Chapter 12
A Note on a Class of Conservative, Well-Posed Linear Control Systems

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Abstract We discuss a class of linear control problems in a Hilbert space setting. The aim is to show that these control problems fit in a particular class of evolutionary equations such that the discussion of well-posedness becomes easily accessible. Furthermore, we study the notion of conservativity. For this purpose we require additional regularity properties of the solution operator in order to allow point-wise evaluations of the solution. We exemplify our findings by a system with unbounded control and observation operators.

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12.1 Introduction

Abstract linear control systems are commonly described by a system of equations of the form

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad y(t) = Cx(t) + Du(t), \quad t \in \mathbb{R}_{>0},$$

with appropriate linear operators A, B, C and D and $\dot{x}$ denoting the time derivative of x in Newton's notation, linking the time development of state x, control u and observation y. The first equation is called *state differential equation* and the second one *observation equation*. The system is formally completed by an initial condition prescribing $x(0+) = x^{(0)}$ for the state trajectory x. As a matter of convenience we will consider this system on the whole real time-line $\mathbb{R}$ in which case the initial data $x^{(0)}$ turns into a Dirac-δ-source at time 0. Writing ∂_0 for time differentiation on the

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full time-line this yields

$$\partial_0 x = Ax + Bu + \delta \otimes x^{(0)}, \qquad y = Cx + Du \quad \text{on } \mathbb{R}.$$

We may formally re-write this into a single block operator matrix equation as

$$\left(\partial_0 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -C & 1 \end{pmatrix} + \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}\right) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & B \\ 0 & D \end{pmatrix} \begin{pmatrix} \delta \otimes x^{(0)} \\ u \end{pmatrix}, \tag{1}$$

which brings our linear control system into the realm of a problem class discussed in [8, 9]. In a suitable setting ∂_0 can be established as a normal operator with continuous inverse so that for continuous linear operators (A, B, C, D) the solution theory is little more than matrix algebra. If (A, B, C, D) contains unbounded linear operators matters are more complicated. If only A is unbounded but such that $\partial_0 + A$ is invertible the solution theory can be largely salvaged. A common instrument here is to express $(\partial_0 + A)^{-1}$ in terms of a semi-group generated by A. Matters become exceedingly complicated if also other operators in the list (A, B, C, D) are also permitted to be unbounded (see [4, 6, 7] for a survey, also [14]). The answer of questions concerning for example well-posedness along this line of reasoning may be quite involved. The classical approach to well-posedness is the concept of so-called admissible control and observation operators, using the theory of strongly continuous semigroups, see for instance [1, 2, 11–13, 15] and [3] for a survey.

Here we want to give a more elementary approach to this issue, by changing the perspective to the above type of space-time operators, which in effect by-passes C_0-semi-groups as a solution tool and at the same time enlarges the class of accessible control problems considerably. On the other hand, we use elementary C_0-semigroup theory as a tool for discussing regularity issues.

We shall consider systems of the general form

$$(\partial_0 M_0 + M_1 + \mathcal{A}) \begin{pmatrix} x \\ y \end{pmatrix} = J \begin{pmatrix} f \\ u \end{pmatrix},$$

$M_0 : X \oplus Y \to X \oplus Y$, $M_1 : X \oplus Y \to X \oplus Y$ continuous linear operators, $\mathcal{A} : D(\mathcal{A}) \subseteq X \oplus Y \to X \oplus Y$ a closed and densely defined operator. Mostly we shall assume that $J : F \oplus U \mapsto X \oplus Y$ is such that

$$J = \begin{pmatrix} E & B \\ 0 & D \end{pmatrix}$$

with $B : U \to X$, $D : U \to Y$, $E : F \to X$ continuous linear operators. Here X, Y, F, U are Hilbert spaces referred to as state, observation, data and control spaces, respectively.

There is little harm in assuming $X = F$ and $U = Y$ and we shall do so.

As the space to model time-dependence we consider the weighted L^2-space $H_{\varrho,0}(\mathbb{R})$, $\varrho \in \mathbb{R}_{>0}$, generated by the completion of $\mathring{C}_\infty(\mathbb{R})$ with respect to the inner product $\langle\cdot|\cdot\rangle_{\varrho,0}$

$$(\varphi, \psi) \mapsto \int_{\mathbb{R}} \varphi(t)^* \psi(t) \exp(-2\varrho t)\, dt.$$

The associated norm will be denoted by $|\cdot|_{\varrho,0}$. The time-derivative ∂_0 can be lifted canonically to corresponding Hilbert-space-valued generalized functions making ∂_0 a normal operator in the resulting Hilbert space $H_{\varrho,0}(\mathbb{R}, H)$, where H is an arbitrary Hilbert space. Thus the linear control system under consideration is a quaternary relation of the form

$$\mathcal{C}_{M_0,M_1,\mathcal{A},J} = \left\{ (x, y, f, u) \middle| (\partial_0 M_0 + M_1 + \mathcal{A}) \begin{pmatrix} x \\ y \end{pmatrix} = J \begin{pmatrix} f \\ u \end{pmatrix} \right\}$$

in spaces derived from this consideration. We say $\mathcal{C}_{M_0,M_1,\mathcal{A},J}$ is *well-posed*, if $\mathcal{C}_{M_0,M_1,\mathcal{A},J}$ considered as the associated binary relation

$$\left\{ \big((x, y), (f, u)\big) \middle| (\partial_0 M_0 + M_1 + \mathcal{A}) \begin{pmatrix} x \\ y \end{pmatrix} = J \begin{pmatrix} f \\ u \end{pmatrix} \right\}$$

induces for all sufficiently large $\varrho \in \mathbb{R}_{>0}$ a continuous linear mapping in a suitable Hilbert space setting linking a solution (x, y) with any given (f, u). Of course we would want the solution operator $(\partial_0 M_0 + M_1 + \mathcal{A})^{-1} J$ also to be causal in the intuitive sense. If there is no danger of confusion and the coefficient operators M_0, M_1, $\mathcal{A}$, J are clear from the context, we simply write $\mathcal{C}$ for $\mathcal{C}_{M_0,M_1,\mathcal{A},J}$.

Another extract of our current linear control system $\mathcal{C}$ is frequently of particular interest. It is the so-called transfer relation T_f which is given for a fixed f by

$$T_f := \left\{ (u, y) \middle| \bigvee_x (\partial_0 M_0 + M_1 + \mathcal{A}) \begin{pmatrix} x \\ y \end{pmatrix} = J \begin{pmatrix} f \\ u \end{pmatrix} \right\}.$$

For a well-posed linear control system this is just reading off the second block component of the solution and yields that T_f is a mapping, the transfer mapping. Frequently, one prefers to consider the unitarily equivalent operator

$$\mathcal{L}_\varrho T_f \mathcal{L}_\varrho^*,$$

where $\mathcal{L}_\varrho$ is the unitary Fourier-Laplace transformation (see Sect. 12.2), as the transfer mapping.

We also address a question approached in [15], namely conservativity of a linear control system. In [15] this notion was defined by means of a certain energy balance equality, that should be fulfilled by state, observation and control.

By considering abstract control system in the above sense we shall show that for reasonable state differential equations it is always possible to construct an observation equation, which leads to a conservative linear control system. Moreover, although in [15] unbounded control and observation operators were considered, we shall see that in the generalized form such system are reduced to the bounded operator case (with $\mathcal{A}$ being the only unbounded linear operator involved).

12.2 Setting

The particular time-derivative defined as a normal and invertible operator in the exponentially weighted space $H_{\varrho,0}(\mathbb{R}) := L_2(\mathbb{R}, \exp(-2\varrho x)\mathrm{d}x)$ (for some $\varrho \in \mathbb{R}_{>0}$) is given in various articles of the authors of this paper. The core issues are discussed in [5]. We state the basic facts as follows. Let $\varrho \in \mathbb{R}_{>0}$. We define ∂_0 as the closure of the operator $\mathring{C}_\infty(\mathbb{R}) \subseteq H_{\varrho,0}(\mathbb{R}) \to H_{\varrho,0}(\mathbb{R}) : f \mapsto f'$, where $\mathring{C}_\infty(\mathbb{R})$ denotes the space of infinitely often differentiable functions with compact support. It can be shown that $\partial_0^{-1} \in L(H_{\varrho,0}(\mathbb{R}), H_{\varrho,0}(\mathbb{R}))$ and $\|\partial_0^{-1}\| \leq 1/\varrho$.

It is well-known that there is an explicit spectral representation as a multiplication operator of the one-dimensional derivative on the real line, which is given by the unitary *Fourier transformation* $\mathcal{F} : L_2(\mathbb{R}) \to L_2(\mathbb{R})$. An analogous representation can be found for ∂_0: Denote by m the multiplication-by-argument-operator in $L_2(\mathbb{R})$ with natural domain and $\exp(-\varrho m) : H_{\varrho,0}(\mathbb{R}) \to L_2(\mathbb{R}) : f \mapsto \exp(-\varrho(\cdot))f(\cdot)$. Then we have the following unitary representation of ∂_0:

$$\mathcal{L}_\varrho^*(im + \varrho)\mathcal{L}_\varrho = \partial_0$$

with the unitary *Fourier-Laplace transformation* $\mathcal{L}_\varrho := \mathcal{F}\exp(-\varrho m)$. This formula can canonically be lifted to the Hilbert-space-valued case. Moreover, the latter unitary representation results in a functional calculus for ∂_0^{-1}. More precisely, let $r > \frac{1}{2\varrho}$ and H be a Hilbert space. Let $M : B(r,r) \to L(H)$ be an element of the Hardy space $\mathcal{H}^\infty(B(r,r), L(H))$ of bounded and analytic functions defined on the open ball $B(r,r) \subseteq \mathbb{C}$ with values in $L(H)$, the set of continuous linear operators within H. Define

$$M\left(\partial_0^{-1}\right) := \mathcal{L}_\varrho^* M\left(\frac{1}{im + \varrho}\right)\mathcal{L}_\varrho,$$

where $M(\frac{1}{im+\varrho})\phi(t) := M(\frac{1}{it+\varrho})\phi(t)$ for all $\phi \in \mathring{C}_\infty(\mathbb{R}, H)$ and $t \in \mathbb{R}$. It is easy to see that $M(\partial_0^{-1}) \in L(H_{\varrho,0}(\mathbb{R}, H))$ and $\partial_0^{-1}M(\partial_0^{-1}) = M(\partial_0^{-1})\partial_0^{-1}$. As it was already mentioned in [5], for $h > 0$ the time-shift τ_{-h} defined as $\tau_{-h} f := f(\cdot - h)$ or the convolution with a $L_1(\mathbb{R})$-function supported on the positive reals yield analytic and bounded functions of ∂_0^{-1} in the above sense.

In the following we shall also make use of the concept of Sobolev lattices, which are related to abstract distribution spaces associated with particular (unbounded) operators in a Hilbert space. The whole set-up is described in [10]. We sketch it as follows. Let C, D be densely defined, closed, linear operators in a Hilbert space H. Furthermore, assume that $0 \in \varrho(C) \cap \varrho(D)$ and $C^{-1}D^{-1} = D^{-1}C^{-1}$. For $k, n \in \mathbb{Z}$ the Hilbert space $H_{k,n}(C, D)$ is defined as the completion of $D(C^{|k|}) \cap D(D^{|n|})$ with respect to the (well-defined) inner product $(\phi, \psi) \mapsto \langle C^k D^n \phi, C^k D^n \psi\rangle$. The family $(H_{k,n}(C, D))_{(k,n)\in\mathbb{Z}^2}$ is called *Sobolev lattice associated with* (C, D). One can show that for $k_1, n_1 \in \mathbb{Z}$ with $k_1 \leq k$ and $n_1 \leq n$ we have dense and continuous embeddings

$$H_{k,n}(C, D) \hookrightarrow H_{k_1,n_1}(C, D).$$

The latter relation justifies the term “lattice”. Indeed, $(H_{k,n}(C,D))_{(k,n)\in\mathbb{Z}^2}$ is a lattice with respect to the order relation $\hookrightarrow$, which is isomorphic to $\mathbb{Z}^2$ endowed with component-wise order.

Moreover, by continuous extension, we have unitary operators

$$C^{k_2}D^{n_2} : H_{k,n}(C,D) \to H_{k-k_2,n-n_2}(C,D)$$

for all $n_2, k_2 \in \mathbb{Z}$. It should be mentioned that any continuous linear operator $B : H \to H$, which commutes with $C^{-1} \in L(H)$, has a unique continuous extension (restriction) to $H_{k,0}(C,D)$. We shall use the construction of Sobolev lattices in the aforementioned situation of linear control systems. For the special case that D is the identity on H, we will write $H_k(C) := H_{k,n}(C,D)$. Moreover, given a densely defined, closed linear operator $\mathcal{A} : D(\mathcal{A}) \subseteq \mathcal{H} \to \mathcal{H}$ with non-empty resolvent set $\varrho(\mathcal{A})$. Then, for $\varrho \in \mathbb{R}_{>0}$ and $\lambda \in \varrho(\mathcal{A})$, we define

$$H_{\varrho,k}\big(\mathbb{R}, \mathcal{H}_n(\mathcal{A}-\lambda)\big) := H_{k,n}(\partial_0, \mathcal{A}-\lambda).$$

If it is clear from the context, which operator $\mathcal{A}$ is under consideration, we shall also write $H_{\varrho,k}(\mathbb{R},\mathcal{H}_n)$ for short. Clearly, the latter set does not depend on the particular choice of $\lambda \in \varrho(\mathcal{A})$. As another short-hand notation we also define

$$H_{\varrho,\infty}(\mathbb{R},\mathcal{H}_n) := \bigcap_{k\in\mathbb{N}} H_{\varrho,k}(\mathbb{R},\mathcal{H}_n).$$

12.3 Solution Theory for Abstract Linear Control Systems

We summarize the core issues of the solution theory used in this paper. In the whole section, we make the following assumptions. Let X and Y be Hilbert spaces and define $\mathcal{H} := X \oplus Y$. Moreover, let $M_0 : \mathcal{H} \to \mathcal{H}$, $M_1 : \mathcal{H} \to \mathcal{H}$, $J : \mathcal{H} \to \mathcal{H}$ be continuous linear operators and let $\mathcal{A} : D(\mathcal{A}) \subseteq \mathcal{H} \to \mathcal{H}$ be a closed linear operator. We assume that

- M_0 is selfadjoint, non-negative and strictly positive on its range, whereas
- $\mathfrak{Re}\, M_1 : \mathcal{H} \to \mathcal{H}$ is strictly positive on the null space of M_0.

To simplify matters, we shall also assume that

- $\mathcal{A}$ is skew-selfadjoint in $\mathcal{H}$, which is a standard case for most problems.

We will use the extension of these operators to the Hilbert space of $\mathcal{H}$-valued $H_{\varrho,0}(\mathbb{R})$ functions. From the 3 aforementioned properties, it is easy to see that the following lemma holds. For a set $S \subseteq \mathbb{R}$, we denote by $\chi_S(m_0)$ the truncation operator, mapping a function $f : \mathbb{R} \to \mathcal{H}$ to the truncated one: $\chi_S(m_0)f := (t \mapsto \chi_S(t)f(t))$.

Lemma 1 *There is a constant $\beta_0 \in \mathbb{R}_{>0}$ such that for all $\xi \in D(\mathcal{A}) \cap D(\partial_0)$ and all sufficiently large $\varrho \in \mathbb{R}_{>0}$*

$$\mathfrak{Re}\big\langle \chi_{\mathbb{R}_{<0}}(m_0)\xi \,|\, (\partial_0 M_0 + M_1 + \mathcal{A})\xi \big\rangle_{\varrho,0,0} \geq \beta_0 \big\langle \chi_{\mathbb{R}_{<0}}(m_0)\xi \,|\, \xi \big\rangle_{\varrho,0,0}. \tag{++a}$$

It follows

$$\mathfrak{Re}\big\langle \xi \,|\, (\partial_0 M_0 + M_1 + \mathcal{A})\xi \big\rangle_{\varrho,0,0} \geq \beta_0 \langle \xi \,|\, \xi \rangle_{\varrho,0,0}. \tag{++b}$$

The proof can be found in Chap. 7 in [10]. It is remarkable that the core of the proof of the solution theory only relies on the positive definiteness as stated in Lemma 1 and the explicit spectral representation of ∂_0.

Theorem 1 *For every sufficiently large $\varrho \in \mathbb{R}_{>0}$ and every $\begin{pmatrix} f \\ u \end{pmatrix} \in H_{\varrho,0}(\mathbb{R}, X \oplus Y)$ there is a unique solution $\begin{pmatrix} x \\ y \end{pmatrix} \in H_{\varrho,0}(\mathbb{R}, X \oplus Y)$ of the problem*

$$\overline{(\partial_0 M_0 + M_1 + \mathcal{A})} \begin{pmatrix} x \\ y \end{pmatrix} = J \begin{pmatrix} f \\ u \end{pmatrix}.$$

Moreover, the solution depends continuously on the data in $H_{\varrho,0}(\mathbb{R}, X \oplus Y)$ and the solution operator $(\partial_0 M_0 + M_1 + \mathcal{A})^{-1} J$ is causal in the sense that

$$\begin{aligned} &\chi_{\mathbb{R}_{<a}}(m_0)\overline{(\partial_0 M_0 + M_1 + \mathcal{A})^{-1}} J \\ &\quad = \chi_{\mathbb{R}_{<a}}(m_0)\overline{(\partial_0 M_0 + M_1 + \mathcal{A})^{-1}} J \chi_{\mathbb{R}_{<a}}(m_0) \end{aligned}$$

for all $a \in \mathbb{R}$.

Remark 1 The assumptions on the operators $\mathcal{A}$, M_1 and M_0 are sharp in the sense that we can easily construct ill-posed systems, if one of the assumptions fails. For instance consider the system

$$\left(\partial_0 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 0 & -C^* \\ C & 0 \end{pmatrix}\right) \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix},$$

where C is an unbounded, closed and densely defined linear operator. Now $\mathfrak{Re}\, M_1 = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$ is not strictly positive definite on the kernel of $M_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Substituting the second equation $v = Cu$ into the first yields

$$\big(\partial_0 - C^* C\big) u = f,$$

which is an abstract heat equation with time reversed and well-known to be ill-posed as a forward causal equation. Even in the ode case, i.e. for $C = 0$, taking now $M_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and considering the resulting system

$$\left(\partial_0 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}$$

would yield

$$u = g,$$
$$\partial_0 u + v = f,$$

which can only have a solution $u, v \in H_{\varrho,0}(\mathbb{R}, H)$ if $g = u = \partial_0^{-1}(f - v) \in H_{\varrho,1}(\mathbb{R}, H)$ and not for general data $f, g \in H_{\varrho,0}(\mathbb{R}, H)$.

Using the Sobolev lattice $(H_{\varrho,k}(\mathbb{R}, \mathcal{H}_n))_{(k,n)\in\mathbb{Z}^2}$, we shall extend the operators ∂_0, M_0, M_1, $\mathcal{A}$ to $H_{\varrho,-\infty}(\mathbb{R}, \mathcal{H}_{-1}) := \bigcup_{k\in\mathbb{Z}} H_{\varrho,k}(\mathbb{R}, \mathcal{H}_{-1})$. This has the effect that we do not need to write the closure bar anymore. However, this has the consequence that, whereas the equation

$$(\partial_0 M_0 + M_1 + \mathcal{A}) \begin{pmatrix} x \\ y \end{pmatrix} = J \begin{pmatrix} f \\ u \end{pmatrix}$$

holds in $H_{\varrho,0}(\mathbb{R}, X \oplus Y)$, the equation

$$\partial_0 M_0 \begin{pmatrix} x \\ y \end{pmatrix} + M_1 \begin{pmatrix} x \\ y \end{pmatrix} + \mathcal{A} \begin{pmatrix} x \\ y \end{pmatrix} = J \begin{pmatrix} f \\ u \end{pmatrix}$$

only holds in $H_{\varrho,-1}(\mathbb{R}, \mathcal{H}_{-1})$. This line of reasoning also yields that $\begin{pmatrix} x \\ y \end{pmatrix} \in H_{\varrho,-1}(\mathbb{R}, \mathcal{H}_1)$. We will use this observation in the forthcoming sections. To incorporate non-vanishing initial data we record the following corollary, where we use the continuous extension of the solution operator—a particular bounded and analytic function of ∂_0^{-1} (cf. Sect. 12.2)—to the space $H_{\varrho,-1}(\mathbb{R}, \mathcal{H})$.

Corollary 1 *For every sufficiently large $\varrho \in \mathbb{R}_{>0}$ and every $\begin{pmatrix} f \\ u \end{pmatrix} \in H_{\varrho,0}(\mathbb{R}, X \oplus Y)$ and $\begin{pmatrix} x^{(0)} \\ y^{(0)} \end{pmatrix} \in M_0[X \oplus Y]$ there is a unique solution $\begin{pmatrix} x \\ y \end{pmatrix} \in H_{\varrho,-1}(\mathbb{R}, X \oplus Y)$ of the problem*

$$(\partial_0 M_0 + M_1 + \mathcal{A}) \begin{pmatrix} x \\ y \end{pmatrix} = J \begin{pmatrix} f \\ u \end{pmatrix} + \delta \otimes M_0 \begin{pmatrix} x^{(0)} \\ y^{(0)} \end{pmatrix}. \tag{2}$$

The solution depends continuously on the data in $H_{\varrho,-1}(\mathbb{R}, X \oplus Y)$.

Proof The existence result follows by applying the previous theorem to

$$(\partial_0 M_0 + M_1 + \mathcal{A}) \begin{pmatrix} \xi \\ \eta \end{pmatrix} = J \begin{pmatrix} \partial_0^{-1} f \\ \partial_0^{-1} u \end{pmatrix} + \chi_{\mathbb{R}_{>0}} \otimes M_0 \begin{pmatrix} x^{(0)} \\ y^{(0)} \end{pmatrix}$$

and then differentiating and letting

$$\begin{pmatrix} x \\ y \end{pmatrix} := \partial_0 \begin{pmatrix} \xi \\ \eta \end{pmatrix}.$$

The uniqueness and continuous dependence part follows conversely by applying ∂_0^{-1} to (2) and using the uniqueness and continuous dependence result of Theorem 1. □

12.4 Regularity

In this section we discuss regularity issues. The method is based on "see-saw"-type arguments and relies on the Sobolev lattice associated with $(\partial_0, \mathcal{A}+1)$, i.e.,

$$\big(H_{\varrho,s}(\mathbb{R}, \mathcal{H}_k)\big)_{(s,k)\in\mathbb{Z}^2}.$$

Our main focus will be initial value problems. We need the following definition.

Definition 1 Let $\mathcal{C}_{M_0,M_1,\mathcal{A},J}$ be a well-posed[1] linear control system. If

$$P_0\big((\partial_0 M_0 + M_1 + \mathcal{A})^{-1}\delta \otimes M_0 - \chi_{\mathbb{R}_{>0}} \otimes P_0\big)[\mathcal{U}] \subseteq H_{\varrho,1}(\mathbb{R}, \mathcal{H})$$

for some subspace $\mathcal{U} \subseteq D(\mathcal{A})$, which is dense in $\mathcal{H}$, then we call $\mathcal{C}_{M_0,M_1,\mathcal{A},J}$ a *globally regularizing* linear control system. If for all $T \in \mathbb{R}$ we have

$$\begin{aligned}\chi_{\mathbb{R}_{<T}}(m_0)P_0\big((\partial_0 M_0 + M_1 + \mathcal{A})^{-1}\delta \otimes M_0 - \chi_{\mathbb{R}_{>0}} \otimes P_0\big)[\mathcal{U}]\\ \subseteq \chi_{\mathbb{R}_{<T}}(m_0)\big[H_{\varrho,1}(\mathbb{R}, \mathcal{H})\big]\end{aligned}$$

we call $\mathcal{C}_{M_0,M_1,\mathcal{A},J}$ a *locally regularizing* linear control system. Here $P_0 := \pi_0^*\pi_0$, where π_0 denotes the orthogonal projector onto $M_0[\mathcal{H}]$.

Obviously, the regularizing property is independent of J. For locally regularizing linear control systems we have according to the Sobolev embedding property (cf. Lemma 3.1.59 in [10]) point-wise evaluation as a continuous operation and we can define, what it means for such a system to be conservative. In the forthcoming sections, we deal with a system studied in [15]. This system may be rewritten into a first order system such that the above theory becomes applicable. Moreover, it can be shown that the respective system is a special case of the system occurring in the next theorem, for which the notion of conservativity can be established.

Theorem 2 *Let $\mathcal{C}_{M_0,M_1,\mathcal{A},J}$ be a linear control system with*

$$M_0 = \begin{pmatrix} M_{00} & 0 \\ 0 & 0 \end{pmatrix}, \qquad M_1 = \begin{pmatrix} M_{11} & 0 \\ \alpha R^{-1}\pi_1 & \alpha \end{pmatrix},$$

$$\mathcal{A} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & 2\,\mathfrak{Re}(M_{11})\pi_1^* R \\ 0 & \alpha \end{pmatrix},$$

[1] In this case

$$\big((\partial_0 M_0 + M_1 + \mathcal{A})^{-1}\delta \otimes M_0\big) : \mathcal{H} \to H_{\varrho,-1}(\mathbb{R}, \mathcal{H}).$$
$$z \mapsto (\partial_0 M_0 + M_1 + \mathcal{A})^{-1}\delta \otimes M_0 z$$

is a continuous linear operator.

where $M_{00} \in L(X)$ is selfadjoint and strictly positive definite on $M_{00}[X]$, $M_{11} \in L(X)$ with $(\mathfrak{Re}\, M_{11}) \geq 0$ and $\mathfrak{Re}\, M_{11}$ is strictly positive definite on $[\{0\}]M_{00}$, $U_1 := (\mathfrak{Re}\, M_{11})[X]$, $R : U_1 \to U_1$ is a continuous linear bijection, $\pi_1 : X \to U_1$ is the orthogonal projector, A is a skew-selfadjoint operator on X and $\alpha \in \mathbb{R} \setminus \{0\}$, is such that

$$4\left\| \left(\sqrt{(\mathfrak{Re}\, M_{11})|_{[\{0\}]M_{00}}}\right)^{-1} \right\|^{-2} \left\| R^{-1} \right\|^{-2} > \alpha > 0.$$

Then $\mathcal{C}_{M_0,M_1,\mathcal{A},J}$ is well-posed. Let $U_0 := M_{00}[X]$ and $\pi_0 : X \to U_0$, $P_0 := \pi_0^* \pi_0$ the corresponding orthogonal projections. Assume in addition that $\mathcal{C}_{M_0,M_1,\mathcal{A},J}$ is locally regularizing. Then $\mathcal{C}_{M_0,M_1,\mathcal{A},J}$ is conservative *in the sense of* [15], *i.e., the solution $\binom{x}{y}$ of*

$$(\partial_0 M_0 + M_1 + \mathcal{A}) \begin{pmatrix} x \\ y \end{pmatrix} = J \begin{pmatrix} 0 \\ u \end{pmatrix} + \delta \otimes \begin{pmatrix} M_{00} x^{(0)} \\ 0 \end{pmatrix}$$

for the initial data $x^{(0)}$ and control u gives rise to mappings

$$\begin{aligned} \Sigma_T : \begin{pmatrix} \sqrt{M_{00}} & 0 \\ 0 & \sqrt{2\,\mathfrak{Re}\, M_{11}}R \end{pmatrix} \begin{pmatrix} P_0 x^{(0)} \\ \chi_{\mathbb{R}_{<T}}(m_0) u \end{pmatrix} \\ \mapsto \begin{pmatrix} \sqrt{M_{00}} & 0 \\ 0 & \sqrt{2\,\mathfrak{Re}\, M_{11}}R \end{pmatrix} \begin{pmatrix} P_0 x(T) \\ \chi_{\mathbb{R}_{<T}}(m_0) y \end{pmatrix}, \end{aligned}$$

which are densely defined isometries on $U_0 \oplus L^2(\mathbb{R}_{>0}, U_1)$ for all $T \in \mathbb{R}_{\geq 0}$.

Remark 2 In the setting of the theorem above, the state space is given by $\mathcal{H} = X \oplus U_1$. Furthermore we shall note here that for the definition of conservativity the parameter $\alpha \in \mathbb{R} \setminus \{0\}$ is irrelevant. However, it is used to adjust for the assumptions of our above solution theory.

Proof of Theorem 2 At first we show well-posedness of $\mathcal{C}_{M_0,M_1,\mathcal{A},J}$. We need to consider the positive definiteness of

$$\mathfrak{Re}\, M_1 = \begin{pmatrix} \mathfrak{Re}\, M_{11} & \frac{1}{2}\alpha \pi_1^* (R^{-1})^* \\ \frac{1}{2}\alpha R^{-1}\pi_1 & \alpha \end{pmatrix}$$

on $[\{0\}]M_0 = [\{0\}]M_{00} \oplus U_1$. Let $z \oplus y \in [\{0\}]M_{00} \oplus U_1$. For $\varepsilon > 0$, we compute

$$\begin{aligned} \left\langle \mathfrak{Re}\, M_1(z \oplus y) | z \oplus y \right\rangle &= \langle z | \mathfrak{Re}\, M_{11} z\rangle + \left\langle z | \alpha \pi_1^* \left(R^{-1}\right)^* y \right\rangle + \alpha \langle y | y\rangle \\ &\geq \langle \sqrt{\mathfrak{Re}\, M_{11}} z | \sqrt{\mathfrak{Re}\, M_{11}} z\rangle - \frac{1}{2\varepsilon}|z|^2 \\ &\quad - \frac{\varepsilon}{2}\alpha^2 \left|\pi_1^* \left(R^{-1}\right)^* y\right|^2 + \alpha \langle y | y\rangle \end{aligned}$$

$$\geq \left(1 - \frac{1}{2\varepsilon}\left\|\left(\sqrt{(\mathfrak{Re}\, M_{11})|_{[\{0\}]M_{00}}}\right)^{-1}\right\|^2\right)|\sqrt{\mathfrak{Re}\, M_{11}}z|^2$$
$$+\alpha\left(1 - \frac{\varepsilon}{2}\alpha\left\|R^{-1}\right\|^2\right)|y|^2.$$

From the first term of the right-hand side of the latter inequality it follows that ε has to be chosen such that

$$\frac{1}{\varepsilon} < 2\left\|\left(\sqrt{(\mathfrak{Re}\, M_{11})|_{[\{0\}]M_{00}}}\right)^{-1}\right\|^{-2}$$

holds. From the second term, we read off that

$$1 - \frac{\varepsilon}{2}\alpha\left\|R^{-1}\right\|^2 > 0$$

should hold. Thus, we want ε to satisfy in addition

$$\frac{1}{\varepsilon} > \frac{\alpha}{2}\left\|R^{-1}\right\|^2.$$

The condition on α ensures that the interval

$$\left]\frac{\alpha}{2}\left\|R^{-1}\right\|^2, 2\left\|\left(\sqrt{(\mathfrak{Re}\, M_{11})|_{[\{0\}]M_{00}}}\right)^{-1}\right\|^{-2}\right[$$

is not empty. Employing Theorem 1, we conclude that the abstract linear control system $\mathcal{C}$ is well-posed. Assume now that $\mathcal{C}$ is locally regularizing. Due to the block structure of the operator matrices M_0, M_1, $\mathcal{A}$ and J there exists a subspace $\mathcal{U} \subseteq D(A)$, dense in X, such that for $x^{(0)} \in \mathcal{U}$, we have

$$\chi_{\mathbb{R}_{<T}}(m_0)\begin{pmatrix} P_0 & 0 \\ 0 & 0 \end{pmatrix}\left((\partial_0 M_0 + M_1 + \mathcal{A})^{-1}\delta \otimes M_0 - \chi_{\mathbb{R}_{>0}} \otimes \begin{pmatrix} P_0 & 0 \\ 0 & 0 \end{pmatrix}\right)\begin{pmatrix} x^{(0)} \\ 0 \end{pmatrix}$$
$$\in \chi_{\mathbb{R}_{<T}}(m_0)\left[H_{\varrho,1}(\mathbb{R}, \mathcal{H})\right]$$

for all $T \in \mathbb{R}$. Let $x^{(0)} \in \mathcal{U}$ and $u \in H_{\varrho,1}(\mathbb{R}_{\geq 0}, U_1)$. Our general solution theory yields the unique existence of $(x, y) \in H_{\varrho,-1}(\mathbb{R}, X \oplus U_1)$ of the problem

$$\left(\partial_0\begin{pmatrix} M_{00} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} M_{11} & 0 \\ \alpha R^{-1}\pi_1 & \alpha \end{pmatrix} + \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}\right)\begin{pmatrix} x \\ y \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 2\,\mathfrak{Re}(M_{11})\pi_1^* R \\ 0 & \alpha \end{pmatrix}\begin{pmatrix} 0 \\ u \end{pmatrix} + \begin{pmatrix} \delta \otimes M_{00}x^{(0)} \\ 0 \end{pmatrix},$$

where $\operatorname{supp} x \subseteq \mathbb{R}_{\geq 0}$ and $\operatorname{supp} y \subseteq \mathbb{R}_{\geq 0}$ due to the causality of the solution operator. This leads to

$$
\begin{aligned}
&\begin{pmatrix} P_0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} - \chi_{\mathbb{R}_{>0}} \otimes \begin{pmatrix} P_0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x^{(0)} \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} P_0 & 0 \\ 0 & 0 \end{pmatrix}(\partial_0 M_0 + M_1 + \mathcal{A})^{-1}\begin{pmatrix} 0 & 2\,\mathfrak{Re}(M_{11})\pi_1^* R \\ 0 & \alpha \end{pmatrix}\begin{pmatrix} 0 \\ u \end{pmatrix} \\
&\quad + \begin{pmatrix} P_0 & 0 \\ 0 & 0 \end{pmatrix}(\partial_0 M_0 + M_1 + \mathcal{A})^{-1}\begin{pmatrix} \delta \otimes M_{00} x^{(0)} \\ 0 \end{pmatrix} \\
&\quad - \chi_{\mathbb{R}_{>0}} \otimes \begin{pmatrix} P_0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x^{(0)} \\ 0 \end{pmatrix}.
\end{aligned}
$$

Since $(\partial_0 M_0 + M_1 + \mathcal{A})^{-1}$ leaves $H_{\varrho,1}(\mathbb{R}, \mathcal{H})$ invariant, we read off that

$$
\begin{aligned}
&\chi_{\mathbb{R}_{<T}}(m_0)\left(\begin{pmatrix} P_0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} - \chi_{\mathbb{R}_{>0}} \otimes \begin{pmatrix} P_0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x^{(0)} \\ 0 \end{pmatrix}\right) \\
&\quad \in \chi_{\mathbb{R}_{<T}}(m_0)\big[H_{\varrho,1}(\mathbb{R}, \mathcal{H})\big]
\end{aligned}
$$

holds for all $T \in \mathbb{R}$. We fix $T \in \mathbb{R}_{>0}$ for the rest of the proof. Let $\varphi \in C_\infty(\mathbb{R})$ be such that $\varphi = 1$ on $\mathbb{R}_{<T+1}$ and $\varphi = 0$ on $\mathbb{R}_{>T+2}$. Using the Sobolev lattice associated with $(\partial_0, \mathcal{A} + 1)$ we get that

$$
\begin{aligned}
\begin{pmatrix} \varphi(m_0) & 0 \\ 0 & 1 \end{pmatrix}(\partial_0 M_0 + M_1 + \mathcal{A})\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} \varphi(m_0) & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 2\,\mathfrak{Re}(M_{11})\pi_1^* R \\ 0 & \alpha \end{pmatrix}\begin{pmatrix} 0 \\ u \end{pmatrix} \\
&\quad + \begin{pmatrix} \varphi(0)\delta \otimes M_{00} x^{(0)} \\ 0 \end{pmatrix},
\end{aligned}
$$

which implies

$$
(\partial_0 M_0 + M_1 + \mathcal{A})\begin{pmatrix} \varphi(m_0) & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \tag{3}
$$

$$
\begin{aligned}
&= \begin{pmatrix} \varphi(m_0) & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 2\,\mathfrak{Re}(M_{11})\pi_1^* R \\ 0 & \alpha \end{pmatrix}\begin{pmatrix} 0 \\ u \end{pmatrix} \\
&\quad + \begin{pmatrix} \varphi(0)\delta \otimes M_{00} x^{(0)} \\ 0 \end{pmatrix} + \begin{pmatrix} M_{00}\varphi'(m_0) & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \\
&= \begin{pmatrix} \varphi(m_0) 2\,\mathfrak{Re}(M_{11})\pi_1^* R u \\ \alpha u \end{pmatrix} + \begin{pmatrix} \delta \otimes M_{00} x^{(0)} \\ 0 \end{pmatrix} \\
&\quad + \begin{pmatrix} M_{00}\varphi'(m_0) x \\ 0 \end{pmatrix}.
\end{aligned} \tag{4}
$$

Define $x_\varphi := \varphi(m_0)x$. Employing the local regularizing property and using that $M_{00}\varphi'(m_0)x \in H_{\varrho,1}(\mathbb{R}, X)$, we deduce that $P_0 x_\varphi - \chi_{\mathbb{R}_{>0}} \otimes P_0 x^{(0)} \in H_{\varrho,1}(\mathbb{R}, X)$.

Moreover, from

$$\begin{aligned}
&(\partial_0 M_0 + M_1 + \mathcal{A}) \begin{pmatrix} x_\varphi \\ y \end{pmatrix} \\
&\quad = \left(\partial_0 M_0 \begin{pmatrix} x_\varphi \\ y \end{pmatrix} + M_1 \begin{pmatrix} x_\varphi \\ y \end{pmatrix} + \mathcal{A} \begin{pmatrix} x_\varphi \\ y \end{pmatrix} \right) \\
&\quad = \begin{pmatrix} \varphi(m_0) 2\,\mathfrak{Re}(M_{11}) \pi_1^* R u \\ \alpha u \end{pmatrix} + \begin{pmatrix} \delta \otimes M_{00} x^{(0)} \\ 0 \end{pmatrix} + \begin{pmatrix} M_{00} \varphi'(m_0) x \\ 0 \end{pmatrix}
\end{aligned}$$

it follows that

$$\begin{aligned}
&\partial_0 M_0 \left(\begin{pmatrix} x_\varphi \\ y \end{pmatrix} - \chi_{\mathbb{R}_{>0}} \otimes \begin{pmatrix} x^{(0)} \\ 0 \end{pmatrix} \right) + M_1 \left(\begin{pmatrix} x_\varphi \\ y \end{pmatrix} - \chi_{\mathbb{R}_{>0}} \otimes \begin{pmatrix} x^{(0)} \\ 0 \end{pmatrix} \right) \\
&\qquad + \mathcal{A} \left(\begin{pmatrix} x_\varphi \\ y \end{pmatrix} - \chi_{\mathbb{R}_{>0}} \otimes \begin{pmatrix} x^{(0)} \\ 0 \end{pmatrix} \right) \qquad (5) \\
&\quad = \begin{pmatrix} \varphi(m_0) 2\,\mathfrak{Re}(M_{11}) \pi_1^* R u \\ \alpha u \end{pmatrix} - \begin{pmatrix} \chi_{\mathbb{R}_{>0}} \otimes A x^{(0)} \\ 0 \end{pmatrix} \\
&\qquad - \chi_{\mathbb{R}_{>0}} \otimes M_1 \begin{pmatrix} x^{(0)} \\ 0 \end{pmatrix} + \begin{pmatrix} M_{00} \varphi'(m_0) x \\ 0 \end{pmatrix}, \qquad (6)
\end{aligned}$$

where equality holds in $H_{\varrho,-1}(\mathbb{R}, \mathcal{H}_{-1})$. However, since the right-hand of the latter equation lies in $H_{\varrho,0}(\mathbb{R}, \mathcal{H}_0)$ we get $x_\varphi - \chi_{\mathbb{R}_{>0}} \otimes x^{(0)} \in H_{\varrho,0}(\mathbb{R}, X)$ and since $P_0 x_\varphi - \chi_{\mathbb{R}_{>0}} \otimes P_0 x^{(0)} \in H_{\varrho,1}(\mathbb{R}, X)$, we deduce that $\begin{pmatrix} x_\varphi \\ y \end{pmatrix} - \chi_{\mathbb{R}_{>0}} \otimes \begin{pmatrix} x^{(0)} \\ 0 \end{pmatrix} \in H_{\varrho,0}(\mathbb{R}, \mathcal{H}_1)$. In particular, this yields $x_\varphi \in H_{\varrho,0}(\mathbb{R}, H_1(A+1))$. We read off the first row equation of (5):

$$\begin{aligned}
&\partial_0 M_{00} \big(x_\varphi - \chi_{\mathbb{R}_{>0}} \otimes x^{(0)} \big) + M_{11} \big(x_\varphi - \chi_{\mathbb{R}_{>0}} \otimes x^{(0)} \big) + A \big(x_\varphi - \chi_{\mathbb{R}_{>0}} \otimes x^{(0)} \big) \\
&\quad = 2\,\mathfrak{Re}(M_{11}) \pi_1^* R \varphi(m_0) u - M_{11} \big(\chi_{\mathbb{R}_{>0}} \otimes x^{(0)} \big) - \chi_{\mathbb{R}_{>0}} \otimes A x^{(0)} + M_{00} \varphi'(m_0) x.
\end{aligned}$$

Thus, we get that

$$\begin{aligned}
&\partial_0 M_{00} \big(x_\varphi - \chi_{\mathbb{R}_{>0}} \otimes x^{(0)} \big) + M_{11} x_\varphi + A x_\varphi \\
&\quad = 2\,\mathfrak{Re}(M_{11}) \pi_1^* R \varphi(m_0) u + M_{00} \varphi'(m_0) x,
\end{aligned}$$

with equality in $H_0(A+1)$ pointwise almost everywhere. Multiplying by $\langle \cdot | x_\varphi \rangle_X$, taking real-parts and using $\mathfrak{Re}\langle A x_\varphi(s) | x_\varphi(s) \rangle = 0$ for almost every $s \in]0, T[$, we deduce that for almost every $t \in]0, T[$ it holds

$$\begin{aligned}
&\mathfrak{Re}\big\langle \partial_0 M_{00} \big(x_\varphi - \chi_{\mathbb{R}_{>0}} \otimes x^{(0)} \big)(t) | x_\varphi(t) \big\rangle + \big\langle \mathfrak{Re}\, M_{11} x_\varphi(t) | x_\varphi(t) \big\rangle \\
&\quad = \mathfrak{Re}\big\langle 2\,\mathfrak{Re}(M_{11}) \pi_1^* R u(t) | x_\varphi(t) \big\rangle.
\end{aligned}$$

We let $\mu_{00} := \pi_0 M_{00} \pi_0^*$, $\mu_{11} := \pi_1(\mathfrak{Re}(M_{11}))\pi_1^*$. Thus, for almost every $t \in]0,T[$ it holds

$$\mathfrak{Re}\big\langle \partial_0\big(\mu_{00}\big(\pi_0 x_\varphi - \chi_{\mathbb{R}_{>0}} \otimes \pi_0 x^{(0)}\big)\big)(t)|\pi_0 x_\varphi(t)\big\rangle + \big\langle \mu_{11}\pi_1 x_\varphi(t)|\pi_1 x_\varphi(t)\big\rangle \\ = \mathfrak{Re}\big\langle 2\mu_{11} Ru(t)|\pi_1 x_\varphi(t)\big\rangle.$$

Hence, we conclude that for almost every $t \in]0,T[$:

$$\frac{1}{2}\Big(s \mapsto \big\langle \mu_{00}\big(\pi_0 x_\varphi - \chi_{\mathbb{R}_{>0}} \otimes \pi_0 x^{(0)}\big)(s)|\big(\pi_0 x_\varphi - \chi_{\mathbb{R}_{>0}} \otimes \pi_0 x^{(0)}\big)(s)\big\rangle\Big)'(t) \\ = -\mathfrak{Re}\big\langle \partial_0\big(\mu_{00}\big(\pi_0 x_\varphi - \chi_{\mathbb{R}_{>0}} \otimes \pi_0 x^{(0)}\big)\big)(t)|\chi_{\mathbb{R}_{>0}} \otimes \pi_0 x^{(0)}(t)\big\rangle \\ - \big\langle \mu_{11}\pi_1 x_\varphi(t)|\pi_1 x_\varphi(t)\big\rangle + \mathfrak{Re}\big\langle 2\mu_{11} Ru(t)|\pi_1 x_\varphi(t)\big\rangle. \tag{7}$$

The second row equation of (5) gives

$$\alpha R^{-1}\pi_1 x_\varphi + \alpha y = \alpha u.$$

Hence,

$$\pi_1 x_\varphi = R(u - y).$$

Since $x_\varphi(t) = x(t)$ for all $t \in]0,T[$, the latter equation put into (7) gives

$$\frac{1}{2}\Big(s \mapsto \big\langle \mu_{00}\big(\pi_0 x - \chi_{\mathbb{R}_{>0}} \otimes \pi_0 x^{(0)}\big)(s)|\big(\pi_0 x - \chi_{\mathbb{R}_{>0}} \otimes \pi_0 x^{(0)}\big)(s)\big\rangle\Big)'(t) \\ = -\mathfrak{Re}\big\langle \partial_0\big(\mu_{00}\big(\pi_0 x - \chi_{\mathbb{R}_{>0}} \otimes \pi_0 x^{(0)}\big)\big)(t)|\chi_{\mathbb{R}_{>0}} \otimes \pi_0 x^{(0)}(t)\big\rangle \\ - \big\langle \mu_{11} R(u-y)(t)|R(u-y)(t)\big\rangle + \mathfrak{Re}\big\langle 2\mu_{11} Ru(t)|R(u-y)(t)\big\rangle \\ = -\mathfrak{Re}\big\langle \partial_0\big(\mu_{00}\big(\pi_0 x - \chi_{\mathbb{R}_{>0}} \otimes \pi_0 x^{(0)}\big)\big)(t)|\chi_{\mathbb{R}_{>0}} \otimes \pi_0 x^{(0)}(t)\big\rangle \\ - \big\langle \mu_{11} Ry(t)|Ry(t)\big\rangle + \big\langle \mu_{11} Ru(t)|Ru(t)\big\rangle.$$

We integrate the latter equation over $]0,T[$. We conclude that

$$\frac{1}{2}\big\langle \mu_{00}\big(\pi_0 x - \chi_{\mathbb{R}_{>0}} \otimes \pi_0 x^{(0)}\big)(T)|\big(\pi_0 x - \chi_{\mathbb{R}_{>0}} \otimes \pi_0 x^{(0)}\big)(T)\big\rangle \\ = -\mathfrak{Re}\big\langle \mu_{00}\big(\pi_0 x - \chi_{\mathbb{R}_{>0}} \otimes \pi_0 x^{(0)}\big)(T)|\pi_0 x^{(0)}\big\rangle \\ - \int_0^T \big\langle \mu_{11} Ry(t)|Ry(t)\big\rangle dt + \int_0^T \big\langle \mu_{11} Ru(t)|Ru(t)\big\rangle dt.$$

Thus, we get that

$$\frac{1}{2}\big\langle \mu_{00}\pi_0 x(T)|\pi_0 x(T)\big\rangle + \int_0^T \big\langle \mu_{11} Ry(t)|Ry(t)\big\rangle dt \\ = \frac{1}{2}\big\langle \mu_{00}\pi_0 x^{(0)}|\pi_0 x^{(0)}\big\rangle + \int_0^T \big\langle \mu_{11} Ru(t)|Ru(t)\big\rangle dt.$$

This shows the conservativity of $\mathcal{C}$. □

Example 1 The heat equation yields a conservative, linear control system. With $G : D(G) \subseteq H_0 \to H_1$ closed and densely defined we consider the heat equation in the abstract form

$$\big(\partial_0 + G^* G\big)\theta = -G^* u,$$

which is equivalent to

$$\left(\partial_0 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & G^* \\ -G & 0 \end{pmatrix}\right) \begin{pmatrix} \theta \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ u \end{pmatrix}.$$

We have $X = H_0 \oplus H_1$ and $Y = U = H_1$. Following the above construction we use

$$2q + y = u$$

with $R = \frac{1}{2}$ as observation equation. So, we get

$$\begin{pmatrix} \left(\partial_0 \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 & G^* \\ -G & 0 \end{smallmatrix}\right)\right) & \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right) \\ (0\ 2) & 1 \end{pmatrix} \begin{pmatrix} \left(\begin{smallmatrix} \theta \\ q \end{smallmatrix}\right) \\ y \end{pmatrix} = \begin{pmatrix} \left(\begin{smallmatrix} 0 \\ u \end{smallmatrix}\right) \\ u \end{pmatrix}.$$

For $\alpha \neq 0$ we have equivalently

$$\begin{pmatrix} \left(\partial_0 \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 & G^* \\ -G & 0 \end{smallmatrix}\right)\right) & \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right) \\ (0\ 2\alpha) & \alpha \end{pmatrix} \begin{pmatrix} \left(\begin{smallmatrix} \theta \\ q \end{smallmatrix}\right) \\ y \end{pmatrix} = \begin{pmatrix} \left(\begin{smallmatrix} 0 \\ u \end{smallmatrix}\right) \\ \alpha u \end{pmatrix},$$

where we choose α suitably to make

$$\mathfrak{Re} \begin{pmatrix} 1 & 0 \\ 2\alpha & \alpha \end{pmatrix} = \begin{pmatrix} 1 & \alpha \\ \alpha & \alpha \end{pmatrix}$$

strictly positive on $H_1 \oplus H_1$. This is the case if

$$0 < \alpha < 1.$$

This makes the example system

$$\begin{aligned} &\begin{pmatrix} \left(\partial_0 \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 & G^* \\ -G & 0 \end{smallmatrix}\right)\right) & \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right) \\ (0\ 1) & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \left(\begin{smallmatrix} \theta \\ q \end{smallmatrix}\right) \\ y \end{pmatrix} \\ &= \begin{pmatrix} \left(\begin{smallmatrix} 0 \\ u \end{smallmatrix}\right) \\ \frac{1}{2} u \end{pmatrix} + \delta \otimes \begin{pmatrix} \left(\begin{smallmatrix} \theta^{(0)} \\ 0 \end{smallmatrix}\right) \\ 0 \end{pmatrix} \end{aligned} \tag{8}$$

a well-posed and at least formally conservative system. It remains to establish the required regularity. To this end put $u = 0$ and let $\theta^{(0)} \in D(G^* G)$ in (8). We compute

$$\begin{aligned}\theta &= \left(\partial_0 + G^*G\right)^{-1}\delta\otimes\theta^{(0)}\\ \theta-\chi_{\mathbb{R}_{>0}}\otimes\theta^{(0)} &= \left(\left(\partial_0 + G^*G\right)^{-1}\delta\otimes\theta^{(0)}-\chi_{\mathbb{R}_{>0}}\otimes\theta^{(0)}\right)\\ &= -\left(\partial_0 + G^*G\right)^{-1}\left(\chi_{\mathbb{R}_{>0}}\otimes G^*G\theta^{(0)}\right).\end{aligned}$$

This shows that the system is globally regularizing. Indeed, for $\phi := \chi_{\mathbb{R}_{>0}} \otimes G^*G\theta^{(0)}$ we estimate

$$\begin{aligned}\left|\left(\partial_0 + G^*G\right)^{-1}\phi\right|^2_{\varrho,1,0} &= \left|\partial_0\left(\partial_0 + G^*G\right)^{-1}\phi\right|^2_{\varrho,0,0}\\ &= \left|\partial_0\left(\partial_0 + |G|^2\right)^{-1}\phi\right|^2_{\varrho,0,0}\\ &= \left|\phi - |G|^2\left(\partial_0 + |G|^2\right)^{-1}\phi\right|^2_{\varrho,0,0}\\ &\leq 2|\phi|^2_{\varrho,0,0}.\end{aligned}$$

Thus, $\theta - \chi_{\mathbb{R}_{>0}} \otimes \theta^{(0)} \in H_{\varrho,1}(\mathbb{R}, H_0)$.

12.5 The Tucsnak-Weiss System

12.5.1 A First Order Formulation

Tucsnak and Weiss suggested the following particular system class, [15], describing a class of linear wave phenomena. In this reference, it is assumed that $H := X = F$, $Y = U$, $E = 1$ and $D = 1$. Let $A_0 : D(A_0) \subseteq H \to H$ be a selfadjoint positive operator. The observation operator C is an unbounded, closed linear operator

$$C : H_1(\sqrt{A_0} + \mathrm{i}) \subseteq H_0(\sqrt{A_0} + \mathrm{i}) \to U.$$

Then

$$\begin{aligned}C_0 : H_1(\sqrt{A_0} + \mathrm{i}) &\to U\\ x &\mapsto Cx\end{aligned}$$

is a continuous linear operator, according to the Closed Graph Theorem. The control operator B is now given as the dual operator $C_0^\diamond$ of C_0, where U and U^* as well as $H_1(\sqrt{A_0} + \mathrm{i})^*$ and $H_{-1}(\sqrt{A_0} + \mathrm{i})$ are identified so that we have $C_0^\diamond : U \to H_{-1}(\sqrt{A_0} + \mathrm{i})$. It is

$$C^* \subseteq C_0^\diamond.$$

The system considered in [15] is formally

$$\partial_0^2 z + A_0 z + \frac{1}{2} C_0^\diamond C_0 \partial_0 z = C_0^\diamond u,$$
$$C\partial_0 z + y = u$$

(C observation operator, $C_0^\diamond$ control operator) on $\mathbb{R}_{>0}$ for a given function $u \in H_{\varrho,0}(\mathbb{R}, U)$. We shall instead consider the first order system

$$\left(\partial_0 \begin{pmatrix} 1 & (0\,0) & 0 \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ 0 & (0\,0) & 0 \end{pmatrix} + \begin{pmatrix} 0 & (0\,0) & 0 \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ 0 & (0\,\sqrt{2}) & 1 \end{pmatrix}\right.$$
$$\left.+ \begin{pmatrix} 0 & \mathbb{DIV} & 0 \\ \mathbb{GRAD} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ 0 & (0\,0) & 0 \end{pmatrix}\right) \begin{pmatrix} v \\ \begin{pmatrix} \zeta \\ w \end{pmatrix} \\ y \end{pmatrix}$$
$$= \begin{pmatrix} 0 \\ \begin{pmatrix} 0 \\ -\sqrt{2}u \end{pmatrix} \\ -u \end{pmatrix} + \delta \otimes \begin{pmatrix} z^{(1)} \\ \begin{pmatrix} \sqrt{A_0} z^{(0)} \\ 0 \end{pmatrix} \\ 0 \end{pmatrix} \tag{9}$$

with

$$\mathbb{GRAD} := \begin{pmatrix} -\sqrt{A_0} \\ -\frac{1}{\sqrt{2}} C \end{pmatrix} : H_1(\sqrt{A_0} + \mathrm{i}) \subseteq H_0(\sqrt{A_0} + \mathrm{i}) \to H_0(\sqrt{A_0} + \mathrm{i}) \oplus U$$

and $\mathbb{DIV} := -(\mathbb{GRAD})^*$. Thus the whole systems acts in the space

$$H_{\varrho,0}\big(\mathbb{R}, H_0(\sqrt{A_0} + \mathrm{i}) \oplus \big(H_0(\sqrt{A_0} + \mathrm{i}) \oplus U\big) \oplus U\big).$$

Here

$$z^{(0)} \in H_1(\sqrt{A_0} + \mathrm{i}), \qquad z^{(1)} \in H_0(\sqrt{A_0} + \mathrm{i})$$

are the implementation of the initial data. Our first observation is that this system is a linear control system in a simple case:

- $$\mathcal{A} := \begin{pmatrix} 0 & \mathbb{DIV} & 0 \\ \mathbb{GRAD} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ 0 & (0\,0) & 0 \end{pmatrix}$$
  is skew-selfadjoint,
- M_0 is the orthogonal projector onto $H_0(\sqrt{A_0} + \mathrm{i}) \oplus (H_0(\sqrt{A_0} + \mathrm{i}) \oplus \{0\}) \oplus \{0\}$,
- $$\mathfrak{Re}\, M_1 = \begin{pmatrix} 0 & (0\,0) & 0 \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} \\ 0 & (0\,\frac{1}{\sqrt{2}}) & 1 \end{pmatrix}$$
  is strictly positive on the null space $\{0\} \oplus (\{0\} \oplus U) \oplus U$ of M_0.

Thus, well-posedness in the above sense is clear. We will show that this system is the appropriate interpretation of the original system. As a first step we compute the adjoint of $\mathbb{GRAD}$ explicitly.

Lemma 2 *Assume* $0 \in \varrho(A_0)$. *Then*

$$\mathbb{DIV} \subseteq \begin{pmatrix} \sqrt{A_0} & \frac{1}{\sqrt{2}} C_0^{\diamond} \end{pmatrix} : H_0(\sqrt{A_0} + \mathrm{i}) \oplus U \to H_{-1}(\sqrt{A_0} + \mathrm{i})$$
$$\begin{pmatrix} \zeta \\ w \end{pmatrix} \mapsto \sqrt{A_0}\zeta + \frac{1}{\sqrt{2}} C_0^{\diamond} w$$

and

$$D(\mathbb{DIV}) = \left\{ \begin{pmatrix} \zeta \\ w \end{pmatrix} \in H_0(\sqrt{A_0} + \mathrm{i}) \oplus U \,\middle|\, \begin{pmatrix} \sqrt{A_0} & \frac{1}{\sqrt{2}} C_0^{\diamond} \end{pmatrix} \begin{pmatrix} \zeta \\ w \end{pmatrix} \in H_0(\sqrt{A_0} + \mathrm{i}) \right\}.$$

Proof We consider

$$\widetilde{\mathbb{DIV}} : D(\widetilde{\mathbb{DIV}}) \subseteq H_0(\sqrt{A_0} + \mathrm{i}) \oplus U \to H_0(\sqrt{A_0} + \mathrm{i})$$
$$\begin{pmatrix} \zeta \\ w \end{pmatrix} \mapsto \begin{pmatrix} \sqrt{A_0} & \frac{1}{\sqrt{2}} C_0^{\diamond} \end{pmatrix} \begin{pmatrix} \zeta \\ w \end{pmatrix}$$

with $D(\widetilde{\mathbb{DIV}})$ being the set

$$\left\{ \begin{pmatrix} \zeta \\ w \end{pmatrix} \in H_0(\sqrt{A_0} + \mathrm{i}) \oplus U \,\middle|\, \begin{pmatrix} \sqrt{A_0} & \frac{1}{\sqrt{2}} C_0^{\diamond} \end{pmatrix} \begin{pmatrix} \zeta \\ w \end{pmatrix} \in H_0(\sqrt{A_0} + \mathrm{i}) \right\}.$$

We want to show that

$$\widetilde{\mathbb{DIV}} = \mathbb{DIV}$$

and we shall do so by showing that

$$\widetilde{\mathbb{DIV}}^* = -\mathbb{GRAD}.$$

Clearly,

$$\begin{pmatrix} \sqrt{A_0} & \frac{1}{\sqrt{2}} C^* \end{pmatrix} : H_1(\sqrt{A_0} + \mathrm{i}) \oplus D\big(C^*\big) \subseteq H_0(\sqrt{A_0} + \mathrm{i}) \oplus U \to H_0(\sqrt{A_0} + \mathrm{i})$$
$$\subseteq \widetilde{\mathbb{DIV}} \subseteq \begin{pmatrix} \sqrt{A_0} & \frac{1}{\sqrt{2}} C_0^{\diamond} \end{pmatrix} : H_0(\sqrt{A_0} + \mathrm{i}) \oplus U \to H_{-1}(\sqrt{A_0} + \mathrm{i})$$

and hence $\widetilde{\mathbb{DIV}}$ is densely defined. So let $v \in D(\widetilde{\mathbb{DIV}}^*)$. Then for some $\begin{pmatrix} f \\ g \end{pmatrix} \in H_0(\sqrt{A_0} + \mathrm{i}) \oplus U$ we have

$$\bigwedge_{\begin{pmatrix} \zeta \\ w \end{pmatrix} \in D(\widetilde{\mathbb{DIV}})} \left\langle \widetilde{\mathbb{DIV}} \begin{pmatrix} \zeta \\ w \end{pmatrix} \middle| v \right\rangle_{H_0(\sqrt{A_0}+\mathrm{i})} = \left\langle \begin{pmatrix} \zeta \\ w \end{pmatrix} \middle| \begin{pmatrix} f \\ g \end{pmatrix} \right\rangle_{H_0(\sqrt{A_0}+\mathrm{i}) \oplus U}.$$

It follows by testing with elements in $H_1(\sqrt{A_0}+\mathrm{i})\oplus\{0\}\subseteq D(\widetilde{\mathbb{DIV}})$ that

$$\bigwedge_{\zeta\in H_1(\sqrt{A_0}+\mathrm{i})}\langle\sqrt{A_0}\zeta\,|\,v\rangle_{H_0(\sqrt{A_0}+\mathrm{i})}=\langle\zeta\,|\,f\rangle_{H_0(\sqrt{A_0}+\mathrm{i})},$$

which implies

$$v\in D(\sqrt{A_0})$$

and

$$\sqrt{A_0}v=f.$$

Let now $w\in U$ be arbitrary. Then with $\zeta=-\frac{1}{\sqrt{2}}\sqrt{A_0}^{-1}C_0^{\diamond}w$ we get[2]

$$\begin{aligned}0&=\left\langle\widetilde{\mathbb{DIV}}\begin{pmatrix}\zeta\\ w\end{pmatrix}\middle|v\right\rangle_{H_0(\sqrt{A_0}+\mathrm{i})}\\&=\left\langle\begin{pmatrix}-\frac{1}{\sqrt{2}}\sqrt{A_0}^{-1}C_0^{\diamond}w\\ w\end{pmatrix}\middle|\begin{pmatrix}\sqrt{A_0}v\\ g\end{pmatrix}\right\rangle_{H_0(\sqrt{A_0}+\mathrm{i})\oplus U}\\&=\left\langle-\frac{1}{\sqrt{2}}\sqrt{A_0}^{-1}C_0^{\diamond}w\middle|\sqrt{A_0}v\right\rangle_{H_0(\sqrt{A_0}+\mathrm{i})}+\langle w|g\rangle_U\\&=\left\langle-\frac{1}{\sqrt{2}}C_0^{\diamond}w\middle|v\right\rangle_{H_0(\sqrt{A_0}+\mathrm{i})}+\langle w|g\rangle_U\\&=\left\langle-\frac{1}{\sqrt{2}}w\middle|C_0v\right\rangle_U+\langle w|g\rangle_U.\end{aligned}$$

This implies

$$\frac{1}{\sqrt{2}}Cv=g$$

and thus, we have

$$\widetilde{\mathbb{DIV}}^*v=\begin{pmatrix}\sqrt{A_0}v\\ \frac{1}{\sqrt{2}}Cv\end{pmatrix}=-\mathbb{GRAD}v,$$

i.e.

$$\widetilde{\mathbb{DIV}}^*\subseteq-\mathbb{GRAD}.$$

[2]Note that in the fourth equality $\langle\cdot|\cdot\rangle_{H_0(\sqrt{A_0}+\mathrm{i})}$ is used not as the inner product in $H_0(\sqrt{A_0}+\mathrm{i})$ but as its continuous extension to the duality pairing between $H_{-1}(\sqrt{A_0}+\mathrm{i})$ and $H_1(\sqrt{A_0}+\mathrm{i})$. This will be utilized throughout without explicit mention.

Moreover, let now $v \in D(\mathbb{GRAD})$. Then for all $\left(\begin{smallmatrix}\zeta\\ w\end{smallmatrix}\right) \in D(\widetilde{\mathbb{DIV}})$

$$\begin{aligned}
&\left\langle \widetilde{\mathbb{DIV}} \begin{pmatrix}\zeta\\ w\end{pmatrix} \middle| v\right\rangle_{H_0(\sqrt{A_0}+\mathrm{i})} + \left\langle \begin{pmatrix}\zeta\\ w\end{pmatrix} \middle| \mathbb{GRAD} v\right\rangle_{H_0(\sqrt{A_0}+\mathrm{i})\oplus U} \\
&= \left\langle \sqrt{A_0}\zeta + \frac{1}{\sqrt{2}} C_0^\diamond w \middle| v\right\rangle_{H_0(\sqrt{A_0}+\mathrm{i})} - \left\langle \begin{pmatrix}\zeta\\ w\end{pmatrix} \middle| \begin{pmatrix}\sqrt{A_0}v\\ \frac{1}{\sqrt{2}}Cv\end{pmatrix}\right\rangle_{H_0(\sqrt{A_0}+\mathrm{i})\oplus U} \\
&= \left\langle \sqrt{A_0}\zeta + \frac{1}{\sqrt{2}} C_0^\diamond w \middle| v\right\rangle_{H_0(\sqrt{A_0}+\mathrm{i})} \\
&\quad - \left\langle \frac{1}{\sqrt{2}} C_0^\diamond w \middle| v\right\rangle_{H_0(\sqrt{A_0}+\mathrm{i})} - \langle \zeta | \sqrt{A_0} v\rangle_{H_0(\sqrt{A_0}+\mathrm{i})} \\
&= \left\langle \sqrt{A_0}\zeta + \frac{1}{\sqrt{2}} C_0^\diamond w \middle| v\right\rangle_{H_0(\sqrt{A_0}+\mathrm{i})} - \left\langle \sqrt{A_0}\zeta + \frac{1}{\sqrt{2}} C_0^\diamond w \middle| v\right\rangle_{H_0(\sqrt{A_0}+\mathrm{i})} = 0,
\end{aligned}$$

from which we see that

$$-\mathbb{GRAD} \subseteq \widetilde{\mathbb{DIV}}^*.$$

Thus, we have shown that

$$\widetilde{\mathbb{DIV}} = -\mathbb{GRAD}^* = \mathbb{DIV}. \qquad \square$$

Noting that the solution

$$\begin{pmatrix} v \\ \left(\begin{smallmatrix}\zeta\\ w\end{smallmatrix}\right) \\ y \end{pmatrix}$$

of (9) is in $H_{\varrho,-1}(\mathbb{R},\mathcal{H}_0) \cap H_{\varrho,-2}(\mathbb{R},\mathcal{H}_1)$, by the results of Sect. 12.3, we can read (9) line by line under the assumption that $0 \in \varrho(A_0)$ and we obtain

$$\begin{aligned}
\partial_0 v + \sqrt{A_0}\zeta + \frac{1}{\sqrt{2}} C_0^\diamond w &= \delta \otimes z^{(1)} \\
\partial_0 \zeta - \sqrt{A_0} v &= \delta \otimes \sqrt{A_0} z^{(0)} \\
w - \frac{1}{\sqrt{2}} C v &= -\sqrt{2} u \\
\sqrt{2} w + y &= -u.
\end{aligned}$$

Since $v, \zeta \in H_{\varrho,-1}(\mathbb{R}, H_0(\sqrt{A_0}+\mathrm{i}))$ and $y, w \in H_{\varrho,-1}(\mathbb{R}, U)$, we see that the first equation holds in $H_{\varrho,-2}(\mathbb{R}, H_{-1}(\sqrt{A_0}+\mathrm{i}))$. Since also $v \in H_{\varrho,-2}(\mathbb{R}, H_1(\sqrt{A_0}+\mathrm{i}))$ and $z^{(0)} \in H_1(\sqrt{A_0}+\mathrm{i})$, the second equation holds in $H_{\varrho,-2}(\mathbb{R}, H_0(\sqrt{A_0}+\mathrm{i}))$ and the third one in $H_{\varrho,-2}(\mathbb{R},U)$. If we let $z := \sqrt{A_0}^{-1}\zeta \in H_{\varrho,-1}(\mathbb{R}, H_1(\sqrt{A_0}+\mathrm{i})) \cap H_{\varrho,0}(\mathbb{R}, H_{-1}(\sqrt{A_0}+\mathrm{i}))$ then $\partial_0 z = v + \delta\otimes z^{(0)}$ and

$$\partial_0^2 z + A_0 z + \frac{1}{\sqrt{2}}C_0^{\diamond} w = \delta\otimes z^{(1)} + \partial_0\delta\otimes z^{(0)}$$
$$w - \frac{1}{\sqrt{2}}Cv = -\sqrt{2}u$$
$$\sqrt{2}w + y = -u.$$

Thus, eliminating w we get

$$\partial_0^2 z + A_0 z + \frac{1}{2}C_0^{\diamond}\big(C\big(\partial_0 z - \delta\otimes z^{(0)}\big) - 2u\big) = \delta\otimes z^{(1)} + \partial_0\delta\otimes z^{(0)}$$
$$y = u - C\big(\partial_0 z - \delta\otimes z^{(0)}\big),$$

or

$$\partial_0^2 z + A_0 z + \frac{1}{2}C_0^{\diamond}C\partial_0 z = C_0^{\diamond}u + \delta\otimes z^{(1)} + \partial_0\delta\otimes z^{(0)} + \frac{1}{2}\delta\otimes C_0^{\diamond}Cz^{(0)}$$
$$y = u - C\partial_0 z + \delta\otimes Cz^{(0)},$$

which is on $\mathbb{R}_{>0}$ formally the equation we started out with. Here the first equality holds in $H_{\varrho,-2}(\mathbb{R}, H_{-1}(\sqrt{A_0}+\mathrm{i}))$ and the second one in $H_{\varrho,-2}(\mathbb{R},U)$.

12.5.2 The Tucsnak-Weiss System as a Conservative Linear Control System

In this section we want to prove that the system considered in the previous part is conservative as it was formulated in Theorem 2 under appropriate assumptions on the initial values $z^{(0)}, z^{(1)}$. In order to formulate pointwise evaluations of the solution, we have to inspect regularity properties for the system. Since the regularization property does not depend on u we may set $u = 0$. By assuming $0 \in \varrho(A_0)$ we arrive at the equations

$$\partial_0 v + \sqrt{A_0}\zeta + \frac{1}{\sqrt{2}}C_0^{\diamond}w = \delta\otimes z^{(1)}$$
$$\partial_0\zeta - \sqrt{A_0}v = \delta\otimes\sqrt{A_0}z^{(0)}$$
$$w - \frac{1}{\sqrt{2}}Cv = 0$$
$$\sqrt{2}w + y = 0$$

and re-assemble them in a different way. As was already pointed out, the first equation holds in the space $H_{\varrho,-2}(\mathbb{R}, H_{-1}(\sqrt{A_0} + \mathrm{i}))$ and the second one in $H_{\varrho,-2}(\mathbb{R}, H_0(\sqrt{A_0} + \mathrm{i}))$, while both the third and fourth one hold in $H_{\varrho,-2}(\mathbb{R}, U)$. Using the third equation to eliminate w in the first one, we get the following system

$$\partial_0 v + \sqrt{A_0}\zeta + \frac{1}{2}C_0^\diamond C_0 v = \delta \otimes z^{(1)},$$
$$\partial_0 \zeta - \sqrt{A_0} v = \delta \otimes \sqrt{A_0} z^{(0)}.$$

Rewriting this in an operator-matrix form we get

$$\partial_0 \begin{pmatrix} \zeta \\ v \end{pmatrix} + \begin{pmatrix} 0 & -\sqrt{A_0} \\ \sqrt{A_0} & \frac{1}{2}C_0^\diamond C_0 \end{pmatrix} \begin{pmatrix} \zeta \\ v \end{pmatrix} = \delta \otimes \begin{pmatrix} \sqrt{A_0} z^{(0)} \\ z^{(1)} \end{pmatrix} \tag{10}$$

as an equation in $H_{\varrho,-2}(\mathbb{R}, H_0(\sqrt{A_0}+\mathrm{i}) \oplus H_{-1}(\sqrt{A_0}+\mathrm{i}))$. We define the following linear operator

$$A : D(A) \subseteq H_0(\sqrt{A_0} + \mathrm{i})^2 \to H_0(\sqrt{A_0} + \mathrm{i})^2,$$

where the domain of A, $D(A)$, is the set

$$\left\{ (\zeta, v) \in H_0(\sqrt{A_0} + \mathrm{i})^2 \middle| v \in H_1(\sqrt{A_0} + \mathrm{i}), \sqrt{A_0}\zeta + \frac{1}{2}C_0^\diamond C_0 v \in H_0(\sqrt{A_0} + \mathrm{i}) \right\}$$

and

$$A \begin{pmatrix} \zeta \\ v \end{pmatrix} := \begin{pmatrix} 0 & -\sqrt{A_0} \\ \sqrt{A_0} & \frac{1}{2}C_0^\diamond C_0 \end{pmatrix} \begin{pmatrix} \zeta \\ v \end{pmatrix}.$$

The density of the domain of A follows by arguing analogously to the proof of Lemma 2.

Lemma 3 *The operator A is closed and continuously invertible. Furthermore the following holds*

$$\mathfrak{Re}\left\langle \begin{pmatrix} \zeta \\ v \end{pmatrix} \middle| A \begin{pmatrix} \zeta \\ v \end{pmatrix} \right\rangle_{H_0(A)} = \frac{1}{2}\left\langle \begin{pmatrix} 0 & C_0 \end{pmatrix} \begin{pmatrix} \zeta \\ v \end{pmatrix} \middle| \begin{pmatrix} 0 & C_0 \end{pmatrix} \begin{pmatrix} \zeta \\ v \end{pmatrix} \right\rangle_U \geq 0$$

and

$$\mathfrak{Re}\left\langle \begin{pmatrix} r \\ s \end{pmatrix} \middle| A^* \begin{pmatrix} r \\ s \end{pmatrix} \right\rangle_{H_0(A)} = \frac{1}{2}\left\langle \begin{pmatrix} 0 & C_0 \end{pmatrix} \begin{pmatrix} r \\ s \end{pmatrix} \middle| \begin{pmatrix} 0 & C_0 \end{pmatrix} \begin{pmatrix} r \\ s \end{pmatrix} \right\rangle_U \geq 0$$

for all $\begin{pmatrix} \zeta \\ v \end{pmatrix} \in D(A)$, $\begin{pmatrix} r \\ s \end{pmatrix} \in D(A^*)$.

Proof The operator A is a restriction of the bounded linear operator

$$\begin{pmatrix} 0 & -\sqrt{A_0} \\ \sqrt{A_0} & \frac{1}{2}C_0^\diamond C_0 \end{pmatrix} : H_0(\sqrt{A_0}+\mathrm{i}) \oplus H_1(\sqrt{A_0}+\mathrm{i}) \to H_0(\sqrt{A_0}+\mathrm{i}) \oplus H_{-1}(\sqrt{A_0}+\mathrm{i}).$$

An easy computation shows that its inverse is given by

$$\begin{pmatrix} \frac{1}{2}A_0^{-1/2}C_0^\diamond C_0 A_0^{-1/2} & A_0^{-1/2} \\ -A_0^{-1/2} & 0 \end{pmatrix} : H_0(\sqrt{A_0}+\mathrm{i}) \oplus H_{-1}(\sqrt{A_0}+\mathrm{i}) \to H_0(\sqrt{A_0}+\mathrm{i}) \oplus H_1(\sqrt{A_0}+\mathrm{i}),$$

which is again bounded. If we consider the restriction

$$\begin{aligned} H_0(\sqrt{A_0}+\mathrm{i})^2 &\to H_0(\sqrt{A_0}+\mathrm{i})^2 \\ \begin{pmatrix} r \\ s \end{pmatrix} &\mapsto \begin{pmatrix} \frac{1}{2}A_0^{-1/2}C_0^\diamond C_0 A_0^{-1/2} & A_0^{-1/2} \\ -A_0^{-1/2} & 0 \end{pmatrix}\begin{pmatrix} r \\ s \end{pmatrix}, \end{aligned}$$

we again obtain a bounded linear operator, whose range is a subset of $D(A)$. Hence it is the inverse of A and thus A^{-1} is a bounded linear operator, which shows that A is closed with $0 \in \varrho(A)$. For $z, v \in H_0(\sqrt{A_0}+\mathrm{i})$ we compute

$$\begin{aligned} \langle A_0^{-1/2}C_0^\diamond C_0 A_0^{-1/2} z | v\rangle_{H_0(\sqrt{A_0}+\mathrm{i})} &= \langle C_0^\diamond C_0 A_0^{-1/2} z | A_0^{-1/2} v\rangle_{H_0(\sqrt{A_0}+\mathrm{i})} \\ &= \langle C_0 A_0^{-1/2} z | C_0 A_0^{-1/2} v\rangle_U \\ &= \langle A_0^{-1/2} z | C_0^\diamond C_0 A_0^{-1/2} v\rangle_{H_0(\sqrt{A_0}+\mathrm{i})} \\ &= \langle z | A_0^{-1/2} C_0^\diamond C_0 A_0^{-1/2} v\rangle_{H_0(\sqrt{A_0}+\mathrm{i})}, \end{aligned}$$

proving that $A_0^{-1/2}C_0^\diamond C_0 A_0^{-1/2}$ is self-adjoint. Thus, we obtain

$$\begin{aligned} (A^*)^{-1} &= (A^{-1})^* \\ &= \begin{pmatrix} \frac{1}{2}A_0^{-1/2}C_0^\diamond C_0 A_0^{-1/2} & -A_0^{-1/2} \\ A_0^{-1/2} & 0 \end{pmatrix} : H_0(\sqrt{A_0}+\mathrm{i})^2 \\ &\to H_0(\sqrt{A_0}+\mathrm{i})^2 \end{aligned}$$

and so the operator $(A^*)^{-1}$ is a restriction of the operator

$$\begin{pmatrix} \frac{1}{2}A_0^{-1/2}C_0^\diamond C_0 A_0^{-1/2} & -A_0^{-1/2} \\ A_0^{-1/2} & 0 \end{pmatrix} : H_0(\sqrt{A_0}+\mathrm{i}) \oplus H_{-1}(\sqrt{A_0}+\mathrm{i}) \to H_0(\sqrt{A_0}+\mathrm{i}) \oplus H_1(\sqrt{A_0}+\mathrm{i}).$$

Using this, we get that

$$
\begin{aligned}
A^* &\subseteq \begin{pmatrix} \frac{1}{2} A_0^{-1/2} C_0^\diamond C_0 A_0^{-1/2} & -A_0^{-1/2} \\ A_0^{-1/2} & 0 \end{pmatrix}^{-1} \\
&= \begin{pmatrix} 0 & A_0^{-1/2} \\ -A_0^{-1/2} & \frac{1}{2} C_0^\diamond C_0 \end{pmatrix} : H_0(\sqrt{A_0} + \mathrm{i}) \oplus H_1(\sqrt{A_0} + \mathrm{i}) \\
&\to H_0(\sqrt{A_0} + \mathrm{i}) \oplus H_{-1}(\sqrt{A_0} + \mathrm{i}).
\end{aligned}
$$

Now we are able to show the two asserted equalities. For $(\zeta, v) \in D(A)$ we have

$$
\begin{aligned}
&\mathfrak{Re}\left\langle \begin{pmatrix} \zeta \\ v \end{pmatrix} \middle| A \begin{pmatrix} \zeta \\ v \end{pmatrix} \right\rangle_{H_0(\sqrt{A_0}+\mathrm{i})^2} \\
&= \mathfrak{Re}\left\langle \begin{pmatrix} \zeta \\ v \end{pmatrix} \middle| \begin{pmatrix} 0 & -\sqrt{A_0} \\ \sqrt{A_0} & \frac{1}{2} C_0^\diamond C_0 \end{pmatrix} \begin{pmatrix} \zeta \\ v \end{pmatrix} \right\rangle_{H_0(\sqrt{A_0}+\mathrm{i})^2} \\
&= \mathfrak{Re}\left\langle v \middle| \sqrt{A_0}\zeta + \frac{1}{2}\left(C_0^\diamond C_0\right) v \right\rangle_{H_0(\sqrt{A_0}+\mathrm{i})} + \mathfrak{Re}\langle \zeta | \sqrt{A_0} v \rangle_{H_0(\sqrt{A_0}+\mathrm{i})} \\
&= \mathfrak{Re}\left\langle v \middle| \frac{1}{2}\left(C_0^\diamond C_0\right) v \right\rangle_{H_0(\sqrt{A_0}+\mathrm{i})} \\
&= \frac{1}{2} \mathfrak{Re}\langle C_0 v | C_0 v \rangle_U \\
&= \frac{1}{2} \left\langle \begin{pmatrix} 0 & C_0 \end{pmatrix} \begin{pmatrix} \zeta \\ v \end{pmatrix} \middle| \begin{pmatrix} 0 & C_0 \end{pmatrix} \begin{pmatrix} \zeta \\ v \end{pmatrix} \right\rangle_U .
\end{aligned}
$$

Analogously we get for $(r, s) \in D(A^*)$

$$
\begin{aligned}
&\mathfrak{Re}\left\langle \begin{pmatrix} r \\ s \end{pmatrix} \middle| A^* \begin{pmatrix} r \\ s \end{pmatrix} \right\rangle_{H_0(\sqrt{A_0}+\mathrm{i})^2} \\
&= \mathfrak{Re}\left\langle \begin{pmatrix} r \\ s \end{pmatrix} \middle| \begin{pmatrix} 0 & \sqrt{A_0} \\ -\sqrt{A_0} & \frac{1}{2} C_0^\diamond C_0 \end{pmatrix} \begin{pmatrix} r \\ s \end{pmatrix} \right\rangle_{H_0(\sqrt{A_0}+\mathrm{i})^2} \\
&= \mathfrak{Re}\langle r | \sqrt{A_0} s \rangle_{H_0(\sqrt{A_0}+\mathrm{i})} - \mathfrak{Re}\left\langle s \middle| \sqrt{A_0} r - \frac{1}{2}\left(C_0^\diamond C_0\right) s \right\rangle_{H_0(\sqrt{A_0}+\mathrm{i})} \\
&= \mathfrak{Re}\left\langle s \middle| \frac{1}{2}\left(C_0^\diamond C_0\right) s \right\rangle_{H_0(\sqrt{A_0}+\mathrm{i})} \\
&= \frac{1}{2} \mathfrak{Re}\langle C_0 s | C_0 s \rangle_U \\
&= \frac{1}{2} \left\langle \begin{pmatrix} 0 & C_0 \end{pmatrix} \begin{pmatrix} r \\ s \end{pmatrix} \middle| \begin{pmatrix} 0 & C_0 \end{pmatrix} \begin{pmatrix} r \\ s \end{pmatrix} \right\rangle_U .
\end{aligned}
$$

□

Remark 3

1. Lemma 3 especially implies that A and A^* are monotone or accretive operators. Hence $-A$ is a generator of a contraction semigroup. Furthermore $(\partial_0 + A)^{-1}$ and $(\partial_0^* + A^*)^{-1}$ are bounded linear operators on $H_{\varrho,0}(\mathbb{R}, H_0(A))$ and can be extended to bounded operators on the associated spaces $H_{\varrho,k}(\mathbb{R}, H_s(A))$ and $H_{\varrho,k}(\mathbb{R}, H_s(A^*))$ respectively, where $k, s \in \mathbb{Z}$.
2. From the equalities we also read off that

$$\begin{pmatrix} 0 & C_0 \end{pmatrix} (\partial_0 + A)^{-1} : H_{\varrho,1}\big(\mathbb{R}, H_1(A)\big) \subseteq H_{\varrho,0}\big(\mathbb{R}, H_0(A)\big) \to H_{\varrho,0}(\mathbb{R}, U)$$

is continuous, since for $u \in H_{\varrho,1}(\mathbb{R}, H_1(A))$ we estimate

$$\begin{aligned} \mathfrak{Re}\big\langle (\mathrm{i}t + \varrho + A)u(t)|u(t)\big\rangle_{H_0(A)} &= \varrho\big|u(t)\big|^2_{H_0(A)} + \big|\begin{pmatrix} 0 & C_0 \end{pmatrix} u(t)\big|^2_U \\ &\geq \big|\begin{pmatrix} 0 & C_0 \end{pmatrix} u(t)\big|^2_U \end{aligned}$$

for every $t \in \mathbb{R}$ and from this we derive the stated continuity. Analogously we get

$$\begin{pmatrix} 0 & C_0 \end{pmatrix} \big(\partial_0^* + A^*\big)^{-1} : H_{\varrho,1}\big(\mathbb{R}, H_1\big(A^*\big)\big) \subseteq H_{\varrho,0}\big(\mathbb{R}, H_0\big(A^*\big)\big) \to H_{\varrho,0}(\mathbb{R}, U)$$

is continuous. Thus we can extend these operators continuously to $H_{\varrho,k}(\mathbb{R}, H_0(A))$ and $H_{\varrho,k}(\mathbb{R}, H_0(A^*))$ respectively taking values in $H_{\varrho,k}(\mathbb{R}, U)$ for all $k \in \mathbb{Z}$. From this it is possible to derive the continuity of the composition operator $(\partial_0 + A)^{-1}\begin{pmatrix} 0 \\ C_0^\diamond \end{pmatrix}$ as a mapping from $H_{\varrho,k}(\mathbb{R}, U)$ to $H_{\varrho,k}(\mathbb{R}, H_0(A))$, which in the terminology of [15] means that $C_0^\diamond$ is admissible. However, in our setting this property is not needed.

Recall that our equation (10) is valid in $H_{\varrho,-2}(\mathbb{R}, H_0(\sqrt{A_0} + \mathrm{i}) \oplus H_{-1}(\sqrt{A_0} + \mathrm{i}))$. We show now that this implies the validity in $H_{\varrho,-2}(\mathbb{R}, H_{-1}(A))$.

Lemma 4 *The Sobolev-chains of $\sqrt{A_0}$ and A^* are related by*

$$H_1\big(A^*\big) \hookrightarrow H_0(\sqrt{A_0} + \mathrm{i}) \oplus H_1(\sqrt{A_0} + \mathrm{i}).$$

Proof Since

$$(A^*)^{-1} \subseteq \begin{pmatrix} \frac{1}{2} A_0^{-1/2} C_0^\diamond C_0 A_0^{-1/2} & -A_0^{-1/2} \\ A_0^{-1/2} & 0 \end{pmatrix}$$

we conclude that the inclusion $H_1(A^*) \subseteq H_0(\sqrt{A_0} + \mathrm{i}) \oplus H_1(\sqrt{A_0} \oplus \mathrm{i})$ holds. The Hilbert spaces $H_0(\sqrt{A_0} + \mathrm{i}) \oplus H_1(\sqrt{A_0} + \mathrm{i})$ and $H_1(A^*)$ are both continuously embedded in $H_0(\sqrt{A_0} + \mathrm{i}) \oplus H_0(\sqrt{A_0} + \mathrm{i}) = H_0(A^*)$ and hence the assertion follows by the Closed Graph Theorem. □

Remark 4 As a direct consequence of Lemma 4 we get

$$H_0(\sqrt{A_0}+\mathrm{i}) \oplus H_{-1}(\sqrt{A_0}+\mathrm{i}) \hookrightarrow H_{-1}(A)$$

since $H_{-1}(A)$ is unitary equivalent to the dual space $H_1(A^*)^*$.

With this we conclude that the equation

$$\partial_0 \begin{pmatrix} \zeta \\ v \end{pmatrix} + A \begin{pmatrix} \zeta \\ v \end{pmatrix} = \delta \otimes \begin{pmatrix} \sqrt{A_0} z^{(0)} \\ z^{(1)} \end{pmatrix}$$

holds in $H_{\varrho,-2}(\mathbb{R}, H_{-1}(A))$. From this we get

$$\begin{pmatrix} \zeta \\ v \end{pmatrix} - \chi_{\mathbb{R}_{>0}} \otimes \begin{pmatrix} \sqrt{A_0} z^{(0)} \\ z^{(1)} \end{pmatrix} = (\partial_0 + A)^{-1} \left(\chi_{\mathbb{R}_{>0}} \otimes A \begin{pmatrix} \sqrt{A_0} z^{(0)} \\ z^{(1)} \end{pmatrix} \right).$$

If we assume that $\begin{pmatrix} \sqrt{A_0} z^{(0)} \\ z^{(1)} \end{pmatrix} \in D(A)$, we get, since $-A$ is the generator of a C_0-semigroup, that $(\partial_0 + A)^{-1}\left(\chi_{\mathbb{R}_{>0}} \otimes A \begin{pmatrix} \sqrt{A_0} z^{(0)} \\ z^{(1)} \end{pmatrix}\right) \in H_{\varrho,1}(\mathbb{R}, H_0(A))$, by employing semigroup theory as a regularity result. This shows that the system (9) is globally regularizing with $\mathcal{U} := D(A)$. Thus Theorem 2 is applicable and we can show the conservativity of the system. We summarize our findings of this section in the following theorem.

Theorem 3 *The system* (9) *is well-posed. If* $0 \in \varrho(A_0)$ *it is globally regularizing and conservative in the sense of Theorem* 2.

Proof The well-posedness was shown in Sect. 12.5.1 and the regularity was proved above. By comparing the system (9) and the setting in Theorem 2 we see that the conservativity follows with $R = \frac{1}{\sqrt{2}}$ and $\alpha = 1$. □

12.6 Main Observations

In this note, we gave a unified approach to a large class of infinite-dimensional control systems. This perspective enabled us, assuming mild regularizing properties of the solution operator, to construct observation equations such that the respective control systems become conservative in the sense of [15]. Moreover, we studied a particular linear control system, which models wave phenomena and consists of unbounded control and observation operators. It turned out that this system may be rewritten into a form introduced in [8], such that the solution theory becomes easily accessible and unbounded control and observation need not to be treated. Surprisingly enough, the system studied in [15] corresponds to the skew-selfadjoint operator case, which might be a rather special one at first glance.

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