Chapter 4
Singular Semilinear Elliptic Equations with Subquadratic Gradient Terms

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Abstract We investigate the semilinear elliptic equation $-\Delta u = a(\delta(x))g(u) + f(x,u) + \lambda|\nabla u|^q$ in a smooth and bounded domain Ω subject to an homogeneous Dirichlet boundary condition. Here g is an unbounded decreasing function, a is positive and continuous, f grows at most linearly at infinity, $\delta(x) = \mathrm{dist}(x, \partial\Omega)$ and $0 < q \le 2$. We emphasize the effect of all these terms in the study of existence, nonexistence and asymptotic behavior of positive solutions.

Mathematics Subject Classification 35J15 · 35J75

4.1 Introduction

We are concerned with semilinear elliptic problems in the form

$$\begin{cases} -\Delta u = a(\delta(x))g(u) + f(x,u) + \lambda|\nabla u|^q & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a smooth and bounded domain in $\mathbb{R}^N$, $N \ge 2$, $\delta(x) = \mathrm{dist}(x, \partial\Omega)$, $\lambda \in \mathbb{R}$ and $0 < q \le 2$.

We assume that $g \in C^1(0,\infty)$ is a positive decreasing function and

$(g1)$ $\lim_{t\to 0^+} g(t) = \infty$.

The function $f : \overline{\Omega} \times [0,\infty) \to [0,\infty)$ is Hölder continuous, nondecreasing with respect to the second variable and f is positive on $\overline{\Omega} \times (0,\infty)$. The analysis we develop in this paper concerns the cases where f is either linear or sublinear with respect to the second variable. This latter case means that f fulfills the hypotheses

$(f1)$ the mapping $(0,\infty) \ni t \longmapsto \frac{f(x,t)}{t}$ is nonincreasing for all $x \in \overline{\Omega}$;
$(f2)$ $\lim_{t\to 0^+} \frac{f(x,t)}{t} = \infty$ and $\lim_{t\to\infty} \frac{f(x,t)}{t} = 0$, uniformly for $x \in \overline{\Omega}$.

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Such singular boundary value problems arise in the context of chemical heterogeneous catalysts and chemical catalyst kinetics, in the theory of heat conduction in electrically conducting materials, singular minimal surfaces, as well as in the study of non-Newtonian fluids or boundary layer phenomena for viscous fluids (we refer for more details to [3–5, 8, 10, 11] and the more recent papers [6, 13, 18–20, 22, 24, 25, 28]). We also point out that, due to the meaning of the unknowns (concentrations, populations, etc.), only the positive solutions are relevant in most cases.

The main features of this paper are the presence of the convection term $|\nabla u|^q$ combined with the singular weight $a : (0, \infty) \to (0, \infty)$ which is assumed to be nonincreasing and Hölder continuous.

Many papers have been devoted to the case $a \equiv 1$ and $\lambda = 0$ (see [7, 9, 13, 23, 24, 27, 29] and the references therein). One of the first works in the literature dealing with singular weights in connection with singular nonlinearities is due to Taliaferro [26]. In [26] the following problem has been considered

$$\begin{cases} -y'' = \varphi(x) y^{-p} & \text{in } (0, 1), \\ y(0) = y(1) = 0, \end{cases} \tag{2}$$

where $p > 0$ and $\varphi(x)$ is positive and continuous on $(0, 1)$. It was proved that problem (2) has solutions if and only if $\int_0^1 t(1-t)\varphi(t)dt < \infty$. Later, Agarwal and O'Regan (Sect. 2 in [1]) studied the more general problem

$$\begin{cases} H''(t) = -a(t) g(H(t)) & \text{in } (0, 1), \\ H > 0 & \text{in } (0, 1), \\ H(0) = H(1) = 0, \end{cases} \tag{3}$$

where g satisfies $(g1)$ and p is positive and continuous on $(0, 1)$. It is shown in [1] that if

$$\int_0^1 t(1-t)a(t)dt < \infty, \tag{4}$$

then (3) has at least one classical solution. In our framework, p is continuous at $t = 1$ so condition (4) reads as

$$\int_0^1 t a(t) dt < \infty. \tag{5}$$

In this paper we prove that the assumption (5) is also necessary for (1) to have solutions.

4.2 Main Results

We start this section by a nonexistence result in which we prove the necessity of condition (5).

Theorem 1 (Nonexistence) *Assume* $\int_0^1 ta(t)dt = \infty$. *Then* (1) *has no solutions.*

We next assume that (5) holds.

Theorem 2 (Sublinear case) *Assume* (5) *and conditions* $(f1)$, $(f2)$, $(g1)$ *hold.*

(i) *If* $0 < a < 1$, *then problem* (1) *has at least one solution, for all* $\lambda \in \mathbb{R}$;
(ii) *If* $1 < a \leq 2$, *then there exists* $\lambda^* > 0$ *such that* (1) *has at least one classical solution for all* $-\infty < \lambda < \lambda^*$ *and no solution exists if* $\lambda > \lambda^*$.

We shall next focus on the case $a = 1$. This case was left as an open question in [14]. We are able here to give a complete answer in the case where Ω is a ball centered at the origin.

Theorem 3 (Case $a = 1$) *Assume* $(f1)$, $(f2)$, (5), $a = 1$ *and* $\Omega = B_R(0)$ *for some* $R > 0$. *Then the problem* (1) *has at least one solution for all* $\lambda \in \mathbb{R}$.

The existence of a solution to (1) is achieved by the sub and super-solution method. In particular, the super-solution of (1) is expressed in terms of the solution H to (3). In some particular cases we are able to describe the asymptotic behavior of solutions near the boundary. This is our next task here.

Let $a(t) = t^{-\alpha}$, $g(t) = t^{-p}$, $\alpha, p > 0$ and consider the following related problem:

$$\begin{cases} -\Delta u = \delta(x)^{-\alpha} u^{-p} + f(x,u) + \lambda |\nabla u|^q & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{6}$$

Then we have:

Theorem 4 (Asymptotic behavior) *Assume* $(g1)$, $(f1)$, $(f2)$.

(i) *If* $\alpha \geq 2$, *then the problem* (6) *has no classical solutions.*
(ii) *If* $\alpha < 2$, *then there exists* $\lambda^* \in (0, \infty]$ (*with* $\lambda^* = \infty$ *if* $0 < a < 1$) *such that problem* (6) *has at least one classical solution* u, *for all* $-\infty < \lambda < \lambda^*$. *Moreover, for all* $0 < \lambda < \lambda^*$, *there exist* $0 < \eta < 1$ *and* $C_1, C_2 > 0$ *such that* u *satisfies*
   (ii1) *If* $\alpha + p > 1$, *then*

$$C_1\delta(x)^{(2-\alpha)/(1+p)} \leq u(x) \leq C_2\delta(x)^{(2-\alpha)/(1+p)}, \quad \textit{for all } x \in \Omega; \tag{7}$$

   (ii2) *If* $\alpha + p = 1$, *then*

$$C_1\delta(x)\ln^{1/(2-\alpha)}\left(\frac{1}{\delta(x)}\right) \leq u(x) \leq C_2\delta(x)\ln^{1/(2-\alpha)}\left(\frac{1}{\delta(x)}\right), \tag{8}$$

for all $x \in \Omega$ *with* $\delta(x) < \eta$;

(ii3) *If* $\alpha + p < 1$, *then*

$$C_1\delta(x) \le u(x) \le C_2\delta(x), \quad \text{for all } x \in \Omega. \tag{9}$$

We have seen that if $a(t) = t^{-\alpha}$ then (1) has no solutions if $\alpha \ge 2$. Motivated by the results in [12], let us now consider the extremal case $a(t) = t^{-2}\ln^{\alpha}(A/t)$ where $A > \operatorname{diam}(\Omega)$ and the corresponding boundary value problem

$$\begin{cases} -\Delta u = \delta(x)^{-2}\ln^{\alpha}(\frac{A}{\delta(x)})u^{-p} + f(x,u) + \lambda|\nabla u|^q & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{10}$$

Theorem 5 (Asymptotic behavior) *Assume* $(g1)$, $(f1)$, $(f2)$.

(i) *If* $\alpha \ge -1$, *then problem* (10) *has no classical solutions.*
(ii) *If* $\alpha < -1$, *then there exists* $\lambda^* \in (0,\infty]$ (*with* $\lambda^* = \infty$ *if* $0 < a < 1$) *such that problem* (6) *has at least one classical solution* u, *for all* $-\infty < \lambda < \lambda^*$. *Moreover, there exist* $C_1, C_2 > 0$ *such that* u *satisfies*

$$C_1 \ln^{(1-\alpha)/(1+p)}\left(\frac{A}{\delta(x)}\right) \le u(x) \le C_2 \ln^{(1-\alpha)/(1+p)}\left(\frac{A}{\delta(x)}\right), \quad \text{for all } x \in \Omega. \tag{11}$$

In the following we study the problem (1) in which we drop out the sublinearity assumptions $(f1)$, $(f2)$ on f but we require in turn that f is linear. More precisely, we assume that $f(x,t) = \mu t$, for some $\mu > 0$. Note that the existence results established in Lemma 4 in [24] or [25] do not apply here since the mapping

$$\Psi(x,t) = a\big(\delta(x)\big)g(t) + \lambda t, \quad (x,t) \in \Omega \times (0,\infty),$$

is not defined on $\partial\Omega \times (0,\infty)$.

Theorem 6 (Linear case) *Assume* (5), $(g1)$, $f(x,u) = \mu u$ *for some* $\mu > 0$ *and* $0 < a < 1$. *Then for any* $\lambda \ge 0$ *problem* (1) *has solutions if and only if* $\mu < \lambda_1$.

4.3 Proof of Theorem 1

The proof of Theorem 1 follows from the following more general result.

Proposition 1 *Assume that $\int_0^1 ta(t)dt = \infty$. Then the inequality boundary value problem*

$$\begin{cases} -\Delta u + \lambda|\nabla u|^2 \geq a(\delta(x))g(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{12}$$

has no classical solutions.

Proof Let (λ_1, φ_1) be the first eigenvalue/eigenfunction of $-\Delta$ in Ω subject to a homogeneous Dirichlet boundary condition. It is known that $\lambda_1 > 0$ and by normalization, one can assume $\varphi_1 > 0$ in Ω. It suffices to prove the result only for $\lambda > 0$. We argue by contradiction and assume that there exists $u \in C^2(\Omega) \cap C(\overline{\Omega})$ a solution of (12). Using $(g1)$, we can find $c_1 > 0$ such $\underline{u} := c_1\varphi_1$ verifies

$$-\Delta\underline{u} + \lambda|\nabla\underline{u}|^2 \leq a\big(\delta(x)\big)g(\underline{u}) \quad \text{in } \Omega.$$

Since g is decreasing, we easily obtain

$$u \geq \underline{u} \quad \text{in } \Omega. \tag{13}$$

We make in (12) the change of variable $v = 1 - e^{-\lambda u}$. Therefore

$$\begin{cases} -\Delta v = \lambda(1-v)(\lambda|\nabla u|^2 - \Delta u) \geq \lambda(1-v)a(\delta(x))g(-\frac{\ln(1-v)}{\lambda}) & \text{in } \Omega, \\ v > 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{14}$$

In order to avoid the singularities in (14) let us consider the approximated problem

$$\begin{cases} -\Delta v = \lambda(1-v)a(\delta(x))g(\varepsilon - \frac{\ln(1-v)}{\lambda}) & \text{in } \Omega, \\ v > 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{15}$$

with $0 < \varepsilon < 1$. Clearly v is a super-solution of (15). By (13) and the fact that $\lim_{t\to 0^+} \frac{1-e^{-\lambda t}}{t} = \lambda > 0$, there exists $c_2 > 0$ such that $v \geq c_2\varphi_1$ in Ω. On the other hand, there exists $0 < c < c_2$ such that $c\varphi_1$ is a sub-solution of (15) and obviously $c\varphi_1 \leq v$ in Ω. Then, by the standard sub- and super-solution method (see, e.g., [16, 21]) the problem (15) has a solution $v_\varepsilon \in C^2(\overline{\Omega})$ such that

$$c\varphi_1 \leq v_\varepsilon \leq v \quad \text{in } \Omega. \tag{16}$$

Multiplying by φ_1 in (15) and integrating we find

$$\lambda_1 \int_\Omega \varphi_1 v_\varepsilon dx = C \int_\Omega (1-v_\varepsilon)\varphi_1 a\big(\delta(x)\big)g\bigg(\varepsilon - \frac{\ln(1-v_\varepsilon)}{\lambda}\bigg)dx.$$

Using (16) we obtain

$$M =: \lambda_1 \int_\Omega \varphi_1 v dx \geq \lambda \int_\Omega (1-v)\varphi_1 a\big(\delta(x)\big) g\left(-\frac{\ln(1-v)}{\lambda}\right) dx$$
$$\geq C_1 \int_{\Omega_\delta} \varphi_1 a\big(\delta(x)\big) dx, \tag{17}$$

where $\Omega_\delta \supset \{x \in \Omega; \delta(x) < \delta\}$, for some $\delta > 0$ sufficiently small. Since $\varphi_1(x)$ behaves like $\delta(x)$ in Ω_δ and $\int_0^1 ta(t)dt = \infty$, by (17) we find a contradiction. Hence, problem (12) has no classical solutions and the proof is now complete. □

4.4 Proof of Theorem 2

The existence part in this result relies on the sub and super-solution method. Basic to our approach is the following comparison result whose proof may be found in [15].

Lemma 1 *Let $\Psi : \overline{\Omega} \times (0, +\infty) \to \mathbb{R}$ be a Hölder continuous function such that the mapping $(0, +\infty) \ni s \longmapsto \frac{\Psi(x,s)}{s}$ is strictly decreasing for each $x \in \Omega$. Assume that there exist $v, w \in C^2(\Omega) \cap C(\overline{\Omega})$ such that*

(a) $\Delta w + \Psi(x, w) \leq 0 \leq \Delta v + \Psi(x, v)$ *in* Ω;
(b) $v, w > 0$ *in* Ω *and* $v \leq w$ *on* $\partial\Omega$;
(c) $\Delta v \in L^1(\Omega)$ *or* $\Delta w \in L^1(\Omega)$.

Then $v \leq w$ *in* Ω.

We shall divide our arguments into two cases according to the values of λ.
(i) Case $\lambda > 0$. By Lemma 4 in [24] there exists $\zeta \in C^2(\overline{\Omega})$ such that

$$\begin{cases} -\Delta\zeta = f(x,\zeta) & \text{in } \Omega, \\ \zeta > 0 & \text{in } \Omega, \\ \zeta = 0 & \text{on } \partial\Omega. \end{cases} \tag{18}$$

Thus, ζ is a sub-solution of (1) provided $\lambda > 0$. We focus now on finding a super-solution $\overline{u}_\lambda$ of (1) such that $\zeta \leq \overline{u}_\lambda$ in Ω.

Let H be the solution of (3). Since H is concave, there exists $H'(0+) \in (0, \infty]$. Taking $0 < b < 1$ small enough, we can assume that $H' > 0$ in $(0, b]$, so H is increasing on $[0, b]$. Multiplying by H' in (3) and integrating on $[t, b]$, we find

$$\big(H'\big)^2(t) - \big(H'\big)^2(b) = 2\int_t^b a(s) g\big(H(s)\big) H'(s) ds \leq 2a(t) \int_{H(t)}^{H(b)} g(\tau) d\tau. \tag{19}$$

Using the monotonicity of g it follows that

$$\big(H'\big)^2(t) \le 2H(b)a(t)g\big(H(t)\big) + \big(H'\big)^2(b), \quad \text{for all } 0 < t \le b. \tag{20}$$

Hence, there exist $C_1, C_2 > 0$ such that

$$\big(H'\big)(t) \le C_1 a(t)g\big(H(t)\big), \quad \text{for all } 0 < t \le b \tag{21}$$

and

$$\big(H'\big)^2(t) \le C_2 a(t)g\big(H(t)\big), \quad \text{for all } 0 < t \le b. \tag{22}$$

Now we can proceed to construct a super-solution for (1). First, we fix $c > 0$ such that

$$c\varphi_1 \le \min\big\{b, \delta(x)\big\} \quad \text{in } \Omega. \tag{23}$$

By Hopf's maximum principle, there exist $\omega \subset\subset \Omega$ and $\delta > 0$ such that

$$|\nabla\varphi_1| > \delta \quad \text{in } \Omega \setminus \omega. \tag{24}$$

Moreover, since

$$\lim_{\delta(x)\to 0^+} \big\{c^2 a(c\varphi_1)g\big(H(c\varphi_1)\big)|\nabla\varphi_1|^2 - 3f\big(x, H(c\varphi_1)\big)\big\} = \infty,$$

we can assume that

$$c^2 a(c\varphi_1)g\big(H(c\varphi_1)\big)|\nabla\varphi_1|^2 \ge 3f\big(x, H(c\varphi_1)\big) \quad \text{in } \Omega \setminus \omega. \tag{25}$$

Let $M > 1$ be such that

$$Mc^2\delta^2 > 3. \tag{26}$$

Since $H'(0+) > 0$ and $0 < a < 1$, we can choose $M > 1$ such that

$$M\frac{(c\delta)^2}{C_1}H'(c\varphi_1) \ge 3\lambda\big(McH'(c\varphi_1)|\nabla\varphi_1|\big)^q \quad \text{in } \Omega \setminus \omega,$$

where C_1 is the constant appearing in (21). By (21), (24) and (26) we derive

$$Mc^2 a(c\varphi_1)g\big(H(c\varphi_1)\big)|\nabla\varphi_1|^2 \ge 3\lambda\big(McH'(c\varphi_1)|\nabla\varphi_1|\big)^q \quad \text{in } \Omega \setminus \omega. \tag{27}$$

Since g is decreasing and $H'(c\varphi_1) > 0$ in $\overline{\omega}$, there exists $M > 0$ such that

$$Mc\lambda_1\varphi_1 H'(c\varphi_1) \ge 3a\big(\delta(x)\big)g\big(H(c\varphi_1)\big) \quad \text{in } \omega. \tag{28}$$

In the same manner, using $(f2)$ and the fact that $\varphi_1 > 0$ in $\overline{\omega}$, we can choose $M > 1$ large enough such that

$$Mc\lambda_1\varphi_1 H'(c\varphi_1) \ge 3\lambda\big(McH'(c\varphi_1)|\nabla\varphi_1|\big)^q \quad \text{in } \omega, \tag{29}$$

and

$$Mc\lambda_1\varphi_1 H'(c\varphi_1) \geq 3f\big(x, MH(c\varphi_1)\big) \quad \text{in } \omega. \tag{30}$$

For M satisfying (26)–(30), we prove that

$$\overline{u}_\lambda(x) := MH\big(c\varphi_1(x)\big), \quad \text{for all } x \in \Omega, \tag{31}$$

is a super-solution of (1). We have

$$-\Delta\overline{u}_\lambda = Mc^2 a(c\varphi_1)g\big(H(c\varphi_1)\big)|\nabla\varphi_1|^2 + Mc\lambda_1\varphi_1 H'(c\varphi_1) \quad \text{in } \Omega. \tag{32}$$

We first show that

$$\begin{aligned} &Mc^2 a(c\varphi_1)g\big(H(c\varphi_1)\big)|\nabla\varphi_1|^2 \\ &\quad \geq a\big(\delta(x)\big)g(\overline{u}_\lambda) + f(x, \overline{u}_\lambda) + \lambda|\nabla\overline{u}_\lambda|^q \quad \text{in } \Omega\setminus\omega. \end{aligned} \tag{33}$$

Indeed, by (23), (24) and (26) we get

$$\begin{aligned} \frac{M}{3}c^2 a(c\varphi_1)g\big(H(c\varphi_1)\big)|\nabla\varphi_1|^2 &\geq a\big(\delta(x)\big)g\big(H(c\varphi_1)\big) \\ &\geq a\big(\delta(x)\big)g\big(MH(c\varphi_1)\big) \\ &= a\big(\delta(x)\big)g(\overline{u}_\lambda) \quad \text{in } \Omega\setminus\omega. \end{aligned} \tag{34}$$

The assumption $(f1)$ and (25) produce

$$\begin{aligned} \frac{M}{3}c^2 a(c\varphi_1)g\big(H(c\varphi_1)\big)|\nabla\varphi_1|^2 &\geq Mf\big(x, H(c\varphi_1)\big) \\ &\geq f\big(x, MH(c\varphi_1)\big) \\ &= f(x, \overline{u}_\lambda) \quad \text{in } \Omega\setminus\omega. \end{aligned} \tag{35}$$

From (27) we obtain

$$\begin{aligned} \frac{M}{3}c^2 a(c\varphi_1)g\big(H(c\varphi_1)\big)|\nabla\varphi_1|^2 &\geq \lambda\big(McH'(c\varphi_1)|\nabla\varphi_1|\big)^q \\ &= \lambda|\nabla\overline{u}_\lambda|^q \quad \text{in } \Omega\setminus\omega. \end{aligned} \tag{36}$$

Now, relation (33) follows by (34), (35) and (36).

Next we prove that

$$Mc\lambda_1\varphi_1 H'(c\varphi_1) \geq a\big(\delta(x)\big)g(\overline{u}_\lambda) + f(x, \overline{u}_\lambda) + \lambda|\nabla\overline{u}_\lambda|^q \quad \text{in } \omega. \tag{37}$$

From (28) and (29) we get

$$\begin{aligned}\frac{M}{3}c\lambda_1\varphi_1 H'(c\varphi_1) &\geq a\big(\delta(x)\big)g\big(H(c\varphi_1)\big)\\ &\geq a\big(\delta(x)\big)g\big(MH(c\varphi_1)\big)\\ &= a\big(\delta(x)\big)g(\overline{u}_\lambda) \quad \text{in } \omega \end{aligned} \tag{38}$$

and

$$\begin{aligned}\frac{M}{3}c\lambda_1\varphi_1 H'(c\varphi_1) &\geq \lambda\big(McH'(c\varphi_1)|\nabla\varphi_1|\big)^q\\ &= \lambda|\nabla\overline{u}_\lambda|^q \quad \text{in } \omega. \end{aligned} \tag{39}$$

Finally, from (30) we derive

$$\frac{M}{3}c\lambda_1\varphi_1 H'(c\varphi_1) \geq f\big(x, MH(c\varphi_1)\big) = f(x,\overline{u}_\lambda) \quad \text{in } \omega. \tag{40}$$

Clearly, relation (37) follows from (38), (39) and (40).

Combining (32) with (33) and (37) we conclude that $\overline{u}_\lambda$ is a super-solution of (1). Thus, by Lemma 1 we obtain $\zeta \leq \overline{u}_\lambda$ in Ω and by sub and super-solution method it follows that (1) has at least one classical solution for all $\lambda > 0$.

CASE $\lambda \leq 0$. We fix $\nu > 0$ and let $u_\nu \in C^2(\Omega)\cap C(\overline{\Omega})$ be a solution of (1) for $\lambda = \nu$. Then u_ν is a super-solution of (1) for all $\lambda \leq 0$. Set

$$m := \inf_{(x,t)\in\overline{\Omega}\times(0,\infty)} \big(a\big(\delta(x)\big)g(t) + f(x,t)\big).$$

Since $\lim_{t\to 0^+} g(t) = \infty$ and the mapping $(0,\infty)\ni t \longmapsto \min_{x\in\overline{\Omega}} f(x,t)$ is positive and nondecreasing, we deduce that m is a positive real number. Consider the problem

$$\begin{cases} -\Delta v = m + \lambda|\nabla v|^q & \text{in } \Omega,\\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{41}$$

Clearly zero is a sub-solution of (41). Since $\lambda \leq 0$, the solution w of the problem

$$\begin{cases} -\Delta w = m & \text{in } \Omega,\\ w = 0 & \text{on } \partial\Omega, \end{cases}$$

is a super-solution of (41). Hence, (41) has at least one solution $v \in C^2(\Omega)\cap C(\overline{\Omega})$. We claim that $v > 0$ in Ω. Indeed, if not, we deduce that $\min_{x\in\overline{\Omega}} v$ is achieved at some point $x_0 \in \Omega$. Then $\nabla v(x_0) = 0$ and

$$-\Delta v(x_0) = m + \lambda\big|\nabla v(x_0)\big|^q = m > 0, \quad \text{contradiction.}$$

Therefore, $v > 0$ in Ω. It is easy to see that v is sub-solution of (1) and $-\Delta v \leq m \leq -\Delta u_\nu$ in Ω, which gives $v \leq u_\nu$ in Ω. Again by the sub and super-solution method we conclude that (1) has at least one classical solution $u_\lambda \in C^2(\Omega)\cap C(\overline{\Omega})$.

(ii) The proof follows the same steps as above. The only difference is that (27) and (29) are no more valid for any $\lambda > 0$. The main difficulty when dealing with estimates like (27) is that $H'(c\varphi_1)$ may blow-up at the boundary. However, combining the assumption $1 < a \le 2$ with (22), we can choose $\lambda > 0$ small enough such that (27) and (29) hold. This implies that the problem (1) has a classical solution provided $\lambda > 0$ is sufficiently small.

Set

$$A = \{\lambda > 0;\ \text{problem (1) has at least one classical solution}\}.$$

From the above arguments, A is nonempty. Let $\lambda^* = \sup A$. We first claim that if $\lambda \in A$, then $(0, \lambda) \subseteq A$. To this aim, let $\lambda_1 \in A$ and $0 < \lambda_2 < \lambda_1$. If u_{λ_1} is a solution of (1) with $\lambda = \lambda_1$, then u_{λ_1} is a super-solution of (1) with $\lambda = \lambda_2$, while ζ defined in (18) is a sub-solution. Using Lemma 1 once more, we get $\zeta \le u_{\lambda_1}$ in Ω so (1) has at least one classical solution for $\lambda = \lambda_2$. This proves the claim. Since $\lambda_1 \in A$ was arbitrary, we conclude that $(0, \lambda^*) \subset A$.

Next, we prove that $\lambda^* < \infty$. To this aim, we use the following result which is a consequence of Theorem 2.1 in [2].

Lemma 2 *Assume that $a > 1$. Then there exists a positive number $\bar{\sigma}$ such that the problem*

$$\begin{cases} -\Delta v \ge |\nabla v|^q + \sigma & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{42}$$

has no solutions for $\sigma > \bar{\sigma}$.

Consider $\lambda \in A$ and let u_λ be a classical solution of (1). Set $v = \lambda^{1/(a-1)} u_\lambda$. Using our assumption $1 < a \le 2$, we deduce that v fulfills

$$\begin{cases} -\Delta v \ge |\nabla v|^q + m\lambda^{1/(a-1)} & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{43}$$

According to Lemma 2, we obtain $m\lambda^{1/(a-1)} \le \bar{\sigma}$, that is, $\lambda \le (\frac{\bar{\sigma}}{m})^{a-1}$. This means that $\lambda^* \le (\frac{\bar{\sigma}}{m})^{a-1}$, hence λ^* is finite. The existence of a solution in the case $\lambda \le 0$ can be achieved in the same manner as above.

This finishes the proof of Theorem 2.

4.5 Proof of Theorem 3

Let us note first that in our setting problem (1) reads

$$\begin{cases} -\Delta u = a(R - |x|)g(u) + f(x, u) + \lambda|\nabla u| & |x| < R, \\ u > 0 & |x| < R, \\ u = 0 & |x| = R. \end{cases} \tag{44}$$

The case $\lambda \le 0$ is the same as in the proof of Theorem 2(i). In what follows, we assume that $\lambda > 0$. Using Theorem 2(i) it is easy to see that there exists $\underline{u} \in C^2(\Omega) \cap C(\overline{\Omega})$ such that

$$\begin{cases} -\Delta \underline{u} = a(R-|x|)g(\underline{u}) & |x| < R, \\ \underline{u} > 0 & |x| < R, \\ \underline{u} = 0 & |x| = R. \end{cases}$$

It is obvious that $\underline{u}$ is a sub-solution of (44) for all $\lambda > 0$. In order to provide a super-solution of (44) we consider the problem

$$\begin{cases} -\Delta u = a(R-|x|)g(u) + 1 + \lambda|\nabla u| & |x| < R, \\ u > 0 & |x| < R, \\ u = 0 & |x| = R. \end{cases} \tag{45}$$

We need the following auxiliary result.

Lemma 3 *Problem* (45) *has at least one solution.*

Proof We are looking for radially symmetric solution u of (45), that is, $u = u(r)$, $0 \le r = |x| \le R$. In this case, problem (45) becomes

$$\begin{cases} -u'' - \frac{N-1}{r}u'(r) = a(R-r)g(u(r)) + 1 + \lambda|u'(r)| & 0 \le r < R, \\ u > 0 & 0 \le r < R, \\ u(R) = 0. \end{cases} \tag{46}$$

This implies

$$-\big(r^{N-1}u'(r)\big)' \ge 0 \quad \text{for all } 0 \le r < R,$$

which yields $u'(r) \le 0$ for all $0 \le r < R$. Then (46) gives

$$-\left(u'' + \frac{N-1}{r}u'(r) + \lambda u'(r)\right) = a(R-r)g\big(u(r)\big) + 1, \quad 0 \le r < R.$$

We obtain

$$-\big(e^{\lambda r}r^{N-1}u'(r)\big)' = e^{\lambda r}r^{N-1}\psi\big(r, u(r)\big), \quad 0 \le r < R, \tag{47}$$

where

$$\psi(r,t) = a(R-r)g(t) + 1, \quad (r,t) \in [0,R) \times (0,\infty).$$

From (47) we obtain

$$u(r) = u(0) - \int_0^r e^{-\lambda t}t^{-N+1}\int_0^t e^{\lambda s}s^{N-1}\psi\big(s,u(s)\big)ds\,dt, \quad 0 \le r < R. \tag{48}$$

On the other hand, in view of Theorem 2 and using the fact that g is decreasing, there exists a unique solution $w \in C^2(B_R(0)) \cap C(\overline{B}_R(0))$ of the problem

$$\begin{cases} -\Delta w = a(R-|x|)g(w)+1 & |x|<R, \\ w>0 & |x|<R, \\ w=0 & |x|=R. \end{cases} \tag{49}$$

Clearly, w is a sub-solution of (45). Due to the uniqueness and to the symmetry of the domain, w is radially symmetric, so, $w=w(r)$, $0 \le r = |x| \le R$. As above we get

$$w(r) = w(0) - \int_0^r t^{-N+1} \int_0^t s^{N-1} \psi\big(s, w(s)\big) ds\, dt, \quad 0 \le r < R. \tag{50}$$

We claim that there exists a solution $v \in C^2[0,R) \cap C[0,R]$ of (48) such that $v>0$ in $[0,R)$.

Let $A = w(0)$ and define the sequence $(v_k)_{k\ge 1}$ by

$$\begin{cases} v_k(r) = A - \int_0^r e^{-\lambda t} t^{-N+1} \int_0^t e^{\lambda s} s^{N-1} \psi(s, v_{k-1}(s)) ds\, dt, \\ \quad 0 \le r < R, k \ge 1, \\ v_0 = w. \end{cases} \tag{51}$$

Note that v_k is decreasing in $[0,R)$ for all $k \ge 0$. From (50) and (51) we easily check that $v_1 \ge v_0$ and by induction we deduce $v_k \ge v_{k-1}$ for all $k \ge 1$. Hence

$$w = v_0 \le v_1 \le \cdots \le v_k \le \cdots \le A \quad \text{in } B_R(0).$$

Thus, there exists $v(r) := \lim_{k\to\infty} v_k(r)$, for all $0 \le r < R$ and $v>0$ in $[0,R)$. We now can pass to the limit in (51) in order to get that v is a solution of (48). By classical regularity results we also obtain $v \in C^2[0,R) \cap C[0,R]$. This proves the claim.

We have obtained a super-solution v of (45) such that $v \ge w$ in $B_R(0)$. So, the problem (45) has at least one solution and the proof of our Lemma is now complete. □

Let u be a solution of the problem (45). For $M>1$ we have

$$\begin{aligned} -\Delta(Mu) &= Ma\big(R-|x|\big)g(u) + M + \lambda\big|\nabla(Mu)\big| \\ &\ge a\big(R-|x|\big)g(Mu) + M + \lambda\big|\nabla(Mu)\big|. \end{aligned} \tag{52}$$

Since f is sublinear, we can choose $M>1$ such that

$$M \ge f\big(x, M|u|_\infty\big) \quad \text{in } B_R(0).$$

Then $\overline{u}_\lambda := Mu$ satisfies

$$-\Delta \overline{u}_\lambda \ge a\big(R-|x|\big)g(\overline{u}_\lambda) + f(x, \overline{u}_\lambda) + \lambda|\nabla \overline{u}_\lambda| \quad \text{in } B_R(0).$$

It follows that $\overline{u}_\lambda$ is a super-solution of (44). Since g is decreasing we easily deduce $\underline{u} \le \overline{u}_\lambda$ in $B_R(0)$ so, problem (1) has at least one solution.

The proof of Theorem 3 is now complete.

4.6 Proof of Theorem 4 and Theorem 5

Proof of Theorem 4 The existence and nonexistence of a solution to (6) follows directly from Theorems 1 and 2. We next prove the boundary estimates (7)–(9).

Recall that if $\int_0^1 ta(t)dt < \infty$ and λ belongs to a certain range, then Theorem 2 asserts that (1) has at least one classical solution u satisfying $u \le MH(c\varphi_1)$ in Ω, for some $M, c > 0$. Here H is the solution of

$$\begin{cases} H''(t) = -t^{-\alpha}H^{-p}(t), & \text{for all } 0 < t \le b < 1, \\ H, H' > 0 & \text{in } (0, b], \\ H(0) = 0. \end{cases} \tag{53}$$

With the same idea as in the proof of Theorem 2, we can show that there exists $m > 0$ small enough such that $v := mH(c\varphi_1)$ satisfies

$$-\Delta v \le \delta(x)^{-\alpha} v^{-p} \quad \text{in } \Omega. \tag{54}$$

Indeed, we have

$$-\Delta v = m\big[c^{2-\alpha}|\nabla\varphi_1|^2\varphi_1^{-\alpha}H^{-p}(c\varphi_1) + \lambda_1 c\varphi_1 H'(c\varphi_1)\big] \quad \text{in } \Omega.$$

Thus, there exist two positive constants $c_1, c_2 > 0$ such that

$$-\Delta v \le m\big[c_1|\nabla\varphi_1|^2 + c_2\varphi_1\big]\delta(x)^{-\alpha}H^{-p}(c\varphi_1) \quad \text{in } \Omega.$$

Clearly (54) holds if we choose $m > 0$ small enough such that $m[c_1|\nabla\varphi_1|^2 + c_2\varphi_1] < 1$ in Ω. Moreover, v is a sub-solution of (6) for all $\mu > 0$ and one can easily see that $v \le u_\mu$ in Ω. Hence

$$mH(c\varphi_1) \le u \le MH(c\varphi_1) \quad \text{in } \Omega. \tag{55}$$

Now, a careful analysis of (53) together with (55) is used in order to obtain boundary estimates for the solution of (6). Our estimates complete the results in Theorem 2.1 in [17] since here the potential $a(\delta(x))$ blows-up at the boundary.

(ii1) Remark that

$$H(t) = \left(\frac{(1+p)^2}{(2-\alpha)(\alpha+p-1)}\right)^{1/(1+p)} t^{(2-\alpha)/(1+p)}, \quad t > 0,$$

is a solution of (53) provided $\alpha + p > 1$. The conclusion in this case follows now from (55).

(ii2) Note that in this case problem (53) becomes

$$\begin{cases} H''(t) = -t^{-\alpha} H^{\alpha-1}(t), & \text{for all } 0 < t \le b < 1, \\ H(0) = 0, & \\ H > 0 & \text{in } (0, b]. \end{cases} \tag{56}$$

Let $w = t \ln^{1/(2-\alpha)}(\frac{1}{t})$, $t > 0$. Then

$$-w''(t) \sim t^{-1} \ln^{(\alpha-1)/(2-\alpha)}\left(\frac{1}{t}\right) \sim t^{-\alpha} w^{-p}$$

in a neighborhood of the origin. Now if $m > 0$ is small enough it follows that w satisfies $-(mw)'' \le t^{-\alpha}(mw)^{\alpha-1}$ in $(0, b)$ and $mw(b) \le H(b)$. By the maximum principle we find $H \ge mw$ in $(0, b)$, that is

$$H(t) \ge c_1 t \ln^{1/(2-\alpha)}\left(\frac{1}{t}\right) \quad \text{in } (0, b).$$

Similarly, if $M > 1$ is large enough we have $-(Mw)'' \le t^{-\alpha}(Mw)^{\alpha-1}$ in $(0, b)$ and $Mw(b) \ge H(b)$. By the maximum principle we find $H \le Mw$ in $(0, b)$, that is

$$H(t) \le c_2 t \ln^{1/(2-\alpha)}\left(\frac{1}{t}\right) \quad \text{in } (0, b).$$

Now the desired estimate follows from (55).

(ii3) Using the fact that $H'(0+) \in (0, \infty]$ we get the existence of $c > 0$ such that

$$H(t) > ct, \quad \text{for all } 0 < t \le b.$$

This yields

$$-H''(t) \le c^{-p} t^{-(\alpha+p)}, \quad \text{for all } 0 < t \le b.$$

Since $\alpha + p < 1$, it follows that $H'(0+) < \infty$, that is, $H \in C^1[0, b]$. Thus, there exists $c_1, c_2 > 0$ such that

$$c_1 t \le H(t) \le c_2 t, \quad \text{for all } 0 < t \le b. \tag{57}$$

The conclusion in Theorem 4(iii) follows directly from (57) and (55).

This completes the proof of Theorem 4. □

Proof of Theorem 5 This follows in the same way as above. The estimate (11) follows by using the approach in Theorem 4(ii2) with $w(t) = \ln^{(1-\alpha)/(1+p)}(A/t)$. □

4.7 Proof of Theorem 6

Fix $\mu \in (0, \lambda_1)$ and $\lambda \geq 0$. By Theorem 2(i) there exists $u \in C^2(\Omega) \cap C(\overline{\Omega})$ a solution of the problem

$$\begin{cases} -\Delta u = a(\delta(x))g(u) + \lambda|\nabla u|^q & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Obviously, $\underline{u}_{\lambda\mu} := u$ is a sub-solution of (1). Since $\mu < \lambda_1$, there exists $v \in C^2(\overline{\Omega})$ such that

$$\begin{cases} -\Delta v = \mu v + 2 & \text{in } \Omega, \\ v > 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Since $0 < a < 1$, we can choose $M > 0$ large enough such that

$$M > \mu|u|_\infty \quad \text{and} \quad M > \lambda(M|\nabla v|)^q \quad \text{in } \Omega.$$

Then $w := Mv$ satisfies

$$-\Delta w \geq \mu(u + w) + \lambda|\nabla w|^q \quad \text{in } \Omega.$$

We claim that $\overline{u}_{\lambda\mu} := u + w$ is a super-solution of (1). Indeed, we have

$$-\Delta\overline{u}_{\lambda\mu} \geq a\big(\delta(x)\big)g(u) + \lambda\overline{u}_{\lambda\mu} + \lambda|\nabla u|^q + \lambda|\nabla w|^q \quad \text{in } \Omega. \tag{58}$$

Using the assumption $0 < a < 1$ one can easily deduce

$$t_1^q + t_2^q \geq (t_1 + t_2)^q, \qquad \text{for all } t_1, t_2 \geq 0.$$

Hence

$$|\nabla u|^q + |\nabla w|^q \geq \big(|\nabla u| + |\nabla w|\big)^q \geq \big|\nabla(u + w)\big|^q \quad \text{in } \Omega. \tag{59}$$

Combining (58) with (59) we obtain

$$-\Delta\overline{u}_{\lambda\mu} \geq a\big(\delta(x)\big)g(\overline{u}_{\lambda\mu}) + \mu\overline{u}_{\lambda\mu} + \lambda|\nabla\overline{u}_{\lambda\mu}|^q \quad \text{in } \Omega.$$

Hence, $(\underline{u}_{\lambda\mu}, \overline{u}_{\lambda\mu})$ is an ordered pair of sub and super-solution of (1), so there exists a classical solution $u_{\lambda\mu}$ of (1), provided $\lambda \geq 0$ and $0 < \mu < \lambda_1$. Assume by contradiction that there exist $\mu \geq \lambda_1$ and $\lambda \geq 0$ such that the problem (1) has a classical solution $u_{\lambda\mu}$. If $m = \min_{x\in\overline{\Omega}} a(\delta(x))g(u_{\lambda\mu}) > 0$ it follows that $u_{\lambda\mu}$ is a super-solution of

$$\begin{cases} -\Delta u = \mu u + m & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{60}$$

Clearly, zero is a sub-solution of (60), so there exists a classical solution u of (60) such that $u \le u_{\lambda\mu}$ in Ω. By maximum principle and elliptic regularity we get $u > 0$ in Ω and $u \in C^2(\overline{\Omega})$. To raise a contradiction, we proceed as in the proof of Theorem 2(ii).

Multiplying by φ_1 in (60) and then integrating over Ω we find

$$-\int_\Omega \varphi_1 \Delta u = \mu \int_\Omega u\varphi_1 + m \int_\Omega \varphi_1.$$

This implies $\lambda_1 \int_\Omega u\varphi_1 = \mu \int_\Omega u\varphi_1 + m \int_\Omega \varphi_1$, which is a contradiction, since $\mu \ge \lambda_1$ and $m > 0$. The proof of Theorem 6 is now complete.

References

1. Agarwal, R., O'Regan, D.: Existence theory for single and multiple solutions to singular positone boundary value problems. J. Differ. Equ. **175**, 393–3414 (2001)
2. Alaa, N.E., Pierre, M.: Weak solutions of some quasilinear elliptic equations with data measures. SIAM J. Math. Anal. **24**, 23–35 (1993)
3. Brauner, C.M., Nicolaenko, B.: On nonlinear eigenvalue problems which extend into free boundaries. In: Bifurcation and Nonlinear Eigenvalue Problems: Proceedings of a Session, Univ. Paris XIII, Villetaneuse, 1978, pp. 61–100. Springer, Berlin (1980)
4. Caffarelli, L., Hardt, R., Simon, L.: Minimal surfaces with isolated singularities. Manuscr. Math. **48**, 1–18 (1984)
5. Callegari, A., Nachman, A.: A nonlinear singular boundary value problem in the theory of pseudoplastic fluids. SIAM J. Appl. Math. **38**, 275–281 (1980)
6. Cîrstea, F.C., Ghergu, M., Rădulescu, V.: Combined effects of asymptotically linear and singular nonlinearities in bifurcation problems of Lane-Emden-Fowler type. J. Math. Pures Appl. **84**, 493–508 (2005)
7. Coclite, M.M., Palmieri, G.: On a singular nonlinear Dirichlet problem. Commun. Partial Differ. Equ. **14**, 1315–1327 (1989)
8. Cohen, D.S., Keller, H.B.: Some positive problems suggested by nonlinear heat generators. J. Math. Mech. **16**, 1361–1376 (1967)
9. Crandall, M.G., Rabinowitz, P.H., Tartar, L.: On a Dirichlet problem with a singular nonlinearity. Commun. Partial Differ. Equ. **2**, 193–222 (1977)
10. Díaz, J.I.: Nonlinear Partial Differential Equations and Free Boundaries. Vol. I: Elliptic Equations. Research Notes in Mathematics, vol. 106. Pitman, Boston (1985)
11. Díaz, J.I., Morel, J.M., Oswald, L.: An elliptic equation with singular nonlinearity. Commun. Partial Differ. Equ. **12**, 1333–1344 (1987)
12. Ghergu, M.: Lane-Emden systems with negative exponents. J. Funct. Anal. **258**, 3295–3318 (2010)
13. Ghergu, M., Rădulescu, V.: Sublinear singular elliptic problems with two parameters. J. Differ. Equ. **195**, 520–536 (2003)
14. Ghergu, M., Rădulescu, V.: Multiparameter bifurcation and asymptotics for the singular Lane-Emden-Fowler equation with convection term. Proc. R. Soc. Edinb., Sect. A, Math. **135**, 61–84 (2005)
15. Ghergu, M., Rădulescu, V.: Singular Elliptic Equations: Bifurcation and Asymptotic Analysis. Oxford University Press, London (2008)
16. Gilbarg, D., Trudinger, N.S.: Elliptic Partial Differential Equations of Second Order. Springer, Berlin (1983)

17. Gui, C., Lin, F.H.: Regularity of an elliptic problem with a singular nonlinearity. Proc. R. Soc. Edinb., Sect. A, Math. **123**, 1021–1029 (1993)
18. Haitao, Y.: Multiplicity and asymptotic behavior of positive solutions for a singular semilinear elliptic problem. J. Differ. Equ. **189**, 487–512 (2003)
19. Hernández, J., Mancebo, F.J., Vega, J.M.: On the linearization of some singular nonlinear elliptic problems and applications. Ann. Inst. Henri Poincaré, Anal. Non Linéaire **19**, 777–813 (2002)
20. Hernández, J., Mancebo, F.J., Vega, J.M.: Nonlinear singular elliptic problems: recent results and open problems. Preprint (2005)
21. Kazdan, J., Warner, F.W.: Remarks on some quasilinear elliptic equations. Commun. Pure Appl. Math. **28**, 567–597 (1975)
22. Meadows, A.: Stable and singular solutions of the equation $\Delta u = 1/u$. Indiana Univ. Math. J. **53**, 1681–1703 (2004)
23. Rhouma, N.B., Mosbah, M.: On the existence of positive solutions for semilinear elliptic equations with indefinite nonlinearities. J. Differ. Equ. **215**, 37–51 (2005)
24. Shi, J., Yao, M.: On a singular nonlinear semilinear elliptic problem. Proc. R. Soc. Edinb., Sect. A, Math. **128**, 1389–1401 (2008)
25. Shi, J., Yao, M.: Positive solutions for elliptic equations with singular nonlinearity. Electron. J. Differ. Equ. **4**, 1–11 (2008)
26. Taliaferro, S.D.: A nonlinear singular boundary value problem. Nonlinear Anal. TMA **3**, 897–904 (1979)
27. Zhang, Z.: Nonexistence of positive classical solutions of a singular nonlinear Dirichlet problem with a convection term. Nonlinear Anal. TMA **27**, 957–961 (1996)
28. Zhang, Z.: The existence and asymptotic behaviour of the unique solution near the boundary to a singular Dirichlet problem with a convection term. Proc. R. Soc. Edinb., Sect. A, Math. **136**, 209–222 (2006)
29. Zhang, Z., Cheng, J.: Existence and optimal estimates of solutions for singular nonlinear Dirichlet problems. Nonlinear Anal. **57**, 473–484 (2004)