Chapter 20
Calculation of Arches and Domes in Fifteenth-Century Samarkand

Yvonne Dold-Samplonius

Introduction

Samarkand, with Bukhara the principal town of Transoxania, is first found in the accounts of Alexander the Great's campaigns in the east as *Maracanda*. Arab legend makes Alexander founder of the city. In ca. 900 AD the Samanid kingdom was founded, the beginning of a century of great prosperity for Transoxania, such as would only be seen 500 years later with Timur and his immediate successors. Although the capital was moved to Bukhara, Samarkand remained the premier centre of commerce and culture, especially in the popular estimation of the Muslim world. Among its native products, the paper of Samarkand, the manufacture of which had been introduced from China, was especially famous. After surrendering to Genghis Khan in 1220, the city was plundered and many of its inhabitants were deported. For the next 150 years it was but a shadow of its former self. The revival of the town's prosperity began when Timur Lang (1336–1405) became supreme in Transoxania after about 1369 and chose Samarkand as the capital of his ever-growing kingdom. It was in his reign that the art called "Timurid" had its origins. Timur enriched Samarkand with magnificent buildings and made it an international market surpassing Tabriz and Baghdad, at least during his lifetime; he transplanted thither the artists and craftsmen from the towns he conquered. The intellectual revival which characterized the fifteenth century is in part the work of the Timurid sovereigns and princes, many of whom were themselves poets, artists and scholars, and attracted to their courts men of genius. Among the former are Timur's son, Shah Rukh, who

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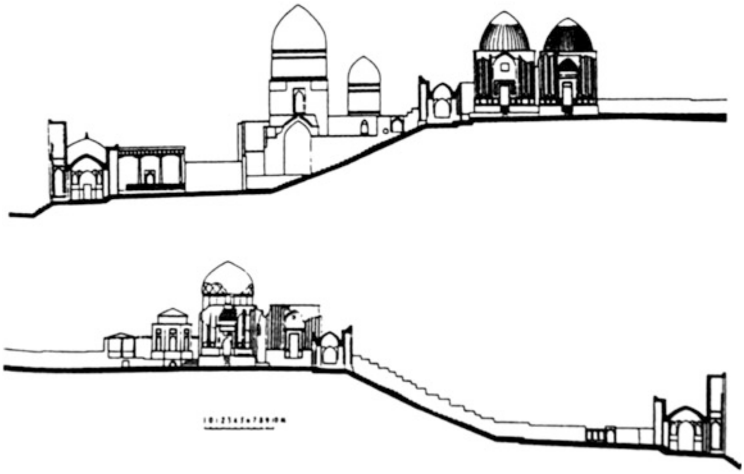

Fig. 20.1 Façades of the mausoleums (qubba) at both sides of the Shah-i Zindeh. Image: author, after Pougatchenkova (1981)

promoted historical studies, and his son, Ulugh Beg (1393–1449), astronomer, poet and theologian, who really made Samarkand what Timur had dreamt of: the centre of Muslim civilization. An artist, Ulugh Beg enriched Samarkand with superb buildings, such as the Timurid burial place, *Shah-iZindeh* (Fig. 20.1), a *madrasa* (high school) and others. A learned mathematician, he could solve the most difficult problems in geometry, but he was above all an astronomer. When Ulugh Beg decided to construct an observatory, he invited Ghiyāth al-Dīn Jamshīd Mas'ūd al-Kāshī to his court—sometime after 1416—as founding director, together with Mo'īn al-Dīn al-Kāshī. Director of the madrasa was Ulugh Beg's mathematics teacher, Qādi Zāde al-Rumi from Bursa (Turkey). The observatory, destroyed in the following century, was regarded in his day as one of the wonders of the world.

Ghiyāih al-Dīn Jamshīd Mas'ūd al-Kāshī ranks among the greatest mathematicians and astronomers in the Islamic world. He was a master computer of extraordinary ability, his wide application of iterative algorithms and his touch in laying out a computation so sure that he controlled the maximum error and maintained a running check at all stages; in short, his talent for optimizing a problem show him to be the first modern mathematician. Al-Kāshī died in June 1429 outside the Samarkand observatory, probably murdered on the command of Ulugh Beg. Two years earlier he had finished the *Key of Arithmetic*, one of his major works. The work is intended for everyday use; al-Kāshī remarks, *I redacted this book and collected in it all that is needed for him who calculates carefully, avoiding tedious length and annoying brevity*. By far the most extensive book is Book IV, *On Measurements*. Its last chapter, *Measuring Structures and Buildings*, is really written for practical purposes:

> The specialists merely spoke about this measuring for the arch and the vault and besides that it was not thought necessary. But I present it among the necessities together with the rest, because it is more often required in measuring buildings than in the rest.

Al-Kāshī uses geometry as a tool for his calculations, not for constructions. Besides arches, vaults and domes (qubba), al-Kāshī calculates here the surface area of a muqarnas (stalactite vault), that is, he establishes approximate values for such a surface. He is able to do so because, although a *muqarnas* is a complex architectural structure, it is based on relatively simple geometrical elements. For the calculation only elementary geometrical rules are used.

Calculation of Arches and Vaults

In this section the terms "arch" and "vault" are interchangeable. The difference between an arch and a vault is that the depth of an arch is not larger than its span, whereas in the case of the vault the depth exceeds the span. The depth of the arch is the distance between the front and back surfaces; that which is called depth in the arch is called length in the vault. Al-Kāshī remarks that,

> The predecessors determined those (i.e., arch and vault) as half a circular hollow cylinder, but we did not see something like it, neither in old nor in new buildings. We have mostly seen ones that are pointed in the middle, and in few cases they are smaller than half a hollow cylinder.

From the Byzantine Empire the Umayyads inherited a system of round arcading that, in the rarest of instances, showed a tendency towards becoming slightly pointed.[1] The innovation of the pointed, or ogival, arch came from the East. Under Umayyad rule the round arch persisted, but developed into the two-centred form showing an increasing tendency towards pointedness. A round arch is struck from a single centre. A pointed arch has more than one centre and can be thought of in its simplest form as being struck from two centres with overlapping arcs; these produce an increasingly pointed arch the further they are moved apart horizontally (Fig. 20.2). In the succeeding two centuries this trend was still apparent, but was complicated by the three- and four-centred arch. Based on this development it is to be expected that in early arithmetic books only hemispherical arches are treated.

First al-Kāshī explains extensively the different elements of an arch and how these are connected, or which part could disappear in a wall. He then gives five methods for drawing the façade of an arch.[2] The first two are three-centred arches. Figure 20.3 shows type 2, a three-centred arch, with point E as a double centre and the other two centres situated in the two lower points Z and H. When the two lower centres move, the arch will change its acuteness. Type 3 (not shown) deals with a

[1] This is essentially Creswell's theory, see Creswell (1960), Warren (1991: 59).

[2] All five constructions are performed on the video "Qubba for al-Kāshī", directed by Yvonne Dold-Samplonius (1995).

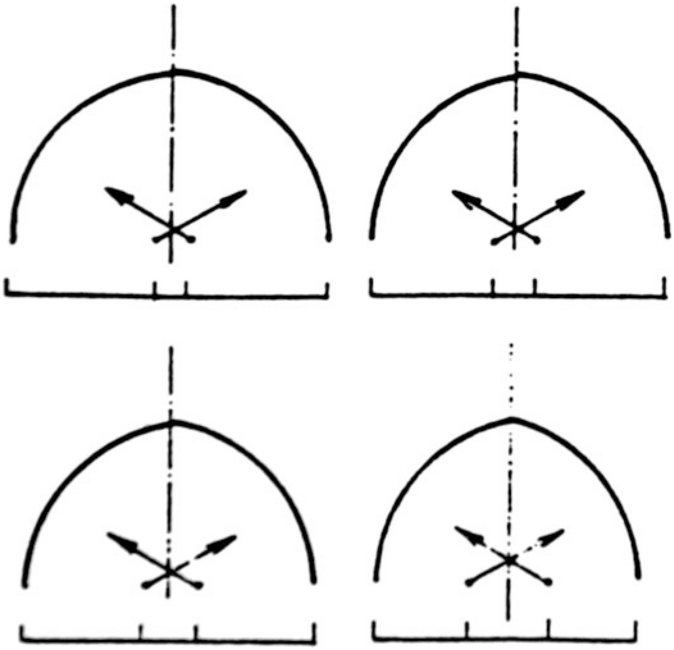

Fig. 20.2 Diagram showing pointed arches formed with constant radii on centres with successive separation of one-tenth, one-seventh, one-fifth, and one-third of a span. Image: author, after Warren (1991)

four-centred arch, which is similar to the three-centred arch except that the centre of the semicircle is split into two points displaced towards the extremes of the span. The greater the displacement, the shallower the profile. Type four (Fig. 20.2, lower right) and five are two-centred. As the second façade was the most common in al-Kāshī's time, he uses it to illustrate his calculation method. This façade is handy, according to al-Kāshī, when you need a span of 5–10, or up to 15, cubits.

Construction of the second façade (Fig. 20.3, taken from the oldest extant manuscript, Tehran, Malek Library 3180/1, with Roman letters added):

(1) Describe a semicircle on AD, the span of the arch;
(2) Extend AD in both ends by the thickness of the arch to the points 1 and M. E is the centre of the semicircle;
(3) Divide this semicircle in four equal parts through the points A, B, C, G, D;
(4) Extend BE and GE by EZ and EH, equal to AC, and by BK and GL, equal to DM, the thickness of the arch;
(5) Describe from the centre E the arcs ML and Kl, from the centre H the arc GT, and from centre Z arc BT;
(6) Connect HT and ZT and extend them by the thickness of the arch to the points O and S;

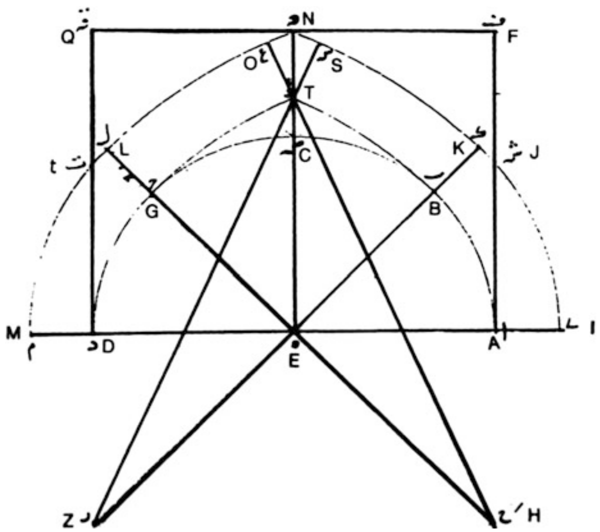

Fig. 20.3 Construction of the second façade. Image: author

(7) Describe arc LO from point H and arc KS from point Z;
(8) Erect the perpendiculars SN and ON on the lines TS and TO.

The sections AK, KT, TN, TL, and LD form together the façade of the arch. When we construct area AFQD with parallel sides and right angles, we obtain the spandrels of the arch.

After al-Kāshī has explained and carried out all five methods for constructing the façade of an arch and has completed the characterization of arch and vault, he continues with surveying them. He explains that he has already found out the relation between some measures of an arch and its span and between some of these and its thickness. He has laid these factors down in a table together with an explanation of the method. These quantities are also transformed into Indian numerals, which he has put down in the table as well. He also informs us about the particulars of finding these quantities. With this table the following parts of the arch (Fig. 20.3) can be found: the interior curve ABTGD, the inner height ET, the upper width TN, and the surface area of the arch as well as the surface area of the concavity, area ABTGDE.

With these values we can then calculate many different parts of the arch. To calculate the *volume* of the arch we proceed in the same way as for round arches: after the surface area of the arch has been found, by means of the table, we multiply this number with the depth of the arch and obtain its volume. Sometimes the arch disappears partly inside a wall and we want to know how much is visible and how large the *segments inside the wall* are, section tDM and section JAI: these segments are calculated by taking the difference of the circle segment MtE and the triangle tDE:

$$\frac{ED}{EM} = \frac{ED}{Et} = \cos \angle tEM, EM = MD + ED,$$
$$tM = \arccos \angle tEM \Rightarrow arctM, arctM \times ME = 2MtE,$$
$$\sin \angle tEM = \frac{tD}{tE} \Rightarrow \sin \angle tEM \times tExDE = 2\Delta tDE,$$
$$2MtE - 2\Delta tDE = 2tDM.$$

When we subtract this amount from the total surface area of the arch we obtain the surface area of the visible part of the arch.

It might be necessary to calculate the *spandrels*, section NQt and section NFJ: in this case we calculate the area AFQD and subtract from this amount the area of the visible part of the arch, calculated above, and the area of the opening of the arch, area ABTGDE, found by means of the table. This yields the surface area of the spandrels. When we multiply this amount with the depth of the spandrels, we obtain the volume of the two spandrels.

Al-Kāshī's book is for practical use, as explained above. Hence he rightly shows how to make life easier by working with rounded off values. More approximation is involved, as the types of arches are more varied than those five given by al-Kāshī. The method for calculating an arch is to select the type of arch nearest to it. Golombek and Wilber (1988: 153–157) have considered existing examples of Timurid arches in the order outlined by al-Kāshī. Examples have been recorded for all but the fifth method, which was, however, certainly common in small windows. In comparing the models described by al-Kāshī with actual examples of Timurid arches, we have to bear in mind that al-Kāshī's purpose was calculating volumes and surfaces, not constructing. This means that an elegant approximation, which leads us to an easy calculation, is the ultimate goal.

Bulatow (1978) has analysed arches from the twelfth to fifteenth centuries in Central Asia and suggests that some pointed arches were constructed as intersections of ellipses. Questioning the reason for using the ellipse, he notes that for spans exceeding 10 m these were easier to construct than four-centred arches. The architects were familiar with the stability of the ellipse as well, for its construction was known from Sassanian examples. According to his analysis, this kind of arch is found in some of the most important Timurid buildings of the period, as in the Gur-i Amir (Fig. 20.4) in Samarkand, and in the mausoleum of Timur Lang and Ulugh Beg, for instance, in the dome, interior niches, arches of the zone of transition, and entrance portal. The same arches have elsewhere been identified as three- and four-centred arches and can be considered as such for all practical purposes (see below).

The section on calculating arches ends in al-Kāshī's *Key of Arithmetic* with the following remark: *I talked a lot about the subject of this section, as this section is very important, and my predecessors did not treat it as they should.*

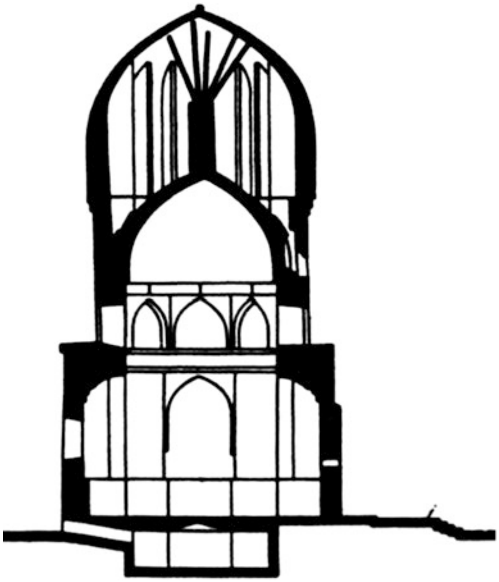

Fig. 20.4 Gur-I Amir, Mausoleum of Timur Lang and Ulugh Beg with a double-shelled dome. Image: author, after Pougatchenkova (1981)

Calculation of Domes

The Arabic word for dome or cupola is *qubba*, plural *qibāb* or *qubab*. By extension *qubba* also means a cupolaed structure or dome-shaped edifice, a domed shrine, a memorial shrine, or *kubba* (especially of a saint). In pre-Islamic times the *qubba* was a small domed leather tent carried by a camel, in which certain tribes kept sacred stones. Also, the dome located in front of the *mihrāb*, a recess in a mosque wall indicating the direction of prayer—as exemplified for instance in the Great Mosques of Damascus, Qayrawan, Cairo, and Cordoba—might have had a special meaning. From the late ninth and the tenth centuries A.D. the building of commemorative structures over certain burial places, especially those of Shī'ī saints, occurred. Throughout the entire Muslim world, all the special names for sepulchral buildings, which vary with country and language as well as with the person interred, come under the generic name of *qubba* (Fig. 20.1). There are basically two types of monuments: the circular, tower-like form, and the often more grandiose square or polygonal type. Both can be covered either by a circular dome or a conical or pyramidal roof. Its original, and later stereotyped, form is a square building covered by a dome. The oldest preserved example is the *Qubba of the Sāmānids* in Bukhara, constructed around 907 but certainly before 943. It consists of a square structure with a large central dome and four small corner ones set over a gallery. As early as the Seljuq period (eleventh century) the construction of domes with double shells was tried, which led to their successful development in Timurid times. The aim of a drum and a double-shelled dome is to give a towering effect to the exterior. A striking example is the *Gur-i Amir* in Samarkand (Fig. 20.4).

As long as domes consisted of cones or sphere-segments, their mensuration was automatically included when measuring solids, and the *qubba* did not have to be mentioned *per se*. At present it seems that, with the exception of al-Kāshī, only *qubbas* in the form of hollow hemispheres have been considered in arithmetic manuals. A hemispherical *qubba* is assumed to consist of the solid shell between two concentric, parallel hemispheres. In praxis, the inner and outer surfaces of the shell are never really parallel, because in the lower part, up to an angle of 61°, the pressure exerts a pulling force in the upper part.

When the inner and outer diameters of a hemisphere *qubba* are known, its volume and the inner and outer surface areas can be calculated as follows (Dold-Samplonius 1998). We know how to compute the surface area of a sphere with diameter equal to the outer diameter. Half of this amount is the outer surface area of the qubba. In the same way, the inner surface area of the *qubba* can be computed. To calculate the volume of the *qubba*, we compute the volumes of the outer and the inner sphere and take each time its half. The difference between these two amounts is the volume of the *qubba*.

The formulas for computing the area and the volume of a sphere are:

$$\mathrm{Area(sphere)} = (2\mathrm{r})^2 \times \pi$$

and

$$\mathrm{Volume(sphere)} = \mathrm{Area\ (sphere)} \times \mathrm{r}/3, 2\mathrm{r} = \mathrm{diameter}$$

Al-Kāshī does not carry out the calculation of the hemisphere *qubba*, but refers to his calculation of the sphere. There he uses, as expected, the right formulas for area and volume expressing л as the ratio between the circumference and the diameter of a circle. He distinguishes the following categories of *qubba*:

> They occur either in the form of a hollow hemisphere, or in the form of a segment of a hollow sphere, or in the form of a polygonal cone, or in the form, which arises by imagining the rotation of the façade of the arch, i.e. of an arch as mentioned in Section 1, around the line of its elevation, that is the line, which connects its upper limit with the middle of the line between its fundaments.

After remarking that the first three categories have already been dealt with earlier in the book, he indicates how to calculate the complicated type of *qubba*, i.e., the dome created by rotating an arch around its vertical axis. The method is illustrated in Fig. 20.5. The dome is divided in parallel slices by drawing circles from the axis on its surface. These circles have to be so close that the curves between two of them equal the corresponding chords. Seven or eight of these circles should normally suffice, according to al-Kāshī. In this way the dome is cut up in a cone and several frusta. We first measure all the circles on the surface of the dome. The next step is to measure the distance from the apex of the dome to the nearest

Fig. 20.5 Qubba, sliced in eight slices. Image: author

circle, i.e. the chord (Fig. 20.5: the segment c) equalling the curve on the circle. By multiplying half the circumference of the nearest circle by this amount we obtain the surface area of the cone. Thereupon we multiply half the sum of every two neighbouring circles by their distance to obtain the surface area of all frusta. The sum of these products yields the surface area of the qubba.

To obtain the volume of the *qubba*, which is a hollow solid, we first measure the volumes of the cone and the frusta, which fill the outer surface of the shell, and add these. From this sum we then subtract the sum of the volumes of the cone and the frusta filling the inside of the shell. The difference between these two sums is the volume of the *qubba*, as we have seen before in the case of the hemispherical *qubba*.

This general method is applied to a *qubba* based on the fourth type of arch, i.e. a two centred arch with its span divided by the two centres in three equal parts (Fig. 20.2, lower right). For practical application just the rules are enough. Hence, "to simplify the procedure", al-Kāshī gives only the calculation method but does not explain how he arrived at these results:

> To obtain the surface area of the interior of the dome we have to multiply the square of the diameter of the base of the hollow (= inner) dome by $1°46'32''$, if we compute sexagesimally, or by 1.775, computing in the decimal system. When we multiply the square of the diameter of the base of the (outer shell of the) dome by the same number, we obtain the exterior curved surface area of the dome, as the inner and outer surfaces are supposed to be parallel to each other. When we multiply the cube of the diameter of the base of the hollow dome as well as the cube of the diameter of the base of the dome by $0°18'23''$, in the sexagesimal system, or by 0.306, in the decimal system, and take the difference of these two products, we obtain the volume of the hollow qubba.

In both cases the results were checked by modern methods and we found that the factors are accurate and that the dome has been cut up in eight slices.

Could this factor be used for all kind of domes, with more or less deviation? Al-Kāshī makes no mention of the elliptical profile for either arches or domes. There are a number of domes for which the profile may be interpreted as the intersection of reflected elliptical curves. These include some of the most important buildings of the period, as the Gur-i Amir in Samarkand. Bulatow has demonstrated that the dome of the Gur-i Amir was probably designed using a pair of foci and string. However, looking at his analysis of the Gur-i Amir (Fig. 20.6) we

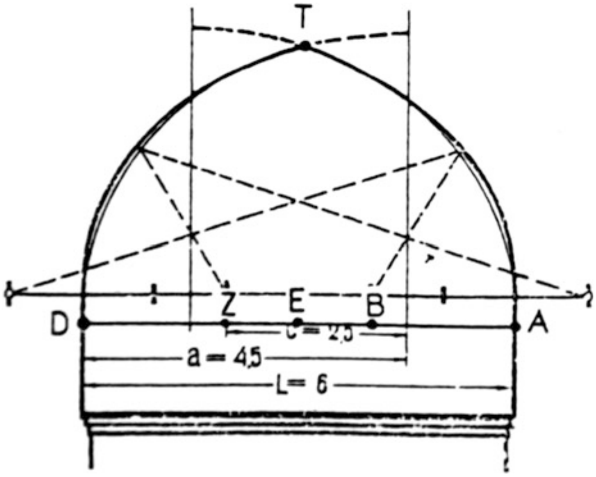

Fig. 20.6 Dome of the Gur-i Amir. Image: author, after the analysis of Bulatow (1978) with additions

see that this dome could also have been originated by the fourth method: with line AD as the span and the points B and Z dividing the span in three equal parts, we obtain the circle segments just inside the curve drawn by Bulatow. The difference between the two curves lies within the margin of error accepted by modern architects. It seems therefore that al- Kāshī's factors can also be used for calculating elliptical domes.

Conclusion

In medieval Italy it was common practice to pay artisans according to the surface area they had completed. Also in seventeenth-century Safavid Iran architects were paid a percentage on each building based on the cubit measure of the height and thickness of the walls:

> The Persians determine the price for masons on the basis of the height and thickness of walls, which they measure by the cubit, like cloth. The king imposes no tax on the sale of buildings, but the Master Architect, that is Chief of Masons, takes two percent of inheritance allotments and sales. This officer also has a right to five percent on all edifices commissioned by the king. These are appraised when they are completed and the Master Architect, who has directed the construction receives as his right and salary as much as five percent of the construction cost of each edifice (Necipoğlu 1995: 44, 159).

The same custom seems to have existed in the Arab world. It is also useful to know, more or less, how much material is needed like gold for gilding, bricks for construction or paint and such things. Payment per cubit was common in Ottoman architectural practice where a team of architects and surveyors had to make cost estimates of projected buildings and supply preliminary drawings for various options. In addition to facilitating estimates of wages and building materials before construction, al-Kāshī's formulas may also have been used in appraising the price of a building after its completion. His sophisticated formulas were, like the simple formulas found in the Arithmetic Books, useful for everyday life. This was al- Kāshī's objective for writing his *Key of Arithmetic*.

Biography Yvonne Dold-Samplonius studied mathematics and Arabic at the University of Amsterdam, specializing in the history of Islamic mathematics. She wrote her thesis with Prof. Bruins, Amsterdam and Prof. Juan Vernet, Barcelona. The academic year 1966/1967 she spent at Harvard studying under Prof. Murdoch. She has published about 40 papers on the history of mathematics. In recent years her interest has shifted to mathematics in Islamic architecture from a historic point of view. Under her supervision two videos concerning this subject, "Qubba for al-Kashi", on arches and vaults, and "Magic of Muqarnas" have been produced at the IWR (Interdisciplinary Center for Scientific Computing), Heidelberg, where she is an associated member. She is an effective member of the International Academy of History of Science.

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