Chapter 12
Stability of Some Social Mathematical Models with Delay Under Stochastic Perturbations

In this chapter we propose a mathematical framework to model some social behavior. To be precise, we propose delayed and stochastic mathematical models to analyze human behaviors related to some addictions: consumption of alcohol and obesity.

12.1 Mathematical Model of Alcohol Consumption

Taking into account the proposal presented in [247], we consider alcohol consumption habit as susceptible to be transmitted by peer pressure or social contact. This fact led us to propose an epidemiologic-type mathematical model to study this social epidemic.

Here we generalize the known nonlinear dynamic model of alcohol consumption [254] by adding distributed delay. We obtain sufficient conditions for the existence of a positive equilibrium point of this system. Similarly to the previous sections, we suppose that this nonlinear system is exposed to additive stochastic perturbations of white noise type that are directly proportional to the deviation of the system state from the equilibrium point. The considered nonlinear system is linearized in the neighborhood of the positive point of equilibrium, and a sufficient condition for asymptotic mean-square stability of the zero solution of the constructed linear system is obtained via the procedure of constructing Lyapunov functionals that is described in Sect. 2.2.2. Since the order of nonlinearity of the considered nonlinear system is higher than one, the obtained condition is also a sufficient one (Sect. 5.3) for stability in probability of the equilibrium point of the initial nonlinear system under stochastic perturbations.

12.1.1 Description of the Model of Alcohol Consumption

Let $A(t)$ be nonconsumers, individuals that have never consumed alcohol or infrequently have alcohol consumption, and $M(t)$ be nonrisk consumers, individuals

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with regular low consumption, to be precise, men consuming less than 50 cc (cubic centimeters) of alcohol every day and women consuming less than 30 cc of alcohol every day. Let $R(t)$ be risk consumers, individuals with regular high consumption, that is, men consuming more than 50 cc of alcohol every day and women who consuming more than 30 cc of alcohol every day.

Considering homogeneous mixing [219], where each individual can contact with any other individual (peer pressure), a dynamic alcohol consumption model is given by the following nonlinear system of ordinary differential equations with distributed delay:

$$\begin{aligned}
\dot{A}(t) &= \mu P(t) + \gamma R(t) - d_A A(t) - \beta A(t) \int_0^\infty \frac{M(t-s)+R(t-s)}{P(t-s)}\, dK(s), \\
\dot{M}(t) &= \beta A(t) \int_0^\infty \frac{M(t-s)+R(t-s)}{P(t-s)}\, dK(s) - dM(t) - \alpha M(t), \\
\dot{R}(t) &= \alpha M(t) - \gamma R(t) - dR(t), \\
P(t) &= A(t) + M(t) + R(t).
\end{aligned} \tag{12.1}$$

Here:

- α the rate at which a nonrisk consumer moves to the risk consumption subpopulation (intensity of transition from the group $M(t)$ to the group $R(t)$).
- β the transmission rate due to social pressure to increase the alcohol consumption, e.g., family, friends, marketing, TV, etc. (intensity of transition from the group $A(t)$ to the group $M(t)$).
- γ the rate at which a risk consumer becomes a nonconsumer (intensity of transition from the group $R(t)$ to the group $A(t)$); so, the scheme of transition from one group to another one is

$$A(t) \to^{\beta}\to M(t) \to^{\alpha}\to R(t) \to^{\gamma}\to A(t).$$

- μ the birth rate.
- d_A the death rate.
- d the augmented death rate due to alcohol consumption (accidents at work, traffic accidents, and diseases derived by alcohol consumption are considered).

We suppose that the parameters $\alpha, \beta, \gamma, \mu, d_A, d$ are nonnegative numbers and $K(s)$ is a nondecreasing function such that

$$\int_0^\infty dK(s) = 1. \tag{12.2}$$

The integral is understood in the Stieltjes sense.

Remark 12.1 In particular, $dK(s) = \delta(s-h)\, ds$, where $h > 0$, $\delta(s)$ is Dirac's function, system (12.1) is a system with discrete delay h. The case of a system without delay ($h = 0$) is considered in [254].

12.1.2 Normalization of the Initial Model

Put

$$a(t) = \frac{A(t)}{P(t)}, \qquad m(t) = \frac{M(t)}{P(t)}, \qquad r(t) = \frac{R(t)}{P(t)}. \tag{12.3}$$

From (12.1) and (12.3) it follows that

$$a(t) + m(t) + r(t) = 1. \tag{12.4}$$

Adding the first three equations in (12.1), by (12.4) we obtain

$$\frac{\dot{P}(t)}{P(t)} = \mu - d + (d - d_A)a(t).$$

From this and from (12.3) we have

$$\begin{aligned}\dot{a}(t) &= \frac{\dot{A}(t)P(t) - A(t)\dot{P}(t)}{P^2(t)} = \frac{\dot{A}(t)}{P(t)} - \frac{A(t)}{P(t)} \times \frac{\dot{P}(t)}{P(t)} \\ &= \frac{\dot{A}(t)}{P(t)} - a(t)\big[\mu - d + (d - d_A)a(t)\big],\end{aligned} \tag{12.5}$$

and, similarly,

$$\begin{aligned}\dot{m}(t) &= \frac{\dot{M}(t)}{P(t)} - m(t)\big[\mu - d + (d - d_A)a(t)\big], \\ \dot{r}(t) &= \frac{\dot{R}(t)}{P(t)} - r(t)\big[\mu - d + (d - d_A)a(t)\big].\end{aligned} \tag{12.6}$$

Thus, putting

$$I(a_t) = \int_0^\infty a(t-s)\,dK(s), \tag{12.7}$$

by (12.5), (12.6), (12.1), (12.2), and (12.4) we obtain

$$\begin{aligned}\dot{a}(t) &= \mu + \gamma r(t) + \beta a(t)I(a_t) - a(t)\big[\beta + \mu - (d - d_A)\big(1 - a(t)\big)\big], \\ \dot{m}(t) &= \beta a(t) - \beta a(t)I(a_t) - m(t)\big[\alpha + \mu + (d - d_A)a(t)\big], \\ \dot{r}(t) &= \alpha m(t) - r(t)\big[\gamma + \mu + (d - d_A)a(t)\big].\end{aligned}$$

In view of (12.4), the last equation can be rejected, and, as a result, we obtain the system of two integro–differential equations

$$\begin{aligned}\dot{a}(t) &= \mu + \gamma - \gamma m(t) + \beta a(t)I(a_t) - a(t)\big[\beta + \mu + \gamma - (d - d_A)\big(1 - a(t)\big)\big], \\ \dot{m}(t) &= \beta a(t) - \beta a(t)I(a_t) - m(t)\big[\alpha + \mu + (d - d_A)a(t)\big].\end{aligned} \tag{12.8}$$

12.1.3 Existence of an Equilibrium Point

By (12.8), (12.2), and (12.4) a point of equilibrium (a^*, m^*, r^*) is defined by the following system of algebraic equations:

$$\begin{aligned}(\mu+\gamma)\big(1-a^*\big) &= a^*(\beta-d+d_A)\big(1-a^*\big)+\gamma m^*,\\ \beta a^*\big(1-a^*\big) &= m^*\big[\alpha+\mu+(d-d_A)a^*\big],\\ a^*+m^*+r^* &= 1.\end{aligned} \tag{12.9}$$

Lemma 12.1 *If $d \in [d_A, \beta+d_A)$, then system (12.9) has a unique positive solution (a^*, m^*, r^*) if and only if*

$$\beta > d-d_A+\mu+\frac{\alpha\gamma}{\alpha+\mu+\gamma+d-d_A}. \tag{12.10}$$

If $d \geq \beta+d_A$, then system (12.9) has no positive solutions.

Proof *Necessity* From the first two equations in (12.9) we have

$$\mu+\gamma = a^*(\beta-d+d_A)+\frac{\gamma\beta a^*}{\alpha+\mu+(d-d_A)a^*}. \tag{12.11}$$

Since $a^* \in (0, 1)$, from (12.11) it follows that

$$\begin{aligned}\mu+\gamma &= a^*(\beta-d+d_A)+\frac{\gamma\beta}{(\alpha+\mu)(a^*)^{-1}+d-d_A}\\ &< \beta-d+d_A+\frac{\gamma\beta}{\alpha+\mu+d-d_A}\\ &= \frac{\alpha+\mu+\gamma+d-d_A}{\alpha+\mu+d-d_A}\beta-d+d_A,\end{aligned}$$

which is equivalent to (12.10) since

$$\begin{aligned}\beta &> \frac{(\mu+\gamma+d-d_A)(\alpha+\mu+d-d_A)}{\alpha+\mu+\gamma+d-d_A}\\ &= d-d_A+\mu+\frac{\alpha\gamma}{\alpha+\mu+\gamma+d-d_A}.\end{aligned}$$

Sufficiency Rewrite (12.11) in the form

$$\begin{aligned}&Q\big(a^*\big)^2+Ba^*-C=0,\\ &B=(\beta-d+d_A)(\alpha+\gamma+\mu)-\mu(d-d_A),\\ &Q=(\beta-d+d_A)(d-d_A),\ \ C=(\mu+\alpha)(\mu+\gamma).\end{aligned} \tag{12.12}$$

Thus, by (12.9) the equilibrium point (a^*, m^*, r^*) is defined by the system of algebraic equations (12.12), and

$$m^* = \frac{\beta a^*(1-a^*)}{\alpha+\mu+(d-d_A)a^*}, \qquad r^* = 1-a^*-m^*. \tag{12.13}$$

It is easy to check that by the condition $d \in [d_A, \beta + d_A)$ (or $Q \geq 0$) the existence of a solution a^* of (12.12) in the interval $(0, 1)$ is equivalent to the condition $C < Q + B$, which is equivalent to (12.10).

If $d \geq \beta + d_A$, then $Q \leq 0$ and $B < 0$. So, (12.12) cannot have positive roots. The proof is completed. □

Example 12.1 Following [254], put

$$\begin{gathered}\alpha = 0.000110247, \qquad \beta = 0.0284534, \qquad \gamma = 0.00144,\\ \mu = 0.01, \qquad d = 0.009, \qquad d_A = 0.008.\end{gathered} \tag{12.14}$$

Then condition (12.10) holds, and the solution of (12.12)–(12.13) is

$$a^* = 0.364739, \qquad m^* = 0.629383, \qquad r^* = 0.00587794, \tag{12.15}$$

or, in percents, $a^* = 36.47\ \%$, $m^* = 62.94\ \%$, $r^* = 0.59\ \%$.

12.1.4 Stochastic Perturbations, Centralization, and Linearization

Let us suppose that system (12.8) is exposed to stochastic perturbations of white noise type $(\dot{w}_1(t), \dot{w}_2(t))$, which are directly proportional to the deviation of system (12.8) state $(a(t), m(t))$ from the equilibrium point (a^*, m^*), i.e.,

$$\begin{aligned}\dot{a}(t) &= \mu+\gamma-\gamma m(t)+\beta a(t)I(a_t)-a(t)\big[\beta+\mu+\gamma-(d-d_A)\big(1-a(t)\big)\big]\\ &\quad +\sigma_1\big(a(t)-a^*\big)\dot{w}_1(t),\\ \dot{m}(t) &= \beta a(t)-\beta a(t)I(a_t)-m(t)\big[\alpha+\mu+(d-d_A)a(t)\big]\\ &\quad +\sigma_2\big(m(t)-m^*\big)\dot{w}_2(t).\end{aligned} \tag{12.16}$$

Here $w_1(t), w_2(t)$ are the mutually independent standard Wiener processes, and the stochastic differential equations (12.16) are understood in the Itô sense (Sect. 2.1.2).

To centralize system (12.16) in the equilibrium point, put now $x_1(t) = a(t) - a^*$, $x_2(t) = m(t) - m^*$. Then from (12.16) it follows that

$$\begin{aligned}\dot{x}_1(t) &= \mu+\gamma-\gamma\big(m^*+x_2(t)\big)+\beta\big(a^*+x_1(t)\big)\big(a^*+I(x_{1t})\big)\\ &\quad -\big(a^*+x_1(t)\big)\big[\beta+\mu+\gamma-(d-d_A)\big(1-a^*-x_1(t)\big)\big]+\sigma_1 x_1(t)\dot{w}_1(t),\\ \dot{x}_2(t) &= \beta\big(a^*+x_1(t)\big)-\beta\big(a^*+x_1(t)\big)\big(a^*+I(x_{1t})\big)\\ &\quad -\big(m^*+x_2(t)\big)\big[\alpha+\mu+(d-d_A)\big(a^*+x_1(t)\big)\big]+\sigma_2 x_2(t)\dot{w}_2(t),\end{aligned}$$

or

$$\begin{aligned}\dot{x}_1(t) &= \mu\bigl(1-a^*\bigr)+\gamma\bigl(1-a^*-m^*\bigr)-a^*\bigl(1-a^*\bigr)(\beta-d+d_A)-\mu x_1(t)\\&\quad+\gamma\bigl(-x_1(t)-x_2(t)\bigr)+x_1(t)\bigl(1-2a^*\bigr)(d-d_A)-\beta x_1(t)\bigl(1-a^*\bigr)\\&\quad+\beta a^* I(x_{1t})-x_1^2(t)(d-d_A)+\beta x_1(t)I(x_{1t})+\sigma_1 x_1(t)\dot{w}_1(t),\\\dot{x}_2(t) &= \beta a^*\bigl(1-a^*\bigr)-m^*\bigl[\alpha+\mu+(d-d_A)a^*\bigr]+\beta x_1(t)\bigl(1-a^*\bigr)\\&\quad-m^* x_1(t)(d-d_A)-\beta a^* I(x_{1t})-x_2(t)\bigl[\alpha+\mu+(d-d_A)a^*\bigr]\\&\quad-\beta x_1(t)I_1(x_{1t})-x_2(t)x_1(t)(d-d_A)+\sigma_2 x_2(t)\dot{w}_2(t).\end{aligned}\tag{12.17}$$

By (12.9) from (12.17) it follows that

$$\begin{aligned}\dot{x}_1(t) &= a_{11}x_1(t)+a_{12}x_2(t)+\beta a^* I(x_{1t})+\beta x_1(t)I(x_{1t})\\&\quad-(d-d_A)x_1^2(t)+\sigma_1 x_1(t)\dot{w}_1(t),\\\dot{x}_2(t) &= a_{21}x_1(t)+a_{22}x_2(t)-\beta a^* I(x_{1t})-\beta x_1(t)I(x_{1t})\\&\quad-(d-d_A)x_1(t)x_2(t)+\sigma_2 x_2(t)\dot{w}_2(t),\end{aligned}\tag{12.18}$$

where

$$\begin{aligned}a_{11} &= -\bigl[\mu+\gamma+(\beta-d+d_A)\bigl(1-a^*\bigr)+(d-d_A)a^*\bigr], \qquad a_{12}=-\gamma,\\a_{21} &= \frac{\beta(\alpha+\mu)(1-a^*)}{\alpha+\mu+(d-d_A)a^*}, \qquad a_{22}=-\bigl[\alpha+\mu+(d-d_A)a^*\bigr].\end{aligned}\tag{12.19}$$

Note that for $d\in[d_A,\beta+d_A)$, the numbers a_{11}, a_{12}, a_{22} are negative, and $a_{21}>0$.

Rejecting the nonlinear terms in (12.18), we obtain the linear part of (12.18):

$$\begin{aligned}\dot{y}_1(t) &= a_{11}y_1(t)+a_{12}y_2(t)+\beta a^* I(y_{1t})+\sigma_1 y_1(t)\dot{w}_1(t),\\\dot{y}_2(t) &= a_{21}y_1(t)+a_{22}y_2(t)-\beta a^* I(y_{1t})+\sigma_2 y_2(t)\dot{w}_2(t).\end{aligned}\tag{12.20}$$

12.1.5 Stability of the Equilibrium Point

Note that the nonlinear system (12.18) has the order of nonlinearity higher than one. Thus, as it is shown in Sect. 5.3, sufficient conditions for the asymptotic mean-square stability of the zero solution of the linear part (12.20) of the nonlinear system (12.18) at the same time are sufficient conditions for the stability in probability of the zero solution of the nonlinear system (12.18) and therefore are sufficient conditions for stability in probability of the solution (a^*,m^*) of (12.16).

To get sufficient conditions for the asymptotic mean-square stability of the zero solution of (12.20), rewrite this system in the form

$$\dot{y}(t)=Ay(t)+B(y_t)+\sigma\bigl(y(t)\bigr)\dot{w}(t),\tag{12.21}$$

where

$$y(t)=\big(y_1(t),y_2(t)\big)',\qquad w(t)=\big(w_1(t),w_2(t)\big)',$$
$$B(y_t)=\big(\beta a^* I(y_{1t}),-\beta a^* I(y_{1t})\big)',\tag{12.22}$$
$$A=\begin{pmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{pmatrix},\qquad \sigma\big(y(t)\big)=\begin{pmatrix}\sigma_1 y_1(t)&0\\0&\sigma_2 y_2(t)\end{pmatrix}.$$

Following the procedure of constructing Lyapunov functionals (Sect. 2.2.2), for stability investigation of (12.21), consider the auxiliary differential equation without memory

$$\dot z(t)=Az(t)+\sigma\big(z(t)\big)\dot w(t).\tag{12.23}$$

By Remark 2.6 the zero solution of the differential equation $\dot z(t)=Az(t)$ is asymptotically stable if and only if conditions (2.62) hold. By Corollary 2.3 conditions (2.66) are sufficient conditions for the asymptotic mean-square stability of the zero solution of (12.23). Below, we suppose that conditions (2.62) and (2.66) hold.

To get stability conditions for (12.20), consider the matrix equation

$$A'P+PA+P_\sigma=-C,\tag{12.24}$$

where

$$P=\begin{pmatrix}p_{11}&p_{12}\\p_{12}&p_{22}\end{pmatrix},\quad P_\sigma=\begin{pmatrix}p_{11}\sigma_1^2&0\\0&p_{22}\sigma_2^2\end{pmatrix},\quad C=\begin{pmatrix}c&0\\0&1\end{pmatrix},$$

$c>0$, and the matrix A is defined in (12.22), (12.19).

If the matrix equation (12.24) has a positive definite solution P, then the function $v(z)=z'Pz$ is a Lyapunov function for (12.23) since

$$Lv=z'\big(A'P+PA+P_\sigma\big)z=-z'Cz.$$

Note that the matrix equation (12.24) can be represented as the system of the equations

$$\begin{aligned}2(p_{11}a_{11}+p_{12}a_{21}+p_{11}\delta_1)&=-c,\\2(p_{12}a_{12}+p_{22}a_{22}+p_{22}\delta_2)&=-1,\\p_{11}a_{12}+p_{12}\,\mathrm{Tr}(A)+p_{22}a_{21}&=0,\end{aligned}\tag{12.25}$$

with the solution

$$p_{11}=-\frac{c+2a_{21}p_{12}}{2\hat a_{11}},\qquad p_{22}=-\frac{1+2a_{12}p_{12}}{2\hat a_{22}},\qquad p_{12}=\frac{a_{21}\hat a_{11}+ca_{12}\hat a_{22}}{2Z},\tag{12.26}$$

where

$$\hat a_{ii}=a_{ii}+\delta_i,\quad \delta_i=\frac{1}{2}\sigma_i^2,\ i=1,2,\tag{12.27}$$

$$Z = \mathrm{Tr}(A)\hat{a}_{11}\hat{a}_{22} - a_{12}a_{21}(\hat{a}_{11} + \hat{a}_{22}). \tag{12.28}$$

Lemma 12.2 *Let conditions* (2.52), (2.56) *hold, and let*

$$\hat{a}_{11} < 0, \qquad \hat{a}_{22} < 0. \tag{12.29}$$

Then the zero solution of (12.23) *is asymptotically mean-square stable.*

Proof It is enough to show that the matrix $P = \|p_{ij}\|$ with the elements (12.26), which are a solution of the matrix equation (12.24), is positive definite for an arbitrary $c > 0$, i.e., $p_{11} > 0$, $p_{22} > 0$, $p_{11}p_{22} > p_{12}^2$. To this aim, note that by (2.62), (12.19), (12.29) we have $Z < 0$. Note also that by (12.27), (12.29), Remark 2.8, and (2.72) we obtain

$$\begin{aligned} \delta_1 < |a_{11}| \le \frac{|\mathrm{Tr}(A)|\det(A)}{A_2} \le \frac{A_1}{|\mathrm{Tr}(A)|}, \\ \delta_2 < |a_{22}| \le \frac{|\mathrm{Tr}(A)|\det(A)}{A_1} \le \frac{A_2}{|\mathrm{Tr}(A)|}, \end{aligned} \tag{12.30}$$

where

$$A_i = \det(A) + a_{ii}^2, \quad i = 1, 2. \tag{12.31}$$

Besides, by (12.28), (12.27), (2.62), and (12.31) we have

$$\begin{aligned} Z + a_{12}a_{21}\hat{a}_{22} &= \mathrm{Tr}(A)\hat{a}_{11}\hat{a}_{22} - a_{12}a_{21}(\hat{a}_{11} + \hat{a}_{22}) + a_{12}a_{21}\hat{a}_{22} \\ &= \big(\mathrm{Tr}(A)\hat{a}_{22} - a_{12}a_{21}\big)\hat{a}_{11} \\ &= \big(A_2 - \big|\mathrm{Tr}(A)\big|\delta_2\big)\hat{a}_{11} \end{aligned} \tag{12.32}$$

and, similarly,

$$Z + a_{12}a_{21}\hat{a}_{11} = \big(A_1 - \big|\mathrm{Tr}(A)\big|\delta_1\big)\hat{a}_{22}. \tag{12.33}$$

From this and from (12.26), (12.32), (12.30) it follows that for an arbitrary $c > 0$,

$$\begin{aligned} p_{11} &= -\frac{cZ + a_{21}(ca_{12}\hat{a}_{22} + a_{21}\hat{a}_{11})}{2Z\hat{a}_{11}} \\ &= -\frac{c(Z + a_{12}a_{21}\hat{a}_{22}) + a_{21}^2\hat{a}_{11}}{2Z\hat{a}_{11}} \\ &= \frac{c(A_2 - |\mathrm{Tr}(A)|\delta_2) + a_{21}^2}{2|Z|} \\ &> 0, \end{aligned} \tag{12.34}$$

and, similarly, by (12.26), (12.33), (12.30) we obtain

$$p_{22} = \frac{A_1 - |\mathrm{Tr}(A)|\delta_1 + ca_{12}^2}{2|Z|} > 0. \tag{12.35}$$

Finally, let us show that $p_{11}p_{22} > p_{12}^2$. Indeed, the inequality

$$\frac{(c+2a_{21}p_{12})(1+2a_{12}p_{12})}{4\hat{a}_{11}\hat{a}_{22}} > p_{12}^2$$

is equivalent to $4Bp_{12}^2 - 2(a_{21}+ca_{12})p_{12} < c$ by $B = \hat{a}_{11}\hat{a}_{22} - a_{12}a_{21} > 0$. Substituting p_{12} from (12.26) into the obtained inequality, we have

$$B(a_{21}\hat{a}_{11}+ca_{12}\hat{a}_{22})^2 - (a_{21}+ca_{12})(a_{21}\hat{a}_{11}+ca_{12}\hat{a}_{22})Z < cZ^2$$

or

$$c^2a_{12}^2\hat{a}_{22}(Z-B\hat{a}_{22}) + c\hat{a}_{11}\hat{a}_{22}\big(Z\,\mathrm{Tr}(A) - 2a_{12}a_{21}B\big) + a_{21}^2\hat{a}_{11}(Z-B\hat{a}_{11}) > 0.$$

Note also that by (12.28), (12.29), (12.19) we obtain

$$\begin{aligned}
\hat{a}_{11}(Z-B\hat{a}_{11}) &= \hat{a}_{11}\big(\mathrm{Tr}(A)\hat{a}_{11}\hat{a}_{22} - a_{12}a_{21}(\hat{a}_{11}+\hat{a}_{22}) \\
&\quad - (\hat{a}_{11}\hat{a}_{22} - a_{12}a_{21})\hat{a}_{11}\big) \\
&= \hat{a}_{11}\big(\mathrm{Tr}(A)\hat{a}_{11}\hat{a}_{22} - a_{12}a_{21}\hat{a}_{22} - \hat{a}_{11}^2\hat{a}_{22}\big) \\
&= \hat{a}_{11}\hat{a}_{22}\big(\mathrm{Tr}(A)\hat{a}_{11} - a_{12}a_{21} - \hat{a}_{11}^2\big) \\
&= \hat{a}_{11}\hat{a}_{22}\big(\big(\mathrm{Tr}(A) - \hat{a}_{11}\big)\hat{a}_{11} - a_{12}a_{21}\big) \\
&= \hat{a}_{11}\hat{a}_{22}\big((a_{22}-\delta_1)\hat{a}_{11} - a_{12}a_{21}\big) > 0
\end{aligned}$$

and, similarly,

$$\begin{gathered}
\hat{a}_{22}(Z-B\hat{a}_{22}) = \hat{a}_{11}\hat{a}_{22}\big((a_{11}-\delta_2)\hat{a}_{22} - a_{12}a_{21}\big) > 0, \\
Z\,\mathrm{Tr}(A) - 2a_{12}a_{21}B > 0.
\end{gathered}$$

So, for an arbitrary $c > 0$, the matrix P with the elements (12.26) is positive definite. The proof is completed. □

Theorem 12.1 *If conditions* (12.29) *hold and, for some* $c > 0$, *the elements* (12.26) *of the matrix* P *satisfy the condition*

$$\big(\beta a^*|p_{12}-p_{22}|\big)^2 + 2\beta a^*|p_{11}-p_{12}| < c, \tag{12.36}$$

then the solution (a^*, m^*) *of system* (12.16) *is stable in probability.*

Proof Note that the order of nonlinearity of system (12.16) is higher than one. Therefore, from Sect. 5.3, to get conditions for stability in probability of the equilibrium point (a^*, m^*) of this system, it is enough to get conditions for the asymptotic mean-square stability of the zero solution of the linear part (12.20) of this system. Following the procedure of constructing Lyapunov functionals, we will construct a Lyapunov functional for system (12.20) in the form $V = V_1 + V_2$, where $V_1 = y'Py$,

$y=(y_1,y_2)'$, P is a positive definite solution of system (12.25) with the elements (12.26), and V_2 will be chosen below.

Let L be the generator (Sect. 2.1.2) of system (12.20). Then by (12.20) and (12.25) we have

$$\begin{aligned}LV_1 &= 2\big(p_{11}y_1(t)+p_{12}y_2(t)\big)\big(a_{11}y_1(t)+a_{12}y_2(t)+\beta a^*I(y_{1t})\big)+p_{11}\sigma_1^2y_1^2(t)\\&\quad+2\big(p_{12}y_1(t)+p_{22}y_2(t)\big)\big(a_{21}y_1(t)+a_{22}y_2(t)-\beta a^*I(y_{1t})\big)+p_{22}\sigma_2^2y_2^2(t)\\&=-cy_1^2(t)-y_2^2(t)+2\beta a^*\big[(p_{11}-p_{12})y_1(t)+(p_{12}-p_{22})y_2(t)\big]I(y_{1t}).\end{aligned}$$

By (12.7), (12.2) we have $2y_1(t)I(y_{1t})\le y_1^2(t)+I(y_{1t}^2)$ and $2y_2(t)I(y_{1t})\le \nu y_2^2(t)+\nu^{-1}I(y_{1t}^2)$ for some $\nu>0$. Using these inequalities, we obtain

$$\begin{aligned}LV_1 &\le -cy_1^2(t)-y_2^2(t)+\beta a^*|p_{11}-p_{12}|\big(y_1^2(t)+I\big(y_{1t}^2\big)\big)\\&\quad+\beta a^*|p_{12}-p_{22}|\big(\nu y_2^2(t)+\nu^{-1}I\big(y_{1t}^2\big)\big)\\&=\big(\beta a^*|p_{11}-p_{12}|-c\big)y_1^2(t)+\big(\beta a^*|p_{12}-p_{22}|\nu-1\big)y_2^2(t)\\&\quad+qI\big(y_{1t}^2\big),\end{aligned}\tag{12.37}$$

where

$$q=\beta a^*\big(|p_{11}-p_{12}|+|p_{12}-p_{22}|\nu^{-1}\big).\tag{12.38}$$

Putting

$$V_2=q\int_0^\infty\int_{t-s}^t y_1^2(\theta)\,d\theta\,dK(s),$$

by (12.2), (12.7) we have $LV_2=q(y_1^2(t)-I(y_{1t}^2))$. Therefore, by (12.37), (12.38) for the functional $V=V_1+V_2$, we have

$$\begin{aligned}LV &\le \big(2\beta a^*|p_{11}-p_{12}|+\beta a^*|p_{12}-p_{22}|\nu^{-1}-c\big)y_1^2(t)\\&\quad+\big(\beta a^*|p_{12}-p_{22}|\nu-1\big)y_2^2(t).\end{aligned}$$

Thus, if

$$2\beta a^*|p_{11}-p_{12}|+\beta a^*|p_{12}-p_{22}|\nu^{-1}<c,\qquad \beta a^*|p_{12}-p_{22}|\nu<1,\tag{12.39}$$

then by Remark 2.1 the zero solution of (12.20) is asymptotically mean-square stable.

From (12.39) it follows that

$$\frac{\beta a^*|p_{12}-p_{22}|}{c-2\beta a^*|p_{11}-p_{12}|}<\nu<\frac{1}{\beta a^*|p_{12}-p_{22}|}.\tag{12.40}$$

Thus, if for some $c>0$, condition (12.36) holds, then there exists $\nu>0$ such that conditions (12.40) (or (12.39)) hold too, and therefore the zero solution of (12.20) is asymptotically mean-square stable. From this it follows also that the zero solution of (12.18) and therefore the equilibrium point of system (12.16) are stable in probability. The proof is completed. □

Example 12.2 Consider system (12.16) with the values of the parameters α, β, γ, μ, d, d_A and the equilibrium point (a^*, m^*) given in (12.14), (12.15). As an example, consider the levels of noises $\sigma_1 = 0.028969$, $\sigma_2 = 0.142252$ or $\delta_1 = 0.000420$, $\delta_2 = 0.010118$. From (12.19) it follows that the values of system (12.20) parameters are $a_{11} = -0.029245$, $a_{12} = -0.001440$, $a_{21} = 0.017446$, $a_{22} = -0.010475$ and the conditions (12.29) hold: $\hat{a}_{11} = -0.028825 < 0$, $\hat{a}_{22} = -0.000357 < 0$. Put $c = 20$. Then by (12.26) $p_{11} = 477.4438$, $p_{12} = 215.6615$, $p_{22} = 530.4124$, and condition (12.36) holds:

$$\big(\beta a^* |p_{12} - p_{22}|\big)^2 + 2\beta a^* |p_{11} - p_{12}| = 16.1036 < 20.$$

Thus, the solution (a^*, m^*) of system (12.16) is stable in probability.

Example 12.3 Consider system (12.16) with the previous values of the all parameters except for the levels of noises that are $\sigma_1 = 0.0075$, $\sigma_2 = 0.0077$ or $\delta_1 = 0.000028$, $\delta_2 = 0.000030$. These values of σ_1 and σ_2 are selected taking into account sample errors of the monitoring of the alcohol consumption in Spain [291]. The parameters a_{11}, a_{21}, a_{22} are the same as in the previous example, and conditions (12.29) hold: $\hat{a}_{11} = -0.029217 < 0$, $\hat{a}_{22} = -0.010445 < 0$. Put $c = 4$. Then by (12.26) $p_{11} = 78.6856$, $p_{12} = 17.1347$, $p_{22} = 45.5060$, and condition (12.36) holds:

$$\big(\beta a^* |p_{12} - p_{22}|\big)^2 + 2\beta a^* |p_{11} - p_{12}| = 1.3643 < 4.$$

Thus, the solution (a^*, m^*) of system (12.16) is stable in probability.

Let us now get three corollaries from Theorem 12.1 that simplify a verification of the stability condition (12.36). By (12.26) and (12.28) we have

$$\begin{aligned} p_{12} - p_{11} &= p_{12} + \frac{c + 2a_{21}p_{12}}{2\hat{a}_{11}} \\ &= \left(1 + \frac{a_{21}}{\hat{a}_{11}}\right)\frac{a_{21}\hat{a}_{11} + ca_{12}\hat{a}_{22}}{2Z} + \frac{c}{2\hat{a}_{11}} \\ &= \frac{(a_{21} + \hat{a}_{11})a_{12}\hat{a}_{22} + Z}{2Z\hat{a}_{11}}c + \frac{(a_{21} + \hat{a}_{11})a_{21}}{2Z} \\ &= B_0 c + B_1, \end{aligned} \tag{12.41}$$

where

$$B_0 = \frac{(\mathrm{Tr}(A) + a_{12})\hat{a}_{22} - a_{12}a_{21}}{2Z}, \qquad B_1 = \frac{(a_{21} + \hat{a}_{11})a_{21}}{2Z} \tag{12.42}$$

and, similarly,

$$
\begin{aligned}
p_{12} - p_{22} &= p_{12} + \frac{1 + 2a_{12} p_{12}}{2\hat{a}_{22}} \\
&= \left(1 + \frac{a_{12}}{\hat{a}_{22}}\right) \frac{a_{21}\hat{a}_{11} + c a_{12}\hat{a}_{22}}{2Z} + \frac{1}{2\hat{a}_{22}} \\
&= \frac{(a_{12} + \hat{a}_{22}) a_{12} c}{2Z} + \frac{(a_{12} + \hat{a}_{22}) a_{21}\hat{a}_{11} + Z}{2Z\hat{a}_{22}} \\
&= D_0 c + D_1, \qquad (12.43)
\end{aligned}
$$

where

$$
D_0 = \frac{(a_{12} + \hat{a}_{22}) a_{12}}{2Z}, \qquad D_1 = \frac{(\mathrm{Tr}(A) + a_{21})\hat{a}_{11} - a_{12} a_{21}}{2Z}. \qquad (12.44)
$$

Remark 12.2 Put

$$
f(c) = \left(\beta a^* D_0\right)^2 \left(c + \frac{D_1}{D_0}\right)^2 + 2\beta a^* |B_0| \left| c + \frac{B_1}{B_0} \right| - c. \qquad (12.45)
$$

From (12.19), (12.29) it follows that $B_0 < 0$. By (12.41)–(12.45) and $B_0 < 0$ condition (12.36) is equivalent to the condition $f(c) < 0$.

Put now

$$
\begin{aligned}
S &= \left(\beta a^* D_0\right)^2 \left(\frac{D_1}{D_0} - \frac{B_1}{B_0}\right)^2 + \frac{B_1}{B_0}, \\
R_+ &= 2\beta a^* |B_0| \left(\frac{1 - 2\beta a^* |B_0|}{2(\beta a^* D_0)^2} - \frac{D_1}{D_0} + \frac{B_1}{B_0} \right), \\
R_- &= -2\beta a^* |B_0| \left(\frac{1 + 2\beta a^* |B_0|}{2(\beta a^* D_0)^2} - \frac{D_1}{D_0} + \frac{B_1}{B_0} \right), \qquad (12.46) \\
Q &= \frac{1}{4(\beta a^* D_0)^2} - \frac{D_1}{D_0} - \frac{B_0^2}{D_0^2}.
\end{aligned}
$$

Corollary 12.1 *If conditions* (12.29) *hold and* $S < 0$, *then the solution* (a^*, m^*) *of system* (12.16) *is stable in probability.*

Proof From $S < 0$ and $B_0 < 0$ it follows that $B_1 > 0$. Putting $c_0 = -B_1 B_0^{-1} > 0$, we obtain $f(c_0) = S < 0$, i.e., condition (12.36) holds. The proof is completed. □

Corollary 12.2 *If conditions* (12.29) *hold and* $0 \le R_+ < Q$, *then the solution* (a^*, m^*) *of system* (12.16) *is stable in probability.*

Proof Let us suppose that $c + B_1 B_0^{-1} \geq 0$. Then the minimum of the function $f(c)$ is reached by

$$c_0 = \frac{1 - 2\beta a^*|B_0|}{2(\beta a^* D_0)^2} - \frac{D_1}{D_0} \geq -\frac{B_1}{B_0}.$$

Substituting c_0 into the function $f(c)$, we obtain that the condition $f(c_0) < 0$ is equivalent to the condition $0 \leq R_+ < Q$. The proof is completed. □

Corollary 12.3 *If conditions* (12.29) *hold and* $0 < R_- < Q$, *then the solution* (a^*, m^*) *of system* (12.16) *is stable in probability.*

Proof Let us suppose that $c + B_1 B_0^{-1} < 0$. Then the minimum of the function $f(c)$ is reached by

$$c_0 = \frac{1 + 2\beta a^*|B_0|}{2(\beta a^* D_0)^2} - \frac{D_1}{D_0} < -\frac{B_1}{B_0}.$$

Substituting c_0 into the function $f(c)$, we obtain that the condition $f(c_0) < 0$ is equivalent to the condition $0 < R_- < Q$. The proof is completed. □

Example 12.4 Consider system (12.16) with the values of the parameters from Example 12.2. Calculating S, R_+, Q, we obtain: $S = 4.50 > 0$, $R_+ = 736 < Q = 1320$. From Corollary 12.2 it follows that the solution (a^*, m^*) of system (12.16) is stable in probability.

Example 12.5 Consider system (12.16) with the values of the parameters from Example 12.3. Calculating S, R_+, Q, we obtain: $S = -0.39 < 0$, $R_+ = 2462 < Q = 4754$. From both Corollary 12.1 and Corollary 12.2 it follows that the solution (a^*, m^*) of system (12.16) is stable in probability.

12.1.6 Numerical Simulation

Let us suppose that in (12.1) $dK(s) = \delta(s - h)\,ds$, where $\delta(s)$ is Dirac's delta-function, and $h \geq 0$ is the delay.

In Fig. 12.1 25 trajectories of the solution of (12.16), (12.4) are shown for the values of the parameters from Examples 12.1 and 12.2: $\alpha = 0.000110247$, $\beta = 0.0284534$, $\gamma = 0.00144$, $\mu = 0.01$, $d = 0.009$, $d_A = 0.008$, the initial values $a_0 = 0.43$, $m_0 = 0.53$, $r_0 = 0.04$, the levels of noises $\sigma_1 = 0.028969$, $\sigma_2 = 0.142252$, and delay $h = 0.1$. We can see that all trajectories go to the equilibrium point $a^* = 0.364739$, $m^* = 0.629383$, $r^* = 0.00587794$.

Note that numerical simulations of the processes $a(t)$, $m(t)$, and $r(t)$ were obtained via the difference analogues of (12.16), (12.4) in the form

$$\begin{aligned} a_{i+1} = a_i &+ \Delta\big[\mu + \gamma - \gamma\mu_i + \beta a_i a_{i-m} - a_i\big(\beta + \mu + \gamma - (d - d_A)(1 - a_i)\big)\big] \\ &+ \sigma_1\big(a_i - a^*\big)(w_{1,i+1} - w_{1i}), \end{aligned}$$

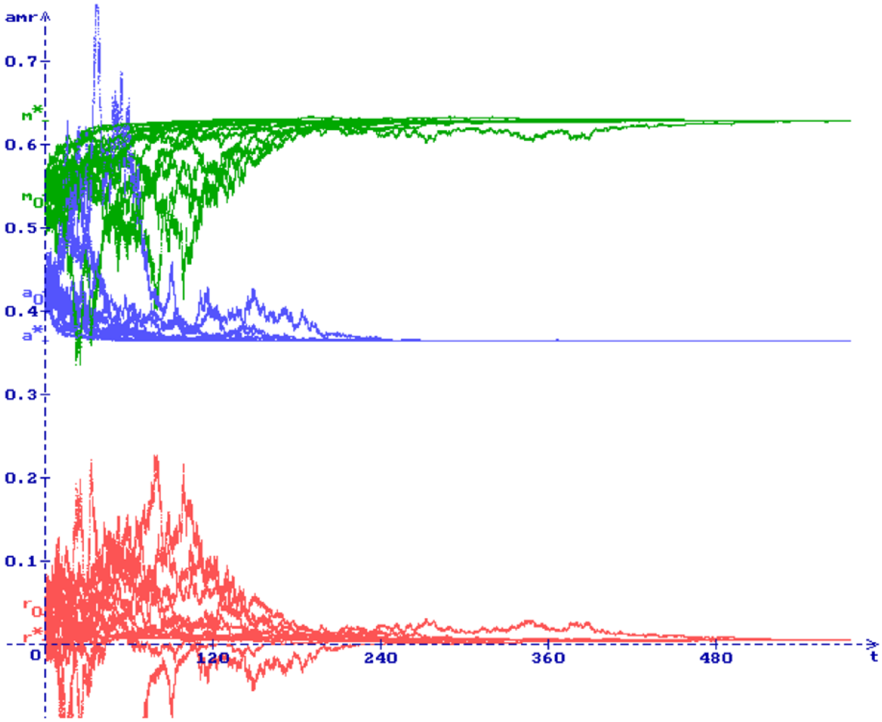

Fig. 12.1 25 trajectories of the processes $a(t)$ (*blue*), $m(t)$ (*green*), $r(t)$ (*red*) with the values of the parameters $\alpha = 0.000110247$, $\beta = 0.0284534$, $\gamma = 0.00144$, $\mu = 0.01$, $d = 0.009$, $d_A = 0.008$, the levels of noises $\sigma_1 = 0.028969$, $\sigma_2 = 0.142252$, the delay $h = 0.1$, the initial values $a(s) = 0.43$, $s \in [-0.1, 0]$, $m(0) = 0.53$, $r(0) = 0.04$, and the equilibrium point $a^* = 0.364739$, $m^* = 0.629383$, $r^* = 0.00587794$

$$\begin{aligned} m_{i+1} &= m_i + \Delta\big[\beta a_i - \beta a_i a_{i-m} - m_i\big(\alpha + \mu + (d - d_A)a_i\big)\big] \\ &\qquad + \sigma_2\big(m_i - m^*\big)(w_{2,i+1} - w_{2i}), \\ r_{i+1} &= 1 - a_{i+1} - m_{i+1}, \\ &\quad i = 0, 1, 2, \ldots, \ a_j = a_0, \ j = -m, \ldots, -1, 0. \end{aligned}$$

Here Δ is the discretization step (which was chosen as $\Delta = 0.01$), $a_i = a(t_i)$, $m_i = m(t_i)$, $r_i = r(t_i)$, $w_{ki} = w_k(t_i)$, $k = 1, 2$, $t_i = i\Delta$, $m = h\Delta^{-1}$, trajectories of the Wiener processes $w_1(t)$ and $w_2(t)$ are simulated by the algorithm described in Sect. 2.1.1.

12.2 Mathematical Model of Social Obesity Epidemic

Social obesity epidemic models are popular enough with researches (see, for instance, [14, 34, 42, 51, 68, 115, 227, 255]). Here the known nonlinear social obesity epidemic model [255] is generalized to the system with distributed delay. It is supposed also that this nonlinear system is exposed to additive stochastic perturbations of white noise type that are directly proportional to the deviation of the system state from the equilibrium point. The research that is similar to the previous one is applied to this model.

12.2.1 Description of the Considered Model

For constructing the mathematical obesity model [255] the 24- to 65-year-old population is divided into three subpopulations based on the so-called body mass index (BMI = Weight/Height^2). The classes or subpopulations are: individuals at a normal weight (BMI < 25 kg/m^2) $N(t)$, people who are overweight (25 kg/m^2 ≤ BMI < 30 kg/m^2) $S(t)$, and obese individuals (BMI ≥ 30 kg/m^2) $O(t)$.

The transition between the different subpopulations is determined as follows: once an adult starts an unhealthy lifestyle, he/she becomes addicted to the unhealthy lifestyle and starts a progression to being overweight $S(t)$ because of this lifestyle. If this adult continues with his/her unhealthy lifestyle, he/she can become an obese individual $O(t)$. In both these classes individuals can stop his/her unhealthy lifestyle and then move to classes $N(t)$ and $S(t)$, respectively.

The transitions between the subpopulations $N(t)$, $S(t)$, and $O(t)$ are governed by terms proportional to the sizes of these subpopulations. Conversely, the transitions from normal to overweight occur through the transmission of an unhealthy lifestyle from the overweight and obese subpopulations to the normal-interactions weight subpopulation, depending on the meet population, depending on the meetings among them. This transition is modeled using the term

$$\beta N(t)\int_0^\infty \big(S(t-s)+O(t-s)\big)\,dK(s),$$

where $K(s)$ is a nondecreasing function that satisfies condition (12.2), and the integral is understood in the Stieltjes sense. The subpopulations' sizes and their behaviors with time determine the dynamic evolution of adulthood excess weight.

Without loss of generality and for the sake of clarity, the 24- to 65-year-old adult population is normalized to unity, and it is supposed for all $t \ge 0$ that

$$N(t)\ge 0, \qquad S(t)\ge 0, \qquad O(t)\ge 0, \tag{12.47}$$

$$N(t)+S(t)+O(t)=1. \tag{12.48}$$

Thus, under the above assumptions, the following nonlinear system of integro–differential equations is obtained:

$$\dot N(t)=\mu N_0-\mu N(t)-\beta N(t)\int_0^\infty \big(S(t-s)+O(t-s)\big)\,dK(s)+\rho S(t),$$

$$\dot{S}(t) = \mu S_0 + \beta N(t) \int_0^\infty \big(S(t-s) + O(t-s)\big)\, dK(s) \\ - (\mu + \gamma + \rho) S(t) + \varepsilon O(t), \tag{12.49}$$
$$\dot{O}(t) = \mu O_0 + \gamma S(t) - (\mu + \varepsilon) O(t), \qquad t \geq 0,$$
$$N(0) = N_0,\ S(s) = S_0,\ O(s) = O_0,\ s \leq 0.$$

The time-invariant parameters of this system of equations are:

- ε the rate at which an obese adult with a healthy lifestyle becomes an overweight individual (intensity of transition from the group $O(t)$ to the group $S(t)$).
- ρ the rate at which an overweight individual moves to the normal-weight subpopulation (intensity of transition from the group $S(t)$ to the group $N(t)$).
- β the transmission rate because of social pressure to adopt an unhealthy lifestyle, e.g., TV, friends, family, job, and so on (intensity of transition from the group $N(t)$ to the group $S(t)$).
- γ the rate at which an overweight 24- to 65-year-old adult becomes an obese individual because of unhealthy lifestyle (intensity of transition from the group $S(t)$ into the group $O(t)$); so, the scheme of transition from one group to another one is

$$O(t) \rightarrow^{\varepsilon}\rightarrow S(t) \rightarrow^{\rho}\rightarrow N(t) \rightarrow^{\beta}\rightarrow S(t) \rightarrow^{\gamma}\rightarrow O(t).$$

- μ the average stay time in the system of 24- to 65-year-old adults ($\mu = 1/$ (65 years – 24 years) $\cdot$ 52 weeks/year).
- N_0 the proportion of normal weight coming from the 23-year age group.
- S_0 the proportion of overweight coming from the 23-year age group.
- O_0 the proportion of obese coming from the 23-year age group.

Here the parameters ε, ρ, β, γ, μ are nonnegative numbers, and N_0, S_0, O_0 satisfy the conditions of type (12.47), (12.48).

By condition (12.48) system (12.49) can be simplified to the following system of two equations:

$$\dot{N}(t) = \mu N_0 - \mu N(t) - \beta N(t) \int_0^\infty \big(1 - N(t-s)\big)\, dK(s) + \rho S(t),$$
$$\dot{S}(t) = \mu S_0 + \beta N(t) \int_0^\infty \big(1 - N(t-s)\big)\, dK(s) - (\mu + \gamma + \rho) S(t) \\ + \varepsilon\big(1 - N(t) - S(t)\big), \quad t \geq 0, \tag{12.50}$$
$$N(s) = N_0,\ s \leq 0,\ S(0) = S_0.$$

12.2.2 Existence of an Equilibrium Point

The equilibrium point (N^*, S^*) of system (12.50) is defined by the conditions $\dot{N}(t) = 0$, $\dot{S}(t) = 0$ and by (12.50), (12.47) is a solution of the following system of algebraic equations:

$$\begin{aligned} \mu N_0 - \mu N^* - \beta N^*(1 - N^*) + \rho S^* = 0, \\ \mu S_0 + \beta N^*(1 - N^*) - (\mu + \gamma + \rho) S^* + \varepsilon(1 - S^* - N^*) = 0. \end{aligned} \tag{12.51}$$

From (12.51) it follows that

$$\begin{aligned} S^* &= \rho^{-1}\left[\mu(N^* - N_0) + \beta N^*(1 - N^*)\right], \\ S^* &= k\rho^{-1}\left[\mu S_0 + (\varepsilon + \beta N^*)(1 - N^*)\right], \end{aligned} \tag{12.52}$$

where

$$k = \rho(\mu + \gamma + \rho + \varepsilon)^{-1} < 1. \tag{12.53}$$

By (12.52), (12.53) we obtain that N^* is a root of the quadratic equation

$$\beta(1-k)(N^*)^2 - (\mu + k\varepsilon + \beta(1-k))N^* + \mu(N_0 + kS_0) + k\varepsilon = 0. \tag{12.54}$$

Lemma 12.3 *Assume that* $N_0 + kS_0 < 1$. *If* $\beta > 0$, *then* (12.54) *has two real roots*, $N_1^* \in (0, 1)$ *and* $N_2^* > 1$. *If* $\beta = 0$ *and* $\mu k \varepsilon > 0$, *then* (12.54) *has one root* $N^* \in (N_0 + kS_0, 1)$.

Proof From $N_0 + kS_0 < 1$ and $\beta > 0$ we have

$$\begin{aligned} D &= \sqrt{(\mu + k\varepsilon + \beta(1-k))^2 - 4\beta(1-k)(\mu(N_0 + kS_0) + k\varepsilon)} \\ &> \sqrt{(\mu + k\varepsilon + \beta(1-k))^2 - 4\beta(1-k)(\mu + k\varepsilon)} \\ &= \left|\mu + k\varepsilon - \beta(1-k)\right|, \end{aligned} \tag{12.55}$$

i.e., $D > |\mu + k\varepsilon - \beta(1-k)| \geq 0$, and therefore the quadratic equation (12.54) has two real roots

$$N_1^* = \frac{\mu + k\varepsilon + \beta(1-k) - D}{2\beta(1-k)}, \qquad N_2^* = \frac{\mu + k\varepsilon + \beta(1-k) + D}{2\beta(1-k)}. \tag{12.56}$$

If $\mu + k\varepsilon < \beta(1-k)$, then

$$N_1^* < \frac{\mu + k\varepsilon}{\beta(1-k)} < 1, \qquad N_2^* > 1.$$

If $\mu + k\varepsilon \geq \beta(1-k)$, then

$$N_1^* < 1, \qquad N_2^* > \frac{\mu + k\varepsilon}{\beta(1-k)} \geq 1.$$

If $\beta = 0$, then from (12.54) it follows that

$$1 > N^* = \frac{\mu(N_0 + kS_0) + k\varepsilon}{\mu + k\varepsilon} > N_0 + kS_0.$$

The proof is completed. □

Lemma 12.4 *Assume that* $N_0 = 1$. *If* $\mu + k\varepsilon < \beta(1-k)$, *then* (12.54) *has two roots on the interval* (0,1]: $N_1^* \in (0, 1)$ *and* $N_2^* = 1$. *If* $\mu + k\varepsilon \geq \beta(1-k)$ *then* (12.54) *has one root only on the interval* (0,1]: $N_1^* = 1$.

Proof From $N_0 = 1$ and (12.47) we have $S_0 = 0$. Then, similarly to (12.55), $D = |\mu + k\varepsilon - \beta(1-k)|$. If $\mu + k\varepsilon < \beta(1-k)$, then $D = \beta(1-k) - (\mu + k\varepsilon)$, and by (12.56) we obtain

$$N_1^* = \frac{\mu + k\varepsilon}{\beta(1-k)} < 1, \qquad N_2^* = 1.$$

If $\mu + k\varepsilon > \beta(1-k)$, then $D = \mu + k\varepsilon - \beta(1-k)$, and by (12.56) we have

$$N_1^* = 1, \qquad N_2^* = \frac{\mu + k\varepsilon}{\beta(1-k)} > 1.$$

If $\mu + k\varepsilon = \beta(1-k)$, then $D = 0$ and $N_1^* = N_2^* = 1$. The proof is completed. □

Example 12.6 Following [255], put

$$\mu = 0.000469, \qquad \gamma = 0.0003, \qquad \varepsilon = 0.000004, \qquad \rho = 0.000035,$$
$$\beta = 0.00085, \qquad N_0 = 0.704, \qquad S_0 = 0.25, \qquad O_0 = 0.046.$$

Then by (12.56), (12.52), (12.48) we obtain

$$N^* = 0.3311, \qquad S^* = 0.3814, \qquad O^* = 0.2875.$$

Putting $\beta = 0$ with the same values of the other parameters, by Lemma 12.3 we obtain

$$N^* = 0.7149 > N_0 + kS_0 = 0.7148, \qquad S^* = 0.1465, \qquad O^* = 0.1386.$$

Put now $N_0 = 1$, $S_0 = O_0 = 0$. By Lemma 12.4, if $\beta = 0.00085$, i.e., if $\beta > 0$, then $\beta > (\mu + k\varepsilon)(1-k)^{-1} = 0.00049$ and $N^* = 0.5770$, $S^* = 0.2588$, $O^* = 0.1642$. If $\beta = 0$, then $N^* = 1$, $S^* = O^* = 0$.

12.2.3 Stochastic Perturbations, Centralization, and Linearization

Let us suppose that system (12.50) is exposed to stochastic perturbations of white noise type $(\dot{w}_1(t), \dot{w}_2(t))$ that are directly proportional to the deviation of system

(12.50) state $(N(t), S(t))$ from the equilibrium point (N^*, S^*), i.e.,

$$\begin{aligned}
\dot{N}(t) &= \mu N_0 - \mu N(t) - \beta N(t) \int_0^\infty \big(1 - N(t-s)\big)\, dK(s) + \rho S(t) \\
&\quad + \sigma_1 \big(N(t) - N^*\big)\dot{w}_1(t), \qquad t \ge 0, \\
\dot{S}(t) &= \mu S_0 + \beta N(t) \int_0^\infty \big(1 - N(t-s)\big)\, dK(s) - (\mu + \gamma + \rho) S(t) \\
&\quad + \varepsilon \big(1 - N(t) - S(t)\big) + \sigma_2 \big(S(t) - S^*\big)\dot{w}_2(t), \quad t \ge 0, \\
N(s) &= N_0,\ s \le 0,\ S(0) = S_0.
\end{aligned} \tag{12.57}$$

Here $w_1(t)$, $w_2(t)$ are mutually independent standard Wiener processes, and the stochastic differential equations of system (12.57) are understood in the Itô sense (Sect. 2.1.2). Note that the equilibrium point (N^*, S^*) of system (12.50) is a solution of (12.57) too.

To centralize system (12.57) at the equilibrium point, put now $x_1 = N - N^*$, $x_2 = S - S^*$. Then by (12.57), (12.53) we have

$$\begin{aligned}
\dot{x}_1(t) &= a_{11}x_1(t) + a_{12}x_2(t) + \beta N^* I(x_{1t}) + \beta x_1(t) I(x_{1t}) \\
&\quad + \sigma_1 x_1(t)\dot{w}_1(t), \\
\dot{x}_2(t) &= a_{21}x_1(t) + a_{22}x_2(t) - \beta N^* I(x_{1t}) - \beta x_1(t) I(x_{1t}) \\
&\quad + \sigma_2 x_2(t)\dot{w}_2(t),
\end{aligned} \tag{12.58}$$

where

$$\begin{gathered}
a_{11} = -\mu - \beta\big(1 - N^*\big), \qquad a_{12} = \rho, \\
a_{21} = -\varepsilon + \beta\big(1 - N^*\big), \qquad a_{22} = -k^{-1}\rho, \\
I(x_{1t}) = \int_0^\infty x_1(t-s)\, dK(s).
\end{gathered} \tag{12.59}$$

Example 12.7 Using the values of the parameters from Example 12.6, by (12.59) we obtain

$$\begin{aligned}
a_{11} &= -0.0010376, \qquad a_{12} = 0.000035, \\
a_{21} &= 0.0005646, \qquad a_{22} = -0.000808.
\end{aligned}$$

It is clear that the stability of the equilibrium point of system (12.57) is equivalent to the stability of the zero solution of (12.58). Rejecting the nonlinear terms in (12.58), we obtain the linear part of system (12.58)

$$\begin{aligned}
\dot{y}_1(t) &= a_{11}y_1(t) + a_{12}y_2(t) + \beta N^* I(y_{1t}) + \sigma_1 y_1(t)\dot{w}_1(t), \\
\dot{y}_2(t) &= a_{21}y_1(t) + a_{22}y_2(t) - \beta N^* I(y_{1t}) + \sigma_2 y_2(t)\dot{w}_2(t).
\end{aligned} \tag{12.60}$$

12.2.4 Stability of an Equilibrium Point

Note that the nonlinear system (12.58) has the order of nonlinearity higher than one. Thus, sufficient conditions for the asymptotic mean-square stability of the zero solution of the linear part (12.60) at the same time are (Sect. 5.3) sufficient conditions for the stability in probability of the zero solution of the nonlinear system (12.58) and therefore are sufficient conditions for the stability in probability of the solution (N^*, S^*) of system (12.57).

To get sufficient conditions for the asymptotic mean-square stability of the zero solution of (12.60), rewrite this system in the form

$$\dot{y}(t) = Ay(t) + B(y_t) + \sigma\big(y(t)\big)\dot{w}(t), \tag{12.61}$$

where

$$\begin{gathered} y(t) = \big(y_1(t), y_2(t)\big)', \qquad w(t) = \big(w_1(t), w_2(t)\big)', \\ B(y_t) = \big(\beta N^* I(y_{1t}), -\beta N^* I(y_{1t})\big)', \\ A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \qquad \sigma\big(y(t)\big) = \begin{pmatrix} \sigma_1 y_1(t) & 0 \\ 0 & \sigma_2 y_2(t) \end{pmatrix}, \end{gathered} \tag{12.62}$$

and a_{ij}, $i, j = 1, 2$, are defined by (12.59).

Following the procedure of constructing Lyapunov functionals, for stability investigation of (12.61), consider the auxiliary differential equation without memory

$$\dot{z}(t) = Az(t) + \sigma\big(z(t)\big)\dot{w}(t). \tag{12.63}$$

Remark 12.3 By (12.62), (12.59) for the matrix A, conditions (2.62) hold:

$$\begin{aligned} \mathrm{Tr}(A) &= -\big[\mu + \rho k^{-1} + \beta\big(1 - N^*\big)\big] < 0, \\ \det(A) &= \rho k^{-1}\big[\mu + k\varepsilon + \beta(1-k)\big(1 - N^*\big)\big] > 0. \end{aligned} \tag{12.64}$$

Example 12.8 Using the values of the parameters from Example 12.6, we have

$$\mathrm{Tr}(A) = -0.0018456, \qquad \det(A) = 0.0000008.$$

Consider $\hat{a}_{ii}$, $i = 1, 2$, defined in (12.27).

Lemma 12.5 *If*

$$a_{21} \le 0 \tag{12.65}$$

and

$$\hat{a}_{11} < 0, \qquad \hat{a}_{22} < 0, \tag{12.66}$$

then the zero solution of (12.63) *is asymptotically mean-square stable.*

Proof By (12.59), (12.65) the matrix A from (12.63) satisfies the condition $a_{12}a_{21} \leq 0$. By (12.19) the same condition is satisfied by the matrix A from (12.23). So, further, the proof coincides with that of Lemma 12.2. □

Lemma 12.6 *If*

$$a_{21} > 0 \tag{12.67}$$

and

$$\max(\delta_1, \delta_2) < \frac{\det(A)}{|\operatorname{Tr}(A)|}, \tag{12.68}$$

then the zero solution of (12.63) *is asymptotically mean-square stable.*

Proof Similarly to Lemma 12.2, it is enough to show that the matrix $P = \|p_{ij}\|$ with the elements (12.26) is positive definite.

Note that by (12.31), (12.59), (12.67) we have

$$A_i = a_{11}a_{22} - a_{12}a_{21} + a_{ii}^2 \leq a_{ii} \operatorname{Tr}(A), \quad i = 1, 2.$$

From this and from (12.59), (12.64), (12.68) it follows that

$$\delta_i < \frac{\det(A)}{|\operatorname{Tr}(A)|} \leq \frac{A_i}{|\operatorname{Tr}(A)|} \leq |a_{ii}|, \quad i = 1, 2. \tag{12.69}$$

By (12.27), (12.28), (12.59), (12.64), (12.67), (12.68) we have

$$\begin{aligned}
Z &= \operatorname{Tr}(A)(a_{11} + \delta_1)(a_{22} + \delta_2) - a_{12}a_{21}\big(\operatorname{Tr}(A) + \delta_1 + \delta_2\big) \\
&= \operatorname{Tr}(A)\det(A) + \operatorname{Tr}(A)\delta_1 a_{22} + \operatorname{Tr}(A)\delta_2 a_{11} \\
&\quad + \operatorname{Tr}(A)\delta_1\delta_2 - a_{12}a_{21}(\delta_1 + \delta_2) \\
&= -\big|\operatorname{Tr}(A)\big|\det(A) + A_2\delta_1 + A_1\delta_2 - \big|\operatorname{Tr}(A)\big|\delta_1\delta_2 \\
&< -\big|\operatorname{Tr}(A)\big|\det(A) + (A_1 + A_2)\frac{\det(A)}{|\operatorname{Tr}(A)|} \\
&= -\big(\big|\operatorname{Tr}(A)\big|^2 - A_1 - A_2\big)\frac{\det(A)}{|\operatorname{Tr}(A)|} \\
&= -\frac{2a_{12}a_{21}\det(A)}{|\operatorname{Tr}(A)|} \\
&< 0.
\end{aligned} \tag{12.70}$$

From (12.34), (12.35), (12.69), (12.70) we obtain that $p_{11} > 0$, $p_{22} > 0$ for arbitrary $c > 0$.

Let us show that $p_{11}p_{22} > p_{12}^2$. Indeed, by (12.26), (12.34), (12.35) this inequality takes the form

$$\big(c\big(A_2 - \big|\operatorname{Tr}(A)\big|\delta_2\big) + a_{21}^2\big)\big(A_1 - \big|\operatorname{Tr}(A)\big|\delta_1 + ca_{12}^2\big) > (ca_{12}\hat{a}_{22} + a_{21}\hat{a}_{11})^2,$$

which is equivalent to the condition

$$
\begin{aligned}
& c^2 a_{12}^2\big(\det(A) - \big|\mathrm{Tr}(A)\big|\delta_2 + a_{22}^2 - \hat{a}_{22}^2\big) \\
&\quad + c\big[\big(A_1 - \big|\mathrm{Tr}(A)\big|\delta_1\big)\big(\det(A) - \big|\mathrm{Tr}(A)\big|\delta_2\big) \\
&\quad + a_{22}^2\big(\det(A) - \big|\mathrm{Tr}(A)\big|\delta_1\big) \\
&\quad + \big(\det(A)\big)^2 + 2a_{12}a_{21}(a_{11}a_{22} - \hat{a}_{11}\hat{a}_{22})\big] \\
&\quad + a_{21}^2\big(\det(A) - \big|\mathrm{Tr}(A)\big|\delta_1 + a_{11}^2 - \hat{a}_{11}^2\big) \\
& > 0. \qquad (12.71)
\end{aligned}
$$

By (12.67), (12.68), (12.69) and $|a_{ii}| \ge |\hat{a}_{ii}|$, $i = 1, 2$, condition (12.71) holds for arbitrary $c > 0$. So, for arbitrary $c > 0$, the matrix P with the elements (12.26) is positive definite. The proof is completed. □

Remark 12.4 If condition (12.65) holds, i.e., $a_{21} \le 0$, then from (12.59) and from the proofs of Lemmas 12.3 and 12.4 it follows that $\beta \in [0, (\mu + \varepsilon)(1 - k)^{-1}]$. On the other hand, if $\beta > (\mu + \varepsilon)(1 - k)^{-1}$, then condition (12.67) holds, i.e., $a_{21} > 0$. For example, by the values of the parameters from Example 12.6 we have $\beta = 0.00085 > (\mu + \varepsilon)(1 - k)^{-1} = 0.0004945$ and $a_{21} = 0.0005646 > 0$.

Theorem 12.2 *If conditions* (12.65), (12.66) *or* (12.67), (12.68) *hold and if, for some* $c > 0$, *the elements* (12.26) *of the matrix* P *satisfy the condition*

$$
\big(\beta N^*|p_{12} - p_{22}|\big)^2 + 2\beta N^*|p_{11} - p_{12}| < c, \qquad (12.72)
$$

then the solution (N^*, S^*) *of system* (12.57) *is stable in probability.*

Proof Note that the stability in probability of the solution (N^*, S^*) of system (12.57) is equivalent to the stability in probability of the zero solution of system (12.58) and the order of nonlinearity of system (12.58) is higher than one. So, to get for this system conditions for stability in probability, it is enough (Sect. 5.3) to get conditions for the asymptotic mean-square stability of the zero solution of the linear part (12.60) of this system. Following the procedure of constructing Lyapunov functionals, we will construct a Lyapunov functional for system (12.60) in the form $V = V_1 + V_2$, where $V_1 = y'Py$, $y = (y_1, y_2)'$, P is the positive definite solution of system (12.25) with the elements (12.26), and V_2 will be chosen below.

Let L be the generator of system (12.60). Then by (12.60), (12.25) we have

$$
\begin{aligned}
LV_1 &= 2\big(p_{11}y_1(t) + p_{12}y_2(t)\big)\big(a_{11}y_1(t) + a_{12}y_2(t) + \beta N^* I(y_{1t})\big) + p_{11}\sigma_1^2 y_1^2(t) \\
&\quad + 2\big(p_{12}y_1(t) + p_{22}y_2(t)\big)\big(a_{21}y_1(t) + a_{22}y_2(t) - \beta N^* I(y_{1t})\big) \\
&\quad + p_{22}\sigma_2^2 y_2^2(t) \\
&= -cy_1^2(t) - y_2^2(t) + 2\beta N^*\big[(p_{11} - p_{12})y_1(t) + (p_{12} - p_{22})y_2(t)\big]I(y_{1t}).
\end{aligned}
$$

By (12.2), (12.59) we have $2y_1(t)I(y_{1t}) \le y_1^2(t) + I(y_{1t}^2)$ and $2y_2(t)I(y_{1t}) \le \nu y_2^2(t) + \nu^{-1}I(y_{1t}^2)$ for some $\nu > 0$. Using these inequalities, we obtain

$$\begin{aligned}
LV_1 &\le -cy_1^2(t) - y_2^2(t) + \beta N^*|p_{11} - p_{12}|\big(y_1^2(t) + I\big(y_{1t}^2\big)\big) \\
&\quad + \beta N^*|p_{12} - p_{22}|\big(\nu y_2^2(t) + \nu^{-1}I\big(y_{1t}^2\big)\big) \\
&= \big(\beta N^*|p_{11} - p_{12}| - c\big)y_1^2(t) + \big(\beta N^*|p_{12} - p_{22}|\nu - 1\big)y_2^2(t) \\
&\quad + qI\big(y_{1t}^2\big),
\end{aligned} \tag{12.73}$$

where

$$q = \beta N^*\big(|p_{11} - p_{12}| + |p_{12} - p_{22}|\nu^{-1}\big). \tag{12.74}$$

Putting

$$V_2 = q\int_0^\infty \int_{t-s}^t y_1^2(\theta)\,d\theta\,dK(s),$$

by (12.2), (12.59) we get $LV_2 = q(y_1^2(t) - I(y_{1t}^2))$. Therefore, by (12.73), (12.74), for the functional $V = V_1 + V_2$, we have

$$\begin{aligned}
LV &\le \big(2\beta N^*|p_{11} - p_{12}| + \beta N^*|p_{12} - p_{22}|\nu^{-1} - c\big)y_1^2(t) \\
&\quad + \big(\beta N^*|p_{12} - p_{22}|\nu - 1\big)y_2^2(t).
\end{aligned}$$

Thus, if

$$2\beta N^*|p_{11} - p_{12}| + \beta N^*|p_{12} - p_{22}|\nu^{-1} < c, \quad \beta N^*|p_{12} - p_{22}|\nu < 1, \tag{12.75}$$

then by Remark 2.1 the zero solution of (12.60) is asymptotically mean-square stable.

From (12.75) it follows that

$$\frac{\beta N^*|p_{12} - p_{22}|}{c - 2\beta N^*|p_{11} - p_{12}|} < \nu < \frac{1}{\beta N^*|p_{12} - p_{22}|}. \tag{12.76}$$

Thus, if for some $c > 0$, condition (12.72) holds, then there exists $\nu > 0$ such that conditions (12.76) (or (12.75)) hold too, and therefore the zero solution of (12.60) is asymptotically mean-square stable. From this it follows that the zero solution of (12.58) and therefore the equilibrium point (N^*, S^*) of system (12.57) is stable in probability. The proof is completed. □

Example 12.9 Consider system (12.50) with the values of the parameters ε, μ, ρ, β, γ and the equilibrium point (N^*, S^*) given in Example 12.6. As an example, consider the levels of noises $\sigma_1 = 0.028256$, $\sigma_2 = 0.029031$. From (12.27) it follows that $\delta_1 = 0.0003992$, $\delta_2 = 0.0004214$ and condition (12.68) holds: $\max(\delta_1, \delta_2) < \det(A)|\operatorname{Tr}(A)|^{-1} = 0.0004436$.

Put $c = 10$. Then by (12.26) $p_{11} = 8335.7$, $p_{12} = 569.4$, $p_{22} = 1344.7$, and condition (12.72) holds:

$$\left(\beta N^*|p_{12} - p_{22}|\right)^2 + 2\beta N^*|p_{11} - p_{12}| = 4.419 < 10.$$

Thus, the solution (N^*, S^*) of system (12.57) is stable in probability.

Using the representations (12.41)–(12.44), we get three corollaries from Theorem 12.2, which simplify a verification of the stability condition (12.72).

Put

$$f(c) = \left(\beta N^* D_0\right)^2 \left(c + \frac{D_1}{D_0}\right)^2 + 2\beta N^*|B_0| \left| c + \frac{B_1}{B_0} \right| - c, \tag{12.77}$$

$$\begin{aligned} S &= \left(\beta N^* D_0\right)^2 \left(\frac{D_1}{D_0} - \frac{B_1}{B_0}\right)^2 + \frac{B_1}{B_0}, \\ R_+ &= 2\beta N^*|B_0| \left(\frac{1 - 2\beta N^*|B_0|}{2(\beta N^* D_0)^2} - \frac{D_1}{D_0} + \frac{B_1}{B_0} \right), \\ R_- &= -2\beta N^*|B_0| \left(\frac{1 + 2\beta N^*|B_0|}{2(\beta N^* D_0)^2} - \frac{D_1}{D_0} + \frac{B_1}{B_0} \right), \\ Q &= \frac{1}{4(\beta N^* D_0)^2} - \frac{D_1}{D_0} - \frac{B_0^2}{D_0^2}, \end{aligned} \tag{12.78}$$

where B_0, B_1, D_0, D_1 are defined by (12.41)–(12.44). So, condition (12.72) is equivalent to the condition $f(c) < 0$.

Corollary 12.4 *If conditions* (12.65), (12.66) *or* (12.67), (12.68) *hold and* $S < 0$, *then the solution* (N^*, S^*) *of system* (12.57) *is stable in probability.*

Proof By (12.78) from $S < 0$ it follows that $B_1 B_0^{-1} < 0$. Substituting $c_0 = -B_1 B_0^{-1} > 0$ into (12.77), we obtain $f(c_0) = S < 0$, i.e., condition (12.72) holds. The proof is completed. □

Corollary 12.5 *If conditions* (12.65), (12.66) *or* (12.67), (12.68) *hold and* $0 \le R_+ < Q$, *then the solution* (N^*, S^*) *of system* (12.57) *is stable in probability.*

Corollary 12.6 *If conditions* (12.65), (12.66) *or* (12.67), (12.68) *hold and* $0 < R_- < Q$, *then the solution* (N^*, S^*) *of system* (12.57) *is stable in probability.*

The proofs of Corollaries 12.5 and 12.6 are similar to those of Corollaries 12.2 and 12.3.

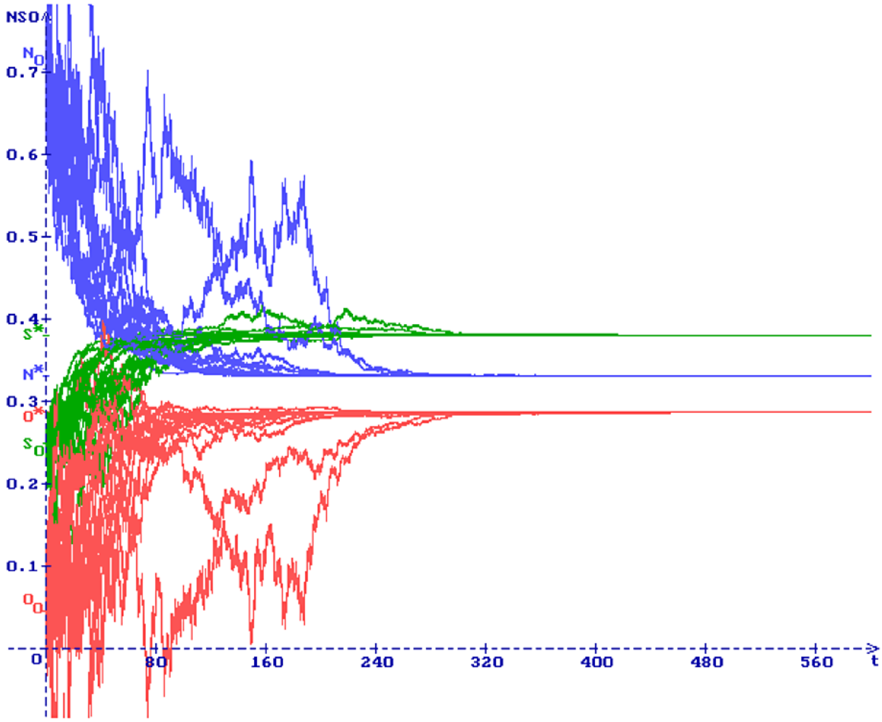

Fig. 12.2 25 trajectories of the processes $N(t)$ (*blue*), $S(t)$ (*green*), $O(t)$ (*red*) with the values of the parameters $\mu = 0.000469$, $\gamma = 0.0003$, $\varepsilon = 0.000004$, $\rho = 0.000035$, $\beta = 0.00085$, $h = 0.1$, $\delta_1 = 0.0003992$, $\delta_2 = 0.0004214$, the initial values $N(s) = 0.704$, $s \in [-0.1, 0]$, $S(0) = 0.25$, $O(0) = 0.046$, and the equilibrium point $N^* = 0.3311$, $S^* = 0.3814$, $O^* = 0.2875$

Example 12.10 Consider system (12.57) with the values of the parameters from Example 12.6 and $\delta_1 = 0.0003992$, $\delta_2 = 0.0002661$. Calculating S, R_+, Q by (12.78), we obtain: $S = -0.0100916 < 0$, $R_+ = 7499 < Q = 18161$. By both Corollaries 12.4 and 12.5 the solution (N^*, S^*) of system (12.57) is stable in probability.

Example 12.11 Consider system (12.57) with the values of the parameters from Example 12.6 and $\delta_1 = 0.0003992$, $\delta_2 = 0.0004214$. Calculating S, R_+, Q by (12.78), we obtain: $S = 0.0051611 > 0$, $R_+ = 7811 < Q = 18914$. The condition of Corollary 12.4 does not hold, but from Corollary 12.5 it follows that the solution (N^*, S^*) of system (12.57) is stable in probability.

12.2.5 Numerical Simulation

Let us suppose that in (12.49) $dK(s) = \delta(s - h)\,ds$, where $\delta(s)$ is Dirac's delta-function, and $h \geq 0$ is a delay.

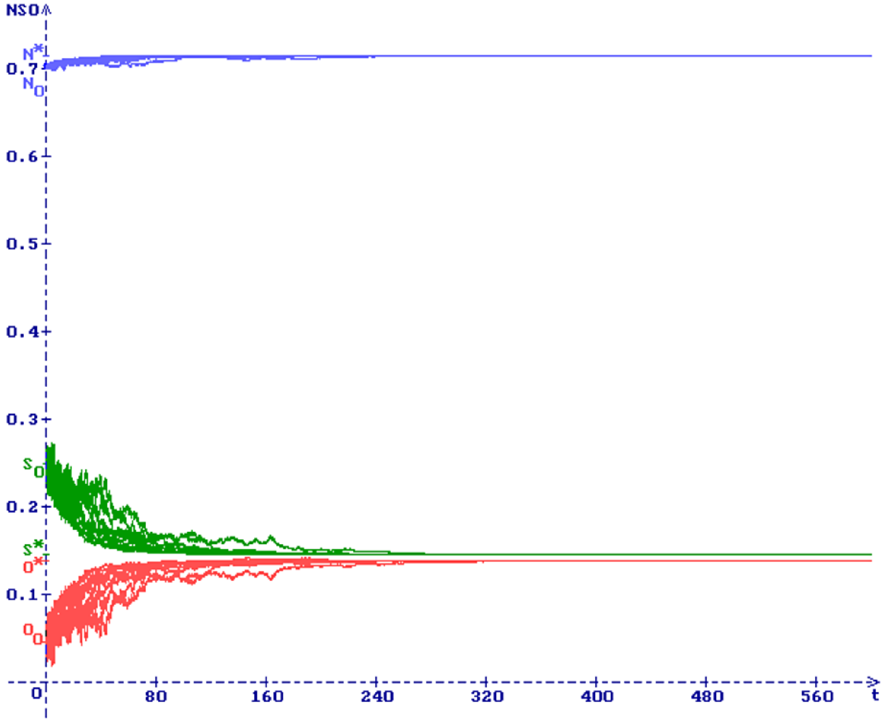

Fig. 12.3 25 trajectories of the processes $N(t)$ (*blue*), $S(t)$ (*green*), $O(t)$ (*red*) with the values of the parameters $\mu = 0.000469$, $\gamma = 0.0003$, $\varepsilon = 0.000004$, $\rho = 0.000035$, $\beta = 0$, $h = 0.1$, $\delta_1 = 0.0003992$, $\delta_2 = 0.0004214$, the initial values $N(s) = 0.704$, $s \in [-0.1, 0]$, $S(0) = 0.25$, $O(0) = 0.046$, and the equilibrium point $N^* = 0.7149$, $S^* = 0.1465$, $O^* = 0.1386$

In Fig. 12.2 25 trajectories of the solution of (12.57), (12.48) are shown for the values of the parameters from Examples 12.6 and 12.9: $\mu = 0.000469$, $\gamma = 0.0003$, $\varepsilon = 0.000004$, $\rho = 0.000035$, $\beta = 0.00085$, the initial values $N_0 = 0.704$, $S_0 = 0.25$, $O_0 = 0.046$, the levels of noises $\sigma_1 = 0.028256$, $\sigma_2 = 0.029031$, and the delay $h = 0.1$. One can see that all trajectories go to the equilibrium point $N^* = 0.3311$, $S^* = 0.3814$, $O^* = 0.2875$.

Putting $\beta = 0$ with the same values of the other parameters, one can see that in accordance with Example 12.6, all trajectories go to another equilibrium point $N^* = 0.7149$, $S^* = 0.1465$, $O^* = 0.1386$ (Fig. 12.3).

Change now the initial values on $N_0 = 1$, $S_0 = O_0 = 0$, and put again $\beta = 0.00085$. In accordance with Example 12.6, corresponding trajectories of the solution go to the equilibrium point $N^* = 0.5770$, $S^* = 0.2588$, $O^* = 0.1642$ (Fig. 12.4).

Note that numerical simulations of the processes $N(t)$, $S(t)$, and $O(t)$ were obtained via the difference analogues of (12.57), (12.48) in the form

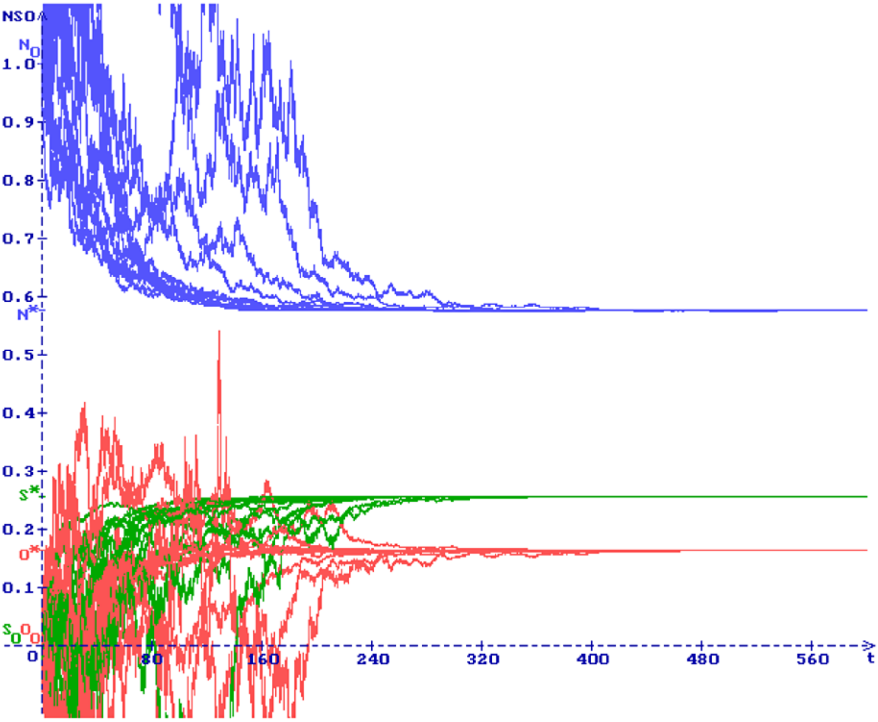

Fig. 12.4 25 trajectories of the processes $N(t)$ (*blue*), $S(t)$ (*green*), $O(t)$ (*red*) with the values of the parameters $\mu = 0.000469$, $\gamma = 0.0003$, $\varepsilon = 0.000004$, $\rho = 0.000035$, $\beta = 0.00085$, $h = 0.1$, $\delta_1 = 0.0003992$, $\delta_2 = 0.0004214$, the initial values $N(s) = 1$, $s \in [-0.1, 0]$, $S(0) = 0$, $O(0) = 0$, and the equilibrium point $N^* = 0.5770$, $S^* = 0.2588$, $O^* = 0.1642$

$$
\begin{aligned}
N_{i+1} &= N_i + \Delta\big[\mu N_0 - \mu N_i - \beta N_i(1 - N_{i-m}) + \rho S_i\big] \\
&\quad + \sigma_1\big(N_i - N^*\big)(w_{1,i+1} - w_{1i}), \\
S_{i+1} &= S_i + \Delta\big[\mu S_0 + \beta N_i N_{i-m} - (\mu + \gamma + \rho)S_i + \varepsilon(1 - N_i - S_i)\big] \\
&\quad + \sigma_2\big(S_i - S^*\big)(w_{2,i+1} - w_{2i}), \\
O_{i+1} &= 1 - N_{i+1} - S_{i+1}, \\
&\quad i = 0, 1, 2, \dots, \ N_j = N_0, \ j = -m, \dots, -1, 0.
\end{aligned}
$$

Here Δ is the discretization step (chosen as $\Delta = 0.01$), $N_i = N(t_i)$, $S_i = S(t_i)$, $O_i = O(t_i)$, $w_{ki} = w_k(t_i)$, $k = 1, 2$, $t_i = i\Delta$, $m = h\Delta^{-1}$, and trajectories of the Wiener processes $w_1(t)$ and $w_2(t)$ are simulated by the algorithm described in Sect. 2.1.1.