

# Chapter 1

## Non-holomorphic Deformations of Special Geometry and Their Applications

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**Abstract** The aim of these lecture notes is to give a pedagogical introduction to the subject of non-holomorphic deformations of special geometry. This subject was first introduced in the context of  $N = 2$  BPS black holes, but has a wider range of applicability. A theorem is presented according to which an arbitrary point-particle Lagrangian can be formulated in terms of a complex function  $F$ , whose features are analogous to those of the holomorphic function of special geometry. A crucial role is played by a symplectic vector that represents a complexification of the canonical variables, i.e. the coordinates and canonical momenta. We illustrate the characteristic features of the theorem in the context of field theory models with duality invariances. The function  $F$  may depend on a number of external parameters that are not subject to duality transformations. We introduce duality covariant complex variables whose transformation rules under duality are independent of these parameters. We express the real Hesse potential of  $N = 2$  supergravity in terms of the new variables and expand it in powers of the external parameters. Then we relate this expansion to the

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one encountered in topological string theory. These lecture notes include exercises which are meant as a guidance to the reader.

## 1.1 Introduction

As is well known, an abelian  $N = 2$  supersymmetric vector multiplet in four dimensions is described by a reduced chiral multiplet, whose gauge covariant degrees of freedom include an (anti-selfdual) field strength  $F_{\mu\nu}^-$  and a complex scalar field  $X$ . The Wilsonian effective Lagrangian for these vector multiplets is encoded in a holomorphic function  $F(X)$  which, when coupled to supergravity, is required to be homogeneous of degree two [1]. The abelian vector multiplets may be further coupled to (scalar) chiral multiplets that describe either additional dynamical fields or background fields. The function  $F$  will then also depend on holomorphic fields that reside in these chiral multiplets. An example thereof is provided by the coupling of vector multiplets to a conformal supergravity background. The multiplet that describes conformal supergravity is the Weyl multiplet, and the chiral background is given by the square of it [2]. In this case the function  $F$ , which now depends on the lowest component field of the chiral background superfield, encodes the couplings of the vector multiplets to the square of the Riemann tensor. These couplings constitute a special class of higher-derivative couplings, namely, they depend on the Riemann tensor but not on derivatives thereof. In this paper we will only consider higher-derivative couplings of this type, i.e. couplings that depend on field strengths but not on their derivatives.<sup>1</sup> We refer to [3] for a discussion on other classes of higher-derivative couplings. When higher-order derivative couplings are absent, we will denote the function  $F$  by  $F^{(0)}(X)$ , which then refers to a Wilsonian action that is at most quadratic in space-time derivatives.

The abelian vector fields in these actions are subject to electric/magnetic duality transformations under which the electric field strengths and their magnetic duals are subjected to symplectic rotations. It is then possible to convert to a different duality frame, by regarding half of the rotated field strengths as the new electric field strengths and the remaining ones as their magnetic duals. The latter are then derivable from a new action. To ensure that the characterization of the new action in terms of a holomorphic function remains preserved, the scalars of the vector multiplets are transformed accordingly. This amounts to rotating the complex fields  $X^I$  and the holomorphic derivatives  $F_I = \partial F / \partial X^I$  of the underlying function  $F$  by the same symplectic rotation as the field strengths and their dual partners [1, 4]. Here the index  $I$  labels the vector multiplets (in supergravity it takes the values  $I = 0, 1, \dots, n$ ). Thus, electric/magnetic duality (which acts on the vector  $(X^I, F_I)$ ), constitutes an equivalence transformation that relates two Lagrangians (based on two different functions) and gives rise to equivalent sets of equations of

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<sup>1</sup> In the language of the theorem that will be presented in Sect. 1.2, this may be rephrased by saying that the Lagrangians we will consider depend on coordinates and velocities, but not on accelerations.

motion and Bianchi identities. A subgroup of these equivalence transformations may constitute a symmetry (an invariance) of the system. For a duality transformation to constitute a symmetry, the substitution  $X^I \rightarrow \tilde{X}^I$  into  $F_I$  must correctly induce the transformation  $(X^I, F_I) \rightarrow (\tilde{X}^I, \tilde{F}_I)$  [5].

At the Wilsonian level, when coupling the  $N = 2$  vector multiplets to supergravity, the scalar fields of the vector multiplets parametrize a non-linear sigma-model whose geometry is called special geometry [6], a name that first arose in the study of the geometry of the effective action of type-II string compactifications on Calabi-Yau threefolds [4]. The sigma-model space is a so-called special-Kähler space, whose Kähler potential is [1],

$$K(z, \bar{z}) = -\ln \left[ \frac{i(X^I \bar{F}_I^{(0)} - \bar{X}^I F_I^{(0)})}{|X^0|^2} \right], \quad (1.1)$$

where  $F^{(0)}(X)$  is the holomorphic function that determines the supergravity action, which is quadratic in space-time derivatives. Because  $F^{(0)}(X)$  is homogeneous of second degree, this Kähler potential depends only on the ‘special’ holomorphic coordinates  $z^i = X^i/X^0$  and their complex conjugates, where  $i = 1, \dots, n$ , so that we are dealing with a special-Kähler space of complex dimension  $n$ . In view of the homogeneity, the symplectic rotations acting on the vector  $(X^I, F_I^{(0)})$ , induce corresponding (non-linear) transformations on the special coordinates  $z^i$ . Up to a Kähler transformation, the Kähler potential transforms as a function under duality.

There actually exist various ways of defining special Kähler geometry. Apart from its definition in terms of special holomorphic coordinates [1], it can also be defined in a coordinate independent way [7]. More recently, the formulation of special geometry in terms of special real instead of special holomorphic coordinates has been emphasized [8–13]. This formulation is based on the real Hesse potential [14–16], which will play an important role below.

In order to pass from the Wilsonian effective action to the 1PI low-energy effective action, one needs to integrate over the massless modes of the model. In the context of  $N = 2$  theories this induces non-holomorphic modifications in the gauge and gravitational couplings of the theory that, at the Wilsonian level, are encoded in the holomorphic function  $F$ . An early example thereof is provided by the computation of the moduli dependence of string loop corrections to gauge coupling constants in heterotic string compactifications [17]. These non-holomorphic modifications of the coupling functions are crucial to ensure that the low-energy effective action possesses the expected duality symmetries. This is therefore a generic feature of the low-energy effective action of  $N = 2$  models with duality symmetries.

Another context where these moduli dependent corrections play an important role is the one of BPS black hole solutions in  $N = 2$  models. Their entropy should exhibit the duality symmetries of the underlying model, and this is achieved by taking into account the non-holomorphic modifications of the low-energy effective action. The need for non-holomorphic modifications of the entropy was established in models with exact S-duality [18], and their presence has been confirmed at the semiclassical

level from microstate counting [19, 20]. The fact that non-holomorphic modifications can be incorporated into the entropy of BPS black holes gave a first indication that the framework of special geometry can be consistently modified by a class of non-holomorphic deformations, to be described below. This can be understood as follows. The free energy of these BPS black holes turns out to be given by a generalized version of the aforementioned Hesse potential [8, 10, 21]. The Hesse potential is related by a Legendre transformation to the function  $F$  that defines the effective action, and thus it can be regarded as the associated ‘Hamiltonian’. The Hamiltonian transforms as a function under electric/magnetic duality transformations. If the  $N = 2$  model under consideration has a duality symmetry, the Hamiltonian will be invariant under symmetry transformations due to the presence of the aforementioned non-holomorphic modifications. Since the Hamiltonian is related to the function  $F$  by an Legendre transformation, these non-holomorphic modifications will also be encoded in  $F$ .

This ‘Hamiltonian’ picture of BPS black holes suggests that special geometry can be consistently modified by a class of non-holomorphic deformations, whereby the holomorphic function  $F(X)$  that characterizes the Wilsonian action is replaced by a non-holomorphic function

$$F(X, \bar{X}) = F^{(0)}(X) + 2i \Omega(X, \bar{X}), \quad (1.2)$$

where  $\Omega$  denotes a real (in general non-harmonic) function. The Wilsonian limit is recovered by taking  $\Omega$  to be harmonic. In Sect. 1.2 we show that the non-holomorphic deformations of special geometry described by (1.2) occur in a generic setting. There we consider general point-particle Lagrangians (that depend on coordinates and velocities) and their associated Hamiltonians. We present a theorem that shows that the dynamics of these models can be reformulated in terms of a symplectic vector  $(X, \partial F/\partial X)$  constructed out of a complex function  $F$  of the form (1.2), and whose real part comprises the canonical variables of the associated Hamiltonian. We show that under duality transformations the transformed symplectic vector is again encoded in a non-holomorphic function of the form (1.2). We illustrate the theorem with various field theory examples with higher-derivative interactions. We give a detailed discussion of these examples in order to illustrate the characteristic features of the theorem. One example consists of the Born-Infeld Lagrangian for an abelian gauge field, which we reformulate in the language of the theorem based on (1.2). We subsequently promote the gauge coupling constant to a dynamical field  $S$  and discuss the duality symmetries of the resulting model. We then turn to more general models with exact S- and T-duality and discuss the restrictions imposed on  $\Omega$  by these symmetries.

The function  $F$  in (1.2) may depend on a number of external parameters which we denote by  $\eta$ . Under duality transformations, the symplectic vector  $(X, \partial F/\partial X)$  transforms into  $(\tilde{X}, \partial \tilde{F}/\partial \tilde{X})$ , while the parameters  $\eta$  are inert. When expressing the transformed variables  $\tilde{X}$  in terms of the  $X$ , the relation will depend on  $\eta$ , i.e.  $\tilde{X} = \tilde{X}(X, \eta)$ . In Sect. 1.3 we introduce covariant complex variables that constitute a complexification of the canonical variables of the Hamiltonian, and whose duality transformation law is independent of  $\eta$ . These variables ensure that when expanding the Hamiltonian

in powers of the external parameters, the resulting expansion coefficients transform covariantly under duality transformations. This expansion can also be studied by employing a modified derivative  $\mathcal{D}_\eta$ , which we construct. The covariant variables introduced in this section have the same duality transformation properties as the ones used in topological string theory and can therefore be identified with the latter. A further indication of the relation with topological string theory is provided by the generating function that relates the canonical variables of the Hamiltonian to the covariant complex variables. This generating function turns out to be the one that is used in the wave function approach to perturbative topological string theory [22–26].

In Sect. 1.4 we turn to supergravity models in the presence of higher-curvature interactions encoded in the square of the Weyl superfield [2, 5]. We consider these models in an  $AdS_2 \times S^2$  background and compute the effective action in this background. This is first done at the level of the Wilsonian effective action [27, 28]. Then we assume that the extension to the low-energy effective action can be implemented by replacing the Wilsonian holomorphic function  $F$  by the non-holomorphic function (1.2). Next, we perform a Legendre transformation of the low-energy effective action in this background and obtain the associated ‘Hamiltonian’, which takes the form of the aforementioned generalized Hesse potential. Using the covariant complex variables introduced in Sect. 1.3, we expand the associated Hesse potential (the Hamiltonian) and work out the first few iterations. This reveals a systematic structure. Namely, the Hesse potential decomposes into two classes of terms. One class consists of combinations of terms, constructed out of derivatives of  $\Omega$ , that transform as functions under electric/magnetic duality. The other class is constructed out of  $\Omega$  and derivatives thereof. Demanding this second class to also exhibit a proper behavior under duality transformations (as a consequence of the transformation behavior of the Hesse potential) imposes restrictions on  $\Omega$ . These restrictions are captured by a differential equation that constitutes half of the holomorphic anomaly equation encountered in the context of perturbative topological string theory. The differential equation is a consequence of the tension between maintaining harmonicity of  $\Omega$  and insisting on a proper behavior under duality transformations [5]. We conclude Sect. 1.4 with a brief discussion of open issues which will be addressed in an upcoming paper. There we will give a detailed discussion of the relation of perturbative topological string theory with the Hesse potential.

In the appendices we have collected various results, as follows. Appendix A discusses the transformation behavior under symplectic transformations of various holomorphic and anti-holomorphic derivatives of  $F$ . We use these expressions to give an alternative proof of the integrability of the resulting structures. In addition, we show that when  $F$  depends on an external parameter  $\eta$ , its derivative  $\partial_\eta F$  transforms as a function under symplectic transformations. In appendix B we show that the modified derivative  $\mathcal{D}_\eta$  of Sect. 1.3 acts as a covariant derivative for symplectic transformations. This is done by showing that when given a quantity  $G(x, \bar{x}; \eta)$  that transforms as a function under symplectic transformations, also  $\mathcal{D}_\eta G$  transforms as a function. In appendix C we review the holomorphic anomaly equation of topological string theory in the big moduli space. Appendix D lists certain combinations that arise in the expansion of the Hesse potential in powers of  $\eta$  and that transform as functions

under electric/magnetic duality. In appendix E we list the transformation properties of various derivatives of  $\Omega$  under duality transformations using the covariant variables of Sect. 1.3.

These lecture notes include exercises which we hope will constitute a guidance to the reader.

## 1.2 Lecture I: Point-Particle Models and $F$ -Functions

We begin by considering a general point-particle Lagrangian that depends on coordinates  $\phi$  and velocities  $\dot{\phi}$ . The associated Hamiltonian will depend on the canonical variables  $\phi$  and  $\pi$ , where  $\pi$  denotes the canonical momentum. After briefly reviewing some of the salient features of the Hamiltonian description, such as canonical transformations in phase space, we present a theorem that shows that the dynamics of these models can be reformulated in terms of a symplectic vector that is complex, and whose real part comprises the canonical variables  $(\phi, \pi)$ . This is achieved by introducing a complex function  $F$  that depends on complex variables  $x$ , with the symplectic vector given by  $(x, \partial F/\partial x)$ . This reformulation exhibits many of the special geometry features that are typical for  $N = 2$  supersymmetric systems. However, it also goes beyond the standard formulation of these systems in that the function  $F$  is of the form (1.2), and hence non-holomorphic in general.

We illustrate the theorem with various field theory examples with higher-derivative interactions. We give a detailed discussion of these examples in order to illustrate the characteristic features of the theorem. One example consists of the Born-Infeld Lagrangian for a Maxwell field, which we reformulate in the language of the theorem. We subsequently promote the gauge coupling constant to a dynamical field  $S$  and discuss the duality symmetries of the resulting model. We turn to more general models with exact S- and T-duality and discuss the restrictions imposed on  $\Omega$  by these symmetries.

The reader not interested in the details of these examples may want to proceed to Sect. 1.2.3, where we discuss the form of the Hamiltonian when the function  $F$  is such that it transforms homogeneously under a real rescaling of the variables involved.

### 1.2.1 Theorem

Let us consider a point-particle model described by a Lagrangian  $L$  with  $n$  coordinates  $\phi^i$  and  $n$  velocities  $\dot{\phi}^i$ . The associated canonical momenta  $\partial L/\partial \dot{\phi}^i$  will be denoted by  $\pi_i$ . The Hamiltonian  $H$  of the system, which follows from  $L$  by Legendre transformation,

$$H(\phi, \pi) = \dot{\phi}^i \pi_i - L(\phi, \dot{\phi}), \quad (1.3)$$

depends on  $(\phi^i, \pi_i)$ , which are called canonical variables, since they satisfy the canonical Poisson bracket relations. The variables  $(\phi^i, \pi_i)$  denote coordinates on a symplectic manifold called the classical phase space of the system. In these coordinates, the symplectic 2-form is  $d\pi_i \wedge d\phi^i$ . This 2-form is preserved under canonical transformations of  $(\phi^i, \pi_i)$  given by

$$\begin{pmatrix} \phi^i \\ \pi_i \end{pmatrix} \longrightarrow \begin{pmatrix} \tilde{\phi}^i \\ \tilde{\pi}_i \end{pmatrix} = \begin{pmatrix} U^i_j & Z^{ij} \\ W_{ij} & V_i^j \end{pmatrix} \begin{pmatrix} \phi^j \\ \pi_j \end{pmatrix}, \quad (1.4)$$

where  $U, V, Z$  and  $W$  denote  $n \times n$  matrices that satisfy the relations

$$\begin{aligned} U^T V - W^T Z &= V^T U - Z^T W = \mathbb{I}, \\ U^T W &= W^T U, \quad Z^T V = V^T Z. \end{aligned} \quad (1.5)$$

These relations are precisely such that the transformation (1.4) constitutes an element of  $\text{Sp}(2n, \mathbb{R})$ . This transformation leaves the Poisson brackets invariant. The Hamiltonian transforms as a function under symplectic transformations, i.e.  $\tilde{H}(\tilde{\phi}, \tilde{\pi}) = H(\phi, \pi)$ . When the Hamiltonian is invariant under a subset of  $\text{Sp}(2n, \mathbb{R})$  transformations, this subset describes a symmetry of the system. This invariance is often called duality invariance. Observe that the Legendre transformation (1.3) also gives rise to the relation  $\partial L / \partial \phi^i = -\partial H / \partial \phi^i$  by virtue of  $\pi_i = \partial L / \partial \dot{\phi}^i$ .

Now we present a theorem that states that the Lagrangian can be reformulated in terms of a complex function  $F(x, \bar{x})$  based on complex variables  $x^i$ , such that the canonical coordinates  $(\phi^i, \pi_i)$  coincide with (twice) the real part of  $(x^i, F_i)$ , where  $F_i = \partial F(x, \bar{x}) / \partial x^i$ .

**Theorem** Given a Lagrangian  $L(\phi, \dot{\phi})$  depending on  $n$  coordinates  $\phi^i$  and  $n$  velocities  $\dot{\phi}^i$ , with corresponding Hamiltonian  $H(\phi, \pi) = \dot{\phi}^i \pi_i - L(\phi, \dot{\phi})$ , there exists a description in terms of complex coordinates  $x^i = \frac{1}{2}(\phi^i + i\dot{\phi}^i)$  and a complex function  $F(x, \bar{x})$ , such that,

$$\begin{aligned} 2 \text{Re } x^i &= \phi^i, \\ 2 \text{Re } F_i(x, \bar{x}) &= \pi_i, \quad \text{where } F_i = \frac{\partial F(x, \bar{x})}{\partial x^i}. \end{aligned} \quad (1.6)$$

The function  $F(x, \bar{x})$  is defined up to an anti-holomorphic function and can be decomposed into a holomorphic and a purely imaginary (in general non-harmonic) function,

$$F(x, \bar{x}) = F^{(0)}(x) + 2i\Omega(x, \bar{x}). \quad (1.7)$$

The relevant equivalence transformations take the form,

$$F^{(0)} \rightarrow F^{(0)} + g(x), \quad \Omega \rightarrow \Omega - \text{Im } g(x), \quad (1.8)$$

which results in  $F(x, \bar{x}) \rightarrow F(x, \bar{x}) + \bar{g}(\bar{x})$ . The Lagrangian and Hamiltonian can then be expressed in terms of  $F^{(0)}$  and  $\Omega$ ,

$$\begin{aligned} L &= 4[\text{Im } F - \Omega], \\ H &= -i(x^i \bar{F}_i - \bar{x}^i F_i) - 4 \text{Im}[F - \frac{1}{2}x^i F_i] + 4\Omega \\ &= -i(x^i \bar{F}_i - \bar{x}^i F_i) - 4 \text{Im}[F^{(0)} - \frac{1}{2}x^i F_i^{(0)}] - 2(2\Omega - x^i \Omega_i - \bar{x}^i \Omega_{\bar{i}}), \end{aligned} \quad (1.9)$$

with  $F_i = \partial F / \partial x^i$ ,  $F_i^{(0)} = \partial F^{(0)} / \partial x^i$ ,  $\Omega_i = \partial \Omega / \partial x^i$ , and similarly for  $\bar{F}_i$ ,  $\bar{F}_i^{(0)}$  and  $\Omega_{\bar{i}}$ .

Furthermore, a crucial observation is that the  $2n$ -vector  $(x^i, F_i)$  denotes a complexification of the phase space coordinates  $(\phi^i, \pi_i)$  that transforms precisely as  $(\phi^i, \pi_i)$  under symplectic transformations, i.e.

$$\begin{pmatrix} x^i \\ F_i(x, \bar{x}) \end{pmatrix} \longrightarrow \begin{pmatrix} \tilde{x}^i \\ \tilde{F}_i(\tilde{x}, \tilde{\bar{x}}) \end{pmatrix} = \begin{pmatrix} U^i_j & Z^{ij} \\ W_{ij} & V_i^j \end{pmatrix} \begin{pmatrix} x^j \\ F_j(x, \bar{x}) \end{pmatrix}. \quad (1.10)$$

Hence, a  $\text{Sp}(2n, \mathbb{R})$  transformation of  $(x^i, F_i)$  is a canonical transformation of  $H(\phi, \pi)$ . The Eq. (1.10) are, moreover, integrable: the symplectic transformation yields a new function  $\tilde{F}(\tilde{x}, \tilde{\bar{x}}) = \tilde{F}^{(0)}(\tilde{x}) + 2i\tilde{\Omega}(\tilde{x}, \tilde{\bar{x}})$ , with  $\tilde{\Omega}$  real.

**Proof** The proof of this theorem proceeds as follows. First we introduce the  $2n$ -vector  $(x^i, y_i)$ ,

$$\begin{aligned} x^i &= \frac{1}{2} \left( \phi^i + i \frac{\partial H}{\partial \pi_i} \right), \\ y_i &= \frac{1}{2} \left( \pi_i - i \frac{\partial H}{\partial \phi^i} \right), \end{aligned} \quad (1.11)$$

which is constructed out of two canonical pairs, one comprising the variables  $(\phi^i, \pi_i)$  and the other one comprising derivatives of  $H(\phi, \pi)$ , namely  $(\partial H / \partial \pi_i, -\partial H / \partial \phi^i)$ . Both pairs transform in the same way under canonical transformations (1.4). Now we relate the vector  $(x^i, y_i)$  to the one given in (1.6), and we show that Lagrangian and the Hamiltonian can be expressed in terms of a complex function  $F(x, \bar{x})$  as in (1.9).

The Legendre transformation (1.3) gives  $\dot{\phi}^i = \partial H / \partial \pi_i$ , where we used  $\pi_i = \partial L / \partial \dot{\phi}^i$ . This equation establishes that the complex  $x^i$  introduced in (1.11) coincide with the  $x^i$  defined above (1.6). Then, expressing the Lagrangian in terms of  $x^i$  and  $\bar{x}^i$ , gives

$$\frac{\partial L(x, \bar{x})}{\partial x^i} = -2iy_i, \quad (1.12)$$



where we used the relation  $\partial L/\partial\phi^i = -\partial H/\partial\phi^i$  mentioned below (1.5). Next we write  $L$  as the sum of a harmonic and a non-harmonic function (which is always possible),

$$L = -2i[F^{(0)}(x) - \tilde{F}^{(0)}(\bar{x})] + 4\Omega(x, \bar{x}). \quad (1.13)$$

By introducing the combination  $F(x, \bar{x}) = F^{(0)}(x) + 2i\Omega(x, \bar{x})$ , we observe that the relation (1.12) can be concisely written as  $y_i = \partial F(x, \bar{x})/\partial x^i$ , while the Lagrangian (1.13) becomes  $L = 4[\text{Im} F - \Omega]$ . Using this as well as (1.11), we obtain that the Hamiltonian  $H(\phi, \pi) = \dot{\phi}^i \pi_i - L(\phi, \dot{\phi})$  can be expressed as in (1.9).

**Exercise 1** Verify that  $H$  can be written as in (1.9).

Thus, we have shown that the vector  $(x^i, y_i)$  equals  $(x^i, F_i)$ , and we have established the validity of (1.9).

Now let us discuss the integrability of  $(x^i, y_i)$  under canonical transformations. The vector  $(x^i, y_i)$ , given in (1.11), consists of two canonical pairs, and hence it transforms as in (1.10) under canonical transformations. We denote the transformed variables by  $(\tilde{x}^i, \tilde{y}_i)$ . The Hamiltonian transforms as a function, i.e.  $\tilde{H}(\text{Re } \tilde{x}, \text{Re } \tilde{y}) = H(\text{Re } x, \text{Re } y)$ , as already mentioned. Since we are dealing with a canonical transformation, the dual quantities  $(\tilde{x}^i, \tilde{y}_i)$  and  $\tilde{H}$  will satisfy the same relations as the original quantities  $(x^i, y_i)$  and  $H$ , so that we can apply the steps (1.11–1.13) to the dual quantities. The dual variables  $(\tilde{x}^i, \tilde{y}_i)$  have the decomposition given in (1.11), but now in terms of the dual quantities. The Lagrangian  $\tilde{L}$  associated to  $\tilde{H}$  is obtained by a Legendre transformation of  $\tilde{H}$ , i.e.  $\tilde{L} = \tilde{\phi}^i \tilde{\pi}_i - \tilde{H}$ . Then, applying the steps given below (1.11) to the dual Lagrangian shows that  $\tilde{L} = 4[\text{Im } \tilde{F} - \tilde{\Omega}]$ , where  $\tilde{F}$  is the sum of a holomorphic function  $\tilde{F}^{(0)}$  and a real function  $\tilde{\Omega}$ , i.e.  $\tilde{F}(\tilde{x}, \tilde{\bar{x}}) = \tilde{F}^{(0)}(\tilde{x}) + 2i\tilde{\Omega}(\tilde{x}, \tilde{\bar{x}})$ . This establishes that  $(\tilde{x}^i, \tilde{y}_i)$  can be obtained from a new function  $\tilde{F}$ , and hence ensures the integrability of  $(\tilde{x}^i, \tilde{y}_i)$  under symplectic transformations.

To complete the proof of the theorem, we need to discuss one more issue, namely the decompositions of  $F(x, \bar{x})$  and  $\tilde{F}(\tilde{x}, \tilde{\bar{x}})$  and their relation. The decomposition of  $F$  into  $F^{(0)}$  and  $\Omega$  suffers from the ambiguity (1.8), and so does the decomposition of  $\tilde{F}$ . Therefore, to be able to relate both decompositions, we need to fix the ambiguity in the decomposition of  $\tilde{F}$ , once a decomposition of  $F$  has been given. To do so, we proceed as follows.

We consider a symplectic transformation (1.10) which, as we just discussed, yields a new function  $\tilde{F}$ . Given a decomposition of  $F$ , we apply the same transformation to the vector  $(x^i, F_i^{(0)})$  alone, where  $F_i^{(0)} = \partial F^{(0)}/\partial x^i$ . This yields the vector  $(\hat{x}^i, \tilde{F}_i^{(0)}(\hat{x}))$ , as explained in appendix A. The transformed vector  $(\hat{x}^i, \tilde{F}_i^{(0)}(\hat{x}))$  can be integrated, i.e.  $\tilde{F}_i^{(0)}$  can be expressed as  $\partial \tilde{F}^{(0)}(\hat{x})/\partial \hat{x}^i$ , where  $\tilde{F}^{(0)}(\hat{x})$  is uniquely determined up to a constant and up to terms linear in  $\hat{x}^i$  (see (1.165)) [5]. The expression for  $\tilde{F}^{(0)}(\hat{x})$  can be readily obtained by using that the combination  $F^{(0)} - \frac{1}{2}x^i F_i^{(0)}$  transforms as a function under symplectic transformations, i.e.  $\delta\left(F^{(0)} - \frac{1}{2}x^i F_i^{(0)}\right) = \frac{1}{2}\left(\delta x^i F_i^{(0)} - x^i \delta F_i^{(0)}\right)$ . One obtains

$\tilde{F}^{(0)}(\hat{x}) = \frac{1}{2} \hat{x}^i \tilde{F}_i^{(0)}(\hat{x}) + F^{(0)} - \frac{1}{2} x^i F_i^{(0)}$ , up to a constant and up to terms linear in  $\hat{x}^i$ . Thus, to relate the decomposition of  $\tilde{F}$  to the decomposition of  $F$ , we demand that  $\tilde{F}^{(0)}$  refers to the combination that follows by applying a symplectic transformation to  $(x^i, F_i^{(0)})$ , as just described. This in turn determines  $\tilde{\Omega} = \frac{1}{4}[\tilde{L} - 4\text{Im} \tilde{F}^{(0)}]$ . This completes the proof of the theorem.

We finish this subsection with a few comments. First, we note that since both  $H$  and  $F^{(0)} - \frac{1}{2} x^i F_i^{(0)}$  transform as functions under symplectic transformations, so does the following combination that appears in (1.9),

$$2\Omega - x^i \Omega_i - \bar{x}^{\bar{i}} \Omega_{\bar{i}}. \quad (1.14)$$

Second, the transformation law of  $2i\Omega_i = F_i - F_i^{(0)}$  under symplectic transformations is determined by the transformation behavior of  $F_i$  and  $F_i^{(0)}$ , as described above. In appendix A we give an equivalent expression for  $\tilde{\Omega}_i$  in terms of a power series in derivatives of  $\Omega$ , see (1.161). The transformation law of  $2i\Omega_{\bar{i}} = F_{\bar{i}}$ , on the other hand, follows from the reality of  $\tilde{\Omega}$ ,

$$\tilde{\Omega}_{\bar{i}} = \overline{(\tilde{\Omega}_i)}. \quad (1.15)$$

Third, as mentioned in the introduction, the function  $F(x, \bar{x})$  may, in general, depend on a number of external parameters  $\eta$  that are inert under symplectic transformations. Without loss of generality, we may take  $\eta$  to be solely encoded in  $\Omega$  and, upon transformation, in  $\tilde{\Omega}$  (we can use the equivalence relation (1.8) to achieve this). In appendix A we show that  $\partial_\eta F = \partial F / \partial \eta$  transforms as a function under symplectic transformations [21]. We will return to this feature in Sect. 1.2.3.

Appendix A also discusses the transformation behavior under symplectic transformations of various holomorphic and anti-holomorphic derivatives of  $F$ . We use these expressions to give an alternative proof of the integrability of (1.10).

## 1.2.2 Examples

We now proceed to illustrate the features of the theorem discussed above in various models that have duality symmetries. To keep the discussion as transparent as possible in all cases, we consider the reduced Lagrangian that is obtained by restricting to spherically symmetric static configurations in flat spacetime. The first model we consider is the Born-Infeld model for an abelian gauge field, which has been known to have an  $\text{SO}(2)$  duality symmetry for a long time [29]. This symmetry may be enlarged to an  $\text{SL}(2, \mathbb{R})$  duality symmetry by coupling the system to a complex scalar field, called the dilaton-axion field [30]. This is the second model we consider. Then we turn to more general models with exact S- and T-duality and discuss the restrictions imposed on  $\Omega$  by these symmetries. We exhibit how the Born-Infeld-dilaton-axion system fits into this class of models. Finally, we focus on the case when

the function  $F(x, \bar{x})$  is taken to be homogeneous, and we discuss the form of the associated Hamiltonian.

### The Born-Infeld Model

The Born-Infeld Lagrangian<sup>2</sup> for an abelian gauge field in a spacetime with metric  $g_{\mu\nu}$  is given by [31]

$$\mathcal{L} = -g^{-2} \left[ \sqrt{|\det[g_{\mu\nu} + g F_{\mu\nu}]|} - \sqrt{|\det g_{\mu\nu}|} \right]. \quad (1.16)$$

It depends on an external parameter  $\eta = g^2$ . In the following we consider spherically symmetric static configurations in flat spacetime given by

$$\begin{aligned} ds^2 &= -dt^2 + dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2), \\ F_{rt} &= e(r), \quad F_{\theta\varphi} = p \sin \theta. \end{aligned} \quad (1.17)$$

Here, the  $\theta$ -dependence of  $F_{\theta\varphi}$  is fixed by rotational invariance, and  $p$  is constant by virtue of the Bianchi identity. Evaluating (1.16) for this configuration gives

$$\mathcal{L} = -g^{-2} r^2 \sin^2 \theta \left[ \sqrt{|1 - g^2 e^2(r)|} \sqrt{1 + g^2 p^2 r^{-4}} - 1 \right]. \quad (1.18)$$

Below we will rewrite (1.18) and bring it into the form (1.9). Since this rewriting does not depend on the angular variables and since it applies to any  $r$ -slice, we integrate over the angular variables and pick the  $r$ -slice  $4\pi r^2 = 1$ , for convenience. The resulting reduced Lagrangian reads,

$$\mathcal{L}(e, p) = -g^{-2} \left[ \sqrt{1 - g^2 e^2} \sqrt{1 + g^2 p^2} - 1 \right], \quad (1.19)$$

where we take  $g^2 e^2 < 1$ .

**Exercise 2** Instead of flat spacetime, consider the  $AdS_2 \times S^2$  line element  $ds^2 = v_1(-r^2 dt^2 + r^{-2} dr^2) + v_2(d\theta^2 + \sin^2 \theta d\varphi^2)$ , where  $v_1$  and  $v_2$  denote constants. Show that the resulting reduced Lagrangian takes the form (1.19) after performing a suitable rescaling of  $g$ ,  $e$  and  $p$ .

In the example (1.19), the role of the coordinate  $\phi$  and of the velocity  $\dot{\phi}$  introduced above (1.6) is played by  $p$  and  $e$ , respectively. The associated Hamiltonian  $\mathcal{H}$  is obtained by Legendre transforming with respect to  $\dot{\phi} = e$ . The conjugate momentum  $\pi$  is given by the electric charge  $q$ , so that

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<sup>2</sup> We will use the notation  $\mathcal{L}$  and  $\mathcal{H}$  when dealing with Lagrangian and Hamiltonian densities, respectively.

$$\mathcal{H}(p, q) = q e - \mathcal{L}(e, p). \quad (1.20)$$

Computing

$$q = \frac{\partial \mathcal{L}}{\partial e} = e \sqrt{\frac{1 + g^2 p^2}{1 - g^2 e^2}}, \quad f \equiv \frac{\partial \mathcal{L}}{\partial p} = -p \sqrt{\frac{1 - g^2 e^2}{1 + g^2 p^2}}, \quad (1.21)$$

where we introduced  $f$  for later convenience, and substituting in (1.20), we obtain for the Hamiltonian,

$$\mathcal{H}(p, q) = g^{-2} \left[ \sqrt{1 + g^2(p^2 + q^2)} - 1 \right]. \quad (1.22)$$

This Hamiltonian is manifestly invariant under  $\text{SO}(2)$  rotations of  $p$  and  $q$  and, in particular, under the discrete symmetry that interchanges the electric and magnetic charges. The external parameter  $\eta = g^2$  is inert under these transformations. These rotations constitute the only continuous symmetry of the system [29]. Their infinitesimal form can be represented by an  $\text{Sp}(2, \mathbb{R})$ -transformation (1.5) with  $U = V = 1$  and  $Z = -W = -c$ , where  $c \in \mathbb{R}$ .

Now, following the construction described in the Sect. 1.2.1, we introduce a complex coordinate  $x$  in terms of the coordinate  $\phi = p$  and the velocity  $\dot{\phi} = e$ , and a complex function  $F(x, \bar{x}; g^2)$ ,

$$x = \frac{1}{2}(p + ie), \quad F(x, \bar{x}; g^2) = F^{(0)}(x) + 2i\Omega(x, \bar{x}; g^2), \quad (1.23)$$

where

$$F^{(0)}(x) = -\frac{1}{2}ix^2, \quad \Omega(x, \bar{x}; g^2) = \frac{1}{8}g^{-2} \left( \sqrt{1 + g^2(x + \bar{x})^2} - \sqrt{1 + g^2(x - \bar{x})^2} \right)^2. \quad (1.24)$$

The split into  $F^{(0)}$  and  $\Omega$  is done in such a way that  $F^{(0)}$  will encode the contribution at the two-derivative level (which corresponds to the term  $\mathcal{L} \approx -\frac{1}{4}F_{\mu\nu}^2 + \mathcal{O}(g^2)$  in (1.16)), while  $\Omega$  will encode the higher-derivative contributions. Indeed, with these definitions the Lagrangian (1.19) can be written as

$$\mathcal{L} = 4[\text{Im}F - \Omega], \quad (1.25)$$

in agreement with the first equation of (1.9). Next, using the first equation of (1.21), we establish

$$p = 2 \text{Re } x, \quad q = 2 \text{Re } F_x, \quad (1.26)$$

in accordance with (1.6), where we recall that the conjugate momentum  $\pi$  equals  $q$ . Then, inserting (1.26) into (1.22) yields

$$\mathcal{H} = i(\bar{x} F_x - x \bar{F}_{\bar{x}}) + 4g^2 \frac{\partial \Omega}{\partial g^2}, \quad (1.27)$$

where  $F_x = \partial F(x, \bar{x}; g^2)/\partial x$ . This is in agreement with the second equation of (1.9), since  $F^{(0)}$  satisfies  $F^{(0)} = \frac{1}{2} x F_x^{(0)}$ , and  $\Omega$  obeys the homogeneity relation

$$2\Omega = x \Omega_x + \bar{x} \Omega_{\bar{x}} - 2g^2 \frac{\partial \Omega}{\partial g^2}, \quad (1.28)$$

which is a consequence of the behavior of  $\Omega$  under the real scaling  $x \rightarrow \lambda x$  and  $g^2 \rightarrow \lambda^{-2} g^2$ .

**Exercise 3** Establish (1.28) by differentiating the relation  $\Omega(\lambda x, \lambda \bar{x}; \lambda^{-2} g^2) = \lambda^2 \Omega(x, \bar{x}; g^2)$ .

**Exercise 4** Verify (1.25), (1.26) and (1.27).

Rather than performing a Legendre transformation of  $\mathcal{L}(e, p)$  with respect to  $e$ , we may instead consider performing a Legendre transformation with respect to  $p$ . The resulting quantity  $\mathcal{S}(e, f)$  will then depend on the canonical pair  $(e, f)$ , rather than on  $(p, q)$ . Using the expression for  $f$  given in (1.21), we obtain

$$\mathcal{S}(e, f) = f p - \mathcal{L}(e, p) = g^{-2} \left[ \sqrt{1 - g^2(e^2 + f^2)} - 1 \right], \quad (1.29)$$

which is invariant under SO(2) rotations of  $e$  and  $f$ . Next, we express  $\mathcal{S}(e, f)$  in terms of  $x$  and  $F_x$  introduced in (1.23). First we establish

$$f = 2 \operatorname{Im} F_x, \quad (1.30)$$

so that

$$x = \frac{1}{2}(p + ie), \quad F_x = \frac{1}{2}(q + if). \quad (1.31)$$

Then, using (1.27) and (1.31), we obtain<sup>3</sup>

$$\mathcal{S} = f p - q e + \mathcal{H} = -i(\bar{x} F_x - x \bar{F}_{\bar{x}}) + 4g^2 \frac{\partial \Omega}{\partial g^2}. \quad (1.32)$$

Let us now return to the discussion about symplectic transformations alluded to below (1.22). A symplectic transformation (1.10) may either constitute a symmetry (an invariance) of the system or correspond to a symplectic reparametrization of the system giving rise to an equivalent set of equations of motion and Bianchi identities [33]. When a symplectic transformation describes a symmetry, a convenient

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<sup>3</sup> In the context of BPS black holes,  $\mathcal{H}$  is the Hesse potential, and the double Legendre transform of  $\mathcal{H}$  yields the entropy function [8, 32].

method for verifying this consists in performing the substitution  $x^i \rightarrow \tilde{x}^i$  in the derivatives  $F_i$ , and checking that this correctly induces the symplectic transformation on  $(x^i, F_i)$  [5].

To elucidate this, let us consider a particular example, namely the discrete symmetry that interchanges the electric and magnetic charges. It can be implemented by the transformation  $(x, F_x) \rightarrow (F_x, -x)$ , which operates on the canonical pairs  $(p, q)$  and  $(e, f)$  through (1.31). This constitutes a symplectic transformation (1.5) with  $U = V = 0$ ,  $Z = 1$ ,  $W = -1$ . To verify that the transformation  $x \rightarrow \tilde{x} = F_x$  correctly induces the transformation of  $F_x$ , we compute

$$F_x = -ix \frac{1 + g^2(x^2 - \bar{x}^2)}{\sqrt{1 + g^2(x + \bar{x})^2} \sqrt{1 + g^2(x - \bar{x})^2}}. \quad (1.33)$$

Also, expressing  $e$  in terms of  $p$  and  $q$  (by using the first relation of (1.21)), we may express  $x$  in terms of  $p = 2\text{Re } x$  and  $q = 2\text{Re } F_x$ ,

$$x = \frac{1}{2} \left( p + \frac{iq}{\sqrt{1 + g^2(p^2 + q^2)}} \right). \quad (1.34)$$

We leave the following exercise to the reader.

**Exercise 5** Using (1.33), show that the transformation  $x \rightarrow F_x$  induces the transformation  $F_x \rightarrow -x$  by inserting the former on the right hand side of  $F_x$ . Similarly, using (1.34), show that the transformation  $(\text{Re } x, \text{Re } F_x) \rightarrow (\text{Re } F_x, -\text{Re } x)$  induces the transformation  $x \rightarrow F_x$ .

Next, let us discuss an example of a symplectic transformation that does not constitute a symmetry of the system, but instead describes a reparametrization of it. Namely, consider the following transformation of the canonical pair  $(p, q)$ ,

$$\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 2\text{Re } x \\ 2\text{Re } F_x \end{pmatrix} \longrightarrow \begin{pmatrix} \tilde{p} \\ \tilde{q} \end{pmatrix} = \begin{pmatrix} 2\text{Re } \tilde{x} \\ 2\text{Re } \tilde{F}_{\tilde{x}} \end{pmatrix} = \begin{pmatrix} p + \alpha q \\ q \end{pmatrix}, \quad \alpha \in \mathbb{R}. \quad (1.35)$$

This constitutes a symplectic transformation (1.5) given by  $U = V = 1$ ,  $Z = \alpha$ ,  $W = 0$ . Since, however, it does not represent an  $\text{SO}(2)$  rotation of  $p$  and  $q$ , it does not leave the Hamiltonian (1.22) invariant. To determine the new function  $\tilde{F}(\tilde{x}, \tilde{\bar{x}}; g^2)$  associated with this reparametrization, we start on the Hamiltonian side and use the fact that  $\mathcal{H}$  transforms as a function under symplectic transformations. Using (1.35) this gives

$$\tilde{\mathcal{H}}(\tilde{p}, \tilde{q}) = \mathcal{H}(p, q) = g^{-2} \left[ \sqrt{1 + g^2[(\tilde{p} - \alpha\tilde{q})^2 + \tilde{q}^2]} - 1 \right]. \quad (1.36)$$

Now we determine the corresponding Lagrangian by Legendre transformation,

$$\tilde{\mathcal{L}}(\tilde{e}, \tilde{p}) = \tilde{e} \tilde{q} - \tilde{\mathcal{H}}(\tilde{p}, \tilde{q}), \quad (1.37)$$

where

$$\tilde{e} = \frac{\partial \tilde{\mathcal{H}}}{\partial \tilde{q}} = \frac{(1 + \alpha^2)\tilde{q} - \alpha \tilde{p}}{\sqrt{1 + g^2(1 + \alpha^2)^{-1}[(1 + \alpha^2)\tilde{q} - \alpha \tilde{p}]^2 + \tilde{p}^2}}. \quad (1.38)$$

This yields,

$$\tilde{q} = \frac{\alpha \tilde{p}}{1 + \alpha^2} + \frac{\tilde{e}}{1 + \alpha^2} \sqrt{\frac{1 + \alpha^2 + g^2 \tilde{p}^2}{1 + \alpha^2 - g^2 \tilde{e}^2}}, \quad (1.39)$$

which, when inserted in (1.37), gives

$$\tilde{\mathcal{L}}(\tilde{e}, \tilde{p}) = \frac{\alpha \tilde{e} \tilde{p}}{1 + \alpha^2} - g^{-2} \left[ \frac{1}{1 + \alpha^2} \sqrt{1 + \alpha^2 - g^2 \tilde{e}^2} \sqrt{1 + \alpha^2 + g^2 \tilde{p}^2} - 1 \right]. \quad (1.40)$$

In order to bring the Lagrangian  $\tilde{\mathcal{L}}$  into the form  $\tilde{\mathcal{L}} = 4 \left[ \text{Im} \tilde{F} - \tilde{\Omega} \right]$ , as in (1.9), we express  $\tilde{\mathcal{L}}$  in terms of the complex coordinate

$$\tilde{x} = \frac{1}{2} (\tilde{p} + i\tilde{e}), \quad (1.41)$$

which is the transformed version of the coordinate  $x$  introduced in (1.23). Then, we consider all the terms in  $\tilde{\mathcal{L}}$  that are independent of  $g^2$ , and we express them in terms of a function  $\tilde{F}^{(0)}(\tilde{x})$ , as follows,

$$\frac{1}{1 + \alpha^2} \left[ \alpha \tilde{e} \tilde{p} + \frac{1}{2} (\tilde{e}^2 - \tilde{p}^2) \right] = 4 \text{Im} \tilde{F}^{(0)}(\tilde{x}). \quad (1.42)$$

This yields

$$\tilde{F}^{(0)}(\tilde{x}) = \frac{\alpha - i}{2(1 + \alpha^2)} \tilde{x}^2, \quad (1.43)$$

up to a real constant. It represents the function that is obtained by applying the symplectic transformation (1.35) to  $F^{(0)}(x)$ , as explained at the end of Sect. 1.2.1. Next, we introduce the function

$$\tilde{F}(\tilde{x}, \tilde{\bar{x}}; g^2) = \tilde{F}^{(0)}(\tilde{x}) + 2i\tilde{\Omega}(\tilde{x}, \tilde{\bar{x}}; g^2), \quad (1.44)$$

with  $\tilde{\Omega}$  real, and we require it to satisfy  $\tilde{\mathcal{L}} = 4 \left[ \text{Im} \tilde{F} - \tilde{\Omega} \right]$ . This implies that all the  $g^2$ -dependent terms will be encoded in  $\tilde{\Omega}(\tilde{x}, \tilde{\bar{x}}; g^2)$ . We obtain

$$\tilde{\Omega}(\tilde{x}, \bar{\tilde{x}}; g^2) = \frac{1}{8(1 + \alpha^2)g^2} \left( \sqrt{1 + \alpha^2 + g^2(\tilde{x} + \bar{\tilde{x}})^2} - \sqrt{1 + \alpha^2 + g^2(\tilde{x} - \bar{\tilde{x}})^2} \right)^2. \quad (1.45)$$

This result gives the function  $\tilde{F}$  associated with the reparametrization (1.35). We now check that it correctly reproduces the relation  $\tilde{q} = 2 \operatorname{Re} \tilde{F}_{\tilde{x}}$ , as required by (1.35). We compute  $\tilde{F}_{\tilde{x}}$  and obtain,

$$\begin{aligned} \tilde{F}_{\tilde{x}} = & \frac{\alpha \tilde{x}}{1 + \alpha^2} \\ & - \frac{i}{2(1 + \alpha^2)} \left\{ (\tilde{x} - \bar{\tilde{x}}) \sqrt{\frac{1 + \alpha^2 + g^2(\tilde{x} + \bar{\tilde{x}})^2}{1 + \alpha^2 + g^2(\tilde{x} - \bar{\tilde{x}})^2}} + (\tilde{x} + \bar{\tilde{x}}) \sqrt{\frac{1 + \alpha^2 + g^2(\tilde{x} - \bar{\tilde{x}})^2}{1 + \alpha^2 + g^2(\tilde{x} + \bar{\tilde{x}})^2}} \right\}. \end{aligned} \quad (1.46)$$

We leave the following exercise to the reader.

**Exercise 6** Using (1.46), verify explicitly that  $2 \operatorname{Re} \tilde{F}_{\tilde{x}}$  equals (1.39).

Now we want to see how  $\tilde{F}_{\tilde{x}}$  is related to  $F_x$ . According to the discussion around (1.10), the symplectic transformation (1.35) of the canonical pair  $(\operatorname{Re} x, \operatorname{Re} F_x)$  induces a corresponding transformation of the vector  $(x, F_x)$ ,

$$\begin{pmatrix} \tilde{x} \\ \tilde{F}_{\tilde{x}} \end{pmatrix} = \begin{pmatrix} x + \alpha F_x \\ F_x \end{pmatrix}. \quad (1.47)$$

This is indeed the case, as can be verified explicitly by expressing the transformed variables  $(\tilde{p}, \tilde{e})$  in terms of the original variables  $(p, e)$  using (1.21), (1.38) and (1.35),

$$\tilde{p} = p + \alpha e \sqrt{\frac{1 + g^2 p^2}{1 - g^2 e^2}}, \quad \tilde{e} = e - \alpha p \sqrt{\frac{1 - g^2 e^2}{1 + g^2 p^2}}, \quad (1.48)$$

and employing the relation

$$\frac{1 + \alpha^2 - g^2 \tilde{e}^2}{1 + \alpha^2 + g^2 \tilde{p}^2} = \frac{1 - g^2 e^2}{1 + g^2 p^2}. \quad (1.49)$$

**Exercise 7** Verify (1.47) explicitly using (1.46).

## Including a Dilaton-Axion Complex Scalar Field

The Born-Infeld system discussed in the previous section possesses a continuous  $SO(2)$  duality symmetry group. It is possible to enlarge this duality symmetry group



to  $\text{Sp}(2, \mathbb{R})$  by coupling the abelian gauge field to a complex scalar field  $S = \Phi + i B$  [30]. This is achieved by replacing  $g F_{\mu\nu}$  in (1.16) with  $g \Phi^{1/2} F_{\mu\nu}$  and adding a term  $B F_{\mu\nu} \tilde{F}^{\mu\nu}$  to the Lagrangian, as follows [30]

$$\mathcal{L} = -g^{-2} \left[ \sqrt{|\det[g_{\mu\nu} + g \Phi^{1/2} F_{\mu\nu}]|} - \sqrt{|\det g_{\mu\nu}|} \right] + \frac{1}{4} B F_{\mu\nu} \tilde{F}^{\mu\nu}. \quad (1.50)$$

Then, the combined system of equations of motion and Bianchi identity for  $F_{\mu\nu}$  is invariant under  $\text{Sp}(2, \mathbb{R})$  transformations, provided that  $S$  transforms in a suitable fashion. The associated Hamiltonian will then be invariant under these transformations. This will be discussed momentarily. The coupling  $g \Phi^{1/2}$  replaces the gauge coupling constant with a dynamical field, customarily called the dilaton field, while the term  $B F_{\mu\nu} \tilde{F}^{\mu\nu}$  introduces a scalar field degree of freedom called the axion. For this reason,  $S$  is also called the dilaton-axion field.

As before, let us consider spherically symmetric static configurations of the form (1.17). Picking again the  $r$ -slice  $4\pi r^2 = 1$ , for convenience, the reduced Lagrangian is now given by

$$\mathcal{L}(e, p, \Phi, B) = -g^{-2} \left[ \sqrt{1 - g^2 \Phi e^2} \sqrt{1 + g^2 \Phi p^2} - 1 \right] + B e p, \quad (1.51)$$

where we take  $g^2 \Phi e^2 < 1$ . This reduces to the previous one in (1.19) when setting  $S = 1$ . To obtain the associated Hamiltonian  $\mathcal{H}$ ,

$$\mathcal{H}(p, q, \Phi, B) = q e - \mathcal{L}(e, p, \Phi, B), \quad (1.52)$$

we first compute  $q = \partial \mathcal{L} / \partial e$ ,

$$q = e \Phi \sqrt{\frac{1 + g^2 \Phi p^2}{1 - g^2 \Phi e^2}} + B p. \quad (1.53)$$

Inverting this relation yields

$$e = \frac{q - B p}{\sqrt{\Phi^2 + g^2 \Phi [\Phi^2 p^2 + (q - B p)^2]}}, \quad (1.54)$$

and substituting in (1.52) gives

$$\mathcal{H}(p, q, \Phi, B) = g^{-2} \left[ \sqrt{1 + g^2 [\Phi p^2 + \Phi^{-1} (q - B p)^2]} - 1 \right]. \quad (1.55)$$

Then, expressing  $\Phi$  and  $B$  in terms of  $S$  and  $\bar{S}$  results in

$$\mathcal{H}(p, q, S, \bar{S}) = g^{-2} \left[ \sqrt{1 + 2g^2 \Sigma(p, q, S, \bar{S})} - 1 \right], \quad (1.56)$$

where

$$\Sigma(p, q, S, \bar{S}) = \frac{q^2 + ipq(S - \bar{S}) + p^2 |S|^2}{S + \bar{S}}. \quad (1.57)$$

**Exercise 8** Verify (1.56).

Now we are in position to discuss the invariance of the Hamiltonian under  $\text{Sp}(2, \mathbb{R})$  transformations. Consider a general  $\text{Sp}(2, \mathbb{R})$  transformation of the canonical pair  $(p, q)$  given by

$$\begin{pmatrix} p \\ q \end{pmatrix} \longrightarrow \begin{pmatrix} \tilde{p} \\ \tilde{q} \end{pmatrix} = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}, \quad (1.58)$$

where  $a, b, c, d \in \mathbb{R}$  and  $ad - bc = 1$ . The latter ensures that the transformation belongs to  $\text{SL}(2, \mathbb{R}) \cong \text{Sp}(2, \mathbb{R})$ . Then,  $\Sigma$  given in (1.57) is invariant under (1.58) provided that  $S$  transforms according to [8]

$$S \longrightarrow \frac{aS - ib}{icS + d}. \quad (1.59)$$

This explains the role of  $S$  in achieving duality invariance. It should be noted that  $S$  does not constitute an additional canonical variable, but instead describes a background field. The external parameter  $g^2$  is inert under these transformations.

**Exercise 9** Show that  $\Sigma$  is invariant under the combined transformation (1.58) and (1.59).

We observe that  $\mathcal{H}$  homogeneously as  $\mathcal{H} \rightarrow \lambda^2 \mathcal{H}$  under the real scaling  $(p, q) \rightarrow \lambda(p, q)$ ,  $g^2 \rightarrow \lambda^{-2} g^2$ ,  $S \rightarrow S$ , with  $\lambda \in \mathbb{R}$ .

Let us now return to the reduced Lagrangian (1.51) and recast it in the form  $\mathcal{L} = 4[\text{Im}F - \Omega]$ , where again we introduce the complex variable  $x = \frac{1}{2}(p + ie)$ . The function  $F$  will now depend on the two complex scalar fields  $x$  and  $S$ ,

$$F(x, \bar{x}, S, \bar{S}; g^2) = F^{(0)}(x, S) + 2i\Omega(x, \bar{x}, S, \bar{S}; g^2), \quad (1.60)$$

and is determined as follows. The holomorphic function  $F^{(0)}$  encodes all the contributions that are independent of  $g^2$ , while  $\Omega$ , which is real, accounts for all the terms in the reduced Lagrangian that depend on  $g^2$ . This yields,

$$\begin{aligned} F^{(0)}(x, S) &= -\frac{1}{2}i S x^2, & (1.61) \\ \Omega(x, \bar{x}, S, \bar{S}; g^2) &= \frac{1}{8} g^{-2} \left( \sqrt{1 + \frac{1}{2}g^2 (S + \bar{S}) (x + \bar{x})^2} \right. \\ &\quad \left. - \sqrt{1 + \frac{1}{2}g^2 (S + \bar{S}) (x - \bar{x})^2} \right)^2. \end{aligned}$$

Observe that under the scaling of  $(p, q)$  and  $g^2$  discussed below (1.59),  $e$  scales as  $e \rightarrow \lambda e$ , and hence  $x$  scales as  $x \rightarrow \lambda x$ . This in turn implies that  $F$  scales as  $F \rightarrow \lambda^2 F$ .

From (1.6) we infer that the canonical pair  $(p, q)$  is given by  $(2\text{Re } x, 2\text{Re } F_x)$ . According to the discussion around (1.10), the symplectic transformation (1.58) of the canonical pair  $(\text{Re } x, \text{Re } F_x)$  induces a transformation of the vector  $(x, F_x)$  given by  $(x, F_x) \rightarrow (d x - c F_x, a F_x - b x)$ . Since (1.58) together with (1.59) constitutes a symmetry of the model, the transformation of  $F_x$  must be induced by the transformation of  $x$  and  $S$  upon substitution. We leave it to the reader to verify this.

**Exercise 10** Show that the transformation of  $x$  and  $S$  (given in (1.58) and (1.59), respectively) induces the transformation  $F_x \rightarrow a F_x - b x$  by substituting  $x$  and  $S$  with  $\tilde{x}$  and  $\tilde{S}$  in  $F_x$ .

The reduced Lagrangian (1.51) describes the system on an  $r$ -slice  $4\pi r^2 = 1$ . Another background leading to a similar reduced Lagrangian, and hence to a similar description in terms of a function  $F$ , is provided by an  $AdS_2 \times S^2$  spacetime.

**Exercise 11** Consider the Born-Infeld-dilaton-axion system in an  $AdS_2 \times S^2$  background and show that, after performing a suitable rescaling of  $g$ ,  $e$  and  $p$ , the resulting reduced Lagrangian is again encoded in (1.61).

## Towards $N = 2$ Supergravity Models

In the Born-Infeld example discussed above, the duality symmetry of the model was enlarged by coupling it to an additional complex scalar field  $S$ . This feature is not an accident. In the context of  $N = 2$  supersymmetric models, it is well known that the presence of complex scalar fields is crucial in order for the model to have duality symmetries. To explore this in more detail, let us broaden the discussion and consider functions  $F$  that depend on three complex scalar fields  $Y^I$  (with  $I = 0, 1, 2$ ), as well as on an external parameter  $\eta$ . They will have the form

$$F(Y, \bar{Y}; \eta) = -\frac{1}{2} \frac{Y^1 (Y^2)^2}{Y^0} + 2i\Omega(Y, \bar{Y}; \eta). \quad (1.62)$$

The function  $F$  describing the Born-Infeld-dilaton-axion system, given in (1.60), is a special case of (1.62). It is obtained by performing the identification  $S = -i Y^1 / Y^0$ ,  $x = Y^2$  and  $\eta = g^2$ . This identification is consistent with the scaling properties of  $x$ ,  $S$  and  $g^2$  discussed below (1.59). Namely, by assigning the uniform scaling behavior  $Y^I \rightarrow \lambda Y^I$  to the  $Y^I$ , we reproduce the scalings of  $x$ ,  $S$  and  $g^2$ . The function (1.62) may, however, also describe other models, such as genuine  $N = 2$  supergravity models and should thus be viewed in a broader context. Depending on the chosen context, the external parameter  $\eta$  will have a different interpretation. Observe that in the description (1.62) based on the  $Y^I$ , duality transformations are

represented by  $\text{Sp}(6, \mathbb{R})$  matrices (which are  $6 \times 6$  matrices of the form (1.5)) acting on  $(Y^I, F_I)$ , where  $F_I = \partial F(Y, \tilde{Y}; \eta) / \partial Y^I$ . The external parameter  $\eta$  is inert under these transformations.

Let us now assume that a model based on (1.62) has a symmetry associated with a subgroup of  $\text{Sp}(6, \mathbb{R})$ . This will impose restrictions on the form of  $\Omega$  [18, 21]. For concreteness, we take the symmetry to be an  $\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$  subgroup. The first  $\text{SL}(2, \mathbb{R})$  subgroup acts as follows on  $(Y^I, F_I)$ ,

$$\begin{aligned} Y^0 &\rightarrow d Y^0 + c Y^1, & F_0 &\rightarrow a F_0 - b F_1, \\ Y^1 &\rightarrow a Y^1 + b Y^0, & F_1 &\rightarrow d F_1 - c F_0, \\ Y^2 &\rightarrow d Y^2 - c F_2, & F_2 &\rightarrow a F_2 - b Y^2, \end{aligned} \quad (1.63)$$

where  $a, b, c, d$  are real parameters that satisfy  $ad - bc = 1$ . This symmetry is referred to as S-duality. Let us describe its action on two complex scalar fields  $S$  and  $T$  that are given by the scale invariant combinations  $S = -iY^1/Y^0$  and  $T = -iY^2/Y^0$ . The field  $S$  is the one we encountered above. The S-duality transformation (1.63) acts as

$$S \rightarrow \frac{aS - ib}{icS + d}, \quad T \rightarrow T + \frac{2ic}{\Delta_S (Y^0)^2} \frac{\partial \Omega}{\partial T}, \quad Y^0 \rightarrow \Delta_S Y^0, \quad (1.64)$$

where we view  $\Omega$  as function of  $S, T, Y^0$  and their complex conjugates, and where

$$\Delta_S = d + icS. \quad (1.65)$$

The second  $\text{SL}(2, \mathbb{R})$  subgroup is referred to as T-duality group. Here we focus on the T-duality transformation that, in the absence of  $\Omega$ , induces the transformation  $T \rightarrow 2/T$ . It is given by the following  $\text{Sp}(6, \mathbb{R})$  transformation,

$$\begin{aligned} Y^0 &\rightarrow F_1, & F_0 &\rightarrow -Y^1, \\ Y^1 &\rightarrow -F_0, & F_1 &\rightarrow Y^0, \\ Y^2 &\rightarrow Y^2, & F_2 &\rightarrow F_2, \end{aligned} \quad (1.66)$$

and yields

$$S \rightarrow S + \frac{2}{\Delta_T (Y^0)^2} \left[ -Y^0 \frac{\partial \Omega}{\partial Y^0} + T \frac{\partial \Omega}{\partial T} \right], \quad T \rightarrow \frac{T}{\Delta_T}, \quad Y^0 \rightarrow \Delta_T Y^0, \quad (1.67)$$

where

$$\Delta_T = \frac{1}{2}T^2 + \frac{2}{(Y^0)^2} \frac{\partial \Omega}{\partial S}. \quad (1.68)$$

As already mentioned below (1.32), when a symplectic transformation describes a symmetry of the system, a convenient method for verifying this consists in performing the substitution  $Y^I \rightarrow \tilde{Y}^I$  in the derivatives  $F_I$ , and checking that this

substitution correctly induces the symplectic transformation of  $F_I$ . This will impose restrictions on the form of  $F$ , and hence also on  $\Omega$ . Imposing that S-duality (1.63) constitutes a symmetry of the model (1.62) results in the following conditions on the transformation behavior of the derivatives of  $\Omega$  [21],

$$\begin{aligned} \left(\frac{\partial\Omega}{\partial T}\right)'_S &= \frac{\partial\Omega}{\partial T}, \\ \left(\frac{\partial\Omega}{\partial S}\right)'_S &= \Delta_S^2 \left(\frac{\partial\Omega}{\partial S}\right) + \frac{\partial(\Delta_S^2)}{\partial S} \left[ -\frac{1}{2}Y^0 \frac{\partial\Omega}{\partial Y^0} - \frac{ic}{2\Delta_S(Y^0)^2} \left(\frac{\partial\Omega}{\partial T}\right)^2 \right], \\ \left(Y^0 \frac{\partial\Omega}{\partial Y^0}\right)'_S &= Y^0 \frac{\partial\Omega}{\partial Y^0} + \frac{2ic}{\Delta_S(Y^0)^2} \left(\frac{\partial\Omega}{\partial T}\right)^2, \end{aligned} \quad (1.69)$$

while requiring (1.66) to constitute a symmetry imposes the transformation behavior [21]

$$\begin{aligned} \left(\frac{\partial\Omega}{\partial S}\right)'_T &= \frac{\partial\Omega}{\partial S}, \\ \left(\frac{\partial\Omega}{\partial T}\right)'_T &= (\Delta_T - T^2) \frac{\partial\Omega}{\partial T} + T Y^0 \frac{\partial\Omega}{\partial Y^0}, \\ \left(Y^0 \frac{\partial\Omega}{\partial Y^0}\right)'_T &= Y^0 \frac{\partial\Omega}{\partial Y^0} + \frac{4}{\Delta_T(Y^0)^2} \frac{\partial\Omega}{\partial S} \left[ -Y^0 \frac{\partial\Omega}{\partial Y^0} + T \frac{\partial\Omega}{\partial T} \right]. \end{aligned} \quad (1.70)$$

These equations allow for various classes of solutions. For instance, if we only impose S-duality invariance, then an exact solution to the S-duality conditions (1.69) is

$$\Omega(S, \bar{S}, Y^0, \bar{Y}^0; \eta) = \eta \left[ \ln Y^0 + \ln \bar{Y}^0 + \ln(S + \bar{S}) \right], \quad (1.71)$$

which is invariant under (1.64). If, on the other hand, we impose both S-duality and T-duality invariance, solutions to both (1.69) and (1.70) may be constructed iteratively by assuming that  $\Omega$  is analytic in  $\eta$  and power expanding in it, so that

$$\Omega(Y, \bar{Y}; \eta) = \sum_{n=1}^{\infty} \eta^n \Omega^{(n)}(Y, \bar{Y}). \quad (1.72)$$

Then, at order  $\eta$ , the differential equations (1.69) reduce to

$$\begin{aligned} \left(\frac{\partial\Omega^{(1)}}{\partial T}\right)'_S &= \frac{\partial\Omega^{(1)}}{\partial T}, \\ \left(\frac{\partial\Omega^{(1)}}{\partial S}\right)'_S &= \Delta_S^2 \left(\frac{\partial\Omega^{(1)}}{\partial S}\right) + \frac{\partial(\Delta_S^2)}{\partial S} \left[ -\frac{1}{2}Y^0 \frac{\partial\Omega^{(1)}}{\partial Y^0} \right], \end{aligned}$$

$$\left( Y^0 \frac{\partial \Omega^{(1)}}{\partial Y^0} \right)'_S = Y^0 \frac{\partial \Omega^{(1)}}{\partial Y^0}, \quad (1.73)$$

while the differential equations (1.70) reduce to

$$\begin{aligned} \left( \frac{\partial \Omega^{(1)}}{\partial S} \right)'_T &= \frac{\partial \Omega^{(1)}}{\partial S}, \\ \left( \frac{\partial \Omega^{(1)}}{\partial T} \right)'_T &= -\frac{1}{2} T^2 \frac{\partial \Omega^{(1)}}{\partial T} + T Y^0 \frac{\partial \Omega^{(1)}}{\partial Y^0}, \\ \left( Y^0 \frac{\partial \Omega^{(1)}}{\partial Y^0} \right)'_T &= Y^0 \frac{\partial \Omega^{(1)}}{\partial Y^0}. \end{aligned} \quad (1.74)$$

Once a solution  $\Omega^{(1)}$  to these equations has been found, the full expression (1.72) can be constructed by solving (1.69) and (1.70) iteratively starting from  $\Omega^{(1)}$ .

As an application, let us return to the Born-Infeld-dilaton-axion model (1.61) which, as we already mentioned, is a model of the form (1.62) that scales as  $F \rightarrow \lambda^2 F$  under  $Y^I \rightarrow \lambda Y^I$  with  $\lambda \in \mathbb{R}$  (see below (1.61)). Let us first check that both S- and T-duality constitute invariances of the model. We recall that  $x = Y^2$ . The S-duality transformation (1.63) precisely induces the transformations (1.58) and (1.59), since  $(p, q) = (2\text{Re } x, 2\text{Re } F_x)$ . The T-duality transformation (1.66) leaves  $(x, F_x)$  invariant. By expressing  $\Omega$  given in (1.61) in terms of  $S, T$  and  $Y^0$  (and their complex conjugates), we see from (1.67) that also  $S$  is invariant under this T-duality transformation, since  $Y^0 \partial \Omega / \partial Y^0 = T \partial \Omega / \partial T$ . Consequently, the Hamiltonian (1.56) is also invariant under (1.66).

Now consider expanding (1.61) in powers of  $g^2$ . To first order we obtain

$$\Omega^{(1)} = \frac{1}{8} |Y^0|^4 (S + \bar{S})^2 |T|^4. \quad (1.75)$$

It is invariant under both (1.63) and (1.66) to lowest order in  $g^2$ , and it is straightforward to check that (1.75) indeed satisfies the differential Eqs. (1.73) and (1.74). We note that under the aforementioned scaling  $Y^I \rightarrow \lambda Y^I$ ,  $\Omega^{(1)}$  scales as  $\Omega^{(1)} \rightarrow \lambda^4 \Omega^{(1)}$ . This scaling behavior is thus very different from the one encountered in supergravity models, such as those considered in [18, 21], where the function  $F$  scaled homogeneously as  $F \rightarrow \lambda^2 F$ , but the associated  $\Omega^{(1)}$  did not scale at all. This difference is due to the fact that in these models, the external parameter  $\eta$  scaled as  $\eta \rightarrow \lambda^2 \eta$ , while in the Born-Infeld-dilaton-axion model it scales as  $\eta \rightarrow \lambda^{-2} \eta$ .

Thus, we see that the actual solutions to (1.69) and (1.70) depend sensitively on the scaling behavior of the  $Y^I$  and  $\eta$ . For instance, the solution (1.71) does not exhibit a homogeneous scaling behavior under  $Y^I \rightarrow \lambda Y^I$ . In the next subsection, we further analyze some of the consequences of this scaling behavior.

### 1.2.3 Homogeneous $F(x, \bar{x}; \eta)$

The theorem in Sect. 1.2.1 did not assume any homogeneity properties for  $F$ . Here we will look at the case when  $F$  is homogeneous of degree two and discuss some of its consequences. As shown in the previous subsections, an example of a model with this feature is the Born-Infeld-dilaton-axion system.

Let us consider a function  $F(x, \bar{x}; \eta) = F^{(0)}(x) + 2i\Omega(x, \bar{x}; \eta)$  that depends on a real external parameter  $\eta$ , and let us discuss its behavior under the scaling  $x \rightarrow \lambda x$ ,  $\eta \rightarrow \lambda^m \eta$  with  $\lambda \in \mathbb{R}$ . We take  $F^{(0)}(x)$  to be quadratic in  $x$ , so that  $F^{(0)}$  scales as  $F^{(0)}(\lambda x) = \lambda^2 F^{(0)}(x)$ . This scaling behavior can be extended to the full function  $F$  if we demand that the canonical pair  $(\phi, \pi)$  given in (1.6) scales uniformly as  $(\phi, \pi) \rightarrow \lambda(\phi, \pi)$ . Then we have

$$F(\lambda x, \lambda \bar{x}; \lambda^m \eta) = \lambda^2 F(x, \bar{x}; \eta), \quad (1.76)$$

which results in the homogeneity relation

$$2F = x^i F_i + \bar{x}^{\bar{i}} F_{\bar{i}} + m \eta F_\eta, \quad (1.77)$$

where  $F_\eta = \partial F / \partial \eta$ . Inspection of (1.11) shows that the associated Hamiltonian  $H$  scales with weight two as

$$H(\lambda \phi, \lambda \pi; \lambda^m \eta) = \lambda^2 H(\phi, \pi; \eta), \quad (1.78)$$

so that  $H$  satisfies the homogeneity relation,

$$2H = \phi \frac{\partial H}{\partial \phi} + \pi \frac{\partial H}{\partial \pi} + m \eta \frac{\partial H}{\partial \eta}. \quad (1.79)$$

Using (1.11) as well as  $y_i = F_i$ , this can be written as

$$H = i \left( \bar{x}^{\bar{i}} F_i - x^i \bar{F}_{\bar{i}} \right) + \frac{m}{2} \eta \frac{\partial H}{\partial \eta}. \quad (1.80)$$

Next, using that the dependence on  $\eta$  is solely contained in  $\Omega$ , we obtain

$$\frac{\partial H}{\partial \eta} \Big|_{\phi, \pi} = - \frac{\partial L}{\partial \eta} \Big|_{\phi, \dot{\phi}} = -4\Omega_\eta, \quad (1.81)$$

where  $\Omega_\eta = \partial \Omega / \partial \eta$ . Thus, we can express (1.80) as

$$H = i \left( \bar{x}^{\bar{i}} F_i - x^i \bar{F}_{\bar{i}} \right) - 2m \eta \Omega_\eta. \quad (1.82)$$

This relation is in accordance with (1.9) upon substitution of the homogeneity relations  $2F^{(0)}(x) = x^i F_i^{(0)}$  and  $2\Omega = x^i \Omega_i + \bar{x}^{\bar{i}} \Omega_{\bar{i}} + m \eta \Omega_\eta$  that follow from (1.77).

The Hamiltonian transforms as a function under symplectic transformations. Since the first term in (1.82) transforms as a function, it follows that  $\Omega_\eta$  also transforms as a function. This is in accordance with the general result quoted at the end of Sect. 1.2.1 which states that  $\partial_\eta F$  transforms as a function.

An application of the above is provided by the Born-Infeld-dilaton-axion system based on (1.61), whose function  $F$  scales according to (1.76) with  $m = -2$  (in this example,  $\eta = g^2$ ).

In certain situations, such as in the study of BPS black holes [34], the discussion needs to be extended to an external parameter  $\eta$  that is complex, so that now we consider a function  $F(x, \bar{x}; \eta, \bar{\eta}) = F^{(0)}(x) + 2i\Omega(x, \bar{x}; \eta, \bar{\eta})$  that scales as follows (with  $\lambda \in \mathbb{R}$ ),

$$F(\lambda x, \lambda \bar{x}; \lambda^m \eta, \lambda^m \bar{\eta}) = \lambda^2 F(x, \bar{x}; \eta, \bar{\eta}). \quad (1.83)$$

For instance, in the case of BPS black holes,  $\eta$  is identified with  $\Upsilon$ , which is complex and denotes the (rescaled) lowest component of the square of the Weyl superfield. The extension to a complex  $\eta$  results in the presence of an additional term on the right hand side of (1.77) and (1.79),

$$\begin{aligned} 2F &= x^i F_i + \bar{x}^{\bar{i}} F_{\bar{i}} + m (\eta F_\eta + \bar{\eta} F_{\bar{\eta}}), \\ 2H &= \phi \frac{\partial H}{\partial \phi} + \pi \frac{\partial H}{\partial \pi} + m \left( \eta \frac{\partial H}{\partial \eta} + \bar{\eta} \frac{\partial H}{\partial \bar{\eta}} \right), \end{aligned} \quad (1.84)$$

and hence

$$H = i \left( \bar{x}^{\bar{i}} F_i - x^i F_{\bar{i}} \right) + \frac{m}{2} \left( \eta \frac{\partial H}{\partial \eta} + \bar{\eta} \frac{\partial H}{\partial \bar{\eta}} \right). \quad (1.85)$$

Then, since the dependence on  $\eta$  and  $\bar{\eta}$  is solely contained in  $\Omega$ , we obtain

$$H = i \left( \bar{x}^{\bar{i}} F_i - x^i F_{\bar{i}} \right) - 2m (\eta \Omega_\eta + \bar{\eta} \Omega_{\bar{\eta}}). \quad (1.86)$$

This is in accordance with (1.9) upon substitution of the homogeneity relations  $2F^{(0)}(x) = x^i F_i^{(0)}$  and  $2\Omega = x^i \Omega_i + \bar{x}^{\bar{i}} \Omega_{\bar{i}} + m (\eta \Omega_\eta + \bar{\eta} \Omega_{\bar{\eta}})$  that follow from (1.84). The case of BPS black holes mentioned above corresponds to  $m = 2$  [8, 21].

The above extends straightforwardly to the case of multiple real external parameters.



### 1.3 Lecture II: Duality Covariant Complex Variables

As already discussed, the function  $F(x, \bar{x})$  may depend on a number of external parameters  $\eta$ . Under duality transformations (1.10), the symplectic vector  $(x^i, F_i(x, \bar{x}))$  transforms into  $(\tilde{x}^i, \tilde{F}_i(\tilde{x}, \tilde{\bar{x}}))$ , while the parameters  $\eta$  are inert. When expressing the transformed variables  $\tilde{x}^i$  in terms of the original  $x^i$ , the relation will depend on  $\eta$ , i.e.  $\tilde{x}^i = \tilde{x}^i(x, \bar{x}, \eta)$ . In this section we introduce duality covariant complex variables  $t^i$  whose duality transformation law is independent of  $\eta$ . These variables constitute a complexification of the canonical variables of the Hamiltonian and ensure that when expanding the Hamiltonian in powers of the external parameters, the resulting expansion coefficients transform covariantly under duality transformations. This expansion can also be organized by employing a suitable covariant derivative, which we construct. The covariant variables introduced here have the same duality transformation properties as the ones used in topological string theory and can therefore be identified with the latter.

We begin by writing the Hamiltonian  $H$  given in (1.9) in the form

$$H = -i(x^i \bar{F}_i^{(0)} - \bar{x}^i F_i^{(0)}) - 4 \operatorname{Im}[F^{(0)} - \frac{1}{2}x^i F_i^{(0)}] - 2 \left[ 2\Omega - (x^i - \bar{x}^i)(\Omega_i - \Omega_{\bar{i}}) \right], \quad (1.87)$$

where we made use of (1.7). We take  $\Omega(x, \bar{x}; \eta)$  to depend on a single real parameter  $\eta$  that is inert under symplectic transformations. The discussion given below can be extended to the case of multiple real external parameters in a straightforward manner. For later convenience, we introduce the notation  $\Omega_\eta = \partial\Omega/\partial\eta$ ,  $F_{\eta j} = \partial^2 F/\partial\eta\partial x^j$ , etc.

The Hamiltonian (1.87) is given in terms of complex fields  $x^i$  and  $\bar{x}^i$  whose transformation law under duality depends on the external parameter  $\eta$ . Now we define complex variables  $t^i$  whose transformation law does not depend on  $\eta$ , as follows. We introduce the complex vector  $(t^i, F_i^{(0)}(t))$  and equate its real part with the vector comprising the canonical variables  $(\phi^i, \pi_i)$  [10],

$$\begin{aligned} 2\operatorname{Re} t^i &= \phi^i, \\ 2\operatorname{Re} F_i^{(0)}(t) &= \pi_i. \end{aligned} \quad (1.88)$$

This definition ensures that the vector  $(t^i, F_i^{(0)}(t))$  describes a complexification of  $(\phi^i, \pi_i)$  that transforms in the same way as  $(\phi^i, \pi_i)$  under duality transformations, namely as in (1.4). This yields the transformation law

$$\tilde{t}^i = U^i_j t^j + Z^{ij} F_j^{(0)}(t), \quad (1.89)$$

which, differently from the one for the  $\tilde{x}^i$ , is independent of  $\eta$ .

Using (1.6), the new variables  $t^i$  are related to the  $x^i$  by

$$\begin{aligned} 2\operatorname{Re} t^i &= 2\operatorname{Re} x^i, \\ 2\operatorname{Re} F_i^{(0)}(t) &= 2\operatorname{Re} F_i(x, \bar{x}; \eta). \end{aligned} \quad (1.90)$$

Now we consider the series expansion of  $H$  in powers of  $\eta$ . If the expansion is performed keeping  $x^i$  and  $\bar{x}^{\bar{i}}$  fixed, the resulting coefficient functions in the expansion do not have a nice behavior under symplectic transformations because of the aforementioned dependence of  $\tilde{x}^i$  on  $\eta$ . This implies that the coefficient functions at a given order in  $\eta$  will transform into coefficient functions at higher order. This can be avoided by performing an expansion in powers of  $\eta$  keeping  $t^i$  and  $\bar{t}^{\bar{i}}$  fixed instead. We obtain

$$H = \sum_{n=0}^{\infty} \frac{\eta^n}{n!} f^{(n)}(t, \bar{t}), \quad (1.91)$$

where the coefficient functions

$$f^{(n)} = \partial_{\eta}^n H(t, \bar{t}; \eta) \Big|_{\eta=0} \quad (1.92)$$

transform as functions under symplectic transformations, i.e.  $\tilde{f}^{(n)}(\tilde{t}, \tilde{\bar{t}}) = f^{(n)}(t, \bar{t})$ . Viewing them as functions of  $\operatorname{Re} t^i$  and of  $\operatorname{Re} F_i^{(0)}(t)$ , we can re-express them in terms of  $x^i$  and  $\bar{x}^{\bar{i}}$  using (1.90), as follows. First we introduce a modified derivative  $\mathcal{D}_{\eta}$  [5, 33] that has the feature that it annihilates the canonical variables  $(\phi^i, \pi_i)$ , so that

$$\mathcal{D}_{\eta}(\operatorname{Re} x^i) = 0, \quad \mathcal{D}_{\eta}(\operatorname{Re} F_i) = 0. \quad (1.93)$$

We then use  $\mathcal{D}_{\eta}$  to expand  $H$  in powers of  $\eta$  while keeping  $\operatorname{Re} x^i$  and  $\operatorname{Re} F_i$  fixed,

$$H = \sum_{n=0}^{\infty} \frac{\eta^n}{n!} H^{(n)}, \quad (1.94)$$

where the coefficient functions are given by

$$H^{(n)} = \mathcal{D}_{\eta}^n H(x, \bar{x}; \eta) \Big|_{\eta=0}. \quad (1.95)$$

By comparing (1.91) with (1.94), we conclude that  $f^{(n)} = H^{(n)}$ , so that the symplectic coefficient functions  $f^{(n)}$  can be expressed as

$$f^{(n)} = \partial_{\eta}^n H(t, \bar{t}; \eta) \Big|_{\eta=0} = \mathcal{D}_{\eta}^n H(x, \bar{x}; \eta) \Big|_{\eta=0}. \quad (1.96)$$

The modified derivative  $\mathcal{D}_{\eta}$  used in the expansion is given by

$$\mathcal{D}_{\eta} = \partial_{\eta} + i \hat{N}^{ij} (F_{\eta j} + \bar{F}_{\eta \bar{j}}) (\partial_i - \partial_{\bar{i}}), \quad (1.97)$$

where  $\hat{N}^{ij}$  denotes the inverse of

$$\hat{N}_{ij} = -i[F_{ij} - \bar{F}_{\bar{i}\bar{j}} - F_{i\bar{j}} + \bar{F}_{\bar{i}j}]. \quad (1.98)$$

Using (1.7), the above can also be written as

$$\mathcal{D}_\eta = \partial_\eta - 2\hat{N}^{ij}(\Omega_{\eta j} - \Omega_{\eta\bar{j}})(\partial_i - \partial_{\bar{i}}), \quad (1.99)$$

with

$$\begin{aligned} \hat{N}_{ij} &= N_{ij} + 4\text{Re}(\Omega_{ij} - \Omega_{i\bar{j}}), \\ N_{ij} &= -i[F_{ij}^{(0)} - \bar{F}_{\bar{i}\bar{j}}^{(0)}]. \end{aligned} \quad (1.100)$$

Observe that  $\hat{N}_{ij}$  is a real symmetric matrix.

**Exercise 12** Verify (1.93).

We now give the first few terms in the expansion of  $H$ . We choose to evaluate them using (1.95). Expanding  $\Omega$  in a power series<sup>4</sup> in  $\eta$ ,

$$\Omega(x, \bar{x}; \eta) = \sum_{n=1}^{\infty} \frac{\eta^n}{n!} \Omega^{(n)}(x, \bar{x}), \quad (1.101)$$

we obtain

$$\begin{aligned} f^{(0)} &= -i(x^i \bar{F}_{\bar{i}}^{(0)} - \bar{x}^{\bar{i}} F_i^{(0)}) - 4\text{Im}[F^{(0)} - \frac{1}{2}x^i F_i^{(0)}], \\ f^{(1)} &= -4\Omega^{(1)}, \\ f^{(2)} &= -4\left[\Omega^{(2)} - 2N^{ij}(\Omega_i^{(1)} - \Omega_{\bar{i}}^{(1)})(\Omega_j^{(1)} - \Omega_{\bar{j}}^{(1)})\right], \\ f^{(3)} &= -4\left[\Omega^{(3)} - 6N^{ij}(\Omega_i^{(2)} - \Omega_{\bar{i}}^{(2)})(\Omega_j^{(1)} - \Omega_{\bar{j}}^{(1)}) \right. \\ &\quad + 12N^{ik}N^{jl}(\Omega_{ij}^{(1)} - \Omega_{i\bar{j}}^{(1)} + \text{c.c.})(\Omega_k^{(1)} - \Omega_{\bar{k}}^{(1)})(\Omega_l^{(1)} - \Omega_{\bar{l}}^{(1)}) \\ &\quad + 4iN^{ip}N^{jl}N^{km}(\Omega_i^{(1)} - \Omega_{\bar{i}}^{(1)})(\Omega_j^{(1)} - \Omega_{\bar{j}}^{(1)})(\Omega_k^{(1)} - \Omega_{\bar{k}}^{(1)}) \\ &\quad \left. \times (F_{plm}^{(0)} + \bar{F}_{\bar{p}\bar{l}\bar{m}}^{(0)})\right]. \end{aligned} \quad (1.102)$$

Observe that at any given order in  $\eta$ , there is no distinction between  $x^i$  and  $t^i$ , so that in (1.102) we may replace  $x^i$  everywhere by  $t^i$ .

The expansion (1.94) yields expansion functions that are symplectic functions. This implies that  $\mathcal{D}_\eta$  acts as a covariant derivative for symplectic transformations.

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<sup>4</sup> Note that here we have chosen a different normalization for the  $\Omega^{(n)}$  compared to the one in (1.72).

This can be verified explicitly and is done in appendix B, where we show that if a quantity  $G(x, \bar{x}; \eta)$  transforms as a function under symplectic transformations, then so does  $\mathcal{D}_\eta G$ . In particular, applying  $\mathcal{D}_\eta$  to  $H$  yields the relation

$$\partial_\eta H(t, \bar{t}; \eta) = \mathcal{D}_\eta H(x, \bar{x}; \eta), \quad (1.103)$$

where the right-hand side defines a symplectic function. More generally, applying multiple derivatives  $\mathcal{D}_\eta^n$  on any symplectic function depending on  $x^i$  and  $\bar{x}^{\bar{i}}$ , will again yield a symplectic function. As an example, consider applying  $\mathcal{D}_\eta$  and  $\mathcal{D}_\eta^2$  on (1.87),

$$\begin{aligned} \mathcal{D}_\eta H(x, \bar{x}; \eta) &= -4 \partial_\eta \Omega(x, \bar{x}; \eta), \\ \mathcal{D}_\eta^2 H(x, \bar{x}; \eta) &= -4 \left[ \partial_\eta^2 \Omega - 2 \hat{N}^{ij} \partial_\eta \omega_i \partial_\eta \omega_j \right], \end{aligned} \quad (1.104)$$

where  $\omega_i = \Omega_i - \Omega_{\bar{i}}$ . According to the above, both these expressions transform as functions under symplectic transformations. For the first expression this is confirmed by the result (1.177) which shows that  $\partial_\eta \Omega$  transforms as a function. The second expression shows that, while  $\partial_\eta^2 \Omega$  does not transform as a function, there exists a modification that can be included such that the result does again transform as a function. Expressions like these were derived earlier in a holomorphic setup [5, 33]. Furthermore, we note that the differential operators  $\mathcal{D}^i$ , defined by

$$\mathcal{D}^i = \hat{N}^{ij} \left( \frac{\partial}{\partial x^j} - \frac{\partial}{\partial \bar{x}^{\bar{j}}} \right), \quad (1.105)$$

are mutually commuting, and they also commute with  $\mathcal{D}_\eta$ ,

$$[\mathcal{D}^i, \mathcal{D}^j] = [\mathcal{D}^i, \mathcal{D}_\eta] = 0. \quad (1.106)$$

**Exercise 13** Verify (1.106).

As already mentioned, it is possible to extend the above to the case of several independent real parameters  $\eta, \eta', \eta'', \dots$ . In that case the additional operators,  $\mathcal{D}_{\eta'}$ , etc., will also commute with the operators considered in (1.106).

Obviously, when imposing the restriction  $\eta = 0$  on the functions  $\mathcal{D}_\eta^n H$ , they reduce to the expressions for the  $f^{(n)}$  obtained in (1.102). This can be explicitly verified for the functions given in (1.104) by comparing them to the expressions in (1.102).

Let us return to the relation (1.88) and discuss it in the light of phase space variables. As mentioned in Sect. 1.2.1, we view  $(\phi^i, \pi_i)$  as coordinates on a classical phase space equipped with the symplectic form  $d\pi_i \wedge d\phi^i$ . Let us express the symplectic form in terms of the  $t^i$  using (1.88),

$$d\pi_i \wedge d\phi^i = i N_{ij} dt^i \wedge d\bar{t}^{\bar{j}}, \quad (1.107)$$

with  $N_{ij}$  given in (1.100). This relation may be interpreted as a canonical transformation from variables  $(\phi^i, \pi_i)$  to  $(t^i, \bar{t}^{\bar{j}})$  which is generated by a function  $S$  that depends on half of all the coordinates. We take  $S$  to depend on  $\phi^i$  and  $t^i$ . We determine it in the linearized approximation by expanding  $N_{ij}$  around a background value  $t_B^i$ . Performing the shift

$$t^i \rightarrow t_B^i + t^i, \quad \bar{t}^{\bar{j}} \rightarrow \bar{t}_B^{\bar{j}} + \bar{t}^{\bar{j}}, \quad (1.108)$$

and keeping only terms linear in the fluctuations  $t^i$  and  $\bar{t}^{\bar{j}}$ , we obtain from (1.88),

$$\begin{aligned} \phi^i &= t^i + \bar{t}^{\bar{j}}, \\ \pi_i &= F_{ij}^{(0)}(t_B) t^j + \bar{F}_{\bar{i}\bar{j}}^{(0)}(\bar{t}_B) \bar{t}^{\bar{j}}, \end{aligned} \quad (1.109)$$

where we absorbed the fluctuation independent pieces into the definition of  $(\phi^i, \pi_i)$ . Then, expressing  $\pi_i$  in terms of  $t^i$  and  $\phi^i$ ,

$$\pi_i = i N_{ij}(t_B, \bar{t}_B) t^j + \bar{F}_{\bar{i}\bar{j}}^{(0)}(\bar{t}_B) \phi^{\bar{j}}, \quad (1.110)$$

and introducing the combination

$$P_i = -i N_{ij}(t_B, \bar{t}_B) (\phi^{\bar{j}} - t^{\bar{j}}), \quad (1.111)$$

yields

$$d\pi_i \wedge d\phi^i = i N_{ij}(t_B, \bar{t}_B) dt^i \wedge d\bar{t}^{\bar{j}} = dP_i \wedge dt^i. \quad (1.112)$$

Hence, the 1-form  $\pi_i d\phi^i - P_i dt^i$  is closed, so that locally,

$$\pi_i d\phi^i - P_i dt^i = dS, \quad (1.113)$$

where  $S(\phi, t)$  is called the generating function of the canonical transformation. Then, integrating this relation yields the following expression for the generating function  $S(\phi, t; t_B, \bar{t}_B)$  [23–25],

$$\begin{aligned} S(\phi, t; t_B, \bar{t}_B) &= \frac{1}{2} \bar{F}_{\bar{i}\bar{j}}^{(0)}(\bar{t}_B) \phi^{\bar{i}} \phi^{\bar{j}} + i N_{ij}(t_B, \bar{t}_B) \phi^{\bar{i}} t^{\bar{j}} - \frac{1}{2} i N_{ij}(t_B, \bar{t}_B) t^{\bar{i}} t^{\bar{j}} \\ &\quad + c(t_B, \bar{t}_B), \end{aligned} \quad (1.114)$$

where  $c$  denotes a background dependent integration constant. Observe that  $S(\phi, t; t_B, \bar{t}_B)$  is holomorphic in the fluctuation  $t$  and non-holomorphic in the background  $t_B$ . The generating function  $S(\phi, t; t_B, \bar{t}_B)$  plays a crucial role in the wave function approach to perturbative topological string theory. This approach represents a concise framework [22–26] for deriving the holomorphic anomaly equation of topological string theory [35, 36], and will be reviewed in appendix C.

## 1.4 Lecture III: The Hesse Potential and the Topological String

In the previous sections we showed that the dynamics of a general class of Lagrangians is encoded in a non-holomorphic function  $F$  of the form given in (1.2). This function  $F$  may depend on a number of external parameters  $\eta$ . We expressed the associated Hamiltonian in terms of duality covariant complex variables and showed that in these variables, the expansion of the Hamiltonian in a power series in  $\eta$  yields expansion coefficients that transform as functions under duality. In this section we apply these techniques to supergravity models in the presence of higher-curvature interactions encoded in the square of the Weyl superfield [2, 5]. We consider these models in an  $AdS_2 \times S^2$  background. The Hamiltonian (1.9) associated to the reduced Lagrangian is a (generalized) Hesse potential. The Hesse potential plays a central role in the formulation of special geometry in terms of real variables [7, 14–16]. The external parameter  $\eta$ , which is now complex, is identified with the lowest component field of the square of the Weyl superfield.

We begin by reviewing the computation of the Wilsonian effective Lagrangian in an  $AdS_2 \times S^2$  background [27, 28] and relate it to the presentation of Sect. 1.2. We then generalize the discussion to the case of a function  $F$  of type (1.7) with a non-harmonic  $\Omega$ . We express the Hesse potential in terms of the aforementioned duality covariant complex variables, and expand it in powers of  $\eta$  and  $\bar{\eta}$ . This reveals a systematic structure. Namely, the Hesse potential decomposes into two classes of terms. One class consists of combinations of terms, constructed out of derivatives of  $\Omega$ , that transform as functions under electric/magnetic duality. The other class is constructed out of  $\Omega$  and derivatives thereof. Demanding this second class to also exhibit a proper behavior under duality transformations (as a consequence of the transformation behavior of the Hesse potential) imposes restrictions on  $\Omega$ . These restrictions are captured by a differential equation that equals half of the holomorphic anomaly equation encountered in perturbative topological string theory.

### 1.4.1 The Reduced Wilsonian Lagrangian in an $AdS_2 \times S^2$ Background

We consider the coupling of  $N = 2$  vector multiplets to  $N = 2$  supergravity in the presence of higher-curvature interactions encoded in the square of the Weyl superfield [2, 5]. We use the conventions of  $N = 2$  supergravity, whereby the vector multiplets are labelled by a capital index  $I = 0, \dots, n$  (instead of the index  $i$  used in the previous sections). The degrees of freedom of a vector multiplet include an abelian gauge field and a complex scalar field, and these will thus carry an index  $I$ . We denote the complex scalar fields by  $X^I$ . The square of the Weyl superfield has various component fields. The highest component field contains the square of the anti-selfdual components of the Riemann tensor, while the lowest one, denoted by

$\hat{A}$ , equals the square of an anti-selfdual tensor field. Below we will find it convenient to work with rescaled complex fields  $Y^I$  and  $\mathcal{Y}$ , which are related to the  $X^I$  and  $\hat{A}$  by a complex rescaling [34].

First we evaluate the Wilsonian effective Lagrangian of these models on a field configuration consistent with the  $\text{SO}(2, 1) \times \text{SO}(3)$  isometry of an  $AdS_2 \times S^2$  background. The spacetime metric  $g_{\mu\nu}$  and the field strengths  $F_{\mu\nu}{}^I$  of the abelian gauge fields are given by

$$\begin{aligned} ds^2 &= v_1 \left( -r^2 dt^2 + \frac{dr^2}{r^2} \right) + v_2 \left( d\theta^2 + \sin^2 \theta d\varphi^2 \right), \\ F_{rt}{}^I &= e^I, \quad F_{\theta\varphi}{}^I = p^I \sin \theta. \end{aligned} \quad (1.115)$$

The  $\theta$ -dependence of  $F_{\theta\varphi}{}^I$  is fixed by rotational invariance and the  $p^I$  denote the magnetic charges. The quantities  $v_1, v_2, e^I$  and  $p^I$  are all constant by virtue of the  $\text{SO}(2, 1) \times \text{SO}(3)$  symmetry.

It is well-known [2] that the Wilsonian Lagrangian  $\mathcal{L}$  is encoded in a holomorphic function  $F(X, \hat{A})$ , which is homogeneous of degree two under the scaling discussed in (1.76), i.e.  $F(\lambda X, \lambda^2 \hat{A}) = \lambda^2 F(X, \hat{A})$ . Evaluating the Wilsonian Lagrangian in the background (1.115) and integrating over  $S^2$  [32],

$$\mathcal{F} = \int d\theta d\varphi \sqrt{|g|} \mathcal{L}, \quad (1.116)$$

yields the reduced Wilsonian Lagrangian which depends on  $e^I$  and  $p^I$ , on the rescaled fields  $Y^I$  and  $\mathcal{Y}$ , and on  $v_1$  and  $v_2$  through the ratio  $U = v_1/v_2$ .

In the following, we will restrict to supersymmetric backgrounds, for simplicity, in which case  $U = 1$  and  $\mathcal{Y} = -64$  [34]. Then, the reduced Wilsonian Lagrangian reads [27, 28],

$$\begin{aligned} \mathcal{F}(e, p, Y, \bar{Y}; \mathcal{Y}, \bar{\mathcal{Y}}) &= -\frac{1}{8}i (F_{IJ} - \bar{F}_{\bar{I}\bar{J}}) \left( e^I e^J - p^I p^J \right) - \frac{1}{4} (F_{IJ} + \bar{F}_{\bar{I}\bar{J}}) e^I p^J \\ &\quad + \frac{1}{2}ie^I \left( F_I + F_{IJ} \bar{Y}^{\bar{J}} - \text{h.c.} \right) - \frac{1}{2}p^I \left( F_I - F_{IJ} \bar{Y}^{\bar{J}} + \text{h.c.} \right) \\ &\quad + i \left( F - Y^I F_I + \frac{1}{2} \bar{F}_{\bar{I}\bar{J}} Y^I Y^{\bar{J}} - \text{h.c.} \right), \end{aligned} \quad (1.117)$$

where  $\mathcal{Y} = \bar{\mathcal{Y}} = -64$  and  $F_I = \partial F / \partial Y^I$ ,  $F_{IJ} = \partial^2 F / \partial Y^I \partial Y^J$ , etc. Introducing the complex scalar fields  $x^I = \frac{1}{2}(p^I + ie^I)$  of Sect. 1.2.2 (see (1.31)), the reduced Lagrangian becomes a function of two types of complex scalar fields, namely the  $x^I$  that incorporate the electromagnetic information, and the moduli fields  $Y^I$ .

Now we recall that in an  $AdS_2 \times S^2$  background the electro/magnetic quantities appearing in (1.115) are related to the moduli fields  $Y^I$ . When the background is supersymmetric, the relation takes the form [37]

$$x^I = i\bar{Y}^I. \quad (1.118)$$

In the context of BPS black holes, the real part of this equation yields the magnetic attractor equation. Then, using (1.118), the reduced Wilsonian Lagrangian becomes equal to

$$\mathcal{F}(Y, \bar{Y}; \gamma, \bar{\gamma}) = -2 \operatorname{Im} F(Y, \gamma), \quad (1.119)$$

with  $\gamma = \bar{\gamma} = -64$ .

**Exercise 14** Verify (1.119).

Let us reformulate the reduced Lagrangian (1.119), which is based on a holomorphic functions  $F(Y, \gamma)$ , in terms of the function  $F(Y, \bar{Y}; \gamma, \bar{\gamma}) = F^{(0)}(Y) + 2i\Omega(Y, \bar{Y}; \gamma, \bar{\gamma})$  introduced in Sect. 1.2. This is achieved by using the equivalence transformation (1.8). Writing the holomorphic function  $F(Y, \gamma)$  as  $F(Y, \gamma) = F^{(0)}(Y) - g(Y, \gamma)$  and applying (1.8), we obtain  $\Omega = -\operatorname{Im} g(Y, \gamma)$ . Thus, at the Wilsonian level,  $\Omega$  is a harmonic function, and the reduced Lagrangian can be expressed as

$$\begin{aligned} \mathcal{F}(Y, \bar{Y}; \gamma, \bar{\gamma}) &= -2 \left[ \operatorname{Im} F^{(0)}(Y) + \Omega(Y, \bar{Y}; \gamma, \bar{\gamma}) \right] \\ &= -2 \left[ \operatorname{Im} F(Y, \bar{Y}; \gamma, \bar{\gamma}) - \Omega(Y, \bar{Y}; \gamma, \bar{\gamma}) \right], \end{aligned} \quad (1.120)$$

with  $\gamma = \bar{\gamma} = -64$ . Both  $F^{(0)}$  and  $\Omega$  are homogeneous functions of degree two, so that  $\mathcal{F}(\lambda Y, \lambda \bar{Y}; \lambda^2 \gamma, \lambda^2 \bar{\gamma}) = \lambda^2 \mathcal{F}(Y, \bar{Y}; \gamma, \bar{\gamma})$ .

The reduced Lagrangian (1.120) agrees with the one in (1.9), up to an overall normalization factor of  $-2$ . In the following, we rescale (1.120) by this factor, so that from now on

$$\mathcal{F}(Y, \bar{Y}; \gamma, \bar{\gamma}) = 4 \left[ \operatorname{Im} F(Y, \bar{Y}; \gamma, \bar{\gamma}) - \Omega(Y, \bar{Y}; \gamma, \bar{\gamma}) \right]. \quad (1.121)$$

Using (1.118), we infer that  $p^I = -i(Y^I - \bar{Y}^I)$  and  $e^I = Y^I + \bar{Y}^I$ . According to (1.6), on the other hand, the real part of  $Y^I$  plays the role of the canonical variable  $\phi^I$ , so that we have  $\phi^I = e^I$ . We may thus view  $\mathcal{F}$  as a function of  $p^I$  and  $\phi^I$ , and consider its Legendre transformation either with respect to  $p^I$  or with respect to  $\phi^I$ . Performing the Legendre transformations with respect to the  $p^I$ , i.e.  $\mathcal{H} = \mathcal{F} - p^I \pi_I$ , results in

$$\pi_I = \frac{\partial \mathcal{F}}{\partial p^I} = F_I + \bar{F}_{\bar{I}}, \quad (1.122)$$

and hence

$$\begin{aligned} \mathcal{H} &= i \left[ Y^I \bar{F}_{\bar{I}} - \bar{Y}^{\bar{I}} F_I \right] + 2 \left[ 2\Omega - Y^I \Omega_I - \bar{Y}^{\bar{I}} \Omega_{\bar{I}} \right] \\ &= i \left[ Y^I \bar{F}_{\bar{I}}^{(0)} - \bar{Y}^{\bar{I}} F_I^{(0)} \right] + 2 \left[ 2\Omega - (Y^I - \bar{Y}^{\bar{I}})(\Omega_I - \Omega_{\bar{I}}) \right], \end{aligned} \quad (1.123)$$

which is the analogue of the Hamiltonian (1.9) (up to an overall sign difference in the definition of both quantities). In the context of BPS black holes,  $\mathcal{H}$  denotes the



BPS free energy of the black hole. When viewed as a function of  $\phi^I$  and  $\pi_I$ ,  $\mathcal{H}(\phi, \pi)$  is called the Hesse potential.

**Exercise 15** Verify (1.123).

On the other hand, performing the Legendre transformations with respect to the  $\phi^I$ , i.e.  $S = \mathcal{F} - \phi^I q_I$ , results in

$$q_I = \frac{\partial \mathcal{F}}{\partial \phi^I} = -i (F_I - \bar{F}_{\bar{I}}), \quad (1.124)$$

and hence

$$\begin{aligned} S &= -i \left[ Y^I \bar{F}_{\bar{I}} - \bar{Y}^{\bar{I}} F_I \right] + 2 \left[ 2\Omega - Y^I \Omega_I - \bar{Y}^{\bar{I}} \Omega_{\bar{I}} \right] \\ &= -i (Y^I \bar{F}_{\bar{I}}^{(0)} - \bar{Y}^{\bar{I}} F_I^{(0)}) + 2 \left[ 2\Omega - (Y^I + \bar{Y}^{\bar{I}})(\Omega_I + \Omega_{\bar{I}}) \right]. \end{aligned} \quad (1.125)$$

In the context of BPS black holes, (1.124) is the electric attractor equation, and  $S$  denotes the black hole entropy when viewed as function of  $p^I$  and  $q_I$  [34].

**Exercise 16** Verify (1.125).

The entropy  $S$  can be obtained from the Hesse potential by a double Legendre transformation with respect to  $(\phi^I, \pi_I)$  [8], i.e.

$$S(p, q) = \mathcal{H}(\phi, \pi) + \pi_I p^I - \phi^I q_I \quad (1.126)$$

with  $p^I = -\partial \mathcal{H} / \partial \pi_I$  and  $q_I = \partial \mathcal{H} / \partial \phi^I$ .

### 1.4.2 The Reduced Low-Energy Effective Action in an $AdS_2 \times S^2$ Background

When passing from the Wilsonian to the low-energy effective action, non-holomorphic terms emerge that are crucial for maintaining duality invariances [17], and that therefore need to be incorporated into the framework of the previous subsection. In the following, we assume that these terms can be incorporated into  $\Omega$  by giving up the requirement that  $\Omega$  is harmonic. We take the reduced low-energy effective Lagrangian and the associated Hesse potential to be given by (1.121) and (1.123), respectively, but now based on a non-harmonic  $\Omega$ .

The Hesse potential (1.123) is given in terms of complex scalar fields  $Y^I$  and  $\bar{Y}^{\bar{I}}$ . Under duality transformations, the scalar fields  $Y^I$  transform into  $\tilde{Y}^I = \tilde{Y}^I(Y, \bar{Y}, \gamma, \tilde{\gamma})$  (and similarly for the  $\bar{Y}^{\bar{I}}$ ), as discussed in Sect. 1.3. In order to obtain expansion coefficients that have a proper behavior under duality when expanding  $\mathcal{H}$  in powers of  $\gamma$  and  $\tilde{\gamma}$ , we first express  $\mathcal{H}$  in terms of the duality covariant complex

coordinates introduced in Sect. 1.3. This can be achieved by iteration, and the result for the Hesse potential in the new coordinates then takes the form of an infinite power series in terms of  $\Omega$  and its derivatives. We explicitly evaluate the first terms in this expansion up to order  $\Omega^5$ . This suffices for appreciating the general structure of the full result. The actual calculations are rather cumbersome, and we have relegated some relevant material to appendices D and E. The expression for the Hesse potential, given in (1.143), consists of a sum of contributions  $\mathcal{H}_i^{(a)}$ , each of which transforms as a function under symplectic transformations. The function  $\mathcal{H}^{(1)}$  is the only one that contains  $\Omega$ , while all the other  $\mathcal{H}_i^{(a)}$  contain derivatives of  $\Omega$ . Using that  $\mathcal{H}^{(1)}$  transforms as a function under symplectic transformations, we determine the transformation law of  $\Omega$ , which is given in (1.146). In the following, we present a detailed derivation of these results. We suppress the superscript in  $F^{(0)}$  for the most part, for simplicity.

The Hesse potential  $\mathcal{H}$  is defined in terms of the real variables  $(\phi^I, \pi_I)$ , whose definition depends on the full effective action. These may be expressed in terms of the duality covariant variables introduced in (1.88), and which will be denoted by  $\mathcal{Y}^I$  in the following. Inspection of (1.90) shows that these new variables are such that they coincide precisely with the fields  $Y^I$  that one would obtain from  $(\phi^I, \pi_I)$  by using only the lowest-order holomorphic function  $F^{(0)}$ ,

$$\begin{aligned} 2 \operatorname{Re} \mathcal{Y}^I &= \phi^I = 2 \operatorname{Re} Y^I, \\ 2 \operatorname{Re} F_I^{(0)}(\mathcal{Y}) &= \pi_I = 2 \operatorname{Re} F_I(Y, \bar{Y}; \Upsilon, \bar{\Upsilon}). \end{aligned} \quad (1.127)$$

Since the relation between the new variables and the real variables  $(\phi^I, \pi_I)$  depends only on  $F^{(0)}$ , their duality transformations will not depend on the details of the full effective action. Under symplectic transformations they transform according to,

$$\tilde{\mathcal{Y}}^I = U^I{}_J \mathcal{Y}^J + Z^{IJ} F_J^{(0)}(\mathcal{Y}) = \mathcal{S}_0^I{}_J(\mathcal{Y}) \mathcal{Y}^J, \quad (1.128)$$

where

$$\mathcal{S}_0^I{}_J(\mathcal{Y}) = U^I{}_J + Z^{IK} F_{KJ}^{(0)}(\mathcal{Y}). \quad (1.129)$$

At the two-derivative level, where  $\Omega = 0$ , we have  $\mathcal{Y}^I = Y^I$ , but in higher orders the relation between these moduli is complicated and will depend on  $\Omega$ . Hence we write  $\mathcal{Y}^I = Y^I + \Delta Y^I$ , where  $\Delta Y^I$  is purely imaginary. Writing  $F = F^{(0)} + 2i\Omega$ , we will express (1.128) in terms of  $F^{(0)}$  and  $\Omega$ , so that we can henceforth suppress the superscript in  $F^{(0)}$ . Hence, in the following,  $F$  will denote a holomorphic homogeneous function of degree two. Therefore it is not necessary to make a distinction between holomorphic and anti-holomorphic derivatives of this function. The Eq. (1.127) can then be written as,

$$\begin{aligned} F_I(\mathcal{Y} - \Delta Y) + \bar{F}_I(\bar{\mathcal{Y}} + \Delta Y) - F_I(\mathcal{Y}) - \bar{F}_I(\bar{\mathcal{Y}}) \\ = -2i [\Omega_I(\mathcal{Y} - \Delta Y, \bar{\mathcal{Y}} + \Delta Y) - \Omega_{\bar{I}}(\mathcal{Y} - \Delta Y, \bar{\mathcal{Y}} + \Delta Y)]. \end{aligned} \quad (1.130)$$

Upon Taylor expanding, this equation will lead to an infinite power series in  $\Delta Y^I$ . Retaining only the term of first order in  $\Delta Y^I$  shows that it is proportional to the first derivative of  $\Omega$ . Proceeding by iteration will then lead to an expression for  $\Delta Y^I$  involving increasing powers of  $\Omega$  and its derivatives taken at  $Y^I = \mathcal{Y}^I$ . Here it suffices to give the result of this iteration up to fourth order in  $\Omega$ ,

$$\begin{aligned}
\Delta Y^I &= 2(\Omega^I - \Omega^{\bar{I}}) \\
&\quad - 2i(F + \bar{F})^{IJK}(\Omega_J - \Omega_{\bar{J}})(\Omega_K - \Omega_{\bar{K}}) - 8\text{Re}(\Omega^{IJ} - \Omega^{I\bar{J}})(\Omega_J - \Omega_{\bar{J}}) \\
&\quad + \frac{4}{3}i\left[(F - \bar{F})^{IJKL} + 3i(F + \bar{F})^{IJM}(F + \bar{F})_M{}^{KL}\right] \\
&\quad \times (\Omega_J - \Omega_{\bar{J}})(\Omega_K - \Omega_{\bar{K}})(\Omega_L - \Omega_{\bar{L}}) \\
&\quad + 8i\left[2(F + \bar{F})^{IJ}{}_K\text{Re}(\Omega^{KL} - \Omega^{K\bar{L}}) + \text{Re}(\Omega^{IK} - \Omega^{I\bar{K}})(F + \bar{F})_K{}^{JL}\right] \\
&\quad \times (\Omega_J - \Omega_{\bar{J}})(\Omega_L - \Omega_{\bar{L}}) \\
&\quad + 32\text{Re}(\Omega^{IJ} - \Omega^{I\bar{J}})\text{Re}(\Omega_{JK} - \Omega_{J\bar{K}})(\Omega^K - \Omega^{\bar{K}}) \\
&\quad + 8i\text{Im}(\Omega^{IJK} - 2\Omega^{I\bar{J}\bar{K}} + \Omega^{I\bar{J}\bar{K}})(\Omega_J - \Omega_{\bar{J}})(\Omega_K - \Omega_{\bar{K}}) + \mathcal{O}(\Omega^4).
\end{aligned} \tag{1.131}$$

Here indices have been raised by making use of  $N^{IJ}$ , which denotes the inverse of

$$N_{IJ} = 2\text{Im}F_{IJ}, \tag{1.132}$$

where we stress that all the derivatives of  $F$  and  $\Omega$  are taken at  $Y^I = \mathcal{Y}^I$  and  $\bar{Y}^I = \bar{\mathcal{Y}}^I$ .

Furthermore, we obtain the following expression for the Hesse potential (1.123),

$$\begin{aligned}
\mathcal{H}(\mathcal{Y}, \bar{\mathcal{Y}}) &= -i[\bar{\mathcal{Y}}^I F_I(\mathcal{Y}) - \mathcal{Y}^I \bar{F}_I(\bar{\mathcal{Y}})] + 4\Omega(\mathcal{Y}, \bar{\mathcal{Y}}) \\
&\quad - i\left[\mathcal{Y}^I (F_I(Y) - F_I(\mathcal{Y})) + \Delta Y^I F_I(Y) - \text{h.c.}\right] \\
&\quad + 4\left[\Omega(Y, \bar{Y}) - \Omega(\mathcal{Y}, \bar{\mathcal{Y}}) + \Delta Y^I (\Omega_I(Y, \bar{Y}) - \Omega_{\bar{I}}(Y, \bar{Y}))\right].
\end{aligned} \tag{1.133}$$

Here we made use of (1.130) at an intermediate stage of the calculation. Again this result must be Taylor expanded upon writing  $Y^I = \mathcal{Y}^I - \Delta Y^I$  and  $\bar{Y}^I = \bar{\mathcal{Y}}^I + \Delta Y^I$ . The last two lines of (1.133) then lead to a power series in  $\Delta Y$ , starting at second order in the  $\Delta Y$ ,

$$\begin{aligned}
\mathcal{H}(\mathcal{Y}, \bar{\mathcal{Y}}) &= -i[\bar{\mathcal{Y}}^I F_I(\mathcal{Y}) - \mathcal{Y}^I \bar{F}_I(\bar{\mathcal{Y}})] + 4\Omega(\mathcal{Y}, \bar{\mathcal{Y}}) \\
&\quad - N_{IJ}\Delta Y^I \Delta Y^J - \frac{2}{3}i(F + \bar{F})_{IJK}\Delta Y^I \Delta Y^J \Delta Y^K \\
&\quad - 4\text{Re}(\Omega_{IJ} - \Omega_{I\bar{J}})\Delta Y^I \Delta Y^J + \frac{1}{4}i(F - \bar{F})_{IJKL}\Delta Y^I \Delta Y^J \Delta Y^K \Delta Y^L \\
&\quad + \frac{8}{3}i\text{Im}(\Omega_{IJK} - 3\Omega_{I\bar{J}\bar{K}})\Delta Y^I \Delta Y^J \Delta Y^K + \dots
\end{aligned} \tag{1.134}$$

Inserting the result of the iteration (1.131) into the expression above leads to the following expression for the Hesse potential, up to terms of order  $\Omega^5$ ,

$$\begin{aligned}
\mathcal{H}(\mathcal{Y}, \bar{\mathcal{Y}}) = & -i[\bar{\mathcal{Y}}^I F_I(\mathcal{Y}) - \mathcal{Y}^I \bar{F}_I(\bar{\mathcal{Y}})] + 4 \Omega(\mathcal{Y}, \bar{\mathcal{Y}}) \\
& - 4 \hat{N}^{IJ} \omega_I \omega_J + \frac{8}{3} i (F + \bar{F})_{IJK} \hat{N}^{IL} \hat{N}^{JM} \hat{N}^{KN} \omega_L \omega_M \omega_N \\
& - \frac{4}{3} i [(F - \bar{F})_{IJKL} + 3i(F + \bar{F})_{IJR} \hat{N}^{RS} (F + \bar{F})_{SKL}] \\
& \times \hat{N}^{IM} \hat{N}^{JN} \hat{N}^{KP} \hat{N}^{LQ} \omega_M \omega_N \omega_P \omega_Q \\
& - \frac{32}{3} i \text{Im}(\Omega_{IJK} - 3\Omega_{IJ\bar{K}}) \hat{N}^{IL} \hat{N}^{JM} \hat{N}^{KN} \omega_L \omega_M \omega_N + \mathcal{O}(\Omega^5),
\end{aligned} \tag{1.135}$$

where  $\omega_I = \Omega_I - \Omega_{\bar{I}}$ , and where we also made use of  $\hat{N}^{IJ}$ , which is the inverse of the real, symmetric matrix  $\hat{N}_{IJ}$  given in (1.100), namely

$$\hat{N}_{IJ} = N_{IJ} + 4 \text{Re}(\Omega_{IJ} - \Omega_{I\bar{J}}). \tag{1.136}$$

Upon expanding  $\hat{N}^{IJ}$  we straightforwardly determine the contributions to the Hesse potential up to fifth order in  $\Omega$ ,

$$\begin{aligned}
\mathcal{H} = \mathcal{H}|_{\Omega=0} + & 4 \Omega - 4 N^{IJ} (\Omega_I \Omega_J + \Omega_{\bar{I}} \Omega_{\bar{J}}) + 8 N^{IJ} \Omega_I \Omega_{\bar{J}} \\
& + 16 \text{Re}(\Omega_{IJ} - \Omega_{I\bar{J}}) N^{IK} N^{JL} (\Omega_K \Omega_L + \Omega_{\bar{K}} \Omega_{\bar{L}} - 2 \Omega_K \Omega_{\bar{L}}) \\
& - \frac{16}{3} (F + \bar{F})_{IJK} N^{IL} N^{JM} N^{KN} \text{Im}(\Omega_L \Omega_M \Omega_N - 3 \Omega_L \Omega_M \Omega_{\bar{N}}) \\
& - 64 N^{IP} \text{Re}(\Omega_{PQ} - \Omega_{P\bar{Q}}) N^{QR} \text{Re}(\Omega_{RK} - \Omega_{R\bar{K}}) N^{KJ} \\
& \times (\Omega_I \Omega_J + \Omega_{\bar{I}} \Omega_{\bar{J}} - 2 \Omega_I \Omega_{\bar{J}}) \\
& + 64 (F + \bar{F})_{IJK} N^{IL} N^{JM} N^{KP} \text{Re}(\Omega_{PQ} - \Omega_{P\bar{Q}}) N^{QN} \\
& \times \text{Im}(\Omega_L \Omega_M \Omega_N - 3 \Omega_L \Omega_M \Omega_{\bar{N}}) \\
& - \frac{8}{3} i [(F - \bar{F})_{IJKL} + 3i(F + \bar{F})_{R(IJ} N^{RS} (F + \bar{F})_{KL)S}] N^{IM} N^{JN} N^{KP} N^{LQ} \\
& \times \text{Re}(\Omega_M \Omega_N \Omega_P \Omega_Q - 4 \Omega_M \Omega_N \Omega_P \Omega_{\bar{Q}} + 3 \Omega_M \Omega_N \Omega_{\bar{P}} \Omega_{\bar{Q}}) \\
& + \frac{64}{3} \text{Im}(\Omega_{IJK} - 3\Omega_{IJ\bar{K}}) N^{IL} N^{JM} N^{KN} \text{Im}(\Omega_L \Omega_M \Omega_N - 3 \Omega_L \Omega_M \Omega_{\bar{N}}) \\
& + \mathcal{O}(\Omega^5).
\end{aligned} \tag{1.137}$$

We stress once more that this expression is taken at  $Y^I = \mathcal{Y}^I$ .

The expression (1.137) gives the Hesse potential in terms of the duality covariant variables  $\mathcal{Y}^I$  and  $\bar{\mathcal{Y}}^I$ , up to order  $\Omega^5$ . It takes a rather complicated form, even at this order of approximation. Nevertheless, it will turn out that there is some systematics here. First of all, the Hesse potential (1.137) transforms as a function under duality transformations acting on the fields  $\mathcal{Y}^I$ . This in turn enables one to determine how  $\Omega$  should transform. Clearly, when  $\Omega = 0$  the Hesse potential transforms manifestly as

a function. In general the transformation behaviour of  $\Omega$  must be rather complicated in view of the non-linear dependence of the Hesse potential on  $\Omega$ . To evaluate this transformation, we have to perform yet another iteration procedure.

To demonstrate how this iteration proceeds, let us have a look at the first few steps. Consider the expression (1.137) at first order in  $\Omega$ . At this order,  $\Omega$  must transform as a function, since both  $\mathcal{H}$  and  $\mathcal{H}|_{\Omega=0}$  transform as functions. This implies that

$$\begin{aligned}\tilde{\Omega}(\tilde{\mathcal{Y}}, \tilde{\bar{\mathcal{Y}}}) &= \Omega(\mathcal{Y}, \bar{\mathcal{Y}}), \\ \tilde{\Omega}_I(\tilde{\mathcal{Y}}, \tilde{\bar{\mathcal{Y}}}) &= [\mathcal{S}_0^{-1}]^J{}_I(\mathcal{Y}) \Omega_J(\mathcal{Y}, \bar{\mathcal{Y}}).\end{aligned}\quad (1.138)$$

Now consider the terms of order  $\Omega^2$  in (1.137). Applying the transformation given in the second line of (1.138) to these terms and demanding  $\mathcal{H}$  to transform as a function, shows that the result given in the first line of (1.138) must be modified to

$$\tilde{\Omega} = \Omega - i(\mathcal{Z}_0^{IJ} \Omega_I \Omega_J - \bar{\mathcal{Z}}_0^{\bar{I}\bar{J}} \Omega_{\bar{I}} \Omega_{\bar{J}}) + \mathcal{O}(\Omega^3), \quad (1.139)$$

which in turn gives rise to the following result for derivatives of  $\Omega$ ,

$$\begin{aligned}\tilde{\Omega}_I &= [\mathcal{S}_0^{-1}]^J{}_I \left[ \Omega_J + iF_{JKL} \mathcal{Z}_0^{KM} \Omega_M \mathcal{Z}_0^{LN} \Omega_N - 2i\Omega_{JK} \mathcal{Z}_0^{KL} \Omega_L \right. \\ &\quad \left. + 2i\Omega_{J\bar{K}} \bar{\mathcal{Z}}_0^{\bar{K}\bar{L}} \Omega_{\bar{L}} \right] + \mathcal{O}(\Omega^3), \\ \tilde{\Omega}_{IJ} &= [\mathcal{S}_0^{-1}]^K{}_I [\mathcal{S}_0^{-1}]^L{}_J \left[ \Omega_{KL} - F_{KLM} \mathcal{Z}_0^{MN} \Omega_N \right] + \mathcal{O}(\Omega^2), \\ \tilde{\Omega}_{I\bar{J}} &= [\mathcal{S}_0^{-1}]^K{}_I [\bar{\mathcal{S}}_0^{-1}]^{\bar{L}}{}_{\bar{J}} \Omega_{K\bar{L}} + \mathcal{O}(\Omega^2),\end{aligned}\quad (1.140)$$

where the symmetric matrix  $\mathcal{Z}_0^{IJ}$  is defined by<sup>5</sup>

$$\mathcal{Z}_0^{IJ} = [\mathcal{S}_0^{-1}]^I{}_K \mathcal{Z}^{KJ}. \quad (1.141)$$

Here we made use of the relations,

$$\begin{aligned}[\mathcal{S}_0^{-1}]^I{}_K [\bar{\mathcal{S}}_0]_{\bar{J}}^{\bar{K}} &= \delta^I{}_J - i\mathcal{Z}_0^{IK} N_{KJ}, \\ \tilde{N}_{IJ} &= [\mathcal{S}_0^{-1}]^K{}_I [\bar{\mathcal{S}}_0^{-1}]^{\bar{L}}{}_{\bar{J}} N_{KL}, \\ \delta \mathcal{Z}_0^{IJ} &= -\mathcal{Z}_0^{IK} \delta F_{KL} \mathcal{Z}_0^{LJ},\end{aligned}\quad (1.142)$$

which are independent of  $\Omega$ .

This iteration can be continued by including the terms of order  $\Omega^3$ , making use of (1.140) for derivatives of  $\Omega$ , to obtain the expression for  $\tilde{\Omega}$  up terms of order  $\Omega^4$ . In the next iterative step one then derives the effect of a duality transformation on  $\Omega$

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<sup>5</sup> This quantity was first defined in [5]. It appeared later in [25], where it was denoted by  $\Delta$ .

up to terms of order  $\Omega^5$ . Before presenting this result, we wish to observe that terms transforming as a proper function under duality, will not contribute to this result. This is precisely what already happened to the  $\Omega$ -independent contribution to the Hesse potential, which decouples from the above equations. As it turns out there actually exists an infinite set of contributions to the Hesse potential that transform as functions under duality. By separating those from (1.137), we do not change the transformation behaviour of  $\Omega$ , but we can extract certain functions from the Hesse potential in order to simplify its structure. We obtain

$$\begin{aligned} \mathcal{H} = & \mathcal{H}^{(0)} + \mathcal{H}^{(1)} + \mathcal{H}^{(2)} + (\mathcal{H}_1^{(3)} + \mathcal{H}_2^{(3)} + \text{h.c.}) + \mathcal{H}_3^{(3)} + \mathcal{H}_1^{(4)} + \mathcal{H}_2^{(4)} + \mathcal{H}_3^{(4)} \\ & + (\mathcal{H}_4^{(4)} + \mathcal{H}_5^{(4)} + \mathcal{H}_6^{(4)} + \mathcal{H}_7^{(4)} + \mathcal{H}_8^{(4)} + \mathcal{H}_9^{(4)} + \text{h.c.}) \dots, \end{aligned} \quad (1.143)$$

where the  $\mathcal{H}_i^{(a)}$  are certain expressions to be defined below, whose leading term is of order  $\Omega^a$ . For higher values of  $a$  it turns out that there exists more than one functions with the same value of  $a$ , and those will be labeled by  $i = 1, 2, \dots$ . Of all the combinations  $\mathcal{H}_i^{(a)}$  appearing in (1.143),  $\mathcal{H}^{(1)}$  is the only that contains  $\Omega$ , while all the other combinations contain derivatives of  $\Omega$ . Obviously,  $\mathcal{H}^{(0)}$  equals,

$$\mathcal{H}^{(0)} = -i[\bar{\mathcal{Y}}^I F_I(\mathcal{Y}) - \mathcal{Y}^I \bar{F}_I(\bar{\mathcal{Y}})], \quad (1.144)$$

whereas  $\mathcal{H}^{(1)}$  at this level of iteration is given by,

$$\begin{aligned} \mathcal{H}^{(1)} = & 4\Omega - 4N^{IJ}(\Omega_I\Omega_J + \Omega_{\bar{I}}\Omega_{\bar{J}}) \\ & + 16\text{Re}\left[(\Omega_{IJ})(N\Omega)^I(N\Omega)^J\right] + 16\Omega_{I\bar{J}}(N\Omega)^I(N\bar{\Omega})^J \\ & - \frac{16}{3}\text{Im}\left[F_{IJK}(N\Omega)^I(N\Omega)^J(N\Omega)^K\right] \\ & - \frac{4}{3}i\left[\left(F_{IJKL} + 3iF_{R(IJ}N^{RS}F_{KL)S}\right)(N\Omega)^I(N\Omega)^J(N\Omega)^K(N\Omega)^L - \text{h.c.}\right] \\ & - \frac{16}{3}\left[\Omega_{IJK}(N\Omega)^I(N\Omega)^J(N\Omega)^K + \text{h.c.}\right] \\ & - 16\left[\Omega_{I\bar{J}\bar{K}}(N\Omega)^I(N\Omega)^J(N\bar{\Omega})^K + \text{h.c.}\right] \\ & - 16i\left[F_{IJK}N^{KP}\Omega_{PQ}(N\Omega)^I(N\Omega)^J(N\Omega)^Q - \text{h.c.}\right] \\ & - 16\left[(N\Omega)^P\Omega_{PQ}N^{QR}\Omega_{RK}(N\Omega)^K + \text{h.c.}\right] \\ & - 32\left[(N\Omega)^P\Omega_{PQ}N^{QR}\Omega_{R\bar{K}}(N\bar{\Omega})^K + \text{h.c.}\right] \\ & - 16\left[(N\Omega)^P\Omega_{P\bar{Q}}N^{QR}\Omega_{\bar{R}K}(N\Omega)^K + \text{h.c.}\right] \\ & - 16i\left[F_{IJK}N^{KP}\Omega_{P\bar{Q}}(N\Omega)^I(N\Omega)^J(N\bar{\Omega})^Q - \text{h.c.}\right] + \mathcal{O}(\Omega^5). \end{aligned} \quad (1.145)$$

Here we have used the notation  $(N\Omega)^I = N^{IJ}\Omega_J$ ,  $(N\bar{\Omega})^I = N^{IJ}\Omega_{\bar{J}}$ . The symmetrization  $F_{R(IJ}N^{RS}F_{KL)S}$  is defined with a symmetrization factor  $1/(4!)$ .

The expressions for the higher-order functions  $\mathcal{H}_i^{(a)}$  with  $a = 2, 3, 4$  are given in appendix D. Each of these higher-order functions transforms as a function under symplectic transformations. Demanding  $\mathcal{H}^{(1)}$  to also transform as a function under these transformations determines the transformation behavior of  $\Omega$ . Proceeding as already explained below (1.138) we obtain for the transformation law of  $\Omega$  (up to order  $\Omega^5$ ),

$$\begin{aligned}
\tilde{\Omega} = & \Omega - i(\mathcal{Z}_0^{IJ} \Omega_I \Omega_J - \bar{\mathcal{Z}}_0^{\bar{I}\bar{J}} \Omega_{\bar{I}} \Omega_{\bar{J}}) \\
& + \frac{2}{3}(F_{IJK} \mathcal{Z}_0^{IL} \Omega_L \mathcal{Z}_0^{JM} \Omega_M \mathcal{Z}_0^{KN} \Omega_N + \text{h.c.}) \\
& - 2(\Omega_{IJ} \mathcal{Z}_0^{IK} \Omega_K \mathcal{Z}_0^{JL} \Omega_L + \text{h.c.}) + 4 \Omega_{I\bar{J}} \mathcal{Z}_0^{IK} \Omega_K \bar{\mathcal{Z}}_0^{\bar{J}\bar{L}} \Omega_{\bar{L}} \\
& + \left[ -\frac{i}{3} F_{IJKL} (\mathcal{Z}_0 \Omega)^I (\mathcal{Z}_0 \Omega)^J (\mathcal{Z}_0 \Omega)^K (\mathcal{Z}_0 \Omega)^L \right. \\
& + \frac{4i}{3} \Omega_{IJK} (\mathcal{Z}_0 \Omega)^I (\mathcal{Z}_0 \Omega)^J (\mathcal{Z}_0 \Omega)^K \\
& + i F_{IJR} \mathcal{Z}_0^{RS} F_{SKL} (\mathcal{Z}_0 \Omega)^I (\mathcal{Z}_0 \Omega)^J (\mathcal{Z}_0 \Omega)^K (\mathcal{Z}_0 \Omega)^L \\
& - 4i \Omega_{IJ\bar{K}} (\mathcal{Z}_0 \Omega)^I (\mathcal{Z}_0 \Omega)^J (\bar{\mathcal{Z}}_0 \bar{\Omega})^K \\
& - 4i F_{IJK} \mathcal{Z}_0^{KP} \Omega_{PQ} (\mathcal{Z}_0 \Omega)^I (\mathcal{Z}_0 \Omega)^J (\mathcal{Z}_0 \Omega)^Q \\
& + 4i F_{IJK} \mathcal{Z}_0^{KP} \Omega_{P\bar{Q}} (\mathcal{Z}_0 \Omega)^I (\mathcal{Z}_0 \Omega)^J (\bar{\mathcal{Z}}_0 \bar{\Omega})^Q \\
& + 4i (\mathcal{Z}_0 \Omega)^P \Omega_{PQ} \mathcal{Z}_0^{QR} \left( \Omega_{RK} (\mathcal{Z}_0 \Omega)^K - 2 \Omega_{R\bar{K}} (\bar{\mathcal{Z}}_0 \bar{\Omega})^K \right) \\
& \left. - 4i (\mathcal{Z}_0 \Omega)^P \Omega_{P\bar{Q}} \bar{\mathcal{Z}}_0^{\bar{Q}\bar{R}} \Omega_{\bar{R}K} (\mathcal{Z}_0 \Omega)^K + \text{h.c.} \right] + \mathcal{O}(\Omega^5). \quad (1.146)
\end{aligned}$$

The transformation laws of the derivatives of  $\Omega$ , such as those in (1.140), are summarized in appendix E.

The transformation law (1.146), which is entirely encoded in  $\mathcal{Z}_0$  and in  $\bar{\mathcal{Z}}_0$ , suggest a systematic pattern, which we now explore. First we observe that (1.146) simplifies when taking  $\Omega$  to be harmonic both in  $\mathcal{Y}^I$  and  $\mathcal{Y}$ ,

$$\Omega(\mathcal{Y}, \bar{\mathcal{Y}}; \mathcal{Y}, \bar{\mathcal{Y}}) = f(\mathcal{Y}, \mathcal{Y}) + \text{h.c.} \quad (1.147)$$

We obtain

$$\begin{aligned}
\tilde{\Omega} = & \Omega + \left[ -i \mathcal{Z}_0^{IJ} \Omega_I \Omega_J \right. \\
& + \frac{2}{3} F_{IJK} \mathcal{Z}_0^{IL} \Omega_L \mathcal{Z}_0^{JM} \Omega_M \mathcal{Z}_0^{KN} \Omega_N \\
& - 2 \Omega_{IJ} \mathcal{Z}_0^{IK} \Omega_K \mathcal{Z}_0^{JL} \Omega_L \\
& - \frac{i}{3} F_{IJKL} (\mathcal{Z}_0 \Omega)^I (\mathcal{Z}_0 \Omega)^J (\mathcal{Z}_0 \Omega)^K (\mathcal{Z}_0 \Omega)^L \\
& + \frac{4i}{3} \Omega_{IJK} (\mathcal{Z}_0 \Omega)^I (\mathcal{Z}_0 \Omega)^J (\mathcal{Z}_0 \Omega)^K \\
& \left. + i F_{IJR} \mathcal{Z}_0^{RS} F_{SKL} (\mathcal{Z}_0 \Omega)^I (\mathcal{Z}_0 \Omega)^J (\mathcal{Z}_0 \Omega)^K (\mathcal{Z}_0 \Omega)^L \right]
\end{aligned}$$

$$\begin{aligned}
& - 4i F_{IJK} Z_0^{KP} \Omega_{PQ} (Z_0 \Omega)^I (Z_0 \Omega)^J (Z_0 \Omega)^Q \\
& + 4i Z_0^{IP} \Omega_{PQ} Z_0^{QR} \Omega_{RK} (Z_0 \Omega)^K \Omega_I + \text{h.c.} \Big] + \mathcal{O}(\Omega^5), \quad (1.148)
\end{aligned}$$

which shows that  $\tilde{\Omega}$  also is harmonic. Hence, the harmonicity of  $\Omega$  is preserved under symplectic transformations. The transformation law (1.148) has a certain resemblance with the one encountered in the context of perturbative topological string theory, where  $Z_0^{IJ}$  plays the role of a propagator [25]. The relation with topological string theory will be discussed below. Next, inserting (1.147) into (1.145), we find that  $\mathcal{H}^{(1)}$  is also almost harmonic, i.e. it equals the real part of a function that contains only purely holomorphic derivatives of  $F$  and  $\Omega$ , contracted with the non-holomorphic tensor  $N^{IJ}$ ,

$$\begin{aligned}
\mathcal{H}^{(1)} = & \left[ 4 f(\mathcal{Y}, \mathcal{Y}) - 4 N^{IJ} \Omega_I \Omega_J \right. \\
& + 8(\Omega_{IJ})(N\Omega)^I (N\Omega)^J + \frac{8}{3}i F_{IJK} (N\Omega)^I (N\Omega)^J (N\Omega)^K \\
& - \frac{4}{3}i \left( F_{IJKL} + 3i F_{R(IJ} N^{RS} F_{KL)S} \right) (N\Omega)^I (N\Omega)^J (N\Omega)^K (N\Omega)^L \\
& - \frac{16}{3} \Omega_{IJK} (N\Omega)^I (N\Omega)^J (N\Omega)^K \\
& - 16i F_{IJK} N^{KP} \Omega_{PQ} (N\Omega)^I (N\Omega)^J (N\Omega)^Q \\
& \left. - 16(N\Omega)^P \Omega_{PQ} N^{QR} \Omega_{RK} (N\Omega)^K + \text{h.c.} \right] + \mathcal{O}(\Omega^5). \quad (1.149)
\end{aligned}$$

Thus, when  $\Omega$  is of the form (1.147),  $\mathcal{H}^{(1)}$  is given in terms of the real part of a function that is holomorphic in  $\mathcal{Y}$ . Moreover, since  $N^{IJ}$  is homogeneous of degree zero, this function is homogeneous of degree two in  $\mathcal{Y}^I$  and homogeneous of degree zero in  $\tilde{\mathcal{Y}}^I$ .

Let us now elucidate the relation of  $\mathcal{H}^{(1)}$  given in (1.149) with topological string theory. We write  $\mathcal{H}^{(1)}$  as

$$\mathcal{H}^{(1)} = h(\mathcal{Y}, \tilde{\mathcal{Y}}, \mathcal{Y}) + \text{h.c.}, \quad (1.150)$$

and we consider two expansions of  $h(\mathcal{Y}, \tilde{\mathcal{Y}}, \mathcal{Y})$ , namely one in powers of  $\Omega$  and the other one in powers of  $\mathcal{Y}$ . First we consider the expansion in powers of  $\Omega$ . Expanding  $h$  as

$$h = \sum_{g=1}^{\infty} h^{(g)} \quad (1.151)$$

and comparing with (1.149), we obtain

$$\begin{aligned}
h^{(1)} &= 4f, \quad h^{(2)} = -4N^{IJ} \Omega_I \Omega_J, \\
h^{(3)} &= 8\Omega_{IJ} (N\Omega)^I (N\Omega)^J + \frac{8}{3}i F_{IJK} (N\Omega)^I (N\Omega)^J (N\Omega)^K, \\
h^{(4)} &= -\frac{4}{3}i \left( F_{IJKL} + 3i F_{R(IJ} N^{RS} F_{KL)S} \right) (N\Omega)^I (N\Omega)^J (N\Omega)^K (N\Omega)^L
\end{aligned}$$



$$\begin{aligned}
& - \frac{16}{3} \Omega_{IJK} (N\Omega)^I (N\Omega)^J (N\Omega)^K \\
& - 16i F_{IJK} N^{KP} \Omega_{PQ} (N\Omega)^I (N\Omega)^J (N\Omega)^Q \\
& - 16(N\Omega)^P \Omega_{PQ} N^{QR} \Omega_{RK} (N\Omega)^K,
\end{aligned} \tag{1.152}$$

where  $(N\Omega)^I = N^{IJ} f_J$ . This shows that all the  $h^{(g)}$  are non-holomorphic in  $\mathcal{Y}^I$  with the exception of  $h^{(1)}$ . Using these expressions, one finds by direct calculation that the following relation holds,

$$\partial_{\bar{I}} h^{(g)} = \frac{1}{4} i \bar{F}_{\bar{I}}^{JK} \sum_{r=1}^{g-1} \partial_J h^{(r)} \partial_K h^{(g-r)}, \quad g \geq 2, \tag{1.153}$$

where  $\bar{F}_{\bar{I}}^{JK} = \bar{F}_{\bar{I}\bar{L}\bar{M}} N^{LJ} N^{MK}$ .

**Exercise 17** Verify (1.153) for  $g = 2, 3$ .

Equation (1.153) captures the  $\bar{\mathcal{Y}}^I$ -dependence of  $h^{(g)}$  (for  $g \geq 2$ ). This dependence is a consequence of requiring  $\mathcal{H}^{(1)}$  to have a proper behavior under symplectic transformations [5]. The differential Eq. (1.153) resembles the holomorphic anomaly equation of perturbative topological string theory. The latter arises in a specific setting, namely in the study of the non-holomorphicity of the genus- $g$  topological free energies  $F^{(g)}$  [36]. To exhibit the relation with the holomorphic anomaly equation, we turn to the second expansion and expand both  $f(\mathcal{Y}, \Upsilon)$  and  $h(\mathcal{Y}, \bar{\mathcal{Y}}, \Upsilon)$  in powers of  $\Upsilon$ ,

$$\begin{aligned}
f(\mathcal{Y}, \Upsilon) &= -\frac{1}{2} i \sum_{g=1}^{\infty} \Upsilon^g f^{(g)}(\mathcal{Y}), \\
h(\mathcal{Y}, \bar{\mathcal{Y}}, \Upsilon) &= -2i \sum_{g=1}^{\infty} \Upsilon^g F^{(g)}(\mathcal{Y}, \bar{\mathcal{Y}}).
\end{aligned} \tag{1.154}$$

Then we obtain

$$\begin{aligned}
F^{(1)}(\mathcal{Y}) &= f^{(1)}(\mathcal{Y}), \quad F^{(2)}(\mathcal{Y}, \bar{\mathcal{Y}}) = f^{(2)}(\mathcal{Y}) + \frac{1}{2} i N^{IJ} F_I^{(1)} F_J^{(1)}, \\
F^{(3)}(\mathcal{Y}, \bar{\mathcal{Y}}) &= f^{(3)}(\mathcal{Y}) + i N^{IJ} f_I^{(2)} F_J^{(1)} - \frac{1}{2} F_{IJ}^{(1)} (NF^{(1)})^I (NF^{(1)})^J \\
&\quad - \frac{1}{6} i F_{IJK} (NF^{(1)})^I (NF^{(1)})^J (NF^{(1)})^K, \\
F^{(4)}(\mathcal{Y}, \bar{\mathcal{Y}}) &= f^{(4)}(\mathcal{Y}) + i N^{IJ} f_I^{(3)} F_J^{(1)} + \frac{1}{2} i N^{IJ} f_I^{(2)} f_J^{(2)} \\
&\quad - \frac{1}{2} f_{IJ}^{(2)} (NF^{(1)})^I (NF^{(1)})^J - F_{IJ}^{(1)} (Nf^{(2)})^I (NF^{(1)})^J \\
&\quad - \frac{1}{2} i F_{IJK} (Nf^{(2)})^I (NF^{(1)})^J (NF^{(1)})^K \\
&\quad + \frac{1}{24} \left( F_{IJKL} + 3i F_{R(IJ} N^{RS} F_{KL)S} \right) (NF^{(1)})^I (NF^{(1)})^J \\
&\quad \quad \times (NF^{(1)})^K (NF^{(1)})^L
\end{aligned}$$

$$\begin{aligned}
& -\frac{1}{6}i F_{IJK}^{(1)} (NF^{(1)})^I (NF^{(1)})^J (NF^{(1)})^K \\
& + \frac{1}{2} F_{IJK} N^{KP} F_{PQ}^{(1)} (NF^{(1)})^I (NF^{(1)})^J (NF^{(1)})^Q \\
& - \frac{1}{2}i (NF^{(1)})^P F_{PQ}^{(1)} N^{QR} F_{RK}^{(1)} (NF^{(1)})^K, \tag{1.155}
\end{aligned}$$

where  $(NF^{(1)})^I = N^{IJ} F_J^{(1)}$  and  $(NF^{(2)})^I = N^{IJ} f_J^{(2)}$ . Observe that all the  $F^{(g)}$  are non-holomorphic except  $F^{(1)}$ . Using these expressions, one again finds by direct calculation,

$$\partial_{\bar{I}} F^{(g)} = \frac{1}{2} \bar{F}_{\bar{I}}^{JK} \sum_{r=1}^{g-1} \partial_J F^{(r)} \partial_K F^{(g-r)}, \quad g \geq 2. \tag{1.156}$$

This is similar to (1.153), except that now the relation holds order by order in  $\mathcal{Y}$ , whereas (1.153) holds order by order in  $\Omega$ . Both expansions are, nevertheless, related. Namely, taking  $f$  in (1.154) to consist of only  $f^{(1)}$ , the expansion (1.155) coincides with the expansion (1.152).

Summarizing, we have found the following. When expressing the Hesse potential, which is a symplectic function, in terms of the duality covariant complex variables (1.127), we obtain an infinite set of contributions  $\mathcal{H}_i^{(a)}$ , all of which transform as functions under symplectic transformations. One of them, namely  $\mathcal{H}^{(1)}$ , has a structure that arises in topological string theory.  $\mathcal{H}^{(1)}$  is the only contribution that contains  $\Omega$ , while all the other combinations contain derivatives of  $\Omega$ . When  $\Omega$  is taken to be harmonic in all the variables (i.e. in both  $\mathcal{Y}^I$  and  $\mathcal{Y}$ ), the resulting  $\mathcal{H}^{(1)}$  is given in terms of the real part of a function that is holomorphic in  $\mathcal{Y}$ , homogeneous of degree two in  $\mathcal{Y}^I$  and homogeneous of degree zero in  $\bar{\mathcal{Y}}^I$ . Then, expanding  $\mathcal{H}^{(1)}$  in powers of  $\mathcal{Y}$  yields expansion functions  $F^{(g)}$ , given in (1.155), that transform as functions under symplectic transformations. The  $F^{(g)}$  are all non-holomorphic, with the exception of  $F^{(1)}$ , and the non-holomorphicity is governed by (1.156). This differential equation equals half of the holomorphic anomaly equation of perturbative topological string theory, which reads [38]

$$\partial_{\bar{I}} F^{(g)} = \frac{1}{2} \bar{F}_{\bar{I}}^{JK} \left( D_J \partial_K F^{(g-1)} + \sum_{r=1}^{g-1} \partial_J F^{(r)} \partial_K F^{(g-r)} \right), \quad g \geq 2, \tag{1.157}$$

where  $D_L V_M = \partial_L V_M + iN^{PI} F_{ILM} V_P$ . This is the holomorphic anomaly equation in the so-called big moduli space [38], and its derivation is reviewed in appendix C following [25]. In the context of topological string theory, the  $F^{(g)}$  denote free energies that arise in the perturbative expansion of the topological free energy  $F_{\text{top}}$  in powers of the topological string coupling  $g_{\text{top}}$ , i.e.  $F_{\text{top}} = \sum_{g=0}^{\infty} g_{\text{top}}^{2g-2} F^{(g)}$ . Whereas  $F^{(0)}$  is holomorphic (it only depends on  $\mathcal{Y}$ ), all the higher  $F^{(g)}$  (with  $g \geq 1$ ) are non-holomorphic. For  $g \geq 2$  this non-holomorphicity is captured by (1.157).

The fact that the first term on the right hand side of (1.157) is missing in (1.156) is due to the holomorphic nature of the expansion function  $F^{(1)}$  appearing in (1.155).

Were it to be non-holomorphic, it would induce a modification of the relation (1.156). The required modification arises by replacing the holomorphic quantity  $F_I^{(1)} = f_I^{(1)}$  with the non-holomorphic combination  $F_I^{(1)} = f_I^{(1)} + \frac{1}{2} i F_{IJK} N^{JK}$  (see (1.206)). This will be addressed in an upcoming paper.

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## A Symplectic Reparametrizations

In Sect. 1.2.1 we introduced the  $2n$ -vector  $(x^i, F_i)$  and discussed its behavior under symplectic transformations. Here we consider derivatives of  $F_i$  and show how they transform under symplectic transformations. We use the resulting expressions to give an alternative proof of integrability of the Eq. (1.10). In addition, we show that  $\partial_\eta F$  transforms as a function under symplectic transformations.

We begin by recalling some of the elements of Sect. 1.2.1. The  $2n$ -vector  $(x^i, F_i)$  is constructed using  $F(x, \bar{x}) = F^{(0)}(x) + 2i\Omega(x, \bar{x})$ . Under symplectic transformations, it transforms as,

$$\begin{aligned}\tilde{x}^i &= U^i_j x^j + Z^{ij} [F_j^{(0)}(x) + 2i\Omega_j(x, \bar{x})], \\ \tilde{F}_i(\tilde{x}, \tilde{\bar{x}}) &= V_i^j [F_j^{(0)}(x) + 2i\Omega_j(x, \bar{x})] + W_{ij} x^j,\end{aligned}\quad (1.158)$$

where  $U, V, Z$  and  $W$  are the  $n \times n$  submatrices (1.5) that define a symplectic transformation belonging to  $\text{Sp}(2n, \mathbb{R})$ . Without loss of generality, we decompose  $\tilde{F}_i$  as

$$\tilde{F}_i(\tilde{x}, \tilde{\bar{x}}) = \tilde{F}_i^{(0)}(\tilde{x}) + 2i\tilde{\Omega}_i(\tilde{x}, \tilde{\bar{x}}).\quad (1.159)$$

This decomposition, which a priori is arbitrary, can be related to the decomposition of  $F_i = F_i^{(0)} + 2i\Omega_i$  in the following way. The symplectic transformation (1.158) is specified by the matrices  $U, V, W$  and  $Z$ . Consider applying the same transformation (specified by these matrices) to the vector  $(x^i, F_i^{(0)})$  alone. This yields the vector  $(\hat{x}^i, \tilde{F}_i^{(0)}(\hat{x}))$ , which is expressed in terms of  $\hat{x}^i = \tilde{x}^i - 2iZ^{ij}\Omega_j(x, \bar{x})$  instead of  $\tilde{x}^i$ ,

$$\begin{aligned}\hat{x}^i &= U^i_j x^j + Z^{ij} F_j^{(0)}(x), \\ \tilde{F}_i^{(0)}(\hat{x}) &= V_i^j F_j^{(0)}(x) + W_{ij} x^j.\end{aligned}\quad (1.160)$$

Thus, by demanding that  $\tilde{F}_i^{(0)}$  follows from the same symplectic transformation applied on  $F_i^{(0)}$  alone, we relate the decomposition of  $\tilde{F}_i$  to the decomposition of  $F_i$ . Then, the second equation of (1.158) can be written as

$$\begin{aligned}\tilde{\Omega}_i(\tilde{x}, \bar{x}) &= V_i^j \Omega_j(x, \bar{x}) - \frac{1}{2} \mathbf{i} [\tilde{F}_i^{(0)}(\hat{x}) - \tilde{F}_i^{(0)}(\bar{x})] \\ &= V_i^j \Omega_j(x, \bar{x}) \\ &\quad + \frac{1}{2} \mathbf{i} \sum_{m=1}^{\infty} \frac{(2\mathbf{i})^m}{m!} Z^{j_1 k_1} \Omega_{k_1}(x, \bar{x}) \cdots Z^{j_m k_m} \Omega_{k_m}(x, \bar{x}) \tilde{F}_{ij_1 \dots j_m}^{(0)}(\hat{x}),\end{aligned}\quad (1.161)$$

where the  $\tilde{F}_{ij_1 \dots j_m}^{(0)}(\hat{x})$  denote multiple derivatives of  $\tilde{F}_i^{(0)}(\bar{x})$  evaluated at  $\hat{x}$ . The right-hand side of (1.161) can be written entirely in terms of functions of  $x$  and  $\bar{x}$ , upon expressing  $\tilde{F}_{ij_1 \dots j_m}^{(0)}(\hat{x})$  in terms of derivatives of  $F_i^{(0)}(x)$  using (1.160). We give the first few derivatives,

$$\begin{aligned}\tilde{F}_{ij}^{(0)}(\hat{x}) &= (V_i^l F_{lk}^{(0)} + W_{ik}) [\mathcal{S}_0^{-1}]^k_j, \\ \tilde{F}_{ijk}^{(0)}(\hat{x}) &= [\mathcal{S}_0^{-1}]^l_i [\mathcal{S}_0^{-1}]^m_j [\mathcal{S}_0^{-1}]^n_k F_{lmn}^{(0)}, \\ \tilde{F}_{ijkl}^{(0)}(\hat{x}) &= [\mathcal{S}_0^{-1}]^m_i [\mathcal{S}_0^{-1}]^n_j [\mathcal{S}_0^{-1}]^p_k [\mathcal{S}_0^{-1}]^q_l \left[ F_{mnpq}^{(0)} - 3 F_{r(mn} \mathcal{Z}_0^{rs} F_{pq)s}^{(0)} \right],\end{aligned}\quad (1.162)$$

where we used the definitions

$$\begin{aligned}\mathcal{S}_0^i_j &= U^i_j + Z^{ik} F_{kj}^{(0)}, \\ \mathcal{Z}_0^{ij} &= [\mathcal{S}_0^{-1}]^i_k Z^{kj}.\end{aligned}\quad (1.163)$$

Let us consider the first expression of (1.162). While  $F_{ij}^{(0)}$  is manifestly symmetric in  $i, j$ , this appears not to be the case for  $\tilde{F}_{ij}^{(0)}$ . However, using the properties (1.5) of the matrices  $U, V, W$  and  $Z$ , it follows that  $\tilde{F}_{ij}^{(0)}$  is symmetric in  $i, j$ . Using this, we obtain

$$\tilde{F}_{ij}^{(0)}(\hat{x}) Z^{jk} = V_i^k - [\mathcal{S}_0^{-1, T}]^i_k. \quad (1.164)$$

**Exercise 18** Verify (1.164) by computing  $V^T \mathcal{S}_0$ .

The symmetry of  $\tilde{F}_{ij}^{(0)}$  implies that  $\tilde{F}_i^{(0)}(\hat{x})$  can be integrated, i.e.  $\tilde{F}_i^{(0)}(\hat{x}) = \partial \tilde{F}^{(0)}(\hat{x}) / \partial \hat{x}^i$ , with  $\tilde{F}^{(0)}(\hat{x})$  given by the well-known expression [5],

$$\begin{aligned}\tilde{F}^{(0)}(\hat{x}) &= F^{(0)}(x) - \frac{1}{2}x^i F_i^{(0)} + \frac{1}{2}(U^T W)_{ij} x^i x^j + \frac{1}{2}(U^T V + W^T Z)_i{}^j x^i F_j^{(0)} \\ &\quad + \frac{1}{2}(Z^T V)^{ij} F_i^{(0)} F_j^{(0)},\end{aligned}\quad (1.165)$$

up to a constant and up to terms linear in  $\hat{x}^i$ .

In addition to (1.163), we will also need the combinations  $\mathcal{S}$  and  $\hat{\mathcal{S}}$  given in (1.167) and (1.169) below, which are related to  $\mathcal{S}_0$  by

$$\begin{aligned}\mathcal{S}^i{}_j &= \mathcal{S}_0^i{}_j + 2iZ^{ik}\Omega_{kj}, \\ \hat{\mathcal{S}}^i{}_j &= \mathcal{S}_0^i{}_j + Z^{ik}[2i\Omega_{kj} - 4\Omega_{k\bar{l}}\bar{Z}^{\bar{l}m}\Omega_{\bar{m}j}], \\ \mathcal{Z}^{ij} &= [\mathcal{S}^{-1}]^i{}_k Z^{kj}.\end{aligned}\quad (1.166)$$

Observe that the matrices  $\mathcal{Z}_0$ ,  $\mathcal{Z}$  and  $\hat{\mathcal{Z}} = \hat{\mathcal{S}}^{-1} \mathcal{Z}$  are symmetric matrices by virtue of the fact that  $ZU^T$  is a symmetric matrix [5].

Next we consider the transformation behavior of the derivatives  $F_{ij} = \partial F_i / \partial x^j$  and  $F_{i\bar{j}} = \partial F_i / \partial \bar{x}^{\bar{j}}$ . First we observe that

$$\frac{\partial \tilde{x}^i}{\partial x^j} \equiv \mathcal{S}^i{}_j = U^i{}_j + Z^{ik} F_{kj}, \quad \frac{\partial \tilde{x}^i}{\partial \bar{x}^{\bar{j}}} \equiv Z^{ik} F_{k\bar{j}}.\quad (1.167)$$

Applying the chain rule to (1.158) yields the relation

$$F_{ij} \rightarrow \tilde{F}_{ij} = \left( V_i{}^l \hat{F}_{lk} + W_{ik} \right) [\hat{\mathcal{S}}^{-1}]^k{}_j, \quad (1.168)$$

where  $\tilde{F}_{ij} = \partial \tilde{F}_i / \partial \tilde{x}^j$  and

$$\begin{aligned}\hat{F}_{ij} &= F_{ij} - F_{i\bar{k}} Z^{\bar{k}l} \bar{F}_{lj} = F_{ij}^{(0)} + 2i\Omega_{ij} - 4\Omega_{i\bar{k}} \bar{Z}^{\bar{k}l} \Omega_{\bar{l}j}, \\ \hat{\mathcal{S}}^i{}_j &= U^i{}_j + Z^{ik} \hat{F}_{kj}.\end{aligned}\quad (1.169)$$

**Exercise 19** Derive (1.168) by differentiating the second equation of (1.158) with respect to either  $x$  or  $\bar{x}$ . Then combine the two resulting equations to arrive at (1.168).

Then, using the first equation of (1.162) as well as (1.164) in (1.168) yields,

$$\begin{aligned}\tilde{\Omega}_{ij}(\tilde{x}, \bar{\tilde{x}}) &= \frac{1}{2}i \left[ \tilde{F}_{ij}^{(0)}(\tilde{x}) - \tilde{F}_{ij}^{(0)}(\tilde{x}^k - 2iZ^{kl}\Omega_l(x, \bar{x})) \right] \\ &\quad + [\hat{\mathcal{S}}^{-1}]^k{}_i [\hat{\mathcal{S}}^{-1}]^l{}_j \left[ \Omega_{kl} + 2i\Omega_{k\bar{m}} \bar{Z}^{\bar{m}\bar{n}} \Omega_{\bar{n}l} \right. \\ &\quad \left. + 2i(\Omega_{km} + 2i\Omega_{k\bar{p}} \bar{Z}^{\bar{p}\bar{r}} \Omega_{\bar{r}m}) \mathcal{Z}_0^{mn} (\Omega_{nl} + 2i\Omega_{n\bar{q}} \bar{Z}^{\bar{q}\bar{s}} \Omega_{\bar{s}l}) \right],\end{aligned}\quad (1.170)$$

which is symmetric by virtue of the symmetry of  $\tilde{F}_{ij}^{(0)}$ ,  $\Omega_{ij}$ ,  $\mathcal{Z}^{mn}$  and  $\mathcal{Z}_0^{mn}$ .

Subsequently we derive the following result from (1.161) [21],

$$\tilde{\Omega}_{i\bar{j}} = [\hat{S}^{-1}]^k{}_i [\bar{S}^{-1}]^{\bar{l}}{}_{\bar{j}} \Omega_{k\bar{l}} = [S^{-1}]^k{}_i [\hat{S}^{-1}]^{\bar{l}}{}_{\bar{j}} \Omega_{k\bar{l}}. \quad (1.171)$$

**Exercise 20** Deduce (1.171) by taking the first line of (1.161) and differentiating it with respect to  $\bar{x}$ . Use the relation (1.164) in the form

$$V_i{}^j = [S_0^{-1,T}]_i{}^k + (V_i{}^l F_{lk}^{(0)} + W_{ik}) Z_0^{kj}, \quad (1.172)$$

together with (1.166).

The relation (1.171) establishes that  $\tilde{\Omega}_{i\bar{j}} = \overline{(\tilde{\Omega}_{\bar{j}i})}$ . Using this as well as (1.15), and recalling that  $\tilde{\Omega}_{i\bar{j}} = \partial\tilde{\Omega}_i/\partial\bar{x}^{\bar{j}}$ , we obtain  $\tilde{\Omega}_{i\bar{j}} = \overline{(\tilde{\Omega}_{\bar{j}i})} = (\partial\tilde{\Omega}_{\bar{j}}/\partial\bar{x}^{\bar{i}}) = \partial(\overline{(\tilde{\Omega}_{\bar{j}})})/\partial\bar{x}^{\bar{i}} = \partial\tilde{\Omega}_{\bar{j}}/\partial\bar{x}^{\bar{i}} \equiv \tilde{\Omega}_{\bar{j}i}$ . This, together with the symmetry of  $\tilde{\Omega}_{ij}$ , ensures the integrability of (1.158), as follows.

We consider the 1-form  $\tilde{A} = \tilde{\Omega}_i d\tilde{x}^i + \tilde{\Omega}_{\bar{i}} d\tilde{x}^{\bar{i}}$ , which is real by virtue of  $\tilde{\Omega}_{\bar{i}} = \overline{(\tilde{\Omega}_i)}$ . Its field strength reads  $\tilde{\mathfrak{F}} = d\tilde{A} = \tilde{\Omega}_{ij} d\tilde{x}^j \wedge d\tilde{x}^i + (\tilde{\Omega}_{i\bar{j}} - \tilde{\Omega}_{\bar{j}i}) d\tilde{x}^{\bar{j}} \wedge d\tilde{x}^i + \tilde{\Omega}_{\bar{i}\bar{j}} d\tilde{x}^{\bar{j}} \wedge d\tilde{x}^{\bar{i}}$ . Then, using  $\tilde{\Omega}_{ij} = \tilde{\Omega}_{ji}$  as well as  $\tilde{\Omega}_{i\bar{j}} = \tilde{\Omega}_{\bar{j}i}$ , we conclude that  $\tilde{\mathfrak{F}} = 0$ , which establishes that locally  $\tilde{A} = d\tilde{\Omega}$ , with a real  $\tilde{\Omega}$ .

Hence we conclude that the Eq. (1.158) are integrable and the decomposition (1.7) is preserved, i.e. the transformed  $2n$ -vector  $(\tilde{x}^i, \tilde{F}_i)$  is constructed from a new function  $\tilde{F}(\tilde{x}, \tilde{\bar{x}}) = \tilde{F}^{(0)}(\tilde{x}) + 2i\tilde{\Omega}(\tilde{x}, \tilde{\bar{x}})$  with a real  $\tilde{\Omega}(\tilde{x}, \tilde{\bar{x}})$ . This was established in Sect. 1.2.1 by relying on the Hamiltonian.

Next, let us assume that the function  $F$  depends on a auxiliary real parameter  $\eta$  that is inert under symplectic transformation, i.e.  $F(x, \bar{x}; \eta)$ , and let us consider partial derivatives with respect to it. A little calculation shows that  $\partial_\eta F_i$  transforms in the following way,

$$\partial_\eta \tilde{F}_i = [\hat{S}^{-1}]^j{}_i \left[ \partial_\eta F_j - F_{j\bar{k}} \bar{Z}^{\bar{k}\bar{l}} \partial_\eta \bar{F}_{\bar{l}} \right], \quad (1.173)$$

where  $\tilde{x}$  and  $\tilde{\bar{x}}$  are kept fixed under the  $\eta$ -derivative in  $\partial_\eta \tilde{F}_i(\tilde{x}, \tilde{\bar{x}}; \eta)$ , while in  $\partial_\eta F_i(x, \bar{x}; \eta)$  the arguments  $x$  and  $\bar{x}$  are kept fixed.

**Exercise 21** Verify (1.173) by differentiating the second equation of (1.158) with respect to  $\eta$ , keeping  $x$  and  $\bar{x}$  fixed. Subsequently, use (1.159), (1.168) and (1.171) to arrive at (1.173).

Let us first consider (1.173) in the case of a holomorphic function  $F$ , so that  $\Omega = 0$ . In that case (1.173) implies that the derivative with respect to  $x^i$  of  $\partial_\eta \tilde{F} - \partial_\eta F$  must vanish. Therefore it follows that  $\partial_\eta F$  transforms as a function under symplectic transformations (possibly up to an  $x$ -independent expression, which is irrelevant in view of the same argument that led to the equivalence (1.8)).

When  $\Omega \neq 0$  one derives the following result using (1.173),

$$\frac{\partial \tilde{x}^j}{\partial x^i} \partial_\eta \tilde{F}_j - \frac{\partial \tilde{x}^{\bar{j}}}{\partial x^i} \partial_\eta (\overline{\tilde{F}_j}) = \partial_\eta F_i. \quad (1.174)$$

**Exercise 22** Deduce (1.174) by suitably combining (1.173) with its complex conjugate, and using the relation

$$\tilde{Z}^{\bar{i}\bar{j}} \bar{F}_{\bar{j}k} [\hat{S}^{-1} \mathcal{S}]^k{}_l = [\tilde{S}^{-1} \tilde{\mathcal{S}}]_{\bar{j}}^{\bar{i}} \tilde{Z}^{\bar{j}\bar{k}} \bar{F}_{\bar{k}l}. \quad (1.175)$$

Next, we assume without loss of generality that the dependence of  $\tilde{F}$  on  $\eta$  is entirely contained in  $\tilde{\Omega}$ . Then, using (1.15), it follows that

$$\partial_\eta (\overline{\tilde{F}_j}) = -\partial_\eta \tilde{F}_{\bar{j}}, \quad (1.176)$$

and the relation (1.174) simplifies. Namely, the left hand side of (1.174) becomes equal to  $\partial(\partial_\eta \tilde{F})/\partial x^i$ , where we used the existence of the new function  $\tilde{F}$ . Thus, we obtain from (1.174),

$$\frac{\partial}{\partial x^i} (\partial_\eta \tilde{F} - \partial_\eta F) = 0. \quad (1.177)$$

This equation, together with its complex conjugate equation, implies that  $\partial_\eta \tilde{F} - \partial_\eta F$  vanishes upon differentiation with respect to  $x$  and  $\bar{x}$ , so that  $\partial_\eta F$  transforms as a function under symplectic transformations (possibly up to an irrelevant term that is independent of  $x$  and  $\bar{x}$ ).

## B The Covariant Derivative $\mathcal{D}_\eta$

The modified derivative (1.97) acts as a covariant derivative for symplectic transformations. Here we verify this explicitly by showing that, given a quantity  $G(x, \bar{x}; \eta)$  that transforms as a function under symplectic transformations, also  $\mathcal{D}_\eta G$  transforms as a function.

To establish this, we need the transformation law of  $\hat{N}^{ij}$  that enters in (1.97). Under symplectic transformations,  $\hat{N}_{ij}$  given in (1.98) transforms as

$$\begin{aligned} \tilde{\hat{N}}_{ij} &= [\hat{S}^{-1}]^k{}_i [\tilde{\hat{S}}^{-1}]_{\bar{j}}^{\bar{l}} \left[ \hat{N}_{kl} + i F_{k\bar{m}} \tilde{Z}^{\bar{m}\bar{n}} \bar{F}_{\bar{n}p} \left( \delta_l^p - \mathcal{Z}^{pq} F_{q\bar{l}} \right) \right. \\ &\quad \left. - i \bar{F}_{\bar{k}m} \mathcal{Z}^{mn} F_{n\bar{p}} \left( \delta_l^{\bar{p}} - \tilde{\mathcal{Z}}^{\bar{p}\bar{q}} \bar{F}_{\bar{q}l} \right) \right] \\ &\quad + i [\hat{S}^{-1}]^k{}_i [\tilde{\hat{S}}^{-1}]_{\bar{j}}^{\bar{l}} \bar{F}_{\bar{k}m} [\mathcal{S}^{-1} \hat{\mathcal{S}}]^m{}_l - i [\tilde{\hat{S}}^{-1}]_{\bar{j}}^{\bar{k}} \bar{F}_{\bar{k}l} [\mathcal{S}^{-1}]^l{}_j, \end{aligned} \quad (1.178)$$

where  $\mathcal{S}$ ,  $\hat{\mathcal{S}}$  and  $\mathcal{Z}$  are defined in (1.166).

**Exercise 23** Verify (1.178) using (1.168) and (1.171).

Then, it follows that the inverse matrix  $\hat{N}^{ij}$  transforms as

$$\tilde{N}^{ij} = \left( S^i_l - Z^{in} F_{nl} \right) \hat{N}^{lk} \left( S^j_k - Z^{jm} F_{mk} \right) - i S^i_k Z^{kl} S^j_l. \quad (1.179)$$

Since the matrix  $Z = S^{-1} Z$  is symmetric [5], so is  $\tilde{N}^{ij}$ . Observe that it can also be written as

$$\tilde{N}^{ij} = \left( \tilde{S}^i_{\bar{l}} - Z^{in} \tilde{F}_{nl} \right) \hat{N}^{lk} \left( S^j_k - Z^{jm} F_{mk} \right) - i Z^{il} Z^{jm} F_{lm}. \quad (1.180)$$

Establishing the transformation behavior (1.179) turns out to be a tedious exercise, which we relegate to end of this appendix.

Now consider a quantity  $G(x, \bar{x}; \eta)$  that transforms as a function under symplectic transformations, i.e.  $G(x, \bar{x}; \eta) = \tilde{G}(\tilde{x}, \tilde{\bar{x}}; \eta)$ . We then calculate the behavior of  $\mathcal{D}_\eta G$  under symplectic transformations. First we establish

$$G_\eta = \tilde{G}_\eta + \tilde{G}_i Z^{ij} F_{\eta j} + \tilde{G}_{\bar{i}} Z^{ij} \tilde{F}_{\eta \bar{j}}, \quad (1.181)$$

where, on the right hand side, the tilde quantities are differentiated with respect to the tilde variables, while those without a tilde are differentiated with respect to the original variables. Similarly,

$$G_i - G_{\bar{i}} = \left( \tilde{G}_j - \tilde{G}_{\bar{j}} \right) \left( S^j_i - Z^{jk} F_{k\bar{i}} \right) + i \tilde{G}_{\bar{j}} Z^{jk} \hat{N}_{ki}, \quad (1.182)$$

as well as

$$F_{\eta j} = \tilde{F}_{\eta i} S^i_j + \tilde{F}_{\eta \bar{i}} Z^{ik} \tilde{F}_{\bar{k}j}, \quad (1.183)$$

where we used that  $F_\eta$  transforms as a symplectic function, as established in (1.177).

**Exercise 24** Verify (1.181) and (1.182) using  $G(x, \bar{x}; \eta) = \tilde{G}(\tilde{x}, \tilde{\bar{x}}; \eta)$ .

Then, inserting (1.181) and (1.182) into (1.97) yields,

$$\mathcal{D}_\eta G = \tilde{G}_\eta + \left( \tilde{G}_i - \tilde{G}_{\bar{i}} \right) Z^{ij} F_{\eta j} + i \hat{N}^{ij} \left( F_{\eta j} + \tilde{F}_{\eta \bar{j}} \right) \left( \tilde{G}_k - \tilde{G}_{\bar{k}} \right) \left( S^k_i - Z^{kl} F_{l\bar{i}} \right). \quad (1.184)$$

Next, using (1.183), we compute

$$\begin{aligned} \left( F_{\eta j} + \tilde{F}_{\eta \bar{j}} \right) &= \left( \tilde{F}_{\eta k} + \tilde{F}_{\eta \bar{k}} \right) \left( S^k_j - Z^{kl} F_{l\bar{j}} \right) - i \tilde{F}_{\eta \bar{l}} Z^{lk} \hat{N}_{kj} \\ &\quad + \left( \tilde{F}_{\eta l} + \tilde{F}_{\eta \bar{l}} \right) Z^{lk} F_{k\bar{j}} + \left( \tilde{F}_{\eta \bar{l}} + \tilde{F}_{\eta l} \right) Z^{lk} \tilde{F}_{\bar{k}j}. \end{aligned} \quad (1.185)$$

Using that  $\tilde{F}$  has the decomposition

$$\tilde{F}(\tilde{x}, \tilde{\bar{x}}; \eta) = \tilde{F}^{(0)}(\tilde{x}) + 2i \tilde{\Omega}(\tilde{x}, \tilde{\bar{x}}; \eta) \quad (1.186)$$



with  $\tilde{\Omega}$  real, it follows that the second line of (1.185) vanishes. Inserting the first line of (1.185) into (1.184) and using  $F_{i\bar{j}} = -\bar{F}_{\bar{j}i}$  as well as  $S Z^T = Z \mathcal{S}^T$ , we obtain

$$\mathcal{D}_\eta G = \tilde{G}_\eta + i \tilde{N}^{ij} \left( \tilde{F}_{\eta j} + \bar{\tilde{F}}_{\eta \bar{j}} \right) \left( \tilde{G}_i - \tilde{G}_{\bar{i}} \right) = \widetilde{(\mathcal{D}_\eta G)}, \quad (1.187)$$

which shows that  $\mathcal{D}_\eta G$  transforms as a function under symplectic transformations.

Now we return to the transformation behavior of  $\hat{N}^{ij}$  given in (1.179) and verify that it is the inverse of (1.178), i.e.  $\hat{N}^{-1} \hat{N} = \mathbb{I}$ . We use the decomposition  $F(x, \bar{x}; \eta) = F^{(0)}(x) + 2i\Omega(x, \bar{x}; \eta)$ . We find it useful to introduce the following matrix notation,

$$\begin{aligned} \bar{S}^{-1} \mathcal{S} &= \mathbb{I} + \bar{Z} (F_{..} - \bar{F}_{--}), \\ S^{-1} \hat{S} &= \mathbb{I} - X, \quad X = Z F_{..} - \bar{Z} \bar{F}_{--} = 4 Z \Omega_{..} - \bar{Z} \Omega_{..}, \\ \hat{S}^{-1} \mathcal{S} &= (\mathbb{I} - X)^{-1} = \sum_{n=0}^{\infty} X^n, \\ \tilde{S} &= \mathcal{S} \left[ \mathbb{I} - X - Z (\hat{F}_{..} - \bar{\hat{F}}_{--}) \right] = [\mathbb{I} - Z (F_{..} - \bar{F}_{--}) - 4 Z \Omega_{..} - Z \Omega_{..}], \\ Z - \bar{Z} &= -\bar{Z} (F_{..} - \bar{F}_{--}) \quad Z = -Z (F_{..} - \bar{F}_{--}) \bar{Z}, \end{aligned} \quad (1.188)$$

where we assume that the power series expansion of  $S^{-1} \hat{S}$  is convergent. Here  $F_{..}$ ,  $F_{--}$ ,  $F_{..}$  denote entries of the type  $F_{ij}$ ,  $F_{\bar{i}\bar{j}}$ ,  $F_{i\bar{j}}$ , respectively. Then, using (1.178), we compute

$$\begin{aligned} S^T \tilde{N} \tilde{S} &= \sum_{n=0}^{\infty} (X^n)^T \left( \hat{N} + 4i \Omega_{..} \bar{Z} \Omega_{..} - 4i \Omega_{..} Z \Omega_{..} + 2\Omega_{..} \bar{X} + 2\Omega_{..} \right) \\ &\quad - 2 (\bar{S}^{-1} S)^T \sum_{n=0}^{\infty} (\bar{X}^n)^T \Omega_{..} [\mathbb{I} - Z (F_{..} - \bar{F}_{--}) - 4 Z \Omega_{..} - Z \Omega_{..}]. \end{aligned} \quad (1.189)$$

Multiplying this with  $\tilde{N}^{-1} S^{-1,T}$  from the left and requiring the resulting expression to equal  $\tilde{S}$  yields the relation

$$\begin{aligned} &\left[ \hat{N}^{-1} - 2i \hat{N}^{-1} \Omega_{..} Z - 2i Z \Omega_{..} \hat{N}^{-1} - 4 Z \Omega_{..} \hat{N}^{-1} \Omega_{..} Z - i Z \right] \\ &\quad \left[ \sum_{n=0}^{\infty} (X^n)^T \left( \hat{N} + 4i \Omega_{..} \bar{Z} \Omega_{..} - 4i \Omega_{..} Z \Omega_{..} + 2\Omega_{..} \bar{X} + 2\Omega_{..} \right) \right] \end{aligned}$$

$$\begin{aligned}
& -2 \left( \bar{S}^{-1} \mathcal{S} \right)^T \sum_{n=0}^{\infty} (\bar{X}^n)^T \Omega_{..} \left[ \mathbb{I} - \mathcal{Z} (F_{..} - \bar{F}_{..}) - 4 \mathcal{Z} \Omega_{..} \mathcal{Z} \Omega_{..} \right] \\
& = \left[ \mathbb{I} - \mathcal{Z} (F_{..} - \bar{F}_{..}) - 4 \mathcal{Z} \Omega_{..} \mathcal{Z} \Omega_{..} \right]. \tag{1.190}
\end{aligned}$$

Thus, checking  $\tilde{N}^{-1} \tilde{N} = \mathbb{I}$  amounts to verifying the relation (1.190). To do so, we write (1.190) as a power series in  $\mathcal{Z}$  by converting  $\bar{\mathcal{Z}}$  into  $\mathcal{Z}$  using the last relation in (1.188). Introducing the expressions

$$\begin{aligned}
\sigma & = 4 \Omega_{..} \mathcal{Z} \Omega_{..} \mathcal{Z}, \\
\Delta & = \sum_{n=1}^{\infty} [(F_{..} - \bar{F}_{..}) \mathcal{Z}]^n, \tag{1.191}
\end{aligned}$$

we obtain

$$\begin{aligned}
\bar{X} & = 4 \mathcal{Z} (\mathbb{I} + \Delta) \Omega_{..} \mathcal{Z} \Omega_{..}, \\
\bar{X}^T \Omega_{..} & = \Omega_{..} X, \\
\sum_{n=0}^{\infty} (\bar{X}^n)^T \Omega_{..} & = \Omega_{..} \sum_{n=0}^{\infty} X^n, \\
X^n & = 4 \mathcal{Z} \Omega_{..} \mathcal{Z} [(\mathbb{I} + \Delta) \sigma]^{n-1} (\mathbb{I} + \Delta) \Omega_{..}, \quad n \geq 1, \\
(X^n)^T & = 4 \Omega_{..} (\mathbb{I} + \Delta^T) \left[ \sigma^T (\mathbb{I} + \Delta^T) \right]^{n-1} \mathcal{Z} \Omega_{..} \mathcal{Z}, \quad n \geq 1, \\
\left( \bar{S}^{-1} \mathcal{S} \right)^T & = \mathbb{I} + (F_{..} - \bar{F}_{..}) \mathcal{Z} (\mathbb{I} + \Delta). \tag{1.192}
\end{aligned}$$

Then, (1.190) becomes

$$\begin{aligned}
& \left[ \mathbb{I} - 2i \Omega_{..} \mathcal{Z} - 2i \hat{N} \mathcal{Z} \Omega_{..} \hat{N}^{-1} - 4 \hat{N} \mathcal{Z} \Omega_{..} \hat{N}^{-1} \Omega_{..} \mathcal{Z} - i \hat{N} \mathcal{Z} \right] \\
& \left[ \sum_{n=0}^{\infty} (X^n)^T \left[ \hat{N} + 4i \Omega_{..} \mathcal{Z} (\mathbb{I} + \Delta) \Omega_{..} - 4i \Omega_{..} \mathcal{Z} \Omega_{..} \right. \right. \\
& \quad \left. \left. + 8 \Omega_{..} \mathcal{Z} (\mathbb{I} + \Delta) \Omega_{..} \mathcal{Z} \Omega_{..} + 2 \Omega_{..} \right] \right. \\
& \quad \left. - 2 \left[ \mathbb{I} + (F_{..} - \bar{F}_{..}) \mathcal{Z} (\mathbb{I} + \Delta) \right] \Omega_{..} \sum_{n=0}^{\infty} X^n \left[ \mathbb{I} - \mathcal{Z} (F_{..} - \bar{F}_{..}) \right. \right. \\
& \quad \left. \left. - 4 \mathcal{Z} \Omega_{..} \mathcal{Z} \Omega_{..} \right] \right] = \hat{N} \left[ \mathbb{I} - \mathcal{Z} (F_{..} - \bar{F}_{..}) - 4 \mathcal{Z} \Omega_{..} \mathcal{Z} \Omega_{..} \right], \tag{1.193}
\end{aligned}$$

where  $X^n$  (with  $n \geq 1$ ) is expressed in terms of  $\mathcal{Z}$  according to (1.192). Now we proceed to check that (1.193) is indeed satisfied, order by order in  $\mathcal{Z}$ . Observe that the right hand side of (1.193) is quadratic in  $\mathcal{Z}$ , so first we check the cancellation of

the terms up to order  $\mathcal{Z}^2$ . Then we proceed to check the terms at order  $n$  with  $n \geq 3$ . Here we use the relations

$$\begin{aligned} F_{..} - \bar{F}_{..} &= i\hat{N} + 2i\Omega_{..} + 2i\Omega_{..}, \\ \Delta^T \mathcal{Z} &= \mathcal{Z} \Delta, \\ \left[ \sigma^T (\mathbb{I} + \Delta^T) \right]^n \mathcal{Z} &= \mathcal{Z} [\sigma (\mathbb{I} + \Delta)]^n, \end{aligned} \quad (1.194)$$

and we organize the terms at order  $n$  into those that end on either  $N$  (introduced in (1.100)),  $\Omega_{..}$  or  $\Omega_{..}$ . It is then straightforward, but tedious, to check that at order  $n$  in  $\mathcal{Z}$  all these terms cancel out. This establishes the validity of the transformation law (1.179).

## C The Holomorphic Anomaly Equation in Big Moduli Space

The holomorphic anomaly Eq. (1.157) of perturbative topological string theory [35, 36] can be succinctly derived in the wave function approach [22] to the latter [23–26]. In this approach, the topological string partition function  $Z$  is represented by a wavefunction,

$$Z(t; t_B, \bar{t}_B) = \int d\phi e^{-S(\phi, t; t_B, \bar{t}_B)/\hbar} Z(\phi), \quad (1.195)$$

where  $S(\phi, t; t_B, \bar{t}_B)$  denotes the generating function (1.114) of canonical transformations<sup>6</sup>. We take the background dependent constant  $c(t_B, \bar{t}_B)$  appearing in  $S$  to be given by [23–26]

$$c(t_B, \bar{t}_B) = -\frac{\hbar}{2} \ln \det N_{IJ}(t_B, \bar{t}_B), \quad (1.196)$$

with  $N_{IJ}$  as in (1.132).

Differentiating (1.195) with respect to the background field  $\bar{t}_B$  on the one hand, and with respect to the fluctuations  $t$  on the other hand, yields the relation [24],

$$\frac{\partial Z(t; t_B, \bar{t}_B)}{\partial \bar{t}_B^L} = \frac{\hbar}{2} \bar{F}_L^{IJ} \frac{\partial}{\partial t^I} \frac{\partial}{\partial t^J} Z(t; t_B, \bar{t}_B). \quad (1.197)$$

Here  $\bar{F}_L^{IJ}$  is evaluated on the background, and is given by  $\bar{F}_L^{IJ} = \bar{F}_{L\bar{M}\bar{O}} N^{MI} N^{OJ}$ . Assigning scaling dimension 1 to both  $t_B$  and  $t$  (and to their complex conjugates) and scaling dimension 2 to  $\hbar$ , we see that (1.197) has scaling dimension  $-1$ . Setting

$$Z(t; t_B, \bar{t}_B) = e^{W(t; t_B, \bar{t}_B)/\hbar}, \quad (1.198)$$

<sup>6</sup> We use the conventions of Sect. 1.4 and suppress the superscript of  $F^{(0)}$ .

we obtain from (1.197)

$$\frac{\partial W(t; t_B, \bar{t}_B)}{\partial \bar{t}_B^L} = \frac{1}{2} \bar{F}_L^{IJ} \left( \hbar \frac{\partial^2 W}{\partial t^I \partial t^J} + \frac{\partial W}{\partial t^I} \frac{\partial W}{\partial t^J} \right), \quad (1.199)$$

which has scaling dimension 1. The BCOV-solution [36] is obtained by making the ansatz [38]

$$W = \sum_{g=0, n=0}^{\infty} \frac{\hbar^g}{n!} C_{I_1 \dots I_n}^{(g)}(t_B, \bar{t}_B) t^{I_1} \dots t^{I_n}, \quad (1.200)$$

with

$$C_{I_1 \dots I_n}^{(g)} = 0, \quad 2g - 2 + n \leq 0. \quad (1.201)$$

The  $C_{I_1 \dots I_n}^{(g)}$  are symmetric in  $I_1, \dots, I_n$  and have scaling dimension  $2 - 2g - n$ . Inserting the ansatz (1.200) into (1.199), equating the terms of order  $\hbar^g$  for  $g \geq 2$  and setting  $t = 0$  gives,

$$\partial_{\bar{L}} C^{(g)}(t_B, \bar{t}_B) = \frac{1}{2} \bar{F}_L^{IJ} \left( C_{IJ}^{(g-1)} + \sum_{r=1}^{g-1} C_I^{(r)} C_J^{(g-r)} \right), \quad g \geq 2. \quad (1.202)$$

**Exercise 25** Verify (1.202).

Now we set [38]

$$C_{I_1 \dots I_n}^{(g)} = D_{I_1} \dots D_{I_n} F^{(g)}, \quad g \geq 1, \quad (1.203)$$

where  $D_L$  is given by

$$D_L V_M = \partial_L V_M + i N^{PI} F_{ILM} V_P. \quad (1.204)$$

$D_L$  acts as a covariant derivative for symplectic reparametrizations  $V_M \rightarrow \left( \mathcal{S}_0^{-1} \right)_M^P V_P$ , since  $N^{IJ}$  transforms as  $N^{IJ} \rightarrow [\mathcal{S}_0 N^{-1} \mathcal{S}_0]^{IJ} - i[\mathcal{S}_0 \mathcal{Z}_0 \mathcal{S}_0]^{IJ}$  (see (1.179)). The  $F^{(g)}$  have scaling dimension  $2 - 2g$  and transform as functions under symplectic transformations. Inserting (1.203) into (1.202) yields the holomorphic anomaly equation in big moduli space [38],

$$\partial_{\bar{L}} F^{(g)}(t_B, \bar{t}_B) = \frac{1}{2} \bar{F}_L^{IJ} \left( D_I \partial_J F^{(g-1)} + \sum_{r=1}^{g-1} \partial_I F^{(r)} \partial_J F^{(g-r)} \right), \quad g \geq 2. \quad (1.205)$$

As an example, consider solving (1.205) for  $g = 2$ . We need  $F_I^{(1)} = \partial_I F^{(1)}(t_B, \bar{t}_B)$ , which is non-holomorphic and given by<sup>7</sup>

$$\partial_I F^{(1)}(t_B, \bar{t}_B) = \partial_I f^{(1)}(t_B) + \frac{1}{2} i F_{IJK} N^{JK}. \quad (1.206)$$

Then, solving (1.205) for  $F^{(2)}$  yields [25, 38]

$$\begin{aligned} F^{(2)}(t_B, \bar{t}_B) &= f^{(2)}(t_B) + \frac{1}{2} i N^{IJ} \left( D_I F_J^{(1)} + F_I^{(1)} F_J^{(1)} \right) \\ &\quad + \frac{1}{2} N^{IJ} N^{KL} \left( \frac{1}{4} F_{IJKL} + \frac{1}{3} i N^{MN} F_{IKM} F_{JLN} + F_{IJK} F_L^{(1)} \right). \end{aligned} \quad (1.207)$$

In this expression, all the terms are evaluated on the background  $(t_B, \bar{t}_B)$ .

**Exercise 26** Verify that (1.207) solves (1.205).

Observe that (1.206) transforms covariantly under symplectic transformations, provided that  $f^{(1)}$  transforms as  $f^{(1)} \rightarrow f^{(1)} - \frac{1}{2} \ln \det \mathcal{S}_0$  in order to compensate for the transformation behavior  $N_{IJ} \rightarrow N_{KL} [\bar{\mathcal{S}}_0^{-1}]^K{}_I [\mathcal{S}_0^{-1}]^L{}_J$  [5], so that

$$\begin{aligned} f_I^{(1)} &\rightarrow \left( f_J^{(1)} - \frac{1}{2} \mathcal{Z}_0^{PQ} F_{PQJ} \right) (\mathcal{S}_0^{-1})^J{}_I, \\ f_{IJ}^{(1)} &\rightarrow (\mathcal{S}_0^{-1})^Q{}_J \partial_Q \left[ \left( f_L^{(1)} - \frac{1}{2} \mathcal{Z}_0^{PQ} F_{PQL} \right) (\mathcal{S}_0^{-1})^L{}_I \right]. \end{aligned} \quad (1.208)$$

**Exercise 27** Determine the transformation behavior of  $f^{(2)}(t_B)$  under symplectic transformations (1.128) that ensures that  $F^{(2)}(t_B, \bar{t}_B)$  transforms as a function. A useful transformation law is,

$$\begin{aligned} F_{IJKL} &\rightarrow (\mathcal{S}_0^{-1})^M{}_I \partial_M \left[ F_{NOP} (\mathcal{S}_0^{-1})^N{}_J (\mathcal{S}_0^{-1})^O{}_K (\mathcal{S}_0^{-1})^P{}_L \right] \\ &= (\mathcal{S}_0^{-1})^M{}_I (\mathcal{S}_0^{-1})^N{}_J (\mathcal{S}_0^{-1})^O{}_K (\mathcal{S}_0^{-1})^P{}_L \left[ F_{MNOP} \right. \\ &\quad \left. - F_{MPS} \mathcal{Z}_0^{SR} F_{RNO} - F_{OPS} \mathcal{Z}_0^{SR} F_{RMN} - F_{NPS} \mathcal{Z}_0^{SR} F_{ROM} \right]. \end{aligned} \quad (1.209)$$

<sup>7</sup>  $F_I^{(1)}$  contains an additional term proportional to the Kähler potential (1.1), but this term drops out of (1.205) due to the special geometry relation  $\bar{F}_{\bar{I}\bar{J}\bar{K}} \bar{t}^{\bar{K}} = 0$ .

## D The Functions $\mathcal{H}_i^{(a)}$ for $a \geq 2$

Here we collect the explicit results for the various functions  $\mathcal{H}_i^{(a)}$  (with  $a \geq 2$ ) that appear in (1.143). These functions can be determined by iteration. We present the functions up to order  $\mathcal{O}(\Omega^4)$ . We use the notation  $(N\Omega)^I = N^{IJ}\Omega_J$ ,  $(N\bar{\Omega})^I = N^{IJ}\bar{\Omega}_{\bar{J}}$ . The symmetrization  $F_{R(IJ}N^{RS}F_{KL)S}$  is defined with a symmetrization factor  $1/(4!)$ .

$$\begin{aligned}
\mathcal{H}^{(2)} = & 8 N^{IJ} \Omega_I \Omega_J - 16 [\Omega_{IJ} (N\bar{\Omega})^I (N\Omega)^J + \Omega_{I\bar{J}} (N\bar{\Omega})^I (N\bar{\Omega})^J + \text{h.c.}] \\
& - 8i [F_{IJK} (N\bar{\Omega})^I (N\Omega)^J (N\Omega)^K - \text{h.c.}] \\
& + \frac{16}{3} i \left[ \left( F_{IJKL} + 3i F_{IJJR} N^{RS} F_{SKL} \right) (N\Omega)^I (N\Omega)^J (N\Omega)^K (N\bar{\Omega})^L \right. \\
& \left. - \text{h.c.} \right] + 16 [\Omega_{IJK} (N\Omega)^I (N\Omega)^J (N\bar{\Omega})^K + \text{h.c.}] \\
& + 16 \left[ \left( \Omega_{IJ\bar{K}} + i F_{IJJ\bar{P}} N^{PQ} \Omega_{Q\bar{K}} \right) \left( (N\Omega)^I (N\Omega)^J (N\Omega)^K \right. \right. \\
& \left. \left. + 2 (N\Omega)^I (N\bar{\Omega})^J (N\bar{\Omega})^K \right) + \text{h.c.} \right] \\
& + 32 [\Omega_{IQ} N^{QR} \Omega_{RJ} (N\Omega)^I (N\bar{\Omega})^J + \text{h.c.}] \\
& + 32 \Omega_{IQ} N^{QR} \Omega_{\bar{R}\bar{J}} (N\Omega)^I (N\bar{\Omega})^J \\
& + 16i [F_{IJK} N^{KP} \Omega_{PQ} \left( (N\Omega)^I (N\Omega)^J (N\bar{\Omega})^Q \right. \\
& \left. + 2 (N\Omega)^Q (N\Omega)^I (N\bar{\Omega})^J \right) - \text{h.c.}] \\
& + 16i [F_{IJK} N^{KP} \Omega_{\bar{P}\bar{Q}} (N\Omega)^I (N\Omega)^J (N\bar{\Omega})^Q - \text{h.c.}] \\
& + 8 (N\Omega)^I (N\Omega)^J F_{IJQ} N^{QR} \bar{F}_{\bar{R}\bar{K}\bar{L}} (N\bar{\Omega})^K (N\bar{\Omega})^L \\
& + 32 [(N\Omega)^I \Omega_{IJ} N^{JK} \Omega_{K\bar{L}} (N\Omega)^L + \text{h.c.}] \\
& + 32 [(N\bar{\Omega})^I \Omega_{IJ} N^{JK} \Omega_{K\bar{L}} (N\bar{\Omega})^L + \text{h.c.}] \\
& + 32 [(N\Omega)^I \Omega_{IJ} N^{JK} \Omega_{\bar{K}\bar{L}} (N\Omega)^L + \text{h.c.}] \\
& + 16i [(N\Omega)^I (N\Omega)^J F_{IJK} N^{KL} \Omega_{\bar{L}\bar{P}} (N\Omega)^P - \text{h.c.}] \\
& + 32 [(N\Omega)^I \Omega_{I\bar{J}} N^{JK} \Omega_{\bar{K}\bar{L}} (N\bar{\Omega})^L + \text{h.c.}] \\
& + 32 [(N\Omega)^I \Omega_{I\bar{J}} N^{JK} \Omega_{K\bar{L}} (N\bar{\Omega})^L], \tag{1.210}
\end{aligned}$$

$$\begin{aligned}
\mathcal{H}_1^{(3)} = & -\frac{8}{3} i F_{IJK} (N\bar{\Omega})^I (N\bar{\Omega})^J (N\bar{\Omega})^K \\
& + 8i F_{IJK} (N\bar{\Omega})^J (N\bar{\Omega})^K N^{IP} \left[ 2\Omega_{\bar{P}\bar{Q}} (N\bar{\Omega})^Q + 2\Omega_{\bar{P}Q} (N\Omega)^Q \right. \\
& \left. - i \bar{F}_{\bar{P}\bar{Q}\bar{R}} (N\bar{\Omega})^Q (N\bar{\Omega})^R \right], \tag{1.211}
\end{aligned}$$

$$\begin{aligned}
\mathcal{H}_2^{(3)} = & 8 \left( \Omega_{IJ} + iF_{IJK}(N\Omega)^K \right) (N\bar{\Omega})^I (N\bar{\Omega})^J \\
& - \frac{4}{3} i \left( F_{IJKL} + 3iF_{R(IJ} N^{RS} F_{KL)S} \right) \left( 6(N\Omega)^I (N\Omega)^J (N\bar{\Omega})^K (N\bar{\Omega})^L \right. \\
& \left. - 4(N\bar{\Omega})^I (N\bar{\Omega})^J (N\bar{\Omega})^K (N\Omega)^L \right) \\
& - \frac{16}{3} \Omega_{IJK} \left( 3(N\bar{\Omega})^I (N\bar{\Omega})^J (N\Omega)^K - (N\bar{\Omega})^I (N\bar{\Omega})^J (N\bar{\Omega})^K \right) \\
& - 16 \Omega_{IJ\bar{K}} (N\bar{\Omega})^I (N\bar{\Omega})^J (N\bar{\Omega})^K \\
& - 16i F_{IJK} N^{KP} \Omega_{PQ} \left[ - (N\bar{\Omega})^I (N\bar{\Omega})^J (N\bar{\Omega})^Q \right. \\
& \left. + (N\bar{\Omega})^I (N\bar{\Omega})^J (N\Omega)^Q + 2(N\bar{\Omega})^I (N\Omega)^J (N\bar{\Omega})^Q \right] \\
& - 16 (N\bar{\Omega})^P \Omega_{PQ} N^{QR} \Omega_{RK} (N\bar{\Omega})^K \\
& - 32 (N\Omega)^I \left( \Omega_{IJ} + iF_{IJP}(N\Omega)^P \right) N^{JK} \left( \Omega_{\bar{K}\bar{L}} - i\bar{F}_{\bar{K}\bar{L}\bar{M}} (N\bar{\Omega})^M \right) (N\Omega)^L \\
& + 16i (N\Omega)^I (N\Omega)^J F_{IJP} N^{PK} \left( \Omega_{\bar{K}\bar{L}} - i\bar{F}_{\bar{K}\bar{L}\bar{Q}} (N\bar{\Omega})^{\bar{Q}} \right) (N\Omega)^L \\
& - 16 (N\Omega)^P \Omega_{\bar{P}\bar{Q}} N^{QR} \Omega_{R\bar{K}} (N\Omega)^K \\
& - 32 (N\bar{\Omega})^I \left( \Omega_{IJ} + iF_{IJK}(N\Omega)^K \right) N^{JL} \Omega_{\bar{L}\bar{M}} (N\Omega)^M \\
& - 16i (N\bar{\Omega})^I (N\bar{\Omega})^J F_{IJK} N^{KP} \Omega_{P\bar{Q}} (N\bar{\Omega})^Q, \tag{1.212}
\end{aligned}$$

$$\begin{aligned}
\mathcal{H}_3^{(3)} = & 16 \Omega_{I\bar{J}} (N\bar{\Omega})^I (N\Omega)^J \\
& - 16 \left[ 2(N\bar{\Omega})^K (N\Omega)^L \left( \Omega_{KM} N^{MN} \Omega_{N\bar{L}} + \Omega_{K\bar{L}Q} (N\Omega)^Q \right) \right. \\
& \left. + (N\bar{\Omega})^K \Omega_{K\bar{L}} N^{LP} \left( iF_{PMN} (N\Omega)^M (N\Omega)^N + 2\Omega_{PJ} (N\Omega)^J \right) \right. \\
& \left. + 2 \Omega_{P\bar{J}} (N\bar{\Omega})^J \right] + 2i (N\bar{\Omega})^I (N\Omega)^J F_{IJK} N^{KP} \Omega_{P\bar{Q}} (N\Omega)^Q + \text{h.c.}], \tag{1.213}
\end{aligned}$$

$$\mathcal{H}_1^{(4)} = 32 (N\bar{\Omega})^I \left( \Omega_{IJ} + iF_{IJK}(N\Omega)^K \right) N^{JP} \left( \Omega_{\bar{P}\bar{Q}} - i\bar{F}_{\bar{P}\bar{Q}\bar{R}} (N\bar{\Omega})^{\bar{R}} \right) (N\Omega)^Q, \tag{1.214}$$

$$\mathcal{H}_2^{(4)} = 32 (N\Omega)^P \Omega_{\bar{P}\bar{Q}} N^{QR} \Omega_{\bar{R}\bar{K}} (N\bar{\Omega})^K \tag{1.215}$$

$$\mathcal{H}_3^{(4)} = 8 F_{IJR} N^{RS} \bar{F}_{\bar{S}\bar{K}\bar{L}} (N\bar{\Omega})^I (N\bar{\Omega})^J (N\Omega)^K (N\Omega)^L, \tag{1.216}$$

$$\mathcal{H}_4^{(4)} = -\frac{4}{3} i \left( F_{IJKL} + 3i F_{IJR} N^{RS} F_{SKL} \right) (N\bar{\Omega})^I (N\bar{\Omega})^J (N\bar{\Omega})^K (N\bar{\Omega})^L, \tag{1.217}$$

$$\mathcal{H}_5^{(4)} = -16i F_{IJK} N^{KL} \Omega_{\bar{L}\bar{Q}} (N\bar{\Omega})^Q (N\bar{\Omega})^I (N\bar{\Omega})^J, \tag{1.218}$$

$$\mathcal{H}_6^{(4)} = -16i F_{IJK} N^{KP} \left( \Omega_{\bar{P}\bar{Q}} - i\bar{F}_{\bar{P}\bar{Q}\bar{R}} (N\bar{\Omega})^{\bar{R}} \right) (N\bar{\Omega})^I (N\bar{\Omega})^J (N\Omega)^Q, \tag{1.219}$$

$$\mathcal{H}_7^{(4)} = 16 \left( \Omega_{IJ\bar{K}} + iF_{IJP} N^P Q \Omega_{Q\bar{K}} \right) (N\bar{\Omega})^I (N\bar{\Omega})^J (N\Omega)^K, \tag{1.220}$$

$$\mathcal{H}_8^{(4)} = 32 (N\bar{\Omega})^I \left( \Omega_{IJ} + iF_{IJK}(N\Omega)^K \right) N^{JP} \Omega_{\bar{P}\bar{Q}} (N\bar{\Omega})^Q, \tag{1.221}$$

$$\mathcal{H}_9^{(4)} = -16i (N\bar{\Omega})^I (N\bar{\Omega})^J F_{IJK} N^{KL} \Omega_{\bar{L}\bar{P}} (N\bar{\Omega})^P. \tag{1.222}$$

## E Transformation Laws by Iteration

The Hesse potential in Sect. 1.4 depends on  $\Omega$ , whose behavior under symplectic transformations can be determined by iteration. Here we summarize the result for the transformation behavior of derivatives of  $\Omega$  (expressed in terms of the covariant variables of Sect. 1.3), up to a certain order. We use the conventions of Sect. 1.4 and suppress the superscript of  $F^{(0)}$ .

$$\begin{aligned}
\tilde{\Omega}_I &= [\mathcal{S}_0^{-1}]^J{}_I \left[ \Omega_J + iF_{JKL} (\mathcal{Z}_0\Omega)^K (\mathcal{Z}_0\Omega)^L - 2i\Omega_{JK} (\mathcal{Z}_0\Omega)^K \right. \\
&\quad + 2i\Omega_{J\bar{K}} (\bar{\mathcal{Z}}_0\bar{\Omega})^{\bar{K}} + \frac{2}{3}F_{JKLP} (\mathcal{Z}_0\Omega)^K (\mathcal{Z}_0\Omega)^L (\mathcal{Z}_0\Omega)^P \\
&\quad + 2F_{KLP} (\mathcal{Z}_0\Omega)^K{}_J (\mathcal{Z}_0\Omega)^L (\mathcal{Z}_0\Omega)^P \\
&\quad + 4F_{JKL} (\mathcal{Z}_0\Omega)^K (\mathcal{Z}_0\Omega)^L{}_P (\mathcal{Z}_0\Omega)^P \\
&\quad - 4F_{JKL} (\mathcal{Z}_0\Omega)^K (\mathcal{Z}_0\Omega)^L{}_{\bar{P}} (\bar{\mathcal{Z}}_0\bar{\Omega})^{\bar{P}} \\
&\quad - 2F_{JKL} \mathcal{Z}_0^{LP} F_{PQS} (\mathcal{Z}_0\Omega)^K (\mathcal{Z}_0\Omega)^Q (\mathcal{Z}_0\Omega)^S \\
&\quad + 2\bar{F}_{\bar{K}\bar{L}\bar{P}} (\bar{\mathcal{Z}}_0\bar{\Omega})^{\bar{K}}{}_J (\bar{\mathcal{Z}}_0\bar{\Omega})^{\bar{L}} (\bar{\mathcal{Z}}_0\bar{\Omega})^{\bar{P}} \\
&\quad - 2\Omega_{JKL} (\mathcal{Z}_0\Omega)^K (\mathcal{Z}_0\Omega)^L - 4\Omega_{KL} (\mathcal{Z}_0\Omega)^K{}_J (\mathcal{Z}_0\Omega)^L \\
&\quad - 2\Omega_{J\bar{K}\bar{L}} (\bar{\mathcal{Z}}_0\bar{\Omega})^{\bar{K}} (\bar{\mathcal{Z}}_0\bar{\Omega})^{\bar{L}} - 4\Omega_{\bar{K}\bar{L}} (\bar{\mathcal{Z}}_0\bar{\Omega})^{\bar{K}}{}_J (\bar{\mathcal{Z}}_0\bar{\Omega})^{\bar{L}} \\
&\quad + 4\Omega_{J\bar{K}\bar{L}} (\mathcal{Z}_0\Omega)^K (\bar{\mathcal{Z}}_0\bar{\Omega})^{\bar{L}} + 4\Omega_{\bar{K}\bar{L}} (\mathcal{Z}_0\Omega)^K{}_J (\bar{\mathcal{Z}}_0\bar{\Omega})^{\bar{L}} \\
&\quad \left. + 4\Omega_{K\bar{L}} (\mathcal{Z}_0\Omega)^K (\bar{\mathcal{Z}}_0\bar{\Omega})^{\bar{L}}{}_J \right] + \mathcal{O}(\Omega^4),
\end{aligned}$$

$$\begin{aligned}
\tilde{\Omega}_{IJ} &= [\mathcal{S}_0^{-1}]^K{}_I [\mathcal{S}_0^{-1}]^L{}_J \left[ \Omega_{KL} - F_{KLM} \mathcal{Z}_0^{MN} \Omega_N \right. \\
&\quad - iF_{KLP} \mathcal{Z}_0^{PM} F_{MQR} (\mathcal{Z}_0\Omega)^Q (\mathcal{Z}_0\Omega)^R + 2iF_{KLP} (\mathcal{Z}_0\Omega)^P{}_Q (\mathcal{Z}_0\Omega)^Q \\
&\quad - 2iF_{KLP} (\mathcal{Z}_0\Omega)^P{}_{\bar{Q}} (\bar{\mathcal{Z}}_0\bar{\Omega})^{\bar{Q}} + iF_{KLMN} (\mathcal{Z}_0\Omega)^M (\mathcal{Z}_0\Omega)^N \\
&\quad + iF_{KMN} (\mathcal{Z}_0\Omega)^M{}_L (\mathcal{Z}_0\Omega)^N + iF_{KMN} (\mathcal{Z}_0\Omega)^N{}_L (\mathcal{Z}_0\Omega)^M \\
&\quad - 2iF_{KMN} \mathcal{Z}_0^{MP} F_{PQL} (\mathcal{Z}_0\Omega)^Q (\mathcal{Z}_0\Omega)^N \\
&\quad - 2i\Omega_{KLP} (\mathcal{Z}_0\Omega)^P - 2i\Omega_{KP} (\mathcal{Z}_0\Omega)^P{}_L + 2i\Omega_{KP} \mathcal{Z}_0^{PQ} F_{QLS} (\mathcal{Z}_0\Omega)^S \\
&\quad \left. + 2i\Omega_{K\bar{L}\bar{P}} (\bar{\mathcal{Z}}_0\bar{\Omega})^{\bar{P}} + 2i\Omega_{K\bar{P}} (\bar{\mathcal{Z}}_0\bar{\Omega})^{\bar{P}}{}_L \right] + \mathcal{O}(\Omega^3),
\end{aligned}$$

$$\begin{aligned}
\tilde{\Omega}_{I\bar{J}} &= [\mathcal{S}_0^{-1}]^K{}_I [\bar{\mathcal{S}}_0^{-1}]^{\bar{L}}{}_{\bar{J}} \left[ \Omega_{K\bar{L}} + 2iF_{KMN} (\mathcal{Z}_0\Omega)^M{}_L (\mathcal{Z}_0\Omega)^N \right. \\
&\quad - 2i\bar{F}_{\bar{L}\bar{P}\bar{N}} (\bar{\mathcal{Z}}_0\bar{\Omega})^{\bar{N}}{}_K (\bar{\mathcal{Z}}_0\bar{\Omega})^{\bar{P}} \\
&\quad - 2i\Omega_{KM\bar{L}} (\mathcal{Z}_0\Omega)^M - 2i\Omega_{KM} (\mathcal{Z}_0\Omega)^M{}_{\bar{L}} + 2i\Omega_{K\bar{L}\bar{M}} (\bar{\mathcal{Z}}_0\bar{\Omega})^{\bar{M}} \\
&\quad \left. + 2i\Omega_{K\bar{M}} (\bar{\mathcal{Z}}_0\bar{\Omega})^{\bar{M}}{}_{\bar{L}} \right] + \mathcal{O}(\Omega^3),
\end{aligned}$$



$$\begin{aligned}
\tilde{\Omega}_{IJL} &= [\mathcal{S}_0^{-1}]^M{}_I [\mathcal{S}_0^{-1}]^N{}_J [\mathcal{S}_0^{-1}]^K{}_L \left[ \Omega_{MNK} - F_{MNKP}(\mathcal{Z}_0\Omega)^P \right. \\
&\quad - F_{MNP}(\mathcal{Z}_0\Omega)^P{}_K - F_{KMP}(\mathcal{Z}_0\Omega)^P{}_N - F_{NKP}(\mathcal{Z}_0\Omega)^P{}_M \\
&\quad + F_{MNP}\mathcal{Z}_0^P{}_Q F_{KQR}(\mathcal{Z}_0\Omega)^R + F_{KMP}\mathcal{Z}_0^P{}_Q F_{QNR}(\mathcal{Z}_0\Omega)^R \\
&\quad \left. + F_{NKP}\mathcal{Z}_0^P{}_Q F_{QMR}(\mathcal{Z}_0\Omega)^R \right] + \mathcal{O}(\Omega^2), \\
\tilde{\Omega}_{IJ\bar{K}} &= [\mathcal{S}_0^{-1}]^M{}_I [\mathcal{S}_0^{-1}]^N{}_J [\bar{\mathcal{S}}_0^{-1}]^{\bar{L}}{}_{\bar{K}} \left[ \Omega_{MN\bar{L}} - F_{MNQ}(\mathcal{Z}_0\Omega)^Q{}_{\bar{L}} \right] \\
&\quad + \mathcal{O}(\Omega^2),
\end{aligned} \tag{1.223}$$

where  $(\mathcal{Z}_0\Omega)^M = \mathcal{Z}_0^{MN}\Omega_N$ ,  $(\bar{\mathcal{Z}}_0\bar{\Omega})^{\bar{M}} = \bar{\mathcal{Z}}_0^{\bar{M}\bar{N}}\Omega_{\bar{N}}$ ,  $(\mathcal{Z}_0\Omega)^M{}_{\bar{L}} = \mathcal{Z}_0^{MN}\Omega_{N\bar{L}}$ ,  $(\bar{\mathcal{Z}}_0\bar{\Omega})^{\bar{P}}{}_{\bar{L}} = \bar{\mathcal{Z}}_0^{\bar{P}\bar{N}}\Omega_{\bar{N}\bar{L}}$ ,  $(\mathcal{Z}_0\Omega)^L{}_{\bar{P}} = \mathcal{Z}_0^{LK}\Omega_{K\bar{P}}$ ,  $(\bar{\mathcal{Z}}_0\bar{\Omega})^{\bar{P}}{}_{\bar{L}} = \bar{\mathcal{Z}}_0^{\bar{P}\bar{N}}\Omega_{\bar{N}\bar{L}}$ .

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