Chapter 39
Quasi-Periodicity in Islamic Geometric Design

Peter Saltzman

Islamic Geometric Design

It is a commonplace assertion that Islamic cultures share the world's oldest and most sophisticated living tradition of geometric ornamental design. Ever since Jules Goury and Owen Jones completed their monumental book on the Alhambra (1842–1845) and Jules Bourgoin published his classic work on Islamic designs (Bourgoin 1879), Western interest in Islamic geometric design has continued unabated.

Symmetry and Its Discontents

In addition to its aesthetic merit, Islamic geometric design is renowned for its mathematical sophistication, constituting the most highly developed chapter in cultural symmetry studies.[1] Dihedral symmetry groups of high order, all seven frieze groups, all seventeen crystallographic groups of plane isometries, and several non-trivial chromatic symmetry groups may be found in abundance in both eastern and western Islamic countries.[2]

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[1] For a general introduction to the cultural applications of symmetry studies, see Crowe and Washburn (1991).

[2] Good introductions to the symmetries of Islamic geometric designs include the delightful book by Fenoll Hach-Alí and Galindo (2003) and the more comprehensive work by Abas and Salman (1995).

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Yet the focus of these symmetry studies has little resonance in the scant historical record documenting the techniques of the Islamic masters, and has limited relevance to the aesthetic complexity of Islamic geometric design. In many cases, one can reconstruct a design as a group orbit of a small motif, but this tells us little about the properties of the design other than its symmetry group.[3] The design within a single periodic unit cell is often quite complex, with symmetry playing a subsidiary role. Grünbaum and Shephard have argued against over reliance on group theory to interpret cultural artefacts, insisting that other mathematical measures of order or disorder—and not just symmetries—are necessary to explain the intrinsic features of the artefacts and better reflect the intentions of those who produced them (Grünbaum and Shepherd 1992).

Responding to this challenge, crystallographers Emil Makovicky and Purificacion Fenoll Hach-Alí have published a series of papers in the *Boletín de la Sociedad Española de Mineralogía* over the past decade or more (Makovicky and Fenoll Hach-Alí 1996, 1997, 1999, 2001), applying a variety of crystallographic structural classification principles to the interpretation of Nasrid designs in Spain. Thus, in addition to symmetries, they have developed informative analyses in terms of crystallographic shear, occupancy of Wyckoff positions, rotation of vortex elements and other crystallographic features that together contribute to the development of a more nuanced grammar of Islamic geometric design. Here, however, we will be concerned not with these crystallographic features, but rather with "quasi-crystallographic" features of certain Islamic designs.

The Presentist Fallacy

Before proceeding further, it is worthwhile to issue a caveat concerning the historical significance of mathematical properties that may attach to certain cultural artefacts. In discussing the elaborate symmetries of Islamic designs, for example, it is tempting to impute mathematical knowledge "ahead of its time" to the architects, artists or others with whom they worked. However, we know little about the medieval artists and scientists who were responsible for these masterful designs—although in the tenth-century text of Abu'l-Wafa' al Buzjani on geometric constructions there is mention of regular meetings between mathematicians and artisans concerning the design of geometric ornament (Özdural 2000). Nevertheless, imputing nascent group theory or a nascent theory of quasi-crystals to medieval artists or mathematicians simply cannot be justified on the basis of the historical record.

Ultimately, attempts to find precursors of contemporary mathematical thought in the cultural production of medieval Islam, or any other period for that matter, fall prey to the *presentist fallacy*—the fallacy of reading the present into the past, or, as

[3] Several works analyse the orbit structure of Islamic designs in this manner, including Grünbaum and Shepherd (1986); Abas and Salman (1995); Ostromoukhov (1998).

Butterfield has expressed it in a different context, the fallacy of using the past as "the ratification if not the glorification of the present" (Butterfield 1931). Therefore, in assessing Islamic geometric designs for their mathematical properties, it is important to keep in mind that we are not addressing the *historical question* of what technical knowledge and concerns motivated the construction of those designs, but rather the *aesthetic question* of how the mathematical properties may help to explain the sensible qualities of the designs themselves.

Islamic Dual Designs

Fortunately, one of the few things we *do* know about the historical practice of Islamic geometric design is the widespread use of grid dualization (sometimes referred to as the "polygonal" or "polygons in contact" technique): the use of an underlying polygonal grid from which the design is derived by a stylized variant of topological dualization.[4] This method is as important to Islamic geometric design as linear perspective is to Renaissance painting. Fig. 39.1, showing two designs from the magnificent Topkapi Scroll (Necipoğlü 1995) of the late fifteenth century, suffices to convey the idea.

In each panel, an underlying polygonal grid is first laid down. Similar points (one or more) are then chosen on each of the edges of the polygons, through which "dual" lines are drawn at specified angles of incidence. The dual lines are then continued (not necessarily linearly) until they meet other dual lines of similar origin.

As can be seen from these examples, grid dualization is a highly versatile technique, with three principal design choices: the type of grid, the method of dualization, and the manner in which the design is rendered. Islamic artists used radial grids, lattice grids and a wide variety of other tiling grids. The dual lines can be drawn through just one edge point—usually the midpoint—or through two or more edge points; and their angles of incidence with the edges can be set at various values. The final design may include the underlying grid along with its dual (an "additive" design) or exclude it; and the lines may be rendered by interweaving (alternating "over" and "under" positions along each line), or with the interlinear regions coloured to form a tiling pattern. Grid dualization has enormous aesthetic value: even quite "ordinary" Archimedean or other tilings have duals that appear far more interesting and dynamic than their progenitors.

According to Jay Bonner, the four most common families of eastern Islamic designs were those whose dual lines were drawn through edge midpoints, with angles of incidence chosen to be 36° ("obtuse"), 54° ("middle") or 72° ("acute"),

[4] Discussions of this technique may be found in Hankin (1925, 1934); Wade (1976); Bonner (2003); Kaplan (2005).

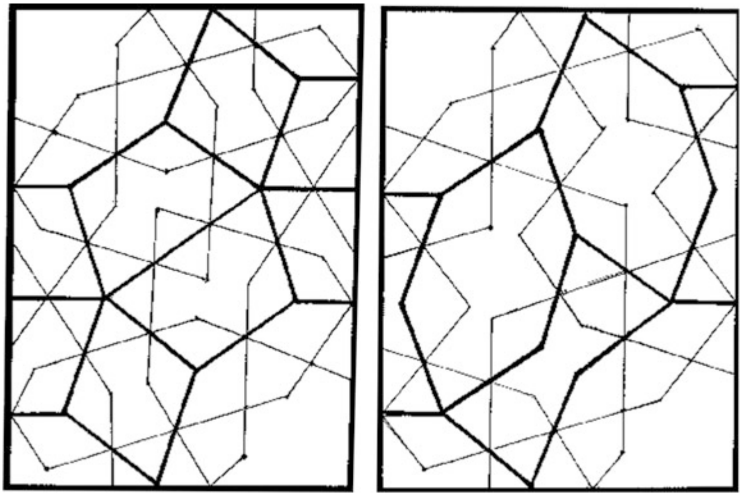


Fig. 39.1 Two dual designs Topkapi Scroll (Necipoğlü 1995). *Thick lines* show the underlying grid. Drawing: author

and those with dual lines drawn through two edge points.[5] Grid dualization (or in Bonner's terminology, the "polygonal technique"),

> is the only method for which there is documented proof that traditional designers used the system widely throughout the Islamic world. The polygonal technique is the only method that allows for the creation of both simple geometric patterns and the most complex compound patterns, often made up of combinations of seemingly irreconcilable symmetries . . . The polygonal technique has the further characteristic of allowing for the creation of all four principal families of Islamic geometric pattern [obtuse, middle, acute and two-point] regularly found throughout the Islamic world (Bonner 2003).

Many writers on Islamic geometric design ignore the grid method, inventing various ad hoc surrogates in its place. For example, Lu and Steinhardt refer to the "direct strapwork method", which they illustrate with a straightedge and compass construction (Lu and Steinhardt 2007a: Figs. 1A–D). They then posit a "paradigm shift" from this direct strap work method to a modular tiling method, whereby a set of five particular *girih* tiles (five of the ten shapes that Bonner includes in what he calls the "5-fold system of geometric pattern generation") decorated with particular dual lines is used to construct a variety of designs. Certainly, it is sometimes useful to develop and construct dual designs in this modular manner, but Lu and Steinhardt's claim that the modular use of these five decorated tiles constituted a "paradigm shift" in medieval Islamic design is unconvincing. Whatever the historical genesis of the *girih* tiles, modular use of decorated tiles is just one facet of dualization, and many other sets of tiles and dual decorations were in constant

[5] An extensive discussion may be found in Bonner (2000).

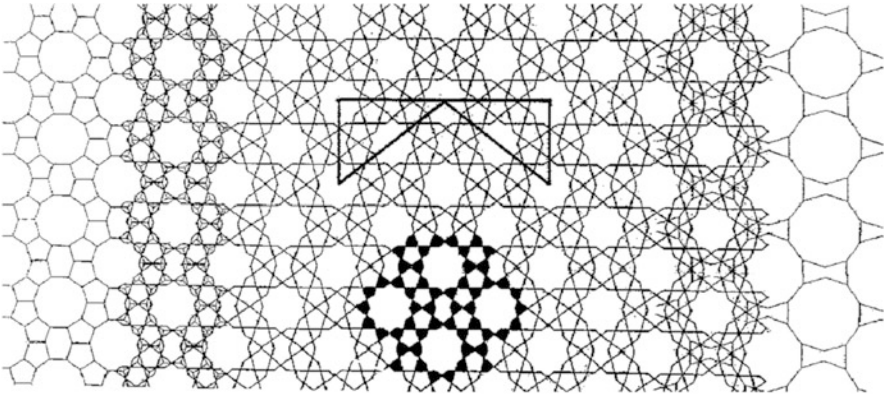


Fig. 39.2 Two different grids which produce the same dual design. The *bracketed triangles* highlight the primary dual design of the spandrel at the Darb-I Imam, Isfahan; the *tiled pattern* below shows one possible rendering of the dual design. Drawings: author, after Kaplan (2005)

use. Moreover, because the same dual design can usually be derived from more than one grid, it is not always clear which tile set was used to generate a given design.

Finally, it should be pointed out that grid dualization is of mathematical interest in its own right. Dual designs inherit many of the symmetry and other properties of the underlying grid, but also introduce certain novel tile shapes, colouring and other combinatorial properties. Bonner (2000) and Kaplan (2005) have noted the fact, already alluded to, that the same dual design can be derived from different grids (the two being connected, in Kaplan's terminology, by a "rosette transform"). For example, the "obtuse" dual of the grid on the left side of Fig. 39.2, consisting of convex pentagons, hexagons and decagons, is the same as the "middle" dual of the grid on the right side of Fig. 39.2, consisting of a decagon packing with non-convex "bowties".

This observation leads to consideration of an equivalence relation on the space of grids (tilings), two grids being "equivalent" if any dual of one is a dual of the other. Equivalent grids behave similarly with respect to symmetry and—to anticipate our main theme—with respect to quasi-periodicity. In fact, grid dualization—in a more modern incarnation due to N. G. de Bruijn—leads directly to quasi-periodicity. These and other mathematical aspects of Islamic grid dualization will be discussed in a sequel to this chapter.

Quasi-Periodicity

Non-Periodic Tilings

One of the frustrations of working with periodic planar grids (i.e., grids with translational symmetries) is the inability to achieve (global) rotational symmetry of orders other than 2, 3, 4 or 6.[6] It is not immediately obvious how to overcome this "crystallographic restriction" with a finite set of tiles, but in 1525, Albrecht Dürer penned an example with just two tiles, a pentagon and a rhomb, achieving global fivefold symmetry through what crystallographers refer to as "pentagonal twinning" (Lück 2000) (Fig. 39.3a). Nearly a century later, Kepler produced another famous example (Fig. 39.3b) (Grünbaum and Shepherd 1987: 52–53, 59). Because they violate the crystallographic restriction, neither of these tilings are periodic.

Prior to Dürer's work, European Renaissance art incorporated only lattice tilings, not pentagonal, non-periodic or other complex tilings. It is certainly conceivable that Dürer derived his interest, and even his tiling, from an Islamic text: there was, after all, an intensive transfer of Arabic scientific works to Europe after the fall of Constantinople to the Ottoman Turks in 1453 (Saliba 2007: 194–195). However, there appear to be no examples of pentagonal twinning in Islamic design, although—as we shall see—Islamic artists did very early develop other—perhaps subtler—examples of fivefold (and tenfold) symmetric designs.

Quasi-Periodic Tilings

In perhaps the most famous example of the "unreasonable effectiveness" of recreational mathematics, in 1974 Roger Penrose constructed a non-periodic tiling of the plane using pentagons, rhombs, pentagrams and partial pentagrams in certain restricted configurations (or satisfying certain "matching rules") (Fig. 39.4); later he found other, essentially equivalent, tilings with fewer tiles (the "kite and dart" and "rhomb" tilings) (Grünbaum and Shepherd 1987: § 10.3, 531–548).

Penrose's tilings, however, are not just non-periodic but are also "quasi-periodic", meaning that any bounded portion of the tiling appears infinitely often in the tiling (and in fact infinitely often in any one of the uncountably many other tilings with the same tile set). Indeed, they have many other properties as well, summed up in the all-encompassing term "quasi-crystalline": the tilings have arbitrarily large bounded fragments with crystallographically forbidden symmetries, they have global "statistical" symmetries, they may be obtained as a projection of slices of higher dimensional lattices, their vertices—considered as complex numbers—possess striking algebraic properties and have "diffractive" Fourier transforms, and more.[7] Thanks largely to the groundbreaking work of N. G. de Bruijn, the class of Penrose tilings has emerged as a mathematical object of great complexity and interest.

After Penrose's discovery, many other families of quasi-periodic tilings of the plane and other spaces were produced, and soon—as with "fractals" and "chaos"—everyone was speaking this new kind of Jourdainian prose. Most famously, in 1985 diffraction

[6] See, for example, Grünbaum and Shepherd (1987: Chap. 1).

[7] Good introductions to this subject may be found in Grünbaum and Shepherd (1987) and Senechal (1995).

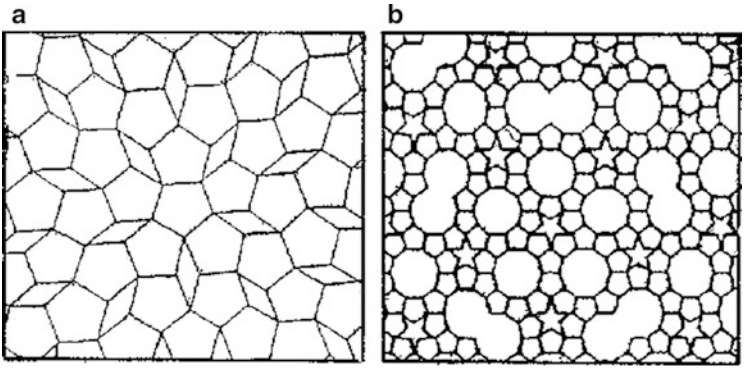


Fig. 39.3 (**a**) Dürer tiling; (**b**) Kepler tiling. Both are non-periodic. Drawing: author

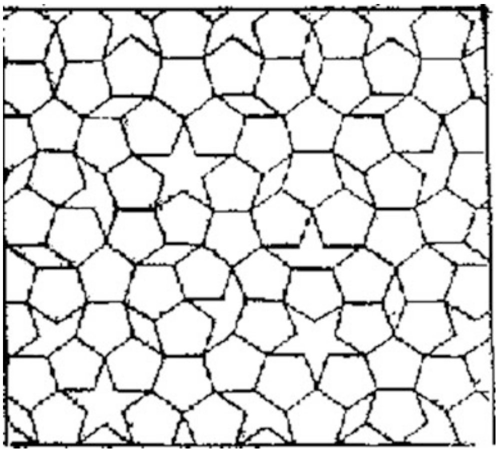


Fig. 39.4 Penrose tiling. Like the Dürer and Kepler tilings, this tiling is non-periodic, but unlike them, it is quasi-periodic. Drawing: author

images of an aluminium manganese alloy produced a quasi-periodic pattern, inaugurating a new era of crystallography and condensed matter physics (Senechal 1995). Of more relevance here is the application of these ideas to aesthetics. Interestingly, in one of his original papers on the subject, Penrose writes about the aesthetic inspiration for his tilings, and proceeds to compare their visual appeal with a design on the window of a mosque in Cairo:

> As one simply stares at the pattern certain regularities seem to jump out. There are a great many regular decagons which tend to overlap at various places . . . Things line up in a surprising way. The appeal of this pattern would seem to have something in common with the appeal of the mosque window . . . (Penrose 1974).

Indeed, if one believes with Birkhoff that aesthetic value may usefully be related to measures of complexity (though not necessarily in the manner Birkhoff promoted), then the mathematical properties of quasi-periodicity summarized above can be used to argue that quasi-periodic designs do indeed have high "aesthetic value".[8] Quasi-periodic patterns occupy an important niche between highly ordered, periodic patterns and highly complex, random ones, and in E. H. Gombrich's words, give us a sense of the delight that "lies somewhere between boredom and confusion" (Gombrich 1979).[9] As Penrose observed, similar sentiments apply to Islamic geometric designs, and surprisingly "Islamic" looking designs can be produced by dualizing quasi-periodic tilings.[10]

Are certain Islamic designs, then, "quasi-periodic"? As it stands, the question is nonsensical. By definition, quasi-periodicity (infinite repetition of bounded fragments) is a property that pertains only to a tiling of infinite extent. So let us define a bounded design as "quasi-periodic" if, first, it can be derived in some systematic manner from a finite tiling (for example, as a dual design or through other kinds of systematic decorations or erasures), and second, the finite tiling from which it is derived can be extended to a quasi-periodic tiling of the entire plane. This definition is consistent with the use of the term "quasi-periodic" or its cognates in the recent literature on Islamic design.

It must be understood, however, that this or any other definition of "quasi-periodicity" in the context of (bounded) designs has several pitfalls. Most important, it is a fact that *any* finite tiling—even one that can be extended to an infinite quasi-periodic tiling—can also be extended to a *periodic* tiling in a variety of ways (Gähler and Rhyner 1986). Most Islamic geometric designs are, in fact, explicitly embedded in periodic frameworks, and arguably *all* were so intended. The definition given here, however, circumvents this rather sterile issue by focusing on a segment of the design (for example, a unit cell), and calling that segment "quasi-periodic" if its underlying grid is a fragment of an (infinite) quasi-periodic tiling. This definition also accords with our interest in the "aesthetic question" (the extent to which the sensible qualities of Islamic designs may be explained, at least in part, by reference to mathematical properties of quasi-periodicity), rather than in the "historical question" (regarding the intentions or motivations of the artists who created the designs and whether they aimed at periodic or non-periodic patterns).

Inflation Tilings

One of the simplest ways to construct a quasi-periodic tiling is to start with a set of tiles that can be inflated and then subdivided into smaller copies of themselves in

[8] An interesting discussion of Birkhoff measures relevant to these remarks is found in Rigau et al. (2007). A study of the Kolmogorov complexity of finite subsets of tilings of the plane is found in Durand et al. (2008).

[9] Similar views are expressed in Arnheim (1971).

[10] Such designs have been produced by Rigby (2006).

such a manner that the process can be iterated. Subject to a variety of alternative sets of conditions, such inflation rules generate quasi-periodic tilings of the plane. As an example, consider the three tiles (decagon, bowtie and long hexagon) with the inflation rules shown in Fig. 39.5. These elegant inflation rules are due to Lu and Steinhardt, who noted that the subdivisions of the decagon and bowtie are implicit in a design from the Darb-I Imam in Isfahan (Lu and Steinhardt 2007a). The tiles themselves are a subset of the *girih* tiles studied by Lu and Steinhardt, and are a variant of the "M2" tile set introduced by Makovicky (1992: n. 30, Fig. 10 and surrounding text) in analysing an earlier design from Margaha, Iran.[11]

Start with an empty decagon centred at the origin, then inflate and subdivide it as shown in Fig. 39.5. This gives the "level 1" tiles, each of which is then inflated and subdivided to obtain the "level 2" tiles. Note that because the borders of the subdivided tiles consist of symmetric half tiles and are all alike, the second level tiles all line up properly. Also note that the level 2 tiles extend the level 1 tiles, which are still centred at the origin. If this process is now iterated, we have a nested sequence of decagons that increase in size; in the limit we obtain an "inflation tiling" of the entire plane. Since each level of the inflation retains 5-fold symmetry, the same is true of the inflation tiling of the entire plane, which is therefore non-periodic. Quasi-periodicity—infinite repetition of each bounded fragment—follows very naturally from the inflation process itself, as any patch of tiles which appears at some stage will be reproduced at each subsequent stage.[12]

Two Designs from Iran

The Gunbad-i Kabud (Maragha)

We are now in a position to look at two designs that have been cited as the prime examples of quasi-periodicity in Islamic art. In 1992 Emil Makovicky analysed the unit cell design on the walls of the Gunbad-i-Kabud (Blue Tomb), an octagonal tower in Maragha, Iran, dating to the late twelfth century (Makovicky 1992) (Fig. 39.6). The unit cell of the primary design spreads over two walls and repeats four times around the tower. The thin lines in Fig. 39.7 show Makovicky's more recent transcription of the primary design over half of its cell unit (the other half is obtained by reflection through the middle vertical line).[13]

[11] Aspects of the M2 tilings later appeared in high resolution transmission electron microscopy of aluminium cobalt nickel alloys; see Cervellino et al. (2002).

[12] For further discussion of inflation tilings, see Senechal (1995: Chap. 5).

[13] The Gunbad-i Kabud has deteriorated, and thus the original design is obscured in parts. Lu and Steinhardt (2007a: Fig. S6) had pointed out that the lower portion of the original reconstruction given in Makovicky (1992) was incorrect. Emil Makovicky has carefully reconstructed the design based on his inspection of the building (Makovicky 2009).

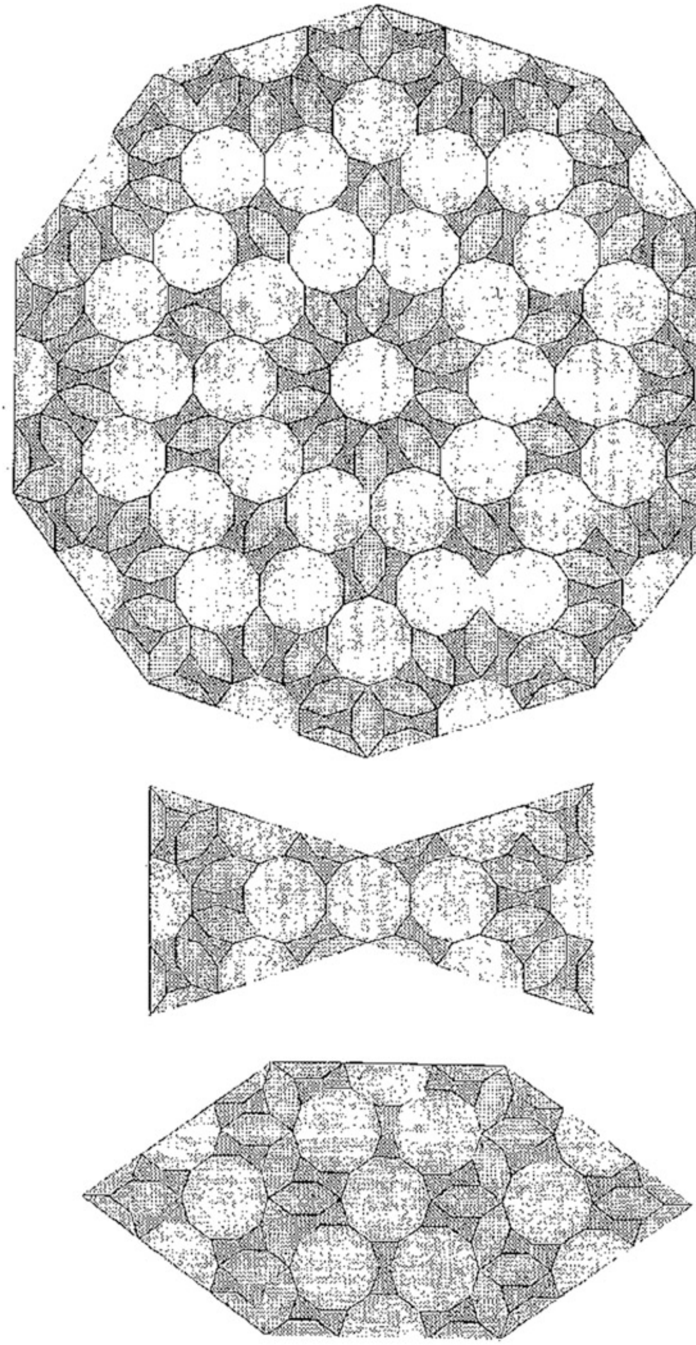


Fig. 39.5 Inflation rules for the decagon, bowtie and long hexagon: Image: Courtesy Peter Lu

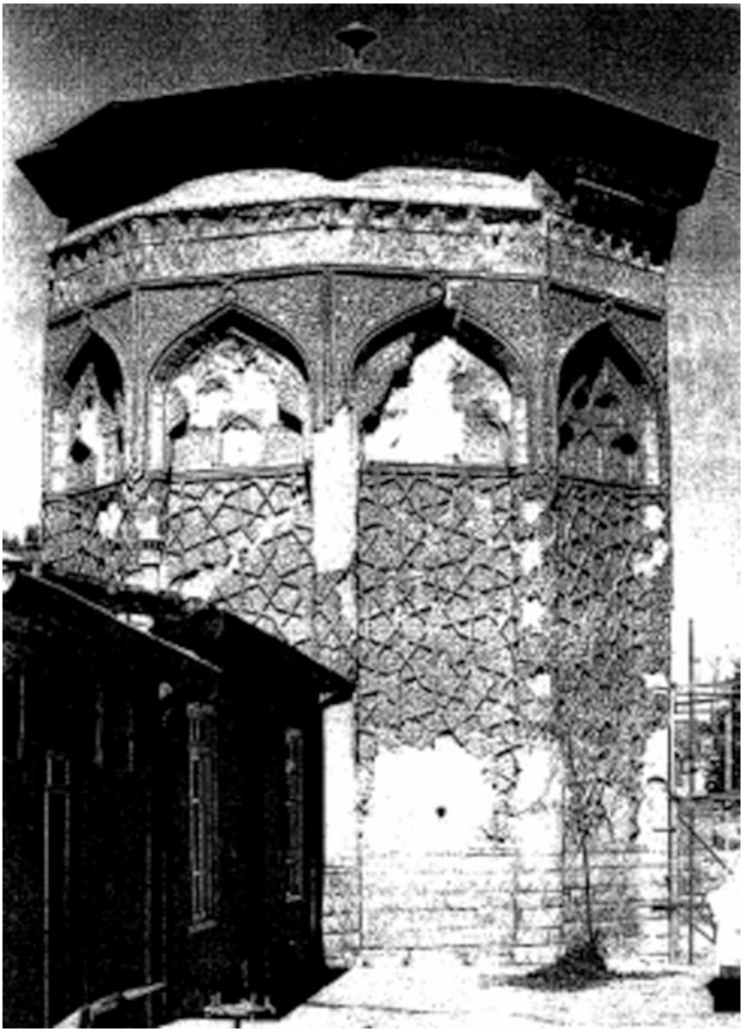


Fig. 39.6 The Gunbad-i Kabud, Maragha. Photo: Courtesy Emil Makovicky

There are a number of ways one might test a design like this for quasi-periodicity. Given the five- and ten-fold symmetry elements, one might, for example, suspect that the Penrose tiles themselves could be used to tile the regions between the primary line elements in a way that could be extended to a Penrose tiling of the entire plane.

This is the approach taken, for example, in a recent paper by Arik and Sancak (2007). Such an analysis is incomplete, however, as generally all that can be done in this regard is to reconstruct the design as far as possible with well-matched Penrose tiles. But a well-matched patch of Penrose tiles is no guaranty of extendability to the

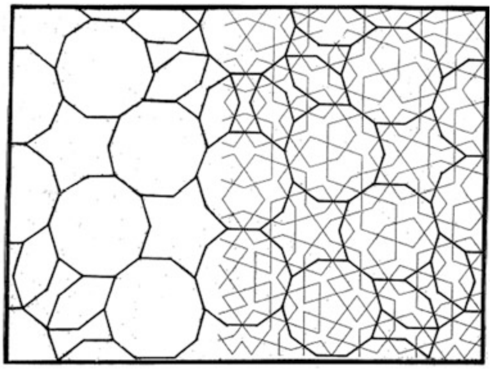


Fig. 39.7 Contours of the underlying grid for the unit cell of the Gunbad-I Kabud, Maragha, with the primary dual design filled in on the *right*. Image: author

entire plane, so without more, this method does not offer a convincing method to establish quasi-periodicity of a design.

A better method—and one implied by the definition of a "quasi-periodic" design here—is taken in Makovicky's paper: find an underlying grid from which the design can be derived, and show that the grid is a fragment of a quasi-periodic tiling. For the grid, Makovicky used the decagons, long hexagons and bowties analysed above, together with five pointed stars. As may be seen from the implied grid in Fig. 39.7, the designers of the Maragha tower decorated these shapes consistently up to rotations of two of the decagons. To show that this grid is a fragment of a quasi-periodic tiling, simply note that it appears in the centre of the subdivided decagon in Fig. 39.5. (The inflation rules in Fig. 39.5 do not aggregate hexagons and bowties into stars, as here, but it is straightforward to incorporate an additional inflation rule for the five-pointed stars.) From the discussion of inflation tilings above, therefore, the grid of Fig. 39.7 is a fragment of a quasi-periodic tiling of the plane, and thus the unit cell of the Gunbad-i Kabud is quasi-periodic in the sense defined here. The fact that the unit cell is repeated four times around the perimeter of the Gunbad-i Kabud, so that the entire design is periodic (with one translational symmetry), does not affect this conclusion.

In addition to the "quasi-periodic" primary design, the Gunbad-i Kabud also features secondary lines within the regions formed by the primary lines. Fig. 39.8 shows a portion of Makovicky's transcription of the complete design with both primary and secondary elements. As may be seen, the secondary design is in effect a two-point dual of the primary design, which itself—as we have seen—is a dual of the implied M2 grid. The result is a masterful example of an "additive" or "double" dual design. The great sophistication and complexity of this design may serve as an appropriate reminder of the fact that Maragha was one of the premier centres of Islamic science: one year after the destruction of Baghdad by the Mongols in 1258, the great astronomer, Nasir al-Din al Tusi—whose key theorem on the "Tusi Couple" was used by Copernicus in *De Revolutionibus*—supervised the construction of the Maragha observatory to which he later brought "the most distinguished company of astronomers ever assembled in one place" (Saliba 2007: 199, 244).

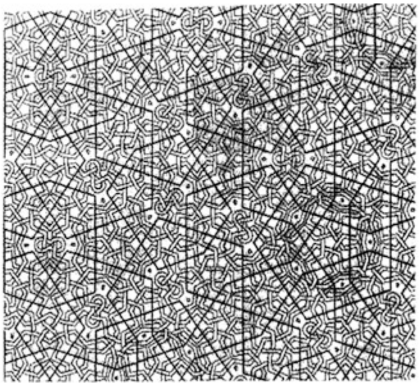


Fig. 39.8 Detail from the *upper right* portion of Fig. 39.7 showing the primary dual design with its (secondary) two-point dual. Image: By permission from Makovicky (2009)

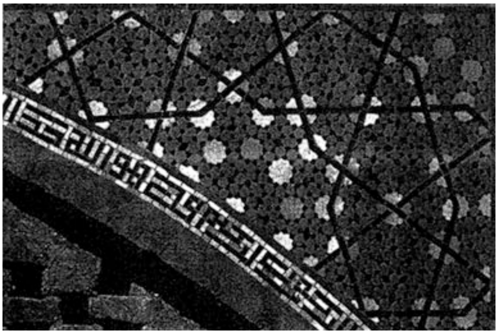


Fig. 39.9 A portion of the right half spandrel at the Darb-I Imam, Isfahan. Image: Courtesy Peter Lu

The Darb-i Imam (Isfahan)

In 2003, Jay Bonner, an architect and leading student and practitioner of Islamic design, published an insightful paper about what he refers to as "self-similar" Islamic designs (2003). In it, he analyses a design found in an arch over a portal at the Darb-i Imam in Isfahan, built in 1453–1454. According to Bonner, this was the work of Sayyid Mahmud-i Naqash, "one of the relatively few architectural ornamentalists in the long history of Islamic art who signed his name to his works." The same design appears also in a spandrel in a different part of the Darb-i Imam, and it is the right half of that spandrel (Fig. 39.9) that Lu and Steinhardt analyse (2007a). For the sake of comparison, I apply Bonner's analysis to the right half spandrel instead of the arch.

Bonner's reconstruction of the Darb-i Imam design proceeds, as did Makovicky's reconstruction of the Maragha design, by imputing an underlying grid—in this case, both a primary and a secondary grid. He first reconstructs the large scale linear design as an obtuse dual of the primary grid on the left side of Fig. 39.2: the two triangles in the centre top of Fig. 39.2 outline the left and right halves of the spandrel with the primary dual lines. Bonner notes that this primary

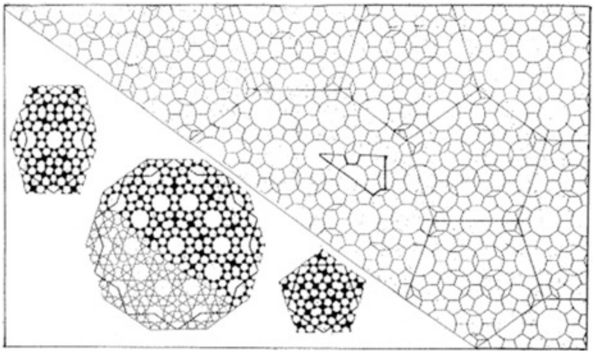
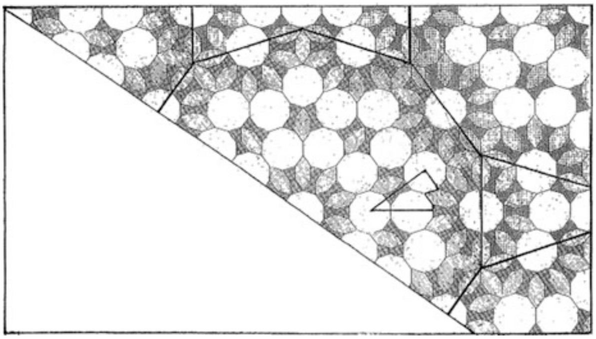


Fig. 39.10 Reconstruction of the primary and secondary grids for the Darb-i Imam spandrel. Individual tiles show the primary (*linear*) and secondary (*tiled*) dual designs; the decagon is partially untiled to show the underlying secondary dual lines. The small region outlined in the central half-decagon is explained in the text. Image: author, after Bonner (2003)

Fig. 39.11 Reconstruction of the primary and secondary grids for the Darb-i Imam spandrel the small region outlined in the central half-decagon is explained in the text. Image: Author, after Lu and Steinhardt (2007a)



grid is the most common fivefold polygonal grid "that can be traced back as far as the year 1000." Next, the primary grid tiles can be subdivided into similar small scale, secondary grid tiles, as shown in Fig. 39.10. The full design then emerges through dualization at both the primary and secondary scales—with the large scale dual design rendered linearly and the small scale dual design rendered in coloured tiles. This procedure reproduces the original design with great accuracy: only a few small tiles in the actual tiling differ from the reconstruction.

Analysing the same design, Lu and Steinhardt impute different, but equivalent, primary and secondary grids—starting with the decagon packing on the right side of Fig. 39.2 rather than the pentagonal tiling on the left (see Fig. 39.11). Furthermore, they show that the secondary grid is a fragment of a quasi-periodic tiling, using the inflation rules discussed above.[14] Thus, with the exception of a few small tiles, the

[14] Although not addressed in Bonner (2003), the subdivisions he uses are in fact part of a set of inflation rules that can also establish the quasi-periodicity of a segment of this design. Those

Darb-i Imam half spandrel design is a fragment of a quasi-periodic tiling of the plane, and so is also quasi-periodic in the sense defined here. Again, the fact that the spandrel design may be embedded in a periodic framework (with two independent translational symmetries)—as is clear from Fig. 39.2—does not affect this conclusion.[15] What is perhaps most compelling about the Darb-i Imam design is the fact that the inflation rules for two of the underlying grid tiles are implicit in the design itself—in effect, a large portion of the design is included in the second level inflation.

It should be emphasized, as Lu and Steinhardt themselves do, that the entire spandrel design is *not* quasi-periodic. Considering the Lu and Steinhardt reconstruction of the right half of the spandrel, the problem is that the large scale grid contains configurations that do not occur in the inflation rules: in particular, the placement of the two large-scale bowties does not occur in any of the inflation rules and cannot occur in the resulting inflation tiling. If, however, the partial bowtie and partial decagon in the lower right corner of the design are removed, what remains is indeed a fragment of the inflation rule for the decagon—as shown in the patch of small tiles outlined in the central half-decagon in Fig. 39.11—and therefore also a fragment of the inflation tiling. The Bonner reconstruction of the design establishes that a different portion of the spandrel is also a quasi-periodic fragment, as shown in the patch of small tiles outlined in the central half-decagon in Fig. 39.10. However, it can be shown that no inflation rules will reproduce the entire design. Lu and Steinhardt attempt to rectify this problem by showing that the entire spandrel can be (approximately) converted to well-matched Penrose tiles, but as already argued, without more this does not establish quasi-periodicity because well-matched Penrose patches are no guarantee of extendability to the entire plane.

Conclusion

As we have seen, the unit cell of the Gunbad-i Kabud design, and a large portion of the Darb-i Imam design, are both quasi-periodic in the sense defined here. In fact, both designs are derived from essentially the same underlying 5-fold symmetric quasi-periodic tiling (M2 or its equivalents.) In both cases, however, the quasi-periodic fragments are embedded in a larger scale periodic framework (in the case of Gunbad-i Kabud, with one translational symmetry) or may be embedded in such a framework (in the case of Darb-i Imam, with two independent translational symmetries).

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inflation rules, however, require a subdivision rule for the narrow rhomb and one for an extra pentagon, and require more complex matching rules than those used by Lu and Steinhardt.

[15] The large scale dual design from Darb-i Imam appears in multiple guises at many other sites with explicit periodic repetition, though it does not do so in the spandrel studied by Lu and Steinhardt. See Arik and Sancak (2007) and Cromwell (2009).

Invoking our earlier caveat, the conclusion that the Gunbad-i Kabud and Darb-i Imam designs are (in whole or in part) "quasi-periodic" is *not* meant to suggest that medieval Islamic artists or scientists understood or were interested in quasi-periodicity in anything like the sense we define the term today. It is clear, however, that these artists developed a remarkable class of tilings to serve as the underlying grids for complex dual designs that are the hallmark of the Islamic geometric aesthetic.

Although we will not survey other examples here, evidence of quasi-periodicity in Islamic designs from Spain and Morocco has been cited as well (Makovicky et al. 1998).[16] Most interestingly, E. Makovicky, and P. Fenoll Hach-Alí have found evidence of the use of a quite different octagonal quasi-periodic design at the Alhambra (1996). Their analysis of quasi-periodicity in that design proceeds directly from a form of dualization rather than from inflation rules.

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