Chapter 6
Themes and Applications of Kinetic Exchange Models: Redux

Asim Ghosh, Anindya S. Chakrabarti, Anjan Kumar Chandra, and Anirban Chakraborti

Abstract In this article, we briefly discuss the general formalism of kinetic exchange models and their various applications in economics and sociology. Inspired from the kinetic theory of gases in statistical physics, the kinetic exchange model for closed economic systems were first proposed by simply considering the agents as gas molecules, and wealth of agents as kinetic energy exchanged amongst the gas molecules. The formalism had been successfully applied to modeling of wealth distributions in 2000s. This has further spurred new research in recent times in various areas of soft sciences—firm dynamics, opinion formation in the society, etc.

6.1 Introduction

The essential theme of the kinetic exchange models is the exchange of energy due to collisions amongst a collection of inanimate particles. Here, we will present that story and its economic and social counterparts to provoke some more collisions between economists and physicists that *may* lead to exchange of ideas (exchanging kinetic energy between these two arrogant groups might not be good idea to begin with!). On a more serious note, the kinetic exchange models have been one of the

A. Ghosh · A.K. Chandra
Theoretical Condensed Matter Physics Division, Saha Institute of Nuclear Physics, 1/AF Bidhannagar, Kolkata 700 064, India

A. Ghosh
e-mail: asimghosh066@gmail.com

A.K. Chandra
e-mail: anjanphys@gmail.com

A.S. Chakrabarti
Economics Department, Boston University, 270 Bay State Road, Boston, MA, USA
e-mail: anindya@bu.edu

A. Chakraborti (✉)
Laboratoire de Mathématiques Appliquées aux Systèmes, École Centrale Paris, 92290 Châtenay-Malabry, France
e-mail: anirban.chakraborti@ecp.fr

F. Abergel et al. (eds.), *Econophysics of Agent-Based models*,
New Economic Windows, DOI 10.1007/978-3-319-00023-7_6,
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most widely used formalisms in the growing interdisciplinary field of econophysics [1–4] and sociophysics [5–7]. The concept of kinetic exchange was taken from statistical physics which was proposed towards the end of the nineteenth century. First, Bernoulli gave a picture of kinetic theory of gases in his paper 'Hydrodynamica.' After that Maxwell and Boltzmann derived energy distribution function for kinetic theory of gas. The other pioneers in this field were Max Planck, Rudolf Clausius, Josiah Willard Gibbs. The kinetic exchange model is one of the simplest models in statistical mechanics, which attempts in deriving the average macroscopic behaviors from the microscopic properties of particles.

The kinetic exchange models[1] had been primarily used to explain income/wealth distributions [8, 9]. More specifically, the target was to write down a minimal set of stochastic equations that gave rise to a distribution mimicking the actual income/wealth distributions. Even though this target has been satisfactorily achieved, as discussed below, the framework suffers from a major drawback in terms of *ad hoc*-ism. The explanations (or the terminology) that has been used to describe the exchange processes is not exactly what one would call 'economic'. The researchers working on this topic essentially took a solvable model from statistical mechanics and made analogies of certain quantities. For example, energy was interpreted as wealth, particles were substituted by agents, etc. and needless to say, such abstraction and *ad hoc* approach attracted its fair share of criticism [10]. However, the same abstraction may also prove to be one of the strongest features of this whole literature. Since the terms are not tied to some specific economic quantity, there is little reason to confine them in the area of income/wealth distributions only. This motivation led to applications of the same basic framework to explain different economic and social phenomena.

What we will discuss in this paper is roughly as follows. We start by describing the simple observation that a simple random scattering-like interaction amongst the agents gives a wealth distribution similar to the 'Boltzmann-Gibbs' type. However in our real society, each of the agents have a "saving propensity." We discuss that when saving is introduced in the model, depending upon distribution of the saving propensity amongst agents, different wealth distributions can be generated. Further, we will review how a kinetic exchange model may give a "phase transition" by introducing a "threshold," where the associated phase transition is of the "active-absorbing" type. Then we will discuss the applications of the same type of formalism in firm dynamics and later in the opinion dynamics in the society. The basic aim of this article is to enthuse the readers in the use of the simple yet powerful formalism of kinetic exchange models in related areas. By no means this is an exhaustive or technical review. We would like to refer the readers to the original books and articles for further details and references.

[1] We shall often use in this article, the full form or the shorter acronym, KEM, interchangeably.

6.2 Kinetic Wealth Exchange Models (KWEM)

Since, the KEM was first applied to explain the origin of inequality that is seen in the income/wealth distributions, it is a natural starting point for us to indicate the regularities of those distributions. The first well known observation was by Pareto who showed that distribution of wealth for the richer section of the economy follows a power law [11]. He observed that roughly 20 % people who were in the tail, owned about 80 % of the total wealth of the economy. After that preliminary but extremely significant study, many others have conducted research to know the exact distribution of wealth or income in an economy, but it remains as one of the elusive problems in economics. For the other end of the income/wealth spectrum it has been observed that the people of low income or wealth in our society follow a log-normal or gamma-like distribution [3] though there is some ambiguity over the fit of the theoretical distributions to the real data. However, one surprising fact is that the general features of the distributions do not change from country to country i.e., they are robust to the exact specification of the economy/country.

To understand the precise origin and nature of these robust features in the income or wealth distribution, concepts of kinetic theory of gas molecules have been used with success. One can easily map the problem with kinetic exchanges, by considering an economic agent to be like a gas molecule and wealth of that agent as similar to the kinetic energy of the molecule. The different exchange dynamics that we discuss below, when combined effectively, produce different features of the empirical wealth distribution.

6.2.1 Basics of Kinetic Wealth Exchange Models

6.2.1.1 Boltzmann-Gibbs Distribution in Economic Systems

Independent of the modeling efforts of social scientists,[2] a distinct approach was taken by physicists Drăgulescu et al. [9, 16] who considered a toy model where the agents simply reshuffled a part of their wealth in a closed economy. The benefit of having a toy model is that everybody knows it is a toy model and it can be changed very easily. The most important departure from the standard economic theory is that they got rid of all microeconomic decision-making processes. That came at the cost of losing all perspective of why should any trade occur at all. However, the benefit exceeds the cost. One can then think of the economic system as comprising of only the agents and their characterizing quantity, wealth instead of keeping track of preferences, beliefs, market mechanisms, etc. (which are the usual burden of most, if not all, neo-classical models in economics).

[2] It was discovered later that economists, like Bennati, and sociologists, like Angle, were independently using similar tools and models since the 1980s; one can refer to Refs. [13–15] for details.

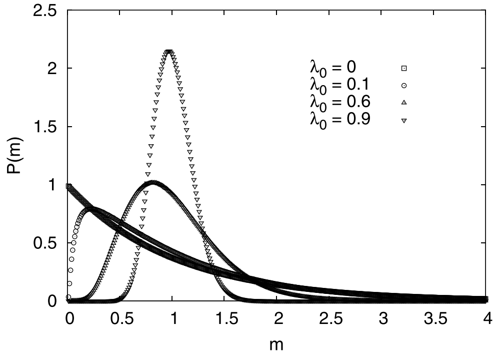

Fig. 6.1 Steady state wealth distribution $P(m)$ vs. m for CC model for $\lambda_0 = 0, 0.1, 0.6, 0.9$. Monte Carlo simulations were done by taking $N = 100$ agents and average wealth $M/N = 1$. Taken from [8]

In the toy model, all the agents traded with each other through pair-wise interaction similar to random energy scattering of gas molecules, thus losing or gaining a certain amount of wealth. Suppose two agents i and j having wealth m_i and m_j respectively, this trade dynamics can be written as

$$m_i' = r(m_i + m_j), \tag{6.1a}$$

$$m_j' = (1 - r)(m_i + m_j) \tag{6.1b}$$

where r is any random number drawn from 0 to 1, and after trading the wealth of the two agents i and j are m_i' and m_j', respectively. From the above equation, it is clear that the total wealth of the two agents before and after trading remains constant i.e., $m_i + m_j = m_i' + m_j'$.

The resultant wealth distribution of this model can be derived analytically.[3] One can also do simple Monte Carlo simulations to find the resulting steady state distribution. The N agents are each given initially 1 unit of wealth (so, total wealth $M = N$ thus fixing the average wealth in the economy),[4] and they trade with each other according to the dynamics given by Eq. (6.1b). It is observed that in steady state, the distribution of wealth is similar to 'Boltzmann-Gibbs' type distribution for kinetic theory of gases, i.e, $P(m) \sim \exp(-m/T)$, where $P(m)$ is the probability of an agent having wealth between m and $m + dm$, and T is average wealth of the model (here $T = M/N = 1$). The Monte Carlo simulation results for this model is shown in Fig. 6.1 ($\lambda_0 = 0$ case).

[3] Standard tools of statistical mechanics like Boltzmann transport equation, Pauli's master equation, maximization of entropy principle, etc. can be used to derive the steady state distribution of the 'Boltzmann-Gibbs' type (see, e.g., Refs. [3, 17]).

[4] It should be noted that the initial wealth distribution does not affect the steady state distribution, as long as the average wealth of the system remains the same. Essentially the system is *ergodic*.

6.2.1.2 CC Model

An important modification of the previous toy model was done by Chakraborti et al. [18] by introducing savings amongst all the agents. In that model, one considers a close economic system where total wealth and total number of agents are conserved. All the agents exchange their wealth through a trading. This much is identical to the basic model. The distinct feature is that before trading both of the participants save a fraction of wealth (this feature kind of mimics the reality that we do not put all of our wealth on the mercy of the market mechanism every now and then!). So the trading equation between two agents i and j can be written as

$$\begin{aligned} m_i' &= \lambda_0 m_i + r(1-\lambda_0)(m_i + m_j), \\ m_j' &= \lambda_0 m_j + (1-r)(1-\lambda_0)(m_i + m_j) \end{aligned} \tag{6.2}$$

where $0 \le \lambda_0 < 1$ is the 'saving propensity' (fraction of wealth that is being saved) of the agents. For simplicity, debt was not permitted in this model. By running simulations, it was observed that the steady state distribution is completely different from Boltzmann-Gibbs like, for any positive λ_0 value. The shape of the distribution looks like Gamma-like distributions [17, 20–22], and the most probable value depends upon the value of λ_0. Results corresponding to different values of the saving propensities are shown in Fig. 6.1. The analytical closed form of the steady state distribution remains an open problem.

6.2.1.3 CCM Model

In the last case, the CC model, all agents had a fixed saving propensity λ_0, i.e., the savings propensity does amongst the (homogeneous) agents. But for modeling purpose and also for the sake of reality, one can go one step forward and assume that the agents are heterogeneous. So a natural generalization is to consider saving propensities to be different for different agents. Precisely this modification was done by Chatterjee et al. [23]. They made the same assumptions in their model as the previous ones, but the only difference was that each agent i had a characteristic saving propensity λ_i, which could take value $0 \le \lambda_i < 1$ drawn randomly from a uniform random distribution. Each λ_i (for $i = 1, \dots, N$) is fixed over time, and is thus a quenched variable. The trading equation of this model can be written as

$$\begin{aligned} m_i' &= \lambda_i m_i + r\big((1-\lambda_i)m_i + (1-\lambda_j)m_j\big), \\ m_j' &= \lambda_j m_j + (1-r)\big((1-\lambda_i)m_i + (1-\lambda_j)m_j\big) \end{aligned} \tag{6.3}$$

where λ_i and λ_j are the saving propensities for agent i and agent j, respectively. By doing this apparently simple modification, an interesting phenomenon emerged—the steady state wealth distribution gave rise to a power law tail with exponent 2

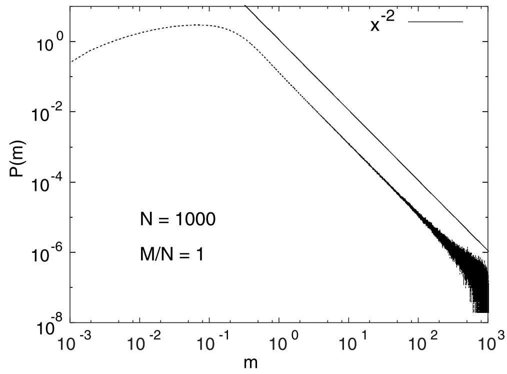

Fig. 6.2 Steady state wealth distribution $P(m)$ vs. m for CCM model ($0 \leq \lambda_i < 1$). *Dashed line* represents the results of Monte Carlo simulations, for $N = 10^3$ and $M/N = 1$. The power law is fitted with x^{-2} (*solid line*). Adapted from [8]

(see Fig. 6.2).[5] The steady state wealth distribution statistics for a single realization of quenched set (fixed propensities λ_i's) is observed to be significantly different in nature with respect to the statistics averaged over a large number of *independent* quenched configurations (variable sets). The peculiarities of the statistics from any one realization is independent of the sample size, as observed in Refs. [14, 24]. This feature of the model suggests that the observed power law tail is essentially a *convolution* of the single member distributions [14, 24]. Thus the power law tail can be explained by a set of overlapping Gamma-distributions arising from the agents with very high propensities ($\lambda_i \to 1$) [14, 19]. Another interesting feature is that the wealth that an agent accumulates is correlated to the saving propensity, first observed numerically in Eq. (14) of Ref. [14]. These observations allow the steady state distribution to be easily derived analytically [19, 37].[6]

As we have pointed out before, the empirical income or wealth distributions do display both the exponential part and the power-law decay. These two models, CC and CCM, and simple other variants [3, 26], are then able to capture (or at least reproduce) the basic features of the whole income/wealth distribution.

6.2.1.4 An Extension of CCM Model

Here we discuss another extension of the kinetic exchange model, studied in [27, 28]. The model can be described as follows: using the same framework as above, the only difference is that a trade takes place between two agents investing the *same* amount of wealth. Therefore in every transaction, the agents take an "effective" saving propensity λ which changes over time. Suppose, any two agents i

[5]Detailed numerical studies [24] showed that while the first two, the toy model studied in Sect. 6.2.1.1 and the CC model in Sect. 6.2.1.2, are *ergodic* and *self-averaging*, the third one (CCM) is not, which makes it very difficult to be studied numerically. This is an advisory note to students and beginners who want to study this numerically.

[6]For another attempt using master equation, see Ref. [25].

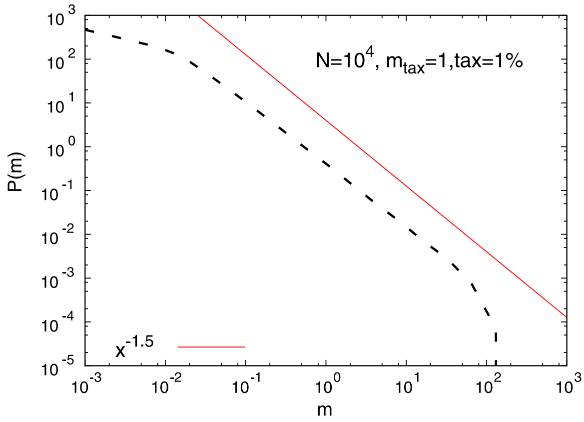

Fig. 6.3 Steady state distribution wealth for the extension model described in Sect. 6.2.1.4. The Monte Carlo simulation was done by taking $N = 10^4$, $M/N = 1$, and 1 % wealth tax was collected from the agents having wealth ($m_{tax} = 1$) greater than the average wealth of the model, after every 10 Monte Carlo time steps. The numerical simulations are plotted using a *dashed line*. The power law is fitted with $x^{-1.5}$ (*solid line*). Taken from [27]

and j have wealth m_i and m_j, respectively, and they go for a trading. If the saving propensity for agent i is assumed to be $\lambda_i = m_i/(m_i + m_j)$ and for agent j, to be $\lambda_j = m_j/(m_i + m_j)$, then it is clear from trading equation Eq. (6.3), that the total wealth is conserved before and after trade, and both agents invest the wealth: $m_i m_j/(m_i + m_j)$. For this model, it is observed that in steady state, the wealth condenses to a single agent, a feature very similar to the results obtained by Chakraborti [29]. The condensation can be avoided, if taxation is introduced into the system. Suppose, the tax is applied for the agents who have wealth greater than the average wealth, and this tax is collected periodically after a constant time interval. For this model it is found that the distribution of wealth again has a power law tail.

In the Monte Carlo simulations, $N = 10^4$ agents were considered and everybody was initially given $M/N = 1$ unit of wealth. All agents traded among themselves according to rules described above. Also, 1 % of total wealth was taken as tax, after every 10 Monte Carlo time steps (one Monte Carlo time step is defined as equivalent to N numbers of random trades among the agents) from the agents who have wealth ($m_{tax} = 1$) greater than the average value. The collected wealth is then *re-distributed uniformly* over all the agents. By doing this, it was observed (see Fig. 6.3) that the wealth distribution follows a power law tail with exponent 1.5. This is another way of recovering the power law using the kinetic exchange framework.

6.2.2 Phase Transitions in Kinetic Exchange Model

Here, we will describe another variant [30] of the above kinetic exchange models by introducing a threshold value, inspired by the concept of 'poverty line' in eco-

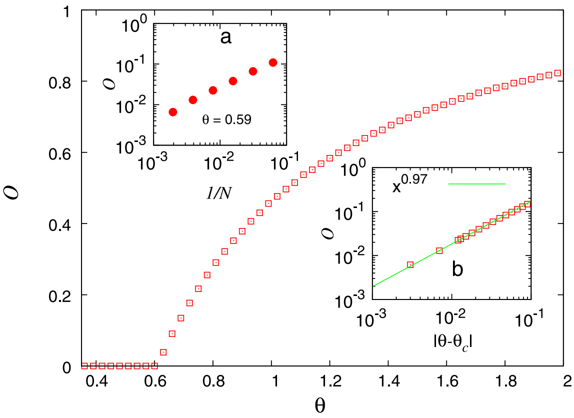

Fig. 6.4 The threshold values versus order parameters plot for $N = 10^4$. (*Right inset*): Near critical point the order parameter fits with scaling form $O = (\theta - \theta_c)^\beta$ with $\beta = 0.97$, $\theta_c = 0.6075$. (*Left inset*): It is shown that below critical point $\theta = 0.59$ the order parameter goes to zero in the thermodynamic limit. Taken from [30]

nomics. The model can be described as follows: Here agents exchange their wealth as described in Eq. (6.1a). But the only difference is that a threshold value of wealth θ is defined, and a trade between two agents occurs, if *at least one* of the two agents has wealth less than θ. Since there is a value of threshold, if all agents accumulate wealth greater than θ, then in such a situation the dynamics stops. The maximum limit of the threshold value θ below which the dynamics is stopped within some finite time, defines as critical value θ_c. The order parameter O is defined as the average total number of agents having wealth less than θ i.e., $O = \int_0^\theta P(m)dm$, where $P(m)$ is the probability distribution function of wealth. To make the system *ergodic*, a perturbation is applied into the system whenever the dynamics is stopped, and a particle having energy above θ is selected at random and its energy fully transfered to any other particle. For characterization, the model was studied for mean field (MF), one dimensional (1D) and two dimensional (2D) square lattices.

The mean field results are discussed here. Suppose after a time step τ, called the "relaxation time", the dynamics reaches a steady state. After the system reaches steady state, the order parameters are measured for different values of θ, and plotted as shown in Fig. 6.4. From the figure, it is observed that after the point $\theta = 0.6075$ (critical point) the order parameter increases. The order parameter near the critical point obeys a scaling form as $O \sim (\theta - \theta_c)^\beta$, where β is order parameter exponent, and the value $\beta = 0.97$ fits well with the scaling form. Also at critical point, the time variation of order parameter fits with the scaling form $O(t) \sim t^\delta$ with $\delta = 0.93$ (see Fig. 6.5). To confirm the existence of the transition, the relaxation times τ are measured for different values of θ. It is observed that there exists a clear time scale divergence behavior with scaling form $\tau \sim |\theta - \theta_c|^{-z}$, with $z = 0.83$ (see Fig. 6.6). All these observations and behaviors suggest that there exists a "phase transition" at $\theta = \theta_c$. To determine the exact universality class, the model was studied for 1D and 2D square lattices too, and the obtained scaling exponents suggested that the universality class is close to the Manna universality [30–32].

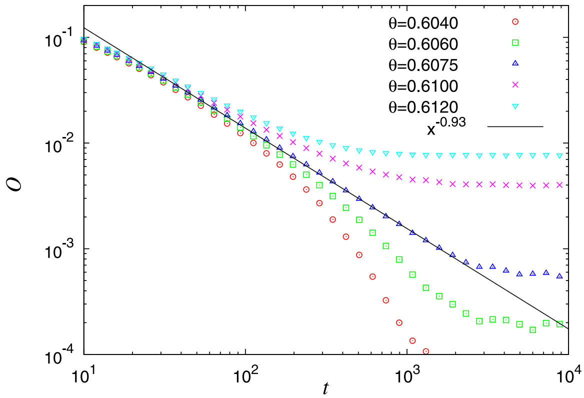

Fig. 6.5 Time variation of order parameters for different values of θ are shown. Near critical point, the order parameter fits the scaling form $O = t^{-\delta}$ with $\delta = 0.93$. Taken from [30]

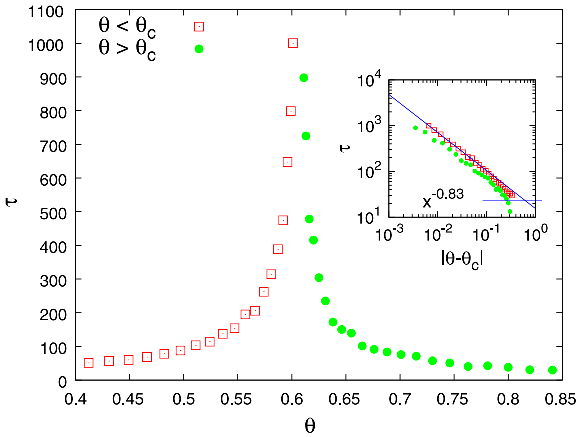

Fig. 6.6 The relaxation times are plotted for different θ values. It is observed that at critical point, the relaxation time diverges clearly. (*Inset*) At critical point, the relaxation time τ diverges as $\tau = (\theta - \theta_c)^{-z}$ with $z = 0.83$. Taken from [30]

6.2.3 A Brief Summary of the KWEM

In this section, we have discussed different kinetic exchange models of wealth for closed economic systems. First we have considered the random reshuffling exchange dynamics and observed that the wealth distribution for such a closed economic system obeys Boltzmann-Gibbs distribution. Next we have discussed that the shape of distribution changes from Boltzmann-Gibbs to Gamma-like if a (homogeneous) saving propensity is introduced in the model [17, 20–22]. We have then discussed a model (CCM) whereby assigning uniformly random saving propensities (i.e., introducing a particular type of heterogeneity in terms of savings propensity) to all the agents, a wealth distribution with a power law tail having exponent 2 is generated. Later an extension of the CCM model has been proposed where it is assumed that both agents invest the same amount wealth for a trading. For this model, if one considers taxation for the agents who have wealth greater than the average wealth in the model, then interestingly the wealth distribution again gives a power law tail, albeit with a different exponent (namely 1.5). These models reproduce features qualitatively and (one might argue) quantitatively similar to those of the empirical wealth distributions in the economy [3, 9]. In short, we have a formalism that reproduces both the basic features of income/wealth distributions (observed since the pioneering studies by Pareto [11] and Gibrat [12]): a low income region that resembles a Gamma-like distribution and the tail region that follows a power law.

Furthermore, we have discussed a kinetic exchange model where the agents who have wealth lower than a threshold value are able to initiate trade with any other agent. In this model, we have observed that there is a certain threshold value of wealth, θ_c, below which there is no agent below that particular threshold value. In other words, there is a critical value of $\theta = \theta_c$ above which this 'absorption' never happens. From critical finite size scaling, it was observed that scaling exponent for this transition is close to Manna universality class.

6.3 Firm Dynamics

In this section, we will present some recent progress made on explaining the firm size distributions in a similar framework. Of course, we cannot literally consider a binary random collision model to be a *replica* of all economic phenomena. But in case of firm dynamics, the essential mechanism that describes a redistribution of a certain quantity (wealth in the standard KEMs) still makes considerable sense. Think of firms as agents (instead of buyers and sellers) and workers in place of wealth. In the wealth exchange models, the buyers and sellers exchange wealth intentionally while carrying out economic transactions, whereas in case of firm-dynamics, there are workers who leave one firm and join others (be it intentional or unintentional). So again we have a redistribution of a certain quantity, workers w (instead of wealth). Assuming no migration, birth or death of workers, the economy remains conserved. The dynamics is technically identical to that of usual KEM except that one has to consider a n-particles collision process instead of binary collision. In case of wealth exchange a binary collision makes sense since usually the number of participants in one particular transaction is two. However, in case of firms, if one worker leaves one firm there is no particular reason to think that (s)he goes to another pre-specified firm. So we need to generalize the standard model to incorporate the possibility of having an n-ary interaction. Apart from this economic reason, there is another important advantage of such a generalization. The usual binary collision model with constant savings propensity λ cannot be solved analytically. At least it has not been solved so far. However, one can easily show that if there is a system-wide redistribution, the system can easily be solved analytically. Reference [38] proposed such a model and gave the solutions.

Before describing the model, we have to explain what we can expect from such an unconventional application of KEM. There are a couple of specific targets. The first one is obvious. Since, the KEM generates a power law very easily and since firm size distribution shows a power law, that is reason enough to apply the model. There is another empirical regularity not as well known. Firm growth process in the developing economies is known to produce divergence in their growth path giving rise to bimodality in the size distribution [40]. A surprising fact is that such bimodality has been observed in wealth distribution as well [41]. Hence, if we have a model which can accommodate binary trading as well as the whole system-side trading, then the same framework can be used to explain the non-standard features of wealth distribution as well as firm size distribution. It appears that KEM is well-suited to explain these features as well (see Ref. [39]).

6.3.1 Regularities in Firm-Size Distribution

The remarkable robustness of the long tail of the size distribution of the firms, is known from the early works by Gibrat [12]. Intuitively, a few very large firms can operate side by side with a large number of small firms. However, it was Axtell [33] who presented clear evidence that the distribution can be characterized very well by a power law. The same paper also remarked on the stability of the power law feature which has survived changes in the political, regulatory and social regimes (e.g., demographic changes in the workforce due to the influx of women in the labor force). Numerous innovations and technological changes in the production process had taken place within the same time period which were unable to affect it. Lastly, the changes in the market structures, policy changes, firm mergers, acquisitions, death and birth of firms, thousand of tweaks in the corporate laws did not affect this feature. This indicates that the statistical features of the firm growth process may well be independent of microeconomic decision-making processes like, why people choose to leave their jobs (or why and how firms decide to lay off for that matter) etc. Hence, any microeconomic foundation for the firm dynamics is not needed (at least at a first approximation). However, the rate at which the firms gain and lose workers is of interest to us as that determines the size of the firm. This rate is called the *turnover rate* in the economics literature. Another way to look at the same thing is that it measures how long the employees stay in their respective jobs.

Apart from the power law, Ref. [40] presents evidence that the developing economies are characterized by a bimodal distribution of firms. There is a bunch of very small firms creating a peak and there is another bunch of very big firms creating another peak with little mass of firms in between. This particular feature is known as the 'missing middle'. A very interesting feature is that as an economy develops usually this 'missing middle' disappears. Hence, this is somehow related to the economic condition of the country under consideration.

A Little Digression on Theory We intend to show that the turnover rates play a crucial role in the firm size distribution. But since turnover rate dictates not only inflow but also outflow, we need another parameter to describe outflow only. Hence, we describe the model in terms of the 'retention rate' which has a role identical to that of savings propensity that we used in KWEM, discussed in the previous section. The retention rate refers to the fraction of workers that stays with the firm at the end of the period. Clearly the turnover rate and the retention rate are related. As is evident, one aspect of job separation and worker hiring is that the process follows the rule of local conservation. If one worker goes from one firm to another then the total workforce remains unchanged but the workers' distribution across the firms change. Since the workers at any given year (or quarter) move around in a very large number of firms, we model this process as a repeated interaction between a large number of agents (firms) which exchanges a finite mass of (number of) workers between themselves. Clearly, the idea of the kinetic exchange model is suitable for this purpose.

Now we can discuss the economic interpretation of the terms used in the model. First, the economy consists of a large number of firms populated by workers. By firms we mean each and every production units capable of producing any kind of goods and services. Therefore there is no formal unemployment in the model. We adopt this idea in order to simplify the model so that we do not have to keep track of the mass of workers who are moving in and out of the employed workers' pool. Note that this does not affect the tail of the distribution (Zipf's law) since the tail is formed only by large firms. Secondly, we have made another simplifying assumption which is that the workers are treated as a continuous variable. While it is certainly true that there is an integer constraint on workers' head-counts, we have an advantage in treating the workers as a continuous flow in and out of the firms as it is easier (mathematically) in this case to derive the size distributions as we do not have to worry about integer constraints. Thirdly, we follow the definition that the firm's size is just the mass of workers working in the firm. There are other measures (like stock valuation, amount of goods or services produced etc.). But the number (mass) of workers has the most unambiguous definition. Hence we stick to it.

6.3.2 A Model with Constant Retention Rate

Reference [38] described the model in the following way. There is exactly one exogenous variable (the retention rate λ) and one endogenous variable (the firm sizes) in the model. We assume that time is discrete. The economy consists of an array of N firms with perfectly elastic demand for labor i.e. any firm can absorb any mass of workers that come to it. At the very beginning of the process, all firms have exactly one unit of workers (more formally, the measure of workers is one for each firm). Technically this means the total (and hence average) mass of w is constant over time. The fraction of workers that decides to stay back in their firm (which we call the *retention rate*), is denoted by λ which may in principle, vary between the firms. For the time being, we treat them as given and constant across the firms. This treatment is pioneered by Ref. [18] in the context of modeling income/wealth distributions as we have discusses above in great details. Let us denote the firm size of the i-th firm (we measure a firm's size by its workforce) by w_i ($i \leq N$ where N is the set of firms). Also, suppose that the number of firms from which the workers are leaving and moving into, is n. At each time point $(1 - \lambda)$ fraction of the workforce of those

n firms wants to leave (or the firms wanted them to leave, whatever appeals to the reader!). As mentioned above, we do not explain why they choose to do so. Hence there would be a total pool of workers that wants to change their workplace. Next, this pool of workers is randomly divided into those n firms. Hence, the dynamics is given by the following set of equations,

$$
\begin{aligned}
w_1(t+1) &= \lambda w_1(t) + \varepsilon_{1(t+1)}(1-\lambda)\sum_j^n w_j(t), \\
&\dots \\
w_i(t+1) &= \lambda w_i(t) + \varepsilon_{i(t+1)}(1-\lambda)\sum_j^n w_j(t), \\
&\dots \\
w_n(t+1) &= \lambda w_n(t) + \varepsilon_{n(t+1)}(1-\lambda)\sum_j^n w_j(t)
\end{aligned}
\tag{6.4}
$$

such that $\sum_j^n \varepsilon_{j(t)} = 1$ for all t. As is evident from above, this is a straight generalization of the usual kinetic exchange models (with $n = 2$) that has primarily been used to study the income/wealth distribution models (see Ref. [8] and the above sections). A little note on the notations: we use t within the first bracket when referring to the endogenous variables like the size of the firm ($w(t)$) and we use the same in subscript when referring to the exogenous random variables (e.g., ε_t).

Construction of the Division Factor ε We impose some meaningful restrictions on ε (as described by Ref. [38]).

1. $\varepsilon_i \geq 0\ \forall i$ and the sum of all ε_is has to be equal to one. Otherwise, the economy would not be conserved.
2. The distributions of all ε_i are identical which implies that the expection $E(\varepsilon_i) = 1/n$ for all i.
3. If $n = 2$, $\varepsilon_i \sim$ uniform [0, 1]. We impose this constraint so that at the lower limit of n, we get back the usual CC-CCM models (see Ref. [8]).

Formally, the problem then boils down to that of sampling uniformly from the unit simplex (see Refs. [35, 36]). We follow the standard algorithm and below we show the corresponding distribution of ε.

1. Create a vector of independent random variables drawn from uniform distribution over [0, 1], $\xi_1, \xi_2, \dots, \xi_n$.
2. Take logarithm of all the elements of the vector.
3. Divide each element by the sum of all the elements. Call the i-th result ε_i for all i.

One can derive the probability density function of the ε_i which is the following:

$$f(\varepsilon_i) = (n-1)(1-\varepsilon_i)^{n-2}, \tag{6.5}$$

that is, ε has a beta pdf with parameters 1 and $n-1$. Clearly, when $n = 2$ the distribution of ε is uniform [0, 1] as expected.

6.3.2.1 Solution of the Model

We follow Ref. [38] to describe the solution. First, we note that the solution to the usual kinetic exchange model with binary interaction is not known yet (see Refs. [8, 15]). However, one can derive an exact result for the case where the number of interacting firms is in the order of the system size N i.e., if one considers the case where $2 \ll n \le N$.

Note that if n is of the order of N, $\sum_j^n w_j$ is well approximated by n (recall that $E(w_j) = 1$ for all j). To make sure, note that $\sum_j^N w_j = N$ by specification of the model. To derive an exact result (instead of approximation), we shall assume that all firms interact at every step, i.e., $n = N$. Evidently the system of equation becomes

$$\begin{aligned} w_1(t+1) &= \lambda w_1(t) + \varepsilon_{1(t+1)}(1-\lambda)N, \\ &\dots \\ w_i(t+1) &= \lambda w_i(t) + \varepsilon_{i(t+1)}(1-\lambda)N, \\ &\dots \\ w_N(t+1) &= \lambda w_N(t) + \varepsilon_{N(t+1)}(1-\lambda)N \end{aligned} \tag{6.6}$$

with each ε_i is beta distributed as has been shown in Eq. (6.5) (see Construction of ε in Sect. 6.3.2). Note that in this form, we get rid of the effects of $w_j(t)$ in the evolution equation of $w_i(t)$ for all $j \neq i$ thus enabling us to uncouple the system of equations describing the coupled system (note that technically this system is still coupled). One more simplification is possible.

Let $\mu = N(1-\lambda)\varepsilon$ ignoring the subscripts. For a given N, it is easy to verify that the probability distribution of μ for large N is

$$\lim_{N\to\infty} f(\mu) \simeq \psi e^{-\psi\mu} \quad \text{where } \psi = \frac{1}{1-\lambda}. \tag{6.7}$$

Therefore, the system effectively reduces to

$$\begin{aligned} w_1(t+1) &= \lambda w_1(t) + \mu_{1(t+1)}, \\ &\dots \\ w_i(t+1) &= \lambda w_i(t) + \mu_{2(t+1)}, \\ &\dots \\ w_N(t+1) &= \lambda w_N(t) + \mu_{N(t+1)}, \end{aligned} \tag{6.8}$$

which is a system of autoregressive type equations with the distribution of errors (μ) given by Eq. (6.7). This is clearly solvable now.

6.3.2.2 Steady State Distributions

Let us now describe the steady state behavior of the system. First, we can consider the moments and show how they differ from the usual binary exchange mechanism. One writes the k-th moment of the distribution (without subscript) as

$$E\big((w-1)^k\big) = E\left(\sum_{l=0}^{k} \binom{k}{l} (-w)^l\right). \tag{6.9}$$

One simplifying assumption we make here is that w_i and w_j are independent variables (technically, they are not since the sum of all w_i's has to be a constant by structure of the model, N in this case; but for large number of interacting firms, this is a good approximation). It is easy to verify that with all firms interacting ($n = N$), the variance is given by

$$V(w) = \frac{(1-\lambda)}{(1+\lambda)}$$

whereas in the case of binary interaction [34]

$$V(w) = \frac{(1-\lambda)}{(1+2\lambda)}.$$

Note that for $\lambda = 0$, variance is unity in both cases which indicates that the distribution is the same (exponential) in both cases (not proven here; please see Ref. [38] for a derivation). To solve the system, let us write it as

$$w(t+1) = \lambda w(t) + \mu_{t+1}$$

which can, in turn, be rewritten with the lag operator L as $(1 - \lambda L)w(t) = \mu_t$ and hence,

$$w(t) = \mu_t + \lambda\mu_{t-1} + \lambda^2\mu_{t-2} + \lambda^3\mu_{t-3} + \cdots .$$

Note that we already know the distribution of μ from Eq. (6.7),

$$f(\mu) \simeq \frac{1}{1-\lambda} e^{-\frac{1}{1-\lambda}\mu}.$$

Therefore by transforming the variable we can write

$$w = \tilde{\mu}_0 + \tilde{\mu}_1 + \tilde{\mu}_2 + \tilde{\mu}_3 + \cdots$$

where $\tilde{\mu}_j = \lambda^j \mu_{t-j}$ is distributed as

$$f(\tilde{\mu}_j) = \frac{1}{\lambda^j(1-\lambda)} e^{-\frac{\tilde{\mu}_j}{\lambda^j(1-\lambda)}}.$$

One can neglect the terms with high powers (more than say $\bar{k}$) of λ. Then firm-size w is the nothing but the sum of $\bar{k}$ exponentially distributed random variables with different parameters. Note that the Laplace transformation $L(s)$ of μ_j is $\phi_j/(\phi_j+s)$ where $\phi_j = 1/(\lambda^j(1-\lambda))$. Since the μ_j's are *i.i.d.* by definition (since the division factor ε was *i.i.d.*), pdf of w would be the convolution of the pdfs of the $\bar{k}$ random variables. By property of Laplace transformation, one can verify that the distribution of w would be (by taking limit on $\bar{k}$)

$$f(w) = \lim_{\bar{k}\to\infty} \sum_{i=1}^{\bar{k}} \phi_i \exp(-\phi_i w) \prod_{j=1, j\neq i}^{\bar{k}} \left(\frac{\phi_j}{\phi_j - \phi_i}\right) \tag{6.10}$$

where ϕ_i defined as $\phi_i = 1/(\lambda^i(1-\lambda))$.

6.3.3 Distributed Retention Rates

So far we have considered only a fixed retention rate λ. In this section we consider distributed λ (i.e., the retention rates differ across firms but they are fixed over time) following Ref. [8]. Specifically, we assume that the retention rates are uniformly distributed over the interval [0, 1] across the firms. The new system of equation is

$$\begin{aligned}
w_1(t+1) &= \lambda_1 w_1(t) + \varepsilon_{1(t+1)} \sum_j^n (1-\lambda_j) w_j(t), \\
&\dots \\
w_i(t+1) &= \lambda_i w_i(t) + \varepsilon_{i(t+1)} \sum_j^n (1-\lambda_j) w_j(t), \\
&\dots \\
w_n(t+1) &= \lambda_n w_n(t) + \varepsilon_{n(t+1)} \sum_j^n (1-\lambda_j) w_j(t).
\end{aligned} \tag{6.11}$$

To solve Eq. (6.11) in the steady state, note that $(1-\lambda_i)E(w_i) = C$, a constant, solves the problem. Hence, by following Ref. [37] one can easily show that the resultant distribution of the above model is a power law. Essentially, the argument is if λ is distributed uniformly across the firms, then the average mass of workers is the inverse of a uniform distribution which is known to be the Zipf's law. We have already encountered this argument in the CCM [8] model discussed at the beginning of this chapter.

6.3.4 A Model with Time-Varying Retention Rate: Emergence of Bimodality

We have already discussed how one can model firm dynamics with the tools provided by KEM. Now we are in a position to tackle heterogeneity in the retention rate both over time and across agents. More precisely, we will describe the retention rate λ as a function of current size and thus induce a non-trivial time-dependence on the retention rate as size of any firm fluctuates over time. Reference [39] defines the dynamics by the following set of equations,

$$
\begin{aligned}
w_1(t+1) &= \lambda\big(w_1(t)\big)w_1(t) + \varepsilon_1(t+1)\sum_j^N \big(1-\lambda\big(w_j(t)\big)\big)w_j(t),\\
&\dots\\
w_i(t+1) &= \lambda\big(w_i(t)\big)w_i(t) + \varepsilon_i(t+1)\sum_j^N \big(1-\lambda\big(w_j(t)\big)\big)w_j(t),\\
&\dots\\
w_n(t+1) &= \lambda\big(w_n(t)\big)w_n(t) + \varepsilon_n(t+1)\sum_j^N \big(1-\lambda\big(w_j(t)\big)\big)w_j(t)
\end{aligned}
$$

such that $\sum_j^n \varepsilon_j(t) = 1$ for all t. As is evident from above, this is a further generalization of the model discussed above. Here, the retention rate λ_i not only characterizes the agents but also explicitly becomes a function of time, $\lambda_i = \lambda_i(t)$ due to dependence on the current level of employment $w_i(t)$. Following Ref. [39], let us assume the following functional form of λ,

$$\lambda(w) = c_1\big(1 - \exp(-c_2 w)\big) \quad c_1 \text{ and } c_2 \text{ being constants.} \tag{6.12}$$

Note that the retention rate increases as the current work-force increases. This equation basically captures the more realistic scenario that as a firm increases its workforce, the more workers it retains; or in the context of wealth distribution, a richer person saves more (see also Ref. [42] for interesting discussions). The exact solution is not known for this system. SO we perform Monte Carlo simulation which shows emergence of bimodality for certain parameter configurations (see Fig. 6.7).

It should be emphasized that this whole exercise distinguishes the KEM approach to the problem of 'missing middle' from other approaches that put the importance either on size-dependent or size-independent dynamics. We take the position that the firm-level dynamics is size-dependent or independent, depending on the level

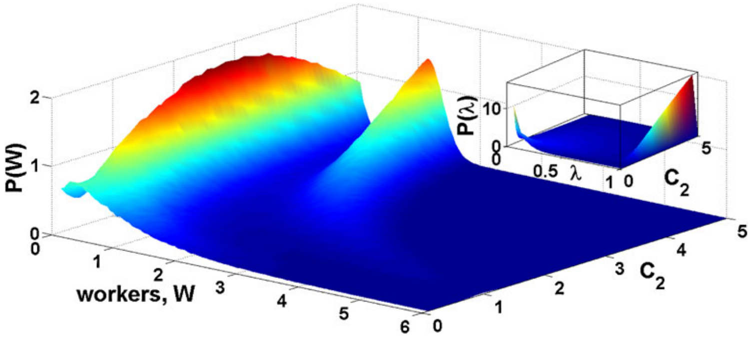

Fig. 6.7 Emergence of bimodality with Eq. (6.12) is shown with the variation of the parameter c_2 with $c_1 = 0.95$. The corresponding distribution of the retention rate λ is shown in the *inset*. Taken from [39]

of development of the economy as a whole. More specifically, we conjecture that the firms in the developed economies have fixed-heterogeneity whereas for poorer economies, the firms have size-dependent heterogeneity in the retention rates. In support of that conjecture, we note that the selection effects are important mostly for the micro firms (mostly unregistered, very small firms in the developing economies) and such effects are less prominent for larger firms (mostly found in the developed economies). The selection effects can produce heterogeneity depending on many factors, e.g., access to credit markets, pool of entrepreneurs, mobility of inputs, etc. Note that such facilities are mostly absent in the poorer economies. Hence, the firms in the poorer economies can have size-dependent dynamics. For example, a large firm can have access to credit market whereas a small firm may not have any access whatsoever; but in the developed economies all firms have access to credit markets, i.e., the access to credit market itself can act as a barrier to a small firm for expansion in size. This justifies the conjecture of scale-dependent heterogeneity for firms in poorer economies. However, we abstract from all such microeconomic details and posit that the heterogeneity is reflected solely in the retention rate which determines the firm's size in the this context. This simplification enables us to economize on the number of variables we study. We see that ex-post heterogeneity (agents are ex-ante identical, but because of the dependence of the retention rate on w, they are ex-post heterogeneous) induces bimodality. However, as the economy develops the heterogeneity becomes ex-ante as in the CCM model [8] giving rise to a power law distribution.

6.4 Opinion Dynamics

6.4.1 Kinetic Opinion Exchange Model

In this section, we deal with the emergence of consensus, which is an important issue in social science problems [5, 43–46]. The key question is of course, how a group of interacting individuals select between different options (vote, language, culture, opinions, etc.), leading to a state of 'consensus' in one such option, or a state of coexistence of many of them. Consensus, e.g., in opinion formation, is an "ordered phase" where the majority of the system is biased to a particular opinion. Though the influence of opinions in society has been an important field of study for a long time, the dynamics of opinion spreading has attracted the attention of physicists only recently, and there has been already several significant attempts to model such behavior in the light of rather well understood topics of physics like phase transitions and critical phenomena. These models have helped us to understand how *global* consensus emerges out of e.g., individual opinions [47–53]. In most of such formulations, opinions are usually modeled as variables, discrete or continuous, and are subject to changes due to many factors—binary interactions, global feedback, etc. or even external factors. Usually the interest in these studies lie in the distinct steady state properties, usually one phase characterized by individuals with widely different opinions and another phase with a finite fraction of individuals with similar opinions.

Here, we intend to focus our attention to a specific class of simple models proposed recently [54, 55], having apparent similarity with kinetic wealth exchange models (KWEMs) discussed earlier [8, 18, 23]. The tuning parameter in these models, analogous to the saving propensity in KEM, is 'conviction' (λ), which determines the extent to which an individual remains biased to its own opinion while interacting with somebody else, and the 'influence' parameter (μ), which is a measure of the influencing power or the ability of an individual to impose its opinion on some other individual. In the original model [54, 55], the two parameters were taken to be identical. The opinions of individuals are continuous variables in $[-1, 1]$ and change due to binary interactions. It was observed that if the conviction parameter was fixed above a threshold, the system reaches a state of consensus ("symmetry-broken" phase or "active" phase) and below this threshold value all individual opinions were equal to zero ("absorbing" phase). Later, a generalised version of this model was studied [56], where two parameters, λ and μ, were taken to be non-identical. It was found that in that case, the symmetric and symmetry-broken phases were separated by a phase boundary given by $\lambda = 1 - \mu/2$. Biswas et al. [57] studied some variants of the above discussed models and estimated the critical points by mean field theory (MFT), which were supported by numerical simulations. The critical exponents associated with the phase transition were also estimated. Later the discretized version of the LCCC model was exactly solved [58], which also showed an "active-absorbing phase transition" as was seen in the continuous version. Apart from the two- agent or binary interaction, the three-agent interaction were also taken into account. While the phase diagram of the two-agent interaction led to a continuous transition line, the three-agent interaction showed a discontinuous transition. Continuous opinion dynamics with both positive and negative mutual interactions were also studied [59].

In the model introduced earlier by Deffaunt et al., opinion exchange between two agents took place only when the difference between their opinions was less than or equal to a pre-assigned quantity δ [51]. This idea of bounded confidence was implemented in the LCCC model (controlled by the only parameter λ) [60]. Three distinct regions were identified in δ–λ phase diagram.

Percolation transitions of geometrical clusters (group of adjacent sites with an opinion value equal to or above a pre-assigned threshold value Ω) in the square lattice LCCC model, had also been studied by varying conviction and influencing parameters [61]. The transition point was different from that found for the transition of the order parameter. Although the transition point was also dependent on Ω, the critical exponents were independent of the threshold opinion value, conviction and influencing parameters. The exponents also suggested that percolation in LCCC model belongs to a separate universality class. We will now discuss in some details the above cases.

6.4.2 LCCC Model

In the original model, a discussion between two persons were viewed as a simple two-body *scattering* process in physics and at any time t, only two persons were allowed to discuss. In a society consisting of N persons each of the i-th person was assigned with an opinion value at a time t as $o_i(t) \in [-1, +1]$. Binary interactions took place as follows:

$$
\begin{aligned}
o_i(t+1) &= \lambda\big[o_i(t) + \varepsilon o_j(t)\big],\\
o_j(t+1) &= \lambda\big[o_j(t) + \varepsilon' o_i(t)\big],
\end{aligned}
\tag{6.13}
$$

where ε and ε' are uncorrelated random numbers uniformly distributed between 0 and 1. λ was the conviction parameter which determines to which extent a person retains his own previous opinion and is independent of time. It was assumed here that the conviction parameter was also equal to the influence parameter which quantified how much an individual influenced another person. The agents were taken to be homogeneous in the sense of having the same or uniform conviction parameter.

Social interactions followed by Eq. (6.13) lead to consensus formation depending upon the value of the conviction parameter λ. The steady state value of the average opinion after a long time t, $O = |\sum_i o_i|/N$ represents the "ordering" in the system. Starting from a random disordered state (o_is are uniformly distributed with positive and negative values and $O \simeq 0$) after a certain time $t = \tau$ (relaxation time) the system either evolves to the "para" state (all individual agents having zero opinion) for $\lambda \leq 2/3$ or continuously changes to a symmetry broken state (all individuals having opinion of same sign) for $\lambda \geq 2/3$ (Fig. 6.8). The variance of O does not diverge but shows a cusp near $\lambda = 2/3$. The fraction of agents (p) having opinion $o_i = \pm 1$ was also measured in the steady state as a function of λ and the growth behaviour was found to be similar to O as discussed above.

The relaxation behaviours were studied for both O and p and a divergence of relaxation time $\tau \sim |\lambda - \lambda_c|^{-z}$ was observed for both. The relaxation behaviour for

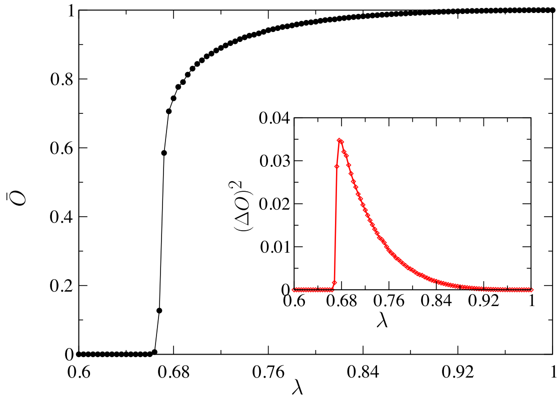

Fig. 6.8 Simulation results for average opinion as a function of λ. (*Inset*) Simulation results for the variance of O with λ. Taken from [54, 55]

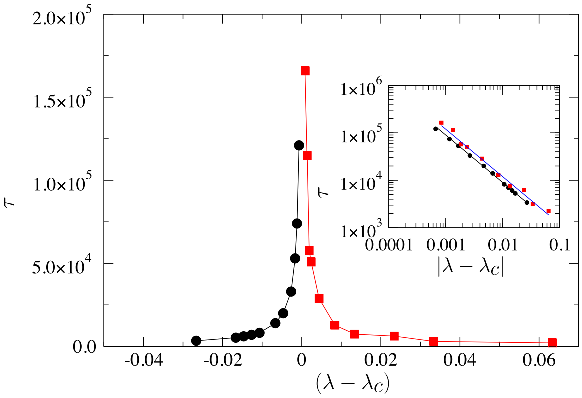

Fig. 6.9 Numerical results for relaxation time behaviors τ versus $\lambda - \lambda_c$, for multi-agent model with O. (*Insets*) Determination of exponent z from numerical fits of $\tau \sim |\lambda - \lambda_c|^{-z}$. Taken from [54, 55]

O has been shown in Fig. 6.9. The value of z corresponding to O and p are 1.0 ± 0.1 and 0.7 ± 0.1 respectively.

Apart from the exponents β and z (both for O and p), two more exponents were measured for 1D LCCC [57]. At the critical point ($\lambda_c = 2/3$), due to critical slowing down the system relaxes algebraically with time

$$O(t) \sim t^{-\delta} \tag{6.14}$$

The order parameter p also shows a similar form at the critical point. The value of δ for O is 1.00 ± 0.05 and p is 1.15 ± 0.01.

From finite size scaling theory, an order parameter X is expected to follow a scaling relation of the form

$$X(t) \approx t^{-\delta} \mathcal{F}\left(t^{1/\nu_{||}} \Delta\right), \tag{6.15}$$

where $\Delta = \lambda - \lambda_c$ and $\mathcal{F}$ is a universal scaling function of a form such that for large argument, the time dependence drops out ($\mathcal{F}(x) \sim x^{\delta \nu_{||}}$). Both O and p follow the scaling relation.

The basic nature of the transition produced by Eq. (6.13) was also obtained from a simple iterative map:

$$y(t+1) = \lambda(1+\varepsilon_t)y(t) \tag{6.16}$$

with the restriction that $y(t) \leq 1$, which is ensured by assuming that if $y(t) \geq 1$, $y(t)$ is set equal to 1. ε_t is a stochastic variable uniformly distributed between 0 and 1. In a mean-field approach, the above equation reduces effectively to a multiplier map like $y(t+1) = \lambda(1+\langle\varepsilon_t\rangle)y(t)$, where $\langle\varepsilon_t\rangle = 1/2$. For $\lambda \leq 2/3$, $y(t)$ converges to zero. An analytical derivation for the critical point was also given where it was found that $\lambda_c = \exp\{-(2\ln 2 - 1)\} \approx 0.6796$ [62].

6.4.3 The Generalised LCCC Model

In the generalised model [56], a second parameter representing the influencing power of an agent was treated distinctly from the conviction parameter, because in most realistic scenarios, a person with a strong retention power may not always have the same power to influence others. Thus, the interaction here is as follows:

$$\begin{aligned} o_i(t+1) &= \lambda_i o_i(t) + \varepsilon\mu_j o_j(t), \\ o_j(t+1) &= \lambda_j o_j(t) + \varepsilon'\mu_i o_i(t), \end{aligned} \tag{6.17}$$

where λ_i and μ_i correspond to the conviction and influencing parameter for the i-th agent. In the simpler version of the model studied, a homogeneous society with uniform λ and μ were assumed. Considering $\lambda = \mu$ gives the original LCCC model, as discussed earlier.

Again, the average opinion was studied both as functions of λ and μ. The average opinion showed spontaneous symmetry breaking in the λ–μ plane. In the steady state for non-zero solution of O the condition is

$$(1-\lambda)^2 = \langle\varepsilon\varepsilon'\rangle\,\mu^2, \tag{6.18}$$

which reduces to

$$\lambda = 1 - \mu/2. \tag{6.19}$$

The phase boundary obtained numerically satisfies Eq. (6.19) (Fig. 6.10). The relaxation behaviour was studied along two paths (A and B), (A) keeping the value of μ constant and changing the value of λ and (B) keeping the value of λ constant and changing the value of μ. The relaxation behaviour of O showed interesting feature along both the paths. The relaxation time along path A diverged algebraically along the phase boundary and more importantly the critical exponent changed with the tuning parameters (λ and μ). For $\mu_c = 0.4$, $z = 1.04 \pm 0.01$, for $\mu_c = 2/3$, $z = 1.10 \pm 0.03$ while for $\mu_c = 0.9$, $z = 1.21 \pm 0.01$ which indicated a

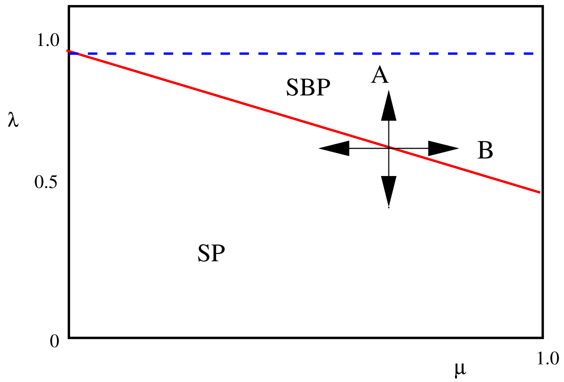

Fig. 6.10 The phase boundary obtained by numerical simulation coincides exactly with that given in Eq. (6.19). The acronyms SP and SBP denote the symmetric phase and the symmetry-broken phase, respectively. The paths A and B are possible trajectories along which the different studies can be made. Along the *dashed line* $\lambda = 1$, the opinions of all the agents are equal and take extreme values in two possible ways, either $o_i = 1$ or $o_i = -1$ for all i. Taken from [56]

non-universal behaviour. The order parameter O also showed power-law behaviour along the phase boundary,

$$O \propto (\lambda - \lambda_c)^{\beta} \tag{6.20}$$

where β also varied strongly with λ_c, μ_c. $\beta = 0.079 \pm 0.001$ at $\mu_c = 0.4$ and $\beta = 0.155 \pm 0.001$ at $\mu_c = 0.9$. The critical exponents corresponding to the condensate fraction p were z_p and β_p, which also showed non-universal behaviour along the phase boundary. When λ was kept constant and μ was varied near the phase boundary line, the magnitude of the time scales were about twice compared to those in path A along the path, although the values of the exponents were very close.

6.4.4 Variants of the LCCC Model

A simpler version of the LCCC model was studied [57], where an individual i upon meeting with another individual j retained his own opinion proportional to his conviction parameter and picked up a random fraction of j's opinion (model C hereafter). The interaction can be written as

$$\begin{aligned} o_i(t+1) &= \lambda o_i(t) + \varepsilon o_j(t), \\ o_j(t+1) &= \lambda o_j(t) + \varepsilon' o_i(t) \end{aligned} \tag{6.21}$$

where the symbols carry their usual meaning, as mentioned earlier.

Numerically, it was observed that below a critical value λ_c, $o_i = 0\ \forall i$ giving $O = 0$ while for $\lambda > \lambda_c$, $O > 0$ and went to 1 as $\lambda \to 1$, a *symmetry broken* phase

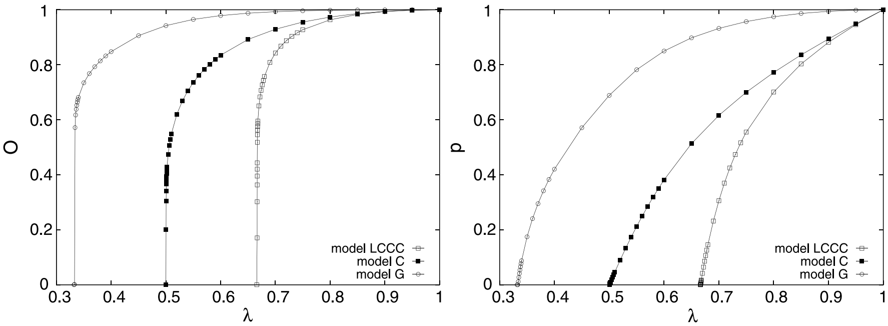

Fig. 6.11 The phase diagrams for the three models: LCCC, C and G, described in the text. *Left*: Behavior of order parameter O. *Right*: Behavior of condensation fraction p. Taken from [57]

with $\lambda_c \approx 1/2$. Mean field estimate gave for the stable value of O,

$$O\big(1 - \lambda - \langle\varepsilon\rangle\big) = 0, \tag{6.22}$$

and hence $\lambda_c = 1/2$.

The effect of global feedback on an agent's personal opinion during an interaction was also investigated [57]. An agent while taking part in a social interaction, apart from being influenced by the other person was also stochastically influenced by the "average opinion" of the entire society at that time (model G). Mathematically, this is represented as,

$$\begin{aligned} o_i(t+1) &= \lambda\big[o_i(t) + \varepsilon o_j(t)\big] + \varepsilon' O(t), \\ o_j(t+1) &= \lambda\big[o_j(t) + \eta o_i(t)\big] + \eta' O(t), \end{aligned} \tag{6.23}$$

where η and η' are random numbers drawn from uniform distribution [0, 1]. In this case, the *symmetry broken* phase $O \neq 0$ appeared for $\lambda > 1/3$, and for $\lambda \leq 1/3$ the system was in a *symmetric* phase, with $o_i = 0\ \forall i$ and all individual agents had the opinion 0. This was also explained by a mean-field approach as O reached a steady state value,

$$O = \lambda\big(1 + \langle\varepsilon\rangle\big)O + \big\langle\varepsilon'\big\rangle O \tag{6.24}$$

which gave $\lambda_c = 1/3$.

The comparative phase diagrams for the three models (1D LCCC, model C and G) according to behaviour of order parameter O and p has been shown in Fig. 6.11. Both for the above discussed models (1D LCCC, model C and G) the critical exponents were measured numerically (Table 6.1).

Table 6.1 Table comparing the different quantities for the 3 models (1D LCCC, model C and G)

Model	λ_c (Mean field)	Measured quantity	β	z	δ	$\nu_{\|\|}$
LCCC	2/3	O	0.10(1)	0.97(1)	1.00(5)	1.2(1)
		p	0.95(2)	1.1(1)	1.2(1)	1.1(1)
C	1/2	O	0.17(1)	1.58(1)	0.500(5)	0.10(1)
		p	0.98(2)	1.34(1)	0.521(5)	2.00(2)
G	1/3	O	0.081(1)	1.2(1)	0.585(1)	1.6(1)
		p	0.85(1)	1.75(1)	0.585(1)	2.0(1)

6.4.5 Discrete LCCC Model

An exact solution for the LCCC model was done for a discretized version (reproduced below in a similar form, from Ref. [58]). The dynamics evolves as

$$o_i(t+1) = \lambda o_i(t) + \mu \varepsilon o_j(t), \tag{6.25}$$

where μ represented the j-th agent's ability to influence others. Note that in the limit $\lambda = \mu$, one recovers the LCCC model. In the discrete version $\lambda = 1$ with probability ϕ and 0 with probability $1-\phi$. The parameter ε is either 1 or 0 with equal probability, and agents could have three possible opinion values ($o_i \in \{-1, 0, +1\} \forall i$). In the generalised case $\mu = 1$ with probability q and 0 with probability $1-q$.

If f_0, f_1 and f_{-1} be the fractions of agents having opinions 0, +1 and −1, then the evolution equation can be written as,

$$\begin{aligned}\frac{dO}{dt} &= f_{-1}^2(1-\phi) + f_{-1}f_1\left(1-\frac{\phi}{2}\right) + \frac{f_0 f_1 \phi}{2} + f_{-1}f_0(1-\phi) \\ &\quad - f_1^2(1-\phi) - f_1 f_{-1}\left(1-\frac{\phi}{2}\right) - \frac{f_0 f_{-1}\phi}{2} - f_1 f_0(1-\phi).\end{aligned} \tag{6.26}$$

In the steady state, the left hand side will be zero. This gives either $f_1 = f_{-1}$, (which implies disorder) or

$$f_0 = \frac{2(1-\phi)}{\phi}. \tag{6.27}$$

It was shown that in the ordered state $f_1 f_{-1} = 0$. This condition and the disordered state condition ($f_1 = f_{-1}$) should both be valid at the critical point. This is possible only when $f_1 = f_{-1} = 0$ at the critical point. This implied, at the critical point $f_0 = 1$. Furthermore, for the sake of continuity of f_1 and f_{-1}, $f_0 = 1$ for the entire disordered phase. This condition along with Eq. (6.27) gave $\phi_c = 2/3$.

Therefore, the order parameter should be (using $f_1 + f_{-1} + f_0 = 1$)

$$O = \pm(1 - f_0) \tag{6.28}$$

where the sign will depend on whether f_1 or f_{-1} is non-zero, in the ordered (symmetry-broken) phase. Using Eq. (6.27), the above expression yields

$$O = \pm \frac{3(\phi - \frac{2}{3})}{\phi}. \tag{6.29}$$

Therefore, Eq. (6.29) gives $\beta = 1$ (since $\phi_c = 2/3$).

Similar calculation in case of the discrete generalised LCCC model yields

$$O = \pm \frac{2(\phi - \phi_c) + (q - q_c)}{q\phi}. \tag{6.30}$$

which gives that the order parameter exponent is $\beta = 1$.

The three body opinion exchange was also solved exactly. Three agents were chosen randomly and an agent changes his opinion only when the other two agree among themselves. If they contradicted, then the first agent considered the group to be neutral and only retained a fraction of his opinion, depending upon his/her conviction parameter. This can be represented mathematically as

$$o_i(t+1) = \lambda o_i(t) + \lambda \varepsilon \theta_{jk}(t), \tag{6.31}$$

where, $\theta_{jk}(t) = o_j(t)$ if $o_j(t) = o_k(t)$, $\theta_{jk}(t) = 0$ otherwise. It was shown that in the ordered state,

$$f_0 = \frac{1}{2} - \frac{3\sqrt{\phi - 8/9}}{2\sqrt{\phi}}, \tag{6.32}$$

and the order parameter takes the form

$$O = \pm \left(\frac{1}{2} + \frac{3\sqrt{\phi - 8/9}}{2\sqrt{\phi}} \right). \tag{6.33}$$

This gives $O = 0$ for $\phi < 8/9$ and in the ordered phase minimum value of O can be $1/2$ which shows that the order-disorder transition is discontinuous.

6.4.6 LCCC Model with Bounded Confidence

In the models discussed so far, there were no restrictions imposed on the interactions between any two agents. A restricted LCCC model was studied, where two agents interact according to Eq. (6.13) only when $|o_i - o_j| \leq 2\delta$ [60], where δ is the parameter that represents the 'confidence' level and can vary from zero to 1. There are two extreme limits corresponding to this model: (a) $\delta = 1$ is identical to the original LCCC model, and (b) $\delta = 0$ is the case when two agents interact only when their opinions are exactly same. Three different states were defined to identify the status of the system. When $o_i = 0$ for all i it was called neutral state, $o_i \neq 0$ for all i, but

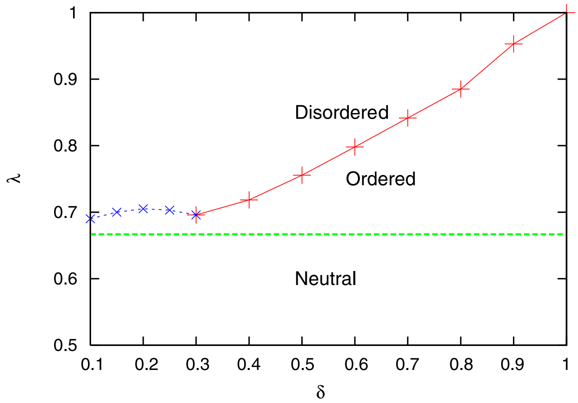

Fig. 6.12 The phase diagram in the δ–λ plane shows the existence of the neutral region (for $\lambda \leq \lambda_{c1} \simeq 2/3$), the ordered region and the disordered region. The ordered and disordered regions are separated by a first order boundary (*continuous line in red*) for $\delta \geq 0.3$ obtained using a finite size scaling analysis. For $\delta < 0.3$, the phase boundary (*broken line in blue*) has been obtained approximately only from the behaviour of the order parameter (see text). Taken from [60]

$O \simeq 0$, it was called disordered state, and when $O \neq 0$ it was an ordered state. The three states were located in the δ–λ plane (Fig. 6.12).

It is quite obvious from the figure that the ordered state appears for $\lambda_{c1} \simeq 2/3$ (as in the original LCCC model) and is independent of the value of δ. For a fixed value of δ, the value of order parameter O increases with λ and decreases to zero as λ is increased further ($\geq \lambda_{c2}$). The decrease becomes steeper with both δ and the system size N. For $\lambda_{c1} \leq \lambda \leq \lambda_{c2}(\delta)$ an ordered region exists where opinions of one sign exist. At λ_{c2} a transition to a disordered state was observed and the transition point was dependent on the value δ. It was found that at least for $\delta > 0.3$, the order-disorder transition is first-order in nature. For $\delta < 0.3$, the ordered phase shrunk to a narrow region of the phase diagram.

6.4.7 Percolation in LCCC Model

The spreading of an opinion through a society is a very important issue. The cluster formation by groups of people acquiring similarity in opinion value is significant regarding this issue. The spreading of opinion among social agents may be compared with the percolation problem in physics. In order to have an insight of the spreading phenomena in LCCC model, percolation of geometrical clusters (comprised by a group of adjacent sites with an opinion value equal to or above a preassigned threshold value (Ω)) was studied on a square lattice where agents were located on the lattice sites [61]. The opinion exchange between pair of agents was same as that of LCCC (Eq. (6.13)). It was observed that the average value of the largest cluster size was controlled by the conviction/influencing parameter λ and for a fixed value

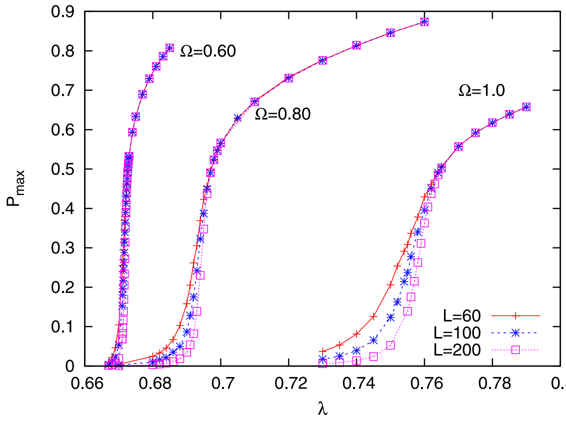

Fig. 6.13 Comparative plots for the largest cluster size with conviction parameter for three different system sizes and at three various values of the opinion threshold ($\Omega = 1.0, 0.80$ and 0.60). Taken from [61]

of Ω, at a critical value of $\lambda = \lambda_c^p$, the percolation transition occurs. One way to determine the percolation transition is to measure the relative size of the largest cluster which is designated by P_{max}. When the steady state is reached, P_{max} is calculated as S_L/L^2, where S_L is the size of the largest cluster and L is the linear size of the 2D system. The value of the critical point (λ_c^p) decreases with Ω (Fig. 6.13) and coincides with that for the transition point $\lambda_c = 2/3$ (as $\Omega \to 0.0$) at which the average opinion transition takes place (discussed in Sect. 6.4.2). Although the system does not show any finite size effect in case of the transition of the average opinion, the percolation transition shows prominent finite size effect for a given threshold opinion value Ω (Fig. 6.13).

The critical exponents were determined from the finite-size scaling relations [63, 64]. The order parameter follows the scaling form

$$P_{max} = L^{-\beta/\nu} \mathcal{F}\big[L^{1/\nu}\big(\lambda_c^p - \lambda\big)\big], \tag{6.34}$$

where $\mathcal{F}$ is a suitable scaling function. $P_{max} L^{\beta/\nu}$ were plotted against λ (at a fixed Ω) for different system sizes and then by tuning the value of β/ν, all the curves were made to cross at a single point which gives the critical conviction parameter (λ_c^p). A typical plot (for $\Omega = 0.80$ and $\lambda = \mu$) has been shown in Fig. 6.14. The finite-size scaling of the reduced fourth-order Binder cumulant of the order parameter defined as

$$U = 1 - \frac{\langle P_{max}^4 \rangle}{3\langle P_{max}^2 \rangle^2}, \tag{6.35}$$

was also studied where $\langle X \rangle$ means ensemble average of the parameter X. The Binder cumulant follows the scaling form

$$U = \mathcal{U}\big(\big(\lambda_c^p - \lambda\big) L^{1/\nu}\big), \tag{6.36}$$

where $\mathcal{U}$ is a suitable scaling function. The critical point corresponding to $\Omega = 0.80$ was $\lambda_c^p = 0.6955 \pm 0.0005$, which varies with the value of Ω. But the critical

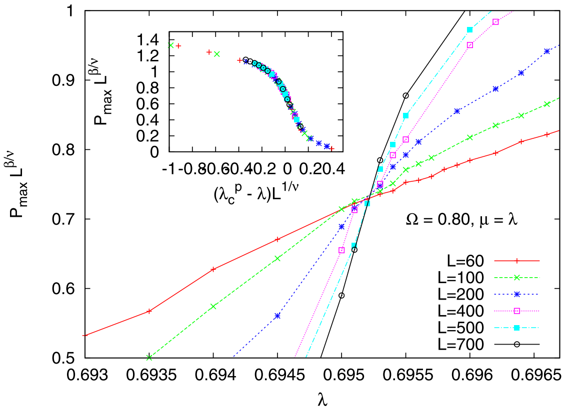

Fig. 6.14 $P_{max} L^{\beta/\nu}$ plotted against the conviction parameter λ where $\Omega = 0.80$ and $\mu = \lambda$. The curves for different system sizes ($L = 60, 100, 200, 400, 500$ and 700) cross at $\lambda_c^p = 0.6955 \pm 0.0005$ for $\beta/\nu = 0.130 \pm 0.005$. In the *inset* the data collapse for P_{max} with $\lambda_c^p - \lambda$ has been shown for $\Omega = 0.80$ giving $1/\nu = 0.80 \pm 0.01$ and $\beta/\nu = 0.130 \pm 0.005$. Taken from [61]

exponents $\beta/\nu = 0.130 \pm 0.005$ and $1/\nu = 0.80 \pm 0.01$ were independent of the value of Ω. They were also different from that obtained for the percolation transition in case of static Ising, dynamic Ising and standard percolation, indicating the LCCC dynamics to belong to a separate universality class.

The percolation transition was also studied in the case of generalised LCCC model (discussed in Sect. 6.4.3). Once again the critical exponents were found to be same as that obtained for the original LCCC model.

6.5 Final Remarks

In this article, we have tried to give a flavor of the many different kinetic exchange models, applied in various contexts such as in modeling of wealth distributions, or firm dynamics, or opinion formation in the society. There has been a flurry of activities in diverse domains, and several interesting observations and explanations have resulted, based on the common framework of simple exchanges of some quantity. It is interesting to see how the kinetic theory of gases which had played a substantial role in the initial development of the field of statistical mechanics, has inspired many more novel approaches in fields far away from the physics of gas molecules. There already exists a number of review articles, books, tutorials, etc. which have dealt with most of these topics. Keeping in mind the quote:

> "Dripping water hollows out stone, not through force but through persistence"—Ovid,

we have made another modest attempt! We would like to emphasize the effectiveness of the kinetic exchange models as serving as a skeleton for many diverse applications and implications. Hopefully, in the near future one will be able to put some more flesh on the skeleton to make it more human-like (or more reasonable, if you want)!

Acknowledgements We thank all our collaborators whose works we have reproduced here, and the critics who continue to motivate us to produce more. The authors would like to give special thanks to A. Chatterjee and S. Biswas for their useful comments and criticisms. AKC acknowledges the financial support from DST (India) under the SERC Fast Track Scheme for Young Scientists [Sanction no. SR/FTP/PS-090/2010(G)].

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