{4.35}$$

where

$$c \equiv \frac{d-2}{(p+1)(\tilde{p}+1)}, \tag{4.36}$$

and the inverse temperature is

$$\beta = \frac{4\pi(-\omega)^{\frac{1}{\tilde{p}+1}}}{(\tilde{p}+1)e^{Cc}}. \tag{4.37}$$

This result for the temperature and the above result for the entropy density lead to the following relation between them and the non-extremality parameter

$$(-\omega)^{\frac{1}{p+1}} = \tfrac{4\pi}{\tilde{p}+1} T \tilde{S}^{\frac{(d-2)}{(p+1)(\tilde{p}+2)}}, \tag{4.38}$$

which generalizes the relation obtained in Ref. [30] for 4-dimensional black holes and justifies in part the definition of the extremality parameter since it shows that ω will vanish whenever the brane's temperature vanishes if the entropy density does not diverge in this limit.

We can use the first integrals of the two equations of motion above to eliminate W and Ψ^M (which only occurs through $\dot{\Psi}^M$) from the remaining three equations of motion, which only involve the variables $\tilde{U}$ and ϕ^i and take the form:

$$\ddot{\tilde{U}} + e^{2\tilde{U}} V_{\rm BB} = 0, \tag{4.39}$$

$$\ddot{\phi}^i + \Gamma_{jk}{}^i \dot{\phi}^j \dot{\phi}^k + \tfrac{d-2}{2(\tilde{p}+1)(p+1)} e^{2\tilde{U}} \partial^i V_{\rm BB} = 0, \tag{4.40}$$

$$(\dot{\tilde{U}})^2 + \tfrac{(p+1)(\tilde{p}+1)}{d-2} \mathcal{G}_{ij} \dot{\phi}^i \dot{\phi}^j + e^{2\tilde{U}} V_{\rm BB} = (\omega/2)^2, \tag{4.41}$$

where $\Gamma_{jk}{}^i(\phi)$ are the components of the Levi-Civita connection of the scalar metric $\mathcal{G}_{ij}(\phi)$, we have defined the negative semidefinite *black-brane potential* (a generalization of the black-hole potential of Ref. [26])

$$V_{\rm BB}(\phi, \mathcal{Q}) \equiv 2\alpha^2 \tfrac{(p+1)(\tilde{p}+1)}{(d-2)} \mathcal{M}_{MN} \mathcal{Q}^M \mathcal{Q}^N. \tag{4.42}$$

The first two equations of motion can be derived from the effective action

$$\mathcal{I}[\tilde{U}, \phi^i] = \int d\rho \left\{(\dot{\tilde{U}})^2 + \tfrac{(p+1)(\tilde{p}+1)}{d-2} \mathcal{G}_{ij} \dot{\phi}^i \dot{\phi}^j - e^{2\tilde{U}} V_{\rm BB} + \hat{\mathcal{B}}^2\right\}. \tag{4.43}$$

The third equation is the Hamiltonian constraint (which follows from the ρ-independence of the Lagrangian) with a particular value for the integration constant related to the non-extremality parameter and the integration constant γ.

Let us summarize the results of this section. We have shown that we can use consistently the ansatz

$$\begin{aligned} ds^2_{(d)} &= e^{\frac{2}{p+1}\tilde{U}}\left[e^{\frac{p}{p+1}\omega\rho}dt^2 - e^{-\frac{1}{p+1}\omega\rho}d\mathbf{y}^2_{(p)}\right] - e^{-\frac{2}{\tilde{p}+1}\tilde{U}}\gamma_{(\tilde{p}+3)\,\underline{mn}}dx^m dx^n. \\ A^M_{(p+1)} &= \Psi^M(\rho)\,dt \wedge dy^1 \wedge \cdots \wedge dy^p, \quad \dot{\Psi}^M = \alpha e^{2\tilde{U}}\mathcal{M}^{MN}\mathcal{Q}_N, \\ \phi^i &= \phi^i(\rho), \end{aligned} \tag{4.44}$$

where $\tilde{U}$ is a function of ρ; γ, $\mathcal{Q}_M$ are constants and $\gamma_{(\tilde{p}+3)\,\underline{mn}}$ is the transverse space metric given in Eq. (4.8) to describe flat, static, regular black-brane solutions of the theories defined by generic family of actions Eq. (4.1). We have also shown that the above ansatz gives these theories if Eqs. (4.39–4.41) are satisfied.[7]

4.2.2 FGK Theorems for Static Flat Branes

The formalism presented in the previous section can be used t derive generalizations of the results obtained in Refs. [20, 26] for black holes.

Let us first consider extremal black branes $\omega = 0$. The general form of their metrics can be obtained by taking the $\omega \longrightarrow 0$ limit of the general metric Eq. (4.34):

$$ds^2_{(d)} = e^{\frac{2\tilde{U}}{p+1}}\left[dt^2 - d\mathbf{y}^2_{(p)}\right] - \frac{e^{-\frac{2\tilde{U}}{\tilde{p}+1}}}{\rho^{\frac{2}{\tilde{p}+1}}}\left[\frac{1}{\rho^2}\frac{d\rho^2}{(\tilde{p}+1)^2} + d\Omega^2_{(\tilde{p}+2)}\right]. \tag{4.45}$$

For the near-horizon ($\rho \to \infty$) limit of this metric to be regular, $\tilde{U}$ must behave as

$$e^{\tilde{U}} \sim \tilde{S}^{-\frac{\tilde{p}+1}{\tilde{p}+2}}\rho^{-1}, \tag{4.46}$$

where $\tilde{S}$ is the entropy density per unit worldvolume defined in Eq. (4.31). Therefore, the near-horizon limit of Eq. (4.45) is the metric of the direct product $AdS_{p+2} \times S^{\tilde{p}+2}$, both with radii equal to $\tilde{S}^{\frac{1}{\tilde{p}+2}}$:

[7] The same result can be obtained by reducing first the action Eq. (4.1) to $(d - p) = (\tilde{p} + 4)$ dimensions in such a way that the action only contains the Einstein-Hilbert term, scalars and 1-forms and then by using the FGK formalism of Ref. [20] for d-dimensional black holes ($p = 0$). See the Appendix in Ref. [19].

$$ds^2_{(d)} = \rho^{\frac{-2}{p+1}} \tilde{S}^{-\frac{2(\tilde{p}+1)}{(p+1)(\tilde{p}+2)}} \left[dt^2 - d\mathbf{y}^2_{(p)}\right] - \tilde{S}^{\frac{2}{\tilde{p}+2}} \left[\frac{1}{\rho^2} \frac{d\rho^2}{(\tilde{p}+1)^2} + d\Omega^2_{(\tilde{p}+2)}\right]. \tag{4.47}$$

To make further progress we need to impose a regularity condition on the scalars which generalizes the one used in Ref. [26] for 4-dimensional black holes. We require that

$$\lim_{\rho\to\infty} \frac{(p+1)(\tilde{p}+1)}{d-2} \mathcal{G}_{ij} \dot{\phi}^i \dot{\phi}^j e^{2\tilde{U}} \rho^4 \equiv \mathcal{X} < \infty. \tag{4.48}$$

from which it follows that the near-horizon limit $\rho \to \infty$ of Eq. (4.41) (the Hamiltonian constraint) is

$$1 + \mathcal{X} \tilde{S}^{\frac{2(\tilde{p}+1)}{\tilde{p}+2}} + \tilde{S}^{-\frac{\tilde{p}+1}{\tilde{p}+2}} V_{\text{BB}}(\phi_H, \mathcal{Q}) = 0. \tag{4.49}$$

Assuming that the near-horizon limit is regular, which implies that the entropy density $\tilde{S}$ does not vanish and that the values of the scalars on the horizon ϕ^i_{h} do not diverge $\phi^i_{\text{h}} < \infty$ so

$$\lim_{\rho\to\infty} \rho \frac{d\phi^i}{d\rho} = 0, \quad \forall i, \tag{4.50}$$

then it can be shown that

$$\mathcal{X} = 0, \tag{4.51}$$

and from Eqs. (4.49) and (4.51) we obtain

$$\tilde{S} = [-V_{\text{BB}}(\phi_{\text{h}}, \mathcal{Q})]^{\frac{\tilde{p}+2}{2(\tilde{p}+1)}}, \tag{4.52}$$

so the entropy of an extremal brane is given by (a power of) the value of the black-brane potential at the horizon.

Furthermore, under the same assumptions, we deduce, from the near-horizon limit of the equations of the scalars, that the values of the scalars on the horizon ϕ^i_{h} are such that

$$\mathcal{G}^{ij}(\phi_{\text{h}}) \partial_i V_{\text{BB}}(\phi_{\text{h}}, \mathcal{Q}) = 0, \tag{4.53}$$

and, if the metric of the scalar manifold $\mathcal{G}_{ij}$ is regular and the values of the scalars on the horizon are admissible so $\mathcal{G}_{ij}(\phi_{\text{h}})$ is also regular, then

$$\partial_i V_{\text{BB}}(\phi_{\text{h}}, \mathcal{Q}) = 0, \tag{4.54}$$

which generalizes the usual attractor mechanism for static extremal black holes to the case of static extremal flat branes.

We would like to stress the fact that the black-brane potential on the horizon does not depend on the moduli (the asymptotic values of the scalars at spatial infinity) even if the values of the scalars on the horizon do (which is what happens in general.[8]

Finally, if we consider the Hamiltonian constraint Eq. (4.41) at spatial infinity ($\rho \to 0^+$) we obtain the generalization of the so-called extremality (or antigravity) bound for black holes

$$\tilde{u}^2 + \frac{(p+1)(\tilde{p}+1)}{d-2}\mathcal{G}_{ij}(\phi_\infty)\Sigma^i\Sigma^j + V_{BB}(\phi_\infty, \mathcal{Q}) = (\omega/2)^2, \tag{4.55}$$

where Σ^i are the scalar charges and we have defined the constant

$$\tilde{u} = -\tilde{U}'(0). \tag{4.56}$$

This constant is a combination of the black p-brane's tension T_p and the non-extremality parameter ω. The relation comes from the definition of the brane's tension:

$$T_p = -\frac{1}{(p+1)(\tilde{p}+2)}\left[(d-2)\tilde{u} + p(\tilde{p}+1)\omega/2\right], \tag{4.57}$$

Then, the brane's antigravity bound differs from the black hole's by terms proportional to $p\omega$ which vanish in the black-hole case $p = 0$.

4.2.3 Inner Horizons

The general metric Eq. (4.34) is designed to cover the exterior of the black brane's event horizon. In general, though, we expect charged black branes to have two horizons that will coincide in the extremal limit, as it happens for charged black holes. The inner horizon, which appears as another place at which the g_{tt} component vanishes, is not an event horizon. In the 4-dimensional Reissner-Nordström black hole, which is the best studied example, the inner horizon is actually a Cauchy horizon.

In Ref. [19] it was shown that, in the 4-dimensional black-hole case ($p = 0$) the same general metric covers the interior of the inner horizon (the region between the curvature singularity and the inner horizon) for the range of the radial coordinate[9]

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[8] Only the supersymmetric attractors, that is, r the values of the scalars of supersymmetric black-brane solutions, are guaranteed to depend only on the charges. The general situation for extremal non-supersymmetric black branes is that the scalars on the horizon keep some dependence on their values at spatial infinity. This situation is sometimes referred to as the existence of a *moduli space of attractors* parametrized by some numbers whose physical meaning (i.e. their expressions in terms of the physical constants) is seldom given in the literature. These parameters are functions of the moduli, as shown explicitly in the $\overline{\mathbb{CP}}^n$ model studied in [19].

Of course, there are some moduli-independent non-supersymmetric attractors but it is important to realize that this is what happens in general.

[9] In that reference the coordinate used was $\tau = -\rho$.

$\rho \in (-\infty, \rho_{\text{sing}})$, where ρ_{sing} denotes the location of the curvature singularity and the inner horizon is placed at $\rho = -\infty$.

In the 5-dimensional black-hole case studied in Ref. [20] it was observed that the general metric Eq. (4.34) is not well defined for negative values of ρ, from which it was concluded that the same metric could not cover the interior of the inner horizon. However, as it has been realized in Ref. [24], one can obtain from a metric of the form Eq. (4.34), regular for $\rho \in (0, +\infty)$ and covering the exterior of the black brane's event horizon, another metric by the simple transformation[10]

$$\rho \longrightarrow -\varrho, \qquad e^{-\tilde{U}(\rho)} \longrightarrow -e^{-U(-\varrho)}. \tag{4.58}$$

The new metric has the same general form, but describes the interior of the inner horizon for $\varrho \in (\varrho_{\text{sing}}, +\infty)$. Observe that, if the original function $e^{-\tilde{U}}$ is always finite[11] for positive values of ρ, the transformed metric will generically hit a singularity before ϱ reaches 0 because, after the transformation, $e^{\tilde{U}}$ one will have a zero for some finite positive value of ϱ, as the explicit examples worked out in the references show.

The reasons to believe that the transformed metric is the metric that covers the interior of the horizon of the same black-brane spacetime are the same that make us believe that the region covered by standard Reissner-Nordström metric for $r < r_-$ corresponds to the interior of the black hole whose exterior is described by that metric for $r > r_+$. Since the Reissner-Nordström metric is singular at $r = r_+$ and $r = r_-$, the standard solution is actually giving us three different metrics which we interpret as covering three different regions of the same black-hole spacetime.

The upshot of this discussion is that the above transformation will allow us to compute the "temperature" and "entropy" of the inner horizon and check the *geometric mean property*. This property has been observed to hold for many different solutions and it has been proven for the charged, rotating, asymptotically flat or anti-De Sitter black-hole solutions of a wide class of theories in [31], following earlier work [32–36].[12] The property consists in the mass-independence (and moduli-independence, when there are scalars present in the theory) of the product of the "entropies" of all the horizons of a black-hole solution. In the asymptotically-flat cases that we are considering, in which the solutions usually have only two horizons, if we denote by $\tilde{S}_+$ the entropy density of the outer (event) horizon by $\tilde{S}_-$ the entropy density of the inner (Cauchy) horizon, the geometry mean property says that $\tilde{S}_+\tilde{S}_-$ is mass and moduli-independent, which means that it only depends on the electric and magnetic charges, which are quantized. This implies that the product only depends on integer numbers, which is a very suggestive property of entropy-related quantities.

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[10] It would be stressed that this a transformation that relates solutions, and not a coordinate transformation.

[11] In the construction of the solution this is achieved by requiring the positivity of certain constants that appear in it, such as the mass or the entropy.

[12] A related result valid for horizons of arbitrary topology has been recently found in [37].

4.2.4 FGK Formalism for the Black Holes of $\mathcal{N} = 2,\ d = 5$ Theories

Let us see how the general formalism developed in the previous sections works in the particular case of the black-hole solutions of theories of $\mathcal{N} = 2,\ d = 5$ supergravity coupled to vector supermultiplets Refs. [38, 39]. We will use the conventions of Refs. [40, 41].

For black-hole solutions (which will only be electrically charged with respect to the vector fields) we can safely ignore the Chern-Simons term in the bosonic action and work with

$$\mathcal{I}[g_{\mu\nu}, A^I{}_\mu, \phi^x] = \int d^5x \left\{R + \tfrac{1}{2}g_{xy}\partial_\mu\phi^x\partial^\mu\phi^y - \tfrac{1}{8}a_{IJ}F^I{}_{\mu\nu}F^{J\,\mu\nu}\right\}, \tag{4.59}$$

where $I, J = 0, 1, \cdots, n$ and $x, y = 1, \cdots, n$. The scalar target spaces are determined by the existence of $n+1$ functions $h^I(\phi)$ of the n physical scalars ϕ^x subject to the constraint

$$C_{IJK}h^I h^J h^K = 1, \tag{4.60}$$

where C_{IJK} is a completely symmetric constant tensor that defines the model. These functions, like the vector fields themselves, transform linearly under the duality group, which must be embedded in SO$(n+1)$, while the physical scalars transform non-linearly, in general. This helps to make the symmetry manifest and it is the main reason why these objects (like the symplectic section of the theories of $\mathcal{N} = 2, d = 4$ supergravity coupled to vector supermultiplets) are introduced.

We also define

$$h_I \equiv C_{IJK}h^J h^K \quad (\text{so } h_I h^I = 1), \tag{4.61}$$

$$a_{IJ} \equiv -2C_{IJK}h^K + 3h_I h_J. \tag{4.62}$$

a_{IJ} can be used to raise and lower the index of the functions h^I, h_I and its derivatives

$$h^I{}_x \equiv -\sqrt{3}\partial_x h^I, \quad h_{Ix} \equiv a_{IJ}h^J = +\sqrt{3}\partial_x h_I. \tag{4.63}$$

These are orthogonal which are orthogonal to the h^I with respect to the metric a_{IJ}:

$$h^I h_{Ix} = h_I h^I{}_x = 0. \tag{4.64}$$

Finally, the target-space metric is related to the matrix a_{IJ} by

$$g_{xy} \equiv a_{IJ}h^I{}_x h^J{}_y, \tag{4.65}$$

which leads to

$$a^{IJ} = h^I h^J + g^{xy} h^I{}_x h^J{}_y. \tag{4.66}$$

The general FGK formalism constructed in the previous section leads, for this particular case and conventions to the general metric ($\tilde{U} \to U$)

$$ds^2 = e^{2U} dt^2 - e^{-U} \left(\frac{\omega/2}{\sinh\left(\frac{\omega}{2}\rho\right)} \right) \left[\left(\frac{\omega/2}{\sinh\left(\frac{\omega}{2}\rho\right)} \right)^2 \frac{d\rho^2}{4} + d\Omega^2_{(3)} \right], \tag{4.67}$$

the effective action

$$\mathcal{I}[U, \phi^x] = \int d\rho \left\{ (\dot{U})^2 + \tfrac{1}{3} g_{xy} \dot{\phi}^x \dot{\phi}^y - e^{2U} V_{\text{bh}} \right\}, \tag{4.68}$$

and the Hamiltonian constraint becomes

$$(\dot{U})^2 + \tfrac{1}{3} g_{xy} \dot{\phi}^x \dot{\phi}^y + e^{2U} V_{\text{bh}} = (\omega/2)^2, \tag{4.69}$$

where the black-brane potential (renamed here black-hole potential) is given by[13]

$$-V_{\text{bh}}(\phi, q) = a^{IJ} q_I q_J = \mathcal{Z}_{\text{e}}^2 + 3 g^{xy} \partial_x \mathcal{Z}_{\text{e}} \partial_y \mathcal{Z}_{\text{e}}, \tag{4.70}$$

where

$$\mathcal{Z}_{\text{e}}(\phi, q) \equiv h^I(\phi) q_I, \tag{4.71}$$

is the *(electric) black-hole central charge* and we have used Eq. (4.65).

A special feature of the FGK formalism for this and other supergravity theories is that the black-brane potential can be written in terms of central charges and that one can prove that the black-brane potential is extremized when the central charge is also extremized:

$$\partial_x \mathcal{Z}|_{\phi_{\text{h}}} = 0, \quad \Rightarrow \quad \partial_x V_{\text{bh}}|_{\phi_{\text{h}}} = 0. \tag{4.72}$$

The converse is not true. The extrema of the central charge are the supersymmetric attractors, the values towards which the scalar fields are attracted when we approach the horizon of supersymmetric extremal black holes.

In some cases the black-hole potential can be written in a similar fashion for other functions of the scalars and charges called *superpotentials* in the literature. the extremization of these superpotentials also leads to the extremization of the black-hole potential, but the extrema are not the supersymmetric attractors and it is not guaranteed that they will only be functions of the charges, as discussed before. Extremal non-supersymmetric black holes are related to these superpotentials, as we will discuss later.

[13] We have chosen, for convenience, the normalization $\alpha^2 = 3/32$.

4.2.5 FGK Formalism for the Black Strings of $\mathcal{N} = 2,\ d = 5$ Theories

The theories of $\mathcal{N} = 2,\ d = 5$ supergravity coupled to vector supermultiplets also admit black string solutions charged with respect to the 2-forms $B_{I\,\mu\nu}$ dual to the vector fields $A^I{}_\mu$. Due to the Chern-Simons term, it is not possible to dualize completely the action, replacing everywhere the vectors by the 2-forms. However, for purely magnetic (string) solutions, electrically charged only with respect to the 2-forms, the Chern-Simons term is, again, irrelevant, and one can work with the bosonic action

$$\mathcal{I} = \int \sqrt{g}\left\{R + \tfrac{1}{2}g_{xy}\partial_\mu\phi^x\partial^\mu\phi^y + \tfrac{1}{2\cdot 3!}a^{IJ}G_{I\,\mu\nu\kappa}G_J^{\mu\nu\kappa}\right\}, \tag{4.73}$$

where $G_I = dB_I$. Observe that the kinetic matrix is in this case the inverse of the kinetic matrix of the black-hole case.

The general formalism can be applied straightforwardly and we arrive to the general for of the metric for non-extremal black strings ($d = 5$, $p = 1$)

$$ds^2 = e^{\tilde{U}}\left[e^{\frac{\omega}{2}\rho}dt^2 - e^{-\frac{\omega}{2}\rho}dy^2\right] - e^{-2\tilde{U}}\left(\frac{\omega/2}{\sinh\left(\frac{\omega}{2}\rho\right)}\right)^2\left[\left(\frac{\omega/2}{\sinh\left(\frac{\omega}{2}\rho\right)}\right)^2 d\rho^2 + d\Omega^2_{(2)}\right], \tag{4.74}$$

to the effective action

$$\mathcal{I}[\tilde{U}, \phi^x] = \int d\rho\left\{(\dot{\tilde{U}})^2 + \tfrac{1}{3}g_{xy}\dot{\phi}^x\dot{\phi}^y - e^{2U}V_{\rm st}\right\}, \tag{4.75}$$

plus the Hamiltonian constraint

$$\dot{\tilde{U}}^2 + \tfrac{1}{3}g_{xy}\dot{\phi}^x\dot{\phi}^y + e^{2U}V_{\rm st} = (\omega/2)^2, \tag{4.76}$$

where we have defined the *black-string potential* as

$$-V_{\rm st}(\phi, p) \equiv a_{IJ}\,p^I p^J = \mathcal{Z}_{\rm m}^2 + 3\partial_x\mathcal{Z}_{\rm m}\partial^x\mathcal{Z}_{\rm m}. \tag{4.77}$$

Here we have introduced the *(magnetic) string central charge*

$$\mathcal{Z}_{\rm m}(\phi, p) = h_I(\phi)p^I, \tag{4.78}$$

which for supersymmetric extremal strings plays the same rôle as the electric one plays for supersymmetric extremal black holes.

4.2.6 FGK Formalism for $\mathcal{N}=2,\ d=4$ Theories

The black-hole solutions of the theories of ungauged $\mathcal{N}=2,\ d=4$ supergravity coupled to n vector supermultiplets[14] have been the ,most studied of all. As mentioned above, they can be electric and magnetically charged with respect to the $\bar{n}=n+1$ vector fields $A^{\Lambda}{}_{\mu}$, $\Lambda=0,1,\ldots,n$, and the n complex scalars of these theories, denoted by Z^i, $i=1,\ldots,n$, which parametrize a special Kähler manifold with Kähler metric $\mathcal{G}_{ij^*}=\partial_i\partial_{j^*}\mathcal{K}$, where $\mathcal{K}(Z,Z^*)$ is the Kähler potential, can have non-trivial profiles.

The bosonic action of these theories is always of the form

$$\begin{aligned}\mathcal{I}=\int d^4x\sqrt{|g|}\,\Big\{&R+2\mathcal{G}_{ij^*}\partial_\mu Z^i\partial^\mu Z^{*\,j^*}\\&+2\Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda}{}_{\mu\nu}F^{\Sigma\,\mu\nu}-2\Re\mathrm{e}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda}{}_{\mu\nu}\star F^{\Sigma\,\mu\nu}\Big\},\end{aligned}\tag{4.79}$$

where $\mathcal{N}_{\Lambda\Sigma}(Z,Z^*)$ is the period matrix mentioned before and which is related to the Kähler metric by the structure of special Kähler geometry.

The general form of the black-hole metrics of these theories is ($\tilde{U}\to U$)

$$ds^2=e^{2U}-e^{-2U}\left(\frac{\omega/2}{\sinh\left(\frac{\omega}{2}\rho\right)}\right)^2\left[\left(\frac{\omega/2}{\sinh\left(\frac{\omega}{2}\rho\right)}\right)^2d\rho^2+d\Omega^2_{(2)}\right],\tag{4.80}$$

the effective action takes the form

$$\mathcal{I}[U,Z^i]=\int d\rho\left\{(U')^2+\mathcal{G}_{ij^*}\dot{Z}^i\dot{Z}^{*\,j^*}-e^{2U}V_{\rm bh}\right\},\tag{4.81}$$

and the Hamiltonian constraint is given by

$$(\dot{U})^2+\mathcal{G}_{ij^*}\dot{Z}^i\dot{Z}^{*\,j^*}+e^{2U}V_{\rm bh}=(\omega/2)^2.\tag{4.82}$$

In these theories the black-hole potential takes the simple form

$$-V_{\rm bh}(Z,Z^*,\mathcal{Q})=|\mathcal{Z}|^2+\mathcal{G}^{ij^*}\mathcal{D}_i\mathcal{Z}\mathcal{D}_{j^*}\mathcal{Z}^*,\tag{4.83}$$

where

$$\mathcal{Z}=\mathcal{Z}(Z,Z^*,\mathcal{Q})\equiv\langle\mathcal{V}\mid\mathcal{Q}\rangle=-\mathcal{V}^M\mathcal{Q}^N\Omega_{MN}=p^{\Lambda}\mathcal{M}_{\Lambda}-q_{\Lambda}\mathcal{L}^{\Lambda},\tag{4.84}$$

[14] For more information on these theories see, for instance, Ref. [42], the review [43], and the original works [44, 45].

is the *central charge* of the theory, $(\mathcal{V}^M) = \begin{pmatrix} \mathcal{L}^\Lambda \\ \mathcal{M}_\Lambda \end{pmatrix}$ is the covariantly holomorphic symplectic section, $(\Omega_{MN}) = \begin{pmatrix} 0 & \\ - & 0 \end{pmatrix}$ is the symplectic metric, and

$$\mathcal{D}_i \mathcal{Z} = e^{-\mathcal{K}/2}\partial_i \left(e^{\mathcal{K}/2}\mathcal{Z}\right), \tag{4.85}$$

is the Kähler-covariant derivative.

The supersymmetric attractors of these theories extremize the absolute value of the central charge

$$\partial_i |\mathcal{Z}||_{Z_\mathrm{h}} = 0. \tag{4.86}$$

4.3 General Solutions and General Ansatzs

The general ansatzs that we are going to use to construct non-extremal black-hole solutions are based on the structure of the supersymmetric extremal ones which are been found in full generality for theories of $\mathcal{N} = 2,\ d = 4, 5$ supergravity coupled to vector supermultiplets using the method pioneered by Tod [46, 47]. Therefore, we are going to start by reviewing them.

4.3.1 General Supersymmetric Solutions

Black Holes of $\mathcal{N} = 2,\ d = 5$

All the supersymmetric solutions of the theories of ungauged $\mathcal{N} = 2,\ d = 5$ supergravity coupled to vector supermultiplets only[15] were found in Refs. [11, 12] We use here the conventions and prescription of Ref. [40], specializing it for the static case.

The supersymmetric, extremal, static black-hole solutions of these theories with n vector supermultiplets are constructed as follows:

1. With the metric function e^U and the scalar functions h_I we define the $\bar{n} = n + 1$ combinations
$$H_I(\rho) \equiv e^{-U} h_I. \tag{4.87}$$
2. These combinations are single-pole harmonic functions in the 4-dimensional transverse space of the general extremal metric Eq. (4.45) which we rewrite here

[15] These results have been extended to theories with hypermultiplets and tensor multiplets in Refs. [40, 48, 49] but these only include regular black-hole solutions when the additional fields vanish and, therefore, we will not consider them here.

for convenience[16] for $d = 5,\ p = 0\ (\tilde{U} \to U)$:

$$ds^2 = e^{2U} dt^2 - e^{-U} \frac{1}{\rho} \left[\frac{1}{4\rho^2} d\rho^2 + d\Omega^2_{(3)} \right]. \tag{4.88}$$

In other words: they are linear functions of the radial coordinate ρ:

$$H_I = A_I + B_I \rho, \tag{4.89}$$

for some constants A_I, B_I.

3. The solutions are completely determined by these harmonic functions. All the physical fields can be constructed in terms of them:

   (a) The vector field strengths are given by

   $$F^I = -\sqrt{3} d(e^U h^I) \wedge dt \tag{4.90}$$

   from which one can identify (in the same normalization we have chosen before) the coefficients B_I with the electric charges q_I:

   $$B_I = q_I. \tag{4.91}$$

   This identification is an important part of the structure of the supersymmetric extremal solutions of these theories.

   (b) The scalar fields can be written, for instance, in the form

   $$\phi^x = h_x / h_0 = H_x / H_0. \tag{4.92}$$

   (c) In order to write the metric in terms of the harmonic functions we first need to solve the (5-dimensional equivalent of the) *stabilization equations*, i.e. we need to find how to write the h^I in terms of the h_I and, therefore, in terms of the H_I and e^U. Then, the constraint $C_{IJK} h^I h^J h^K = 1$ gives a relation between e^U and the harmonic functions,

   $$e^U(H)$$

   which we will describe in detail when we review the H-FGK formalism.

4. The expressions of the physical fields can be use to determine completely the constants A_I in terms of their asymptotic values (basically only the moduli since the metric function is normalized to 1 at spatial infinity), as explained in Ref. [24].

[16] The change $\rho = r^{-2}$ brings the metric to the standard form.

Black Strings of $\mathcal{N}=2,\ d=5$

The supersymmetric, extremal, static black-string solutions of these theories with n vector supermultiplets are constructed following a very similar recipe [12, 48, 50]:

1. With the metric function e^{U} and the scalar functions h^{I} we define the $\bar{n}=n+1$ combinations
$$K^{I}(\rho)\equiv e^{-U}h^{I}. \tag{4.93}$$
2. These combinations are single-pole harmonic functions in the 3-dimensional transverse space of the general extremal metric Eq. (4.45) which we rewrite here for convenience[17] for $d=5,\ p=1$ $(\tilde{U}\to U)$:
$$ds^{2}=e^{U}[dt^{2}-dy^{2}]-e^{-2U}\frac{1}{\rho^{2}}\left[\frac{1}{\rho^{2}}d\rho^{2}+d\Omega^{2}_{(2)}\right]. \tag{4.94}$$
In other words: again, they are linear functions of the radial coordinate ρ:
$$K^{I}=A^{I}+B^{I}\rho, \tag{4.95}$$
for some constants A^{I}, B^{I}.
3. The solutions are completely determined by these harmonic functions. All the physical fields can be constructed in terms of them:
   (a) The field strengths are given by
   $$F^{I}=\sqrt{3}\star_{(3)}dH^{I}. \tag{4.96}$$
   from which one can identify the coefficients B^{I} with the magnetic charges (the electric charges of the dual field strengths) p^{I}:
   $$B^{I}=p^{I}. \tag{4.97}$$
   Again, this identification is a feature of the supersymmetric extremal solutions.
   (b) The scalar fields can be written as in the black-hole case, which requires that we solve the stabilization equations, or we can use a different parametrization f the scalar manifold and write
   $$\phi^{x}=h^{x}/h^{0}=K^{x}/K^{0}. \tag{4.98}$$
   (c) The metric function e^{U} is found by substituting the definition of the variables K^{I} in the constraint $C_{IJK}h^{I}h^{J}h^{K}=1$, which yields

[17] The change $\rho=r^{-1}$ brings the metric to the standard form.

$$e^{-3U}(K) = C_{IJK} K^I K^J K^K . \tag{4.99}$$

4. The expressions of the physical fields can be use to determine completely the constants A^I in terms of their asymptotic values (basically only the moduli since the metric function is normalized to 1 at spatial infinity ($\rho = 0$)):

$$\phi^x_\infty = A^x/A^0, \quad e^{-3U}(A) = 1, \quad (A^0)^{-3} = e^{-3U}(A/A^0) = e^{-3U}(\phi_\infty), \tag{4.100}$$

where we defined, for convenience $\phi^0 \equiv 1$. Then

$$A^0 = e^U(\phi_\infty), \quad A^x = \phi^x_\infty e^U(\phi_\infty). \tag{4.101}$$

Black Holes of $\mathcal{N} = 2,\ d = 4$

All the timelike[18] supersymmetric solutions of the most general, gauged matter-coupled theories have been classified in Refs. [9, 14–16, 46, 51–54]. The supersymmetric extremal black holes of the ungauged theories[19] were constructed in [6–9] and we are going to give the recipe of Ref. [9] to construct the static ones: all the supersymmetric solutions of a theory of $\mathcal{N} = 2,\ d = 4$ supergravity coupled to vector supermultiplets and defined by the covariantly-holomorphic section $\mathcal{V}^M$ can be constructed as follows:

1. We introduce an auxiliary function of Kähler weight 1 (like $\mathcal{V}$) which, as we will see later (and we can safely ignore here) is related to the metric function e^U by $e^{2U} = 2|X|^2$.
2. We define the Kähler-neutral real symplectic vectors $\mathcal{R}^M$ and $\mathcal{I}^M$

$$\mathcal{R}^M + i\mathcal{I}^M \equiv \mathcal{V}^M/X. \tag{4.102}$$

   No Kähler gauge-fixing are necessary with this construction.
3. The components of $\mathcal{I}^M$ are real functions H^M which are single-pole harmonic functions in the 3-dimensional transverse space of the general extremal metric Eq. (4.45) which we rewrite here for convenience[20] for $d = 4$, $p = 0$ ($\tilde{U} \to U$):

$$ds^2 = e^{2U} dt^2 - e^{-2U} \frac{1}{\rho^2} \left[\frac{1}{\rho^2} d\rho^2 + d\Omega^2_{(2)} \right]. \tag{4.103}$$

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[18] These are the supersymmetric solutions such that the vector constructed as a bilinear from its Killing spinor is timelike. In particular, it is a timelike Killing vector. The other possible class is the null class. The supersymmetric extremal black-hole solutions belong to the timelike class.

[19] In the gauged theories there are asymptotically-AdS black holes [55] and also asymptotically-flat, regular black holes with non-Abelian hair [14–16], but here we are not going to consider these cases.

[20] The change $\rho = r^{-1}$ brings the metric to the standard form.

Yet again, they are linear functions of the radial coordinate ρ:

$$H^M = A^M + B^M \rho, \tag{4.104}$$

for some constants A^M, B^M.

4. In this case, the constants must satisfy the constraint

$$A^M B_M = \langle A \mid B \rangle = 0. \tag{4.105}$$

This constraint is equivalent to the requirement that there is no NUT charge [56]. A solution with NUT charge is, first of all, not static, and second of all, it would generically have either singularities or closed timelike curves.

5. The solutions are completely determined by these harmonic functions. All the physical fields can be constructed in terms of them. The construction requires finding the $\mathcal{R}^M$s in terms of the $\mathcal{I}^M$s, and, hence, of the harmonic functions H^M. This is always possible due to the redundancy of the description provided by $\mathcal{V}$ which implies the existence of relations between $\mathcal{R}$s and $\mathcal{I}$s known as *stabilization equations*. These h may be very difficult to solve in practice.

   (a) The vector field strengths are given by

$$\mathcal{F}^M = -\sqrt{2}d(\mathcal{R}^M|X|^2) \wedge dt - \sqrt{2}|X|^2 \star (dt \wedge d\mathcal{I}^M), \tag{4.106}$$

   which allows us to identify the constants B^M with the electric and magnetic charges collected in the symplectic vector $\mathcal{Q}^M$:

$$(B^M) = (\mathcal{Q}^M) = \begin{pmatrix} p^\Lambda \\ q_\Lambda \end{pmatrix}. \tag{4.107}$$

   This is a characteristic feature of the supersymmetric extremal solutions.

   (b) The physical scalars Z^i are given by the quotients

$$Z^i = \frac{\mathcal{V}^i/X}{\mathcal{V}^0/X} = \frac{\mathcal{R}^i + i\mathcal{I}^i}{\mathcal{R}^0 + i\mathcal{I}^0}. \tag{4.108}$$

   (c) The metric function is given by

$$e^{-2U} = \frac{1}{2|X|^2} = \langle \mathcal{R} \mid \mathcal{I} \rangle. \tag{4.109}$$

6. The expressions of the physical fields can be use to determine completely the constants A^M in terms of their asymptotic values (basically only the moduli since the metric function is normalized to 1 at spatial infinity $\rho = 0$). The asymptotic conditions take the form

$$e^{2U}(A) = 1, \tag{4.110}$$

$$Z^i_\infty = \frac{\mathcal{R}^i(A) + iA^i}{\mathcal{R}^0(A) + iA^0}. \tag{4.111}$$

Now, let us write X as

$$X = \tfrac{1}{\sqrt{2}} e^{U+i\alpha}, \tag{4.112}$$

where α is some function. Then, the definition of $\mathcal{I}^M$ implies that

$$H^M = \sqrt{2} e^{-U}\, \Im\mathrm{m}(e^{-i\alpha}\mathcal{V}^M), \tag{4.113}$$

and, at spatial infinity $\rho = 0$, using the asymptotic flatness conditions Eq. (4.110), we find

$$A^M = \sqrt{2}\, \Im\mathrm{m}(e^{-i\alpha_\infty}\mathcal{V}^M_\infty). \tag{4.114}$$

α_∞ can be found using Eq. (4.105) and the definition of the central charge Eq. (4.84). Observe that

$$A_M B^M = \langle\, H \mid B\,\rangle = \Im\mathrm{m}\langle\, \mathcal{V}/X \mid B\,\rangle = \Im\mathrm{m}(\tilde{\mathcal{Z}}/X) = e^{-U}\Im\mathrm{m}(e^{-i\alpha}\tilde{\mathcal{Z}}) = 0. \tag{4.115}$$

Then

$$e^{i\alpha} = \pm\tilde{\mathcal{Z}}/|\tilde{\mathcal{Z}}|, \tag{4.116}$$

and the general expression of the constants A^M as functions of the charges $\mathcal{Q}^M$ and the asymptotic values of the scalar fields Z^i_∞ is

$$A^M = \pm\sqrt{2}\, \Im\mathrm{m}\left(\frac{\mathcal{Z}^*}{|\mathcal{Z}|}\mathcal{V}^M_\infty\right). \tag{4.117}$$

It can be seen that only the upper sign gives a positive value of the mass and a regular black-hole metric.

4.3.2 General Ansatzs for Non-extremal Solutions

In the previous section we have seen how, the supersymmetric extremal black-hole and black-string solutions of $\mathcal{N} = 2,\ d = 4, 5$ theories can be constructed by following a simple recipe. The main ingredient in the recipe is the expression of the physical fields (the metric function e^U and the scalar fields Z^i in $d = 4$ and ϕ^x in $d = 5$, which are the only fields that need to be determined in the FGK formalism) in terms of some functions H^M, H_I and K^I in the different theories and cases. In the supersymmetric solutions these functions are linear in the radial coordinate ρ.

Based on these recipes we can make the following ansatz for the non-extremal solutions: the physical fields are given by the same expressions in terms of the functions H^M, H_I and K^I as in the supersymmetric case but, now, these function are no longer linear in ρ. It seems that in all cases [19, 20, 24] these functions are linear combinations of hyperbolic sines and cosines of $\frac{\omega}{2}\rho$:

$$H^M = A^M \cosh\left(\frac{\omega}{2}\rho\right) + \frac{2B^M}{\omega} \sinh\left(\frac{\omega}{2}\rho\right), \tag{4.118}$$

etc.

We are assuming that there is a universal way to express the physical fields of this kind of solutions in terms of the variables H^M, H_I and K^I, and this probably needs some justification, beyond the examples for which this seems to be the case.[21] It can be argued that the duality-invariance of e^U and the duality-covariance of the scalars can only be achieved by very specific combinations of functions and that we roughly expect as many independent functions as electric and magnetic charges can be carried by the black objects. There is, however, a better argument: for the cases considered, the functions H^M, H_I and K^I can be used as independent variables in the FGK formalism. In other words: the general expressions for e^U and the scalars as functions of the H^M, H_I or K^I can always be used to change the variables in the FGK effective action and Hamiltonian constraints. In this way, one gets an equivalent formulation of the FGK system in which the fundamental variables are the functions H^M, H_I or K^I that we have called *H-FGK formalism* [22].[22] Solving the new equations of motion and Hamiltonian constraint for the new variables one can reconstruct the physical fields using always the same expressions.

This proves the first assumption. As for the second assumption in our ansatz (the hyperbolicity of the functions H^M, H_I and K^I in the non-extremal cases), there is no complete proof, although in the H-FGK formalism it arises as a most natural possibility.

In the next section we review the H-FGK formalism.

4.4 A Better Framework: The H-FGK Formalism

4.4.1 For the Black-Hole Solutions of $\mathcal{N} = 2,\ d = 5$

Here we are going to show how the metric function e^U and the n real scalars ϕ^x can be replaced in the FGK action by the $\bar{n} = n + 1$ variables denoted by H_I. We will also need to define $\bar{n}$ dual variables $\tilde{H}^I$ for intermediate calculations.

[21] Apart from the examples studied in Refs. [19, 20], the assumption is true in all the supersymmetric solutions of $\mathcal{N} = 2,\ d = 4, 5$ theories, for all matter couplings and gaugings.

[22] A similar, more general, formalism that reduces to the H-FGK one for single, static, spherically-symmetric black holes of $\mathcal{N} = 2,\ d = 4, 5$ has been given in Refs. [17, 18, 21]. The $\mathcal{N} = 2,\ d = 5$ string case has not been treated with this method.

A very important ingredient of the ensuing calculations will be the homogeneity of the functions that occur in the supergravity theories and in the formalism. To start with, we define $\mathcal{V}(h^{\cdot})$, homogeneous of third degree in the h^Is

$$\mathcal{V}(h^{\cdot}) \equiv C_{IJK} h^I(\phi) h^J(\phi) h^K(\phi). \tag{4.119}$$

This function defines the scalar manifold as the hypersurface $\mathcal{V} = 1$. The dual scalar functions h_I, defined in Eq. (4.61) can also be defined by

$$h_I(h^{\cdot}) \equiv \tfrac{1}{3} \frac{\partial \mathcal{V}}{\partial h^I}. \tag{4.120}$$

They are, obviously, homogenous of second degree in the h^I. This relation can be inverted to express the h^I as functions of the h_I, $h^I(h_{\cdot})$ (finding this relations is the same as solving the *stabilization equations*). It is evident that $h^I(h_{\cdot})$ is homogeneous of degree 1/2 which implies that, in its turn, $\mathcal{V}(h_{\cdot})$ is homogeneous of degree 3/2.

It is useful to define the Legendre transform of $\mathcal{V}(h^{\cdot})$ $\mathcal{W}(h_{\cdot})$ by

$$\mathcal{W}(h_{\cdot}) \;\equiv\; 3 h_I h^I(h_{\cdot}) - \mathcal{V}(h^{\cdot}) = 2\mathcal{V}[h^{\cdot}(h_{\cdot})], \tag{4.121}$$

which is homogenous of degree 3/2. From the standard properties of the Legendre transform we get

$$h^I \;\equiv\; \tfrac{1}{3} \frac{\partial \mathcal{W}}{\partial h_I}. \tag{4.122}$$

The next step in this construction is the introduction of two sets of variables H_I and $\tilde{H}^I$ which are related to the physical fields (U, ϕ^x) by

$$H_I \equiv e^{-U} h_I(\phi), \tag{4.123}$$

$$\tilde{H}^I \equiv e^{-U/2} h^I(\phi), \tag{4.124}$$

and two new functions V and W, which have the same form in the new variables as $\mathcal{V}$ and $\mathcal{W}$ had in the old ones, that is

$$\mathsf{V}(\tilde{H}) \equiv C_{IJK} \tilde{H}^I \tilde{H}^J \tilde{H}^K, \tag{4.125}$$

$$\mathsf{W}(H) \equiv 3\tilde{H}^I H_I \;-\; \mathsf{V}(\tilde{H}) \;=\; 2\mathsf{V}. \tag{4.126}$$

These functions are not constrained as $\mathcal{V}$ and $\mathcal{W}$ are.

The properties that we proved for $\mathcal{V}$ and $\mathcal{W}$ (in particular, the homogeneity properties) implies the following properties for $\mathcal{V}$ and $\mathcal{W}$:

$$H_I \equiv \tfrac{1}{3} \frac{\partial \mathsf{V}}{\partial \tilde{H}^I}, \tag{4.127}$$

$$\tilde{H}^I \equiv \tfrac{1}{3}\frac{\partial \mathsf{W}}{\partial H_I} \equiv \tfrac{1}{3}\,\partial^I \mathsf{W}, \tag{4.128}$$

$$e^{-\frac{3}{2}U} = \tfrac{1}{2}\mathsf{W}(H), \tag{4.129}$$

$$h_I = (\mathsf{W}/2)^{-2/3} H_I, \tag{4.130}$$

$$h^I = (\mathsf{W}/2)^{-1/3}\tilde{H}^I. \tag{4.131}$$

Having defined the $\bar{n}$ variables H_I in terms of the metric function e^U and the n scalar fields ϕ^x (through h_I), we can view the above formulae Eqs. (4.129) and (4.130) as the inverse relations and we can use these relations and the rest of the auxiliary formulae to rewrite the FGK action Eq. (4.68) in terms of the new variables H^I.

First, we rewrite that action in the equivalent form

$$\mathcal{I}_{\mathrm{FGK}}[U,\phi^x] = \int d\rho\left\{(\dot{U})^2 + a^{IJ}\dot{h}_I\dot{h}_J + e^{2U}a^{IJ}q_Iq_J\right\}, \tag{4.132}$$

so that we only need to express U, h_I and a^{IJ} in terms of the new variables. For U and h_I this is, trivial, using the above formulae. For the inverse metric a^{IJ} one can show that the relation between a^{IJ} and the new variables is

$$a^{IJ} = -\tfrac{2}{3}\,(\mathsf{W}/2)^{4/3}\,\partial^I\partial^J \log \mathsf{W}, \tag{4.133}$$

and, therefore, after the change of variables, the effective FGK action becomes

$$-\tfrac{3}{2}\mathcal{I}_{\mathrm{H-FGK}}[H] = \int d\rho\left\{\partial^I\partial^J \log\,\mathsf{W}\left(\dot{H}_I\dot{H}_J + q_Iq_J\right)\right\}, \tag{4.134}$$

while the Hamiltonian constraint becomes

$$\mathcal{H} \equiv \partial^I\partial^J \log\,\mathsf{W}\left(\dot{H}_I\dot{H}_J - q_Iq_J\right) = -\tfrac{3}{2}(\omega/2)^2. \tag{4.135}$$

Observe that $\partial^I\partial^J \log \mathrm{W}$ plays the role of a metric in a σ-model with coordinates H_I. Manifolds whose metrics can be written as the Hessian of a function are called *Hessian manifolds* and the function ($\log \mathsf{W}$ in this case) is known as *Hessian potential*. The problem of finding black-hole solutions becomes, thus, a mechanical problem on a Hessian manifold.

The equations of motion derived from the effective action (4.134) are

$$\partial^K\partial^I\partial^J \log\,\mathsf{W}\left(\dot{H}_I\dot{H}_J - q_Iq_J\right) + 2\partial^K\partial^I \log \mathsf{W}\,\ddot{H}_I = 0. \tag{4.136}$$

Multiplying by H_K and using the homogeneity properties of W and the Hamiltonian constraint we get

$$\partial^I \log \mathsf{W}\, \ddot{H}_I \;=\; \tfrac{3}{2}(\omega/2)^2. \tag{4.137}$$

This is just the equation of motion of U after the change of variables.

Observe that in the extremal case $\mathcal{B} = 0$, the equations of motion can be always satisfied by harmonic functions $\dot{H}_I = q_I$. This proves that the supersymmetric configurations constructed according to the recipe give in previous sections are always solutions of the equations of motion.

On the other hand, observe that, since W is homogenous of degree $3/2$ on the H_I

$$H_I \partial^I \log \mathsf{W} = \frac{3}{2}, \tag{4.138}$$

we can rewrite the Eq. (4.137) in the form

$$\partial^I \log \mathsf{W} \left[\ddot{H}_I - (\omega/2)^2 H_I\right], \tag{4.139}$$

which is generically solved by functions H_I satisfying

$$\ddot{H}_I - (\omega/2)^2 H_I = 0, \tag{4.140}$$

that is: by linear combinations of hyperbolic sines and cosines of $\frac{\omega}{2}\rho$. Thus justifies our ansatz for non-extremal black holes.

The application of this formalism to extremal non-supersymmetric and non-extremal black-hole solutions has been studied in detail in Ref. ([22]) showing the power of this formalism to obtain general results concerning the entropy, first-order flow equations for extremal and non-extremal black holes etc.

4.4.2 For the Black-String Solutions of $\mathcal{N} = 2,\ d = 5$

An analogous formalism can be developed for string-like solutions, taking into account that, even though we are in the same theory, we are interested in different solutions which are naturally given in terms of different variables: the functions K^I [12, 48, 50], related to the $h^I(\phi)$. We will also introduce auxiliary dual functions $\tilde{K}_I$.

The new variables are defined by

$$K^I \equiv e^{-U} h^I(\phi), \tag{4.141}$$

and we also define the function

$$\mathsf{V}(K) \equiv C_{IJK} K^I K^J K^K, \tag{4.142}$$

which is homogenous of third degree on the K^I. The equation that defines the scalar manifold implies that the metric function is related to the new variables by

$$e^{-3U} = \mathsf{V}(K). \tag{4.143}$$

The dual variables $\tilde{K}_I$ can be defined either by

$$\tilde{K}_I \equiv e^{-2U} h_I(\phi) \tag{4.144}$$

or by

$$\tilde{K}_I \equiv \tfrac{1}{3}\partial_I \mathsf{V}(K). \tag{4.145}$$

Following essentially the same steps as in the black-hole case, we arrive to the H-FGK action

$$-3\,\mathcal{I}_{\mathrm{H-FGK}}[K] = \int d\rho \left\{\partial_I\partial_J\mathsf{V}\left(\dot{K}^I\dot{K}^J + p^I p^J\right)\right\}, \tag{4.146}$$

and the Hamiltonian constraint

$$\mathcal{H} \equiv \partial_I\partial_J\mathsf{V}\left(\dot{K}^I\dot{K}^J - p^I p^J\right) = -3(\omega/2)^2. \tag{4.147}$$

The equations of motion that follow from the H-FGK action are

$$\partial_I\partial_K\partial_L\mathsf{V}\left(\dot{K}^K\dot{K}^L - K^K\ddot{K}^L - p^K p^L\right) = 0. \tag{4.148}$$

Contracting these equations with K^I one gets

$$\ddot{K}^I\partial_I \log\mathsf{V} = 3(\omega/2)^2, \tag{4.149}$$

which can be rewritten in the form

$$\partial_I\mathsf{V}\left[\ddot{K}^I - (\omega/2)^2 K^I\right] = 0, \tag{4.150}$$

which is, again, solved generically by linear combinations of hyperbolic sines and cosines of $\frac{\omega}{2}\rho$.

4.4.3 $\mathcal{N} = 2,\ d = 4$

The 4-dimensional case is more complicated. To start with, there is a mismatch between the number of original variables in the FGK formalism: e^U and the Z^i represent $2n + 1$ real degrees of freedom and the variables of the H-FGK formalism

H^M are $2n+2$. This should not be a problem, because we can always perform a change of variables that increases the number of variables, since the change will introduce constraints in the system. However, defining the change of variables will be more complicated. It is convenient to start with the complex variable of Kähler weight one (as the covariantly holomorphic symplectic section)

$$X = \tfrac{1}{\sqrt{2}} e^{U+i\alpha}, \tag{4.151}$$

where the phase α is a variable that does not occur in the original FGK formalism.

As in the 5-dimensional case, the homogeneity properties of the functions that appear in the supergravity theory are essential in this construction. They are simpler to find if we assume that the theory is specified by the *prepotential* $\mathcal{F}$ which is a homogeneous function of second degree in the complex coordinates $\mathcal{X}^\Lambda$. Defining

$$\mathcal{F}_\Lambda \equiv \frac{\partial \mathcal{F}}{\partial \mathcal{X}^\Lambda}, \quad \mathcal{F}_{\Lambda\Sigma} \equiv \frac{\partial^2 \mathcal{F}}{\partial \mathcal{X}^\Lambda \partial \mathcal{X}^\Sigma}, \tag{4.152}$$

we find that

$$\mathcal{F}_\Lambda = \mathcal{F}_{\Lambda\Sigma} \mathcal{X}^\Sigma. \tag{4.153}$$

The coordinates $\mathcal{X}^\Lambda$ and the dual coordinates $\mathcal{F}_\Lambda$ are related to the components of t he covariantly holomorphic section by

$$\left(\mathcal{V}^M\right) = \begin{pmatrix} \mathcal{L}^\Lambda \\ \mathcal{M}_\Lambda \end{pmatrix} = e^{\mathcal{K}/2} \begin{pmatrix} \mathcal{X}^\Lambda \\ \mathcal{F}_\Lambda \end{pmatrix}, \tag{4.154}$$

where $\mathcal{K}$ is the Kähler potential. Then, the above relation implies this relation between the components of $\mathcal{V}^M$ (dividing by X):

$$\frac{\mathcal{M}_\Lambda}{X} = \mathcal{F}_{\Lambda\Sigma} \frac{\mathcal{L}^\Sigma}{X}. \tag{4.155}$$

Splitting this relation into its real and imaginary parts and using the definitions Eq. (4.102) we get

$$\mathcal{R}_M = -\mathcal{M}_{MN}(\mathcal{F}) \mathcal{I}^N, \tag{4.156}$$

where the $2\bar{n} \times 2\bar{n}$ symmetric symplectic matrix $\mathcal{M}_{MN}(\mathcal{A})$ is defined for any complex symmetric $\bar{n} \times \bar{n}$ matrix $\mathcal{A}_{\Lambda\Sigma}$ with non-degenerate imaginary part by

$$\mathcal{M}(\mathcal{A}) \equiv \begin{pmatrix} \Im\mathrm{m}\,\mathcal{A}_{\Lambda\Sigma} + \Re\mathrm{e}\,\mathcal{A}_{\Lambda\Omega}\, \Im\mathrm{m}\,\mathcal{A}^{-1|\,\Omega\Gamma}\, \Re\mathrm{e}\,\mathcal{A}_{\Gamma\Sigma} & -\Re\mathrm{e}\,\mathcal{A}_{\Lambda\Omega}\, \Im\mathrm{m}\,\mathcal{A}^{-1|\,\Omega\Sigma} \\ -\Im\mathrm{m}\,\mathcal{A}^{-1|\,\Lambda\Omega}\, \Re\mathrm{e}\,\mathcal{A}_{\Omega\Sigma} & \Im\mathrm{m}\,\mathcal{A}^{-1|\,\Lambda\Sigma} \end{pmatrix}. \tag{4.157}$$

In the above expression $\mathcal{A}_{\Lambda\Sigma} = \mathcal{F}_{\Lambda\Sigma}$. Later on we will use the matrix $\mathcal{M}_{MN}(\mathcal{N})$ where $\mathcal{N}_{\Lambda\Sigma}$ is the period matrix. Both matrices are related by[23]

$$-\mathcal{M}_{MN}(\mathcal{N}) = \mathcal{M}_{MN}(\mathcal{F}) + 4\mathcal{V}_{(M}\mathcal{V}^*_{N)}. \tag{4.158}$$

The inverse of $\mathcal{M}_{MN}$, denoted by $\mathcal{M}^{MN}$, can be obtained by raising the indices with the inverse symplectic metric.

It is also immediate to prove the relation

$$d\mathcal{R}_M = -\mathcal{M}_{MN}(\mathcal{F})\, d\mathcal{I}^N, \tag{4.159}$$

from which one can derive the following relation between partial derivatives [56]:

$$\frac{\partial \mathcal{I}^M}{\partial \mathcal{R}_N} = \frac{\partial \mathcal{I}^N}{\partial \mathcal{R}_M} = -\frac{\partial \mathcal{R}^M}{\partial \mathcal{I}_N} = -\frac{\partial \mathcal{R}^N}{\partial \mathcal{I}_M} = -\mathcal{M}^{MN}(\mathcal{F}). \tag{4.160}$$

We are now ready to introduce two dual sets of variables H^M and $\tilde{H}_M$ and replace the original $\bar{n}$ complex fields X, Z^i by the $2\bar{n}$ real variables H^M:

$$H^M \equiv \mathcal{I}^M(X, Z, X^*, Z^*), \tag{4.161}$$

$$\tilde{H}_M \equiv \mathcal{R}^M(H). \tag{4.162}$$

Observe that the definition of the dual variables $\tilde{H}_M(H)$ implies that the stabilization equations have been solved. Knowing both sets of variables, we can reconstruct the physical fields :

$$e^{-2U} = \frac{1}{2|X|^2} = \mathcal{R}_M \mathcal{I}^M, \tag{4.163}$$

$$Z^i = \frac{\tilde{H}^i(H) + iH^i}{\tilde{H}^0(H) + iH^0}. \tag{4.164}$$

The phase of X (α) can be found[24] by solving the differential equation (*cf.* Eqs. (3.8), (3.28) in Ref. [58])

$$\dot{\alpha} = 2|X|^2 \dot{H}^M H_M - \mathcal{Q}_\star, \tag{4.165}$$

where

$$\mathcal{Q}_\star = \tfrac{1}{2i} \dot{Z}^i \partial_i \mathcal{K} + \text{c.c.}, \tag{4.166}$$

is the pullback of the Kähler connection 1-form.

[23] This relation can be derived from the identities in Ref. [57].

[24] Observe that we do not really need it, since it does not appear in the original FGK action anyway.

We are now almost ready to perform the change of variables in the FGK action. First, we need to introduce the function $\mathsf{W}(H)$

$$\mathsf{W}(H) \equiv \tilde{H}_M(H)H^M = e^{-2U} = \frac{1}{2|X|^2}, \tag{4.167}$$

which is homogenous of second degree in the H^M. Using the properties (4.160) one can show that

$$\partial_M \mathsf{W} \equiv \frac{\partial \mathsf{W}}{\partial H^M} = 2\tilde{H}_M, \tag{4.168}$$

$$\partial^M \mathsf{W} \equiv \frac{\partial \mathsf{W}}{\partial \tilde{H}_M} = 2H^M, \tag{4.169}$$

$$\partial_M \partial_N \mathsf{W} = -2\mathcal{M}_{MN}(\mathcal{F}), \tag{4.170}$$

$$\mathsf{W}\partial_M \partial_N \log \mathsf{W} = 2\mathcal{M}_{MN}(\mathcal{N}) + 4\mathsf{W}^{-1} H_M H_N, \tag{4.171}$$

where the last property is based on Eq. (4.158).

We also need the special geometry identity

$$\mathcal{G}_{ij^*} = -i\mathcal{D}_i \mathcal{V}_M \mathcal{D}_{j^*}\mathcal{V}^{*\,M} \tag{4.172}$$

to deal with the scalars' kinetic term.

Using all these results, after some work, we can rewrite the FGK effective action in the form

$$-I_{\text{H}-\text{FGK}}[H] = \int d\tau \left\{ \tfrac{1}{2}\partial_M \partial_N \log \mathsf{W} \left(\dot{H}^M \dot{H}^N + \tfrac{1}{2}\mathcal{Q}^M \mathcal{Q}^N \right) - \Lambda \right\}, \tag{4.173}$$

where we have defined

$$\Lambda \equiv \left(\frac{\dot{H}^M H_M}{\mathsf{W}} \right)^2 + \left(\frac{\mathcal{Q}^M H_M}{\mathsf{W}} \right)^2, \tag{4.174}$$

and the Hamiltonian constraint in the form

$$\mathcal{H} \equiv -\tfrac{1}{2}\partial_M \partial_N \log \mathsf{W} \left[\dot{H}^M \dot{H}^N - \tfrac{1}{2}\mathcal{Q}^M \mathcal{Q}^N \right] + \left(\frac{\dot{H}^M H_M}{\mathsf{W}} \right)^2 - \left(\frac{\mathcal{Q}^M H_M}{\mathsf{W}} \right)^2 = r_0^2, \tag{4.175}$$

where we are using the more conventional form of the non-extremality parameter $r_0 = \omega/2$ in $d = 4$.

The equations of motion for the H^P can be written in the form

$$\frac{1}{2}\partial_P\partial_M\partial_N \log \mathsf{W}\left[\dot{H}^M\dot{H}^N - \frac{1}{2}\mathcal{Q}^M\mathcal{Q}^N\right] + \partial_P\partial_M \log \mathsf{W}\,\ddot{H}^M - \frac{d}{d\tau}\left(\frac{\partial\Lambda}{\partial\dot{H}^P}\right) + \frac{\partial\Lambda}{\partial H^P} = 0. \tag{4.176}$$

Contracting these equations with H^P and using the homogeneity properties of the different terms as well as the Hamiltonian constraint above, we find the equation (*cf.* Eq. (3.31) of Ref. [58] for the stationary extremal case)

$$\frac{1}{2}\partial_M \log \mathsf{W}\left(\ddot{H}^M - r_0^2 H^M\right) + \left(\frac{\dot{H}^M H_M}{\mathsf{W}}\right)^2 = 0, \tag{4.177}$$

which corresponds to the equation of motion of the variable U in the standard FGK formulation.

If we impose the constraint

$$\dot{H}^M H_M = 0, \tag{4.178}$$

which implies the absence of NUT charge in the supersymmetric extremal case, we find that the above equation is solved, quite naturally, by H^Ms which are linear combinations of hyperbolic sines and cosines of $r_0\rho$. Furthermore, in the extremal case ($r_0 = 0$) the equations of motion are solved by linear functions of ρ such that $\dot{H}^M = \mathcal{Q}^M$ [56]. We recover, in this way, the supersymmetric extremal functions reviewed before. A more general study of the extremal non-supersymmetric and non-extremal solutions will be presented elsewhere [25].

4.5 Conclusions

As promised in the introduction, we have constructed a formalism that justifies the general ansatzs proposed in Refs. [19, 20] to find non-extremal black-hole and black-string solutions in theories of ungauged $\mathcal{N} = 2,\ d = 4, 5$ supergravity coupled to vector supermultiplets. The formalism turns out to be most useful in the study of general classes of solutions [24, 25] and, to a certain extent, closes the problem of finding the most general static, spherically symmetric black-hole and black-string solutions of those theories. At this point, the use of this formalism to find solutions of complicated theories that have resisted other methods remains a challenge, since more of the examples studied so far correspond to simple theories.

The extension of this formalism to handle Abelian gaugings via Fayet-Iliopoulos terms is straightforward and will be studied in [25].[25]

How about other 4- and 5-dimensional supergravities? It might seem that, since they are quite different (in particular the relations between the numbers of scalar fields and the possible electric and magnetic charges) an H-FGK formulation is

[25] It has also been studied via the analogous method mentioned before in Ref. [59].

simply not possible. However, the general form of the black-hole solutions of all the 4-dimensional supergravities is known [13] and structures similar to those of the $\mathcal{N} = 2$ case arise quite naturally there and these similarities have been recently used to construct the metrics and the vector field strengths of the supersymmetric, extremal, (single- or multi-center) black holes of $\mathcal{N} = 8,\ d = 4$ supergravity have been constructed in terms of a set of harmonic functions H^M [60]. It is not known how to construct explicitly the scalar fields, though. However, as we have seen, the explicit expressions of the fields are not always needed to perform a change of variables, since they tend to appear in combinations that we do know how to express in the new variables. Therefore, it is not ruled out that such formulations are possible. If found, they would give us a handle on the non-supersymmetric extremal solutions and on the non-extremal ones that we are now missing. Work in this direction is under way.

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