Chapter 91
Equiangular Numbers

Henry Crapo and Claude Le Conte De Poly-Barbut

Some mathematical problems are resolutely geometric. No matter what you do to them, subjecting them to different sorts of manipulations and calculations, their 'geometric content' persists even in the tiniest parts of what remains, even in the numbers used to express their solution, like the parts of an image residing 'everywhere' in a hologram, or like the smile of a Cheshire cat. We want to tell you of one such problem, and of a delightful series of real numbers starting with 0,1, . . . and tending toward 2, that does its best to recall the struggles along its path into existence. We maintain that it is because of these ancient struggles (which are bound to recur when one tries to 'construct' them) that these numbers are of architectural and artistic significance.

σ_2	$\sigma 3$	σ_4	σ_5	σ_6	σ_7	σ_8	$\ldots \sigma_8$
0	1	1.41421	1.61803	1.73205	1.80194	1.84776	. . . 2

You will recognize the first few even in this inappropriate form, rounded off to five decimal places: (σ_4 is $\sqrt{2}$, while σ_5 is τ, the Golden Mean, and σ_6 is $\sqrt{3}$). We call the sequence $\{\sigma_n\}$ the *equiangular numbers*.

The story begins with one of Donald Coxeter's masterpieces, his algebraic characterization of groups generated by reflections (Coxeter 1935). His formulation is simple: you insist that your group be generated by a finite set of

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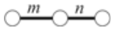
elements of order 2, say $\{s_i \mid i = 1 \ldots n\}$, and that the defining relations be all of the form

$$\left(s_i s_j\right)^{c_{ij}} = \varepsilon$$

for extended integer values c_{ij}, where $i < j$ and $2 \leq c_{ij} \leq \infty$, and ε is the identity element of the group. These values are recorded in a *graph* whose vertices are the generators, and where the edges ij are labelled c_{ij} whenever this value is at least 3. (By $c_{ij} = \infty$ we mean simply that there is no corresponding relation imposed; all the powers $(s_i s_j)^n$ are distinct.) For instance the graph  denotes the group with three generators, and relations

$$s_1^2 = s_2^2 = s_3^2 = \left(s_1 s_2\right)^m = \left(s_2 s_3\right)^n = \left(s_1 s_3\right)^2 = \varepsilon.$$

The hard part was then to show that every such group is geometrically representable as a group generated by reflections.

Say you have a group generated by reflections in n mirrors, which we call the *generators*, surrounding a fundamental region in a space. These generators are reflected in each other to form virtual reflectors, which we call *mirrors*; algebraically they are conjugates xsx^{-1} of a generator s by an element x of the group. The space is divided up into *cells*, each an image of the fundamental region under a succession of reflections, and representing the element of the group that carries the fundamental region to that location (Fig. 91.1). Since each element x of the group is expressible as the product of a *word* in generators, it has a *length* $\ell(x)$, equal to the minimum length of a word $s_1 \ldots s_n$ with product $\pi(s_1 \ldots s_n) = x$ x. A word of this length is called a *short* word for x. Geometrically, the length is the number of mirrors you have to cross in order to get from the identity (fundamental region) to the cell representing the element x. Every mirror is a conjugate xsx^{-1} of a generator s by an element x; take this element x to be of minimum length among such expressions, and choose any short expression for x. You find a short *palindrome* $s_1 \ldots s_n \ldots s_1$ for the mirror.

Under the partial order

$$x \leq y \quad \text{if and only if some short word for } x \\ \text{is a prefix of some short word for } y$$

the group becomes a *semilattice* (to be precise: a complete meet-semilattice: every subset of the group has a greatest lower bound), or simply a *lattice*, if the group is finite. Each *step*, or covering pair $[x,y]$, where $x < y$ and $\ell(x) + 1 = \ell(y)$, has an associated generator $s = x^{-1}y$ and an associated mirror $m = yx^{-1}$, which we call the *generator label* and *mirror label*, respectively, of the step. It is a nice surprise to find that there are consistent drawings in which *steps with the same mirror label are drawn parallel*. Perhaps even more surprisingly, if these vector directions x_m, one for each mirror m, are *very* carefully chosen in n-dimensional space, the resulting

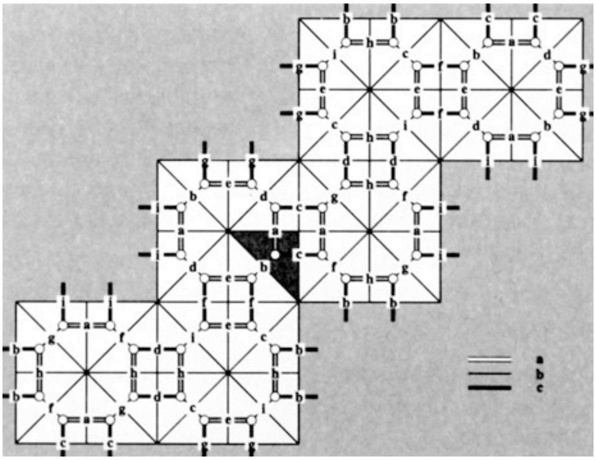


Fig. 91.1 The group generated by three reflections of the cube

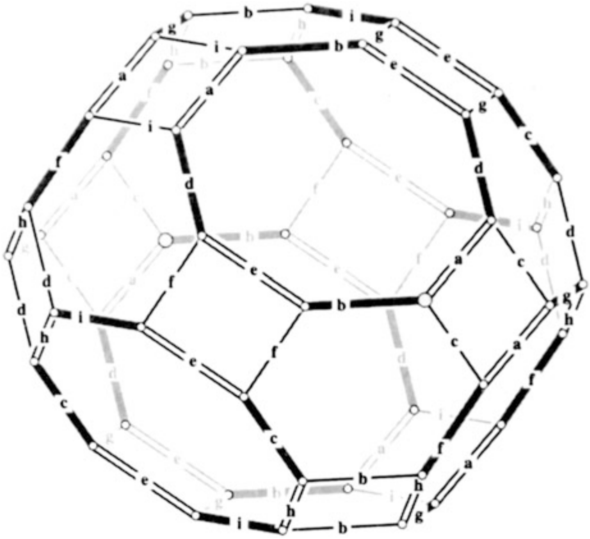


Fig. 91.2 The group as zonohedron

figure becomes the 1-skeleton of a *zonotope* (Fig. 91.2), the convex figure formed as the Minkowski sum of the line segments $[0, x_m]$, or of a zonotopal tiling.

The question remains, for a Coxeter group, given, say, by its graph of generators and relations, how do we choose the vectors x_m in order correctly to draw the corresponding zonotope or zonotopal tiling? In his charming article on zonotopes (1962), Coxeter showed how it suffices to cut all the vectors by a hyperplane, and so

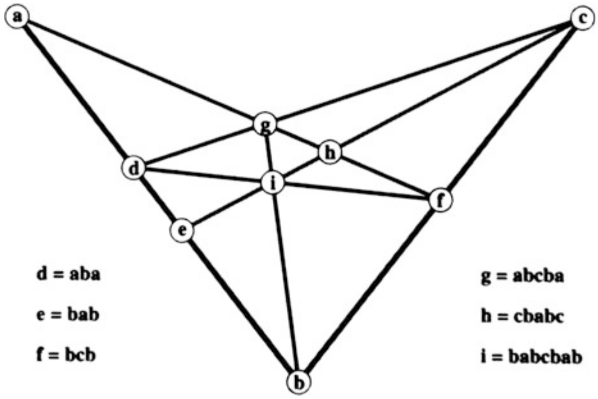


Fig. 91.3 The mirror configuration for

work with figures of projective points. This will be our approach, to construct these *mirror diagrams*.

Look at Fig. 91.1 in detail. There is a cell for each element of the group generated by reflections of the cube, with fundamental region shown in grey. The three types of edges encode the corresponding generator labels, while the letters are the mirror labels. The outer edges should be identified in pairs so that the sheet forms a polyhedral surface (here, the cube). Figure 91.2 is a correct projection of the corresponding zonotope, with flat octagonal, hexagonal, and square faces in 3-space. Figure 91.3 shows the mirror diagram for this group. Note that for any pair xy of mirrors, their successive conjugates

$$x, xyx, xyxyx, xyxyxyx, \ldots$$

are collinear, and that the mirrors at the ends of each line are of minimal lengths for that line. Here are nine mirrors, arranged on seven major lines. This simple figure already possesses a non-trivial *projective property*. Since the four mirrors a, $d = aba$, $e = bab$, b lie at the four points of intersection of a line ab with the six edges of a plane tetrahedron (vertices $cfgi$), these four mirrors are *harmonic*. If we assign coordinates (1, 0, 0) to a, (0, 1, 0) to b, and place the points d and e symmetrically relative to the midpoint (1, 1, 0), then the point d will have coordinates $(\sqrt{2}, 1, 0) = (\sigma_4, 1, 0)$.

Mirror diagrams for the groups of permutations of an n-element set are generalized Desargues configurations, as in Fig. 91.4, formed by the intersection of n hyperplanes in general position in a space of dimension $n-1$. For S_4 this is a complete quadrilateral in the plane, for S_5, the usual Desargues configuration in 3-space.

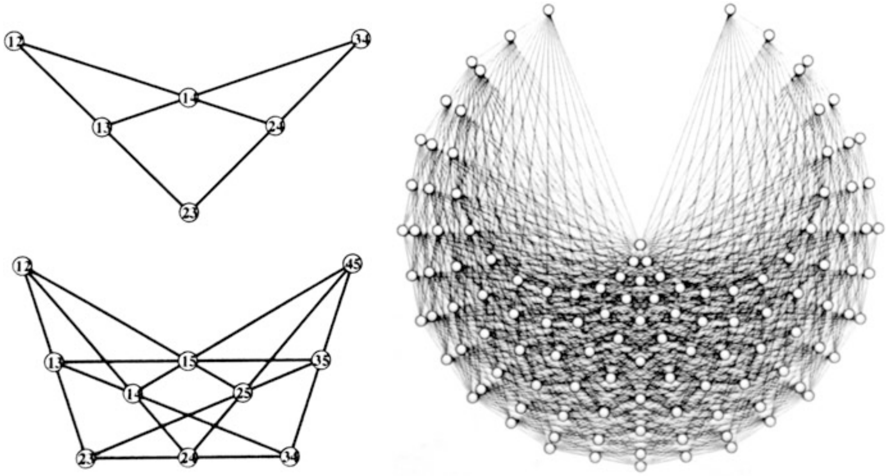
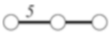


Fig. 91.4 Mirror diagrams for the symmetric S_4, S_5, S_{16}

For the group  generated by reflections of the icosohedron or dodecahedron, the mirrors configuration (Fig. 91.5) already has *no possible projective construction*! Any such construction would be a projective construction of the golden mean, which is known to be impossible using straightedge only. Try drawing this figure, just looking at a list of sets of points that are supposed to be collinear. You will quickly see why we are making a fuss about equiangular points! (You can get it right quickly by trial and error, but trial and error has no standing as a projective construction.) We have shaded some triangular regions of the diagram in order to emphasize that this is a *projective regular pentagon chgij*, with inner and outer stars: take the line *ab* to be the line at infinity.

The problem of drawing mirror diagrams for Coxeter groups has a simple general solution if we are willing to impose appropriate restrictions on the positions of those mirrors on lines joining pairs of generators. If these choices are made in a natural way, there is a *straightforward* construction of the remaining positions; everything just falls into place. We must take a closer look at the case of two generators.

For two generators *a,b*, the simplest such group is that for $(ab)^{\infty} = \varepsilon$. This is the group you see in the barbershop (Fig. 91.6) with parallel mirrors on opposite walls. You see not only infinitely many chairs, but infinitely many mirrors, each making its own faithful reflected image of the entire infinite scene. The generators are the two mirrors bounding region ε, with real silvered glass.

If the two generators are not quite parallel, the series of images will bend along a circular path of large diameter. Whenever the angle between them is a rational multiple of π, the images will pile up in a finite number of distinct positions. For mirrors at an angle of $\frac{\pi}{n}$ we find $2n$ images, one for each element of the dihedral group D_n, and n concurrent mirrors at equal angles (Fig. 91.7).

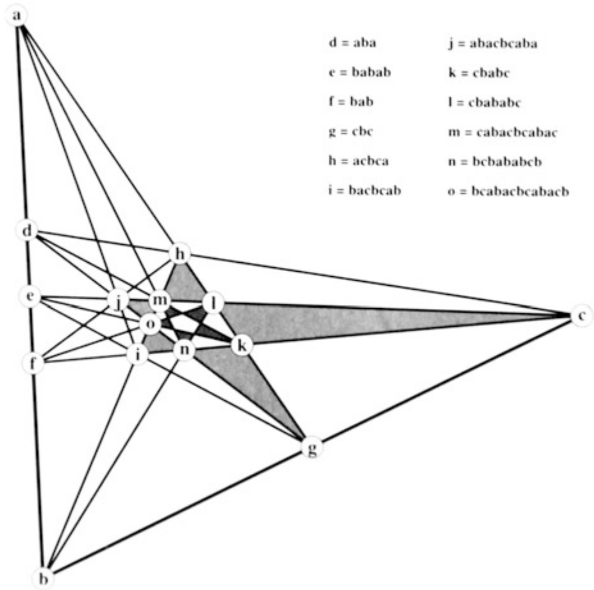


Fig. 91.5 Mirror diagram for the group of the icosahedron

Fig. 91.6 A portion of the group

Consider a family P of n coplanar and concurrent equiangular lines $L_1, \ldots, L_n$ through a point c in the plane. Intersect this family of lines with a line L parallel to the bisector of a pair of consecutive lines, say of L_1, and L_n. By $\wedge$ and $\vee$ we denote the operators join (of a pair of points, to form a line) and meet (of a pair of lines, to form a point), respectively, in the projective plane. Let $p_i = L_i \wedge L$, for $i = 1 \ldots n$. We call such a set $E_n = \{p_1, \ldots, p_n\}$ a centrally symmetric set of n *equiangular* points (Fig. 91.8).

Without loss of generality we may select homogeneous (projective) coordinates

$$p_1 \to (1,0)$$
$$p_n \to (0,1) p_n \to (0,1)$$
$$\text{midpoint of the segment} \quad [p_0, p_n] \to (1,1).$$

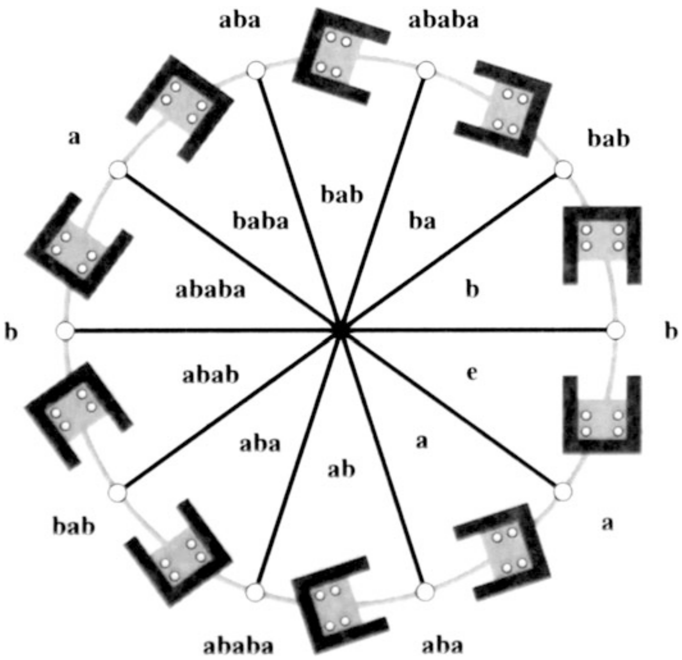


Fig. 91.7 The group $\circ\overset{5}{\text{---}}\circ$ (barbershop quintet)

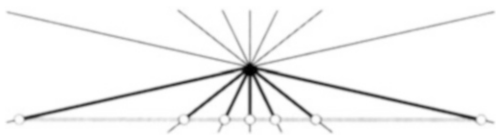


Fig. 91.8 A line of seven equiangular points

Then the projective coordinates of all points p_i are determined, each up to a non-zero scalar multiple. Let $(\sigma_n, 1)$ be the coordinates of p_2, given n equiangular lines.

These values σ_n can be computed as roots of a sequence of polynomials, as follows. Let r_1 be reflection of the plane in mirror L_1. The mapping

$$A : p \to L \wedge \left(r_1\left(p \bigvee c\right)\right)$$

is a projective map, an involution of the line L fixing the point p_1 and inducing the permutation

$$(p_1)(p_2p_n)(p_2p_{n-1})\dots\left(p_{k+1}\right) \quad \text{if } n = 2k$$
$$(p_1)(p_2p_n)(p_2p_{n-1})\dots\left(p_kp_{k+1}\right) \quad \text{if } n = 2k-1$$

of the points p_i. This mapping A can be expressed as right-multiplication by the (2×2)-matrix

$$\begin{pmatrix} -1 & 0 \\ \sigma & 1 \end{pmatrix}$$

since this linear transformation and its non-zero scalar multiples are the only linear maps that send (1,0) to a scalar multiple of itself, exchanging (σ,1) and (0,1) with scalar multiples of each other.

Our symmetric choice of projective coordinates (1,0) for p_1 and (0,1) for p_n, permits us to express the central symmetry D of the figure by the linear transformation with matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

This transformation induces the permutation

$$(p_1p_n)(p_2p_{n-1})\dots\left(p_kp_{k+1}\right) \quad \text{if } n = 2k$$
$$(p_1p_n)(p_2p_{n-1})\dots(p_k) \quad \text{if } n = 2k-1.$$

Composing the maps D, then A, we obtain a map that induces the cyclic permutation $(p_1\ p_2 \dots p_n)$ which advances the points along the line (a turn by one of the $2n$ cogs of the wheel), and has matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ \sigma & 1 \end{pmatrix} = \begin{pmatrix} \sigma & 1 \\ -1 & 0 \end{pmatrix}.$$

Using multiplication by this matrix DA to compute the coordinates of the successive points p_i, we find

$$\begin{gathered} p_1 = (1,0) \\ p_2 = (\sigma, 1) \\ p_3 = (\sigma_2, -1, \sigma) \\ p_4 = (\sigma_2 - 2\sigma, \sigma_2 - 1) \\ p_5 = (\sigma_4 - 3\sigma_2 + 1, \sigma_2 - 2\sigma) \\ \vdots \\ p_m = (f_m(\sigma), f_{m-1}(\sigma)), \\ \vdots \end{gathered}$$

for $m = 1, \dots, n$, where the f_m form a sequence of polynomials

$$\begin{aligned} f_0 &= 0 \\ f_1 &= 1 \\ f_2 &= x \\ f_3 &= x^2 - 1 \\ f_4 &= x^3 - 2x \\ f_5 &= x^4 - 3x^2 + 1 \\ f_6 &= x^5 + 4x^3 + 3x \\ f_7 &= x^6 - 5x^4 + 6x^2 - 1 \\ &\vdots \end{aligned}$$

determined by initial values $f_0(x) = 0, f_1(x) = 1$ and the simple recursion

$$f_m = xf_m - 1 - f_m - 2,$$

for all $m \geq 2$. In closed form:

$$f_m(x) = \sum_{i=0}^{\lfloor (m-1)/2 \rfloor} (-1)^i \binom{m-i-1}{i} x^{m-2i-1}$$

The terminal condition $f_n = 0$ applies, and permits us to determine the correct value of σ_n, the largest positive root of f_n. Factorizations of these polynomials, with integer coefficients, and exact expressions for their roots in terms of radicals, begin as follows:

0:	0	
1:	1	
2:	x	0
3:	$(x-1)(x+1)$	± 1
4:	$x(x^2-2)$	$0, \pm\sqrt{2}$
5:	$(x^2-x-1)(x^2+x-1)$	$\left(\pm 1 \pm \sqrt{5}\right)/2$
6:	$x(x^2-3)(x^2-1)$	$0, \pm 1, \sqrt{3}$
7:	$(x^3-x^2-2x+1)(x^3+x^2-2x-1)$.	

Our attempts to use computer algebra systems to solve the polynomial equations for $f_n = 0$ yielded useful results only for $n \leq 6$, a difficult expression in radicals for $n = 7$, and no results at all for $n > 7$.

A trigonometric solution, however, to the equations $f_n(x) = 0$ exists, and takes the form:

$$2\cos\frac{k\pi}{n} \quad \text{for} \quad k = 1, \ldots, n-1,$$

with largest positive root

$$\sigma_n = 2\cos\frac{\pi}{n}.$$

Using these values of σ_n we can construct a linear representation of the group. Each generator s_i is given by standard unit vector,

$$s_i = (0, \ldots, 1 \ldots 0),$$

while the linear operator 'conjugation by s_i' is given by the matrix

$$\begin{pmatrix} 1 & 0 & \cdots & \sigma_{1i} & \cdots & 0 & 0 \\ 0 & 1 & \cdots & \sigma_{2i} & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & \sigma_{n-1,i} & \cdots & 1 & 0 \\ 0 & 0 & \cdots & \sigma_{ni} & \cdots & 0 & 1 \end{pmatrix}$$

Extend this matrix representation multiplicatively, first representing each element x in the group as a product of generators, then 'conjugation by x' as the corresponding product of matrices. It is now an easy matter to compute projective coordinates for all the mirrors, since each is the conjugate of a generator by an element of the group, and is thus the image of a standard basis vector under multiplication by one of these product matrices. For instance, each mirror in the symmetric group , being a permutation with cycle structure (ij), gets coordinates (0,. . .,0,1,. . .,1,0,. . .,0), where the 1s are in positions i through j–1.

In the limit, with $\sigma = 2$, the 'translation' map DA has matrix

$$\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$$

and creates an infinite sequence of points

$$p_1 = (1,0), p_2 = (2,1), p_3 = (3,2), \ldots p_k = (k, k-1) \ldots$$

reaching a projective limit at the midpoint (1,1) (Fig. 91.9). Mirror reflection in the line L_1 the linear map A, permutes pairs of points on opposite sides of this midpoint (Fig. 91.10):

$$(k, k-1)\begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix} = (k-2, k-1).$$

In closing, we should notice that a geometric situation gave rise to a difficult (yea, impossible) construction problem in projective geometry, then to a problem in

Fig. 91.9 The limiting case of infinitely many equiangular points (Venice harbour entrance)

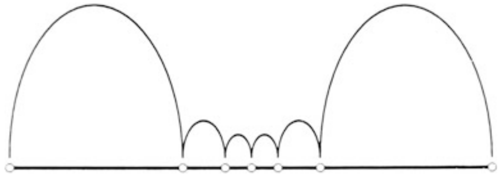


Fig. 91.10 Pairs of points on opposite sides of the midpoint

polynomial algebra that taxes the powers of the best modern computer algebra systems, but which had a simple solution in terms of trigonometry. It is fair to ask whether these further values of σ_n, for $n = 7,8,\ldots$ occur already in nature, for the simple reason that they are the natural coordinates of *equiangular points*. Finally, since the merits of the golden mean are well recognized in artistic matters (planning of paintings, design of building façades, or choice of relative dimensions for European paper stock), where the aspect of 5-equiangularity is thoroughly disguised, surely the subsequent values of s_n for $n > 5$ can give rise to analogous aesthetic feelings in similar situations. Can our readers point to any instances of the use of s_7 in ancient or contemporary architecture?

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Biography Henry Crapo has a Ph.D. in mathematics from MIT (1964). He was a professor at the University of Waterloo in Ontario, Canada. He carried out his research in geometry at INRIA-Rocquencourt, France and at Centre d'analyse et de mathématique sociales (CAMS), EHESS in Paris.
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