

# Chapter 13

## Recent Progress in Smoothing Estimates for Evolution Equations

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**Abstract** This paper is a survey article of results and arguments from authors' papers (Ruzhansky and Sugimoto in Proc. Lond. Math. Soc. 105:393–423, 2012; Ruzhansky and Sugimoto in Smoothing properties of non-dispersive equations; Ruzhansky and Sugimoto in Smoothing properties of inhomogeneous equations via canonical transforms), and describes a new approach to global smoothing problems for dispersive and non-dispersive evolution equations based on ideas of comparison principle and canonical transforms. For operators  $a(D_x)$  of order  $m$  satisfying the dispersiveness condition  $\nabla a(\xi) \neq 0$ , the smoothing estimate

$$\|\langle x \rangle^{-s} |D_x|^{(m-1)/2} e^{ita(D_x)} \varphi(x)\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)} \leq C \|\varphi\|_{L^2(\mathbb{R}_x^n)} \quad (s > 1/2)$$

is established, while it is known to fail for general non-dispersive operators. Especially, time-global smoothing estimates for the operator  $a(D_x)$  with lower order terms are the benefit of our new method. For the case when the dispersiveness breaks, we suggest a form

$$\|\langle x \rangle^{-s} |\nabla a(D_x)|^{1/2} e^{ita(D_x)} \varphi(x)\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)} \leq C \|\varphi\|_{L^2(\mathbb{R}_x^n)} \quad (s > 1/2)$$

which is equivalent to the usual estimate in the dispersive case and is also invariant under canonical transformations for the operator  $a(D_x)$ . It does continue to hold for a variety of non-dispersive operators  $a(D_x)$ , where  $\nabla a(\xi)$  may become zero on some set. It is remarkable that our method allows us to carry out a global microlocal reduction of equations to the translation invariance property of the Lebesgue measure.

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### 13.1 Introduction

This survey article is a collection of results and arguments from authors' papers [18, 19], and [20].

Let us consider the following Cauchy problem to the Schrödinger equation:

$$\begin{cases} (i\partial_t + \Delta_x)u(t, x) = 0 & \text{in } \mathbb{R}_t \times \mathbb{R}_x^n, \\ u(0, x) = \varphi(x) & \text{in } \mathbb{R}_x^n. \end{cases}$$

By Plancherel's theorem, the solution  $u(t, x) = e^{it\Delta_x}\varphi(x)$  preserves the  $L^2$ -norm of the initial data  $\varphi$ , that is, we have  $\|u(t, \cdot)\|_{L^2(\mathbb{R}_x^n)} = \|\varphi\|_{L^2(\mathbb{R}^n)}$  for any fixed time  $t \in \mathbb{R}$ . But if we integrate the solution in  $t$ , we get an extra gain of regularity of order  $1/2$  in  $x$ . For example we have the estimate

$$\|\langle x \rangle^{-s} |D_x|^{1/2} e^{it\Delta_x} \varphi\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)} \leq C \|\varphi\|_{L^2(\mathbb{R}_x^n)} \quad (s > 1/2)$$

for  $u = e^{it\Delta_x}\varphi$ , where  $\langle x \rangle = \sqrt{1 + |x|^2}$ , and (a sharper version of) this estimate was first given by Kenig, Ponce and Vega [12]. This type of estimate is called a smoothing estimate, and its local version was first proved by Sjölin [23], Constantin and Saut [6], and Vega [26]. We remark that, historically, such a smoothing estimate was first shown to Korteweg-de Vries equation

$$\begin{cases} \partial_t u + \partial_x^3 u + u\partial_x u = 0, \\ u(0, x) = \varphi(x) \in L^2(\mathbb{R}), \end{cases}$$

and Kato [10] proved that the solution  $u = u(t, x)$  ( $t, x \in \mathbb{R}$ ) satisfies

$$\int_{-T}^T \int_{-R}^R |\partial_x u(x, t)|^2 dx dt \leq c(T, R, \|\varphi\|_{L^2}).$$

Similar smoothing estimates have been observed for generalised equations

$$\begin{cases} (i\partial_t + a(D_x))u(t, x) = 0, \\ u(0, x) = \varphi(x) \in L^2(\mathbb{R}^n), \end{cases}$$

which come from equations of fundamental importance in mathematical physics as their principal parts:

- $a(\xi) = |\xi|^2 \dots$  Schrödinger

$$i\partial_t u - \Delta_x u = 0$$

- $a(\xi) = \sqrt{|\xi|^2 + 1} \dots$  Relativistic Schrödinger

$$i\partial_t u + \sqrt{-\Delta_x + 1}u = 0$$

- $a(\xi) = \xi^3$  ( $n = 1$ )  $\dots$  Korteweg-de Vries (shallow water wave)

$$\partial_t u + \partial_x^3 u + u\partial_x u = 0$$

- $a(\xi) = |\xi|\xi$  ( $n = 1$ )  $\cdots$  Benjamin-Ono (deep water wave)

$$\partial_t u - \partial_x |D_x|u + u\partial_x u = 0$$

- $a(\xi) = \xi_1^2 - \xi_2^2$  ( $n = 2$ )  $\cdots$  Davey-Stewartson (shallow water wave of 2D)

$$\begin{cases} i\partial_t u - \partial_x^2 u + \partial_y^2 u = c_1|u|^2u + c_2u\partial_x v \\ \partial_x^2 v - \partial_y^2 v = \partial_x|u|^2 \end{cases}$$

- $a(\xi) = \xi_1^3 + \xi_2^3, \xi_1^3 + 3\xi_2^2, \xi_1^2 + \xi_1\xi_2^2$   $\cdots$  Shrira (deep water wave of 2D)
- $a(\xi) =$  quadratic form ( $n \geq 3$ )  $\cdots$  Zakharov-Schulman (interaction of sound wave and low amplitudes high frequency wave)

There has already been a lot of literature on this subject from various points of view. See, Ben-Artzi and Devinatz [2, 3], Ben-Artzi and Klainerman [4], Chihara [5], Hoshiro [7, 8], Kato and Yajima [11], Kenig, Ponce and Vega [12–16], Linares and Ponce [17], Simon [22], Sugimoto [24, 25], Walther [27, 28], and many others. We note that for a given operator  $A$  the following are equivalent to each other based on classical works by Agmon [1] and Kato [9]:

- Smoothing estimate

$$\|Ae^{-it\Delta_x}\varphi(x)\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)} \leq C\|\varphi\|_{L^2(\mathbb{R}_x^n)} \quad \text{where } A = A(X, D_x),$$

- Restriction estimate

$$\|\widehat{A^*f}|_{S_\rho^{n-1}}\|_{L^2(S_\rho^{n-1})} \leq C\sqrt{\rho}\|f\|_{L^2(\mathbb{R}^n)}, \quad \text{where } S_\rho^{n-1} = \{\xi; |\xi| = \rho\} \ (\rho > 0),$$

- Resolvent estimate

$$\sup_{\text{Im}\zeta > 0} |(R(\zeta)A^*f, A^*f)| \leq C\|f\|_{L^2(\mathbb{R}^n)}^2, \quad \text{where } R(\zeta) = (-\Delta - \zeta)^{-1}.$$

Most of the literature so far use the above equivalence to show smoothing estimates for dispersive equations by showing restriction or resolvent estimates instead.

But here we develop a completely different strategy. We investigate smoothing estimates by using methods of comparison and canonical transform which are quite efficient for this problem:

1. *Comparison principle*  $\cdots$  comparison of symbols implies that of estimates,
2. *Canonical transform*  $\cdots$  transform an equation to another simple one.

They work not only for all the dispersive equations (that is, the case  $\nabla a \neq 0$ ) but also for some non-dispersive equations, and induce smoothing estimates of an invariant form. Smoothing estimates for inhomogeneous equations can be also discussed by a similar treatment. We will explain them in due order.

### 13.2 Comparison Principle

Here we list theorems exemplifying the comparison principle, which have been established in Sect. 2 in [18]:

**Theorem 1** (1D case) *Let  $f, g \in C^1(\mathbb{R})$  be real-valued and strictly monotone. If  $\sigma, \tau \in C^0(\mathbb{R})$  satisfy*

$$\frac{|\sigma(\xi)|}{|f'(\xi)|^{1/2}} \leq A \frac{|\tau(\xi)|}{|g'(\xi)|^{1/2}}$$

then we have

$$\|\sigma(D_x)e^{itf(D_x)}\varphi(x)\|_{L^2(\mathbb{R}_t)} \leq A \|\tau(D_x)e^{itg(D_x)}\varphi(x)\|_{L^2(\mathbb{R}_t)}$$

for all  $x \in \mathbb{R}$ .

**Theorem 2** (2D case) *Let  $f(\xi, \eta), g(\xi, \eta) \in C^1(\mathbb{R}^2)$  be real-valued and strictly monotone in  $\xi \in \mathbb{R}$  for each fixed  $\eta \in \mathbb{R}$ . If  $\sigma, \tau \in C^0(\mathbb{R}^2)$  satisfy*

$$\frac{|\sigma(\xi, \eta)|}{|f_\xi(\xi, \eta)|^{1/2}} \leq A \frac{|\tau(\xi, \eta)|}{|g_\xi(\xi, \eta)|^{1/2}}$$

then we have

$$\begin{aligned} & \|\sigma(D_x, D_y)e^{itf(D_x, D_y)}\varphi(x, y)\|_{L^2(\mathbb{R}_t \times \mathbb{R}_y)} \\ & \leq A \|\tau(D_x, D_y)e^{itg(D_x, D_y)}\varphi(x, y)\|_{L^2(\mathbb{R}_t \times \mathbb{R}_y)} \end{aligned}$$

for all  $x \in \mathbb{R}$ .

**Theorem 3** (Radially Symmetric case) *Let  $f, g \in C^1(\mathbb{R}_+)$  be real-valued and strictly monotone. If  $\sigma, \tau \in C^0(\mathbb{R}_+)$  satisfy*

$$\frac{|\sigma(\rho)|}{|f'(\rho)|^{1/2}} \leq A \frac{|\tau(\rho)|}{|g'(\rho)|^{1/2}}$$

then we have

$$\|\sigma(|D_x|)e^{itf(|D_x|)}\varphi(x)\|_{L^2(\mathbb{R}_t)} \leq A \|\tau(|D_x|)e^{itg(|D_x|)}\varphi(x)\|_{L^2(\mathbb{R}_t)}$$

for all  $x \in \mathbb{R}^n$ .

### 13.3 Canonical Transforms

Next we will review the idea of canonical transforms discussed in Sect. 4 in [18]. It is based on the so-called Egorov's theorem.

Let  $\psi : \Gamma \rightarrow \tilde{\Gamma}$  be a  $C^\infty$ -diffeomorphism between open sets  $\Gamma \subset \mathbb{R}^n$  and  $\tilde{\Gamma} \subset \mathbb{R}^n$ . We always assume that

$$C^{-1} \leq |\det \partial \psi(\xi)| \leq C \quad (\xi \in \Gamma),$$

for some  $C > 0$ . We set formally

$$I_\psi u(x) = \mathcal{F}^{-1}[\mathcal{F}u(\psi(\xi))](x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x \cdot \xi - y \cdot \psi(\xi))} u(y) dy d\xi.$$

The operators  $I_\psi$  can be justified by using cut-off functions  $\gamma \in C^\infty(\Gamma)$  and  $\tilde{\gamma} = \gamma \circ \psi^{-1} \in C^\infty(\tilde{\Gamma})$  which satisfy  $\text{supp } \gamma \subset \Gamma$ ,  $\text{supp } \tilde{\gamma} \subset \tilde{\Gamma}$ . We set

$$\begin{aligned} I_{\psi,\gamma} u(x) &= \mathcal{F}^{-1}[\gamma(\xi) \mathcal{F}u(\psi(\xi))](x) \\ &= (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\Gamma} e^{i(x \cdot \xi - y \cdot \psi(\xi))} \gamma(\xi) u(y) dy d\xi. \end{aligned} \tag{1}$$

In the case that  $\Gamma, \tilde{\Gamma} \subset \mathbb{R}^n \setminus 0$  are open cones, we may consider the homogeneous functions  $\psi$  and  $\gamma$  which satisfy  $\text{supp } \gamma \cap \mathbb{S}^{n-1} \subset \Gamma \cap \mathbb{S}^{n-1}$  and  $\text{supp } \tilde{\gamma} \cap \mathbb{S}^{n-1} \subset \tilde{\Gamma} \cap \mathbb{S}^{n-1}$ , where  $\mathbb{S}^{n-1} = \{\xi \in \mathbb{R}^n : |\xi| = 1\}$ . Then we have the expressions for compositions

$$I_{\psi,\gamma} = \gamma(D_x) \cdot I_\psi = I_\psi \cdot \tilde{\gamma}(D_x)$$

and also the formula

$$I_{\psi,\gamma} \cdot \sigma(D_x) = (\sigma \circ \psi)(D_x) \cdot I_{\psi,\gamma}. \tag{2}$$

We also introduce the weighted  $L^2$ -spaces. For a weight function  $w(x)$ , let  $L^2(\mathbb{R}^n; w)$  be the set of measurable functions  $f : \mathbb{R}^n \rightarrow \mathbb{C}$  such that the norm

$$\|f\|_{L^2(\mathbb{R}^n; w)} = \left( \int_{\mathbb{R}^n} |w(x) f(x)|^2 dx \right)^{1/2}$$

is finite. Then, on account of the relations (2), we obtain the following fundamental theorem (Theorem 4.1 in [18]):

**Theorem 4** *Assume that the operator  $I_{\psi,\gamma}$  defined by (1) is  $L^2(\mathbb{R}^n; w)$ -bounded. Suppose that we have the estimate*

$$\|w(x) \rho(D_x) e^{it\sigma(D_x)} \varphi(x)\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)} \leq C \|\varphi\|_{L^2(\mathbb{R}_x^n)}$$

for all  $\varphi$  such that  $\text{supp } \widehat{\varphi} \subset \text{supp } \tilde{\gamma}$ , where  $\tilde{\gamma} = \gamma \circ \psi^{-1}$ . Assume also that the function

$$q(\xi) = \frac{\gamma \cdot \zeta}{\rho \circ \psi}(\xi)$$

is bounded. Then we have

$$\|w(x)\zeta(D_x)e^{ita(D_x)}\varphi(x)\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^d)} \leq C\|\varphi\|_{L^2(\mathbb{R}_x^d)}$$

for all  $\varphi$  such that  $\text{supp } \widehat{\varphi} \subset \text{supp } \gamma$ , where  $a(\xi) = (\sigma \circ \psi)(\xi)$ .

Note that  $e^{ita(D_x)}\varphi(x)$  and  $e^{it\sigma(D_x)}\varphi(x)$  are solutions to

$$\begin{cases} (i\partial_t + a(D_x))u(t, x) = 0, \\ u(0, x) = \varphi(x), \end{cases} \quad \text{and} \quad \begin{cases} (i\partial_t + \sigma(D_x))v(t, x) = 0, \\ v(0, x) = g(x), \end{cases}$$

respectively. Theorem 4 means that smoothing estimates for the equation with  $\sigma(D_x)$  implies those with  $a(D_x)$  if the canonical transformations which relate them are bounded on weighted  $L^2$ -spaces.

As for the  $L^2(\mathbb{R}^n; w)$ -boundedness of the operator  $I_{\psi, \gamma}$ , we have criteria for some special weight functions. For  $\kappa \in \mathbb{R}$ , let  $L^2_\kappa(\mathbb{R}^n)$  be the set of measurable functions  $f$  such that the norm

$$\|f\|_{L^2_\kappa(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} |\langle x \rangle^\kappa f(x)|^2 dx \right)^{1/2}$$

is finite. Then we have the following mapping properties (Theorems 4.2, 4.3 in [18]).

**Theorem 5** *Let  $\Gamma, \widetilde{\Gamma} \subset \mathbb{R}^n \setminus \{0\}$  be open cones. Suppose  $|\kappa| < n/2$ . Assume  $\psi(\lambda\xi) = \lambda\psi(\xi)$ ,  $\gamma(\lambda\xi) = \gamma(\xi)$  for all  $\lambda > 0$  and  $\xi \in \Gamma$ . Then the operator  $I_{\psi, \gamma}$  defined by (1) is  $L^2_\kappa(\mathbb{R}^n)$ -bounded.*

**Theorem 6** *Suppose  $\kappa \in \mathbb{R}$ . Assume that all the derivatives of entries of the  $n \times n$  matrix  $\partial\psi$  and those of  $\gamma$  are bounded. Then the operator  $I_{\psi, \gamma}$  defined by (1) are  $L^2_\kappa(\mathbb{R}^n)$ -bounded.*

### 13.4 Smoothing Estimates for Dispersive Equations

We consider smoothing estimates for solutions  $u(t, x) = e^{ita(D_x)}\varphi(x)$  to general equations

$$\begin{cases} (i\partial_t + a(D_x))u(t, x) = 0, \\ u(0, x) = \varphi(x) \in L^2(\mathbb{R}^n). \end{cases}$$

Let  $a_m(\xi)$  be the principal term of  $a(\xi)$  satisfying

$$a_m(\xi) \in C^\infty(\mathbb{R}^n \setminus \{0\}), \quad \text{real-valued}, \quad a_m(\lambda\xi) = \lambda^m a_m(\xi) \quad (\lambda > 0, \xi \neq 0).$$

We assume that  $a(\xi)$  is *dispersive* in the following sense:

$$a(\xi) = a_m(\xi), \quad \nabla a_m(\xi) \neq 0 \quad (\xi \in \mathbb{R}^n \setminus \{0\}), \tag{H}$$

or, otherwise, we assume

$$\begin{aligned}
 a(\xi) \in C^\infty(\mathbb{R}^n), \quad \nabla a(\xi) \neq 0 \quad (\xi \in \mathbb{R}^n), \quad \nabla a_m(\xi) \neq 0 \quad (\xi \in \mathbb{R}^n \setminus 0), \\
 |\partial^\alpha (a(\xi) - a_m(\xi))| \leq C_\alpha |\xi|^{m-1-|\alpha|} \quad \text{for all multi-indices } \alpha \text{ and all } |\xi| \geq 1.
 \end{aligned}
 \tag{L}$$

*Example 1*  $a(\xi) = \xi_1^3 + \dots + \xi_n^3 + \xi_1$  satisfies (L).

The dispersiveness means that the classical orbit, that is, the solution of the Hamilton-Jacobi equations

$$\begin{cases} \dot{x}(t) = (\nabla a)(\xi(t)), & \dot{\xi}(t) = 0, \\ x(0) = 0, & \xi(0) = k, \end{cases}$$

does not stop, and the singularity of  $u(t, x) = e^{ita(D_x)}\varphi(x)$  travels to infinity along this orbit. Hence we can expect the smoothing, and indeed we have the following result (Theorem 5.1, Corollary 5.5 in [18]):

**Theorem 7** *Assume (H) or (L). Suppose  $m \geq 1$  and  $s > 1/2$ . Then we have*

$$\left\| \langle x \rangle^{-s} |D_x|^{(m-1)/2} e^{ita(D_x)} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^d)} \leq C \|\varphi\|_{L^2(\mathbb{R}^n)}.$$

*Remark 1* Theorem 7 with polynomials  $a(\xi)$  follows immediately from a sharp version of local smoothing estimate proved by Theorem 4.1 of Kenig, Ponce and Vega [12], and any polynomial  $a(\xi)$  which satisfies the estimate in Theorem 7 has to be dispersive, that is  $\nabla a_m(\xi) \neq 0$  ( $\xi \neq 0$ ) (see Hoshiro [8]). Theorem 7 with  $a(\xi) = |\xi|^2$  and  $n \geq 3$  was also stated by Ben-Artzi and Klainerman [4], and with the case (H) and  $m > 1$  by Chihara [5] in different contexts.

### 13.5 Proof by New Methods

We explain how to prove Theorem 7 under the condition (H) by our new method. The main strategy is that we obtain estimates for low dimensional model cases from some trivial estimate by the comparison principle, and reduce general case to such model cases by the method of canonical transforms.

#### 13.5.1 Low Dimensional Model Estimates

By the comparison principle, we can show the equivalence of low dimensional estimates of various type. In the 1D case, we have (for  $l, m > 0$ )

$$\sqrt{m} \left\| |D_x|^{(m-1)/2} e^{it|D_x|^m} \varphi(x) \right\|_{L^2(\mathbb{R}_t)} = \sqrt{l} \left\| |D_x|^{(l-1)/2} e^{it|D_x|^l} \varphi(x) \right\|_{L^2(\mathbb{R}_t)} \tag{3}$$

for all  $x \in \mathbb{R}$ . Here  $\text{supp } \widehat{\varphi} \subset [0, +\infty)$  or  $(-\infty, 0]$ .

In the 2D case, we have (for  $l, m > 0$ )

$$\begin{aligned} & \left\| |D_y|^{(m-1)/2} e^{itD_x|D_y|^{m-1}} \varphi(x, y) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_y)} \\ &= \left\| |D_y|^{(l-1)/2} e^{itD_x|D_y|^{l-1}} \varphi(x, y) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_y)} \end{aligned} \tag{4}$$

for all  $x \in \mathbb{R}$ . On the other hand, in 1D case, we have trivially

$$\left\| e^{itD_x} \varphi(x) \right\|_{L^2(\mathbb{R}_t)} = \left\| \varphi(x + t) \right\|_{L^2(\mathbb{R}_x)} = \|\varphi\|_{L^2(\mathbb{R}_x)} \tag{5}$$

for all  $x \in \mathbb{R}$ . Using the equality (5), the right hand sides of (3) and (4) with  $l = 1$  can be estimated, and we have for all  $x \in \mathbb{R}$ :

- (1D Case)

$$\left\| |D_x|^{(m-1)/2} e^{it|D_x|^m} \varphi(x) \right\|_{L^2(\mathbb{R}_t)} \leq C \|\varphi\|_{L^2(\mathbb{R}_x)},$$

- (2D Case)

$$\left\| |D_y|^{(m-1)/2} e^{itD_x|D_y|^{m-1}} \varphi(x, y) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_y)} \leq C \|\varphi\|_{L^2(\mathbb{R}_x^2, \mathbb{R}_y)}.$$

*Remark 2* In the case  $m = 2$ , these estimates were proved by Kenig, Ponce & Vega [12] (1D case) and Linares & Ponce [17] (2D case).

The following is a straightforward consequence from these estimates:

**Proposition 1** *Suppose  $m > 0$  and  $s > 1/2$ . Then for  $n \geq 1$  we have*

$$\left\| \langle x \rangle^{-s} |D_n|^{(m-1)/2} e^{it|D_n|^m} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)} \leq C \|\varphi\|_{L^2(\mathbb{R}_x^n)}$$

and for  $n \geq 2$  we have

$$\left\| \langle x \rangle^{-s} |D_n|^{(m-1)/2} e^{itD_1|D_n|^{m-1}} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)} \leq C \|\varphi\|_{L^2(\mathbb{R}_x^n)},$$

where  $D_x = (D_1, \dots, D_n)$ .

### 13.5.2 Reduction to Model Estimates

On account of the method of canonical transform (Theorem 4), smoothing estimates for dispersive equations (Theorem 7) can be reduced to low dimensional model estimates (Proposition 1) by the canonical transformation if we find a homogeneous change of variable  $\psi$  such that

$$a(\xi) = (\sigma \circ \psi)(\xi), \quad \sigma(D) = |D_n|^m \quad \text{or} \quad \sigma(D) = D_1|D_n|^{m-1}.$$



We show how to select such  $\psi$  under the assumption (H). The argument for the case (L) is similar. By microlocalisation and rotation, we may assume that the initial data  $\varphi$  satisfies  $\text{supp } \hat{\varphi} \subset \Gamma$ , where  $\Gamma \subset \mathbb{R}^n \setminus 0$  is a sufficiently small conic neighbourhood of  $e_n = (0, \dots, 0, 1)$ . Furthermore, we have Euler’s identity

$$a(\xi) = a_m(\xi) = \frac{1}{m} \xi \cdot \nabla a(\xi),$$

and the dispersiveness  $\nabla a(e_n) \neq 0$  implies the following two cases:

- (I)  $\partial_n a(e_n) \neq 0 \dots$  (elliptic). By Euler’s identity, we have  $a(e_n) \neq 0$ . Hence, in this case, we may assume  $a(\xi) > 0$  ( $\xi \in \Gamma$ ),  $\partial_n a(e_n) \neq 0$ .
- (II)  $\partial_n a(e_n) = 0 \dots$  (non-elliptic). By assumption  $\nabla a(e_n) \neq 0$ , there exists  $j \neq n$  such that  $\partial_j a(e_n) \neq 0$ . Hence, in this case, we may assume  $\partial_1 a(e_n) \neq 0$ .

In the elliptic case (I), we take

$$\sigma(\eta) = |\eta_n|^m, \quad \psi(\xi) = (\xi_1, \dots, \xi_{n-1}, a(\xi)^{1/m}).$$

Then we have  $a(\xi) = (\sigma \circ \psi)(\xi)$ , and  $\psi$  is surely a change of variables on  $\Gamma$  since

$$\det \partial \psi(e_n) = \begin{vmatrix} E_{n-1} & 0 \\ * & \frac{1}{m} a(e_n)^{1/m-1} \partial_n a(e_n) \end{vmatrix} \neq 0$$

where  $E_{n-1}$  is the identity matrix. In the non-elliptic case (II), we take

$$\sigma(\eta) = \eta_1 |\eta_n|^{m-1}, \quad \psi(\xi) = \left( \frac{a(\xi)}{|\xi_n|^{m-1}}, \xi_2, \dots, \xi_n \right).$$

Then we have again  $a(\xi) = (\sigma \circ \psi)(\xi)$  and

$$\det \partial \psi(e_n) = \begin{vmatrix} \partial_1 a(e_n) & * \\ 0 & E_{n-1} \end{vmatrix} \neq 0.$$

Thus, we successfully showed Theorem 7 in both cases.

### 13.6 Non-dispersive Case

Now we consider what happens if the equation does not satisfy the dispersiveness assumption  $\nabla a(\xi) \neq 0$  ( $\xi \in \mathbb{R}^n$ ). All the precise results and arguments in this section are to appear in our forthcoming paper [19].

Although we cannot have smoothing estimates (see Remark 1), such case appears naturally in physics. For example, let us consider a coupled system of Schrödinger equations

$$i \partial_t v = \Delta_x v + b(D_x)w, \quad i \partial_t w = \Delta_x w + c(D_x)v,$$

which represents a linearised model of wave packets with two modes. Assume that this system is diagonalised and regard it as a single equations for the eigenvalues:

$$a(\xi) = -|\xi|^2 \pm \sqrt{b(\xi)c(\xi)}.$$

Then there could exist points  $\xi$  such that  $\nabla a(\xi) = 0$  because of the lower order terms  $b(\xi), c(\xi)$ . Another interesting examples are Shrira equations, in which case:

$$a(\xi) = \xi_1^3 + \xi_2^3, \quad \xi_1^3 + 3\xi_2^2, \quad \xi_1^2 + \xi_1\xi_2^2.$$

Although  $a(\xi) = \xi_1^3 + \xi_2^3$  satisfies assumption (H),  $a(\xi) = \xi_1^3 + 3\xi_2^2$  and  $a(\xi) = \xi_1^2 + \xi_1\xi_2^2$  do not satisfy assumption (L) because  $\nabla a(0) = 0$ .

We suggest an estimate which we expect to hold for non-dispersive equations:

$$\| \langle x \rangle^{-s} |\nabla a(D_x)|^{1/2} e^{ita(D_x)} \varphi(x) \|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)} \leq C \|\varphi\|_{L^2(\mathbb{R}_x^n)} \quad (s > 1/2) \quad (6)$$

and let us call it *invariant estimate*. This estimate has a number of advantages:

- in the dispersive case  $\nabla a(\xi) \neq 0$ , it is equivalent to Theorem 7;
- it is invariant under canonical transformations for the operator  $a(D_x)$ ;
- it does continue to hold for a variety of non-dispersive operators  $a(D_x)$ , where  $\nabla a(\xi)$  may become zero on some set and when the usual estimate fails;
- it does take into account zeros of the gradient  $\nabla a(\xi)$ , which is also responsible for the interface between dispersive and non-dispersive zone (e.g. how quickly the gradient vanishes).

### 13.6.1 Secondary Comparison

By using comparison principle again to the smoothing estimates obtained from the comparison principle, we can have new estimates. This is a powerful tool to induce the invariant estimates (6) for non-dispersive equations. For example, we have just obtained the estimate

$$\| \langle x \rangle^{-s} |D_x|^{(m-1)/2} e^{it|D_x|^m} \varphi \|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)} \leq C \|\varphi\|_{L^2(\mathbb{R}_x^n)}$$

(Theorem 7 with  $a(\xi) = |\xi|^m$ ) from comparison principle and canonical transformation. If we set  $g(\rho) = \rho^m, \tau(\rho) = \rho^{(m-1)/2}$ , then we have  $|\tau(\rho)|/|g'(\rho)|^{1/2} = 1/\sqrt{m}$ . Hence by the comparison result again for the radially symmetric case (Theorem 3), we have

**Theorem 8** *Suppose  $s > 1/2$ . Let  $f \in C^1(\mathbb{R}_+)$  be real-valued and strictly monotone. If  $\sigma \in C^0(\mathbb{R}_+)$  satisfy*

$$|\sigma(\rho)| \leq A |f'(\rho)|^{1/2},$$

then we have

$$\| \langle x \rangle^{-s} \sigma(|D_x|) e^{itf(|D_x|)} \varphi(x) \|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^d)} \leq C \|\varphi\|_{L^2(\mathbb{R}_x^d)}.$$

From this secondary comparison, we obtain immediately the following invariant estimate since a radial function  $a(\xi) = f(|\xi|)$  always satisfies  $|\nabla a(\xi)| = |f'(|\xi|)|$ .

**Theorem 9** *Suppose  $s > 1/2$ . Let  $a(\xi) = f(|\xi|)$  and  $f \in C^\infty(\mathbb{R}_+)$  be real-valued. Then we have*

$$\| \langle x \rangle^{-s} |\nabla a(D_x)|^{1/2} e^{ita(D_x)} \varphi(x) \|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^d)} \leq C \|\varphi\|_{L^2(\mathbb{R}_x^d)}.$$

*Example 2*  $a(\xi) = (|\xi|^2 - 1)^2$  is non-dispersive because

$$\nabla a(\xi) = 4(|\xi|^2 - 1)\xi = 0$$

if  $|\xi| = 0, 1$ . But we have the invariant estimate by Theorem 9.

For the non-radially symmetric case, we compare again to the low dimensional model estimates (Proposition 1) and obtain

**Theorem 10** (1D secondary comparison) *Suppose  $s > 1/2$ . Let  $f \in C^1(\mathbb{R})$  be real-valued and strictly monotone. If  $\sigma \in C^0(\mathbb{R})$  satisfies*

$$|\sigma(\xi)| \leq A |f'(\xi)|^{1/2},$$

then we have

$$\| \langle x \rangle^{-s} \sigma(D_x) e^{itf(D_x)} \varphi(x) \|_{L^2(\mathbb{R}_t \times \mathbb{R}_x)} \leq AC \|\varphi(x)\|_{L^2(\mathbb{R}_x)}.$$

**Theorem 11** (2D secondary comparison) *Suppose  $s > 1/2$ . Let  $f \in C^1(\mathbb{R}^2)$  be real-valued and  $f(\xi, \eta)$  be strictly monotone in  $\xi \in \mathbb{R}$  for every fixed  $\eta \in \mathbb{R}$ . If  $\sigma \in C^0(\mathbb{R}^2)$  satisfies*

$$|\sigma(\xi, \eta)| \leq A |\partial f / \partial \xi(\xi, \eta)|^{1/2},$$

then we have

$$\| \langle x \rangle^{-s} \sigma(D_x, D_y) e^{itf(D_x, D_y)} \varphi(x, y) \|_{L^2(\mathbb{R}_t \times \mathbb{R}_{x,y}^2)} \leq AC \|\varphi(x, y)\|_{L^2(\mathbb{R}_{x,y}^2)}.$$

*Example 3* By using secondary comparison for non-radially symmetric case, we have invariant estimates for Shrira equations. In fact, for  $a(\xi) = \xi_1^3 + 3\xi_2^2$ , we have by 1D secondary comparison (Theorem 10)

$$\| \langle x_1 \rangle^{-s} |D_1| e^{itD_1^3} \varphi(x) \|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^2)} \leq C \|\varphi\|_{L^2(\mathbb{R}_x^2)},$$

$$\| \langle x_2 \rangle^{-s} |D_2|^{1/2} e^{it^3 D_2^2} \varphi(x) \|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^2)} \leq C \|\varphi\|_{L^2(\mathbb{R}_x^2)},$$

for  $s > 1/2$ . Hence by  $\langle x \rangle^{-s} \leq \langle x_k \rangle^{-s}$  ( $k = 1, 2$ ) we have

$$\| \langle x \rangle^{-s} (|D_1| + |D_2|^{1/2}) e^{ita(D_x)} \varphi(x) \|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^2)} \leq C \|\varphi\|_{L^2(\mathbb{R}_x^2)}$$

and hence we have

$$\| \langle x \rangle^{-s} |\nabla a(D_x)|^{1/2} e^{ita(D_x)} \varphi(x) \|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^2)} \leq C \|\varphi\|_{L^2(\mathbb{R}_x^2)}.$$

For  $a(\xi) = \xi_1^2 + \xi_1 \xi_2^2$ , we have by 2D secondary comparison (Theorem 11)

$$\| \langle x_1 \rangle^{-s} |2D_1 + D_2^2|^{1/2} e^{ita(D_1, D_2)} \varphi(x) \|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^2)} \leq C \|\varphi\|_{L^2(\mathbb{R}_x^2)},$$

$$\| \langle x_2 \rangle^{-s} |D_1 D_2|^{1/2} e^{ita(D_1, D_2)} \varphi(x) \|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^2)} \leq C \|\varphi\|_{L^2(\mathbb{R}_x^2)},$$

for  $s > 1/2$ , hence we have similarly

$$\| \langle x \rangle^{-s} |\nabla a(D_x)|^{1/2} e^{ita(D_x)} \varphi(x) \|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^2)} \leq C \|\varphi\|_{L^2(\mathbb{R}_x^2)}.$$

### 13.6.2 Non-dispersive Case Controlled by Hessian

We will show that in the non-dispersive situation the rank of  $\nabla^2 a(\xi)$  still has a responsibility for smoothing properties.

First let us consider the case when dispersiveness (L) is true only for large  $\xi$ :

$$\begin{aligned} |\nabla a(\xi)| &\geq C \langle \xi \rangle^{m-1} \quad (|\xi| \gg 1), \\ |\partial^\alpha (a(\xi) - a_m(\xi))| &\leq C \langle \xi \rangle^{m-1-|\alpha|} \quad (|\xi| \gg 1). \end{aligned} \tag{L'}$$

**Theorem 12** *Suppose  $n \geq 1, m \geq 1$ , and  $s > 1/2$ . Let  $a \in C^\infty(\mathbb{R}^n)$  be real-valued and assume that it has finitely many critical points. Assume (L') and*

$$\nabla a(\xi) = 0 \quad \Rightarrow \quad \det \nabla^2 a(\xi) \neq 0.$$

Then we have

$$\| \langle x \rangle^{-s} |\nabla a(D_x)|^{1/2} e^{ita(D_x)} \varphi(x) \|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^q)} \leq C \|\varphi\|_{L^2(\mathbb{R}_x^q)}.$$

*Example 4*  $a(\xi) = \xi_1^4 + \dots + \xi_n^4 + |\xi|^2$  satisfies assumptions in Theorem 12.

We outline the proof of Theorem 12. For the region where  $\nabla a(\xi) \neq 0$ , we can use a smoothing estimate for dispersive equations. Near the points  $\xi$  where  $\nabla a(\xi) = 0$ , there exists a change of variable  $\psi$  by Morse's lemma such that  $a(\xi) = (\sigma \circ \psi)(\xi)$

where  $\sigma(\eta)$  is a non-degenerate quadratic form, and satisfies dispersiveness (H). Hence the estimate can be reduced to the dispersive case by the method of canonical transformation.

Next we consider the case when  $a(\xi)$  is homogeneous (of order  $m$ ). Then, by Euler's identity, we have

$$\nabla a(\xi) = \frac{1}{m-1} \xi \nabla^2 a(\xi) \quad (\xi \neq 0),$$

hence

$$\nabla a(\xi) = 0 \implies \det \nabla^2 a(\xi) = 0 \quad (\xi \neq 0).$$

Therefore assumption in Theorem 12 does not make any sense in this case, but we can have the following result if we use the idea of canonical transform again:

**Theorem 13** *Suppose  $n \geq 2$  and  $s > 1/2$ . Let  $a \in C^\infty(\mathbb{R}^n \setminus 0)$  be real-valued and satisfy  $a(\lambda\xi) = \lambda^2 a(\xi)$  ( $\lambda > 0, \xi \neq 0$ ). Assume that*

$$\nabla a(\xi) = 0 \implies \text{rank } \nabla^2 a(\xi) = n - 1 \quad (\xi \neq 0).$$

Then we have

$$\| \langle x \rangle^{-s} |\nabla a(D_x)|^{1/2} e^{ita(D_x)} \varphi(x) \|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)} \leq C \|\varphi\|_{L^2(\mathbb{R}_x^n)}.$$

*Example 5*  $a(\xi) = \frac{\xi_1^2 \xi_2^2}{\xi_1^2 + \xi_2^2} + \xi_3^2 + \dots + \xi_n^2$  satisfies the assumptions in Theorem 13. In the case  $n = 2$ , this is an illustration of a smoothing estimate for the Cauchy problem for an equation like

$$i \partial_t u + D_1^2 D_2^2 \Delta^{-1} u = 0$$

which is regarded as a mixture of Davey-Stewartson and Benjamin-Ono type equations.

## 13.7 Concluding Remarks

### 13.7.1 Summary

Finally we summarise what is explained in this article in a diagram below. It is remarkable that all the results of smoothing estimates so far is derived from just the translation invariance of Lebesgue measure:

- Trivial estimate  $\|\varphi(x + t)\|_{L^2(\mathbb{R}_t)} = \|\varphi\|_{L^2(\mathbb{R}_x)}$

$\Downarrow$  (comparison principle)

- Low dimensional model estimates (Proposition 1)

↓ (canonical transform)

- Smoothing estimates for dispersive equations (Theorem 7)

↓ (secondary comparison & canonical transform)

- Invariant estimates for non-dispersive equations at least for

- \* radially symmetric  $a(\xi) = f(|\xi|)$ ,  $f \in C^1(\mathbb{R}_+)$ ,
- \* Shrira equation  $a(\xi) = \xi_1^3 + 3\xi_2^2, \xi_1^2 + \xi_1\xi_2^2$ ,
- \* non-dispersive  $a(\xi)$  controlled by its Hessian.

### 13.7.2 Smoothing Estimates for Inhomogeneous Equations

We finish this article by mentioning some results for inhomogeneous equations. Let us consider the solution

$$u(t, x) = -i \int_0^t e^{i(t-\tau)a(D_x)} f(\tau, x) d\tau$$

to the equation

$$\begin{cases} (i\partial_t + a(D_x))u(t, x) = f(t, x) & \text{in } \mathbb{R}_t \times \mathbb{R}_x^n, \\ u(0, x) = 0 & \text{in } \mathbb{R}_x^n. \end{cases}$$

Although smoothing estimates for such equation are necessary for nonlinear applications (see [21] for example), there are considerably less results on this topic available in the literature. But the method of canonical transform also works to this problem, and we will list here some recent achievement given in our forthcoming paper [20]. The following result is a counter part of Theorem 7. Especially, this kind of time-global estimate for the operator  $a(D_x)$  with lower order terms are the benefit of our new method:

**Theorem 14** *Assume (H) or (L). Suppose  $n \geq 2, m \geq 1$ , and  $s > 1/2$ . Then we have*

$$\left\| \langle x \rangle^{-s} |D_x|^{m-1} \int_0^t e^{i(t-\tau)a(D_x)} f(\tau, x) d\tau \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)} \leq C \left\| \langle x \rangle^s f(t, x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)}.$$

The proof of Theorem 14 is carried out by reducing it to model estimates below via canonical transform:

**Proposition 2** *Suppose  $n = 1$  and  $m > 0$ . Let  $a(\xi) \in C^\infty(\mathbb{R} \setminus 0)$  be a real-valued function which satisfies  $a(\lambda\xi) = \lambda^m a(\xi)$  for all  $\lambda > 0$  and  $\xi \neq 0$ . Then we have*

$$\left\| a'(D_x) \int_0^t e^{i(t-\tau)a(D_x)} f(\tau, x) d\tau \right\|_{L^2(\mathbb{R}_t)} \leq C \int_{\mathbb{R}} \|f(t, x)\|_{L^2(\mathbb{R}_t)} dx$$

for all  $x \in \mathbb{R}$ . Suppose  $n = 2$  and  $m > 0$ . Then we have

$$\begin{aligned} & \left\| |D_x|^{m-1} \int_0^t e^{i(t-\tau)|D_x|^{m-1}D_y} f(\tau, x, y) d\tau \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x)} \\ & \leq C \int_{\mathbb{R}} \|f(t, x, y)\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x)} dy \end{aligned}$$

for all  $y \in \mathbb{R}$ .

*Remark 3* Proposition 2 with the case  $n = 1$  is a unification of the results by Kenig, Ponce and Vega who treated the cases  $a(\xi) = \xi^2$  (p. 258 in [14]),  $a(\xi) = |\xi|\xi$  (p. 160 in [15]), and  $a(\xi) = \xi^3$  (p. 533 in [13]).

Since we unfortunately do not know the comparison principle for inhomogeneous equations, we gave a direct proof to Proposition 2 in [20].

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