Chapter 78
Aperiodic Tiling, Penrose Tiling and the Generation of Architectural Forms

Michael J. Ostwald

Introduction

In September of 1995 the Australian architectural practice Ashton Raggatt McDougall (ARM) invited the eminent mathematician Roger Penrose to open their soon to be completed refurbishment of the historic Storey Hall complex of buildings at the Royal Melbourne Institute of Technology. Penrose, who admitted that the design seemed "extremely exciting" (Penrose 1996), regretfully declined on the grounds that he was already overcommitted to too many projects to visit Australia at the required time. He concluded his response to the invitation with an enigmatic postscript which records that he is currently working on "the single tile problem" and recently "found a tile set consisting of one tile together with complicated matching rule that can be enforced with two small extra pieces" (Penrose 1996). This postscript contains the first clue to understanding the mysterious connection between Penrose and Storey Hall, between a scientist and a controversial, award-winning, building.

Storey Hall is significant for many reasons but only one prompted ARM to invite Penrose to open it. The newly completed Storey Hall is literally covered in a particular set of giant, aperiodic tiles that were discovered by Roger Penrose in the 1970s and have since become known as Penrose tiles. While architecture has, historically, always been closely associated with the crafts of tiling and patterning, Storey Hall represents a resurrection of that tradition.

First published as: Michael J. Ostwald, "Aperiodic Tiling, Penrose Tiling and the Generation of Architectural Forms", pp. 99–111 in *Nexus II: Architecture and Mathematics,* ed. Kim Williams, Fucecchio (Florence): Edizioni dell'Erba, 1998.

M.J. Ostwald (✉)
School of Architecture and Built Environment, The University of Newcastle, Callaghan, NSW 2308, Australia
e-mail: michael.ostwald@newcastle.edu.au

K. Williams and M.J. Ostwald (eds.), *Architecture and Mathematics from Antiquity to the Future*, DOI 10.1007/978-3-319-00143-2_31,

But what is Penrose tiling and what does it have to do with architecture in general and Storey Hall in particular? This chapter provides an overview of the special properties and characteristics of Penrose tilings before describing the way in which they are used in ARM's Storey Hall. The purpose of this binary analysis is not to critique Storey Hall or its use of aperiodic tiling but to use ARM's design as a catalyst for taking the first few steps in a greater analysis of Penrose tiling in the context of architectural form generation.

Periodic and Aperiodic Tilings

The geometers Grünbaum and Shephard record that the "art of tiling must have originated very early in the history of civilization" because, with the very first attempt to "use stones to cover the floors", humanity "could be said to have begun tiling" (Grünbaum and Shephard 1987: 1). Throughout history tiling has always been associated with architecture. From the moment the earliest primitive hut was floored with woven matting, walled with masonry, or carved with geometric patterns, it became, in mathematical terms, a tiled structure. The reason for this is that mathematical tiling is not defined by the craft of combining materials but by the repetitious creation of patterns formed through the application of a set of usually polygonal shapes. For this reason cut-stone mosaics are as much examples of mathematical tiling as painted frescos or carved Celtic knot-work. Any system of geometric patterning that covers or fills a surface using a finite set of shapes is considered tiling in a mathematical sense.[1] One of the reasons Grünbaum and Shephard suggest that the art of architecture has always been closely associated with the craft of tiling is that tiling or patterning adds richness, through ornamentation, to the surface of a building. Such ornamentation is not simply valued for aesthetic reasons but also for symbolic, practical and monetary purposes. At any point in history, Grünbaum and Shephard argue, "whatever kind of tiling was in favour, its art and technology always attracted skilful artisans, inventive practitioners and magnanimous patrons" (1987: 1). However, while many treatises have been written throughout history on the formation of architecture through geometric principles, such works rarely consider the relationship between tiling and architecture.[2] Further, despite Johannes Kepler's analysis of tiling patterns in his 1619 work *Harmonice Mundi*, a rigorous and scientific approach to understanding the properties of tilings has been formulated only in the last few decades.

[1] In theory it does not matter how large the set of tiles is. An infinite set of different shapes that fills a plane is still a form of tiling although an unconventional kind.

[2] Whereas minor or subtle references to the art of tiling may be discerned in various translations of the works of Vitruvious as well as in those of Alberti, Vignola and Serlio, even such minor references are increasingly rare in the treatises that followed; see Kruft (1994).

Charting the rise in enthusiasm for geometric tiling patterns to the works of Hao Wang in the early 1960s has become a veritable truism in mathematics (Penrose 1990: 174). In 1961 the philosopher Hao Wang became interested in questions of pattern recognition in the use of geometry as a tool for symbolic logic. Wang wanted to determine if, given a set of polygonal shapes, there is a procedure for determining whether or not they will tile a plane in such a way that they will necessarily repeat their configuration. Tilings that repeat their configuration or that display multiple lines of symmetry are usually called *periodic tilings*. The most recognisable periodic tilings are based on sets of squares, rectangles, trapezoids or parallelograms. In order to examine the question of whether or not a procedure exists for determining if a set of shapes will tile periodically, Wang developed a set of square tiles, each with different coloured edges. The edges of Wang tiles are only allowed to join other edges of identical colour—they may not be rotated or reflected, only translated. Wang conjectured that if *aperiodic tiles* (tiles that do not repeat their patterns) exist, then he could not derive a decision procedure whereby a given set of tiles will periodically tile a plane. Conversely, if a decision procedure could be determined, then there was no such thing as an aperiodic system of tiling (Rubinsteim 1996: 20–21).

In order to clarify Wang's agenda, it must be understood that there are two types of aperiodic tiles. There are sets of tiles that can fill a plane both periodically and non-periodically, and there are sets of tiles that only fill a plane non-periodically.[3] An example of the former is Gardner's set of quadrilaterals that tile both periodically and non-periodically; the choice is up to the person placing the tiles (Figs. 78.1 and 78.2). There are countless examples of geometric sets of this kind; Penrose frequently uses Marjorie Rice's 1976 single tile set to explain this idea (Fig. 78.3). Despite this, the sets of shapes that are conventionally referred to in mathematics as aperiodic are usually those that can fill a plane only non-periodically, or those that are necessarily non-periodic. This latter category of shapes is the one with which Wang was primarily concerned.

In 1965 Robert Berger developed Wang's thesis to prove that there is no decision procedure for tiling surfaces periodically and, thus, there must be a set of aperiodic tiles in existence. Following this realisation Berger set about finding the first set of aperiodic tiles. The tiling system he discovered, comprising a set of more than 20,000 different shapes, was exhibited the following year. However, Berger's tiling system was based on a peculiarity of logic, and in the following years a number of mathematicians produced increasingly less numerous sets of tiles that would fill a plane aperiodically. In 1967 Berger himself lowered the number of tiles from 20,426 to 104 and, in 1968, Donald Knuth further reduced the set to 92. Yet, in

[3] It must be noted that there is some confusion surrounding the terminology "aperiodic" and "non-periodic" as both terms are used interchangeably in popular mathematics and science. This chapter generally conforms to the wording used in Grünbaum and Shephard's encyclopedic work *Tilings and Patterns* (1987) and uses "aperiodic" as an accurate description of the properties of a tile set and "non-periodic" only when quoting from another work or when using the term as a broad, non-definitive, descriptor.

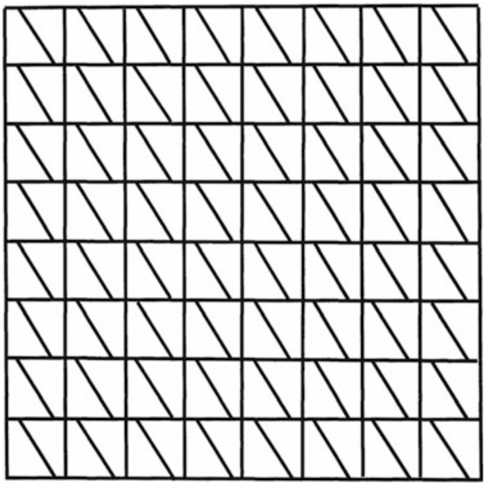

Fig. 78.1 Gardner's set of quadrilateral tiles (with a square period) used periodically. Image: author

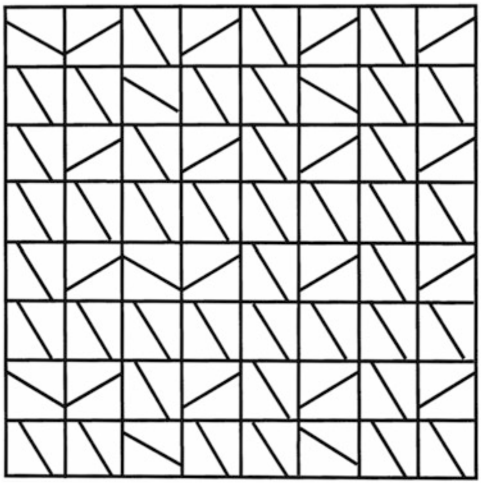

Fig. 78.2 Gardner's set of quadrilateral tiles (still with a square period) used aperiodically. Image: author

1971, a more dramatic reduction occurred when Raphael Robinson both allowed the set of tiles to be rotated and reflected and then removed colour altogether from the tiles. Instead of colour, Robinson used a series of geometric additions and indentations to ensure that certain edges could be combined while others couldn't. In this way Robinson reduced the set of tiles, from Knuth's 92 to just 6 (Figs. 78.4 and 78.5). In essence, Robinson's tiles are still Wang tiles because they are still based on a square tiling pattern, and for this reason they represent the minimum possible set of aperiodic tiles founded on an underlying square period. However, while the square tiling period has a minimum limit of six tiles, Penrose proposed in 1973 that by using a parallelogram tiling period the set could be

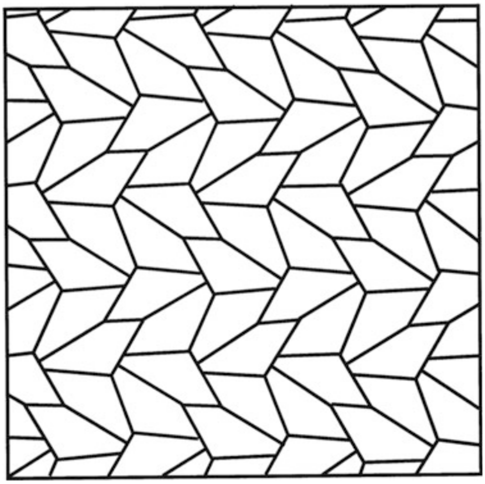

Fig. 78.3 Rice's single polygon set tiling a surface periodically. Image: author

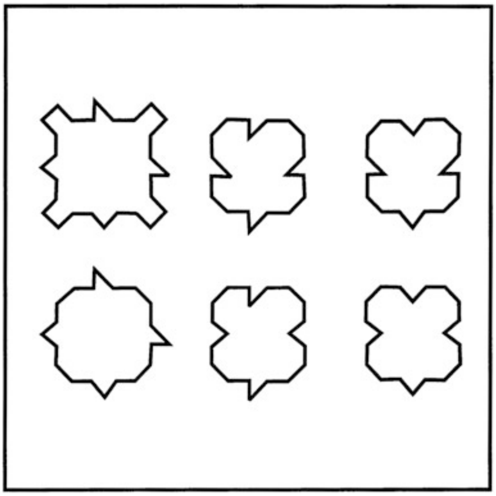

Fig. 78.4 Robinson's six tiles set. Image: author

reduced to just two tiles. Moreover, Penrose then proposed two sets of two tiles that each could be tiled only aperiodically.

The first set of "Penrose tiles", named by the mathematician John Conway the "darts and kites" set, is derived from a rhombus (or parallelogram) with four sides of equal length ($length = \phi$) with obtuse angles of 108° and acute of 72°. A line is then drawn between the acute corners of the rhombus (bisecting each of these into two angles of 32°) and a length equal to the length of a typical side of the rhombus (i.e., ϕ) measured along this line (Figs. 78.6 and 78.7). The new point created in this way is connected to the remaining obtuse corners of the rhombus. The rhombus is then cut along these two lines creating a kite form (with angles of 72°, 72°, 72°, and 144°) and a dart form (with angles of 36°, 72°, 36° and 216°). Then, if the two

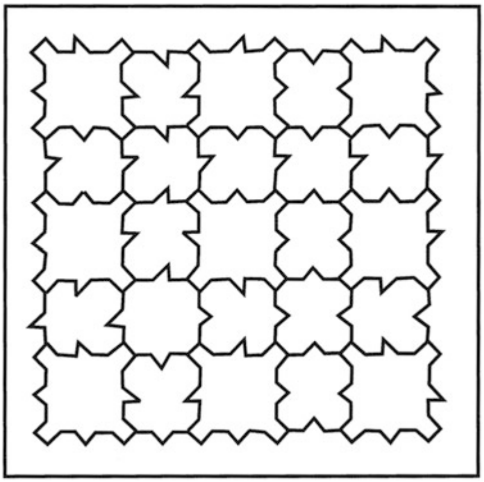

Fig. 78.5 Robinson's tiles displaying forced aperiodicity. Image: author

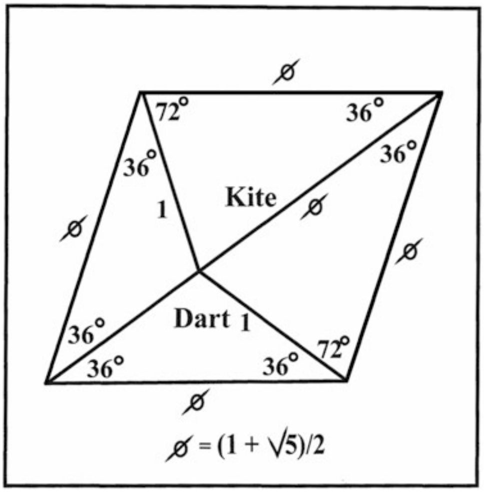

Fig. 78.6 Construction of Penrose dart and kite tiles. Image: author

cutting lines that connect the obtuse corners are of length 1, the other lengths created are reflections, proportionally, of the golden mean (i.e., $\phi = \frac{1+\sqrt{5}}{2}$). Finally, the two forms thus created, the kite and the dart, are coloured or indented to ensure that they may only be connected to certain other surfaces and thus tile only aperiodically.

An intriguing property of the "darts and kites" set of Penrose tiles is that if an infinitely large surface is to be tiled, $\frac{1+\sqrt{5}}{2}$ (approximately 1.618) times as many kites as darts is required; in other words, the ratio of darts to kites is the golden mean.

The first tile of the second Penrose set of aperiodic tiles is identical to the starting rhombus used to construct the first pair. That is, the first tile is a rhombus with sides

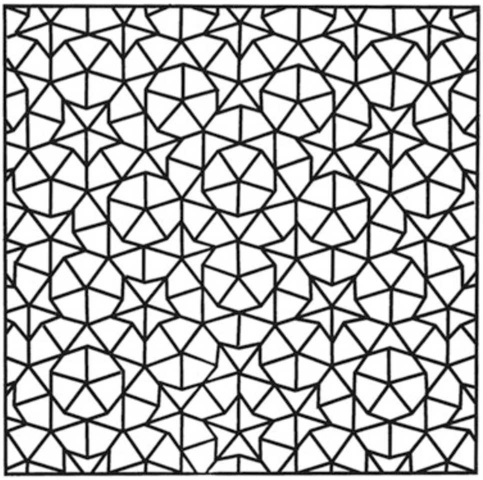

Fig. 78.7 Penrose dart and kite tiles displaying forced aperiodicity (note that the *coloured patches* and indents forcing the patterns are removed for clarity). Image: author

of equal length (*length* $=\phi$), with obtuse angles of 108° and acute of 72°. The second rhombus tile also has four sides of length equal to those of the first but with obtuse angles of 144° and acute of 36°. These are then modified with colours, shades or indentations to ensure that they tile aperiodically (Figs. 78.8 and 78.9). This second set similarly has proportions and ratios that reflect the characteristics of the golden mean. This close relationship between both sets of Penrose tiles and the golden mean may be more readily appreciated by closely examining a Pythagorean pentagram at multiple scales (Fig. 78.10).

One special characteristic of Penrose tiling patterns is that they exhibit *quasi-symmetry*. Normally any type of symmetry in a tiling pattern would render a set periodic, but Penrose tiles display partial, not complete, symmetry through rotation at 72°. For this reason Penrose tilings are said to be quasi-symmetrical. This characteristic is important because until 1984 it was believed that all crystalline materials must be based on lattices with conventional periodic symmetries. Crystals exhibit rotational symmetry at only 2, 3, 4 and 6 rotations. However, in 1984 Dany Schechtman discovered an aperiodic crystalline structure in aluminium-manganese by electron micrography. This crystalline structure, which was called a *quasicrystal*, almost possessed fivefold symmetry in much the same way that Penrose tiling patterns are almost symmetrical. Although, as Cracknell records, fivefold symmetry in crystals had been discovered as early as 1966, it was commonly believed that crystals could not have fivefold symmetry as such was unknown in morphological crystallography (Cracknell 1969). For Gardner, the discovery of the quasicrystal had great repercussions in science:

> Among physicists, chemists and crystallographers the effect of this discovery was explosive. Similar nonperiodic structures were soon being induced in other alloys, and dozens of papers began to appear. It became clear that solid matter could exhibit nonperiodic lattices with any kind of rotational symmetry. Wide varieties of solid tiles in sets of two or more

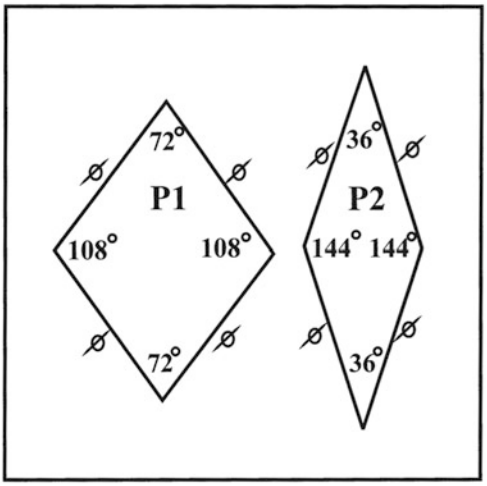

Fig. 78.8 Construction of Penrose twin parallelogram tiles. Image: author

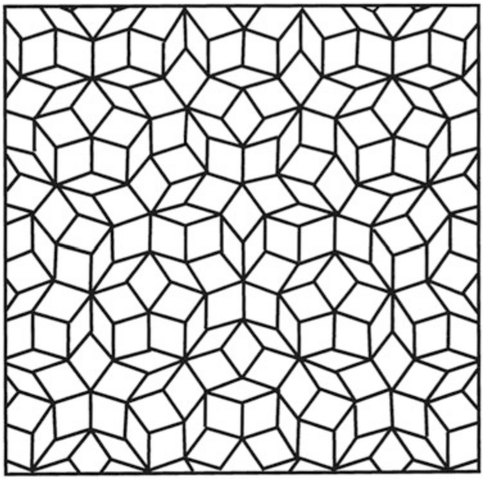

Fig. 78.9 Penrose twin parallelogram tiles displaying forced aperiodicity (note that the coloured patches and indents forcing the patterns are removed for clarity). Image: author

were proposed as models, some forcing nonperiodicity, some merely allowing it (Gardner 1989: 25).

Fivefold quasi-symmetry defies the laws of crystalline restriction that suggests that crystalline lattices cannot posses fivefold symmetry. However, as Stewart and Golubitsky maintain, "quasicrystals 'almost repeat' their structure, and *can* have axes of fivefold symmetry" (Stewart and Golubitsky 1993: 95). Nevertheless, despite these recent developments in the geometry and mathematics of aperiodic tiling patterns, Penrose tiles are still widely regarded as examples of "recreational mathematics" (Gardner 1989). While Penrose tilings might possess some latent ability to describe crystal anomalies or "symmetry-breaking" in natural inorganic

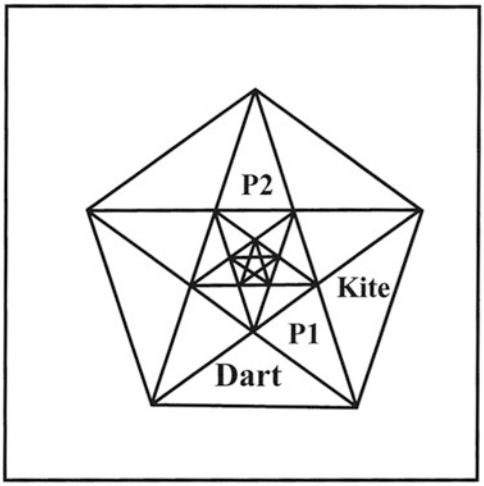

Fig. 78.10 The Pythagorean Pentagram displaying all four Penrose tiles at different scales. Image: author

forms (Stewart and Golubitsky 1993), they are still largely without clear application in any specific field.

Storey Hall

Storey Hall is a large auditorium with ancillary spaces that are built into the shell of an existing, historic structure (Figs. 78.11 and 78.12). The building was opened in late 1996 and has since been awarded a number of state and national design awards. The most contentious aspect of the design is the way in which ARM have carefully restored and retained parts of the original Victorian building, only to combine them with a glaringly modern, brightly coloured, geometric addition. The architectural designer and critic Norman Day describes the original Victorian detailing as being almost completely overwhelmed by a complex applique of "white, green, pink, purple and red panels, with inset green neon lights and great panels of white translucent and reflected light fittings" (Day 1995: 36). These tiles, which overlay both the façade of the building in Swanston street as well as most of the interior spaces, are all the second set of Penrose tiles (the twin parallelograms) marked in accordance with Penrose colouring (not Penrose indentations). In the words of ARM, on the exterior of the building the Penrose tiles are "shrouded in veil and drapery, folded sash, delicious lace, and strong rope lines, marking the inner boundaries of Penrose's mysterious geometry" (Ashton Raggatt McDougall 1996: 9). On the inside of the building, particularly in the auditorium space, "the tile is made to iterate its own pattern as if in multiple dimensions."

ARM assert that in the design of Storey Hall they are using Penrose tiles as part of a twofold strategy of transfiguring the dominant Euclidean geometry of the existing structure and as a symbol for the power of the "new sciences." For ARM

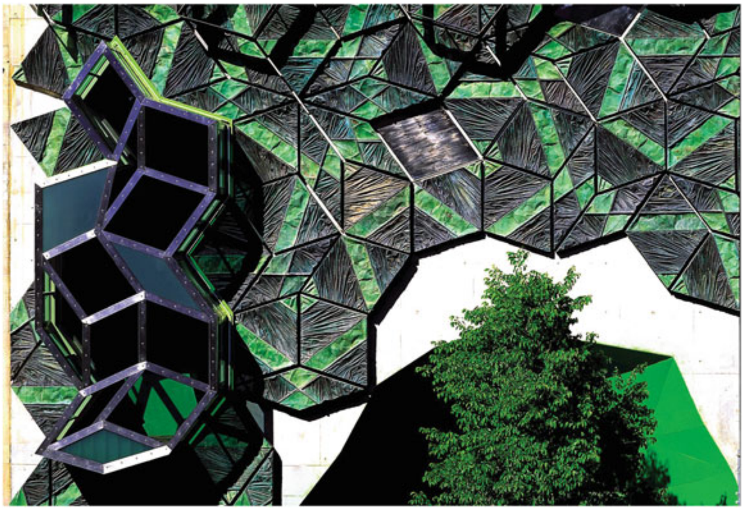

Fig. 78.11 Storey Hall, designed by ARM. Photo: courtesy John Gollings

Fig. 78.12 Storey Hall, designed by ARM. Photo: courtesy John Gollings

the use of Penrose tiles is not simply a reference to plane-filling geometry but to the greater conceptual shift associated with what they describe as the "new mathematics." This is a claim that various critics read as suggesting that there is a connection between Penrose tiling and fractal geometry. For example, Day claims that the Penrose tiles must be read in concert with the form of the facade itself, which is a "site-poured concrete wall" that is twisted "according to the new geometry of complexity" (Day 1995: 40). Inside the building the act of reading the meaning behind the giant Penrose tiles is further complicated because the tiles are layered with a range of other icons from science and geometry (Kohane 1996: 8–15). Norman Day suggests that amidst the geometry of Penrose there are also signs that lead to "chaos theory, Walter Burley-Griffin, urbanism, the sexual revolution, feminism, Einstein's grotto, Plato's cave, X and Y chromosomes, the Vault sculpture, paradoxes [and] contextualism" (Day 1995: 37).

But are the Penrose tiles significant in any way or are they simply icons for geometric knowledge? When the author of this chapter questioned Charles Jencks on the validity of applying mathematical forms to the exterior of an historic building, Jencks replied that the more "transformational" the use of iconography, the more interesting the outcome. For Jencks, while Storey Hall "doesn't really use the Penrose tiling pattern in section or plan ... it still uses it importantly in wall depth, in all-over ornament, and iconographically" (Ostwald et al. 1996: 30). All of these uses, Jencks argues, are appropriate means of creating architectonic forms. But doesn't this type of use border on simple "applique" or the "purely ornamental"?—seemingly not for Jencks. There is more happening in Storey Hall, according to Jencks, than simple conversion of Penrose tiles to architectural ornamentation. "Look at the lighting," he says, "and the way it relates to the old Victorian building. Its musculature is similar to the Victorian building and it has the same information density". If it were simply "applique, or applied ornament" it would be "less interesting than something that has greater organisational depth" (Ostwald et al. 1996: 30).

ARM's use of Penrose tiles as a generator of architectural form seems to reflect not so much a close reading of topographical Mathematics but an awareness of the role of these ideas in the understanding of quasicrystals and, less directly, as a connection to fractal geometry. Leon van Schaik (1996: 5) describes Storey Hall as "an architecture which works through the contemporary mathematics of surface". For van Schaik, the Penrose tiles refer not only to topographical mathematics but to an "unfolding symphony of forms that envelop us in an encounter with the spatiality of the new mathematics". When viewed in this way, the architectural forms generated through the use of Penrose tiles are simply extensions of the historic relationship between architecture and tiling, a relationship that otherwise seems, in recent years, to be lacking in creativity.

Penrose Tiling and Architecture

Despite their possible use in the interpretation of quasicrystals, Penrose tiles are still simply plane-filling patterns with a few unusual properties. Moreover, these unusual properties are not in any obvious way particularly relevant to architecture.[4] Ultimately, in the context of architecture, a periodic tile set is the same as an aperiodic set; the choice of using one or the other is simply aesthetic. Yet there are two recent developments in tiling geometry that have occurred as a result of Penrose's discoveries that seem potentially more profitable for the development

[4] Although Robbin's arguments to the contrary are very intriguing, this author remains sceptical; see Robbin (1990: 140–142).

of architectural forms. The reason these two developments seem more useful is that they promote the understanding of a spatial dimension in aperiodic tiling as well as a topological one. In 1976 the mathematician Robert Ammann proposed that a two-component set could be devised that would tile space aperiodically. This means that instead of being a "plane-filling" system, Ammann's tiles are "space-filling." Significantly, this same system was independently discovered at around the same time by the Japanese architect and geometer Koji Miyazaki (Miyazaki 1977). Ammann's aperiodic space-filling tiles are a pair of rhombohedra formed by creating two solids, each of which have six sides that are all the same as Penrose's starting rhombus for the formation of the dart and kite set. That is, each of the surfaces of space-filling tiles is a rhombus with sides of equal length and with obtuse angles of 108° and acute of 72°. The two solids produced in this way bear an uncanny resemblance to the basic geometry of Peter Eisenman's axonometric model for House X as well as his House El Even Odd project (Eisenman 1982, 1995). When coloured or modified in a certain way, Ammann's tiles will only fit together in three dimensions, aperiodically.

One final discovery in the geometry of tiling—a discovery that is rather more complex and is thus necessarily described in a very superficial manner here—concerns forced holes in tiled planes. Significantly, in order to describe the presence of forced holes in Penrose aperiodic tilings, mathematicians have resorted to the use of architectural metaphors (Ostwald and Moore 1995; Ostwald et al. 1997). For instance, John Conway describes the discovery of hole theory as akin to imagining "a vast temple with a floor tessellated by Penrose tiles and a circular column exactly in the centre. The tiles seem to go under the column. Actually, the column covers a hole that can't be tessellated" (Gardner 1989: 26–27). Certain combinations of Penrose tiles (and indeed any necessarily aperiodic tilings) can force areas that are unable to be tiled. Conventionally this type of error is rectified by removing a number of surrounding tiles and reworking the pattern until there are no holes. But if holes are formed, they impact on the greater pattern in many subtle and significant ways. Holes in tessellated planes, like space-filling aperiodic tiles, are emphatically spatial systems of geometry that broach many possible connections between architecture, Penrose tilings and other aperiodic tilings. These two aspects of aperiodic tiling warrant further investigation in architecture.

Biography Michael J. Ostwald is Professor and Dean of Architecture at the University of Newcastle (Australia) and a visiting Professor at RMIT University. He has previously been a Professorial Research Fellow at Victoria University Wellington, an Australian Research Council (ARC) Future Fellow at Newcastle and a visiting fellow at UCLA and MIT. He has a PhD in architectural history and theory and a DSc in design mathematics and computing. He completed post-doctoral research on baroque geometry at the CCA (Montreal) and at the Loeb Archives (Harvard). He is Co-Editor-in-Chief of the *Nexus Network Journal* and on the editorial boards of *ARQ* and *Architectural Theory Review*. He has authored more

than 300 scholarly publications including 20 books and his architectural designs have been published and exhibited internationally.

References

ASHTON RAGGATT McDOUGALL. 1996. New Patronage. In Ashton Raggatt McDougall, eds. *RMIT Storey Hall*, Melbourne: Faculty of Environmental Design and Construction, RMIT.

CRACKNELL, Arthur. 1969. *Crystals and their Structure*. London: Pergamon.

DAY, Norman. 1995. Storey Hall. *Architecture Australia* **85**, 1 (1995): 36.

EISENMAN, Peter. 1982. *House X*. New York: Rizzoli.

———. 1995. *Eisenman Architects: Selected And Current Works*. Mulgrave: Images Publishing.

GARDNER, Martin. 1989. *Penrose Tiles To Trapdoor Ciphers*. New York: W. H. Freeman.

GRÜNBAUM, Branko and Geoffrey Colin SHEPHARD. 1987. *Tilings and Patterns*. New York: W. H. Freeman.

KOHANE, Peter. 1996. Ashton Raggatt and McDougall's Imitative Architecture: Real and Imaginary Paths through Storey Hall. *Transition* **52/53** (1996): 8–15.

KRUFT, Hanno Walter. 1994. *A History of Architectural Theory from Vitruvius to the Present*. In R. Taylor, E. Callander, and A. Wood, trans. New York: Princeton Architectural Press.

MIYAZAKI, Koji. 1977. *On Some Periodical and Non-Periodical Honeycombs*. Kobe: University Monograph.

OSTWALD, Michael J., and R. John MOORE. 1995. Mathematical Misreadings in Nonlinearity: Architecture as Accessory/Theory. In Mike Linzey, ed. *Accessory/Architecture*. Vol. 1. Auckland: University of Auckland. 1995.

———. 1997. Unravelling the Weave: an Analysis of Architectural Metaphors in Nonlinear Dynamics. *Interstices*. Vol. 4 (1997) CD-ROM.

OSTWALD, Michael J., Peter ZELLNER and Charles JENCKS. 1996. An Architecture Of Complexity. *Transition* **52/53** (1996): 30.

PENROSE, Roger. 1990. *The Emperor's New Mind: Concerning Computers, Minds, and the Laws of Physics*. London: Vintage.

———. 1996. Fax to Howard Raggatt. In Ashton Raggatt McDougall, eds. *RMIT Storey Hall*. Melbourne: Faculty of Environmental Design and Construction, RMIT.

RUBINSTEIM, Heim. 1996. Penrose Tiling. *Transition* **52/53** (1996): 20–21.

ROBBIN, Tony. 1990. Quasicrystals for Architecture: The Visual Properties of Three-Dimensional Penrose Tessellations. *Leonardo* **23**, 1: 140–141.

STEWART, Ian and Martin GOLUBITSKY. 1993. *Fearful Symmetry: Is God A Geometer?* London: Penguin.

VAN SCHAIK, Leon. 1996. Preface. In Ashton Raggatt McDougall, eds. *RMIT Storey Hall*. Melbourne: Faculty of Environmental Design and Construction, RMIT.