Chapter 6
H^∞ Well-Posedness for Degenerate p-Evolution Models of Higher Order with Time-Dependent Coefficients

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Abstract In this paper we deal with time dependent p-evolution Cauchy problems. The differential operators have characteristics of variable multiplicity. We consider a degeneracy only in $t = 0$. We shall prove a well-posedness result in the scale of Sobolev spaces using a C^1-approach. In this way we will prove H^∞ well-posedness with an (at most) finite loss of regularity.

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6.1 Introduction

In this paper we are interested in well-posedness results in Sobolev spaces for p-evolution Cauchy problems. Starting point of our considerations is the monograph [11]. The author gives a well-posedness result for the Cauchy problem for 1-evolution (hyperbolic) equation

$$D_t^l u - \sum_{\substack{0\le j+k\le l\\ j<l}} a_{j,k}(t,x) D_x^k D_t^j u = 0, \qquad (1)$$
$$D_t^m u(0,x) = u_m(x) \quad \text{for } m = 0,\dots,l-1 \text{ and } l \ge 2.$$

For analytic functions $a_{j,k}(t,x)$ the Cauchy problem is H^∞ well-posed. In other words, for data $u_m \in H^s$ with $m = 0,\dots,l-1$ there exists a unique solution

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$u \in C([0,T], H^{s-s_0}) \cap C^1([0,T], H^{s-s_0-p}) \cap \ldots \cap C^{l-1}([0,T], H^{s-s_0-(l-1)p})$ for some s, s_0 and $T > 0$. Cauchy problem (1) is a special case of the Cauchy problem for the p-evolution equation introduced by Petrowsky, see [13]. It can be written as follows:

$$D_t^l u - \sum_{\substack{j+k/p=l \\ j<l}} a_{j,k}(t) D_x^k D_t^j u - \sum_{0 \le j+k/p<l} a_{j,k}(t,x) D_x^k D_t^j u = 0, \tag{2}$$
$$D_t^m u(0,x) = u_m(x) \quad \text{for } m = 0, \ldots, l-1 \text{ and } l \ge 2.$$

For this Cauchy problem there exist only a few results about well-posedness in scales of Sobolev spaces. But as stated in [11] the Cauchy problem is no longer of Cauchy-Kovalevskaya type. In [3] the authors proved H^∞ well-posedness for Cauchy problem (2) with $l = 2$ and $p = 2$ and for complex coefficients. They had to assume some conditions on the coefficients $a_{j,k}(t,x)$. At the moment it is important that they had to pose decay conditions on the imaginary part of $a_{j,k}(t,x)$ as x tends to infinity. Furthermore, they posed decay conditions on the derivatives of some of the real parts of $a_{j,k}(t,x)$. In this paper we do not have such decay conditions for the coefficients in the spatial variables. So we want to consider Cauchy problem (2) with real coefficients. Now also from [5] we see that we have to pose decay conditions with respect to x for some t or x-derivatives of the coefficients $a_{j,k}(t,x)$ even if they are real. In this paper we are not interested to take into consideration this effect. For this reason we will restrict ourselves to the Cauchy problem

$$D_t^l u - \sum_{\substack{0 \le j+k/p \le l \\ j<l}} a_{j,k}(t) D_x^k D_t^j u = 0, \tag{3}$$
$$D_t^m u(0,x) = u_m(x) \quad \text{for } m = 0, \ldots, l-1 \text{ and } l \ge 2$$

with real-valued time dependent coefficients in the 'extended principle part', see (5). For a statement about well-posedness we need a certain regularity of the coefficients and, furthermore, separated characteristic roots. Our goal is to consider coefficients which vanish at $t = 0$. So the roots can only be expected to be separated on $(0, T]$. We will use the so-called C^1-approach and pose assumptions on the coefficients and their first derivatives to prove H^∞ well-posedness. This is an at most finite loss of derivatives in scales of Sobolev spaces. We are going to prove a statement of the following type.

"*We consider Cauchy problem (3) under assumptions on the coefficients $a_{j,k} = a_{j,k}(t)$ and their first derivatives. Furthermore, we pose assumptions on the characteristic roots of the problem. Then for initial data u_m with $m = 0, \ldots, l-1$ given in certain scales of Sobolev spaces there exists in some evolution spaces a unique solution u of (3). The solution has an (at most) finite loss of derivatives in comparison with the given regularity of the data (see Theorem 1).*"

6.2 General Notation and Main Theorem

In this section we will give the precise assumptions we need to prove our main result. Different parts of the operator given in (2) will play a different role. In order to emphasize this distinction for the special case (3) we split the coefficients into the following three groups.

The *principal part in the sense of Petrowsky* of the p-evolution operator for (3) is given by

$$D_t^l - \sum_{\substack{j+k/p=l \\ j<l}} a_{j,k}(t) D_x^k D_t^j. \tag{4}$$

The *extended principal part* for (3) is given by

$$D_t^l - \sum_{\substack{l-1<j+k/p\le l \\ j<l}} a_{j,k}(t) D_x^k D_t^j \tag{5}$$

and, finally, the *terms of lower order* for (3) are given by

$$- \sum_{0\le j+k/p\le l-1} a_{j,k}(t) D_x^k D_t^j. \tag{6}$$

Furthermore, the *terms of Levi condition* for (3) are given by

$$- \sum_{j\le l-1} a_{j,(l-1-j)p}(t) D_x^{(l-1-j)p} D_t^j. \tag{7}$$

Remark 1 Due to the Lax-Mizohata condition for H^∞ well-posedness for p-evolution equations from [12] the coefficients of the principal part in the sense of Petrowsky have to be real. If we restrict ourselves to time-dependent coefficients, then also the coefficients of the extended principle part have to be real. If we would assume complex-valued coefficients, then we need some decay behavior in x for the imaginary parts. Our assumptions for the coefficients of the extended principal part guarantee a dominance condition (see Lemma 2). The coefficients of the terms of lower order are allowed to be complex-valued.

To get a better feeling for this classification we introduce Table 6.1.

In the following we pose assumptions for the coefficients of our starting equation. We introduce the shape function $\lambda(t)$, which satisfies the assumptions

$$\begin{aligned} &\lambda(0)=0, \qquad \lambda'(t)>0 \quad \text{for } t>0, \\ &d_0\frac{\lambda(t)}{\Lambda(t)} \le \frac{\lambda'(t)}{\lambda(t)} \le d_1\frac{\lambda(t)}{\Lambda(t)}, \quad 0<d_0. \end{aligned} \tag{8}$$

As mentioned before we can see that our strategy is to assume only a degeneracy in $t=0$. Let us give some examples. A shape function of finite degeneracy is given

Table 6.1 Classification of coefficients

$a_{0,lp}$	$a_{0,lp-1}\dots a_{0,(l-1)p+1}$	$a_{0,(l-1)p}$	$a_{0,(l-1)p-1}\dots a_{0,0}$
$a_{1,(l-1)p}$	$a_{1,(l-1)p-1}\dots a_{1,(l-2)p+1}$	$a_{1,(l-2)p}$	$a_{1,(l-2)p-1}\dots a_{1,0}$
$a_{2,(l-2)p}$	$\ddots$		
$\vdots$			
$a_{l-2,lp}$	$a_{l-2,2p-1}\dots a_{l-2,p+1}$	$a_{l-2,p}$	$a_{l-2,p-1}\dots a_{l-2,0}$
$a_{l-1,lp}$	$a_{l-1,p-1}\dots a_{l-1,1}$	$a_{l-1,0}$	
Petrowsky principal part		Terms of Levi size	
Extended principal part		Lower order terms	
Real coefficients		Complex coefficients	

by $\lambda(t) = t^{\beta}$ with $\beta > 0$. An example for infinite or exponential type degeneracy is given by $\lambda(t) = t^{-2}\exp(-t^{-1})$ and for super exponential type degeneracy by $\lambda(t) = \frac{\exp(-\exp^{[n]}(1)/(t))}{t^2}\prod_{k=1}^{n}\exp^{[k]}\frac{1}{t}$. For a logarithmic type degeneracy we do not have any example which satisfies (8). With these examples for the degeneracy in $t = 0$ in mind we want to formulate assumptions on the roots of the principal part in the sense of Petrowsky. The roots are defined as solutions of the characteristic equation

$$\widehat{\tau}^l - \sum_{\substack{j+k/p=l\\ j<l}} a_{j,k}(t)\xi^k\widehat{\tau}^j = 0. \tag{9}$$

We assume that the roots are real and, furthermore, that they satisfy the following conditions:

$$\begin{aligned} &\text{separation condition:} && \left|\widehat{\tau}_i(t,\xi) - \widehat{\tau}_j(t,\xi)\right| \geq C\lambda(t)|\xi|^p \quad \text{for } i \neq j,\\ &\text{control of oscillations:} && \left|D_t^m D_\xi^k \widehat{\tau}_j(t,\xi)\right| \leq C_m\lambda(t)|\xi|^{p-k}\left(\frac{\lambda(t)}{\Lambda(t)}\right)^m, \end{aligned} \tag{10}$$

for all $(t,\xi) \in (0,T] \times \mathbb{R}$ with $i, j = 1, 2, \dots, l$, $k \in \mathbb{N}$ and $m = 0, 1$, where $\Lambda(t) = \int_0^t \lambda(t)\mathrm{d}t$ and $\Lambda(t) < 1$. In the following statement we are only interested to describe the oscillation condition by the coefficients of the operator.

Lemma 1 *The conditions* (10) *are equivalent to the following behavior of the coefficients of the principal part in the sense of Petrowsky*:

$$\begin{aligned} &\textit{separation condition}: && \left|\widehat{\tau}_i(t,\xi) - \widehat{\tau}_j(t,\xi)\right| \geq C\lambda(t)|\xi|^p \quad \textit{for } i \neq j,\\ &\textit{control of oscillations}: && \left|D_t^m a_{j,p(l-j)}(t)\right| \leq C_m\lambda(t)^{l-j}\left(\frac{\lambda(t)}{\Lambda(t)}\right)^m \end{aligned} \tag{11}$$

for $m = 0, 1$.

Proof Using Vieta's formulas we get the following:

$$D_t^m D_\xi^\beta \sum_{i_1<\ldots<i_{l-j}} \widehat{\tau}_{i_1}(t,\xi)\ldots\widehat{\tau}_{i_{l-j}}(t,\xi) = (-1)^j D_t^m D_\xi^\beta a_{j,k}(t)\xi^k$$

for $k = p(l-j)$ and $j = 0,\ldots,l-1$. This already yields the control of oscillations of (11) if we assume (10). To prove the other direction of the statement we get the following system from Vieta's formulas:

$$\underbrace{\begin{pmatrix} 1 & \ldots & 1 \\ \sum_{j\neq 1}\widehat{\tau}_j & \ldots & \sum_{j\neq l}\widehat{\tau}_j \\ \vdots & \vdots & \vdots \\ \prod_{j\neq 1}\widehat{\tau}_j & \ldots & \prod_{j\neq l}\widehat{\tau}_j \end{pmatrix}}_{=:A} \begin{pmatrix} D_t\widehat{\tau}_1(t,\xi) \\ D_t\widehat{\tau}_2(t,\xi) \\ \vdots \\ D_t\widehat{\tau}_l(t,\xi) \end{pmatrix} = \begin{pmatrix} (-1)^{l-1}D_t a_{l-1,p}(t)\xi^p \\ (-1)^{l-2}D_t a_{l-2,2p}(t)\xi^{2p} \\ \vdots \\ D_t a_{0,lp}(t)\xi^{lp} \end{pmatrix}.$$

This can be solved for the derivatives $D_t\widehat{\tau}_k$ of the roots of the principal symbol in the sense of Petrowsky if the matrix A is invertible. The determinant of the matrix is given by

$$\det A = \prod_{k<j}(\widehat{\tau}_k - \widehat{\tau}_j).$$

Due to the separation condition the matrix is invertible and we can control the oscillations of (10) from the assumptions (11). This completes our proof. □

For all coefficients we assume

$$\left|a_{j,k}(t)\right| \leq C\lambda(t)^{l-j}\left(\frac{|\log\Lambda(t)|}{\Lambda(t)}\right)^{l-j-k/p}. \tag{12}$$

This coincides with the behavior of the coefficients of the principal part in the sense of Petrowsky coming from the assumptions on the roots. For the coefficients of the extended principal part and for the real part of the coefficients of Levi size we assume additionally

$$\left|D_t a_{j,k}(t)\right| \leq C\lambda(t)^{l-j}\left(\frac{|\log\Lambda(t)|}{\Lambda(t)}\right)^{l-j-k/p}\left(\frac{\lambda(t)}{\Lambda(t)}\right). \tag{13}$$

For some of the coefficients of the lower order terms we need additional assumptions.

- For $a_{j,0}(t)$ with $0 \leq j < l$ we assume

$$a_{j,0}(t) \in L^1(0,T). \tag{14}$$

- For $a_{j,k}(t)$ with $l-1-j-\frac{k}{p} \geq d_0(l-1-j)$ and $k \neq 0$ we assume

$$a_{j,k}(t) \in B[0,T]. \tag{15}$$

The space $B[0,T]$ is the space of all bounded functions on $[0,T]$.

- For the terms of Levi size we assume the Levi conditions

$$\left|\Im a_{l-1-k/p,k}(t)\right| \le C\lambda(t)^{k/p}\left(\frac{\lambda(t)}{\Lambda(t)}\right). \tag{16}$$

Remark 2 We want to remark that our goal is to assume $d_0 > 0$. If we would assume $d_0 > \frac{l-1}{l}$ as in [14] instead, then we can omit assumptions (14) and (15). But, as a consequence, this narrows the set of admissible shape functions.

Theorem 1 *Let us consider the Cauchy problem* (3) *under the assumptions* (8) *and* (10) *to* (16). *Then there exists non-negative constants* s_0 *and* C *such that for all initial data* $u_m \in H^{s-mp}(\mathbb{R})$, $m = 0, \dots, l-1$ *there is a unique solution* $u \in C([0,T], H^{s-s_0}(\mathbb{R})) \cap C^1([0,T], H^{s-s_0-p}(\mathbb{R})) \cap \dots \cap C^{l-1}([0,T], H^{s-s_0-(l-1)p}(\mathbb{R}))$. *An a priori estimate is given by*

$$\left\| D_t^m u(t,\cdot)\right\|_{H^{s-s_0-mp}} \le C\left(\|u_0\|_{H^s} + \dots + \|u_{l-1}\|_{H^{s-(l-1)p}}\right)$$

for $m = 0, \dots, l-1$.

Remark 3 Let us give some comments to the assumptions (12) to (16). One can only understand assumption (12) together with assumption (14) and (15). For the real parts of Levi size coefficients we can allow an additional $\log \Lambda(t)$ term in opposite to the imaginary parts. This was already observed in [14], where among other things the conditions (12), (13) and (16) are proposed for $p = 1$.

The model equation with $l = p = 2$ was studied in [1] for a finite degeneracy. Our conditions (12), (13) and (16) are in line with the assumptions which are used there apart from the fact that no $\log \Lambda(t)$ term is allowed.

Remark 4 We have an *at most* finite loss of derivatives but we can not expect optimality of the statement. The at most difference of regularity between the initial data and the solution is given by s_0. This yields H^∞ well-posedness. Using the C^1-approach implies an *at most* finite loss of derivatives but it does not explain if the loss really appears. In opposite, if we apply C^2-approach, then we are able to study the precise loss of regularity and to show its optimality [7].

6.3 Proof

We can apply partial Fourier transformation and get an ordinary differential equation with parameter ξ. We divide the extended phase space into a pseudo-differential and an evolution zone. Then, we consider in each one different micro-energies. The goal is to get a priori estimates for the micro-energies in each zone. Our techniques to get these estimates differ from the pseudo-differential to the evolution zone.

6.3.1 First Step of the Proof

At first we apply the partial Fourier transform with respect to x and obtain

$$D_t^l v(t,\xi) - \sum_{\substack{0\le j+k/p\le l\\ j<l}} a_{j,k}(t)\xi^k D_t^j v = 0, \qquad (17)$$
$$\text{with } v = F_{x\to\xi}(u), \quad v_m = F_{x\to\xi}(u_m) \text{ for } m = 0,\dots,l-1.$$

6.3.2 Symbol Classes and Zones

By analogy with [14] we introduce the following zones:

Definition 1 (Zones) We divide the extended phase space into two zones. We need the pseudo-differential zone $Z_{pd}(M,N)$ and the $Z_{evo}(M,N)$. They are defined as follows:

$$Z_{pd}(M,N) = \Big\{(t,\xi)\in[0,T]\times\Big\{|\xi|\ge M>1\Big\} : \Lambda(t)|\xi|^p \le N\Big|\log\Lambda(t)\Big|\Big\},$$
$$Z_{evo}(M,N) = \Big\{(t,\xi)\in[0,T]\times\Big\{|\xi|\ge M>1\Big\} : \Lambda(t)|\xi|^p \ge N\Big|\log\Lambda(t)\Big|\Big\}.$$

And accordingly, we define t_ξ to be the solution of $\Lambda(t)|\xi|^p = N|\log\Lambda(t)|$.

Definition 2 (Symbols in $Z_{evo}(M,N)$) By $S_n\{l_1,l_2,l_3,l_4\}$ we denote the class of all amplitudes $a = a(t,\xi)\in C(Z_{evo}(M,N))$ satisfying for all $k, j\in\mathbb{N}$ with $j\le n$ the estimates

$$\Big|D_t^j D_\xi^k a(t,\xi)\Big| \le C_{j,k}|\xi|^{pl_1-k}\lambda(t)^{l_2}\left(\frac{\lambda(t)}{\Lambda(t)}\right)^{l_3+j}\left(\frac{\log(1/\Lambda(t))}{\Lambda(t)}\right)^{l_4}.$$

These symbol classes satisfy the following properties:

$$a\in S_n\{l_1,l_2,l_3,l_4\} \to D_\xi^k a \in S_n\left\{l_1-\frac{k}{p},l_2,l_3,l_4\right\},$$
$$a\in S_n\{l_1,l_2,l_3,l_4\} \to D_t^k a \in S_{n-k}\{l_1,l_2,l_3+k,l_4\} \quad \text{if } k\le n,$$
$$a\in S_n\{l_1,l_2,l_3,l_4\}, \widetilde{a}\in S_{\widetilde{n}}\{\widetilde{l}_1,\widetilde{l}_2,\widetilde{l}_3,\widetilde{l}_4\}$$
$$\to a\cdot\widetilde{a}\in S_{\min(n,\widetilde{n})}\{l_1+\widetilde{l}_1,l_2+\widetilde{l}_2,l_3+\widetilde{l}_3,l_4+\widetilde{l}_4\},$$

and generate symbol hierarchies

$$S_n\{l_1,l_2,l_3,l_4\} \subset S_{n-1}\{l_1,l_2,l_3,l_4\},$$
$$S_n\{l_1,l_2,l_3+k,l_4\} \subset S_n\{l_1,l_2+k,l_3,l_4+k\} \quad \text{for } k\ge 0,$$
$$S_n\{l_1,l_2,l_3,l_4\} \subset S_n\{l_1+k,l_2,l_3,l_4-k\} \quad \text{for } k\ge 0.$$

Our strategy is to have a dominance condition for the extended principal part, that is, the principal part in the sense of Petrowsky dominates the other terms of the extended principal part. By assumption (12) and the definition of zones we have the following lemma.

Lemma 2 (Dominance condition) *For all* $(t,\xi) \in Z_{evo}(M,N)$ *it holds*

$$\left|a_{j,k}(t)\right||\xi|^k \leq \frac{C}{N^{l-j-k/p}}\lambda(t)^{l-j}|\xi|^{p(l-j)}. \tag{18}$$

Proof We use the first inequality of assumption (12) and the definition of the evolution zone. It holds:

$$\begin{aligned} \left|a_{j,k}(t)\right||\xi|^k &\leq C\lambda(t)^{l-j}\left(\frac{|\log(1/\Lambda(t))|}{\Lambda(t)}\right)^{l-j-k/p}|\xi|^k \\ &\leq C\lambda(t)^{l-j}|\xi|^{p(l-j)}\left(\frac{|\log(1/\Lambda(t))|}{\Lambda(t)}\right)^{l-j-k/p}\frac{1}{|\xi|^{-k+p(l-j)}} \\ &\leq C\lambda(t)^{l-j}|\xi|^{p(l-j)}\frac{1}{N^{l-j-k/p}}. \end{aligned} \tag{19}$$

This yields the desired statement. □

Remark 5 The last line of the estimate shows that the coefficients of the extended principal part, which do not belong to the principal part in the sense of Petrowsky are always small in comparison to the used estimate of the coefficients of the principal part in the sense of Petrowsky. This holds true because the exponent of the large constant N in (19) disappears for the coefficients of the principal part in the sense of Petrowsky and this yields together with assumption (10) the dominance of those terms.

6.3.3 Treatment in the Pseudo-differential Zone

In the pseudo-differential zone we define the micro-energy

$$V(t,\xi) = \left(\rho(t,\xi)^{l-1}v, \rho(t,\xi)^{l-2}D_t v, \ldots, D_t^{l-1}v\right)^T.$$

The choice of $\rho(t,\xi)$ is important for our calculus, see [14]. There are different ways to do this. Sometimes authors propose micro-energies which depend only on ξ. But we are interested to study general degeneracies (of finite or infinite order). For this reason we follow [14] and introduce

$$\rho(t,\xi) := \sqrt[l]{1 + \frac{\lambda(t)^l}{\Lambda(t)^\alpha}\left(\log\frac{1}{\Lambda(t)}\right)^\alpha |\xi|^{p(l-\alpha)}} \tag{20}$$

for a suitable positive α. This α is connected to the minimal speed of degeneracy given by d_0. We introduce the notation $\alpha_{j,k} := l\frac{l-1-j-k/p}{l-1-j}$ and with this

$$\alpha_{j^*,k^*} = \max\left\{\alpha_{j,k} \text{ with } \frac{\alpha_{j,k}}{l} < d_0\right\} \quad \text{for } j < l-1.$$

Now we define

$$\alpha := ld_0 - \varepsilon \quad \text{with } \varepsilon < \min\left\{ld_0, ld_0 - \alpha_{j^*,k^*}, \frac{1}{1+l^2}\right\}. \tag{21}$$

In (20) we use $\log\frac{1}{\Lambda(t)}$. This is always positive in the pseudo-differential zone for $|\xi|$ large. And for the proof of our regularity statement we need only to consider $|\xi|$ large (see Definition 1).

Remark 6 In the 1-evolution (hyperbolic) case with a minimal speed of finite degeneracy determined by $d_0 > \frac{l-1}{l}$, so the shape function is t^β with $\beta > l-1$, it is sufficient to choose $\alpha = (l-1)d_0$.

In the next lemma we state all the properties of $\rho(t,\xi)$ that we will use in this section.

Lemma 3 *We have the following properties for the weight $\rho(t,\xi)$ for $t \in [0, t_\xi]$:*

$$1 \le \rho(t,\xi) \le C|\xi|^p, \qquad \rho(0,\xi) = 1, \qquad \int_0^t \rho(\tau,\xi)\mathrm{d}\tau \le C\big(1+\log|\xi|\big),$$

$$\log\rho(t_\xi,\xi) \le C\log|\xi|,$$

and for $\frac{\partial_t\rho(t,\xi)}{\rho(t,\xi)}$ it holds

$$\frac{\partial_t\rho(t,\xi)}{\rho(t,\xi)} \ge 0 \quad \textit{and} \quad \int_0^t \frac{\partial_\tau\rho(\tau,\xi)}{\rho(\tau,\xi)}\mathrm{d}\tau \le C\log|\xi|$$

provided that M and N are large.

Proof At first we need the non-negativity of $\frac{\partial_t\rho(t,\xi)}{\rho(t,\xi)}$. It holds:

$$\begin{aligned}\frac{\partial_t\rho(t,\xi)}{\rho(t,\xi)} = \frac{1}{l}\Bigg(\bigg(l\frac{\lambda'(t)\lambda(t)^{l-1}}{\Lambda(t)^\alpha}\Big(\log\frac{1}{\Lambda(t)}\Big)^\alpha \\ -\alpha\frac{\lambda(t)^{l+1}}{\Lambda(t)^{\alpha+1}}\bigg(\Big(\log\frac{1}{\Lambda(t)}\Big)^\alpha + \Big(\log\frac{1}{\Lambda(t)}\Big)^{\alpha-1}\bigg)\bigg) \\ \Big/\bigg(|\xi|^{-p(l-\alpha)} + \frac{\lambda(t)^l}{\Lambda(t)^\alpha}\Big(\log\frac{1}{\Lambda(t)}\Big)^\alpha\bigg)\Bigg)\end{aligned}$$

and this is non-negative if the following condition holds:

$$d_0 - \frac{\alpha}{l} - \frac{\alpha}{l}\left(\log\frac{1}{\Lambda(t)}\right)^{-1} \geq 0 \quad \to \quad d_0 \geq \frac{\alpha+\varepsilon}{l} \quad \to \quad d_0 > \frac{\alpha}{l}, \tag{22}$$

respectively. For $|\xi|$ large $\log\frac{1}{\Lambda(t)}$ is larger than $\frac{\alpha}{\varepsilon}$ for an arbitrary small $\varepsilon > 0$ and $T \leq T_0(\alpha, \varepsilon)$ in the pseudo-differential zone. So estimate (22) holds true for our choice of α. The non-negativity of $\frac{\partial_t \rho(t,\xi)}{\rho(t,\xi)}$ together with the positivity of $\rho(t,\xi)$ yields the monotonicity of $\rho(t,\xi)$. Furthermore, we get

$$\begin{aligned}\lim_{t\to 0+}\rho(t,\xi) &= \lim_{t\to 0+}\sqrt[l]{1+\frac{\lambda(t)^l}{\Lambda(t)^\alpha}\left(\log\frac{1}{\Lambda(t)}\right)^{\alpha}|\xi|^{p(l-\alpha)}}\\ &= \lim_{t\to 0+}\sqrt[l]{1+\frac{\lambda(t)^l}{\Lambda(t)^{ld_0-\varepsilon}}\left(\log\frac{1}{\Lambda(t)}\right)^{ld_0-\varepsilon}|\xi|^{p(l-ld_0+\varepsilon)}}.\end{aligned}$$

For the finite degenerate case $\lambda(t) = t^\beta$ we have

$$\lim_{t\to 0+}\frac{\lambda(t)^l}{\Lambda(t)^{ld_0-\varepsilon}}\left(\log\frac{1}{\Lambda(t)}\right)^{ld_0-\varepsilon} = 0$$

with $d_0 = \frac{\beta}{\beta+1}$ which brings $\lim_{t\to 0+} t^\nu = 0$ with a suitable $\nu > 0$. For the infinite degenerate case $\frac{\lambda(t)^l}{\Lambda(t)^{ld_0-\varepsilon}}$ yields a term which tends to zero of infinite order for any $d_0 < 1$. This brings $\rho(0,\xi) = 1$ for both cases.

With this we can estimate as follows:

$$1 \leq \rho(t,\xi) \leq \rho(t_\xi,\xi) \leq \sqrt[l]{1+\lambda(t_\xi)|\xi|^{pl}\left(\frac{\log(1/\Lambda(t_\xi))}{\Lambda(t_\xi)|\xi|^p}\right)^{\alpha}} \leq C_N|\xi|^p.$$

For the integrals we get

$$\int_0^t \frac{\partial_t\rho(\tau,\xi)}{\rho(\tau,\xi)}\mathrm{d}\tau \leq C\log\rho(\tau,\xi)|_0^t \leq C\log\rho(t_\xi,\xi) \leq C_N\log|\xi| \tag{23}$$

and

$$\begin{aligned}\int_0^t \rho(\tau,\xi)\mathrm{d}\tau &\leq C\left(\int_0^t \mathrm{d}\tau + \int_0^t \frac{\lambda(t)}{\Lambda(t)^{\alpha/l}}\left(\log\frac{1}{\Lambda(t)}\right)^{\alpha/l}|\xi|^{p((l-\alpha)/l)}\right)\\ &\leq C\left(T + \Lambda(t_\xi)^{(l-\alpha)/l}\left(\log\frac{1}{\Lambda(t_\xi)}\right)^{\alpha/l}|\xi|^{p((l-\alpha)/l)}\right)\\ &\leq C\left(1+\left(\log\frac{1}{\Lambda(t_\xi)}\right)^{\alpha/l}\left(N\log\frac{1}{\Lambda(t_\xi)}\right)^{(l-\alpha)/l}\right)\\ &\leq C_N\left(1+\log\frac{1}{\Lambda(t_\xi)}\right) \leq C_N\big(1+\log|\xi|\big).\end{aligned} \tag{24}$$

This completes the proof of Lemma 3. □

Lemma 4 *For all* $(t,\xi) \in Z_{pd}(M,N)$ *it holds*

$$\begin{cases} |v(t,\xi)| \lesssim \rho(t,\xi)^{-l+1}|\xi|^{C_{pd}}(|v_0(\xi)| + \ldots + |v_{l-1}(\xi)|), \\ |D_t v(t,\xi)| \lesssim \rho(t,\xi)^{-l+2}|\xi|^{C_{pd}}(|v_0(\xi)| + \ldots + |v_{l-1}(\xi)|), \\ \ldots \\ |D_t^{l-1} v(t,\xi)| \lesssim |\xi|^{C_{pd}}(|v_0(\xi)| + \ldots + |v_{l-1}(\xi)|). \end{cases}$$

Proof Using the micro-energy in the pseudo-differential zone for our Fourier transformed Cauchy problem (17) this leads to the system of first order $D_t V = A(t,\xi)V$ with

$$A(t,\xi) := \begin{pmatrix} (l-1)\frac{D_t\rho(t,\xi)}{\rho(t,\xi)} & \rho(t,\xi) & 0 & 0 & \ldots & 0 \\ 0 & (l-2)\frac{D_t\rho(t,\xi)}{\rho(t,\xi)} & \rho(t,\xi) & 0 & \ldots & 0 \\ \vdots & & \ddots & \ddots & & \vdots \\ 0 & \ldots & 0 & 2\frac{D_t\rho(t,\xi)}{\rho(t,\xi)} & \rho(t,\xi) & 0 \\ 0 & \ldots & 0 & 0 & \frac{D_t\rho(t,\xi)}{\rho(t,\xi)} & \rho(t,\xi) \\ \frac{\sum_{0\le k/p\le l} a_{0,k}(t)\xi^k}{\rho(t,\xi)^{l-1}} & \frac{\sum_{0\le k/p\le l-1} a_{1,k}(t)\xi^k}{\rho(t,\xi)^{l-2}} & & \ldots & & \sum_{0\le k/p\le 1} a_{l-1,k}(t)\xi^k \end{pmatrix}.$$

We are interested in the fundamental solution $E = E(t,s,\xi)$ to the system $D_t V - AV = 0$, that is, the solution of

$$D_t E - AE = 0, \quad E(s,s,\xi) = I, \quad \text{thus } V(t,\xi) = E(t,0,\xi)V(0,\xi).$$

The matrix $E(t,s,\xi)$ can be estimated by

$$\|E(t,s,\xi)\| \le \exp\Big(\int_0^t \|A(\tau,\xi)\| d\tau\Big), \quad 0 \le s \le t \le t_\xi. \tag{25}$$

Due to Lemma 3 we can estimate $\|A(t,\xi)\|$ in the following way:

$$\|A(t,\xi)\| \lesssim \frac{\partial_t\rho(t,\xi)}{\rho(t,\xi)} + \rho(t,\xi) + \sum_{\substack{0\le j+k/p\le l \\ j<l}} \frac{|a_{j,k}(t)||\xi|^k}{\rho(t,\xi)^{l-1-j}}. \tag{26}$$

The integrals of $\rho(t,\xi)$ and $\frac{\partial_t\rho(t,\xi)}{\rho(t,\xi)}$ over $[0,t]$, $t \le t_\xi$, are discussed in Lemma 3. Left is the estimate of $\int_0^t \frac{|a_{j,k}(\tau)||\xi|^k}{\rho(\tau,\xi)^{l-1-j}} d\tau$. It depends on the structure of $a_{j,k}(t)$. We begin with $a_{j,0}(t)$. Using condition (14) we can estimate

$$\int_0^t \frac{|a_{j,0}(\tau)|}{\rho(\tau,\xi)^{l-1-j}}\mathrm{d}\tau \le \int_0^t \big|a_{j,0}(\tau)\big|\mathrm{d}\tau \le C.$$

For the terms $a_{j,k}(t)$ with $l-1-j-\frac{k}{p} \ge d_0(l-1-j)$ we introduce another sub-zone to distinguish which part of $\rho(t,\xi)$ is dominant. Here we want to remember that only a shape function $\lambda(t)=t^\beta$ with finite degeneracy has to be considered, because for flat degeneracies, this assumption is meaningless. Let $t_{\xi,1}$ solve

$$1 = \frac{\lambda(t)^l}{\Lambda(t)^\alpha}\left(\log\frac{1}{\Lambda(t)}\right)^\alpha |\xi|^{p(l-\alpha)},$$

where α is the same as in (20). Then $0 \le t_{\xi,1} \le t_\xi$ for $|\xi|$ large. This follows from the following calculations:

$$1 = \frac{\lambda(t_{\xi,1})^l}{\Lambda(t_{\xi,1})^\alpha}\left(\log\frac{1}{\Lambda(t_{\xi,1})}\right)^\alpha |\xi|^{p(l-\alpha)}, \quad \Lambda(t_\xi)|\xi|^p = N\log\frac{1}{\Lambda(t_\xi)},$$

$$t_{\xi,1} = |\xi|^{-p/(\beta-\alpha/(l-\alpha))}\underbrace{\left(\log\frac{1}{\Lambda(t_{\xi,1})}\right)^{-\alpha/(l\beta-\alpha(\beta+1))}}_{<1},$$

$$t_\xi = |\xi|^{-p/(\beta+1)}\underbrace{N^{1/(\beta+1)}\left(\log\frac{1}{\Lambda(t_\xi)}\right)^{1/(\beta+1)}}_{>1}.$$

The definition of $t_{\xi,1}$ yields that for $0 \le t \le t_{\xi,1}$ the number 1 is dominant in the definition of $\rho(t,\xi)$ whereas for $t_{\xi,1} \le t \le t_\xi$ the second part $\frac{\lambda(t)^l}{\Lambda(t)^\alpha}(\log\frac{1}{\Lambda(t)})^\alpha|\xi|^{p(l-\alpha)}$ is dominant. With this it holds

$$\int_0^t \frac{|a_{j,k}(\tau)||\xi|^k}{\rho(\tau,\xi)^{l-1-j}}\mathrm{d}\tau = \int_0^{t_{\xi,1}} \frac{|a_{j,k}(\tau)||\xi|^k}{\rho(\tau,\xi)^{l-1-j}}\mathrm{d}\tau + \int_{t_{\xi,1}}^t \frac{|a_{j,k}(\tau)||\xi|^k}{\rho(\tau,\xi)^{l-1-j}}\mathrm{d}\tau.$$

As remarked before, we only have to consider the case of finite degeneracy. For $\lambda(t)=t^\beta$ we get $d_0=\frac{\beta}{\beta+1}$. Now we consider the first integral on the right-hand side. With assumption (15) it holds

$$\begin{aligned}\int_0^{t_{\xi,1}} \frac{|a_{j,k}(\tau)||\xi|^k}{\rho(t,\xi)^{l-1-j}}\mathrm{d}\tau &\le C\int_0^{t_{\xi,1}} |\xi|^k\mathrm{d}\tau = Ct_{\xi,1}|\xi|^k \\ &\le Ct_{\xi,1}\left(\frac{\lambda(t_{\xi,1})^l}{\Lambda(t_{\xi,1})^\alpha}\left(\log\frac{1}{\Lambda(t_{\xi,1})}\right)^\alpha\right)^{-k/(p(l-\alpha))}\end{aligned}$$

and with $\alpha = l\frac{\beta}{\beta+1} - \varepsilon$ we get

$$\int_0^{t_{\xi,1}} \frac{|a_{j,k}(\tau)||\xi|^k}{\rho(t,\xi)^{l-1-j}}\mathrm{d}\tau \le C t_{\xi,1}^{(pl-p\alpha-\beta kl+(\beta+1)k\alpha)/(p(l-\alpha))}\big(\log|\xi|\big)^{-\alpha k/(p(l-\alpha))}.$$

Now with $\varepsilon < \frac{1}{1+l^2}$, see (21), the exponent of $t_{\xi,1}$ is positive. Because of the negative exponent of $\log|\xi|$ it holds

$$\int_0^{t_{\xi,1}} \frac{|a_{j,k}(\tau)||\xi|^k}{\rho(t,\xi)^{l-1-j}}\mathrm{d}\tau \le C.$$

For the second integral we get

$$\int_{t_{\xi,1}}^{t_\xi} \frac{|a_{j,k}(\tau)||\xi|^k}{\rho(\tau,\xi)^{l-1-j}}\mathrm{d}\tau \le C\int_{t_{\xi,1}}^{t_\xi} \frac{|a_{j,k}(\tau)||\xi|^k}{((\lambda(\tau)^l/\Lambda(\tau)^\alpha)(\log(1/\Lambda(\tau)))^\alpha)^{l-1-j/(l)}}\mathrm{d}\tau$$

and for $d_0 = \frac{\beta}{\beta+1}$ it holds

$$\begin{aligned}
&= C\int_{t_{\xi,1}}^{t_\xi} \tau^{(\beta+1)(\alpha(l-1-j)/l)-\beta(l-1-j)}\left(\log\frac{1}{\tau}\right)^{-\alpha((l-1-j)/l)}|\xi|^{k-p((l-\alpha)(l-1-j)/l)}\mathrm{d}\tau\\
&\le C t_\xi^{1+(\beta+1)(\alpha(l-1-j)/l)-\beta(l-1-j)}\left(\log\frac{1}{t_{\xi,1}}\right)^{-\alpha((l-1-j)/l)}|\xi|^{k-p((l-\alpha)(l-1-j)/l)}\\
&\le C t_\xi^{1+(\beta+1)(\alpha(l-1-j)/l)-\beta(l-1-j)-k((\beta+1)/p)+(\beta+1)(l-\alpha)(l-1-j)/l}\\
&\quad\times\big(\log|\xi|\big)^{-\alpha((l-1-j)/l)+k/p-(l-\alpha)(l-1-j)/l}\\
&\le C t_\xi^{l-j-(k/p)(\beta+1)}\big(\log|\xi|\big)^{k/p-l+1+j}\\
&\le C t_\xi^{l-j-(k/p)(\beta+1)}\log|\xi|.
\end{aligned}$$

This gives an estimate for an at most finite loss of derivatives if the exponent of t_ξ is non negative. So, we have to guarantee

$$l - j - \frac{k}{p}(\beta+1) \ge 0$$

which is always satisfied for $a_{j,k}(t)$ with $d_0(l-1-j) \le l-1-j-\frac{k}{p}$. Consequently, we have shown that

$$\int_0^t \frac{|a_{j,k}(\tau)||\xi|^k}{\rho(\tau,\xi)^{l-1-j}}\mathrm{d}\tau \le C\big(1+\log|\xi|\big) \tag{27}$$

for all $0 \le t \le t_\xi$ and all coefficients $a_{j,k}(t)$ with $d_0(l-1-j) \le l-1-j-\frac{k}{p}$. This completes the explanations for the part of lower order terms satisfying assumption (15). Left is the procedure for the other part. We need to estimate

$$\int_0^t \frac{|a_{j,k}(\tau)||\xi|^k}{\rho(\tau,\xi)^{l-1-j}} \mathrm{d}\tau \le C\big(1+\log|\xi|\big) \tag{28}$$

by using assumption (12). We can estimate as follows:

$$\begin{aligned}
\frac{|a_{j,k}(t)||\xi|^k}{\rho(t,\xi)^{l-1-j}} &\le C \frac{\lambda(t)^{l-j}(\log(1/\Lambda(t))/\Lambda(t))^{l-j-k/p}|\xi|^k}{(1+(\lambda(t)^l/\Lambda(t)^\alpha)(\log(1/\Lambda(t)))^\alpha|\xi|^{p(l-\alpha)})^{(l-1-j)/l}} \\
&\le C \frac{\lambda(t)^{l-j-(l-1-j)}}{\Lambda(t)^{-\alpha(l-1-j)/l+l-j-k/p}} \left(\log\frac{1}{\Lambda(t)}\right)^{l-j-k/p-\alpha((l-1-j)/l)} \\
&\quad \times |\xi|^{k-p(l-\alpha)(l-1-j)/l} \\
&\le C \frac{\lambda(t)}{\Lambda(t)^{l-j-k/p-\alpha+\alpha/l+\alpha j/l}} \left(\log\frac{1}{\Lambda(t)}\right)^{l-j-k/p-\alpha+\alpha/l+\alpha j/l} \\
&\quad \times |\xi|^{k-pl+p+pj+\alpha p-\alpha p/l-\alpha jp/l},
\end{aligned} \tag{29}$$

which leads to

$$\begin{aligned}
\int_0^t \frac{|a_{j,k}(\tau)||\xi|^k}{\rho(\tau,\xi)^{l-1-j}} \mathrm{d}\tau &\le \Lambda(t)^{1-l+j+k/p+\alpha(1-1/l-j/l)}|\xi|^{p(1-l+j+k/p+\alpha(1-1/l-j/l))} \\
&\quad \times \left(\log\frac{1}{\Lambda(t)}\right)^{l-j-k/p-\alpha(1-1/l-j/l)} \\
&\le C_N\big(\log|\xi|\big)
\end{aligned} \tag{30}$$

for all $0 \le t \le t_\xi$ by using the definition of the pseudo-differential zone. The last step only holds true for $1-l+j+\frac{k}{p}+\alpha(1-\frac{1}{l}-\frac{j}{l}) \ge 0$. With our definition of α and $\varepsilon < ld_0 - \alpha_{j^*,k^*}$, see (21), the condition is always satisfied. So we obtain an estimate for (25)

$$\begin{aligned}
\big\|E(t,s,\xi)\big\| &\le \exp\bigg(\int_0^t \big\|A(\tau,\xi)\big\| d\tau\bigg) \\
&\lesssim \exp\bigg(C\bigg(\int_0^t \frac{\partial_t\rho(\tau,\xi)}{\rho(\tau,\xi)}\mathrm{d}\tau + \int_0^t \rho(\tau,\xi)\mathrm{d}\tau \\
&\qquad + \int_0^t \sum_{\substack{j+k/p\le l\\ j<l}} \bigg|\frac{a_{j,k}(\tau)\xi^k}{\rho(\tau,\xi)^{l-1-j}}\bigg| \mathrm{d}\tau\bigg)\bigg) \\
&\lesssim \exp\big(C\big(1+\log|\xi|\big)\big).
\end{aligned}$$

We complete the proof by using our fundamental solution E

$$\begin{aligned}
V(t,\xi) &= E(t,0,\xi)V(0,\xi), \\
\rho(t,\xi)^{l-1}\big|v(t,\xi)\big| &\lesssim \exp\big(C\big(1+\log|\xi|\big)\big)\big(\big|v_0(\xi)\big| + \big|v_1(\xi)\big| + \ldots + \big|v_{l-1}(\xi)\big|\big),
\end{aligned}$$

$$\rho(t,\xi)^{l-2}\big|D_t v(t,\xi)\big| \lesssim \exp\big(C\big(1+\log|\xi|\big)\big)\big(\big|v_0(\xi)\big|+\big|v_1(\xi)\big|+\ldots+\big|v_{l-1}(\xi)\big|\big),$$
$$\ldots$$
$$\big|D_t^{l-1} v(t,\xi)\big| \lesssim \exp\big(C\big(1+\log|\xi|\big)\big)\big(\big|v_0(\xi)\big|+\big|v_1(\xi)\big|+\ldots+\big|v_{l-1}(\xi)\big|\big).$$

Here we used $\rho(0,\xi)=1$. In this way the proof of Lemma 4 is completed. □

6.3.4 Treatment in the Evolution Zone

In the evolution zone $Z_{evo}(M,N)$ we define the micro-energy

$$V=\Big(\big(\lambda(t)|\xi|^p\big)^{l-1}v,\ \big(\lambda(t)|\xi|^p\big)^{l-2}D_t v,\ldots,D_t^{l-1}v\Big)^T.$$

Lemma 5 *For all $(t,\xi)\in Z_{evo}(M,N)$ it holds*

$$\begin{cases}(\lambda(t)|\xi|^p)^{l-1}|v(t,\xi)| \\ \quad\lesssim \exp(C(1+\log|\xi|))(\sum_{j=1}^{l}(\lambda(t_\xi)|\xi|^p)^{l-j}|D_t^{j-1}v(t_\xi,\xi)|), \\ (\lambda(t)|\xi|^p)^{l-2}|D_t v(t,\xi)| \\ \quad\lesssim \exp(C(1+\log|\xi|))(\sum_{j=1}^{l}(\lambda(t_\xi)|\xi|^p)^{l-j}|D_t^{j-1}v(t_\xi,\xi)|), \\ \ldots \\ |D_t^{l-1}v(t,\xi)| \lesssim \exp(C(1+\log|\xi|))(\sum_{j=1}^{l}(\lambda(t_\xi)|\xi|^p)^{l-j}|D_t^{j-1}v(t_\xi,\xi)|).\end{cases}$$

Proof First we want to consider the roots of the symbol containing the transformed extended principal part together with the real part of the terms of Levi size. They are given as the solutions to the characteristic equation

$$\tau^l-\sum_{\substack{l-1\le j+k/p\le l\\ j<l}}\Re a_{j,k}(t)\xi^k\tau^j=0. \tag{31}$$

The following proposition shows how the roots of (31) inherit the properties for the roots of (9).

Proposition 1 *We consider the roots $\tau_1,\ldots,\tau_l$ of (31). With assumption (10) for the roots of the principal part in the sense of Petrowsky and with the definition of the zone we get real roots satisfying*

$$\begin{aligned}\big|\tau_i(t,\xi)-\tau_j(t,\xi)\big| &\ge C\lambda(t)|\xi|^p \quad \text{for } i\neq j,\\ \big|D_t^m D_\xi^k \tau_j(t,\xi)\big| &\le C_m\lambda(t)|\xi|^{p-k}\left(\frac{\lambda(t)}{\Lambda(t)}\right)^m,\end{aligned} \tag{32}$$

for all $(t,\xi)\in Z_{evo}(M,N)$ and for $i,j=1,2,\ldots,l$, $k\in\mathbb{N}$ and $m=0,1$.

Proof We rewrite the assumption for the coefficients in the following way:

$$a_{j,k}(t) = \lambda(t)^{l-j}\left(\frac{\log(1/\Lambda(t))}{\Lambda(t)}\right)^{l-j-k/p}\widetilde{a}_{j,k}(t)$$

with $\widetilde{a}_{j,k}(t) \in B(0,T]$. We apply the transformation $\tau = \lambda(t)\xi^p z$. The transformation yields

$$z^l - \sum_{\substack{j+k/p=l\\ j<l}} \widetilde{a}_{j,k}(t)z^j - \sum_{\substack{l-1\le j+k/p<l\\ j<l}} \Re\widetilde{a}_{j,k}(t)\left(\frac{\log(1/\Lambda(t))}{\Lambda(t)\xi^p}\right)^{l-j-k/p} z^j = 0. \tag{33}$$

If we consider the transformation $\widehat{\tau} = \lambda(t)\xi^p\widehat{z}$ for (9) we obtain

$$\widehat{z}^l - \sum_{\substack{j+k/p=l\\ j<l}} \widetilde{a}_{j,k}(t)\widehat{z}^j = 0 \tag{34}$$

and from assumption (10) we know that equation (34) has real and distinct roots. It holds

$$\left|\widehat{z}_i(t,\xi) - \widehat{z}_j(t,\xi)\right| \ge C \quad \text{for } i \neq j,\ (t,\xi) \in [0,T] \times \big(\mathbb{R}\setminus\{0\}\big).$$

Equation (33) is a perturbed equation (34), so the roots $\tau_1,\ldots,\tau_l$ are in a small neighborhood of the respective roots $\widehat{\tau}_1,\ldots,\widehat{\tau}_l$ if the perturbation is sufficiently small. We know that the coefficients of the extended principal part are real. This and the distinctness of the roots $\widehat{\tau}_1,\ldots,\widehat{\tau}_l$ yields that roots $z_1,\ldots,z_l$ are real and distinct, because the smallness of the real perturbations is given by

$$\left|\Re\widetilde{a}_{j,k}(t)\right|\left(\frac{\log(1/\Lambda(t))}{\Lambda(t)\xi^p}\right)^{l-j-k/p} \le \frac{1}{C^*(N)} \quad \text{with } C^*(N) \to \infty \text{ for } N \to \infty.$$

And this holds true for any sufficiently large constant N in the definition of the zones. Backward transformation yields the first statement of the proposition. Furthermore, due to Vieta's formulas we have

$$\left|D_t^m D_\xi^\beta \sum_{i_1<\ldots<i_{l-j}} \tau_{i_1}(t,\xi)\ldots\tau_{i_{l-j}}(t,\xi)\right| = \left|D_t^m D_\xi^\beta a_{j,k}(t)\xi^k\right|$$

$$\le C_m\lambda(t)^{l-j}|\xi|^{k-\beta}\left(\frac{\lambda(t)}{\Lambda(t)}\right)^m$$

for $k = p(l-j)$ and $j = 0,\ldots,l-1$.

So we know that the roots of the extended principal part satisfy Proposition 1. □

Using the micro-energy in the evolution zone for our Fourier transformed Cauchy problem (17) this leads to the system of first order $D_t V = A(t,\xi)V$ with

$$A(t,\xi) := \begin{pmatrix} \frac{(l-1)}{i}\frac{\lambda'(t)}{\lambda(t)} & \lambda(t)|\xi|^p & 0 & 0 & \dots & 0 \\ 0 & \frac{(l-2)}{i}\frac{\lambda'(t)}{\lambda(t)} & \lambda(t)|\xi|^p & 0 & \dots & 0 \\ \vdots & & \ddots & \ddots & & \vdots \\ 0 & \dots & 0 & \frac{2}{i}\frac{\lambda'(t)}{\lambda(t)} & \lambda(t)|\xi|^p & 0 \\ 0 & \dots & 0 & 0 & \frac{1}{i}\frac{\lambda'(t)}{\lambda(t)} & \lambda(t)|\xi|^p \\ \frac{\sum_{0\le k/p\le l} a_{0,k}(t)\xi^k}{(\lambda(t)|\xi|^p)^{l-1}} & \frac{\sum_{0\le k/p\le l-1} a_{1,k}(t)\xi^k}{(\lambda(t)|\xi|^p)^{l-2}} & & \dots & & \sum_{0\le k/p\le 1} a_{l-1,k}(t)\xi^k \end{pmatrix}.$$

Now we split matrix $A(t,\xi)$ into several parts. We introduce

$$A_1(t,\xi) := \begin{pmatrix} 0 & \lambda(t)|\xi|^p & & & 0 \\ \vdots & & \ddots & & \\ 0 & \dots & 0 & \lambda(t)|\xi|^p & \\ 0 & \dots & 0 & 0 & \lambda(t)|\xi|^p \\ \frac{\sum_{l-1\le k/p\le l} \Re a_{0,k}(t)\xi^k}{(\lambda(t)|\xi|^p)^{l-1}} & \frac{\sum_{l-2\le k/p\le l-1} \Re a_{1,k}(t)\xi^k}{(\lambda(t)|\xi|^p)^{l-2}} & & \dots & \sum_{0\le k/p\le 1} \Re a_{l-1,k}(t)\xi^k \end{pmatrix},$$

$$A_2(t,\xi) := \begin{pmatrix} \frac{(l-1)}{i}\frac{\lambda'(t)}{\lambda(t)} & 0 & 0 & 0 & \dots & 0 \\ 0 & \frac{(l-2)}{i}\frac{\lambda'(t)}{\lambda(t)} & 0 & 0 & \dots & 0 \\ \vdots & & \ddots & \ddots & & \vdots \\ 0 & \dots & 0 & \frac{2}{i}\frac{\lambda'(t)}{\lambda(t)} & 0 & 0 \\ 0 & \dots & 0 & 0 & \frac{1}{i}\frac{\lambda'(t)}{\lambda(t)} & 0 \\ 0 & 0 & & \dots & & 0 \end{pmatrix},$$

$$A_3(t,\xi) := \begin{pmatrix} 0 & 0 & \dots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \dots & 0 \\ \frac{\Im a_{0,p(l-1)}(t)\xi^{p(l-1)}+\sum_{k/p<l-1} a_{0,k}(t)\xi^k}{(\lambda(t)|\xi|^p)^{l-1}} & \frac{\Im a_{1,p(l-2)}(t)\xi^{p(l-2)}+\sum_{k/p<l-2} a_{1,k}(t)\xi^k}{(\lambda(t)|\xi|^p)^{l-2}} & \dots & \Im a_{l-1,0}(t) \end{pmatrix}.$$

We are interested in the symbol classes for $A_2(t,\xi)$ and $A_3(t,\xi)$. It is obvious that $A_2(t,\xi) \in S_0\{0,0,1,0\}$ and for $A_3(t,\xi)$ the assumptions (12) and (16) and straight forward calculations yield $A_3(t,\xi) \in S_0\{0,0,0,0\} + S_0\{-\frac{1}{p},1,0,1+\frac{1}{p}\}$.

Remark 7 Let us come back to the assumptions (12) and (16) for the terms of Levi size. The real parts are included in the matrix A_1, this allows a $\log \Lambda(t)$ term. The imaginary parts are included in the matrix A_3. To stay in the correct symbol classes we are not able to allow a $\log \Lambda(t)$ term for the imaginary parts.

Using the system $\frac{\tau_1}{\lambda(t)|\xi|^p}, \dots, \frac{\tau_l}{\lambda(t)|\xi|^p}$ we form the Vandermonde matrix

$$M(t,\xi) := \begin{pmatrix} 1 & 1 & \dots & 1 \\ \frac{\tau_1(t,\xi)}{\lambda(t)|\xi|^p} & \frac{\tau_2(t,\xi)}{\lambda(t)|\xi|^p} & \dots & \frac{\tau_l(t,\xi)}{\lambda(t)|\xi|^p} \\ \vdots & \vdots & \vdots & \vdots \\ (\frac{\tau_1(t,\xi)}{\lambda(t)|\xi|^p})^{l-1} & (\frac{\tau_2(t,\xi)}{\lambda(t)|\xi|^p})^{l-1} & \dots & (\frac{\tau_l(t,\xi)}{\lambda(t)|\xi|^p})^{l-1} \end{pmatrix}$$

and apply the transformation $V := M(t,\xi)V_1$ to our system

$$D_t V = A_1 V + A_2 V + A_3 V. \tag{35}$$

The matrix M is chosen as a diagonalizer of A_1. The determinant of M is given by

$$\det\big(M(t,\xi)\big) = \prod_{1\le i<j\le n} \frac{\tau_j(t,\xi)-\tau_i(t,\xi)}{\lambda(t)|\xi|^p}.$$

Because of the separation condition from (32) the determinant of $M(t,\xi)$ satisfies $|\det(M(t,\xi))| \ge C > 0$ and so the inverse matrix $M^{-1}(t,\xi)$ exists for all $(t,\xi) \in Z_{evo}(M,N)$.

Lemma 6 *After the first step of diagonalization we obtain from system* (35) *the new system*

$$D_t V_1 = DV_1 + RV_1, \quad V_1(t_\xi,\xi) = V_{1,0}(\xi) := M^{-1}V(t_\xi,\xi) \tag{36}$$

with a diagonal matrix

$$D = D(t,\xi) = \begin{pmatrix} \tau_1(t,\xi) & & 0 \\ & \ddots & \\ 0 & & \tau_l(t,\xi) \end{pmatrix}$$

and a matrix

$$R = R(t,\xi) \in S_0\{0,0,0,0\} + S_0\{0,0,1,0\} + S_0\left\{-\frac{1}{p}, 1, 0, 1+\frac{1}{p}\right\}. \tag{37}$$

Proof System (35) transforms to

$$D_t V_1 = M^{-1}A_1 M V_1 + M^{-1}A_2 M V_1 + M^{-1}A_3 M V_1 - M^{-1}(D_t M)V_1 \tag{38}$$

with the diagonal matrix $D = M^{-1}A_1 M$. The matrix R is defined by

$$R := M^{-1}A_2 M - M^{-1}(D_t M) + M^{-1}A_3 M.$$

For the entries of M it holds

$$\left| \left(\frac{\tau_k(t,\xi)}{\lambda(t)|\xi|^p} \right)^j \right| \le C$$

for $j = 0,\dots,l-1$ and $k = 1,\dots,l$. With this $M(t,\xi)$ and its inverse $M^{-1}(t,\xi) \in S_0\{0,0,0,0\}$. So the calculus of the symbol classes yields the statement of the lemma. □

The function

$$E_2(t,r,\xi) := \begin{pmatrix} e^{i\int_r^t \tau_1(s,\xi)ds} & & 0 \\ & \ddots & \\ 0 & & e^{i\int_r^t \tau_l(s,\xi)ds} \end{pmatrix}$$

solves the Cauchy problem $(D_t - D)E(t,r,\xi) = 0$, $E(r,r,\xi) = I$. It holds for $r \ge t_\xi$

$$\left\| E_2(t,r,\xi) \right\| \le \max_{k=1,\dots,l} \left| \exp\left(i \int_r^t \sum_{k=1}^{l} \tau_k(s,\xi)\mathrm{d}s \right) \right| = 1,$$

because the roots of (31) are all real. Here we feel the dispersive character of our Cauchy problem and the dominance condition from Lemma 2. We define the matrix-valued function $H = H(t,r,\xi)$ with $t, r \ge t_\xi$:

$$H(t,r,\xi) := E_2(r,t,\xi)R(t,\xi)E_2(t,r,\xi).$$

Because $E_2(r,t,\xi) = E_2^{-1}(t,r,\xi)$, $\|E_2(r,t,\xi)\| = \|E_2^{-1}(t,r,\xi)\| = 1$, and due to (37) the following estimate holds:

$$\|H(t,r,\xi)\| \leq C + C\frac{\lambda(t)}{\Lambda(t)} + C\frac{\lambda(t)}{\Lambda(t)^{1+1/p}|\xi|}\left(\log\frac{1}{\Lambda(t)}\right)^{1+1/p}. \tag{39}$$

We will consider $\log\frac{1}{\Lambda(t)}$ to be positive for all $t \leq T$, because we are only interested in times close to the degeneracy $t = 0$. Now

$$V_1(t,\xi) := E_2(t,t_\xi,\xi)Q(t,t_\xi,\xi)V_{1,0}(\xi)$$

solves (36) if $D_t Q = H(t,r,\xi)Q$. This follows from

$$\begin{aligned} &D_t(E_2 Q) - DE_2 Q - RE_2 Q = 0, \\ &\underbrace{(D_t E_2)Q - DE_2 Q}_{=0} + E_2 D_t Q = RE_2 Q. \end{aligned}$$

Knowing that $H(t,r,\xi)$ can be estimated by (39) we are able to estimate $Q = Q(t,r,\xi)$. We see that

$$\begin{aligned} \int_{t_\xi}^{t}\|H(s,t_\xi,\xi)\|ds &\lesssim \int_{t_\xi}^{t} 1 + \frac{\lambda(s)}{\Lambda(s)} + \frac{\lambda(s)}{\Lambda(s)^{1+1/p}|\xi|}\left(\log\frac{1}{\Lambda(s)}\right)^{1+1/p} ds \\ &\lesssim 1\Big|_{t_\xi}^{t} + \log\frac{1}{\Lambda(s)}\Big|_{t}^{t_\xi} - \Lambda(s)^{-1/p}\left(\log\frac{1}{\Lambda(s)}\right)^{1+1/p}|\xi|^{-1}\Big|_{t_\xi}^{t} \\ &\leq C\left(1 + \log\frac{1}{\Lambda(t_\xi)}\right) \leq C_{evo}\log|\xi|. \end{aligned} \tag{40}$$

This leads to

$$\|Q(t,t_\xi,\xi)\| \lesssim \exp\left(C\left(1+\log\frac{1}{\Lambda(t_\xi)}\right)\right) \leq C|\xi|^{C_{evo}}.$$

Now we will estimate $|V_1(t,\xi)|$ and with the backward transformation we obtain an estimate for $|V(t,\xi)|$:

$$\begin{aligned} V_1(t,\xi) &= E_2(t,t_\xi,\xi)Q(t,t_\xi,\xi)V_{1,0}(\xi), \\ |V_1(t,\xi)| &\leq C\exp\left(C\left(1+\log\frac{1}{\Lambda(t_\xi)}\right)\right)|V_{1,0}(\xi)|, \\ |V(t,\xi)| &= |M(t,\xi)V_1(t,\xi)| \\ &\leq C\exp\left(C\left(1+\log\frac{1}{\Lambda(t_\xi)}\right)\right)|M^{-1}(t_\xi,\xi)V(t_\xi,\xi)| \\ &\leq C\exp\left(C\left(1+\log\frac{1}{\Lambda(t_\xi)}\right)\right)|V(t_\xi,\xi)|. \end{aligned}$$

Summarizing we arrive in the evolution zone at the following estimates:

$$\begin{aligned}
\big|V(t,\xi)\big| &\le C|\xi|^{C_{evo}}\big|V(t_\xi,\xi)\big|,\\
\big(\lambda(t)|\xi|^p\big)^{l-1}\big|v(t,\xi)\big| &\le C|\xi|^{C_{evo}}\left(\sum_{j=1}^{l}\big(\lambda(t_\xi)|\xi|^p\big)^{l-j}\big|D_t^{j-1}v(t_\xi,\xi)\big|\right),\\
&\dots\\
\big|D_t^{l-1}v(t,\xi)\big| &\le C|\xi|^{C_{evo}}\left(\sum_{j=1}^{l}\big(\lambda(t_\xi)|\xi|^p\big)^{l-j}\big|D_t^{j-1}v(t_\xi,\xi)\big|\right).
\end{aligned} \tag{41}$$

With this Lemma 5 is proved. □

6.3.5 Verification

Now we want to use the estimates of both zones to get an estimate for an arbitrary $t \in [0,T]$. For $t \le t_\xi$ we get an estimate in the pseudo-differential zone. Using the initial conditions we obtain

$$\begin{aligned}
\big|D_t^m v(t,\xi)\big| \le C\rho(t,\xi)^{-l+1+m}\exp\big(C\big(1+\log|\xi|\big)\big)&\\
\times\big(\big|v_0(\xi)\big|+\ldots+\big|v_{l-1}(\xi)\big|\big)&
\end{aligned} \tag{42}$$

for $m=0,\ldots,l-1$. In the case $t \ge t_\xi$ we use the estimates from the evolution zone

$$\begin{aligned}
&\big|D_t^m v(t,\xi)\big|\\
&\quad\le C\big(\lambda(t)|\xi|^p\big)^{-l+m+1}\exp\big(C\big(1+\log|\xi|\big)\big)\\
&\qquad\times\left(\sum_{j=1}^{l}\big(\lambda(t_\xi)|\xi|^p\big)^{l-j}\big|D_t^{j-1}v(t_\xi,\xi)\big|\right)\\
&\quad\le C\big(\lambda(t)|\xi|^p\big)^{-l+m+1}\exp\big(C\big(1+\log|\xi|\big)\big)\\
&\qquad\times\left(\sum_{j=1}^{l}\big(\lambda(t_\xi)|\xi|^p\big)^{l-j}\rho(t_\xi,\xi)^{-l+j}\big(\big|v_0(\xi)\big|+\ldots+\big|v_{l-1}(\xi)\big|\big)\right)
\end{aligned} \tag{43}$$

for $m=0,\ldots,l-1$. Now we use that $\rho(t,\xi)$ is larger 1 and the monotonicity of $\lambda(t)$. So it holds

$$\begin{aligned}
\big|D_t^m v(t,\xi)\big| \le C\exp\big(C\big(1+\log|\xi|\big)\big)&\\
\times\sum_{j=1}^{l}\frac{(\lambda(t_\xi)|\xi|^p)^{l-j}}{(\lambda(t_\xi)|\xi|^p)^{l-m-1}}\big(\big|v_0(\xi)\big|+\ldots+\big|v_{l-1}(\xi)\big|\big)&
\end{aligned}$$

$$
\begin{aligned}
&\leq C|\xi|^{s_0-(l-1)p}\frac{(\lambda(t_\xi)|\xi|^p)^{l-1}}{(\lambda(t_\xi)|\xi|^p)^{l-m-1}}\big(\big|v_0(\xi)\big|+\ldots+\big|v_{l-1}(\xi)\big|\big)\\
&\leq C|\xi|^{s_0-(l-1)p}\big(\lambda(t_\xi)|\xi|^p\big)^m\big(\big|v_0(\xi)\big|+\ldots+\big|v_{l-1}(\xi)\big|\big)\\
&\leq C|\xi|^{s_0-(l-1)p+mp}\big(\big|v_0(\xi)\big|+\ldots+\big|v_{l-1}(\xi)\big|\big)\\
&\leq C|\xi|^{s_0+mp}\big(|\xi|^{-(l-1)p}\big|v_0(\xi)\big|+\ldots+|\xi|^{-(l-1)p}\big|v_{l-1}(\xi)\big|\big)\\
&\leq C|\xi|^{s_0+mp}\big(\big|v_0(\xi)\big|+\ldots+|\xi|^{-(l-1)p}\big|v_{l-1}(\xi)\big|\big)
\end{aligned}
$$

for $m = 0, \ldots, l-1$ and a constant s_0 which gives an (at most) finite loss of regularity. So our solution $D_t^m u(t,\cdot)$ is in $H^{s-s_0-mp}(\mathbb{R})$ if and only if

$$\int_{\mathbb{R}} \big|D_t^m v(t,\xi)\big|^2 |\xi|^{2(s-s_0-mp)} \mathrm{d}\xi < \infty.$$

It holds

$$
\begin{aligned}
&\int_{\mathbb{R}} \big|D_t^m v(t,\xi)\big|^2 |\xi|^{2(s-s_0-mp)} \mathrm{d}\xi\\
&\quad\lesssim \int_{\mathbb{R}} |\xi|^{2s}\big(\big|v_0(\xi)\big|^2+\ldots+|\xi|^{-2(l-1)p}\big|v_{l-1}(\xi)\big|^2\big)\mathrm{d}\xi < \infty
\end{aligned}
$$

by taking account of the regularity of the data. The continuity of solutions and their derivatives with respect to t follows from the continuity of $V = V(t,\xi)$ with respect to t in suitable function spaces in the phase space. This completes the proof of Theorem 1.

6.4 Outlook

This last section gives an outlook about further research and open problems.

6.4.1 About Optimality—C^1-Theory

One could pose the question, whether the assumptions on the degeneracy or the assumptions on the behavior of coefficients of the extended principal part near to $t = 0$ or on their oscillating behavior are sharp. Whether a loss really appears, whether this result is optimal. But there is not much to say about optimality results in C^1-theory. There are no results to prove the sharpness of the assumptions and there are no examples that show that this loss really appears. The control of the first derivative in t allows us to diagonalize the Fourier transformed system once. This yields a diagonal part and a remainder. But this remainder belongs to a symbol class which does not allow to apply methods for proving optimality. Another approach to show optimality for the C^1-theory is the a priori knowledge of reflection points or maximum points to get some kind of classification of oscillations. This is an attempt by Prof. Hirosawa from Yamaguchi University ([8], unpublished notes). For the x-dependent case there are no results about the sharpness of the decay rates for a p-evolution Cauchy problem. In [10] sharpness for decay rates has only been shown for the Cauchy problem to Schrödinger equations with time-independent coefficients of the form

$$i\partial_t u + \partial_x^2 u - a(x)\partial_x u = 0, \quad u(0,x) = u_0(x). \tag{44}$$

An open problem that might be attackable is the sharpness of the decay rates using the ideas of the mentioned paper.

6.4.2 About Optimality—C^2-theory

The advantage of a C^2-theory would be that we can diagonalize twice so that we get a remainder which is better in some hierarchies of symbol classes. A paper about C^2-theory for the p-evolution Cauchy problem of second order in D_t is in preparation, see [6] and [7].

6.4.3 About x-Dependence—C^1-Theory

Here we want to consider the p-evolution Cauchy problem (2), where the coefficients $a_{j,k}$ may depend on space and time. The first thing we can do is to try to include x-dependence in a way that we can generalize the result for the pure time-dependent model without the need of more assumptions on the coefficients except the boundedness of the coefficients and of its derivatives with respect to the spatial variable. This is only possible for the coefficients $a_{j,k}$ of the extended principal part with the lowest order $j + \frac{k}{p} = l - 1 + \frac{1}{p}$ and for the terms of lower order. We consider the p-evolution Cauchy problem of higher order in D_t with coefficients depending on space and time as follows:

$$\begin{aligned} D_t^l u - \sum_{\substack{l-1+1/p<j+k/p\le l \\ j<l}} a_{j,k}(t) D_x^k D_t^j u \\ - \sum_{0\le j+k/p\le l-1+1/p} a_{j,k}(t,x) D_x^k D_t^j u = 0, \\ D_t^m u(0,x) = u_m \quad \text{for } m = 0,\dots,l-1 \text{ and } l \ge 2. \end{aligned} \tag{45}$$

All coefficients are real and in $B^\infty(\mathbb{R})$ with respect to x.

Theorem 2 *Let us consider the Cauchy problem* (45) *under the assumptions* (8) *and* (10) *to* (16). *For initial data* $u_m \in H^{s-mp}(\mathbb{R})$, $m = 0, \ldots, l-1$, *there exists a non-negative constant* s_0 *and a unique solution* $u \in C([0,T], H^{s-s_0}(\mathbb{R})) \cap C^1([0,T], H^{s-s_0-p}(\mathbb{R})) \cap \ldots \cap C^{l-1}([0,T], H^{s-s_0-(l-1)p}(\mathbb{R}))$. *An a priori estimate for the solution is given by*

$$\left\| D_t^m u(t,\cdot) \right\|_{H^{s-s_0-mp}} \leq C\big(\|u_0\|_{H^s} + \ldots + \|u_{l-1}\|_{H^{s-(l-1)p}}\big)$$

for $m = 0, \ldots, l-1$.

Remark 8 It is important to understand that the only difference in the Theorems 1 and 2 is the x-dependence of some coefficients, but this brings a complete change in the proof. We can not use the partial Fourier transformation with respect to x. We need cut-off functions techniques which help to localize the considerations to the needed zones. Moreover, we should apply methods basing on a pseudo-differential calculus.

If we include decay conditions of the coefficients with respect to x, then we can consider x-dependence for almost all coefficients. We can consider

$$\begin{aligned} &D_t^l u - \sum_{\substack{j+k/p=l \\ j<l}} a_{j,k}(t) D_x^k D_t^j u - \sum_{0 \leq j+k/p \leq l-1/p} a_{j,k}(t,x) D_x^k D_t^j u = 0, \\ &D_t^m u(0,x) = u_m(x) \quad \text{for } m = 0, \ldots, l-1 \text{ and } l \geq 2. \end{aligned} \tag{46}$$

We propose the following decay conditions which are related to the conditions in [2]:

$$\begin{aligned} \left| D_x a_{j,(l-j)p-k}(t,x) \right| &\leq C\lambda(t)^{l-j} \langle x \rangle^{-(p-k-1)/(p-1)}, \\ \left| D_x^\beta a_{j,(l-j)p-k}(t,x) \right| &\leq C\lambda(t)^{l-j} \langle x \rangle^{-(p-k-[\beta/2])/(p-1)} \end{aligned} \tag{47}$$

for $2 \leq \beta < 2(p-k)$, $j = 0, \ldots, l-1$ and $k = 1, \ldots, p-2$.

Hypothesis Let us consider the Cauchy problem (46) under the assumptions (8), (10) to (16) and (47). For initial data $u_m \in H^{s-mp}(\mathbb{R})$, $m = 0, \ldots, l-1$ there exists a non-negative constant s_0 and a unique solution $u \in C([0,T], H^{s-s_0}(\mathbb{R})) \cap C^1([0,T], H^{s-s_0-p}(\mathbb{R})) \cap \ldots \cap C^{l-1}([0,T], H^{s-s_0-(l-1)p}(\mathbb{R}))$. An a priori estimate for the solution is given by

$$\left\| D_t^m u(t,\cdot) \right\|_{H^{s-s_0-mp}} \leq C\big(\|u_0\|_{H^s} + \ldots + \|u_{l-1}\|_{H^{s-(l-1)p}}\big)$$

for $m = 0, \ldots, l-1$.

Remark 9 We can also extend the calculus to Cauchy problem (46) with complex-valued coefficients depending on t and x. For the theorem to hold we need a decay for the imaginary part. We would propose the following assumptions:

$$\left|\Im a_{j,(l-j)p-k}(t,x)\right| \le C\lambda(t)^{l-j}\langle x\rangle^{-(p-k)/(p-1)} \tag{48}$$

for $j = 0, \ldots, l-1$ and $k = 2, \ldots, p-1$. Furthermore we pose an assumption for the imaginary part of $a_{j,(l-j)p-1}$ in the following way:

$$\left|\Im a_{j,(l-j)p-1}(t,x)\right| \le C\lambda(t)^{l-j} g\big(\langle x\rangle\big), \tag{49}$$

where the function $g = g(s) \in L^1(\mathbb{R}_+) \cap C[0,\infty)$ is a strictly decreasing function.

For a better understanding of the influence coming from the imaginary parts of the coefficients see [4].

6.4.4 About *x*-Dependence—C^2-Theory

If we merge the last results we can get a result for a Cauchy problem similar to (46). We want to propose a hypothesis for the following Cauchy problem:

$$\begin{aligned} &D_t^l u - \lambda(t)^l b(t)^l D_x^{lp} u - \sum_{0\le j+k/p\le l-1/p} a_{j,k}(t,x) D_x^k D_t^j u = 0, \\ &D_t^m u(0,x) = u_m(x) \quad \text{for } m = 0,\ldots,l-1 \text{ and } l \ge 2. \end{aligned} \tag{50}$$

We have a special choice for the principal part in the sense of Petrowsky due to the interactions in the principal part in the sense of Petrowsky shown in [9] for a strictly hyperbolic problem. The coefficients $a_{j,k}(t,x)$ are considered to be complex. We consider a shape function $\lambda(t)$ which satisfies

$$\begin{aligned} &\lambda(0) = 0, \qquad \lambda'(t) > 0 \quad \text{for } t > 0, \\ &d_0 \frac{\lambda(t)}{\Lambda(t)} \le \frac{\lambda'(t)}{\lambda(t)} \le d_1 \frac{\lambda(t)}{\Lambda(t)}, \quad d_0 > \frac{l-1}{l}, \\ &\left|D_t^2 \lambda(t)\right| \le d_2 \lambda(t) \left(\frac{\lambda(t)}{\Lambda(t)}\right)^2. \end{aligned} \tag{51}$$

The function $b(t)$ describes the oscillating behavior of the coefficient and we assume

$$\begin{aligned} &c_0 := \inf_{t\in(0,T]} b(t) \le b(t) \le c_1 := \sup_{t\in(0,T]} b(t), \quad t \in (0,T], c_0, c_1 > 0, \\ &\left|D_t^m b(t)\right| \le C\left(\frac{\lambda(t)}{\Lambda(t)}\nu(t)\right)^m, \quad m = 1, 2. \end{aligned} \tag{52}$$

For the coefficients we pose the assumptions

$$\left|D_t^m a_{j,k}(t,x)\right| \le C_m \lambda(t)^{l-j}\left(\frac{\nu(t)}{\Lambda(t)}\right)^{l-j-k/p}\left(\frac{\lambda(t)}{\Lambda(t)}\nu(t)\right)^m \tag{53}$$

for $m = 0, 1, 2$. For the terms of Levi size we need the additional Levi conditions

$$\left|D_t^m \Im a_{l-1-k/p,k}(t)\right| \le C_m \lambda(t)^{k/p}\left(\frac{\lambda(t)}{\Lambda(t)}\nu(t)\right)^{m+1} \tag{54}$$

for $m = 0, 1, 2$. In some of the assumptions we used a function $\nu = \nu(t)$, which is a positive and strictly decreasing function. Furthermore, for the function $\nu(t)$ we need the assumption

$$\frac{\lambda(t)}{\Lambda(t)}\nu(t) \gg -\nu'(t). \tag{55}$$

Furthermore, we propose decay conditions

$$\left|D_t^m D_x a_{j,(l-j)p-k}(t,x)\right| \le C\lambda(t)^{l-j}\langle x\rangle^{-(p-k-1)/(p-1)}, \tag{56}$$

$$\left|D_t^m D_x^\beta a_{j,(l-j)p-k}(t,x)\right| \le C\lambda(t)^{l-j}\langle x\rangle^{-(p-k-[\beta/2])/(p-1)} \tag{57}$$

for $2 \le \beta < 2(p-k)$, $j = 0, \dots, l-1$, $k = 1, \dots, p-2$, $m = 0, 1$ and

$$\left|D_t^m \Im a_{j,(l-j)p-k}(t,x)\right| \le C\lambda(t)^{l-j}\langle x\rangle^{-(p-k)/(p-1)}, \tag{58}$$

$$\left|D_t^m \Im a_{j,(l-j)p-1}(t,x)\right| \le C\lambda(t)^{l-j} g\big(\langle x\rangle\big) \tag{59}$$

for $j = 0, \dots, l-1$, $k = 2, \dots, p-1$, $m = 0, 1$, where the function $g = g(s) \in L^1(\mathbb{R}_+) \cap C[0,\infty)$ is a strictly decreasing function.

Hypothesis Let us consider the Cauchy problem (50) under the assumptions (51) to (59). For initial data $u_0 \in H^s$ and u_m, $m = 1, \dots, l-1$ in appropriate spaces, then there exists a unique solution $u = u(t,x)$ with the properties

$$u(t,\cdot) \in \exp\left(C\nu\left(\left(\frac{\Lambda}{\nu}\right)^{(-1)}\left(\frac{N}{\langle D_x\rangle^p}\right)\right)\right)H^s(\mathbb{R}),$$

where N is a suitable positive constant. The loss of regularity of the solution is described by

$$\exp\left(C\nu\left(\left(\frac{\Lambda}{\nu}\right)^{(-1)}\left(\frac{N}{\langle D_x\rangle^p}\right)\right)\right).$$

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