Problems in Fast Moving Non-Holonomic Elastic Systems

Hartmut Bremer

Abstract The Central Equation of Dynamics allows a unified view on existing methods and reveals them as a specific view on one and the same classical mechanics. Thereby, the particular methods exhibit special advantages and disadvantages according to the aim of investigation. For the derivation of the motion equations, the analytical methods display some drawbacks: the use of non-holonomic velocities needs an enormous effort and non-holonomic constraints can not à-priori be considered. Due to the directional derivatives w.r.t. the angular velocities, the obtained linearized equations do not represent the motions w.r.t. the co-rotational frame (and any orthogonal frame, resp.) as usually requested. This fact may lead to severe misinterpretations. In elastic multi body systems, the calculation effort increases dramatically. All these drawbacks are removed when using the Projection Equation.

1 Basics: The Central Equation of Dynamics

The principle of virtual work in dynamics,

$$\int_{(S)} (dm\ddot{\mathbf{r}} - d\mathbf{f}^e)^T \delta\mathbf{r} = 0, \tag{1}$$

was established by J.L. de Lagrange in 1764. Two years before he explained "I have to emphasize that I introduced a new characteristic δ; here, $\delta\mathbf{r}$[1] shall express a differential w.r.t. $\mathbf{r}$ which is not the same as $d\mathbf{r}$ but which is nevertheless built with the same rules". This statement obviously misled many people, in the past as well as in the present, ["obscure" (L.Poinsot,

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[1] Z from his original contribution is replaced here with $\mathbf{r}$.

1837); "black magic" (Th. Kane (1986)]. However, considering $d\mathbf{r}/dt$ instead of $d\mathbf{r}$ sheds light on the brilliant background of LAGRANGE's concept: $d\mathbf{r}/dt = \dot{\mathbf{r}}$ and $\delta\mathbf{r}$ are tangent vectors w.r.t. the constraint plane $\boldsymbol{\Phi}(\mathbf{r}) = 0 : (\partial\boldsymbol{\Phi}/\partial\mathbf{r})\dot{\mathbf{r}} = 0 \bigwedge (\partial\boldsymbol{\Phi}/\partial\mathbf{r})\delta\mathbf{r} = 0$. (LAGRANGE himself calls $\delta\mathbf{r}$ virtual *velocities*). Hence, $\delta\mathbf{r}$ is kept arbitrary (in direction and in magnitude) while $\dot{\mathbf{r}}$ represents the real solution. Adopting this intepretation leads with a few steps of calculation to the Central Equation of Dynamics

$$\frac{d}{dt}\left[\left(\frac{\partial T}{\partial \dot{\mathbf{s}}}\right)\delta\mathbf{s}\right] - \delta T - \delta W = 0, \quad \begin{cases} T : \text{kinetic energy} \\ \dot{\mathbf{s}} = \mathbf{H}(\mathbf{q})\dot{\mathbf{q}} : \text{minimal velocities,} \\ \qquad\qquad \text{non-holonomic} \\ \mathbf{q} \in \mathbb{R}^f : \text{minimal coordinates} \\ \mathbf{H} \in \mathbb{R}^{f,f} : \text{regular,} \end{cases} \tag{2}$$

from which a considerable body of methods in dynamics is derived (HELMHOLTZ, GIBBS, APPELL, HAMILTON, LAGRANGE, TZENOFF, NIELSEN, MAGGI, HAMEL $\cdots$ and the Projection Equation). The Central Equations thereby states that all these methods represent one and the same (classical) mechanics but looked at from different view-points. The Central Equation allows thus a fair comparison of methods.

2 Non-Holonomicity

The motion of a (fast moving) elastic system is composed of "rigid body coordinates" and of superimposed "elastic coordinates" (the combination of which has been introduced as "hybrid coordinates" by P.W. LIKINS in the 1970ies). The corresponding "elastic velocities" are assumed to move with small amplitudes and are therefore integrable. Thus, non-holonomicity can only arise from the rigid body motion. It is essential to emphasize that non-holonomic velocities have à-priori nothing in common with non-holonomic constraints while, the other way round, non-holonomic constraints need non-holonomic velocities for description.

2.1 Analytical Methods

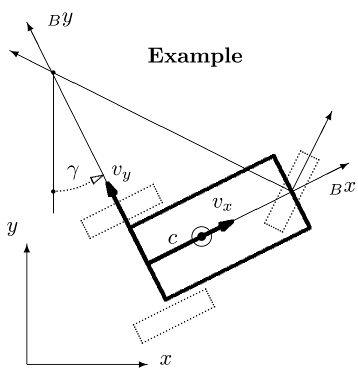
One of the most common procedures for the treatment of non-holonomic systems is due to G. HAMEL. His (explicit form of) equations read

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{s}_n} - \frac{\partial T}{\partial s_n} - Q_n + \sum_{\nu,\mu} \frac{\partial T}{\partial \dot{s}_\nu}\dot{s}_\mu \beta_\nu^{\mu,n} = 0\,; \quad n = 1 \cdots f, \tag{3}$$

$$\beta_\nu^{\mu,n} = \sum_{i,k} \frac{\partial \dot q_k}{\partial \dot s_\mu} \frac{\partial \dot q_i}{\partial \dot s_n} \left(\frac{\partial^2 s_\nu}{\partial q_i \partial q_k} - \frac{\partial^2 s_\nu}{\partial q_k \partial q_i} \right) = -\beta_\nu^{n,\mu}, \quad i,k = 1 \cdots f, \tag{4}$$

where $\beta_\nu^{\mu,n}$ represent his famous coefficients; as can be seen from Eq.(4), they are zero for s_ν being holonomic (fulfillment of H. SCHWARZ's rule). For nonholonomic s_ν, the term in parentheses vanishes for $i = k$, thus $2f(f-1)$ summation terms remain for each $\beta_\nu^{\mu,n}$.

Let us consider a simplified model of a (rigid) car neglecting the wheel masses. It moves in the (inertial) x-y-plane with velocities v_x, v_y, γ. The front wheel is a suspension wheel with arbitrary motion while the rear wheels are not allowed to slide, i.e. $v_y = 0$ w.r.t. the body-fixed frame (index B). This is a non-holonomic constraint. It is, however, not allowed to set $v_y = 0$ in advance, since then T would not more depend on v_y yielding wrong results. Thus, the calculation has first to be done for the whole set of variables:



$$T = \frac{1}{2} \dot{\mathbf{s}}^T \begin{bmatrix} m & 0 & 0 \\ 0 & m & mc \\ 0 & mc & C^o \end{bmatrix} \dot{\mathbf{s}},$$

$$\dot{\mathbf{s}} = \begin{pmatrix} v_x \\ v_y \\ \omega_z \end{pmatrix}, \quad \mathbf{q} = \begin{pmatrix} x \\ y \\ \gamma \end{pmatrix}, \tag{5}$$

$$\begin{pmatrix} \dot s_1 \\ \dot s_2 \\ \dot s_3 \end{pmatrix} = \begin{bmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} \dot q_1 \\ \dot q_2 \\ \dot q_3 \end{pmatrix} \Rightarrow \begin{pmatrix} \dot q_1 \\ \dot q_2 \\ \dot q_3 \end{pmatrix} = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} \dot s_1 \\ \dot s_2 \\ \dot s_3 \end{pmatrix}, \tag{6}$$

(m: mass, c: mass center distance, C^o: moment of inertia w.r.t. the B-frame origin). The determination of the β's is tedious[2] even for this simple example. They turn out $+1, -1, 0$, e.g.

$$\beta_1^{1,3} = -\frac{\partial q_1}{\partial s_1} \frac{\partial^2 s_1}{\partial q_1 \partial q_3} - \frac{\partial q_2}{\partial s_1} \frac{\partial^2 s_1}{\partial q_2 \partial q_3} = (-\cos\gamma)(-\sin\gamma) - (\sin\gamma)(\cos\gamma) = 0 \tag{7}$$

etc. From Eq.(3) one obtains the equations

[2] For practical purposes, HAMEL himself prefers a direct calculation of $d\delta\mathbf{s} - \delta d\mathbf{s}$: "it is perhaps not always convenient to calculate the table of the β's ... but in fact easier to look for the δs_μ from the expression $\delta d s_\rho - d\delta s_\rho$" [Hamel 1949], p. 483. (Hamel's original ϑ is replaced here with s).

$$(\frac{d}{dt}\frac{\partial T}{\partial v_x} + \underbrace{\beta_2^{3,1}}_{-1}\omega_z\frac{\partial T}{\partial v_y} - Q_1)\delta s_1 = \left[m\dot{v}_x - mc\omega_z^2 - m\omega_z v_y - f_x\right]\delta s_1 = 0,$$

$$(\frac{d}{dt}\frac{\partial T}{\partial v_y} + \underbrace{\beta_1^{3,2}}_{+1}\omega_z\frac{\partial T}{\partial v_x} - Q_2]\delta s_2 = [m\dot{v}_y + mc\dot{\omega}_z + m\omega_z v_x - f_y)\,\delta s_2 = 0,$$

$$(\frac{d}{dt}\frac{\partial T}{\partial \omega_z} + \underbrace{\beta_1^{2,3}}_{-1}v_y\frac{\partial T}{\partial v_x} + \underbrace{\beta_2^{1,3}}_{+1}v_x\frac{\partial T}{\partial v_y} - Q_3)\delta s_3$$
$$= [C^o\dot{\omega}_z + mc\dot{v}_y + mcv_x\omega_z - M_z]\,\delta s_3 = 0, \tag{8}$$

which, after inserting the non-holonomic constraint $v_y = 0$, yields the equations of motion

$$\begin{bmatrix} m & 0 \\ 0 & C^o \end{bmatrix}\begin{pmatrix} \dot{v}_x \\ \dot{\omega}_z \end{pmatrix} + \begin{bmatrix} 0 & -mc\omega_z \\ mc\omega_z & 0 \end{bmatrix}\begin{pmatrix} v_x \\ \omega_z \end{pmatrix} - \begin{pmatrix} f_x \\ M_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{9}$$

2.2 Projection Equation

On the other hand, applying the Projection Equation

$$\sum_{i=1}^{N}\left\{\left[\left(\frac{\partial \mathbf{v}_c}{\partial \dot{\mathbf{s}}}\right)^T \left(\frac{\partial \boldsymbol{\omega}_c}{\partial \dot{\mathbf{s}}}\right)^T\right]\begin{bmatrix} \dot{\mathbf{p}} + \widetilde{\boldsymbol{\omega}}_{IR}\mathbf{p} - \mathbf{f}^e \\ \dot{\mathbf{L}} + \widetilde{\boldsymbol{\omega}}_{IR}\mathbf{L} - \mathbf{M}^e \end{bmatrix}\right\}_i = 0 \tag{10}$$

(index c: mass center, index IR: reference frame R w.r.t. inertial frame I; $\mathbf{v}, \boldsymbol{\omega}$: velocity and angular velocity.; $\mathbf{p}, \mathbf{L}$: momentum and momentum of momentum; $\mathbf{f}^e, \mathbf{M}^e$: impressed force and torque; $\widetilde{(\,)}$: spin tensor; all terms represented in the reference coordinate system R) leads directly to the desired results. Once the cartesian velocities are calculated, all the remainder is known. Especially, the functional matrix $[(\partial\mathbf{v}_c/\partial\dot{\mathbf{s}})^T(\partial\boldsymbol{\omega}_c/\partial\dot{\mathbf{s}})^T]$ is nothing but the coefficient matrix of the cartesian velocities w.r.t. the (chosen or calculated) minimal velocities $\dot{\mathbf{s}}$. For the car model we have

$$\begin{pmatrix} v_{cx} \\ v_{cy} \\ \omega_{cz} \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & c \\ 0 & 1 \end{bmatrix}\dot{\mathbf{s}}\,, \quad \dot{\mathbf{s}} = \begin{pmatrix} v_x \\ \omega_z \end{pmatrix}. \tag{11}$$

The chosen reference frame is the body-fixed one and the non-holonomic constraint $v_y = 0$ is already inserted. The matrix in square brackets represents the requested functional matrix $[(\partial v_{cx}/\partial\dot{\mathbf{s}})^T(\partial v_{cy}/\partial\dot{\mathbf{s}})^T(\partial\omega_{cz}/\partial\dot{\mathbf{s}})^T]^T$. The momenta are obtained by multiplication with m (mass) and C^c (moment of inertia w.r.t. the mass center c), respectively. These ingredients are combined according to Eq.(10) by simple matrix multiplications to obtain the motion equations without any detour,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & c & 1 \end{bmatrix} \left\{ \begin{bmatrix} m & 0 \\ 0 & mc \\ 0 & C^c \end{bmatrix} \ddot{\mathbf{s}} + \begin{bmatrix} 0 & -\omega_z & 0 \\ \omega_z & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} m & 0 \\ 0 & mc \\ 0 & C^c \end{bmatrix} \dot{\mathbf{s}} - \begin{pmatrix} f_x \\ f_y \\ M_z \end{pmatrix} \right\} = 0, \quad (12)$$

yielding Eq.(9) directly.

As a conclusion one may state that for non-holonomic systems the use of the Projection Equation is preferable to the use of any of the analytical methods which come into question. The calculation requirements are simple and the non-holonomic constraints may be inserted already at the beginning. This is because directional derivations are not requested. The required effort is minimal.

3 Rigid Multibody Systems (MBS)

For a later comparison with fast moving elastic systems we may stay for a short while with MBS. The over all sum in Eq.(10) may be split into a double sum where the first one denotes a number of considered subsystems to be chosen. The second one then characterizes the number of bodies N_n within the actual subsystem n. Along with the chain rule of differentiation one obtains

$$\sum_{n=1}^{N_{sub}} \left(\frac{\partial \dot{\mathbf{y}}_n}{\partial \dot{\mathbf{s}}} \right)^T \sum_{i=1}^{N_n} \left\{ \left[\left(\frac{\partial \mathbf{v}_c}{\partial \dot{\mathbf{y}}_n} \right)^T \left(\frac{\partial \boldsymbol{\omega}_c}{\partial \dot{\mathbf{y}}_n} \right)^T \right] \begin{bmatrix} \dot{\mathbf{p}} + \widetilde{\boldsymbol{\omega}}_{IR}\mathbf{p} - \mathbf{f}^e \\ \dot{\mathbf{L}} + \widetilde{\boldsymbol{\omega}}_{IR}\mathbf{L} - \mathbf{M}^e \end{bmatrix} \right\}_i = 0 \quad (13)$$

in terms of describing velocities $\dot{\mathbf{y}}_n$ for each subsystem. Carrying out the calculation for the second sum, from 1 to N_n, leads to the typical structure of mechanical systems in the form $[\mathbf{M}_n\ddot{\mathbf{y}}_n + \mathbf{G}_n\dot{\mathbf{y}}_n - \mathbf{Q}_n]$. In matrix notation one has then for Eq.(13)

$$\left[\left(\frac{\partial \dot{\mathbf{y}}_1}{\partial \dot{\mathbf{s}}} \right)^T \left(\frac{\partial \dot{\mathbf{y}}_2}{\partial \dot{\mathbf{s}}} \right)^T \cdots \left(\frac{\partial \dot{\mathbf{y}}_N}{\partial \dot{\mathbf{s}}} \right)^T \right] \begin{bmatrix} \mathbf{M}_1\ddot{\mathbf{y}}_1 + \mathbf{G}_1\dot{\mathbf{y}}_1 - \mathbf{Q}_1 \\ \mathbf{M}_2\ddot{\mathbf{y}}_2 + \mathbf{G}_2\dot{\mathbf{y}}_2 - \mathbf{Q}_2 \\ \vdots \\ \mathbf{M}_N\ddot{\mathbf{y}}_N + \mathbf{G}_N\dot{\mathbf{y}}_N - \mathbf{Q}_N \end{bmatrix} = 0 \quad (14)$$

where N_{sub} is abbreviated N for brevity. The describing velocities follow from the kinematic chain $\dot{\mathbf{y}}_n = \mathbf{T}_{np}\dot{\mathbf{y}}_p + \mathbf{F}_n\dot{\mathbf{s}}_n$ (index p: predecessor). Starting with the first subsystem which does not have a predecessor yields $\dot{\mathbf{y}}_1 = \mathbf{F}_1\dot{\mathbf{s}}_1$. Insertion into $\dot{\mathbf{y}}_2$ then yields $\dot{\mathbf{y}}_2 = \mathbf{T}_{21}\mathbf{F}_1\dot{\mathbf{s}}_1 + \mathbf{F}_2\dot{\mathbf{s}}_2$, hence $\dot{\mathbf{y}}_3 = \mathbf{T}_{31}\mathbf{F}_1\dot{\mathbf{s}}_1 + \mathbf{T}_{32}\mathbf{F}_2\dot{\mathbf{s}}_2 + \mathbf{F}_2\dot{\mathbf{s}}_3$ where $\mathbf{T}_{31} = \mathbf{T}_{32}\mathbf{T}_{21}$ etc. Using $\dot{\mathbf{s}} = (\dot{\mathbf{s}}_1^T \dot{\mathbf{s}}_2^T \cdots \dot{\mathbf{s}}_N^T)^T$ for minimal velocities leads Eq.(14) to

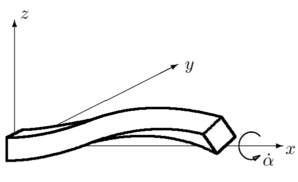
$$\begin{bmatrix} \mathbf{F}_1^T & \mathbf{F}_1^T\mathbf{T}_{21}^T & \cdots & \mathbf{F}_1^T\mathbf{T}_{\mathrm{N}1}^T \\ & \mathbf{F}_2^T & \cdots & \mathbf{F}_2^T\mathbf{T}_{\mathrm{N}2}^T \\ & & \ddots & \vdots \\ & & & \mathbf{F}_{\mathrm{N}}^T \end{bmatrix} \begin{bmatrix} \mathbf{M}_1\ddot{\mathbf{y}}_1 + \mathbf{G}_1\dot{\mathbf{y}}_1 - \mathbf{Q}_1 \\ \mathbf{M}_2\ddot{\mathbf{y}}_2 + \mathbf{G}_2\dot{\mathbf{y}}_2 - \mathbf{Q}_2 \\ \vdots \\ \mathbf{M}_{\mathrm{N}}\ddot{\mathbf{y}}_{\mathrm{N}} + \mathbf{G}_{\mathrm{N}}\dot{\mathbf{y}}_{\mathrm{N}} - \mathbf{Q}_{\mathrm{N}} \end{bmatrix} = 0 \tag{15}$$

which gives access to either a minimal representation (by inserting $\dot{\mathbf{y}}_n, \ddot{\mathbf{y}}_n$ explicitly) or to a recursive alorithm in the sense of a Gaussian elimination procedure. Obviously, the use of the Projection Equation once more leads to minimum effort when compared to the analytical procedures.

4 Orthogonality

4.1 Hamilton's Principle

One of the most popular procedures in the field of elastic body oscillations is the use of Hamilton's Principle to derive the equations of motion. Its direct use in fast moving elastic systems, however, may cause problems. This is demonstrated by a simple example: consider an elastic beam [two bending variables $v(x,t)$ and $w(x,t)$] which rotates quickly around its x-axis with $\dot{\alpha}$. Hamilton's Principle requires the variation of the kinetic energy T and of the elastic potential V,



$$\delta T = \int_0^L \left(\mathbf{v}_c^T \rho A \delta \mathbf{v}_c + \boldsymbol{\omega}_c^T \rho \mathbf{I} \delta \boldsymbol{\omega}_c\right) dx,$$

$$\delta V = \int_0^L \left(EI_z v'' \delta v'' + EI_y w'' \delta w''\right) dx,$$

$$\delta W = M_x \delta\alpha \tag{16}$$

where: ρ: mass density, A: cross sectional area, $\mathbf{I} = diag\{I_x, I_y, I_z\}$: tensor of area moments of inertia, E: Young's modulus, M_x: driving torque. The mass center velocities of an element are

$$\mathbf{v}_c = \begin{pmatrix} 0 \\ \dot{v} - \dot{\alpha} w \\ \dot{w} + \dot{\alpha} v \end{pmatrix}, \quad \omega_c = \begin{bmatrix} \left(1 - \frac{v'^2}{2} - \frac{w'^2}{2}\right) & v' & 0 \\ -v' & \left(1 - \frac{v'^2}{2}\right) & 0 \\ -w' & 0 & 1 \end{bmatrix} \begin{pmatrix} \dot{\alpha} \\ -\dot{w}' \\ \dot{v}' \end{pmatrix} \tag{17}$$

where a Cardan–sequence $-w', v'$ has been chosen for transformation from the reference frame (rotating with $\dot{\alpha}$) to the element-fixed frame (sometimes referred to as Tait-Bryan-sequence in the english speaking area). The prime denotes spatial derivation, $(\,)' = \partial(\,)/\partial x$. Without going into the details of (the tedious) calculations, one obtains, after carrying out the required integrations by parts, a result in the form $\int_0^L\{\delta\alpha[\cdots] + \delta v[\cdots] + \delta w[\cdots]\}$ plus boundary terms. This is correct. However, usually one takes the square brackets as motion equations, setting these individually equal to zero. Then, in the present case, one obtains a rather strange result. Considering a circular cross sectional area ($I_x = 2I_y = 2I_z := 2I$) yields

$$\int_0^L (\rho I_x\ddot{\alpha} - M_x)dx = 0 \quad (\Rightarrow \alpha(t) \text{ known function}),$$
$$\rho A(\ddot{v} - 2\dot{\alpha}\dot{w}\underline{-\ddot{\alpha}w} - \dot{\alpha}^2 v) - \rho I(\ddot{v}'' + v''\dot{\alpha}^2\underline{\underline{-w''\ddot{\alpha}}}) + (EIv'')'' = 0,$$
$$\rho A(\ddot{w} + 2\dot{\alpha}\dot{v}\underline{+\ddot{\alpha}v} - \dot{\alpha}^2 w) - \rho I(\ddot{w}'' + w''\dot{\alpha}^2\underline{\underline{-v''\ddot{\alpha}}}) + (EI_y w'')'' = 0. \quad (18)$$

Here, the (generalized) circulatory forces due to the angular acceleration $\ddot{\alpha}$ are, as expected, skew-symmetric for the translational part (ρA), but they are symmetric for the rotational part (ρI). The reason is, that the rotation axes which are assigned to the CARDAN angular velocities ($\dot{\alpha}, -\dot{w}', \dot{v}'$ in the present example) are not orthogonal. Because the analytical methods require directional derivatives w.r.t. these, Eq.(18) represents the motion equations in a non-orthogonal coordinate system which depends on the choice of the sequence of deformations even in the case of small deformations. An interpretation as motion equations w.r.t. the co-rotating reference frame, as usually requested, is wrong.

4.2 The Projection Equation

The same as in the case of non-holonomic systems, the Projection Equation does not need directional derivations and will therefore avoid such difficulties. Considering elastic multibody systems (EMBS), the number of bodies of a subsystem (e.g. beam slices) goes to infinity and the summation is replaced with an integral, yielding the same equation structure as in the rigid body case:

$$\sum_{n=1}^{N_{sub}} \int_{B_n} \left(\frac{\partial \dot{\mathbf{y}}_n}{\partial \dot{\mathbf{s}}}\right)^T \underbrace{\left\{ \left[\left(\frac{\partial \mathbf{v}_c}{\partial \dot{\mathbf{y}}_n}\right)^T \left(\frac{\partial \boldsymbol{\omega}_c}{\partial \dot{\mathbf{y}}_n}\right)^T \right] \begin{bmatrix} d\dot{\mathbf{p}} + \widetilde{\boldsymbol{\omega}}_{IR} d\mathbf{p} - d\mathbf{f}^e \\ d\dot{\mathbf{L}} + \widetilde{\boldsymbol{\omega}}_{IR} d\mathbf{L} - d\mathbf{M}^e \end{bmatrix} \right\}_n}_{[d\mathbf{M}_n \ddot{\mathbf{y}}_n + d\mathbf{G}_n \dot{\mathbf{y}}_n - d\mathbf{Q}_n]} = 0 \quad (19)$$

However, since the describing velocities now require the consideration of partial derivatives w.r.t. the spatial variables (arising from bending angles and curvatures), the functional matrix $(\partial\dot{\mathbf{y}}_n/\partial\dot{\mathbf{s}})$ can not directly be calculated. We therefore pass to the corresponding virtual work expression,

$$\sum_{n=1}^{N_{sub}} \int_{B_n} \delta \mathbf{y}_n^T \left[d\mathbf{M}_n \ddot{\mathbf{y}}_n + d\mathbf{G}_n \dot{\mathbf{y}}_n - d\mathbf{Q}_n \right] = 0. \tag{20}$$

The solution steps are as follows: Consider $N_{sub} = 1$ for simplicity. Then $\dot{\mathbf{y}}$ is calculated with the aid of a differential operator, $\dot{\mathbf{y}} = \overline{\mathcal{D}} \circ \dot{\mathbf{s}}$, yielding $\delta\mathbf{y} = \overline{\mathcal{D}} \circ \delta\mathbf{s}$. Integration by parts yields

$$\int_{B_n} \delta \mathbf{s}^T \mathcal{D}^T \circ \left[d\mathbf{M}_n \ddot{\mathbf{y}}_n + d\mathbf{G}_n \dot{\mathbf{y}}_n - d\mathbf{Q}_n \right] + \delta W_{bound} = 0 \tag{21}$$

with a new differential operator $\mathcal{D}$. This seemingly costly procedure results extremely simple: The operator $\overline{\mathcal{D}}$ follows from $\dot{\mathbf{y}}$ which contains the deviations, the bending angles and the curvatures,

$$\dot{\mathbf{y}} = \begin{pmatrix} \dot{\alpha} \\ \dot{v} \\ \dot{w} \\ -\dot{w}' \\ \dot{v}' \\ \dot{v}'' \\ \dot{w}'' \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -\frac{\partial}{\partial x} \\ 0 & \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial^2}{\partial x^2} & 0 \\ 0 & 0 & \frac{\partial^2}{\partial x^2} \end{bmatrix} \circ \begin{pmatrix} \dot{\alpha} \\ \dot{v} \\ \dot{w} \end{pmatrix} = \bar{\mathcal{D}} \circ \dot{\mathbf{s}}. \tag{22}$$

The requested operator $\mathcal{D}$ is the same as $\overline{\mathcal{D}}$ with the only difference that odd derivatives change their sign. (Simultaneously one obtains the operators $\mathcal{B}_0$ and $\mathcal{B}_1$ for the (kinetic) boundary conditions by successive degeneration of the differentiation grade with once more changing sign – this reflects the consecutive integrations by parts with its sign changes). Applying $\mathcal{D}^T$ to $d\mathbf{M}\ddot{\mathbf{y}} + d\mathbf{G}\dot{\mathbf{y}} - d\mathbf{Q}$ yields, for the present example,

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{\partial}{\partial x} & \frac{\partial^2}{\partial x^2} & 0 \\ 0 & 0 & 1 & \frac{\partial}{\partial x} & 0 & 0 & \frac{\partial^2}{\partial x^2} \end{bmatrix} \circ \begin{pmatrix} \rho I \ddot{\alpha} \\ \rho A(-\ddot{\alpha} w + \ddot{v} - \dot{\alpha}^2 v - 2\dot{\alpha}\dot{w}) \\ \rho A(+\ddot{\alpha} v + \ddot{w} - \dot{\alpha}^2 w + 2\dot{\alpha}\dot{v}) \\ \rho I(v'\ddot{\alpha} - \ddot{w}' - \dot{\alpha}^2 w') \\ \rho I(w'\ddot{\alpha} + \ddot{v}' + \dot{\alpha}^2 v') \\ EIv'' \\ EIw'' \end{pmatrix} dx =$$

$$\begin{pmatrix} \rho I \ddot{\alpha} \\ \rho A(\ddot{v} - 2\dot{\alpha}\dot{w} \underline{- \ddot{\alpha} w} - \dot{\alpha}^2 v) - \rho I(\ddot{v}'' + \dot{\alpha}^2 v'' \underline{\underline{+ w''\ddot{\alpha}}}) + (EIv'')'' \\ \rho A(\ddot{w} + 2\dot{\alpha}\dot{v} \underline{+ \ddot{\alpha} v} - \dot{\alpha}^2 w) - \rho I(\ddot{w}'' + \dot{\alpha}^2 w'' \underline{\underline{- v''\ddot{\alpha}}}) + (EIw'')'' \end{pmatrix} = 0. \tag{23}$$

As expected, one obtains automatically the correct signs for a representation in the co-rotating frame, along with much less effort in calculation.

5 Elastic Multibody System (EMBS)

5.1 Partial Differential Equations

Clearly, one might proceed this way to generate the partial differential equations (along with the corresponding boundary conditions) for an elastic multi body system. The result is a GAUSS form for the rigid body variables and a set of differential operators for the elastic variables, and a combination of these for the boundary conditions (in detail reported in [1]). However, such a foregoing seems to lead to a dead end, because an analytical solution is virtually impossible to achieve.

5.2 Approximative Solution

When looking for an approximative solution, it is not advisable to expand the equations into partial differential equations and the corresponding boundary conditions. This is simply because one will obviously never find admissible shape functions which fulfill all the boundary conditions as requested by GALERKIN's (original) method, for instance. With an interpretation of GALERKIN's method as a result from the virtual work one might think of adding the work which is accomplished by the boundary forces and torques, thus reducing the requirements for the shape functions to pure geometrical ones. But formulating the boundary terms explicitly is, for the approximative motion equations, unnecessary because the spatial coordinates do not appear as independent variables any more. Therefore, we go back to Eq.(20) along with $\dot{\mathbf{y}}_n = \overline{\mathcal{D}}_n \circ \dot{\mathbf{s}}_n$ [see Eq.(22)]. A RITZ series expansion $\dot{\mathbf{s}}_n = \mathbf{\Phi}_n(\mathbf{x})^T \dot{\mathbf{y}}_{n\mathrm{Ritz}}(t)$ yields $\dot{\mathbf{y}}_n = [\overline{\mathcal{D}} \circ \mathbf{\Phi}(\mathbf{x})^T]_n \dot{\mathbf{y}}_{n\mathrm{Ritz}}(t) := [\mathbf{\Psi}(\mathbf{x})^T]_n \dot{\mathbf{y}}_{n\mathrm{Ritz}}(t)$ where $\mathbf{\Psi}(\mathbf{x})$ comprises the shape functions along with their spatial derivatives as far as they are needed. The virtual displacements are then $\delta \mathbf{y}_n = [\mathbf{\Psi}]_n^T (\partial \dot{\mathbf{y}}_{n\mathrm{Ritz}} / \dot{\mathbf{s}}) \delta \mathbf{s}$. Since $\delta \mathbf{s}$ is arbitrary, one obtains from Eq.20)

$$\begin{bmatrix} \mathbf{F}_1^T & \mathbf{F}_1^T \mathbf{T}_{21}^T & \cdots & \mathbf{F}_1^T \mathbf{T}_{\mathrm{N}1}^T \\ & \mathbf{F}_2^T & \cdots & \mathbf{F}_2^T \mathbf{T}_{\mathrm{N}2}^T \\ & & \ddots & \vdots \\ & & & \mathbf{F}_{\mathrm{N}}^T \end{bmatrix} \begin{bmatrix} \mathbf{M}_1 \ddot{\mathbf{y}}_{1\mathrm{Ritz}} + \mathbf{G}_1 \dot{\mathbf{y}}_{1\mathrm{Ritz}} - \mathbf{Q}_1 \\ \mathbf{M}_2 \ddot{\mathbf{y}}_{2\mathrm{Ritz}} + \mathbf{G}_2 \dot{\mathbf{y}}_{2\mathrm{Ritz}} - \mathbf{Q}_2 \\ \vdots \\ \mathbf{M}_{\mathrm{N}} \ddot{\mathbf{y}}_{N\mathrm{Ritz}} + \mathbf{G}_{\mathrm{N}} \dot{\mathbf{y}}_{N\mathrm{Ritz}} - \mathbf{Q}_{\mathrm{N}} \end{bmatrix} = 0 \tag{24}$$

where

$$\mathbf{M}_n = \int\limits_{B_n} \left[\mathbf{\Psi} d\mathbf{M} \mathbf{\Psi}^T \right]_n \;, \quad \mathbf{G}_n = \int\limits_{B_n} \left[\mathbf{\Psi} d\mathbf{G} \mathbf{\Psi}^T \right]_n \;, \quad \mathbf{Q}_n = \int\limits_{B_n} \left[\mathbf{\Psi} d\mathbf{Q} \right]_n . \tag{25}$$

One has thus once more the same GAUSS form as in Eq.(15).

6 Conclusions

When non-holonomic constraints come into play, then the analytical methods require at first a calculation for the f ull set of variables. The non-holonomic constraint may be inserted afterward. This is because the analytical methods need directional derivations of the kinetic energy w.r.t. the minimal velocities. Inserting the non-holonomic constraint in advance would lead to a loss of information and yields wrong results. This is avoided with the Projection Equation which does not require directional derivatives. Here, the constraints may be inserted in advance. This goes along with a considerable reduction of calculation effort.

Directional derivations are also the reason that fast moving (accelerated) systems have to be considered with care. At least when using CARDAN-like transformations, the resulting rotational axes which refer to the generalized angular velocities are not orthogonal. As a consequence, the resulting equations are not independent when seen from the co-rotating coordinate system, for instance. This difficulty is avoided with the Projection Equation. Its use once more goes along with considerable effort savings.

Considering elastic *multi* body systems, the use of the analytical methods requires an enormous effort in calculation. Here, one really runs into problems. Once more, the Projection Equation reduces this effort to a minimum. Along with a direct RITZ approach one obtains the afore mentioned GAUSS form for approximative solution. Its evaluation leads to an order-n-formalism which seems the only reasonable way to come around with such challenging systems.

It should not remain unnoticed that in case of fast moving elastic systems the corresponding zero order reaction forces (as well as zero order impressed forces) need the consideration of second order displacement fields. These lead to the so-called "dynamical stiffening effects". They are, in the present context, assumed to be taken into account with $d\boldsymbol{Q}$.

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For space requirements, we refer to only one (recently published) reference which contains nearer explanations as well as a literature overview.

References

1. H.Bremer: Elastic Multibody Dynamics – A Direct Ritz Approach. Springer, ISBN 978-1-4020-8679-3, August 2008.