Chapter 9
Stability of Equilibrium Points of Nicholson's Blowflies Equation with Stochastic Perturbations

We consider the Nicholson blowflies equation (one of the most known models in ecology) with stochastic perturbations. We obtain sufficient conditions for stability in probability of the trivial and positive equilibrium points of this nonlinear differential equation with delay.

9.1 Introduction

Consider the nonlinear differential equation with exponential nonlinearity

$$\dot{x}(t) = ax(t-h)e^{-bx(t-h)} - cx(t). \tag{9.1}$$

It describes a population dynamics of the well-known Nicholson blowflies [224]. Here $x(t)$ is the size of the population at time t, a is the maximum per capita daily egg production rate, $1/b$ is the size at which the population reproduces at the maximum rate, c is the per capita daily adult death rate, and h is the generation time.

Nicholson's blowflies model is popular enough with researchers [7, 8, 30, 31, 38, 40, 61, 73, 99, 102, 129, 166, 173–176, 178, 184, 250–252, 278, 284–287, 297, 307, 312, 316, 320]. The majority of the results on (9.1) deal with the global attractiveness of the positive point of equilibrium and oscillatory behaviors of solutions [73, 102, 129, 166, 173, 174, 178, 250–252, 278, 285].

Below we will obtain sufficient conditions for stability in probability of the trivial and positive equilibrium points of (9.1) by stochastic perturbations. The basic stages of the proposed research are the following. It is assumed that the considered nonlinear differential equation has an equilibrium point and exposed to white-noise-type stochastic perturbations that are proportional to the deviation of the system current state from the considered equilibrium point. In this case the equilibrium point is a solution of the stochastic differential equation too. The constructed stochastic differential equation is centered around the considered equilibrium point and linearized in the neighborhood of this equilibrium point. Necessary and sufficient conditions for the asymptotic mean-square stability of the linear part of the considered equation are

L. Shaikhet, *Lyapunov Functionals and Stability of Stochastic Functional Differential Equations*, DOI 10.1007/978-3-319-00101-2_9,

obtained. Since the order of nonlinearity in (9.1) is higher than one (see Sect. 5.3), these conditions are sufficient for stability in probability of the equilibrium point of the initial nonlinear equation by stochastic perturbations.

9.2 Two Points of Equilibrium, Stochastic Perturbations, Centering, and Linearization

The points of equilibrium of (9.1) are defined by the condition $\dot{x}(t) = 0$ that can be represented in the form

$$ae^{-bx^*}x^* = cx^*. \tag{9.2}$$

From (9.2) it follows that (9.1) has two points of equilibrium

$$x_1^* = 0, \qquad x_2^* = \frac{1}{b}\ln\frac{a}{c}. \tag{9.3}$$

Similarly to Sect. 5.4, let us assume that (9.1) is exposed to stochastic perturbations that are of white noise type and are directly proportional to the deviation of $x(t)$ from the point of equilibrium x^* and influence $\dot{x}(t)$ immediately. In this way, (9.1) takes the form

$$\dot{x}(t) = ax(t-h)e^{-bx(t-h)} - cx(t) + \sigma\big(x(t) - x^*\big)\dot{w}(t). \tag{9.4}$$

Let us center (9.4) at the point of equilibrium x^* using the new variable $y(t) = x(t) - x^*$. By this way from (9.4) via (9.2) we obtain

$$\dot{y}(t) = -cy(t) + ae^{-bx^*}\big[y(t-h)e^{-by(t-h)} + x^*\big(e^{-by(t-h)} - 1\big)\big] + \sigma y(t)\dot{w}(t). \tag{9.5}$$

It is clear that the stability of an equilibrium point x^* of (9.4) is equivalent to the stability of the trivial solution of (9.5).

Along with (9.5), we will consider the linear part of this equation. Using the representation $e^y = 1 + y + o(y)$ (where $o(y)$ means that $\lim_{y\to 0}\frac{o(y)}{y} = 0$) and neglecting $o(y)$, we obtain the linear part (process $z(t)$) of (9.5) in the form

$$\dot{z}(t) = -cz(t) - ae^{-bx^*}\big(bx^* - 1\big)z(t-h) + \sigma z(t)\dot{w}(t). \tag{9.6}$$

As it follows from Remark 5.3, if the order of nonlinearity of the equation under consideration is higher than one, then a sufficient condition for the asymptotic mean-square stability of the linear part of the initial nonlinear equation is also a sufficient condition for the stability in probability of the initial equation. So, we will investigate sufficient conditions for the asymptotic mean-square stability of the linear part (9.6) of the nonlinear equation (9.5).

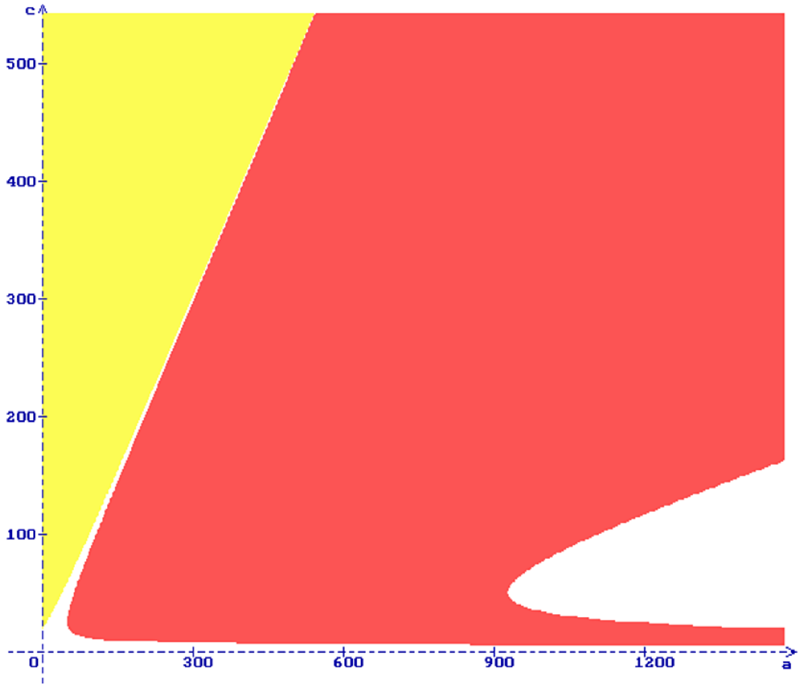

Fig. 9.1 Regions of stability in probability for zero equilibrium point (*yellow*) and positive equilibrium point (*red*) of (9.4) for $h = 0.02,\ p = 20$

9.3 Sufficient Conditions for Stability in Probability for Both Equilibrium Points

By (9.2)–(9.3) the nonlinear and linear equations (9.5)–(9.6) for the equilibrium points $x_1^* = 0$ respectively are

$$\dot{y}(t) = -cy(t) + ay(t-h)e^{-by(t-h)} + \sigma y(t)\dot{w}(t), \tag{9.7}$$

$$\dot{z}(t) = -cz(t) + az(t-h) + \sigma z(t)\dot{w}(t). \tag{9.8}$$

By Lemma 2.1 a necessary and sufficient condition for the asymptotic mean-square stability of the trivial solution of (9.8) is

$$G^{-1} > p, \quad p = \frac{1}{2}\sigma^2, \tag{9.9}$$

where

$$G = \frac{1 - aq^{-1}\sinh(qh)}{c - a\cosh(qh)}, \quad c > a,\ q = \sqrt{c^2 - a^2}. \tag{9.10}$$

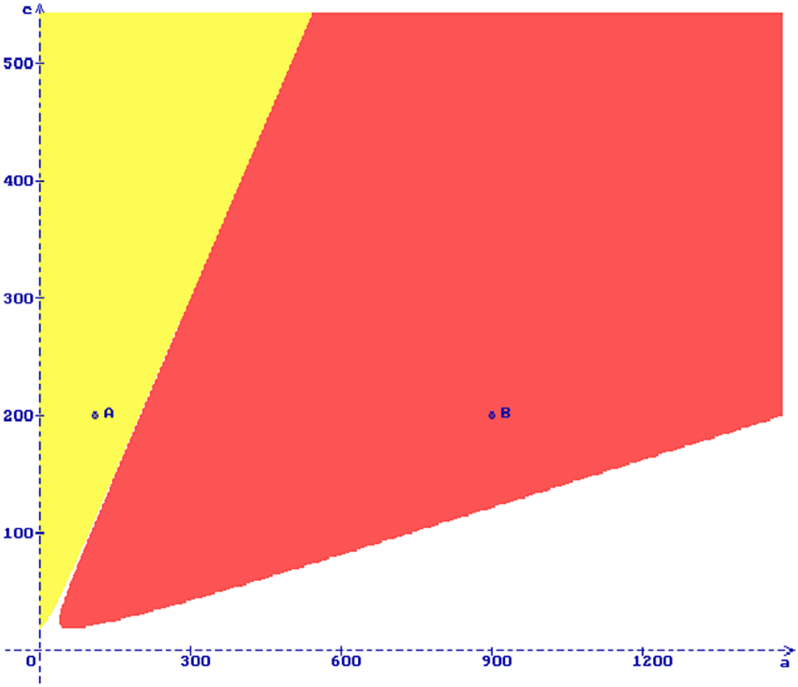

Fig. 9.2 Regions of stability in probability for zero equilibrium point (*yellow*) and positive equilibrium point (*red*) of (9.4) for $h = 0.1,\ p = 20$

In particular, if $p \geq 0$ and $h = 0$, then the stability condition (9.9)–(9.10) takes the form $c > a + p$.

Condition (9.9)–(9.10) is also a sufficient condition for the stability in probability of the zero equilibrium point $x^* = 0$ of (9.4).

Similarly to (9.9)–(9.10), for the equilibrium points $x_2^* = \frac{1}{b} \ln \frac{a}{c}$, the nonlinear and linear equations (9.5)–(9.6) respectively are

$$\dot{y}(t) = -cy(t) + cy(t-h)e^{-by(t-h)} + \frac{c}{b} \ln \frac{a}{c} \left(e^{-by(t-h)} - 1\right) + \sigma y(t)\dot{w}(t), \quad (9.11)$$

$$\dot{z}(t) = -cz(t) - c\left(\ln \frac{a}{c} - 1\right) z(t-h) + \sigma z(t)\dot{w}(t). \quad (9.12)$$

By Lemma 2.1 a necessary and sufficient condition for the asymptotic mean-square stability of the trivial solution of (9.12) is (9.9), where

$$G = \begin{cases} \frac{1+cq^{-1}(\ln(ac^{-1})-1)\sinh(qh)}{c[1+(\ln(ac^{-1})-1)\cosh(qh)]}, & c < a < ce^2,\ q = c\sqrt{\ln \frac{a}{c}(2 - \ln \frac{a}{c})}, \\ \frac{1+ch}{2c}, & a = ce^2, \\ \frac{1+cq^{-1}(\ln(ac^{-1})-1)\sin(qh)}{c[1+(\ln(ac^{-1})-1)\cos(qh)]}, & a > ce^2,\ q = c\sqrt{\ln \frac{a}{c}(\ln \frac{a}{c} - 2)}. \end{cases} \quad (9.13)$$

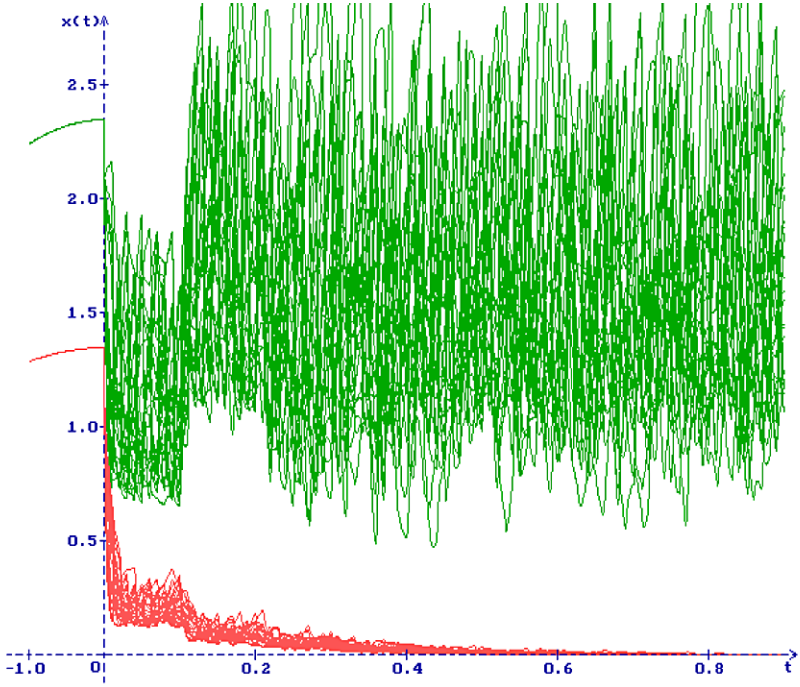

Fig. 9.3 Zero equilibrium point is stable at the point $A = (110, 200)$ (25 *red trajectories*) and is unstable in the point $B = (900, 200)$ (25 *green trajectories*)

In particular, if $p > 0$ and $h = 0$, then the stability condition takes the form $c \ln \frac{a}{c} > p$; if $p = 0$ and $h > 0$, then the region of stability is bounded by the lines $c = 0$, $c = a$, and $1 + (\ln \frac{a}{c} - 1)\cos(qh) = 0$ for $a > ce^2$.

Condition (9.9), (9.13) is also a sufficient condition for the stability in probability of the positive equilibrium point $x^* = \frac{1}{b} \ln \frac{a}{c}$ of (9.4).

Remark 9.1 Note that the stability conditions (9.9), (9.10) and (9.9), (9.13) have the following property: if the point (a, c) belongs to the stability region with some p and h, then for arbitrary positive α, the point $(a_0, c_0) = (\alpha a, \alpha c)$ belongs to the stability region with $p_0 = \alpha p$ and $h_0 = \alpha^{-1} h$.

9.4 Numerical Illustrations

In Fig. 9.1 the stability regions for (9.4) given by conditions (9.9), (9.10) for the zero equilibrium point (yellow) and (9.9), (9.13) for the positive equilibrium point (red) are shown in the space of the parameters (a, c) for $h = 0.02$ and $p = 20$. In Fig. 9.2 the similar regions of stability are shown for $h = 0.1$ and $p = 20$.

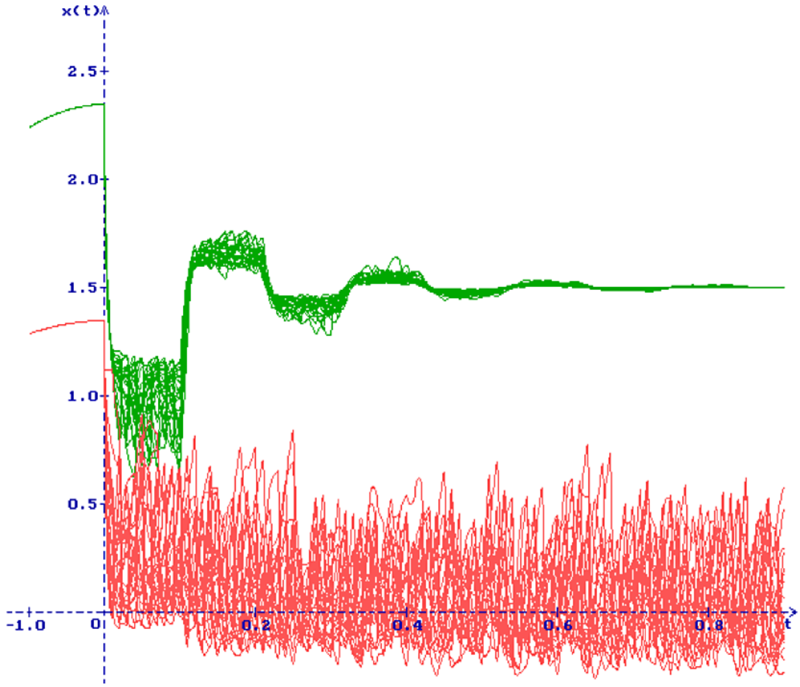

Fig. 9.4 Positive equilibrium point is unstable a the point $A = (110, 200)$ (25 *red trajectories*) and is stable at the point $B = (900, 200)$ (25 *green trajectories*)

For numerical simulation of the solution of (9.4), one uses the algorithm of numerical simulation of trajectories of the Wiener process (Chap. 2) and the Euler–Maruyama scheme [200]. Note that the stability of the difference analogue of (9.4) was investigated in detail in [38, 278].

Numerical simulation of the solution of (9.4) with $x^* = 0$ is shown in Fig. 9.3. At the point A with coordinates $a = 110$, $c = 200$ (see Fig. 9.2) the zero equilibrium point is stable in probability, so, all 25 trajectories (red) of the solution with the initial function $x(s) = 1.35\cos(3s)$ converge to zero. At the point B with coordinates $a = 900$, $c = 200$ (see Fig. 9.2) the zero equilibrium point is unstable, so, 25 trajectories (green) of the solution with the initial function $x(s) = 2.35\cos(3s)$ fill the whole space.

In Fig. 9.4 numerical simulation of the solution of (9.4) with the positive equilibrium point $x^* = \frac{1}{b}\ln\frac{a}{c}$ is shown by $b = 1$. At the point B with coordinates $a = 900$, $c = 200$ (see Fig. 9.2) the positive equilibrium point is stable in probability, so, all 25 trajectories (green) of the solution converge to $x^* = \ln(900/200) = 1.504$. At the point A with coordinates $a = 110$, $c = 200$ (see Fig. 9.2) the positive equilibrium point is unstable, and the trajectories (red) of the solution do not go to zero.