Chapter 10
Stability of Positive Equilibrium Point of Nonlinear System of Type of Predator–Prey with Aftereffect and Stochastic Perturbations

Here we consider a system of two nonlinear differential equations that is destined to unify different known mathematical models, in particular, very often investigated models of predator–prey type [47, 53, 60, 65, 72, 82, 83, 94, 107, 108, 112, 113, 127, 128, 153, 180, 235, 249, 267, 283, 288, 304, 305, 311, 314, 317, 321, 325]. The system under consideration is exposed to stochastic perturbations and is linearized in a neighborhood of the positive point of equilibrium. Asymptotic mean-square stability conditions for the trivial solution of the constructed linear system are at the same time sufficient conditions for the stability in probability of the positive equilibrium point of the initial nonlinear system by stochastic perturbations.

10.1 System Under Consideration

Consider the system of two nonlinear differential equations

$$\begin{aligned}
\dot{x}_1(t) &= x_1(t)\big(a - F_0(x_{1t}, x_{2t})\big) - F_1(x_{1t}, x_{2t}),\\
\dot{x}_2(t) &= -x_2(t)\big(b + G_0(x_{1t}, x_{2t})\big) + G_1(x_{1t}, x_{2t}),\\
x_i(s) &= \phi_i(s), \quad s \le 0,\ i = 1, 2.
\end{aligned} \tag{10.1}$$

Here $x_i(t)$, $i = 1, 2$, is the value of the process x_i at time t, and $x_{it} = x_i(t+s)$, $s \le 0$, is a trajectory of the process x_i to the point of time t.

Put, for example,

$$\begin{aligned}
F_0(x_{1t}, x_{2t}) &= \int_0^\infty f_0\big(x_1(t-s)\big)\,dK_0(s),\\
F_1(x_{1t}, x_{2t}) &= \prod_{i=1}^{2} \int_0^\infty f_i\big(x_i(t-s)\big)\,dK_i(s),
\end{aligned} \tag{10.2}$$

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$$G_0(x_{1t}, x_{2t}) = \int_0^\infty g_0\big(x_1(t-s)\big)\,dR_0(s),$$
$$G_1(x_{1t}, x_{2t}) = \prod_{i=1}^{2}\int_0^\infty g_i\big(x_i(t-s)\big)\,dR_i(s),$$

where $K_i(s)$ and $R_i(s)$, $i = 0, 1, 2$, are nondecreasing functions such that

$$\begin{aligned} K_i &= \int_0^\infty dK_i(s) < \infty, \qquad & R_i &= \int_0^\infty dR_i(s) < \infty, \\ \hat{K}_i &= \int_0^\infty s\,dK_i(s) < \infty, \qquad & \hat{R}_i &= \int_0^\infty s\,dR_i(s) < \infty, \end{aligned} \tag{10.3}$$

and all integrals are understood in the Stieltjes sense.

In the case (10.2)–(10.3) system (10.1) takes the form

$$\begin{aligned} \dot{x}_1(t) =& x_1(t)\bigg(a - \int_0^\infty f_0\big(x_1(t-s)\big)\,dK_0(s)\bigg) - \prod_{i=1}^{2}\int_0^\infty f_i\big(x_i(t-s)\big)\,dK_i(s), \\ \dot{x}_2(t) =& -x_2(t)\bigg(b + \int_0^\infty g_0\big(x_1(t-s)\big)\,dR_0(s)\bigg) + \prod_{i=1}^{2}\int_0^\infty g_i\big(x_i(t-s)\big)\,dR_i(s). \end{aligned} \tag{10.4}$$

Systems of type (10.4) are investigated in some biological problems. Put here, for example,

$$\begin{gathered} f_0(x) = f_1(x) = f_2(x) = g_1(x) = g_2(x) = x, \\ g_0(x) = 0, \qquad dK_1(s) = \delta(s)\,ds, \qquad dR_0(s) = 0 \end{gathered} \tag{10.5}$$

($\delta(s)$ is Dirac's function). If a and b are positive constants, $x_1(t)$ and $x_2(t)$ are respectively the densities of prey and predator populations, then (10.4) is transformed to the mathematical predator–prey model [267] with distributed delay

$$\begin{aligned} \dot{x}_1(t) &= x_1(t)\bigg(a - \int_0^\infty x_1(t-s)\,dK_0(s) - \int_0^\infty x_2(t-s)\,dK_2(s)\bigg), \\ \dot{x}_2(t) &= -bx_2(t) + \int_0^\infty x_1(t-s)\,dR_1(s)\int_0^\infty x_2(t-s)\,dR_2(s). \end{aligned} \tag{10.6}$$

Putting in (10.6)

$$\begin{aligned} dK_0(s) &= a_1\delta(s)\,ds, \qquad & dK_2(s) &= a_2\delta(s)\,ds, \\ dR_1(s) &= b_1\delta(s-h_1)\,ds, \qquad & dR_2(s) &= \delta(s-h_2)\,ds, \end{aligned} \tag{10.7}$$

we obtain the known predator–prey mathematical model with fixed delays

$$\begin{aligned} \dot{x}_1(t) &= x_1(t)\big(a - a_1x_1(t) - a_2x_2(t)\big), \\ \dot{x}_2(t) &= -bx_2(t) + b_1x_1(t-h_1)x_2(t-h_2). \end{aligned} \tag{10.8}$$

If here $h_1 = h_2 = 0$, we have the classical Lotka–Volterra model

$$\begin{aligned} \dot{x}_1(t) &= x_1(t)\big(a - a_1x_1(t) - a_2x_2(t)\big), \\ \dot{x}_2(t) &= x_2(t)\big(-b + b_1x_1(t)\big). \end{aligned}$$

Many authors [15, 19, 23, 50, 69, 70, 116, 306, 309] consider the so-called ratio-dependent predator–prey models with delays of type

$$\begin{aligned} \dot{x}_1(t) &= x_1(t)\left(a - \int_0^\infty x_1(t-s)\,dK_0(s)\right) \\ &\quad - \int_0^\infty \frac{x_1^k(t-s)x_2(t)}{x_1^k(t-s) + a_2x_2^k(t-s)}\,dK_1(s), \\ \dot{x}_2(t) &= -bx_2(t) + \int_0^\infty \frac{x_1^m(t-s)x_2(t)}{x_1^m(t-s) + b_2x_2^m(t-s)}\,dR_1(s). \end{aligned} \tag{10.9}$$

Here it is supposed that m and k are positive constants.

System (10.9) follows from (10.1) if

$$\begin{aligned} F_0(x_{1t}, x_{2t}) &= \int_0^\infty x_1(t-s)\,dK_0(s), \qquad G_0(x_{1t}, x_{2t}) = 0, \\ F_1(x_{1t}, x_{2t}) &= \int_0^\infty f\big(x_1(t-s), x_2(t-s)\big)x_2(t)\,dK_1(s), \\ G_1(x_{1t}, x_{2t}) &= \int_0^\infty g\big(x_1(t-s), x_2(t-s)\big)x_2(t)\,dR_1(s), \\ f(x_1, x_2) &= \frac{x_1^k}{x_1^k + a_2x_2^k}, \qquad g(x_1, x_2) = \frac{x_1^m}{x_1^m + b_2x_2^m}. \end{aligned} \tag{10.10}$$

Putting in (10.9), for example,

$$\begin{aligned} dK_0(s) &= a_0\delta(s)\,ds, \qquad dK_1(s) = a_1\delta(s)\,ds, \\ dR_1(s) &= b_1\delta(s-h)\,ds, \qquad k = m = 1, \end{aligned} \tag{10.11}$$

we obtain the system

$$\begin{aligned} \dot{x}_1(t) &= x_1(t)\left(a - a_0x_1(t) - \frac{a_1x_2(t)}{x_1(t) + a_2x_2(t)}\right), \\ \dot{x}_2(t) &= x_2(t)\left(-b + \frac{b_1x_1(t-h)}{x_1(t-h) + b_2x_2(t-h)}\right), \end{aligned} \tag{10.12}$$

which was considered in [23, 50].

10.2 Equilibrium Points, Stochastic Perturbations, Centering, and Linearization

10.2.1 Equilibrium Points

Let in system (10.1) $F_i = F_i(\phi, \psi)$ and $G_i = G_i(\phi, \psi)$, $i = 0, 1$, be functionals defined on $H \times H$, where H is a set of functions $\phi = \phi(s)$, $s \le 0$, with the norm $\|\phi\| = \sup_{s\le 0} |\phi(s)|$, the functionals F_i and G_i are nonnegative for nonnegative functions ϕ and ψ. Let us suppose also that system (10.1) has a positive equilibrium point (x_1^*, x_2^*). This point is obtained from the conditions $\dot{x}_1(t) \equiv 0$, $\dot{x}_2(t) \equiv 0$ and is defined by the system of algebraic equations

$$\begin{aligned} x_1^*\big(a - F_0\big(x_1^*, x_2^*\big)\big) &= F_1\big(x_1^*, x_2^*\big), \\ x_2^*\big(b + G_0\big(x_1^*, x_2^*\big)\big) &= G_1\big(x_1^*, x_2^*\big). \end{aligned} \tag{10.13}$$

From (10.13) it follows that system (10.1) has a positive solution by the condition

$$a > F_0\big(x_1^*, x_2^*\big) \tag{10.14}$$

only. For example, if $a > K_0 f_0(x_1^*)$, a positive equilibrium point of system (10.4) is defined by the system of algebraic equations

$$\begin{aligned} x_1^*\big(a - K_0 f_0\big(x_1^*\big)\big) &= K_1 K_2 f_1\big(x_1^*\big) f_2\big(x_2^*\big), \\ x_2^*\big(b + R_0 g_0\big(x_1^*\big)\big) &= R_1 R_2 g_1\big(x_1^*\big) g_2\big(x_2^*\big). \end{aligned} \tag{10.15}$$

In particular, from (10.5), (10.14), (10.15) it follows that system (10.6) has a positive equilibrium point

$$x_1^* = \frac{b}{R_1 R_2}, \qquad x_2^* = \frac{a - K_0 x_1^*}{K_2} = \frac{a - (R_1 R_2)^{-1} K_0 b}{K_2}, \tag{10.16}$$

provided that $a > (R_1 R_2)^{-1} K_0 b$. For system (10.8), from (10.7), (10.16) we obtain

$$x_1^* = \frac{b}{b_1}, \qquad x_2^* = \frac{A}{a_2}, \quad A = a - b\frac{a_1}{b_1} > 0. \tag{10.17}$$

From (10.13), (10.10) it follows that the positive equilibrium point for system (10.9) is

$$x_1^* = \frac{A}{K_0}, \qquad x_2^* = \frac{A}{B K_0}, \quad A = a - \frac{K_1}{B + a_2 B^{1-k}} > 0, \ B = \left(\frac{b b_2}{R_1 - b}\right)^{\frac{1}{m}} > 0.$$

In particular, by (10.11), for system (10.12), it is

$$x_1^* = \frac{A}{a_0}, \qquad x_2^* = \frac{A}{B a_0}, \quad A = a - \frac{a_1}{B + a_2} > 0, \ B = \frac{b b_2}{b_1 - b} > 0. \tag{10.18}$$

10.2.2 Stochastic Perturbations and Centering

Similarly to Sect. 9.2, we will assume that system (10.1) is exposed to stochastic perturbations that are of white noise type and are directly proportional to the deviations of the system state $(x_1(t), x_2(t))$ from the equilibrium point (x_1^*, x_2^*) and influence $\dot{x}_1(t)$, $\dot{x}_2(t)$, respectively. In this way system (10.1) is transformed to the form

$$\begin{aligned} \dot{x}_1(t) &= x_1(t)\big(a - F_0(x_{1t}, x_{2t})\big) - F_1(x_{1t}, x_{2t}) + \sigma_1\big(x_1(t) - x_1^*\big)\dot{w}_1(t), \\ \dot{x}_2(t) &= -x_2(t)\big(b + G_0(x_{1t}, x_{2t})\big) + G_1(x_{1t}, x_{2t}) + \sigma_2\big(x_2(t) - x_2^*\big)\dot{w}_2(t). \end{aligned} \tag{10.19}$$

Here σ_1, σ_2 are constants, and $w_1(t)$, $w_2(t)$ are independent standard Wiener processes.

Centering system (10.19) at the positive point of equilibrium via the new variables $y_1 = x_1 - x_1^*$, $y_2 = x_2 - x_2^*$, we obtain

$$\begin{aligned} \dot{y}_1(t) &= \big(y_1(t) + x_1^*\big)\big(a - F_0\big(y_{1t} + x_1^*, y_{2t} + x_2^*\big)\big) \\ &\quad - F_1\big(y_{1t} + x_1^*, y_{2t} + x_2^*\big) + \sigma_1 y_1(t)\dot{w}_1(t), \\ \dot{y}_2(t) &= -\big(y_2(t) + x_2^*\big)\big(b + G_0\big(y_{1t} + x_1^*, y_{2t} + x_2^*\big)\big) \\ &\quad + G_1\big(y_{1t} + x_1^*, y_{2t} + x_2^*\big) + \sigma_2 y_2(t)\dot{w}_2(t). \end{aligned} \tag{10.20}$$

It is clear that the stability of equilibrium point (x_1^*, x_2^*) of system (10.19) is equivalent to the stability of the trivial solution of system (10.20).

For system (10.4), the representations (10.19) and (10.20) respectively take the forms

$$\begin{aligned} \dot{x}_1(t) &= x_1(t)\left(a - \int_0^\infty f_0\big(x_1(t-s)\big)\,dK_0(s)\right) \\ &\quad - \prod_{i=1}^2 \int_0^\infty f_i\big(x_i(t-s)\big)\,dK_i(s) + \sigma_1\big(x_1(t) - x_1^*\big)\dot{w}_1(t), \\ \dot{x}_2(t) &= -x_2(t)\left(b + \int_0^\infty g_0\big(x_1(t-s)\big)\,dR_0(s)\right) \\ &\quad + \prod_{i=1}^2 \int_0^\infty g_i\big(x_i(t-s)\big)\,dR_i(s) + \sigma_2\big(x_2(t) - x_2^*\big)\dot{w}_2(t) \end{aligned} \tag{10.21}$$

and

$$\begin{aligned} \dot{y}_1(t) &= \big(y_1(t) + x_1^*\big)\left(a - \int_0^\infty f_0\big(y_1(t-s) + x_1^*\big)\,dK_0(s)\right) \\ &\quad - \prod_{i=1}^2 \int_0^\infty f_i\big(y_i(t-s) + x_i^*\big)\,dK_i(s) \end{aligned}$$

$$
\begin{aligned}
&+\sigma_1 y_1(t)\dot{w}_1(t), \\
\dot{y}_2(t) &= -\big(y_2(t)+x_2^*\big)\bigg(b+\int_0^\infty g_0\big(x_1(t-s)+x_1^*\big)\,dR_0(s)\bigg) \\
&+\prod_{i=1}^2\int_0^\infty g_i\big(y_i(t-s)+x_i^*\big)\,dR_i(s)+\sigma_2 y_2(t)\dot{w}_2(t).
\end{aligned}
\tag{10.22}
$$

In particular, for system (10.6), from (10.21), (10.22) by (10.5), (10.16) we obtain

$$
\begin{aligned}
\dot{x}_1(t) &= x_1(t)\bigg(a-\int_0^\infty x_1(t-s)\,dK_0(s)-\int_0^\infty x_2(t-s)\,dK_2(s)\bigg) \\
&+\sigma_1\big(x_1(t)-x_1^*\big)\dot{w}_1(t), \\
\dot{x}_2(t) &= -bx_2(t)+\int_0^\infty x_1(t-s)\,dR_1(s)\int_0^\infty x_2(t-s)\,dR_2(s) \\
&+\sigma_2\big(x_2(t)-x_2^*\big)\dot{w}_2(t)
\end{aligned}
\tag{10.23}
$$

and

$$
\begin{aligned}
\dot{y}_1(t) &= -\big(y_1(t)+x_1^*\big)\bigg(\int_0^\infty y_1(t-s)\,dK_0(s)+\int_0^\infty y_2(t-s)\,dK_2(s)\bigg) \\
&+\sigma_1 y_1(t)\dot{w}_1(t), \\
\dot{y}_2(t) &= -by_2(t)+R_2x_2^*\int_0^\infty y_1(t-s)\,dR_1(s) \\
&+R_1x_1^*\int_0^\infty y_2(t-s)\,dR_2(s) \\
&+\prod_{i=1}^2\int_0^\infty y_i(t-s)\,dR_i(s)+\sigma_2 y_2(t)\dot{w}_2(t).
\end{aligned}
\tag{10.24}
$$

For (10.8), systems (10.23) and (10.24) take respectively the forms

$$
\begin{aligned}
\dot{x}_1(t) &= x_1(t)\big(a-a_1x_1(t)-a_2x_2(t)\big)+\sigma_1\big(x_1(t)-x_1^*\big)\dot{w}_1(t), \\
\dot{x}_2(t) &= -bx_2(t)+b_1x_1(t-h_1)x_2(t-h_2)+\sigma_2\big(x_1(t)-x_2^*\big)\dot{w}_2(t)
\end{aligned}
\tag{10.25}
$$

and

$$
\begin{aligned}
\dot{y}_1(t) &= -\big(y_1(t)+x_1^*\big)\big(a_1y_1(t)+a_2y_2(t)\big)+\sigma_1 y_1(t)\dot{w}_1(t), \\
\dot{y}_2(t) &= -by_2(t)+b_1\big(x_2^*y_1(t-h_1)+x_1^*y_2(t-h_2)\big) \\
&+b_1y_1(t-h_1)y_2(t-h_2)+\sigma_2 y_2(t)\dot{w}_2(t).
\end{aligned}
\tag{10.26}
$$

10.2.3 Linearization

Along with the considered nonlinear system, we will use the linear part of this system. Let us suppose that the functionals in (10.19) have the representations (10.2) with differentiable functions $f_i(x)$, $g_i(x)$, $i = 0, 1, 2$. Using for all these functions the representation

$$f\big(z + x^*\big) = f_0 + f_1 z + o(z), \quad f_0 = f\big(x^*\big), \; f_1 = \frac{df}{dx}\big(x^*\big),$$

and neglecting $o(z)$, we obtain the linear part (process $(z_1(t), z_2(t))$) of system (10.22)

$$\begin{aligned}
\dot z_1(t) &= (a - K_0 f_{00}) z_1(t) - \int_0^\infty z_1(t-s)\, dK(s) \\
&\quad - K_1 f_{10} f_{21} \int_0^\infty z_2(t-s)\, dK_2(s) + \sigma_1 z_1(t) \dot w_1(t), \\
\dot z_2(t) &= -(b + R_0 g_{00}) z_2(t) + \int_0^\infty z_1(t-s)\, dR(s) \\
&\quad + R_1 g_{10} g_{21} \int_0^\infty z_2(t-s)\, dR_2(s) + \sigma_2 z_2(t) \dot w_2(t),
\end{aligned} \tag{10.27}$$

where

$$\begin{aligned}
dK(s) &= K_2 f_{20} f_{11}\, dK_1(s) + f_{01} x_1^*\, dK_0(s), \\
dR(s) &= R_2 g_{20} g_{11}\, dR_1(s) - g_{01} x_2^*\, dR_0(s).
\end{aligned} \tag{10.28}$$

Below we will speak about system (10.27) as about the linear part corresponding to system (10.22) or, for brevity, as about the linear part of system (10.22).

In particular, by conditions (10.5), (10.16), and (10.28) from (10.27) we obtain the linear part of system (10.24)

$$\begin{aligned}
\dot z_1(t) &= -x_1^* \left(\int_0^\infty z_1(t-s)\, dK_0(s) + \int_0^\infty z_2(t-s)\, dK_2(s) \right) + \sigma_1 z_1(t) \dot w_1(t), \\
\dot z_2(t) &= -b z_2(t) + R_2 x_2^* \int_0^\infty z_1(t-s)\, dR_1(s) + R_1 x_1^* \int_0^\infty z_2(t-s)\, dR_2(s) \\
&\quad + \sigma_2 z_2(t) \dot w_2(t).
\end{aligned} \tag{10.29}$$

From (10.26) or, via (10.7), from (10.29) we have the linear part of system (10.26)

$$\begin{aligned}
\dot z_1(t) &= -x_1^* \big(a_1 z_1(t) + a_2 z_2(t) \big) + \sigma_1 z_1(t) \dot w_1(t), \\
\dot z_2(t) &= -b z_2(t) + b_1 \big(x_2^* z_1(t-h_1) + x_1^* z_2(t-h_2) \big) + \sigma_2 z_2(t) \dot w_2(t).
\end{aligned} \tag{10.30}$$

As it is shown in Sect. 5.3, if the order of nonlinearity of the system under consideration is higher than one, then a sufficient condition for the asymptotic mean-square stability of the linear part of the considered nonlinear system is also a sufficient condition for the stability in probability of the initial system. So, below we will obtain sufficient conditions for the asymptotic mean-square stability of the linear part of considered nonlinear systems.

10.3 Stability of Equilibrium Point

Obtain now sufficient conditions for the asymptotic mean-square stability of the trivial solution of system (10.27) as the linear part of (10.22). The obtained conditions will be at the same time sufficient conditions for the stability in probability of the equilibrium point of (10.21).

Following the procedure of constructing Lyapunov functionals (Sect. 2.2.2), rewrite (10.27) in the form

$$\begin{aligned}\dot{Z}_1(t) &= a_{11}z_1(t) + a_{12}z_2(t) + \sigma_1 z_1(t)\dot{w}_1(t),\\ \dot{Z}_2(t) &= a_{21}z_1(t) + a_{22}z_2(t) + \sigma_2 z_2(t)\dot{w}_2(t),\end{aligned} \tag{10.31}$$

where

$$\begin{aligned}Z_1(t) &= z_1(t) - \int_0^\infty \int_{t-s}^t z_1(\theta)\,d\theta\,dK(s) - K_1 f_{10} f_{21} \int_0^\infty \int_{t-s}^t z_2(\theta)\,d\theta\,dK_2(s),\\ Z_2(t) &= z_2(t) + \int_0^\infty \int_{t-s}^t z_1(\theta)\,d\theta\,dR(s) + R_1 g_{10} g_{21} \int_0^\infty \int_{t-s}^t z_2(\theta)\,d\theta\,dR_2(s),\end{aligned} \tag{10.32}$$

and, by (10.15), (10.28),

$$\begin{aligned}a_{11} &= a - K - K_0 f_{00} = K_1 K_2 f_{20}\left(\frac{f_{10}}{x_1^*} - f_{11}\right) - K_0 f_{01} x_1^*,\\ a_{12} &= -K_1 K_2 f_{10} f_{21}, \qquad a_{21} = R = R_1 R_2 g_{20} g_{11} - R_0 g_{01} x_2^*,\\ a_{22} &= R_1 R_2 g_{10} g_{21} - b - R_0 g_{00} = -R_1 R_2 g_{10}\left(\frac{g_{20}}{x_2^*} - g_{21}\right).\end{aligned} \tag{10.33}$$

System (10.31), (10.32) is a system of stochastic differential equations of neutral type, so, following (2.10), we have to suppose that

$$\begin{aligned}\int_0^\infty s\,dK(s) + K_1|f_{10}f_{21}| \int_0^\infty s\,dK_2(s) < 1,\\ \int_0^\infty s\,dR(s) + R_1|g_{10}g_{21}| \int_0^\infty s\,dR_2(s) < 1,\end{aligned}$$

or, by (10.3), (10.28), that

$$
\begin{aligned}
&|f_{01}|x_1^*\hat{K}_0 + K_2|f_{20}f_{11}|\hat{K}_1 + K_1|f_{10}f_{21}|\hat{K}_2 < 1,\\
&|g_{01}|x_2^*\hat{R}_0 + R_2|g_{20}g_{11}|\hat{R}_1 + R_1|g_{10}g_{21}|\hat{R}_2 < 1.
\end{aligned}
\tag{10.34}
$$

10.3.1 First Way of Constructing a Lyapunov Functional

Let $\hat{A} = \|a_{ij}\|$ be the matrix with the elements defined by (10.33), and $P = \|p_{ij}\|$ be the matrix with the elements defined by (1.29) for some $q > 0$. Represent p_{11}, p_{22} in the form

$$
p_{ii} = \frac{1}{2}\big(qp_{ii}^{(0)} + p_{ii}^{(1)}\big), \quad i = 1, 2, \tag{10.35}
$$

where

$$
\begin{aligned}
p_{11}^{(0)} &= \frac{a_{22}^2 + \det(\hat{A})}{|\operatorname{Tr}(\hat{A})|\det(\hat{A})}, & p_{11}^{(1)} &= \frac{a_{21}^2}{|\operatorname{Tr}(\hat{A})|\det(\hat{A})},\\
p_{22}^{(0)} &= \frac{a_{12}^2}{|\operatorname{Tr}(\hat{A})|\det(\hat{A})}, & p_{22}^{(1)} &= \frac{a_{11}^2 + \det(\hat{A})}{|\operatorname{Tr}(\hat{A})|\det(\hat{A})},
\end{aligned}
\tag{10.36}
$$

and put

$$
d\mu_{ij}(s) = q\,d\mu_{ij}^{(0)}(s) + d\mu_{ij}^{(1)}(s), \quad i, j = 1, 2, \tag{10.37}
$$

where

$$
\begin{aligned}
d\mu_{11}^{(0)} &= dK(s) - \frac{a_{12}}{|\operatorname{Tr}(\hat{A})|}dR(s), \qquad d\mu_{11}^{(1)} = \frac{a_{21}}{|\operatorname{Tr}(\hat{A})|}dR(s),\\
d\mu_{12}^{(0)} &= K_1 f_{10} f_{21}\, dK_2(s) - \frac{a_{12}}{|\operatorname{Tr}(\hat{A})|}R_1 g_{10} g_{21}\, dR_2(s),\\
d\mu_{12}^{(1)} &= \frac{a_{21}}{|\operatorname{Tr}(\hat{A})|}R_1 g_{10} g_{21}\, dR_2(s),\\
d\mu_{21}^{(0)} &= -\frac{a_{12}}{|\operatorname{Tr}(\hat{A})|}dK(s), \qquad d\mu_{21}^{(1)} = \frac{a_{21}}{|\operatorname{Tr}(\hat{A})|}dK(s) - dR(s),\\
d\mu_{22}^{(0)} &= -\frac{a_{12}}{|\operatorname{Tr}(\hat{A})|}K_1 f_{10} f_{21}\, dK_2(s),\\
d\mu_{22}^{(1)} &= \frac{a_{21}}{|\operatorname{Tr}(\hat{A})|}K_1 f_{10} f_{21}\, dK_2(s) - R_1 g_{10} g_{21}\, dR_2(s),
\end{aligned}
\tag{10.38}
$$

and $dK(s)$, $dR(s)$ are defined by (10.28).

Put also

$$\delta_i = \frac{1}{2}\sigma_i^2, \quad i = 1, 2,$$
$$\nu_{ij}^{(m)} = \int_0^\infty s\left|d\mu_{ij}^{(m)}(s)\right|, \quad i, j = 1, 2, \ m = 0, 1, \tag{10.39}$$

and

$$\begin{aligned} A_1 &= 1 - \nu_{11}^{(0)} - p_{11}^{(0)}\delta_1, & A_2 &= 1 - \nu_{22}^{(1)} - p_{22}^{(1)}\delta_2, \\ B_1 &= \nu_{11}^{(1)} + p_{11}^{(1)}\delta_1, & B_2 &= \nu_{22}^{(0)} + p_{22}^{(0)}\delta_2, \\ C_1 &= \nu_{12}^{(1)} + \nu_{21}^{(1)}, & C_2 &= \nu_{12}^{(0)} + \nu_{21}^{(0)}. \end{aligned} \tag{10.40}$$

Theorem 10.1 *If* $A_1 > 0$, $A_2 > 0$, *and conditions* (10.34) *and*

$$\sqrt{(A_1C_1 + B_1C_2)(A_2C_2 + B_2C_1)} + B_1B_2 < A_1A_2 \tag{10.41}$$

hold, then the trivial solution of system (10.27) *is asymptotically mean-square stable and the equilibrium point of system* (10.21) *is stable in probability.*

Proof We will consider now system (10.31)–(10.33) and suppose that the trivial solution of the appropriate auxiliary system without delays of type (2.60) with a_{ij}, $i, j = 1, 2$, defined by (10.33) is asymptotically mean-square stable, and so conditions (2.62) hold.

Consider the functional

$$V_1(t) = p_{11}Z_1^2(t) + 2p_{12}Z_1(t)Z_2(t) + p_{22}Z_2^2(t) \tag{10.42}$$

with p_{ij}, $i, j = 1, 2$, defined by (1.29). Let L be the generator of system (10.31). Then, by (10.31), (10.42),

$$\begin{aligned} LV_1(t) &= 2\big(p_{11}Z_1(t) + p_{12}Z_2(t)\big)\big(a_{11}z_1(t) + a_{12}z_2(t)\big) + p_{11}\sigma_1^2 z_1^2(t) \\ &\quad + 2\big(p_{12}Z_1(t) + p_{22}Z_2(t)\big)\big(a_{21}z_1(t) + a_{22}z_2(t)\big) + p_{22}\sigma_2^2 z_2^2(t) \\ &= 2(p_{11}a_{11} + p_{12}a_{21})Z_1(t)z_1(t) + 2(p_{12}a_{11} + p_{22}a_{21})Z_2(t)z_1(t) \\ &\quad + 2(p_{11}a_{12} + p_{12}a_{22})Z_1(t)z_2(t) + p_{11}\sigma_1^2 z_1^2(t) \\ &\quad + 2(p_{12}a_{12} + p_{22}a_{22})Z_2(t)z_2(t) + p_{22}\sigma_2^2 z_2^2(t). \end{aligned} \tag{10.43}$$

Putting

$$\rho = \frac{a_{21} - a_{12}q}{|\operatorname{Tr}(\hat{A})|} \tag{10.44}$$

and using (1.29), (2.62), we obtain

$$
\begin{aligned}
2(p_{11}a_{11}+p_{12}a_{21}) &= -q, \qquad 2(p_{12}a_{12}+p_{22}a_{22}) = -1, \\
2(p_{12}a_{11}+p_{22}a_{21}) &= \frac{-(a_{12}a_{22}q+a_{21}a_{11})a_{11}+(a_{11}^2+\det(\hat{A})+a_{12}^2q)a_{21}}{|\operatorname{Tr}(\hat{A})|\det(\hat{A})} \\
&= \frac{\det(\hat{A})a_{21}-(a_{11}a_{22}-a_{12}a_{21})a_{12}q}{|\operatorname{Tr}(\hat{A})|\det(\hat{A})} = \rho, \\
2(p_{11}a_{12}+p_{12}a_{22}) &= \frac{((a_{22}^2+\det(\hat{A}))q+a_{21}^2)a_{12}-(a_{12}a_{22}q+a_{21}a_{11})a_{22}}{|\operatorname{Tr}(\hat{A})|\det(\hat{A})} \\
&= \frac{\det(A)a_{12}q-a_{21}(a_{11}a_{22}-a_{21}a_{12})}{|\operatorname{Tr}(\hat{A})|\det(\hat{A})} = -\rho.
\end{aligned}
$$

So, (10.43) takes the form

$$
\begin{aligned}
LV_1(t) &= -qZ_1(t)z_1(t)+\rho Z_2(t)z_1(t)+p_{11}\sigma_1^2z_1^2(t) \\
&\quad -\rho Z_1(t)z_2(t)-Z_2(t)z_2(t)+p_{22}\sigma_2^2z_2^2(t).
\end{aligned}
\tag{10.45}
$$

Substituting (10.32) into (10.45), we have

$$
\begin{aligned}
LV_1 = {} & \big(-q+p_{11}\sigma_1^2\big)z_1^2(t)+\big(-1+p_{22}\sigma_2^2\big)z_2^2(t) \\
& + q\int_0^\infty\int_{t-s}^t z_1(t)z_1(\theta)\,d\theta\,dK(s) \\
& + qK_1f_{10}f_{21}\int_0^\infty\int_{t-s}^t z_1(t)z_2(\theta)\,d\theta\,dK_2(s) \\
& + \rho\int_0^\infty\int_{t-s}^t z_1(t)z_1(\theta)\,d\theta\,dR(s) \\
& + \rho R_1g_{10}g_{21}\int_0^\infty\int_{t-s}^t z_1(t)z_2(\theta)\,d\theta\,dR_2(s) \\
& + \rho\int_0^\infty\int_{t-s}^t z_2(t)z_1(\theta)\,d\theta\,dK(s) \\
& + \rho K_1f_{10}f_{21}\int_0^\infty\int_{t-s}^t z_2(t)z_2(\theta)\,d\theta\,dK_2(s) \\
& - \int_0^\infty\int_{t-s}^t z_2(t)z_1(\theta)\,d\theta\,dR(s) \\
& - R_1g_{10}g_{21}\int_0^\infty\int_{t-s}^t z_2(t)z_2(\theta)\,d\theta\,dR_2(s).
\end{aligned}
$$

By (10.37), (10.38), (10.44), it can be written in the form

$$
\begin{aligned}
LV_1 &= \big(-q + p_{11}\sigma_1^2\big) z_1^2(t) + \big(-1 + p_{22}\sigma_2^2\big) z_2^2(t) \\
&\quad + \int_0^\infty \int_{t-s}^t z_1(t) z_1(\theta)\, d\theta\, d\mu_{11}(s) + \int_0^\infty \int_{t-s}^t z_1(t) z_2(\theta)\, d\theta\, d\mu_{12}(s) \\
&\quad + \int_0^\infty \int_{t-s}^t z_2(t) z_1(\theta)\, d\theta\, d\mu_{21}(s) + \int_0^\infty \int_{t-s}^t z_2(t) z_2(\theta)\, d\theta\, d\mu_{22}(s).
\end{aligned}
\tag{10.46}
$$

Using (10.35), (10.37), (10.39) and some positive number γ from (10.46), we obtain

$$
\begin{aligned}
LV_1 &\le \big(-q + qp_{11}^{(0)}\delta_1 + p_{11}^{(1)}\delta_1\big) z_1^2(t) + \big(-1 + qp_{22}^{(0)}\delta_2 + p_{22}^{(1)}\delta_2\big) z_2^2(t) \\
&\quad + \frac{1}{2}\int_0^\infty \int_{t-s}^t \big(z_1^2(t) + z_1^2(\theta)\big)\, d\theta\, \big(q\big|d\mu_{11}^{(0)}(s)\big| + \big|d\mu_{11}^{(1)}(s)\big|\big) \\
&\quad + \frac{1}{2}\int_0^\infty \int_{t-s}^t \big(\gamma^{-1} z_1^2(t) + \gamma z_2^2(\theta)\big)\, d\theta\, \big(q\big|d\mu_{12}^{(0)}(s)\big| + \big|d\mu_{12}^{(1)}(s)\big|\big) \\
&\quad + \frac{1}{2}\int_0^\infty \int_{t-s}^t \big(\gamma z_2^2(t) + \gamma^{-1} z_1^2(\theta)\big)\, d\theta\, \big(q\big|d\mu_{21}^{(0)}(s)\big| + \big|d\mu_{21}^{(1)}(s)\big|\big) \\
&\quad + \frac{1}{2}\int_0^\infty \int_{t-s}^t \big(z_2^2(t) + z_2^2(\theta)\big)\, d\theta\, \big(q\big|d\mu_{22}^{(0)}(s)\big| + \big|d\mu_{22}^{(1)}(s)\big|\big).
\end{aligned}
$$

From this by (10.39) we have

$$
\begin{aligned}
LV_1 &\le \big(-q + qp_{11}^{(0)}\delta_1 + p_{11}^{(1)}\delta_1\big) z_1^2(t) + \big(-1 + qp_{22}^{(0)}\delta_2 + p_{22}^{(1)}\delta_2\big) z_2^2(t) \\
&\quad + \frac{1}{2}\big(q\nu_{11}^{(0)} + \nu_{11}^{(1)}\big) z_1^2(t) + \frac{1}{2}\int_0^\infty \int_{t-s}^t z_1^2(\theta)\, d\theta\, \big(q\big|d\mu_{11}^{(0)}(s)\big| + \big|d\mu_{11}^{(1)}(s)\big|\big) \\
&\quad + \frac{\gamma^{-1}}{2}\big(q\nu_{12}^{(0)} + \nu_{12}^{(1)}\big) z_1^2(t) \\
&\quad + \frac{\gamma}{2}\int_0^\infty \int_{t-s}^t z_2^2(\theta)\, d\theta\, \big(q\big|d\mu_{12}^{(0)}(s)\big| + \big|d\mu_{12}^{(1)}(s)\big|\big) \\
&\quad + \frac{\gamma}{2}\big(q\nu_{21}^{(0)} + \nu_{21}^{(1)}\big) z_2^2(t) \\
&\quad + \frac{\gamma^{-1}}{2}\int_0^\infty \int_{t-s}^t z_1^2(\theta)\, d\theta\, \big(q\big|d\mu_{21}^{(0)}(s)\big| + \big|d\mu_{21}^{(1)}(s)\big|\big) \\
&\quad + \frac{1}{2}\big(q\nu_{22}^{(0)} + \nu_{22}^{(1)}\big) z_2^2(t) + \frac{1}{2}\int_0^\infty \int_{t-s}^t z_2^2(\theta)\, d\theta\, \big(q\big|d\mu_{22}^{(0)}(s)\big| + \big|d\mu_{22}^{(1)}(s)\big|\big)
\end{aligned}
$$

$$
\begin{aligned}
&= \Big[q\Big(-1 + \frac{1}{2}\big(\nu_{11}^{(0)} + \gamma^{-1}\nu_{12}^{(0)}\big) + p_{11}^{(0)}\delta_1 \Big) \\
&\quad + \frac{1}{2}\big(\nu_{11}^{(1)} + \gamma^{-1}\nu_{12}^{(1)}\big) + p_{11}^{(1)}\delta_1 \Big] z_1^2(t) \\
&\quad + \Big[-1 + \frac{1}{2}\big(\gamma\nu_{21}^{(1)} + \nu_{22}^{(1)}\big) + p_{22}^{(1)}\delta_2 + q\Big(\frac{1}{2}\big(\gamma\nu_{21}^{(0)} + \nu_{22}^{(0)}\big) + p_{22}^{(0)}\delta_2 \Big) \Big] z_2^2(t) \\
&\quad + \sum_{i=1}^{2} \int_0^\infty \int_{t-s}^{t} z_i^2(\theta)\,d\theta\,dF_i(s),
\end{aligned}
\tag{10.47}
$$

where

$$
\begin{aligned}
dF_i(s) &= \frac{1}{2}\big(q\,dF_i^{(0)}(s) + dF_i^{(1)}(s)\big), \quad i = 1, 2, \\
dF_1^{(0)}(s) &= \big|d\mu_{11}^{(0)}(s)\big| + \gamma^{-1}\big|d\mu_{21}^{(0)}(s)\big|, \\
dF_1^{(1)}(s) &= \big|d\mu_{11}^{(1)}(s)\big| + \gamma^{-1}\big|d\mu_{21}^{(1)}(s)\big|, \\
dF_2^{(0)}(s) &= \big|\gamma\,d\mu_{12}^{(0)}(s)\big| + \big|d\mu_{22}^{(0)}(s)\big|, \\
dF_2^{(1)}(s) &= \big|\gamma\,d\mu_{12}^{(1)}(s)\big| + \big|d\mu_{22}^{(1)}(s)\big|.
\end{aligned}
$$

Note that for the functional

$$
V_2(t) = \sum_{i=1}^{2} \int_0^\infty \int_{t-s}^{t} (\theta - t + s) z_i^2(\theta)\,d\theta\,dF_i(s),
$$

we have

$$
LV_2(t) = \hat{F}_1 z_1^2(t) + \hat{F}_2 z_2^2(t) - \sum_{i=1}^{2} \int_0^\infty \int_{t-s}^{t} z_i^2(\theta)\,d\theta\,dF_i(s),
\tag{10.48}
$$

where

$$
\begin{aligned}
\hat{F}_1 &= \frac{1}{2}\big[q\big(\nu_{11}^{(0)} + \gamma^{-1}\nu_{21}^{(0)}\big) + \nu_{11}^{(1)} + \gamma^{-1}\nu_{21}^{(1)}\big], \\
\hat{F}_2 &= \frac{1}{2}\big[q\big(\gamma\nu_{12}^{(0)} + \nu_{22}^{(0)}\big) + \gamma\nu_{12}^{(1)} + \nu_{22}^{(1)}\big].
\end{aligned}
$$

From (10.47), (10.48), for the functional $V = V_1 + V_2$, by (10.40) we obtain

$$
\begin{aligned}
LV(t) \le{}& \Big[q\Big(-A_1 + \frac{\gamma^{-1}}{2} C_2 \Big) + B_1 + \frac{\gamma^{-1}}{2} C_1 \Big] z_1^2(t) \\
&+ \Big[-A_2 + \frac{\gamma}{2} C_1 + q\Big(B_2 + \frac{\gamma}{2} C_2 \Big) \Big] z_2^2(t).
\end{aligned}
\tag{10.49}
$$

By Theorem 2.1, if there exist positive numbers q and γ such that

$$q\left(-A_1+\frac{\gamma^{-1}}{2}C_2\right)+B_1+\frac{\gamma^{-1}}{2}C_1<0,$$
$$-A_2+\frac{\gamma}{2}C_1+q\left(B_2+\frac{\gamma}{2}C_2\right)<0, \tag{10.50}$$

then the trivial solution of system (10.27) is asymptotically mean-square stable.

Rewrite (10.50) in the form

$$\left(B_1+\frac{\gamma^{-1}}{2}C_1\right)\left(A_1-\frac{\gamma^{-1}}{2}C_2\right)^{-1}<q<\left(A_2-\frac{\gamma}{2}C_1\right)\left(B_2+\frac{\gamma}{2}C_2\right)^{-1}. \tag{10.51}$$

So, if

$$\left(B_1+\frac{\gamma^{-1}}{2}C_1\right)\left(A_1-\frac{\gamma^{-1}}{2}C_2\right)^{-1}<\left(A_2-\frac{\gamma}{2}C_1\right)\left(B_2+\frac{\gamma}{2}C_2\right)^{-1}, \tag{10.52}$$

then there exists $q>0$ such that (10.51) holds.

Rewriting (10.52) in the form

$$\frac{\gamma}{2}(A_1C_1+B_1C_2)+\frac{\gamma^{-1}}{2}(A_2C_2+B_2C_1)<A_1A_2-B_1B_2$$

and calculating the infimum of the left-hand part of the obtained inequality with respect to $\gamma>0$, we obtain (10.41). So, if (10.41) holds, then there exist positive numbers q and γ such that (10.50) holds, and therefore the trivial solution of system (10.27) is asymptotically mean-square stable. The proof is completed. □

Put now

$$D_1=\frac{a_1}{b_1}-Ah_1-\frac{\delta_1}{b},\qquad D_2=1-bh_2-\frac{a_1\delta_2}{Ab_1}, \tag{10.53}$$

and note that the first condition (10.34) for system (10.30) is a trivial one and the second condition takes the form $Aa_2^{-1}b_1h_1+bh_2<1$ or, via the representation (10.17) for A,

$$b_1h_1a+(a_2h_2-a_1h_1)b<a_2. \tag{10.54}$$

Corollary 10.1 *If* $D_1>0$, $D_2>0$, *and conditions* (10.54) *and*

$$\sqrt{A(D_1h_1+h_2)(\delta_2h_1+D_2bh_2)}+\frac{\delta_2}{b}<D_1D_2 \tag{10.55}$$

hold, then the trivial solution of system (10.30) *is asymptotically mean-square stable, and the equilibrium point of system* (10.25) *is stable in probability.*

Proof Calculating for (10.30) the parameters (10.33), (10.36), (10.38), (10.39), (10.40), we obtain

$$a_{11} = -\frac{a_1 b}{b_1}, \qquad a_{12} = -\frac{a_2 b}{b_1}, \qquad a_{21} = \frac{Ab_1}{a_2}, \qquad a_{22} = 0,$$

$$p_{11}^{(0)} = \frac{b_1}{a_1 b}, \qquad p_{11}^{(1)} = \frac{Ab_1^3}{a_1 a_2^2 b^2}, \qquad p_{22}^{(0)} = \frac{a_2^2}{Aa_1 b_1}, \qquad p_{22}^{(1)} = \frac{a_1}{Ab_1} + \frac{b_1}{a_1 b},$$

$$\nu_{11}^{(0)} = \frac{Ab_1 h_1}{a_1}, \qquad \nu_{11}^{(1)} = \frac{A^2 b_1^3 h_1}{a_1 a_2^2 b}, \qquad \nu_{12}^{(0)} = \frac{a_2 b h_2}{a_1}, \qquad \nu_{12}^{(1)} = \frac{Ab_1^2 h_2}{a_1 a_2},$$

$$\nu_{21}^{(0)} = 0, \qquad \nu_{21}^{(1)} = \frac{Ab_1 h_1}{a_2}, \qquad \nu_{22}^{(0)} = 0, \qquad \nu_{22}^{(1)} = bh_2,$$

$$A_1 = \frac{b_1}{a_1} D_1, \qquad A_2 = D_2 - \frac{b_1 \delta_2}{a_1 b},$$

$$B_1 = \frac{Ab_1^3}{a_1 a_2^2 b}\left(Ah_1 + \frac{\delta_1}{b}\right), \qquad B_2 = \frac{a_2^2 \delta_2}{Aa_1 b_1},$$

$$C_1 = \frac{Ab_1}{a_1 a_2}(a_1 h_1 + b_1 h_2), \qquad C_2 = \frac{a_2 b h_2}{a_1}.$$

From this it follows that

$$\begin{aligned} A_1 C_1 + B_1 C_2 &= \frac{Ab_1^2}{a_1 a_2}(D_1 h_1 + h_2), \\ A_2 C_2 + B_2 C_1 &= \frac{a_2}{a_1}(\delta_2 h_1 + D_2 b h_2), \\ A_1 A_2 - B_1 B_2 &= \frac{b_1}{a_1}\left(D_1 D_2 - \frac{\delta_2}{b}\right). \end{aligned} \tag{10.56}$$

From the representations for a_{ij}, $i, j = 1, 2$, it follows also that conditions (2.62) hold. Substituting (10.56) into (10.41), we obtain (10.55). The proof is completed. □

Remark 10.1 Note that condition (10.55) does not depend on a_2. The dependence on a_2 is included in condition (10.54).

Remark 10.2 By the absence of the delays, i.e., by $h_1 = h_2 = 0$, condition (10.54) is trivial, and condition (10.55) can be written in the form

$$\delta_1 < b\frac{a_1}{b_1}, \qquad \delta_2 < \frac{Ab_1(a_1 b - b_1 \delta_1)}{Ab_1^2 + a_1(a_1 b - b_1 \delta_1)}.$$

The same conditions can be obtained immediately from Corollary 2.3.

10.3.2 Second Way of Constructing a Lyapunov Functional

Let us consider another way of constructing of a Lyapunov functional for system (10.30).

Theorem 10.2 *If* $D_1 > 0$, $D_2 > 0$, *and conditions* (10.54) *and*

$$\left(\sqrt{Abh_2^2 + 4\delta_2 b^{-1} D_1} + \sqrt{Ab}h_2\right)\left(\sqrt{Abh_1^2 + 4D_2} + \sqrt{Ab}h_1\right) < 4D_1 D_2 \quad (10.57)$$

hold, where A *and* D_1, D_2 *are defined by* (10.17) *and* (10.53), *respectively, then the trivial solution of system* (10.30) *is asymptotically mean-square stable, and the equilibrium point of system* (10.25) *is stable in probability.*

Proof Using (10.17), rewrite system (10.30) in the form

$$\begin{aligned}\dot{z}_1(t) =& -\frac{a_1 b}{b_1} z_1(t) - \frac{a_2 b}{b_1} z_2(t) + \sigma_1 z_1(t)\dot{w}_1(t),\\ \dot{Z}_2(t) =& \frac{Ab_1}{a_2} z_1(t) + \sigma_2 z_2(t)\dot{w}_2(t),\end{aligned} \quad (10.58)$$

where

$$\begin{aligned}Z_2(t) =& z_2(t) + \frac{Ab_1}{a_2} J_1(z_{1t}) + bJ_2(z_{2t}),\\ J_i(z_{it}) =& \int_{t-h_i}^{t} z_i(s)\,ds, \quad i = 1, 2.\end{aligned} \quad (10.59)$$

Consider now the functional

$$V_1(t) = z_1^2(t) + 2\mu z_1(t) Z_2(t) + \gamma Z_2^2(t), \quad (10.60)$$

where the parameters μ and γ will be chosen below. Then by (10.60), (10.58) we have

$$\begin{aligned}LV_1(t) =& -2\frac{b}{b_1}\big(z_1(t) + \mu Z_2(t)\big)\big(a_1 z_1(t) + a_2 z_2(t)\big) + \sigma_1^2 z_1^2(t)\\ &+ 2\frac{Ab_1}{a_2}\big(\mu z_1(t) + \gamma Z_2(t)\big) z_1(t) + \gamma\sigma_2^2 z_2^2(t)\\ =& -2\left(\frac{a_1 b}{b_1} - \mu\frac{Ab_1}{a_2} - \delta_1\right) z_1^2(t) - 2\left(\mu\frac{a_2 b}{b_1} - \gamma\delta_2\right) z_2^2(t)\\ &+ 2\left(\gamma\frac{Ab_1}{a_2} - \mu\frac{a_1 b}{b_1} - \frac{a_2 b}{b_1}\right) z_1(t) z_2(t)\\ &+ 2\left(\gamma\frac{Ab_1}{a_2} - \mu\frac{a_1 b}{b_1}\right) z_1(t)\left(\frac{Ab_1}{a_2} J_1(z_{1t}) + bJ_2(z_{2t})\right)\end{aligned}$$

$$-2\mu\frac{a_2b}{b_1}z_2(t)\left(\frac{Ab_1}{a_2}J_1(z_{1t})+bJ_2(z_{2t})\right). \tag{10.61}$$

Defining now γ by the equality

$$\gamma\frac{Ab_1}{a_2}=\mu\frac{a_1b}{b_1}+\frac{a_2b}{b_1}, \tag{10.62}$$

from (10.61) we obtain

$$\begin{aligned}LV_1(t)=&-2\left(\frac{a_1b}{b_1}-\mu\frac{Ab_1}{a_2}-\delta_1\right)z_1^2(t)-2\left(\mu\frac{a_2b}{b_1}-\gamma\delta_2\right)z_2^2(t)\\&+2\frac{a_2b}{b_1}z_1(t)\left(\frac{Ab_1}{a_2}J_1(z_{1t})+bJ_2(z_{2t})\right)\\&-2\mu\frac{a_2b}{b_1}z_2(t)\left(\frac{Ab_1}{a_2}J_1(z_{1t})+bJ_2(z_{2t})\right).\end{aligned}$$

By (10.59) from this, for some positive γ_1, γ_2, we have

$$\begin{aligned}LV_1(t)\le&-2\left(\frac{a_1b}{b_1}-\mu\frac{Ab_1}{a_2}-\delta_1\right)z_1^2(t)-2\left(\mu\frac{a_2b}{b_1}-\gamma\delta_2\right)z_2^2(t)\\&+\frac{a_2b}{b_1}\left(\frac{Ab_1}{a_2}\int_{t-h_1}^{t}\left(z_1^2(t)+z_1^2(s)\right)ds+b\int_{t-h_2}^{t}\left(\gamma_1z_1^2(t)+\gamma_1^{-1}z_2^2(s)\right)ds\right)\\&+\mu\frac{a_2b}{b_1}\left(\frac{Ab_1}{a_2}\int_{t-h_1}^{t}\left(\gamma_2z_2^2(t)+\gamma_2^{-1}z_1^2(s)\right)ds\right.\\&\left.+b\int_{t-h_2}^{t}\left(z_2^2(t)+z_2^2(s)\right)ds\right).\end{aligned} \tag{10.63}$$

By the representations (10.53) for D_1, D_2 and (10.62) for γ inequality (10.63) can be written in the form

$$\begin{aligned}LV_1(t)\le&\left(-2bD_1-Abh_1+2\mu\frac{Ab_1}{a_2}+\gamma_1\frac{a_2b^2h_2}{b_1}\right)z_1^2(t)\\&+\left(-2\mu\frac{a_2bD_2}{b_1}-\frac{\mu a_2b^2h_2}{b_1}+\frac{2a_2^2b\delta_2}{Ab_1^2}+\gamma_2\mu Abh_1\right)z_2^2(t)\\&+Ab\left(1+\mu\gamma_2^{-1}\right)\int_{t-h_1}^{t}z_1^2(s)\,ds+\frac{b^2a_2}{b_1}\left(\gamma_1^{-1}+\mu\right)\int_{t-h_2}^{t}z_2^2(s)\,ds.\end{aligned}$$

Put now

$$V_2=Ab\left(1+\mu\gamma_2^{-1}\right)\int_{t-h_1}^{t}(s-t+h_1)z_1^2(s)\,ds$$

$$+\frac{b^2 a_2}{b_1}\left(\gamma_1^{-1}+\mu\right)\int_{t-h_2}^{t}(s-t+h_2)z_2^2(s)\,ds.$$

Then

$$\begin{aligned}LV_2 &= Ab\left(1+\mu\gamma_2^{-1}\right)\left(h_1 z_1^2(t)-\int_{t-h_1}^{t} z_1^2(s)\,ds\right)\\&\quad+\frac{b^2 a_2}{b_1}\left(\gamma_1^{-1}+\mu\right)\left(h_2 z_2^2(t)-\int_{t-h_2}^{t} z_2^2(s)\,ds\right),\end{aligned}$$

and as a result, for the functional $V=V_1+V_2$, we obtain

$$\begin{aligned}LV &\le \left(-2bD_1+2\mu\frac{Ab_1}{a_2}+\gamma_1\frac{a_2 b^2 h_2}{b_1}+\gamma_2^{-1}\mu Abh_1\right)z_1^2(t)\\&\quad+\left(-2\mu\frac{a_2 b}{b_1}D_2+2\frac{a_2^2 b}{Ab_1^2}\delta_2+\gamma_1^{-1}\frac{a_2 b^2 h_2}{b_1}+\gamma_2\mu Abh_1\right)z_2^2(t).\end{aligned}$$

By Theorem 2.1, if

$$\begin{aligned}&-2bD_1+2\mu\frac{Ab_1}{a_2}+\gamma_1\frac{a_2 b^2 h_2}{b_1}+\gamma_2^{-1}\mu Abh_1<0,\\&-2\mu\frac{a_2 b}{b_1}D_2+2\frac{a_2^2 b}{Ab_1^2}\delta_2+\gamma_1^{-1}\frac{a_2 b^2 h_2}{b_1}+\gamma_2\mu Abh_1<0,\end{aligned}\tag{10.64}$$

then the trivial solution of system (10.30) is asymptotically mean-square stable.

Rewrite (10.64) in the form

$$\frac{2\frac{a_2^2 b}{Ab_1^2}\delta_2+\gamma_1^{-1}\frac{a_2 b^2 h_2}{b_1}}{2\frac{a_2 b}{b_1}D_2-\gamma_2 Abh_1}<\mu<\frac{2bD_1-\gamma_1\frac{a_2 b^2 h_2}{b_1}}{2\frac{Ab_1}{a_2}+\gamma_2^{-1}Abh_1}.\tag{10.65}$$

So, if the inequality

$$\frac{2\frac{a_2^2 b}{Ab_1^2}\delta_2+\gamma_1^{-1}\frac{a_2 b^2 h_2}{b_1}}{2\frac{a_2 b}{b_1}D_2-\gamma_2 Abh_1}<\frac{2bD_1-\gamma_1\frac{a_2 b^2 h_2}{b_1}}{2\frac{Ab_1}{a_2}+\gamma_2^{-1}Abh_1}\tag{10.66}$$

holds, then there exists μ such that (10.65) holds too.

It is easy to check that from (10.65), (10.62) the condition $\mu^2<\gamma$ follows, which ensures the positivity of the functional (10.60).

Representing (10.66) in the form

$$\frac{\frac{a_2^2}{Ab_1^2}\delta_2+\gamma_1^{-1}\frac{a_2 bh_2}{2b_1}}{D_1-\gamma_1\frac{a_2 bh_2}{2b_1}}\times\frac{\frac{Ab_1^2}{a_2^2 b}+\gamma_2^{-1}\frac{Ab_1 h_1}{2a_2}}{D_2-\gamma_2\frac{Ab_1 h_1}{2a_2}}<1$$

and using Lemma 2.4 twice, we obtain (10.57) □

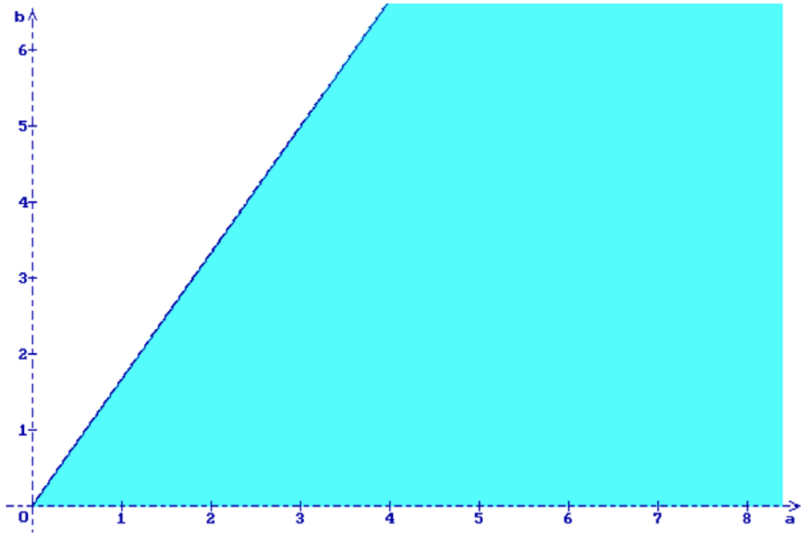

Fig. 10.1 Region of stability in probability for (10.25): $a_1 = 0.6$, $b_1 = 1$, $h_1 = 0$, $h_2 = 0$, $\delta_1 = 0$, $\delta_2 = 0$

Remark 10.3 Note that the representation (10.31)–(10.33) for system (10.30) coincides with (10.58), (10.59). So, conditions (10.55) and (10.57) are equivalent and give the same stability region. For simplicity, let us check this statement by the condition $h_1 = 0$. Indeed, in this case from (10.57) we have

$$Abh_2^2 + 4\delta_2 b^{-1} D_1 < (2D_1\sqrt{D_2} - \sqrt{Ab}h_2)^2 = 4D_1^2 D_2 - 4D_1 h_2\sqrt{AbD_2} + Abh_2^2$$

or $\delta_2 b^{-1} < D_1 D_2 - h_2\sqrt{AbD_2}$, which is equivalent to (10.55) by $h_1 = 0$. Similarly, it is easy to get that (10.55) coincides with (10.57) by the condition $h_2 = 0$ or by the condition $\delta_2 = 0$. In the general case the necessary transformation is bulky enough.

The regions of stability in probability for a positive point of equilibrium of system (10.25), obtained by condition (10.55) (or (10.57)), are shown in the space of the parameters (a, b) for $a_1 = 0.6$, $b_1 = 1$ and different values of the other parameters: in Fig. 10.1 for $h_1 = 0$, $h_2 = 0$, $\delta_1 = 0$, $\delta_2 = 0$, in Fig. 10.2 for $h_1 = 0$, $h_2 = 0$, $\delta_1 = 0.2$, $\delta_2 = 0.3$, in Fig. 10.3 for $a_2 = 0.6$, $h_1 = 0.1$, $h_2 = 0.15$, $\delta_1 = 0$, $\delta_2 = 0$, and in Fig. 10.4 for $a_2 = 0.07$, $h_1 = 0.01$, $h_2 = 0.15$, $\delta_1 = 0.05$, $\delta_2 = 0.1$.

The equation of the straight line in Figs. 10.1 and 10.2 is $ab_1 = ba_1$, which corresponds to the condition $A = 0$. In Figs. 10.3 and 10.4 the straight line 1 also corresponds to this equation and the straight line 2 is defined by the equation $b_1 h_1 a + (a_2 h_2 - a_1 h_1)b = a_2$, which follows from condition (10.54).

Note that the stability of the positive equilibrium point of the difference analogue of system (10.25) is investigated in [278].

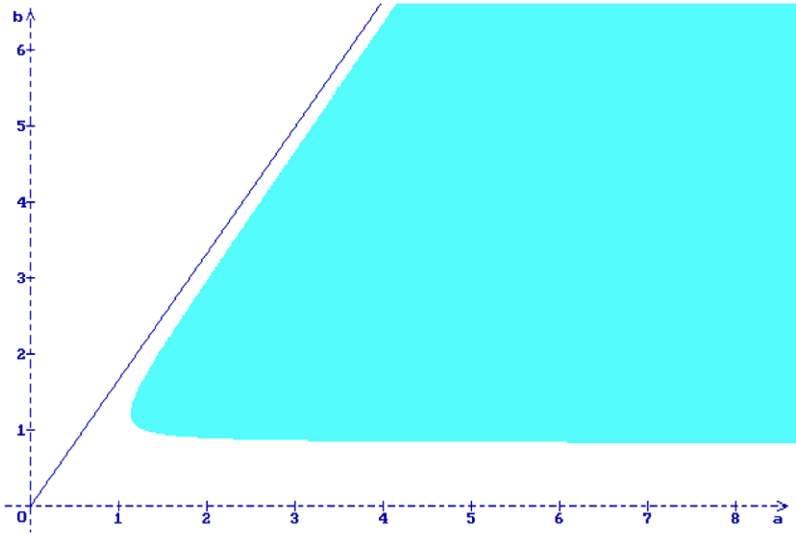

Fig. 10.2 Region of stability in probability for (10.25): $a_1 = 0.6$, $b_1 = 1$, $h_1 = 0$, $h_2 = 0$, $\delta_1 = 0.2$, $\delta_2 = 0.3$

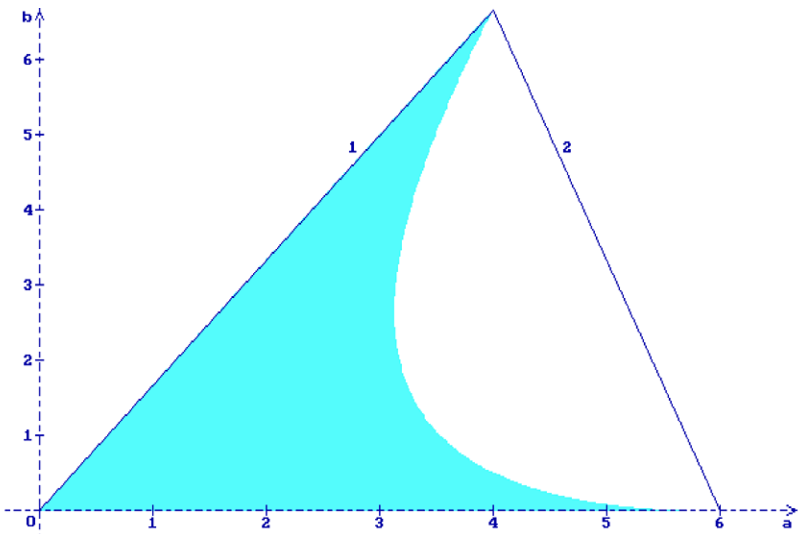

Fig. 10.3 Region of stability in probability for (10.25): $a_1 = 0.6$, $a_2 = 0.6$, $b_1 = 1$, $h_1 = 0.1$, $h_2 = 0.15$, $\delta_1 = 0$, $\delta_2 = 0$

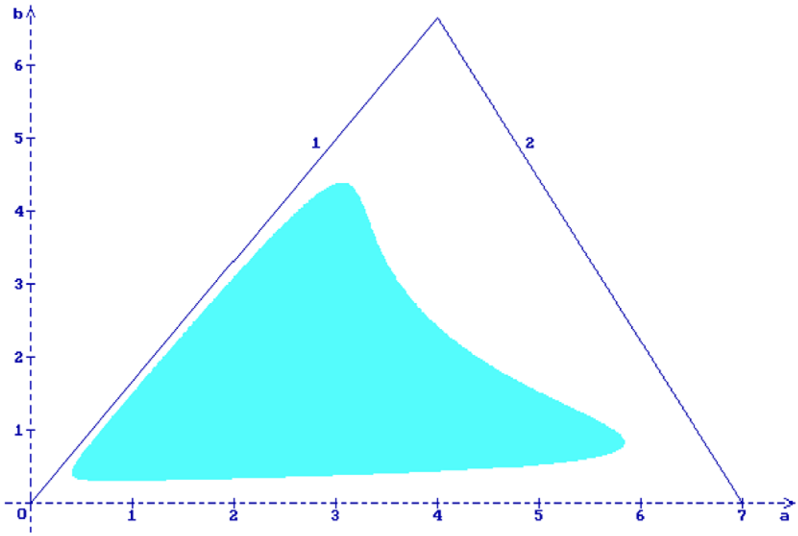

Fig. 10.4 Region of stability in probability for (10.25): $a_1 = 0.6$, $a_2 = 0.07$, $b_1 = 1$, $h_1 = 0.01$, $h_2 = 0.15$, $\delta_1 = 0.05$, $\delta_2 = 0.1$

10.3.3 Stability of the Equilibrium Point of Ratio-Dependent Predator–Prey Model

Consider now system (10.9) with stochastic perturbations, i.e.,

$$\begin{aligned}\dot{x}_1(t) &= x_1(t)\left(a - \int_0^\infty x_1(t-s)\,dK_0(s)\right) - \int_0^\infty \frac{x_1^k(t-s)x_2(t)}{x_1^k(t-s) + a_2 x_2^k(t-s)}\,dK_1(s) \\ &\quad + \sigma_1\big(x_1(t) - x_1^*\big)\dot{w}_1(t), \qquad (10.67)\\ \dot{x}_2(t) &= -bx_2(t) + \int_0^\infty \frac{x_1^m(t-s)x_2(t)}{x_1^m(t-s) + b_2 x_2^m(t-s)}\,dR_1(s) + \sigma_2\big(x_1(t) - x_2^*\big)\dot{w}_2(t).\end{aligned}$$

System (10.9) was obtained from (10.1) by conditions (10.10). So, by (10.13), (10.14) the positive equilibrium point (x_1^*, x_2^*) of system (10.9) (and also (10.67)) is defined by the conditions

$$\begin{aligned}&x_1^*\big(a - K_0 x_1^*\big) = K_1 f\big(x_1^*, x_2^*\big)x_2^*,\\ &b = R_1 g\big(x_1^*, x_2^*\big), \qquad a > K_0 x_1^*.\end{aligned} \qquad (10.68)$$

Suppose that the functions $f(x_1, x_2)$ and $g(x_1, x_2)$ in (10.10) are differentiable and can be represented in the form

$$\begin{aligned}f\big(y_1 + x_1^*, y_2 + x_2^*\big) &= f_0 + f_1 y_1 - f_2 y_2 + o(y_1, y_2),\\ g\big(y_1 + x_1^*, y_2 + x_2^*\big) &= g_0 + g_1 y_1 - g_2 y_2 + o(y_1, y_2),\end{aligned}$$

where $\lim_{|y|\to 0}\frac{o(y_1,y_2)}{|y|}=0$ for $|y|=\sqrt{y_1^2+y_2^2}$, and

$$f_0=f\left(x_1^*,x_2^*\right),\qquad f_1=x_2^*\hat{f},\qquad f_2=x_1^*\hat{f},\qquad \hat{f}=\frac{ka_2(x_1^*x_2^*)^{k-1}}{((x_1^*)^k+a_2(x_2^*)^k)^2},$$
$$g_0=g\left(x_1^*,x_2^*\right),\qquad g_1=x_2^*\hat{g},\qquad g_2=x_1^*\hat{g},\qquad \hat{g}=\frac{mb_2(x_1^*x_2^*)^{m-1}}{((x_1^*)^m+b_2(x_2^*)^m)^2}.$$

So, the functionals $F_0(x_{1t},x_{2t})$, $F_1(x_{1t},x_{2t})$, $G_1(x_{1t},x_{2t})$ in (10.10) have the representations

$$\begin{aligned}
F_0\left(y_{1t}+x_1^*,y_{2t}+x_2^*\right)&=K_0x_1^*+\int_0^\infty y_1(t-s)\,dK_0(s),\\
F_1\left(y_{1t}+x_1^*,y_{2t}+x_2^*\right)&=K_1f_0x_2^*+f_1x_2^*\int_0^\infty y_1(t-s)\,dK_1(s)+K_1f_0y_2(t)\\
&\quad-f_2x_2^*\int_0^\infty y_2(t-s)\,dK_1(s)+o(y_1,y_2),\\
G_1\left(y_{1t}+x_1^*,y_{2t}+x_2^*\right)&=R_1g_0x_2^*+g_1x_2^*\int_0^\infty y_1(t-s)\,dR_1(s)+R_1g_0y_2(t)\\
&\quad-g_2x_2^*\int_0^\infty y_2(t-s)\,dR_1(s)+o(y_1,y_2).
\end{aligned}\tag{10.69}$$

By (10.68), (10.69) the linear part of system (10.67) has the form

$$\begin{aligned}
\dot{z}_1(t)&=\left(a-K_0x_1^*\right)z_1(t)-K_1f_0z_2(t)-\int_0^\infty z_1(t-s)\,dK(s)\\
&\quad+f_2x_2^*\int_0^\infty z_2(t-s)\,dK_1(s)+\sigma_1z_1(t)\dot{w}_1(t),\\
\dot{z}_2(t)&=g_1x_2^*\int_0^\infty z_1(t-s)\,dR_1(s)-g_2x_2^*\int_0^\infty z_2(t-s)\,dR_1(s)+\sigma_2z_2(t)\dot{w}_2(t),
\end{aligned}\tag{10.70}$$

where $dK(s)=x_1^*\,dK_0(s)+f_1x_2^*\,dK_1(s)$. Rewrite system (10.70) in the form (10.31) with

$$\begin{aligned}
Z_1(t)&=z_1(t)-\int_0^\infty\int_{t-s}^t z_1(\theta)\,d\theta\,dK(s)+f_2x_2^*\int_0^\infty\int_{t-s}^t z_2(\theta)\,d\theta\,dK_1(s),\\
Z_2(t)&=z_2(t)+g_1x_2^*\int_0^\infty\int_{t-s}^t z_1(\theta)\,d\theta\,dR_1(s)\\
&\quad-g_2x_2^*\int_0^\infty\int_{t-s}^t z_2(\theta)\,d\theta\,dR_1(s),
\end{aligned}\tag{10.71}$$
$$a_{11}=K_1x_2^*\left(\frac{f_0}{x_1^*}-f_1\right)-K_0x_1^*,\qquad a_{12}=K_1\left(f_2x_2^*-f_0\right),$$
$$a_{21}=R_1g_1x_2^*,\qquad a_{22}=-R_1g_2x_2^*.$$

Further investigation is similar to the previous sections.

For short, consider system (10.67) by conditions (10.11). The point of equilibrium in this case is defined by (10.18). From (10.11), (10.18), (10.70) and (10.71) it follows that system (10.67) and the linear part of this system respectively take the forms

$$\begin{aligned}\dot{x}_1(t) &= x_1(t)\left(a - a_0x_1(t) - \frac{a_1x_2(t)}{x_1(t)+a_2x_2(t)}\right) + \sigma_1\big(x_1(t)-x_1^*\big)\dot{w}_1(t),\\ \dot{x}_2(t) &= x_2(t)\left(-b + \frac{b_1x_1(t-h)}{x_1(t-h)+b_2x_2(t-h)}\right) + \sigma_2\big(x_1(t)-x_2^*\big)\dot{w}_2(t)\end{aligned} \tag{10.72}$$

and

$$\begin{aligned}\dot{z}_1(t) &= a_{11}z_1(t) + a_{12}z_2(t) + \sigma_1z_1(t)\dot{w}_1(t),\\ \dot{z}_2(t) &= a_{21}z_1(t-h) + a_{22}z_2(t-h) + \sigma_2z_2(t)\dot{w}_2(t),\end{aligned} \tag{10.73}$$

where

$$\begin{aligned}&a_{11} = Ba_1\alpha^2 - A, \qquad a_{12} = -B^2a_1\alpha^2,\\ &a_{21} = b_1b_2\beta^2, \qquad a_{22} = -Bb_1b_2\beta^2,\\ &A = a - a_1\alpha,\ B = \frac{bb_2}{b_1-b},\ \alpha = \frac{1}{B+a_2},\ \beta = \frac{1}{B+b_2}.\end{aligned} \tag{10.74}$$

Let $\hat{A} = \|a_{ij}\|$ be the matrix with the elements defined by (10.74). Suppose that

$$\begin{gathered} b \in (0, b_1),\\ a > \begin{cases} a_1\alpha & \text{if } a_1\alpha^2 \le b_1b_2\beta^2,\\ a_1\alpha + B(a_1\alpha^2 - b_1b_2\beta^2) & \text{if } a_1\alpha^2 > b_1b_2\beta^2.\end{cases}\end{gathered} \tag{10.75}$$

By conditions (10.75) conditions (2.62) for the matrix $\hat{A}$ hold. Indeed,

$$\mathrm{Tr}(\hat{A}) = B\big(a_1\alpha^2 - b_1b_2\beta^2\big) - A < 0, \qquad \det(\hat{A}) = ABb_1b_2\beta^2 > 0. \tag{10.76}$$

Let $P = \|p_{ij}\|$ be the matrix with the elements defined by (1.29) for some $q > 0$ and represented in the form (10.35), (10.36). Using (10.35), (10.36), (10.44), (10.76), put

$$\rho = \rho^{(0)}q + \rho^{(1)}, \quad \rho^{(0)} = -\frac{a_{12}}{|\,\mathrm{Tr}(\hat{A})|},\ \rho^{(1)} = \frac{a_{21}}{|\,\mathrm{Tr}(\hat{A})|}, \tag{10.77}$$

and

$$\begin{gathered} A_1 = 1 - p_{11}^{(0)}\delta_1 - \rho^{(0)}|a_{21}|h, \qquad A_2 = 1 - p_{22}^{(1)}\delta_2 - |a_{22}|h,\\ B_1 = p_{11}^{(1)}\delta_1 + \rho^{(1)}|a_{21}|h, \qquad B_2 = p_{22}^{(0)}\delta_2, \quad \delta_i = \frac{1}{2}\sigma_i^2,\ i = 1, 2,\\ C_1 = \big(|a_{21}| + \rho^{(1)}|a_{22}|\big)h, \qquad C_2 = \rho^{(0)}|a_{22}|h.\end{gathered} \tag{10.78}$$

Rewrite system (10.73) in the form

$$\begin{aligned}\dot{z}_1(t) &= a_{11}z_1(t) + a_{12}z_2(t) + \sigma_1 z_1(t)\dot{w}_1(t),\\ \dot{Z}_2(t) &= a_{21}z_1(t) + a_{22}z_2(t) + \sigma_2 z_2(t)\dot{w}_2(t),\end{aligned} \tag{10.79}$$

where

$$Z_2(t) = z_2(t) + \int_{t-h}^{t} \big(a_{21}z_1(s) + a_{22}z_2(s)\big)\,ds, \tag{10.80}$$

and following condition (2.10), suppose that the parameters a_{21} and a_{22} in (10.74) satisfy the condition $h\sqrt{a_{21}^2 + a_{22}^2} < 1$ or, via (10.74), $b_1 b_2 \beta^2 h\sqrt{1+B^2} < 1$, which is equivalent to

$$(b_1 - b)\sqrt{(b_1 - b)^2 + b^2 b_2^2} < \frac{b_1 b_2}{h}. \tag{10.81}$$

Theorem 10.3 *Let conditions* (10.75), (10.81) *hold. If* $A_1 > 0$, $A_2 > 0$, *and*

$$\sqrt{(A_1C_1 + B_1C_2)(A_2C_2 + B_2C_1)} + B_1B_2 < A_1A_2, \tag{10.82}$$

then the trivial solution of system (10.73) *is asymptotically mean-square stable*, *and the equilibrium point of system* (10.72) *is stable in probability.*

Proof Consider the functional

$$V_1(t) = p_{11}z_1^2(t) + 2p_{12}z_1(t)Z_2(t) + p_{22}Z_2^2(t)$$

with p_{ij}, $i, j = 1, 2$, defined by (1.29). Let L be the generator of system (10.79). Then, using (10.77), similarly to (10.45), for system (10.79), we obtain

$$\begin{aligned}LV_1(t) = {}& -qz_1^2(t) + \rho Z_2(t)z_1(t) + p_{11}\sigma_1^2 z_1^2(t)\\ & - \rho z_1(t)z_2(t) - Z_2(t)z_2(t) + p_{22}\sigma_2^2 z_2^2(t).\end{aligned} \tag{10.83}$$

Substituting (10.80) into (10.83) and using some positive γ, we obtain

$$\begin{aligned}LV_1(t) = {}& -qz_1^2(t) + \rho z_1(t)\left(z_2(t) + \int_{t-h}^{t}\big(a_{21}z_1(s) + a_{22}z_2(s)\big)\,ds\right) + p_{11}\sigma_1^2 z_1^2(t)\\ & - \rho z_1(t)z_2(t) - z_2(t)\left(z_2(t) + \int_{t-h}^{t}\big(a_{21}z_1(s) + a_{22}z_2(s)\big)\,ds\right)\\ & + p_{22}\sigma_2^2 z_2^2(t)\\ \le {}& \big(-q + p_{11}\sigma_1^2\big)z_1^2(t) + \frac{\rho}{2}|a_{21}|\int_{t-h}^{t}\big(z_1^2(t) + z_1^2(s)\big)\,ds\\ & + \frac{\rho}{2}|a_{22}|\int_{t-h}^{t}\big(\gamma^{-1}z_1^2(t) + \gamma z_2^2(s)\big)\,ds\end{aligned}$$

$$
\begin{aligned}
&+\left(-1+p_{22}\sigma_2^2\right)z_2^2(t)+\frac{1}{2}|a_{21}|\int_{t-h}^{t}\left(\gamma z_2^2(t)+\gamma^{-1}z_1^2(s)\right)ds \\
&+\frac{1}{2}|a_{22}|\int_{t-h}^{t}\left(z_2^2(t)+z_2^2(s)\right)ds \\
=&\left(-q+p_{11}\sigma_1^2+\frac{\rho}{2}|a_{21}|h+\gamma^{-1}\frac{\rho}{2}|a_{22}|h\right)z_1^2(t) \\
&+\left(-1+p_{22}\sigma_2^2+\frac{1}{2}|a_{22}|h+\frac{\gamma}{2}|a_{21}|h\right)z_2^2(t) \\
&+\frac{|a_{21}|}{2}\left(\rho+\gamma^{-1}\right)\int_{t-h}^{t}z_1^2(s)\,ds+\frac{|a_{22}|}{2}(1+\rho\gamma)\int_{t-h}^{t}z_2^2(s)\,ds.
\end{aligned}
$$

Putting

$$
\begin{aligned}
V_2=&\frac{|a_{21}|}{2}\left(\rho+\gamma^{-1}\right)\int_{t-h}^{t}(s-t+h)z_1^2(s)\,ds \\
&+\frac{|a_{22}|}{2}(1+\rho\gamma)\int_{t-h}^{t}(s-t+h)z_2^2(s)\,ds,
\end{aligned}
$$

for the functional $V=V_1+V_2$, we have

$$
\begin{aligned}
LV(t)\le&\left[-q+p_{11}\sigma_1^2+\rho|a_{21}|h+\frac{\gamma^{-1}}{2}h\left(|a_{21}|+\rho|a_{22}|\right)\right]z_1^2(t) \\
&+\left[-1+p_{22}\sigma_2^2+|a_{22}|h+\frac{\gamma}{2}h\left(|a_{21}|+\rho|a_{22}|\right)\right]z_2^2(t). \qquad (10.84)
\end{aligned}
$$

Using the representations (10.35), (10.36), (10.77), (10.78), we can rewrite (10.84) in the form

$$
\begin{aligned}
LV(t)\le&\left[q\left(-A_1+\frac{\gamma^{-1}}{2}C_2\right)+B_1+\frac{\gamma^{-1}}{2}C_1\right]z_1^2(t) \\
&+\left[-A_2+\frac{\gamma}{2}C_1+q\left(B_2+\frac{\gamma}{2}C_2\right)\right]z_2^2(t),
\end{aligned}
$$

which coincides with (10.49). So, from this (10.82) follows, which coincides with (10.41). The proof is completed. □

The regions of stability in probability for a positive point of equilibrium of system (10.72), obtained by conditions (10.81), (10.82), are shown in the space of the parameters (a,b) for $a_0=0.3$, $a_1=5$ $a_2=0.5$, $b_1=6$, $b_2=2$, $h=0.4$ and different values of δ_1, δ_2: in Fig. 10.5 for $\delta_1=1.5$, $\delta_2=0.05$, in Fig. 10.6 for $\delta_1=1$, $\delta_2=0.55$.

In the both figures the thick line shows the stability region given by conditions (10.75) that corresponds to the values of the parameters $h=\delta_1=\delta_2=0$.

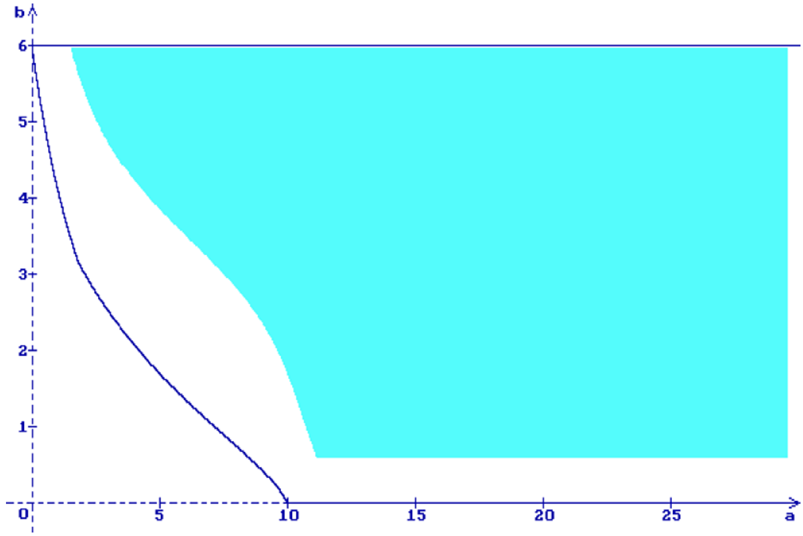

Fig. 10.5 Region of stability in probability for (10.69): $a_0 = 0.3$, $a_1 = 5$, $a_2 = 0.5$, $b_1 = 6$, $b_2 = 2$, $h = 0.4$, $\delta_1 = 1.5$, $\delta_2 = 0.05$

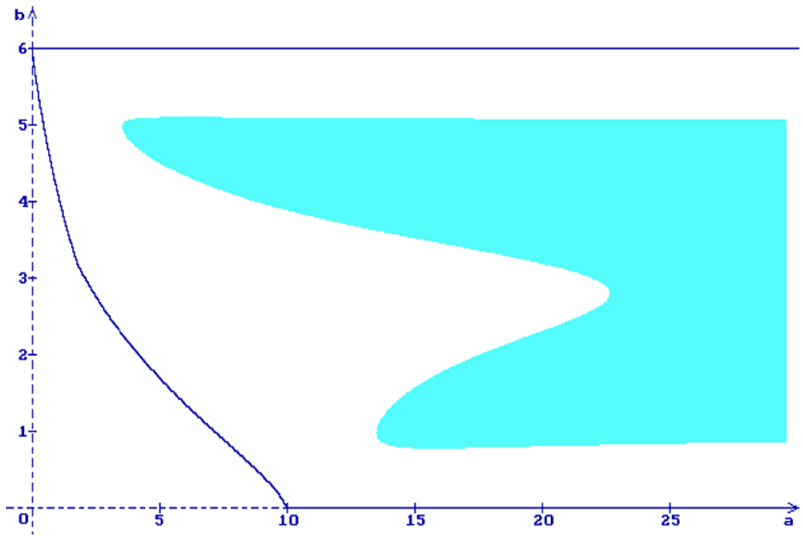

Fig. 10.6 Region of stability in probability for (10.69): $a_0 = 0.3$, $a_1 = 5$, $a_2 = 0.5$, $b_1 = 6$, $b_2 = 2$, $h = 0.4$, $\delta_1 = 1$, $\delta_2 = 0.55$