The basis of optimal active (static and dynamic) shape- and stress-control by means of smart materials

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Abstract Vibrations may shorten the lifetime of structures and machines, cause discomfort in many cases (noise radiation) and are totally unwanted in precision engineering. The latter requires also static shape control. The method of unique decomposition of eigenstrains into two constituents, namely in impotent eigenstrains, that do not cause stress and in the complementary nilpotent eigenstrains that do not induce any deformation in the linear elastic solid is considered in detail. These two complete classes of eigenstrains render optimal solutions by keeping shape and stress-control problems well separated. Assuming a common time function of the dynamic load, a novel approach is addressed to annihilate the forced vibrations. This optimal benchmark solution may serve the purpose in practical application to select properly shaped actuator patches and the control current.

1 Introduction

The state-of-the-art of active structural control up to 1990 is available in book form [1], the current one is reflected in [2], with general views collected in [3] and [4]. Haftka and Adelman [5] used transient thermal strains (eigenstrains) imposed on the supporting structure for the first time to minimize deviations of large space structures from their original shape. In [6] an adaptive wing of a fighter plane is considered. Vibration suppression of rotary wings is analyzed in [7]. An early summary is provided in [8]. In [9], reviews with emphasis on piezoelectricity and its application in disturbance sensing and control of flexible structures are provided, however, the sources of eigenstrain are not within the scope of this short paper. Nonlinear optimization routines destroy

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the comfortable settings in the control of linear vibrations. Linear solutions of the inverse problem are presented in [10] with a recent review given in [11]. It is shown that dynamic shape control when based on the annihilation of the quasi-static portion of the force-induced deformations renders the optimal distribution and intensity level of (shaped) actuator patches thereby fully quieting the vibrations. The controlled structure at rest finally carries the quasi-static force-induced stresses only. Such benchmark solutions with unlimited intensity of the actuators understood, serve for the best possible practical design. In [12] the crucial, unique decomposition of an eigenstrain tensor, e.g. of piezoelectric strain, is performed by means of the scalar product measure in Hilbert functional space, rendering its impotent part that does not produce stress, see also [13] and [14], and its complement, the nilpotent eigenstrain that renders stresses but does not produce deformation, see [10], [12] and [15].

2 Suppression of force-induced small vibrations about an equilibrium state

The generalization of static shape control by means of imposed eigenstrains, denoted $\boldsymbol{\varepsilon}^*$, to the dynamic shape control can be based on the dynamic generalization of Maysel's formula, [16], where the dynamic Green's stress dyadic of the structure is applied and a convolution in time must be considered. It is shown below that such a separate solution of the actuator problem is superfluous, see also [17] and [18].

2.1 Force-induced small vibrations about an equilibrium state

The dynamic shape control problem is solved by linear methods in [19], by assigning proper actuator stresses, i.e., transient eigenstresses, assuming the forced vibrations to be known. They elegantly use an extension of Neumann's method in [20], to define directly the properly distributed actuators. The force load must be considered first. Since mass inertia is taken into account, conservation of momentum renders the Euler-Cauchy equation of motion, [21], $\boldsymbol{b}$ is the given transient body force load, if any

$$div\boldsymbol{\sigma} + \boldsymbol{b} = \rho\boldsymbol{a}\ , \quad \boldsymbol{a} = \boldsymbol{u}_{,tt} \tag{1}$$

On part of the boundary, kinematic boundary conditions apply, on the remaining part of the surface, the transient traction of the force load is prescribed,

$$\Gamma_u : \boldsymbol{u} = \boldsymbol{0} , \quad \Gamma_\sigma : \boldsymbol{\sigma} \cdot \boldsymbol{n} = \boldsymbol{t}^{(n)} \tag{2}$$

Within the validity of both, linearized geometric relations and Hooke's law, [21],

$$\varepsilon_{ij(F)} = \frac{1}{2}(u_{i,j} + u_{j,i}) , \quad \varepsilon_{ij(F)} = C_{ijlm}\sigma_{lm} \tag{3}$$

the solution of the force-displacements $\boldsymbol{u}(\boldsymbol{x}, t) = \boldsymbol{u}^{(F)}$ is determined by Eqs. (1)-(3). It will be shown that dynamic control of these deformations, produced by prescribed body forces and surface traction, is achieved by the control of the much simpler quasi-static solution of the force problem, determined by the successive equilibrium states of the reduced Eq. (1),

$$div\boldsymbol{\sigma}^{(F)}_{(s)} + \boldsymbol{b} = \boldsymbol{0} \tag{4}$$

taking into account the prescribed dynamic boundary condition of Eq. (2). To fully relate the solution to the static shape control, the kinematic boundary condition, see again Eq. (2), is applied as well. With the simplifying assumption for both, body force and traction to be separable in space and time, Eq. (4) has to be solved only once and/or all available static solutions become candidates for dynamic shape control. The directions of the principal strain axes become time invariant! Since it is common practice in structural dynamics and modal analysis, to split the response to the force load into its quasi-static part, already posed by the boundary value problem of Eq. (4),

$$\boldsymbol{u}^{(F)} = \boldsymbol{u}^{(F)}_{(s)} + \boldsymbol{u}^{(F)}_{(d)} \tag{5}$$

the complementary dynamic part in the solution, $\boldsymbol{u}^{(F)}_{(d)}$, is considered further. Subtracting Eq. (4) from Eq. (1) renders the latter in a reduced form. Note both, the "body force" $\boldsymbol{b}^*_{(s)}$, determined by substituting Eq. (5) and, consequently recognized as the inertia force of the quasi-static force solution, and the remaining homogeneous dynamic boundary condition; the kinematic b.c. still holds true,

$$div\boldsymbol{\sigma}^{(F)}_{(d)} + \boldsymbol{b}^*_{(s)} = \rho\boldsymbol{u}^{(F)}_{(d),tt} , \quad \boldsymbol{b}^*_{(s)} = -\rho\boldsymbol{u}^{F}_{(s),tt} , \quad \Gamma_\sigma : \boldsymbol{\sigma}^{(F)}_{(d)} \cdot \boldsymbol{n} = \boldsymbol{0} \tag{6}$$

2.2 Eigenstrain-induced small vibrations: dynamic shape control

If impotent eigenstrains (with positive sign) equaling the quasi-static force-induced strains $\bar{\varepsilon}^*_{ij(s)}(\boldsymbol{x}) = \varepsilon^{(F)}_{ij(s)}(\boldsymbol{x})$ are imposed, no additional stresses are produced since the eigenstrain is a compatible one, $\boldsymbol{\sigma}^{(\varepsilon)}_{(s)} = \boldsymbol{0}$. That is a trivial solution of the homogeneous Eq. (4) with (ε) substituted for the superscript (F). The remaining dynamic boundary value problem of the eigenstrain load is detemined by (homogeneous dynamic boundary conditions are understood),

$$div\boldsymbol{\sigma}_{(d)}^{(\varepsilon)} + \boldsymbol{b}_{(s)}^{*} = \rho \boldsymbol{u}_{(d),tt}^{(\varepsilon)}\ , \quad \Gamma_\sigma :\ \boldsymbol{\sigma}_{(d)}^{(\varepsilon)} \cdot \boldsymbol{n} = \boldsymbol{0},$$
$$\boldsymbol{u}^{(\varepsilon)} = \boldsymbol{u}_{(s)}^{(\varepsilon)} + \boldsymbol{u}_{(d)}^{(\varepsilon)}\ , \quad \boldsymbol{u}_{(s)}^{(\varepsilon)} = \boldsymbol{u}_{(s)}^{(F)} \tag{7}$$

Since these impotent eigenstrains reproduce one and the same body force distribution $\boldsymbol{b}_s^*$ as defined in Eq. (6), they render the solution of the dynamic part of the force problem, Eq. (6). Hence, it can be concluded that the quasi-static impotent eigenstrain with reversed sign annihilates also the dynamic part of the force-displacements and, in addition, counteracts the dynamic stress portion of the force problem. The ideal dynamic shape control thus totally suppresses force-induced vibrations (quiet initial conditions have been assumed throughout) and leaves the quasi-static force-produced stresses unchanged. The simple example of the dynamic shape control of a redundant planar smart ideal truss, Fig. 1, by impotent eigenstrains illustrates this solution technique based on the quasi-static force response. The mass is lumped to the nodes, stiffness EA of the member rods is assumed to be constant. Since the load case is prescribed, we can directly calculate the quasi-static strains in the smart member rods, [21], and impose these strains as impotent eigenstrains, thus preserving the quasi-static member forces $\boldsymbol{N}_{(s)}^{(F)}$,

$$\boldsymbol{\varepsilon}_{(s)}^{(F)T} = -\bar{\boldsymbol{\varepsilon}}_{(s)}^{*} = \frac{F(t)}{100EA}\left[-11, 89, -37, 15, -74, -49, 61, 54, -87, -39, -39\right]. \tag{8}$$

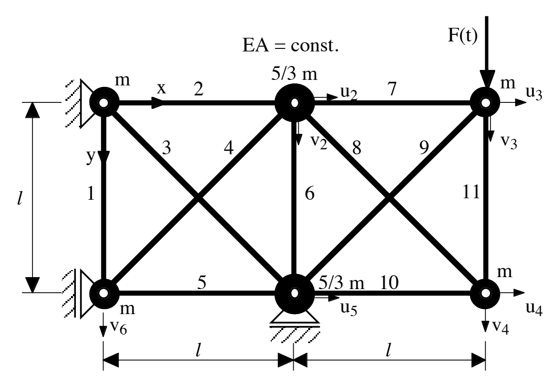

Fig. 1 Smart truss with internal and external redundancy (rank 3)

3 The basis of impotent and nilpotent eigenstrain in Hilbert (energy) space

For a rigorous formulation of the main theorem on eigenstress and eigenstrain, the Hilbert function space H is introduced [12], [14], as the space of second rank symmetric tensors where the components are real functions of spatial coordinates in the function space L_2. Assume, that the eigenstrain tensors are the elements of the Hilbert space. The inner product defines the norm, - of the eigenstrain $\bar{\boldsymbol{\varepsilon}}$,

$$(\boldsymbol{\alpha},\boldsymbol{\beta}) = \int_{\Omega} \boldsymbol{\alpha}\cdot\cdot\,\boldsymbol{C}^{-1}\cdot\cdot\,\boldsymbol{\beta} dV\ , \quad \|\boldsymbol{\alpha}\|_H = \sqrt{(\boldsymbol{\alpha},\boldsymbol{\alpha})}\ , \quad \boldsymbol{\alpha} = \boldsymbol{\beta} = \bar{\boldsymbol{\varepsilon}} \tag{9}$$

Mikhlin [22] and [23] identified the first part of Eq. (9) as twice the strain energy, stored internally in the elastic body if $\boldsymbol{\alpha} = \boldsymbol{\beta} = \bar{\boldsymbol{\varepsilon}}$. Accordingly, the space is thus called energy space. For discretized structures, two mutual orthogonal finite dimensional sub-spaces exist, i. e., any tensor of eigenstrain $\bar{\boldsymbol{\varepsilon}} \in H$ existing in a body can be uniquely decomposed into its impotent $\bar{\boldsymbol{\varepsilon}}^{*}$ and nilpotent $\bar{\boldsymbol{\varepsilon}}^{**}$ constituents, see [12] and [14],

$$\bar{\boldsymbol{\varepsilon}} = \bar{\boldsymbol{\varepsilon}}^{*} + \bar{\boldsymbol{\varepsilon}}^{**}\ , \quad (\bar{\boldsymbol{\varepsilon}}^{*},\bar{\boldsymbol{\varepsilon}}^{**}) = 0\ , \quad \boldsymbol{\sigma} = -\boldsymbol{C}^{-1}\cdot\cdot\,\bar{\boldsymbol{\varepsilon}}^{**}\ , \quad \boldsymbol{\varepsilon} = \bar{\boldsymbol{\varepsilon}}^{*} \tag{10}$$

Equation (10) implies: there exists the orthogonal decomposition of the Hilbert (energy) space H into subspaces H_u and H_σ, see [12], $H = H_u \oplus H_\sigma$. Further, the unique decomposition of the space of eigenstrains allows us to establish the significant properties of eigenstress and deformation induced by eigenstrain, see again Eq. (10). Consequently, determination of impotent and nilpotent constituents of eigenstrain imposed on the structure allows in a general manner the determination of eigenstress and deformation caused by eigenstrain without straightforwardly solving the appropriate boundary value problem in linearized elasticity with eigenstrain. Consequently, the static or quasi-static control problems for load stress and deformation (or displacement) are kept apart just by selecting the proper class of eigenstrains. The general solution that may be called the basis of all possible impotent strains in H_u is easily derived by inverting the stiffness matrix, $\boldsymbol{K}^{-1}$ is the flexibility matrix. Each column renders, by means of a proper transformation, [24], a strain distribution in the finite elements or simply in the member rods that constructs such a base vector. For shells and FEM see [25]. Linear shape functions in triangular or tetrahedral finite elements render the candidates of impotent strain tensors constant. In case of higher order elements, a proper mean strain should be determined, see again [24].

3.1 Determination of the dimensions of the subspaces H_u and H_σ of the energy space

The discretized system is composed of N variable types of elements. The number of independent scalar parameters determining the deformation of an element of type "k" is denoted by m_k. The number of elements of type "k" is denoted by n_k. Since $H_\sigma \subset H$, it can be concluded that elastic strain and eigenstrain have the identical approximation in any given element (e.g., given by the shape functions selected for the finite element), thus, it remains valid for deformation and stress as well. The total number of independent scalar parameters defining the deformation of the discrete system thus defines the dimension of space H,

$$dimH = dimH_u + dimH_\sigma = \sum_{k=1}^{N} n_k m_k \tag{11}$$

The determination of $dimH_u$ requires the application of the theorem on eigenstrain, [12]: the eigenstrain in space H belongs to the subspace H_u iff there exist such (fictitious) body forces and surface traction that produce in the same elastic body a deformation that equals the given eigenstrain. Consequently, the number of independent variants of external nodal forces determines the dimension of subspace H_u. It is obvious, that in the case of a three-dimensional discrete system, the dimension of subspace H_u is thus given by: N_n, number of nodes; N_R, number of support reactions ($N_R \geq 6$), cf. with the size of the flexibility matrix,

$$dimH_u = 3N_n - N_R \tag{12}$$

Using Eqs. (11) and (12) renders at once

$$dimH_\sigma = dimH - dimH_u = \sum_{k=1}^{N} n_k m_k + N_R - 3N_n = s \tag{13}$$

equal to the rank of redundancy of the discrete system: The rank of redundancy s is defined by the number of internal and external forces that cannot be determined from the system of nodal equilibrium equations, [21]. Note, for discrete statically determinate structures the dimension of subspace H_σ is zero. Consequently, stress control by eigenstrain can be performed for redundant systems only, [5].

3.2 Construction of the basis of nilpotent eigenstrain

For a given truss it is advisable to use the principle of constraint release rendering forces $\boldsymbol{R}_j$. Then the equilibrium conditions result for arbitrary admissible node displacements $\boldsymbol{w}$ in the generalized form, $S_i^k = S_j^k = S_k$ is the axial member force,

$$\sum_{k=1}^{N_m} S_k \varepsilon_k(\boldsymbol{w}) l_k - \sum_{j=1}^{N_R} \boldsymbol{R_j} \cdot \boldsymbol{w}_j = 0\,, \quad \forall \boldsymbol{w} \in \left(W_2^1(\Omega)\right)^3\,, \quad \boldsymbol{w}_j = \boldsymbol{w}(\boldsymbol{r}_j) \quad (14)$$

With a unit vector $\boldsymbol{e}_k$the strain in a member rod k between nodes numbered i and j becomes $\varepsilon_k(\boldsymbol{w}) = \Delta l_k/l = (1/l_k)(\boldsymbol{w}_j - \boldsymbol{w}_i) \cdot \boldsymbol{e}_k$. Hence, denoting the force acting on the node "m" on the source side of the attached members by $\boldsymbol{F}_m$, Eq. (14) becomes the equilibrium equation in the free-body-diagram,

$$\sum_{m=1}^{N_n} \boldsymbol{F}_m \cdot \boldsymbol{w}_m - \sum_{j=1}^{N_R} \boldsymbol{R}_j \cdot \boldsymbol{w}_j = 0 \quad (15)$$

Therefore, the determination of statically admissible stresses reduces to the solution of the nodal equilibrium equations. Subsequently, the nilpotent eigenstrain can be obtained from the uniaxial Eq. (3). In the course of analysis of the redundant truss, an appropriate basic statically determinate system is selected, [21]. It means that it is necessary to release the redundant supports and member rods. We designate the magnitudes of redundant forces by X_j, $j = 1...s$, where s is the rank of system redundancy. These forces in the selected redundant member rods and supports X_j, $j = 1...s$, form the $(3N_n + s) \times 1$ column matrix: $\boldsymbol{F}^T = \{0...0, X_1...X_s\}$. Thus, constructing the influence function of strain $\varepsilon_i^{X_j} = S_i^{X_j}/E_i A_i$, $i = 1...N_m$, $j = 1...s$, i.e., of eigenstrains due to redundant forces, yields the desired basis of nilpotent eigenstrains.

Considering a simply supported single field of the truss in Fig. 1, $s = 1$, the single nilpotent unit basis is given by

$$\boldsymbol{\phi}_\sigma^{(1)} = \frac{\bar{\boldsymbol{\varepsilon}}^{**}}{\|\bar{\boldsymbol{\varepsilon}}^{**}\|} = \frac{1}{\sqrt{2EAl}\sqrt{1+\sqrt{2}}} \left(\frac{1}{\sqrt{2}}\,,\ -1\,,\ \frac{1}{\sqrt{2}}\,,\ \frac{1}{\sqrt{2}}\,,\ -1\,,\ \frac{1}{\sqrt{2}} \right) \quad (16)$$

4 Conclusion

Vibration annihilation is shown to rely on the quasi-static shape control by imposed impotent eigenstrains. A novel and efficient solution method for modelling and control of static or quasi-static stress and deformation by eigenstrain is ilustrated based on the theorem on decomposition of eigenstrain, [12] and [14]. A straightforward method for the determination of the dimensions in energy space of eigenstrain, subspaces of impotent eigenstrain and nilpotent eigenstrain, for discrete (trusses) or discretized structures (FEM) is discussed.

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