Chapter 15
Solution of the Cauchy Problem for Generalized Euler-Poisson-Darboux Equation by the Method of Fractional Integrals

A.K. Urinov and S.T. Karimov

Abstract In this work the singular Cauchy problem for the multi-dimensional Euler-Poisson-Darboux equation with spectral parameter has been investigated with the help of the generalized Erdelyi-Kober fractional operator. Solution of the considered problem is found in explicit form for various values of the parameter p of the equation.

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15.1 Introduction

For the first time the equation

$$u_{xy} - \frac{\alpha}{x-y}u_x + \frac{\beta}{x-y}u_y + \frac{\gamma}{(x-y)^2}u = 0, \tag{1}$$

where $\alpha, \beta, \gamma = \text{const}$, was obtained by Euler [1] in connection with the study of the air flow in pipes of different cross sections and the vibrations of strings of variable thickness. He gave a solution of this equation for $\alpha = \beta = m$, $\gamma = n$ (m, n are natural numbers).

The same equation, but in another form

$$E^{-}_{q,p}(u) \equiv u_{xx} - u_{yy} - \frac{2q}{y}u_x - \frac{2p}{y}u_y = 0, \tag{2}$$

where $q, p = \text{const}$, was solved by Poisson [2] for $q = 0$. He found a hyperbolic analogue of the representation of solution for this equation. In the same work he

A.K. Urinov (✉) · S.T. Karimov
Ferghana State University, 19 Murabbiylar street, Ferghana city, 150100, Republic of Uzbekistan
e-mail: urinovak@mail.ru

S.T. Karimov
e-mail: shkarimov09@rambler.ru

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considered the equation

$$L_p(u) \equiv \sum_{k=1}^{n} \frac{\partial^2 u}{\partial x_k^2} - \frac{\partial^2 u}{\partial t^2} - \frac{2p}{t}\frac{\partial u}{\partial t} = 0, \tag{3}$$

with $n = 3,\ p = 1$.

The general solution of (1) with $\alpha = \beta$ was found by Riemann [3]. He constructed the solution of the Cauchy problem with the help of auxiliary function using the method which is now called after him.

Much later, (2) with $q = 0,\ 0 < p < 1$ appeared in the monograph by Darboux [4] in connection with studying curvature of surfaces, where it was called the Euler-Poisson equation. Subsequently, many authors began to cite equations of the forms (1), (2), (3) and their elliptic analogs, as the equations of Euler-Poisson-Darboux.

After the publication of the first issue of the book by Tricomi [5], where the problem for mixed elliptic-hyperbolic equation $yu_{xx} + u_{yy} = 0$, later called as Tricomi equation, was formulated and investigated, the interest in such equations greatly increased. When studying this problem the key role is played by the equation of the form (2) and

$$E^{+}_{q,p}(u) \equiv u_{xx} + u_{yy} + \frac{2q}{y}u_x + \frac{2p}{y}u_y = 0, \tag{4}$$

where $q = 0,\ p = (1/6)$.

More bibliography in this direction can be found in the monographs by Bitsadze [6] and Smirnov [7].

The theory of equations with singular coefficients is directly connected to the theory of equation degenerating on the boundary. Using a change of variables, a wide class of degenerate equations can be reduced to equations with singular coefficients. For instance, the equation with degeneration of type and order,

$$y^m \sum_{k=1}^{n} \frac{\partial^2 u}{\partial x_k^2} - y^k \frac{\partial^2 u}{\partial y^2} - \alpha y^{k-1}\frac{\partial u}{\partial y} - \lambda^2 y^k u = 0$$

by the change of variables $t = \frac{2}{m-k-2} y^{(m-k+2)/2}$ can be reduced to the equation

$$L_p^{\lambda}(u) \equiv \sum_{k=1}^{n} \frac{\partial^2 u}{\partial x_k^2} - \frac{\partial^2 u}{\partial t^2} - \frac{2p}{t}\frac{\partial u}{\partial t} - \lambda^2 u = 0. \tag{5}$$

The main role in creating the theory of Euler-Poisson-Darboux equations was played by works of Weinstein [8–11]. In these works Weinstein investigated the Cauchy problem for (3) with various values of the parameter p, with half-homogeneous initial conditions

$$u(x,0) = \tau(x), \qquad u_t(x,0) = 0, \quad x \in R^n, \tag{6}$$

and found its solution in an explicit form.

There he showed also the matching formulae of the form

$$E^{+}_{q,p}\big(y^{1-2p}u\big) = y^{1-2p}E^{+}_{q,1-p}(u), \tag{7}$$

considering (4) with $q = 0$, $0 < p < (1/2)$. Note that the formula of the form (7) can be found in the work of Darboux [4].

In the work by Young [12] one can find the survey of the investigations of the singular Cauchy problem {(3), (6)}. In the works of Diaz, Weinberger [13], Blum [14], the problem {(3), (6)} was studied for various values of the parameter p.

Kapilevich [15] investigated the Cauchy problem with initial conditions

$$u(x,0) = \tau(x), \qquad \lim_{t\to+0} t^{2p}u_t(x,t) = \nu(x), \quad x \in R^n \tag{8}$$

for (5), when $\lambda \neq 0$, $0 < p < (1/2)$ and $n = 1, 2$.

The uniqueness of the solution of the Cauchy problem {(5), (8)} was proved in the works by Fox [16], Blum [17], Bresters [18]. However, as it was shown by Bresters [18], the solution is not unique when $p < 0$.

In the present work, using fractional integrals, we investigate the Cauchy problem {(5), (8)} for various values of the parameters $p \geq 0$ and $\lambda \neq 0$.

15.2 Generalized Erdelyi-Kober Operator

In the paper [10] Weinstein found a formulae in which the connection of the solution of (2) for $q = 0$ with fractional integrals was made for various values of the parameter p. This idea was substantially developed in the work of Erdelyi [19–22], who continued investigations by Weinstein [11], and studied properties of the differential operator

$$B^{(x)}_{\eta} = x^{-2\eta-1}\frac{d}{dx}x^{2\eta+1}\frac{d}{dx} = \frac{d^2}{dx^2} + \frac{2\eta+1}{x}\frac{d}{dx}. \tag{9}$$

In the work of Erdelyi [22] the apparatus of fractional integration was used for developing the result by Friedlander and Heins [23], where (2) was considered for $q = 0$.

The results of Erdelyi were generalized by Lowndes [24–26], where a generalized Erdelyi-Kober operator

$$J_\lambda(\eta,\alpha)f(x) = 2^{\alpha}\lambda^{1-\alpha}x^{-2\alpha-2\eta} \times \int_0^x t^{2\eta+1}\big(x^2-t^2\big)^{(\alpha-1)/2}J_{\alpha-1}\big(\lambda\sqrt{x^2-t^2}\big)f(t)dt \tag{10}$$

was introduced and studied. Here $\eta, \alpha, \lambda \in R$, such that $\alpha > 0$, $\eta \geq -(1/2)$, and $J_\nu(z)$ is the Bessel function of the first kind of order ν [27–29].

Further we need the following properties of the operator (10), which were proved in [25]:

1. It is obvious that for $\lambda \to 0$ the operator (10) coincides with the regular Erdelyi-Kober operator

$$I_{\eta,\alpha} f(x) = \frac{2x^{-2(\eta+\alpha)}}{\Gamma(\alpha)} \int_0^x \left(x^2 - t^2\right)^{\alpha-1} t^{2\eta+1} f(t) dt,$$

   where $\Gamma(\alpha)$ is Euler's Gamma function.
2. The following equalities hold true:

$$J_{i\lambda}(\eta+\alpha,\beta) J_\lambda(\eta,\alpha) = J_\lambda(\eta+\alpha,\beta) J_{i\lambda}(\eta,\alpha) = I_{\eta,\alpha+\beta},$$

   where i is the imaginary unit, $\alpha, \beta, \lambda \in R$.
3. From the latter equality, using the property $J_0(\eta, 0) = E$, where E is unique operator, one can pre-define the operator $J_\lambda(\eta,\alpha)$ for $\alpha < 0$ in the following way:

$$J_\lambda(\eta,\alpha) f(x) = x^{-2(\eta+\alpha)} \left(\frac{d}{2xdx}\right)^m x^{2(\eta+\alpha+m)} J_\lambda(\eta, \alpha+m) f(x), \qquad (11)$$

   where $-m < \alpha < 0$, $m = 1, 2, \ldots$.
4. From the property 3, the relations for inverse operator

$$J_{i\lambda}^{-1}(\eta,\alpha) = J_\lambda(\eta+\alpha, -\alpha), \qquad J_\lambda^{-1}(\eta,\alpha) = J_{i\lambda}(\eta+\alpha,-\alpha)$$

   follow.

In the work [26] Lowndes proved the following lemma:

Lemma 1 *Let $\alpha > 0$, $f(x) \in C^2(0,b)$, $b > 0$, let the function $x^{2\eta+1} f(x)$ be integrable in a neighborhood and let $x^{2\eta+1} f'(x) \to 0$ as $x \to 0$. Then*

$$J_\lambda^{(x)}(\eta,\alpha) B_\eta^{(x)} f(x) = \left(B_{\eta+\alpha}^{(x)} + \lambda^2\right) J_\lambda^{(x)}(\eta,\alpha) f(x), \qquad (12)$$

where $B_\eta^{(x)}$ is the operator of Bessel which is defined by (9).

Using this lemma Lowndes solved the Cauchy problem {(5), (8)} for $p = 0$. Further we need the following form of the formula (10):

$$J_\lambda(\eta,\alpha) f(x) = \frac{2x^{-2(\alpha+\eta)}}{\Gamma(\alpha)} \int_0^x t^{2\eta+1} \left(x^2 - t^2\right)^{\alpha-1} \bar{J}_{\alpha-1}\left(\lambda\sqrt{x^2-t^2}\right) f(t) dt, \quad (13)$$

where $\bar{J}_\nu(z)$ is the Bessel-Clifford function, which can be written by the Bessel function as: $\bar{J}_\nu(z) = \Gamma(\nu+1)(z/2)^{-\nu} J_\nu(z)$.

15.3 Application of the Erdelyi-Kober Operator for Solving the Cauchy Problem

For the construction of the solution of the problem {(5), (8)}, corresponding to various values of the parameter p, first we give some properties of the solution of (5), [9].

We denote by $u(x,t;p)$, $w(x,t;p)$ the solutions of (5) for a given value of p.

1. If $u(x,t;1-p)$ is a solution of the equation $L^{\lambda}_{1-p}(u)=0$, then the function $w(x,t;p)=t^{1-2p}u(x,t;1-p)$ will be a solution of the equation $L^{\lambda}_{p}(w)=0$ and vice versa, if $w(x,t;p)$ is a solution of the equation $L^{\lambda}_{p}(w)=0$, then $u(x,t;1-p)=t^{2p-1}w(x,t;p)$ will be a solution of the equation $L^{\lambda}_{1-p}(u)=0$.
2. If $u(x,t;p)$ is a solution of the equation $L^{\lambda}_{p}(u)=0$, then the function

$$u(x,t;1+p)=\left(\frac{1}{t}\frac{\partial}{\partial t}\right)u(x,t;p)$$

will be a solution of the equation $L^{\lambda}_{1+p}(u)=0$ and vice versa, if $u(x,t;1+p)$ is a solution of the equation $L^{\lambda}_{1+p}(u)=0$, then there exists always a solution $u(x,t;p)$ of the equation $L^{\lambda}_{p}(u)=0$.

Now we begin the investigation of the problem {(5), (8)}. Assume that the solution of the problem {(5), (6)} exists. We look for this solution as a generalized Erdelyi-Kober operator:

$$\begin{aligned} u(x,t) &= J^{(t)}_{\lambda}(\eta,\alpha)V(x,t) \\ &= \frac{2t^{-2(\eta+\alpha)}}{\Gamma(\alpha)}\int_0^t s^{2\eta+1}\left(t^2-s^2\right)^{\alpha-1}\bar{J}_{\alpha-1}\left(\lambda\sqrt{t^2-s^2}\right)V(x,s)ds, \end{aligned} \quad (14)$$

where $\alpha,\eta\in R$ are numbers to be specified later and, moreover, $\alpha>0$, $\eta\geq-(1/2)$, $V(x,t)$ is a twice continuously differentiable unknown function.

Substituting (14) into (5) and initial condition (6), and applying Lemma 1 we find the unknown function $V(x,s)$, so that it satisfies the equation

$$\sum_{k=1}^{n}\frac{\partial^2 V}{\partial x_k^2}-\frac{\partial^2 V}{\partial s^2}-\frac{2\eta+1}{s}\frac{\partial V}{\partial s}=0 \quad (15)$$

and the initial conditions

$$\begin{aligned} &V(x,0)=k_0\tau(x), \qquad V_s(x,0)=0, \\ &x\in R^n, k_0=\Gamma(\alpha+\eta+1)/\Gamma(\eta+1). \end{aligned} \quad (16)$$

Further, we choose parameters α, η such that the function $u(x,t)$ defined by (14) satisfies (5) and the initial conditions (8).

Let $\eta = (n/2) - 1$, $\alpha = p - (n-1)/2$ and $p > (n-1)/2$. Then (15) is transformed to the Darboux equation. It is known from [30] that the solution of the problem $\{(15), (16)\}$ in this case is unique and represented by $M_n(x,t;\tau)$, which is the spherical mean of the function $\tau(x)$ in the space R^n, by the formula

$$V(x,s) = k_0 M_n(x,s;\tau) = \frac{k_0}{\omega_n s^{n-1}} \int_{|\xi - x| = s} \tau(\xi) d\sigma_\xi = \frac{k_0}{\omega_n} \int_{|y|=1} \tau(x+sy) d\omega, \qquad (17)$$

where $|\xi - x|^2 = \sum_{k=1}^n (\xi_k - x_k)^2$, $d\omega$ is the area-element of the unit sphere, and $\omega_n = 2\pi^{n/2}/\Gamma(n/2)$ is the area of its surface.

It is easy to verify that the function $M_n(x,s;\tau)$ satisfies the initial conditions

$$\lim_{s\to 0} M_n(x,s;\tau) = \tau(x), \qquad \lim_{s\to 0} \frac{\partial M_n(x,s;\tau)}{\partial s} = 0, \quad x \in R^n. \qquad (18)$$

Substituting (17) into the equality (14) we obtain

$$u(x,t) = \frac{\Gamma(p+1/2) t^{1-2p}}{\pi^{n/2} \Gamma(p-(n-1)/2)} \times \int_{|\xi - x| \le t} \tau(\xi) \frac{\bar{J}_{p-(n+1)/2}(\lambda\sqrt{t^2 - |\xi - x|^2})}{[t^2 - |\xi - x|^2]^{(n+1)/2 - p}} d\xi. \qquad (19)$$

If $\tau(x) \in C^2(R^n)$, then by virtue of Lemma 1, the function (19) will be a regular solution of (5) satisfying the initial conditions (6).

Note that in the case when $p < (n-1)/2$, the function (19) will be the solution of the problem $\{(5), (6)\}$, if one uses the pre-definition of the operator (14) for $\alpha < 0$ based on (11):

$$J_\lambda^{(t)}(\eta,\alpha)V(x,t) = t^{-2(\eta+\alpha)} \left(\frac{\partial}{2t\partial t}\right)^m t^{2(\eta+\alpha+m)} J_\lambda^{(t)}(\eta, \alpha+m)$$
$$= \frac{2t^{-2(\eta+\alpha)}}{\Gamma(\alpha+m)} \left(\frac{\partial}{2t\partial t}\right)^m \times \int_0^t s^{2\eta+1} \left(t^2 - s^2\right)^{\alpha+m-1} J_{\alpha+m-1}\left(\lambda\sqrt{t^2-s^2}\right) V(x,s) ds,$$

where $-m < \alpha < 0$, $m = 1,2,3\ldots.$ In this case we choose m to be the smallest positive integer satisfying the inequality $p + m > (n-1)/2$.

Here one can see that the range of the parameter p depends on the dimension of the space R^n. There is a question: how to find the solution of the considered problem for any n, if the range of the parameter p is fixed in advance, for instance, $0 < p < 1/2$?

In this case we choose the parameter η so that the function $u(x,t)$ which is defined by (14) satisfies (5) and the initial conditions (6).

Let $\eta = -1/2$, then the parameter $\alpha = p$ and (15) can be transformed to the n-dimensional wave equation.

In this case the solution of the problem $\{(15), (16)\}$ for odd n has the form ([30]):

$$V(x,s) = \gamma_1 k_0 \frac{\partial}{\partial s}\left(\frac{1}{s}\frac{\partial}{\partial s}\right)^{(n-3)/2} \left(s^{n-2} M_n(x,s;\tau)\right), \tag{20}$$

where $\gamma_1 = 1/[1 \cdot 3 \cdot \dots \cdot (n-2)]$.

Let $n = 2m+1$, then the solution (20) has the form

$$V(x,s) = \gamma_1 k_0 \frac{\partial}{\partial s}\left(\frac{1}{s}\frac{\partial}{\partial s}\right)^{m-1} \left(s^{2m-1} M_{2m+1}(x,s;\tau)\right). \tag{21}$$

The solution of the problem $\{(15), (16)\}$ for even n can be written as ([30]):

$$V(x,s) = \gamma_2 k_0 \frac{\partial}{\partial s}\left(\frac{1}{s}\frac{\partial}{\partial s}\right)^{(n-2)/2} \left(\int_0^s M_n(x,\rho;\tau) \frac{\rho^{n-1} d\rho}{\sqrt{s^2-\rho^2}}\right), \tag{22}$$

where $\gamma_2 = 1/[2 \cdot 4 \cdot \dots \cdot (n-2)]$.

Let $n = 2m$, then the solution (22) will have the form

$$V(x,s) = \gamma_2 k_0 \frac{\partial}{\partial s}\left(\frac{1}{s}\frac{\partial}{\partial s}\right)^{m-1} \left(\int_0^s M_{2m}(x,\rho;\tau) \frac{\rho^{2m-1} d\rho}{\sqrt{s^2-\rho^2}}\right). \tag{23}$$

Combining solutions (21) and (23), we obtain

$$V(x,s) = \gamma \frac{\partial}{\partial s}\left(\frac{1}{s}\frac{\partial}{\partial s}\right)^{m-1} T(x,s), \tag{24}$$

where

$$\gamma = \begin{cases} \gamma_1 k_0, & n = 2m+1, \\ \gamma_2 k_0, & n = 2m, \end{cases}$$

$$T(x,s) = \begin{cases} s^{2m-1} M_{2m+1}(x,s;\tau), & n = 2m+1, \\ \int_0^s M_{2m}(x,\rho;\tau) \frac{\rho^{2m-1} d\rho}{\sqrt{s^2-\rho^2}}, & n = 2m. \end{cases}$$

Substituting (24) into the formula (14) we have

$$u(x,t) = \frac{2\gamma t^{1-2p}}{\Gamma(p)} \int_0^t \frac{\bar{J}_{p-1}(\lambda\sqrt{t^2-s^2})}{(t^2-s^2)^{1-p}} \frac{\partial}{\partial s}\left(\frac{1}{s}\frac{\partial}{\partial s}\right)^{m-1} T(x,s) ds. \tag{25}$$

The following lemmas hold true:

Lemma 2 *If $\tau(x)$ is m times continuously differentiable, then*

$$\lim_{s\to 0}\left(\frac{1}{s}\frac{\partial}{\partial s}\right)^{m-1} T(x,s) = 0, \quad m = 1, 2, \dots.$$

Proof Considering (17) and (18) we rewrite the function $T(x,s)$ as $T(x,s) = s^{2m-1}T_0(x,s)$, where

$$T_0(x,s) = \begin{cases} M_{2m+1}(x,s;\tau), & n = 2m+1, \\ \int_0^1 M_{2m}(x,s\zeta;\tau)\frac{\zeta^{2m-1}d\zeta}{\sqrt{1-\zeta^2}}, & n = 2m. \end{cases}$$

Then

$$\left(\frac{1}{s}\frac{\partial}{\partial s}\right)^{m-1} T(x,s) = T_0(x,s)\left(\frac{1}{s}\frac{\partial}{\partial s}\right)^{m-1} s^{2m-1} + s^{2m-1}\left(\frac{1}{s}\frac{\partial}{\partial s}\right)^{m-1} T_0(x,s).$$

Further, considering the equality

$$\left(\frac{1}{s}\frac{\partial}{\partial s}\right)^{m-1} s^{2m-1} = s\prod_{k=1}^{m-1}\left[2m-(2k-1)\right],$$

$$\left(\frac{1}{s}\frac{\partial}{\partial s}\right)^{m-1} T_0(x,s) = O\left(s^{-2m+3}\right),$$

we obtain $(\frac{1}{s}\frac{\partial}{\partial s})^{m-1}T(x,s) = O(s)$, from which the statement of the Lemma 2 follows. □

Lemma 3 *Under the conditions of Lemma 2 the equality*

$$\int_0^t \left(t^2-s^2\right)^{p-1}\bar{J}_{p-1}\left(\lambda\sqrt{t^2-s^2}\right)\left[\left(\frac{1}{s}\frac{\partial}{\partial s}\right)^m T(x,s)\right]sds$$
$$= \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^m \int_0^t \left(t^2-s^2\right)^{p-1}\bar{J}_{p-1}\left(\lambda\sqrt{t^2-s^2}\right)T(x,s)sds \qquad (26)$$

holds true.

Proof We prove this lemma using the method of mathematical induction. First, we prove that (25) is true for $m = 1$.

Consider the function

$$u_\varepsilon(x,t) = \int_0^{t-\varepsilon} \left(t^2-s^2\right)^{p-1}\bar{J}_{p-1}\left(\lambda\sqrt{t^2-s^2}\right)\frac{\partial}{\partial s}T(x,s)ds,$$

where ε is a small enough positive real number.

Applying integration by parts to the latter integral and considering statements of the Lemma 2, we obtain

$$u_\varepsilon(x,t) = \left[t^2-(t-\varepsilon)^2\right]^{p-1}\bar{J}_{p-1}\left(\lambda\sqrt{t^2-(t-\varepsilon)^2}\right)T(x,t-\varepsilon)$$
$$-\int_0^{t-\varepsilon}\frac{\partial}{\partial s}\left[\left(t^2-s^2\right)^{p-1}\bar{J}_{p-1}\left(\lambda\sqrt{t^2-s^2}\right)\right]T(x,s)ds.$$

Further, taking into account the following easily checkable equalities

$$\frac{\partial}{\partial s}\left[\left(t^2-s^2\right)^{p-1}\bar{J}_{p-1}\left(\lambda\sqrt{t^2-s^2}\right)\right]=-\frac{s}{t}\frac{\partial}{\partial t}\left[\left(t^2-s^2\right)^{p-1}\bar{J}_{p-1}\left(\lambda\sqrt{t^2-s^2}\right)\right],$$

$$\int_0^{t-\varepsilon}\frac{1}{t}\frac{\partial}{\partial t}\left[\left(t^2-s^2\right)^{p-1}\bar{J}_{p-1}\left(\lambda\sqrt{t^2-s^2}\right)\right]T(x,s)sds$$
$$=\left(\frac{1}{t}\frac{\partial}{\partial t}\right)\int_0^{t-\varepsilon}\left(t^2-s^2\right)^{p-1}\bar{J}_{p-1}\left(\lambda\sqrt{t^2-s^2}\right)T(x,s)sds$$
$$-\frac{t-\varepsilon}{t}\left[t^2-(t-\varepsilon)^2\right]^{p-1}\bar{J}_{p-1}\left(\lambda\sqrt{t^2-(t-\varepsilon)^2}\right)T(x,t-\varepsilon),$$

we have

$$u_\varepsilon=\frac{1}{t}\left[t-(t-\varepsilon)\right]^p\left[t+(t-\varepsilon)\right]^{p-1}\bar{J}_{p-1}\left(\lambda\sqrt{t^2-(t-\varepsilon)^2}\right)T(x,t-\varepsilon)$$
$$+\left(\frac{1}{t}\frac{\partial}{\partial t}\right)\int_0^{t-\varepsilon}\left(t^2-s^2\right)^{p-1}\bar{J}_{p-1}\left(\lambda\sqrt{t^2-s^2}\right)T(x,s)sds.$$

From here, by virtue of $p>0$, after $\varepsilon\to 0$ we get

$$\int_0^t\left(t^2-s^2\right)^{p-1}\bar{J}_{p-1}\left(\lambda\sqrt{t^2-s^2}\right)\left(\frac{1}{s}\frac{\partial}{\partial s}\right)T(x,s)sds$$
$$=\left(\frac{1}{t}\frac{\partial}{\partial t}\right)\int_0^t\left(t^2-s^2\right)^{p-1}\bar{J}_{p-1}\left(\lambda\sqrt{t^2-s^2}\right)T(x,s)sds. \qquad (27)$$

Assume that formula (26) holds for $m=k-1$. We prove that it is valid also for $m=k$:

$$\int_0^t\left(t^2-s^2\right)^{p-1}\bar{J}_{p-1}\left(\lambda\sqrt{t^2-s^2}\right)\left(\frac{1}{s}\frac{\partial}{\partial s}\right)^k T(x,s)sds$$
$$=\int_0^t\left(t^2-s^2\right)^{p-1}\bar{J}_{p-1}\left(\lambda\sqrt{t^2-s^2}\right)\left(\frac{1}{s}\frac{\partial}{\partial s}\right)^{k-1}\left[\frac{1}{s}\frac{\partial}{\partial s}T(x,s)\right]sds$$
$$=\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{k-1}\int_0^t\left(t^2-s^2\right)^{p-1}\bar{J}_{p-1}\left(\lambda\sqrt{t^2-s^2}\right)\left[\frac{1}{s}\frac{\partial}{\partial s}T(x,s)\right]sds.$$

Further, considering (27) we get the statement of Lemma 3. □

Now, applying Lemma 3 to (25) we obtain

$$u(x,t)=\frac{2\gamma t^{1-2p}}{\Gamma(p)}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^m\int_0^t\left(t^2-s^2\right)^{p-1}\bar{J}_{p-1}\left(\lambda\sqrt{t^2-s^2}\right)T(x,s)sds. \qquad (28)$$

Let $n = 2m+1$. Then, substituting the value of the function $T(x,s)$ into (28) we deduce

$$u(x,t) = \frac{\gamma_1 \Gamma(p+1/2)\Gamma(n/2)}{\pi^{(n+1)/2}\Gamma(p)} t^{1-2p} \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{(n-1)/2} \times \int_{|\xi - x| \le t} \tau(\xi)\big(t^2 - |x-\xi|^2\big)^{p-1} \bar{J}_{p-1}\big(\lambda\sqrt{t^2 - |x-\xi|^2}\big) d\xi. \quad (29)$$

We now construct a solution of (5) for odd n satisfying the conditions

$$w(x,0) = 0, \qquad \lim_{t\to+0} t^{2p} w_t(x,t) = \nu(x), \quad x \in R^n, \quad (30)$$

where $\nu(x) \in C^{[n/2]+1}(R^n)$ is a given function, and $[n/2]$ means the integer part of the number $n/2$.

Let function $u(x,t;1-p)$ be a solution of the equation $L^{\lambda}_{1-p}(u) = 0$ satisfying conditions (6). Then by virtue of the property 1 of (5), the function $w(x,t;p) = t^{1-2p}u(x,t;1-p)$ will be a solution of the equation $L^{\lambda}_{p}(w) = 0$, satisfying conditions (30). Further, substituting $(1-2p)\tau(x)$ to $\nu(x)$, we get

$$\begin{aligned} w(x,t;p) &= t^{1-2p}u(x,t;1-p) \\ &= \frac{\gamma_1 \Gamma[(1/2)-p]\Gamma(n/2)}{\pi^{(n+1)/2}\Gamma(1-p)} \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{(n-1)/2} \\ &\quad \times \int_{|\xi - x| \le t} \nu(\xi)\big(t^2 - |x-\xi|^2\big)^{-p} \bar{J}_{-p}\big(\lambda\sqrt{t^2 - |x-\xi|^2}\big) d\xi. \quad (31) \end{aligned}$$

Thus, if $\tau(x) \in C^{[n/2]+2}(R^n)$, $\nu(x) \in C^{[n/2]+1}(R^n)$, then the sum of the functions (29) and (31) for odd n is a solution of (5), satisfying conditions (8).

Let $n = 2m$. Then, substituting the value of the function $T(x,s)$ into formula (28) we obtain

$$u(x,t) = \frac{2\gamma_2 k_0 t^{1-2p}}{\Gamma(p)} \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{m} \int_0^t \big(t^2 - s^2\big)^{p-1} \bar{J}_{p-1}\big(\lambda\sqrt{t^2 - s^2}\big) \times \left\{\int_0^s M_{2m}(x,\rho;\tau)\frac{\rho^{2m-1}d\rho}{\sqrt{s^2-\rho^2}}\right\} s ds.$$

Changing the order of integration by the Dirichlet formula we have

$$u(x,t) = \frac{2\gamma_2 k_0 t^{1-2p}}{\Gamma(p)} \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{m} \int_0^t M_{2m}(x,\rho;\tau)\rho^{2m-1}d\rho \times \int_\rho^t \big(s^2-\rho^2\big)^{-(1/2)}\big(t^2 - s^2\big)^{p-1} \bar{J}_{p-1}\big(\lambda\sqrt{t^2 - s^2}\big) s ds. \quad (32)$$

We now evaluate the inner integral. Using the expansion of the Bessel-Clifford function into a series, and calculating the obtained integral we get

$$\int_{\rho}^{t}\left(s^2-\rho^2\right)^{-(1/2)}\left(t^2-s^2\right)^{p-1}\bar{J}_{p-1}\left(\lambda\sqrt{t^2-s^2}\right)sds$$
$$=\frac{1}{2}\frac{\Gamma(p)\Gamma(1/2)}{\Gamma[p+(1/2)]}\left(t^2-\rho^2\right)^{p-1/2}\bar{J}_{p-(1/2)}\left(\lambda\sqrt{t^2-\rho^2}\right). \tag{33}$$

Substituting (33) into (32) we obtain

$$u(x,t)=\frac{\gamma_2}{\omega_n}t^{1-2p}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{(n/2)}\int_{|\xi-x|\le t}\tau(\xi)\left[t^2-|\xi-x|^2\right]^{p-1/2}$$
$$\times\bar{J}_{p-(1/2)}\left(\lambda\sqrt{t^2-|\xi-x|^2}\right)d\xi. \tag{34}$$

Similarly, as in the case when n is odd, for even n we get a solution of the problem (5), (30) as

$$w(x,t)=\frac{\gamma_2}{\omega_n(1-2p)}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{(n/2)}\int_{|\xi-x|\le t}\nu(\xi)\left[t^2-|\xi-x|^2\right]^{(1/2)-p}$$
$$\times\bar{J}_{(1/2)-p}\left(\lambda\sqrt{t^2-|\xi-x|^2}\right)d\xi. \tag{35}$$

Thus, if $\tau(x)\in C^{[n/2]+2}(R^n)$, $\nu(x)\in C^{[n/2]+1}(R^n)$, then the sum of the functions (34) and (35) for even n will be the solution of (5) satisfying conditions (8).

The formulae (29) and (31) for odd n, and formulae (34) and (35) for even n were obtained for $0<p<(1/2)$. For other values of the parameter $p\neq(1/2),(3/2),\ldots,$ the solution will be defined by the analytic continuation of the operator $J_\lambda(\eta,\alpha)$ in the parameter $\alpha=p$.

When $\tau(x)$ and $\nu(x)$ are arbitrary functions, then the sum of the functions (29) and (31) for odd n, and formulae (34), (35) for even n, respectively, give the general solution of (5). Assume that $p=(1/2)$. Then these sums contain only one arbitrary function. Therefore, it is not a general solution of (5) for $p=(1/2)$.

Naturally, it is interesting to find a general solution of (5) for $p=(1/2)$, because with the help of the general solution for any equation one can find information on correct initial and boundary problems for this equation.

Let n be odd. Then by virtue of $\bar{J}_{(-1/2)}(z)=\cos(z)$ from the formula (29) for arbitrary $\varphi(x)\in C^{[n/2]+2}(R^n)$, it follows that the function $u(x,t)$ defined by the formula

$$u(x,t)=\frac{\gamma_1\Gamma(n/2)}{\pi^{(n+3)/2}}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{(n-1)/2}$$
$$\times\int_{|\xi-x|\le t}\varphi(\xi)\frac{\cos(\lambda\sqrt{t^2-|\xi-x|^2})}{\sqrt{t^2-|\xi-x|^2}}d\xi \tag{36}$$

will be a solution of (5).

In order to construct a second linear-independent solution of (5), we replace in formulae (29) and (31) the functions $\tau(x)$ and $\nu(x)$ by an arbitrary function $g(x) \in C^{[n/2]+2}(R^n)$ and rewrite it as

$$u(x,t) = \frac{2\Gamma[(1/2)+p]}{\sqrt{\pi}\Gamma(p)} \times \int_0^1 \left(1-z^2\right)^{p-1} \bar{J}_{p-1}\left(\lambda t\sqrt{1-z^2}\right) P_n(x,tz;g)dz, \tag{37}$$

$$w(x,t) = \frac{2\Gamma[(3/2)-p]t^{1-2p}}{\sqrt{\pi}\Gamma(1-p)(1-2p)} \times \int_0^1 \left(1-z^2\right)^{-p} \bar{J}_{-p}\left(\lambda t\sqrt{1-z^2}\right) P_n(x,tz;g)dz, \tag{38}$$

where

$$P_n(x,s;g) = \gamma_1 \frac{\partial}{\partial s}\left(\frac{1}{s}\frac{\partial}{\partial s}\right)^{(n-3)/2} \left(s^{n-2} M_n\left(x,s;g(x)\right)\right). \tag{39}$$

Calculating the derivatives appearing in equality (39) we deduce

$$P_n(x,s;g) = M_n(x,s;g) + A_1 s\frac{\partial M_n}{\partial s} + A_2 s^2 \frac{\partial^2 M_n}{\partial s^2} + \cdots + A_n s^{n-3} \frac{\partial^{(n-3)/2} M_n}{\partial s^{(n-3)/2}},$$

where A_k $(k=\overline{1,n})$ are some constants.

By virtue of (18), from the latter equality it follows that the function $P_n(x,s;g)$ satisfies the conditions

$$\lim_{s\to 0} P_n(x,s;g) = g(x), \qquad \lim_{s\to 0} \frac{\partial P_n(x,s;g)}{\partial s} = 0, \quad x \in R^n.$$

It is obvious that the linear combination of the expressions (37) and (38) of the form

$$W(x,t) = \frac{u(x,t)}{1-2p} - \frac{\Gamma(1-p)\Gamma[(1/2)+p]}{\Gamma[(3/2)-p]\Gamma(p)} w(x,t)$$

will be a solution of (5). We rewrite this combination as

$$W(x,t) = \frac{2\Gamma[(1/2)+p]}{\sqrt{\pi}\Gamma(p)} \int_0^1 \left(1-z^2\right)^{p-1} \frac{1}{1-2p} \times \left\{\bar{J}_{p-1}\left(\lambda t\sqrt{1-z^2}\right) - \left[t\left(1-z^2\right)\right]^{1-2p} \bar{J}_{-p}\left(\lambda t\sqrt{1-z^2}\right)\right\} \times P_n(s,tz;g)dz. \tag{40}$$

Considering

$$\begin{aligned}
&\frac{\bar{J}_{p-1}(\lambda t\sqrt{1-z^2}) - [t(1-z^2)]^{1-2p}\bar{J}_{-p}(\lambda t\sqrt{1-z^2})}{1-2p} \\
&\quad = \frac{\bar{J}_{p-1}(\lambda t\sqrt{1-z^2}) - \bar{J}_{-p}(\lambda t\sqrt{1-z^2})}{1-2p} \\
&\qquad + \frac{1-[t(1-z^2)]^{1-2p}}{1-2p}\bar{J}_{-p}\big(\lambda t\sqrt{1-z^2}\big),
\end{aligned}$$

passing to the limit as $p \to (1/2)$, and taking into consideration the equalities

$$\lim_{p\to(1/2)} \frac{1-[t(1-z^2)]^{1-2p}}{1-2p}\bar{J}_{-p}\big(\lambda t\sqrt{1-z^2}\big) = -\cos\big(\lambda t\sqrt{1-z^2}\big)\ln\big[t\big(1-z^2\big)\big],$$

$$\lim_{p\to(1/2)} \frac{\bar{J}_{p-1}(\lambda t\sqrt{1-z^2}) - \bar{J}_{-p}(\lambda t\sqrt{1-z^2})}{1-2p} = -B_{-(1/2)}\big(\lambda t\sqrt{1-z^2}\big),$$

we deduce from (40) that

$$\begin{aligned}
W_1(x,t;g) &= \lim_{p\to(1/2)} W(x,t) \\
&= -\frac{2}{\pi}\int_0^1 \big\{\cos\big(\lambda t\sqrt{1-z^2}\big)\ln\big[t\big(1-z^2\big)\big] + B_{-(1/2)}\big(\lambda t\sqrt{1-z^2}\big)\big\} \\
&\quad \times \big(1-z^2\big)^{-(1/2)} P_n(x,tz;g)dz. \qquad (41)
\end{aligned}$$

Here

$$B_\nu(\sigma) = \Gamma(\nu+1)\sum_{k=1}^{\infty} \frac{(-1)^k(\sigma/2)^{2k}}{k!\Gamma(\nu+k+1)}\big[\psi(\nu+1) - \psi(k+\nu+1)\big], \qquad (42)$$

and $\psi(z) = [\Gamma'(z)/\Gamma(z)]$ is the logarithmic derivative of the Gamma-function ([27]).

Consequently, in the case $p = (1/2)$ and odd n, the general solution of (5), in accordance with (36) and (41), has the form

$$\begin{aligned}
u(x,t) &= \frac{2}{\pi}\int_0^1 \big(1-z^2\big)^{-(1/2)}\cos\big(\lambda t\sqrt{1-z^2}\big)P_n\big(x,tz;\varphi(x)\big)dz \\
&\quad - \frac{2}{\pi}\int_0^1 \big\{\cos\big(\lambda t\sqrt{1-z^2}\big)\ln\big[t\big(1-z^2\big)\big] + B_{-(1/2)}\big(\lambda t\sqrt{1-z^2}\big)\big\} \\
&\quad \times \big(1-z^2\big)^{-(1/2)} P_n\big(x,tz;g(x)\big)dz, \qquad (43)
\end{aligned}$$

where $P_n(x,s;f)$ is the function, defined by (39), and $\varphi(x)$, $g(x)$ are arbitrary functions from the class of functions $C^{[n/2]+2}(R^n)$.

Now consider the case when n is even, $p=(1/2)$. In this case one of the solutions of (5) will be the function:

$$u(x,t)=\frac{\gamma_2}{\omega_n}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{(n/2)}\int_{|\xi-x|\le t}\varphi(\xi)J_0\big(\lambda\sqrt{t^2-|\xi-x|^2}\big)d\xi, \tag{44}$$

which follows from (34) at $p=(1/2)$, here $\varphi(x)\in C^{[n/2]+2}(R^n)$ is an arbitrary function.

With the aim to find a second linearly-independent solution of (5), we replace in formulae (34), (35), the functions $\tau(x)$ and $\nu(x)$ by an arbitrary function $g(x)\in C^{[n/2]+2}(R^n)$, and rewrite them as

$$\begin{aligned}u(x,t)=&\int_0^1\big[2\bar{J}_{p-(1/2)}(\sigma)-(1-2p)\bar{J}_{p-(3/2)}(\sigma)\big]\\&\times Q_n(x,tz;g)\big(1-z^2\big)^{p-(1/2)}zdz\\&+\int_0^1\bar{J}_{p-(1/2)}(\sigma)t\frac{\partial Q_n(x,tz;g)}{\partial t}\big(1-z^2\big)^{p-(1/2)}zdz,\end{aligned}\tag{45}$$

$$\begin{aligned}w(x,t)=&\frac{t^{1-2p}}{1-2p}\int_0^1\big[2\bar{J}_{(1/2)-p}(\sigma)+(1-2p)\bar{J}_{-p-(1/2)}(\sigma)\big]\\&\times Q_n(x,tz;g)\big(1-z^2\big)^{(1/2)-p}zdz\\&+\frac{t^{1-2p}}{1-2p}\int_0^1\bar{J}_{(1/2)-p}(\sigma)t\frac{\partial Q_n(x,tz;g)}{\partial t}\big(1-z^2\big)^{(1/2)-p}zdz,\end{aligned}\tag{46}$$

where $\sigma=\lambda t\sqrt{1-z^2}$,

$$Q_n(x,s;g)=\gamma_2\frac{\partial}{\partial s}\left(\frac{1}{s}\frac{\partial}{\partial s}\right)^{(n-2)/2}\big(s^{n-2}M_n\big(x,s;g(x)\big)\big). \tag{47}$$

Calculating all the necessary derivatives in (47) we obtain

$$\begin{aligned}Q_n(x,s;g)=&M_n(x,s;g)+C_1 s\frac{\partial M_n}{\partial s}\\&+C_2 s^2\frac{\partial^2 M_n}{\partial s^2}+\cdots+C_n s^{n-2}\frac{\partial^{(n-2)/2}M_n}{\partial s^{(n-2)/2}},\end{aligned}$$

where C_k $(k=\overline{1,n})$ are some well-defined constants.

By virtue of (18), from the latter equality it follows that the function $Q_n(x,s;g)$ satisfies the conditions

$$\lim_{s\to 0}Q_n(x,s;g)=g(x),\qquad \lim_{s\to 0}\frac{\partial Q_n(x,s;g)}{\partial s}=0,\quad x\in R^n.$$

The following linear combination of the functions (45), (46),

$$W^*(x,t) = \frac{u(x,t)}{1-2p} - w(x,t)$$
$$= \int_0^1 \frac{(1-z^2)^{p-(1/2)}}{1-2p} \left\{ \bar{J}_{p-(1/2)}(\sigma) - \left[t\left(1-z^2\right)\right]^{1-2p} \bar{J}_{(1/2)-p}(\sigma) \right\}$$
$$\times \left[2Q_n(x,tz;g) + t\frac{\partial Q_n(x,tz;g)}{\partial t} \right] zdz$$
$$- \int_0^1 \left\{ \bar{J}_{p-(3/2)}(\sigma) + \left[t\left(1-z^2\right)\right]^{1-2p} \bar{J}_{-p-(1/2)}(\sigma) \right\}$$
$$\times Q_n(x,tz;g)\left(1-z^2\right)^{p-(1/2)} zdz \tag{48}$$

will be a solution of (5).

In the equality (48) we pass to the limit as $p \to (1/2)$, and we have

$$W_2(x,t;g) = \lim_{p \to (1/2)} W^*(x,t)$$
$$= -\int_0^1 \left\{ J_0(\sigma) \ln\left[t\left(1-z^2\right)\right] + B_0(\sigma) \right\}$$
$$\times \left[2Q_n(x,tz;g) + t\frac{\partial Q_n(x,tz;g)}{\partial t} \right] zdz$$
$$-2\int_0^1 \left\{ 1 + B^*(\sigma) \right\} Q_n(x,tz;g) zdz, \tag{49}$$

where $B_0(\sigma)$ is the function defined by (42), $\sigma = \lambda t\sqrt{1-z^2}$,

$$B^*(\sigma) = \sum_{k=1}^{\infty} \frac{(-1)^k (\sigma/2)^{2k}}{k!\Gamma(k)} \left\{ \psi(1) - \psi(k) + \ln\left[t\left(1-z^2\right)\right] \right\},$$

such that $B^*(\sigma) = O(\sigma^2[C + \ln\sigma])$, $C = \text{const}$.

Consequently, in the case $p = (1/2)$ and even n, the general solution of (5), in accordance with (44), (49), has the form

$$u(x,t) = \int_0^1 J_0(\sigma) \left[2Q_n(x,tz;\varphi) + t\frac{\partial Q_n(x,tz;\varphi)}{\partial t} \right] zdz$$
$$- \int_0^1 \left\{ J_0(\sigma) \ln\left[t\left(1-z^2\right)\right] + B_0(\sigma) \right\}$$
$$\times \left[2Q_n(x,tz;g) + t\frac{\partial Q_n(x,tz;g)}{\partial t} \right] zdz$$
$$-2\int_0^1 \left\{ 1 + B^*(\sigma) \right\} Q_n(x,tz;g) zdz, \tag{50}$$

where $Q_n(x, s; \varphi)$ is the function defined by (47), and $\varphi(x)$, $g(x)$ are arbitrary functions from $C^{[n/2]+2}(R^n)$.

From formulae (43) and (50), which give the general solution of (5) for $p = (1/2)$, it follows that the Cauchy problem for this equation with initial conditions (8) is not correctly formulated. In this case the initial conditions should be given in a modified form. Precisely, in the case when n is odd, they should be given in the form of

$$\lim_{t\to+0} \frac{u(x,t)}{(-\ln t)} = \tau(x), \qquad \lim_{t\to+0} t(\ln t)^2 \frac{\partial}{\partial t}\left[\frac{u(x,t) - W_1(x,t;\tau)}{(-\ln t)}\right] = \nu(x), \quad (51)$$

and in the case when n is even, in the form of

$$\lim_{t\to+0} \frac{u(x,t)}{(-\ln t)} = \tau(x), \qquad \lim_{t\to+0} t(\ln t)^2 \frac{\partial}{\partial t}\left[\frac{u(x,t) - W_2(x,t;\tau)}{(-\ln t)}\right] = \nu(x). \quad (52)$$

Here W_1 and W_2 are functions which are defined by (41) and (49), respectively.

Remark Using property 2 of (5) in the case when $p = l + (1/2)$, $l = 1, 2, \ldots,$ one can find a formula for a general solution of this equation.

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