Infinite Row of Parallel Cracks in a Piezoelectric Material Strip under Mechanical and Transient Thermal Loadings

Sei Ueda

Abstract In this paper, the problem of an infinite row of parallel cracks in a piezoelectric material strip is analyzed under static mechanical and transient thermal loadings. The crack faces are supposed to be completely insulated. By using the Laplace and Fourier transforms, the thermoelectromechanical problem is reduced to a singular integral equation, which is solved numerically. The stress intensity factors for both the embedded and edge cracks are computed. The results for the crack contact problem are also included.

1 Introduction

Due to the rapid growth in application for smart or intelligent systems [1-5], the fracture problems of homogeneous piezoelectric materials under thermal loading conditions have attracted many research activities in recent years [6-12]. Especially, the overshooting phenomena of the stress and electric displacement intensity factors were observed in piezoelectric strips under thermal shock loading condition with a normal crack [6] and a parallel crack [12].

However, in spite of the fact that piezoelectric materials involve multiple cracks, most of the existing contributions are concerned with the fracture behavior of a single crack except for the dynamic interaction between the two coplanar cracks in homogeneous piezoelectric materials under electromechanical loadings [13]. Then, one of the remaining problems that need to be fully understood is that of interaction between cracks in such media subjected to thermal loading, and the present author investigated the thermoelectromechanical interaction of piezoelectric strips under thermoelectric loading with two parallel cracks [14] and with two coplanar cracks [15].

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In this paper, the problem of an infinite row of parallel cracks in a piezoelectric material strip is analyzed under static mechanical and transient thermal loadings. The crack faces are supposed to remain thermally and electrically insulated [16,17]. The superposition technique is used to solve the governing equations. The transient temperature and thermal stress in an uncracked strip are the same as the previous results [18]. This thermal stress is used as the crack surface traction with opposite sign to formulate the mixed boundary value problem. By using the Fourier transform [19,20], the electromechanical problem is reduced to a singular integral equation, which is solved numerically [21]. The stress intensity factors for both the embedded and edge cracks are computed. The results for the crack contact problem are also included.

2 Formulation of the problem

As shown in Fig. 1, suppose a piezoelectric material strip with the thickness h containing an infinite row of parallel cracks of equal length $2c = b - a \quad (0 \leq a < b < h)$ being spaced at equal distance $2d$ perpendicular to the free boundaries. The system of rectangular Cartesian coordinates (x, y, z) is introduced in the material in such a way that one of the crack is located along the z-axis, and the x-axis is parallel to the boundaries. The piezoelectric material is under a mechanical stress σ_0 in the x-direction and is poled in the z-direction. It is assumed that initially the medium is at the uniform temperature T_I (stress free temperature) and is suddenly subjected to a uniform temperature change $T_0 H(t)$ along the bottom surface $(z = 0)$, where $H(t)$ is the Heaviside step function and t denotes time. The temperature at the top surface $(z = h)$ is maintained at T_I. The crack faces remain thermally and electrically insulated [16,17]. The crack problem may be solved by superposition. In

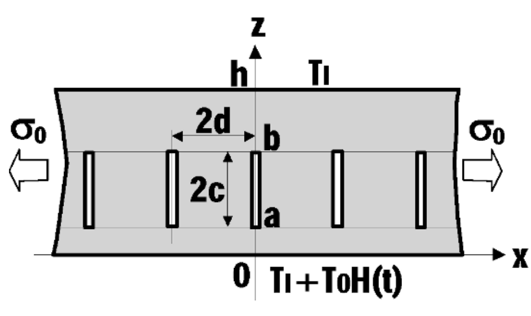


Fig. 1 An infinite row of parallel cracks in a piezoelectric strip.

the problem considered here, since the heat conduction is one-dimensional, straight cracks do not obstruct the heat flow in this arrangement, determination of the temperature distribution and the resulting thermal stress would be quite straightforward and the related crack problem would be one of mode

I. The lines of the centers of the cracks $x = \pm(2n+1)d, (n = 0, 1, 2, ...)$ are the axes of symmetry of the configuration and we suppose that each crack is opened under the action of the same distribution of the internal pressure $\sigma_0 + \sigma_0^T(z,t)$ where t is the time, σ_0 is the static mechanical stress and $\sigma_0^T(z,t)$ is the thermal stress induced by the time-dependent temperature change. The thermal stress $\sigma_0^T(z,t)$ has been already obtained in [18]. In the following, the subscripts x, y, z will be used to refer to the direction of coordinates.

The constitutive equations are given by

$$\left.\begin{aligned}
&\sigma_{xx} = c_{11}\frac{\partial u_x}{\partial x} + c_{13}\frac{\partial u_z}{\partial z} + e_{31}\frac{\partial \phi}{\partial z},\ D_x = e_{15}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) - \varepsilon_{11}\frac{\partial \phi}{\partial x}\\
&\sigma_{zz} = c_{13}\frac{\partial u_x}{\partial x} + c_{33}\frac{\partial u_z}{\partial z} + e_{33}\frac{\partial \phi}{\partial z},\ \ D_z = e_{31}\frac{\partial u_x}{\partial x} + e_{33}\frac{\partial u_z}{\partial z} - \varepsilon_{33}\frac{\partial \phi}{\partial z}\\
&\sigma_{zx} = c_{44}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) + e_{15}\frac{\partial \phi}{\partial x}
\end{aligned}\right\} \tag{1}$$

where $\phi(x,z,t)$ is the electric potential, $u_x(x,z,t)$, $u_z(x,z,t)$ are the displacement components, $\sigma_{xx}(x,z,t)$, $\sigma_{zz}(x,z,t)$, $\sigma_{zx}(x,z,t)$ are the stress components and $D_x(x,z,t)$, $D_z(x,z,t)$ are the electric displacement components.

The governing equations for the electromechanical fields may be expressed as follows:

$$\left.\begin{aligned}
&c_{11}\frac{\partial^2 u_x}{\partial x^2} + c_{44}\frac{\partial^2 u_x}{\partial z^2} + (c_{13}+c_{44})\frac{\partial^2 u_z}{\partial x \partial z} + (e_{31}+e_{15})\frac{\partial^2 \phi}{\partial x \partial z} = 0\\
&c_{44}\frac{\partial^2 u_z}{\partial x^2} + c_{33}\frac{\partial^2 u_z}{\partial z^2} + (c_{13}+c_{44})\frac{\partial^2 u_x}{\partial x \partial z} + e_{15}\frac{\partial^2 \phi}{\partial x^2} + e_{33}\frac{\partial^2 \phi}{\partial z^2} = 0\\
&e_{15}\frac{\partial^2 u_z}{\partial x^2} + e_{33}\frac{\partial^2 u_z}{\partial z^2} + (e_{15}+e_{31})\frac{\partial^2 u_x}{\partial x \partial z} - \varepsilon_{11}\frac{\partial^2 \phi}{\partial x^2} - \varepsilon_{33}\frac{\partial^2 \phi}{\partial z^2} = 0
\end{aligned}\right\} \tag{2}$$

From the symmetry conditions it follows that the described problem may be reduced to that of a piezoelectric material rectangle with one crack loaded by the pressure $\sigma_0 + \sigma_0^T(z,t)$. Then the boundary conditions for the rectangular region of $0 \le x \le d$ and $0 \le z \le h$ can be stated as follows:

$$\left.\begin{aligned}
&\sigma_{xx}(0,z,t) = -\sigma_0 - \sigma_0^T(z,t) && (a < z < b)\\
&u_x(0,z,t) = 0 && (0 \le z \le a, b \le z \le h)
\end{aligned}\right\} \tag{3}$$

$$\sigma_{zx}(0,z,t) = 0,\ D_x(0,z,t) = 0 \quad (0 \le z \le h) \tag{4}$$

$$\sigma_{zx}(d,z,t) = 0,\ D_x(d,z,t) = 0,\ \frac{\partial}{\partial z}u_x(d,z,t) = 0 \quad (0 \le z \le h) \tag{5}$$

$$\left.\begin{aligned}
&\sigma_{zx}(x,0,t) = 0,\ \sigma_{zz}(x,0,t) = 0,\ D_z(x,0,t) = 0,\\
&\sigma_{zx}(x,h,t) = 0,\ \sigma_{zz}(x,h,t) = 0,\ D_z(x,h,t) = 0
\end{aligned}\right\} (0 \le x \le d) \tag{6}$$

3 Analysis

The general solutions of Eq.(2) are obtained by using the Fourier integral transform techniques [19] :

$$\left.\begin{aligned}
u_x(x,z,t) &= \frac{i}{2\pi}\sum_{j=1}^{6}\int_{-\infty}^{\infty}\frac{|s|}{s}a_{1j}A_{1j}(s,t)\exp(|s|\gamma_{1j}x)\exp(-isz)ds \\
&+\sum_{j=1}^{6}\sum_{n=1}^{\infty}a_{2j}A_{2jn}(t)\exp(\mu_n\gamma_{2j}z)\sin(\mu_n x)+\frac{d-x}{d}F_0(t) \\
u_z(x,z,t) &= \frac{1}{2\pi}\sum_{j=1}^{6}\int_{-\infty}^{\infty}A_{1j}(s,t)\exp(|s|\gamma_{1j}x)\exp(-isz)ds \\
&+\sum_{j=1}^{6}\sum_{n=1}^{\infty}A_{2jn}(t)\exp(\mu_n\gamma_{2j}z)\sin(\mu_n x)+\frac{z}{h}F_1(t) \\
\phi(x,z,t) &= \frac{1}{2\pi}\sum_{j=1}^{6}\int_{-\infty}^{\infty}b_{1j}A_{1j}(s,t)\exp(|s|\gamma_{1j}x)\exp(-isz)ds \\
&+\sum_{j=1}^{6}\sum_{n=1}^{\infty}b_{2j}A_{2jn}(t)\exp(\mu_n\gamma_{2j}z)\sin(\mu_n x)+\frac{z}{h}F_2(t)
\end{aligned}\right\} \tag{7}$$

where $A_{1j}(s,t)$, $A_{2jn}(t)$ $(j=1,2,...,6, n=1,2,...)$ and $F_i(t)$ $(i=1,2,3)$ are the unknown functions to be solved, and $\mu_n = n\pi/d$ $(n=1,2,...)$. The constants γ_{ij}, a_{ij} and $b_{ij}(i=1,2, j=1,2,...,6)$ can be obtained by setting $s\to\infty$ of the functions $\gamma_{1j}(s)$, $a_{1j}(s)$ and $b_{1j}(s)$ $(j=1,2,...,6)$ in Appendix A of the previous paper [18] and the functions $\gamma_{2j}(s)$, $a_{2j}(s)$ and $b_{2j}(s)$ $(j=1,2,...,6)$ in Appendix B of the previous paper [22], respectively. Substituting the displacements and electric potential solutions (7) into the constitutive equations (1), one can obtain the stresses and electric displacement components.

The problem may be reduced to a singular integral equation by defining the following new unknown function $G(z,t)$ [20]:

$$G(z,t)=\left\{\begin{array}{cc}\dfrac{\partial}{\partial z}u_x(0,z,t) & (a<z<b)\\ 0 & (0\le z\le a, b\le z\le h)\end{array}\right\} \tag{8}$$

Making use of the first boundary condition (3) with Eqs.(4)-(6), we have the following singular integral equation for the determination of the unknown function $G(\xi,t)$:

$$\int_b^a G(\xi,t)\left[\frac{1}{\xi-z}+\sum_{i=1}^{4}M_i(\xi,z)+\frac{C}{\Re[Z^\infty]d}\right]d\xi$$
$$=\frac{\pi}{\Re[Z^\infty]}\left[\sigma_0+\sigma_0^T(z,t)\right]\quad (a<z<b) \tag{9}$$

where Z^∞ and $M_i(\xi,z)$ $(i=1,2,3,4)$ are the known constant and kernel functions. The singular integral equation (9) for $a>0.0$ is to be solved with the following subsidiary conditions obtained from the second boundary condition (3).

$$\int_b^a G(\xi,t)d\xi=0 \tag{10}$$

For the case of $a=0.0$, the constant C can be determined from the second boundary condition (3) as follows:

$$C=\frac{1}{2}\left[\frac{c_{13}^2\varepsilon_{33}+2c_{13}e_{31}e_{33}-c_{33}e_{31}^2}{c_{33}\varepsilon_{33}+e_{33}^2}-c_{11}\right] \tag{11}$$

To solve the singular integral equation (9) and the additional equation (10) by using the Gauss-Jacobi integration formula [21], we introduce the following function $\Phi(u,t)$:

$$G(\xi,t)=\frac{c\Phi(u,t)}{(1+u)^\alpha(1-u)^{1/2}} \tag{12}$$

where $\alpha=1/2$ for $(a+b)/2>1$ (embedded crack) and $\alpha=-1/2$ for $(a+b)/2=1$ (edge crack), and $u=(2\xi-a-b)/(b-a)\quad(-1<u<1, a<\xi<b)$. The stress intensity factors $K_{Ia}(t)$ at $z=a$ and $K_{Ib}(t)$ at $z=b$ may be defined, and evaluated as

$$K_{Ia}(t)=\lim_{z\to a^-}\{2\pi(z-a)\}^{1/2}\sigma_{xx}(0,z,t)$$
$$=\begin{Bmatrix}-\Re[Z^\infty](\pi c)^{1/2}\Phi(-1,t) & (a>0)\\ 0 & (a=0)\end{Bmatrix} \tag{13}$$

$$K_{Ib}(t)=\lim_{z\to b^+}\{2\pi(b-z)\}^{1/2}\sigma_{xx}(0,z,t)$$
$$=\begin{Bmatrix}\Re[Z^\infty](\pi c)^{1/2}\Phi(1,t) & (a>0)\\ \Re[Z^\infty](2\pi b)^{1/2}\Phi(1,t) & (a=0)\end{Bmatrix} \tag{14}$$

4 Numerical results and discussion

To examine the effect of thermoelectroelastic interactions on the stress intensity factors, the solutions of the singular integral equation have been computed numerically. For the numerical calculations, the thermoelectroelastic properties of cadmium selenide are used [4]. Since the values of the coefficients of heat conduction for cadmium selenide could not be found in the literature, the value $\kappa^2 = \kappa_x/\kappa_z = 1/1.5$ is used.

4.1 The Stress Intensity Factors under Pure Mechanical Load

First, we consider the case of $T_0 = 0.0$. In this case, the stress intensity factors are independent of the time t. Fig. 2 shows the effect of $(a+b)/2h$ on

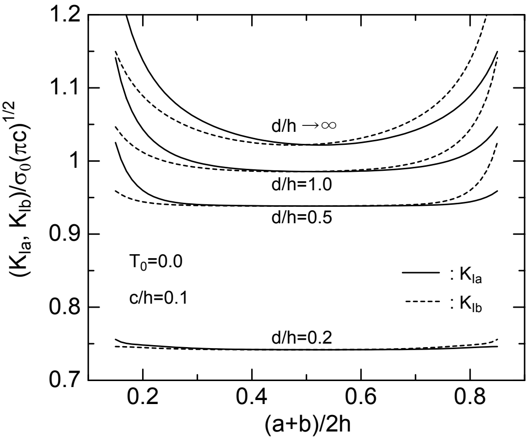
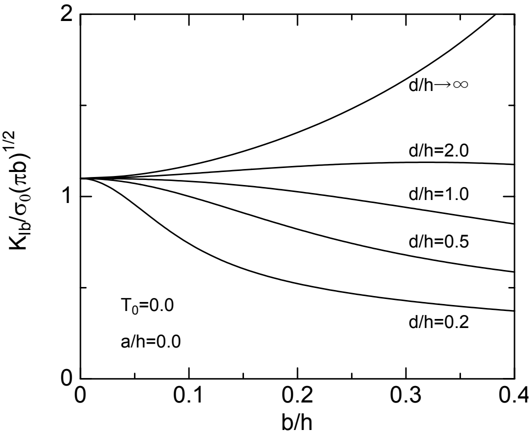


Fig. 2 The effect of the crack location on the stress intensity factors K_{Ia} and K_{Ib} of the embedded crack under pure mechanical load.

$(K_{Ia}, K_{Ib})/\sigma_0(\pi c)^{1/2}$ for various values of d/h with $c/h = 0.1$, respectively. The results for $d/h \to \infty$ are obtained in [18]. As d/h decreases, the values of the stress intensity factors decrease. The stress intensity factors of the crack tips near the free boundary ($(a+b)/2h \to 0.1$ or 0.9) become very large. The influence of the crack location on the stress intensity factors decreases with decreasing d/h.

Fig. 3 displays $K_{Ib}/\sigma_0(\pi b)^{1/2}$ of the edge crack versus b/h for various values of d/h. Different from the cases for $d/h \to \infty$, in which $K_{Ib}/\sigma_0(\pi b)^{1/2}$ increases monotonically with increasing b/h, $K_{Ib}/\sigma_0(\pi b)^{1/2}$ for $d/h = 2.0$ initially increases and then gradually decreases with increasing b/h and $K_{Ib}/\sigma_0(\pi b)^{1/2}$ for $d/h \leq 1.0$ decreases monotonically with increasing b/h.

Fig. 3 The effect of the crack length on the stress intensity factor K_{Ib} of the edge crack under pure mechanical load.



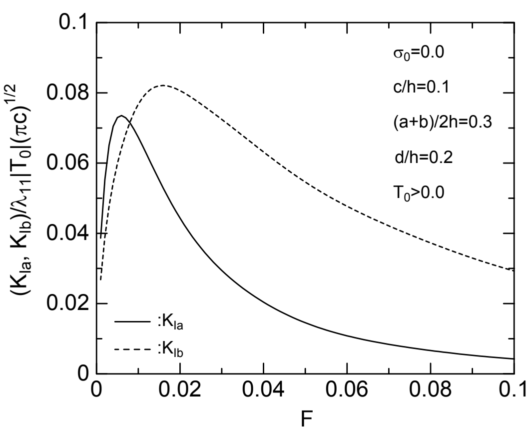
4.2 The Stress Intensity Factors of the Embedded Crack under Pure Thermal Load

Next, we consider the case of $\sigma_0 = 0.0$. Assume the bottom surface of the strip is suddenly heated from initial temperature T_I to $T_I + T_0(T_0 > 0.0)$, the calculated normalized stress intensity factors $(K_{Ia}, K_{Ib})/\lambda_{11}|T_0|(\pi c)^{1/2}$ versus time for $c/h = 0.1$, $(a + b)/2h = 0.3$ and $d/h = 0.2$ are shown in Fig. 4. In the figure, the time t is represented through the dimensionless Fourier number defined by

$$F = \frac{\lambda_0 t}{h^2} \tag{15}$$

Since the thermal stress $\sigma_0^T(z, t)$ is statically self-equilibrating, large com-

Fig. 4 The transient stress intensity factors K_{Ia} and K_{Ib} of the embedded crack under pure thermal load for $(a + b)/2h = 0.3$ and $d/h = 0.2$.



pressive stress occurs near the surface, and the tensile stress appears inside the strip [18]. Accordingly, the crack will open and the stress intensity factor will be positive. The values of the stress intensity factors increase at first, go through maxima, and then decrease with increasing F. As was expected, the stress intensity factors approach zero when F goes to infinity. The maximum value of $K_{Ia}/\lambda_{11}|T_0|(\pi c)^{1/2}$ is smaller and occurs faster than that of $K_{Ib}/\lambda_{11}|T_0|(\pi c)^{1/2}$.

4.3 The Stress Intensity Factor of the Edge Crack under Pure Thermal Load

Finally, we consider the case of $a/h = 0.0$ (edge crack). As mentioned above, if the strip is heated ($T_0 > 0.0$) suddenly on its bottom surface, large compressive stress will occur near the surface, and the small edge crack will be fully closed and the stress intensity factor will be negative. This is the crack contact problem and this phenomenon would be considered in this section. In this problem it is assumed that the strip contains pre-existing edge cracks of length $b/h = 0.2 \quad (a/h = 0.0)$ and the crack spacing $d/h = 0.1$, and the bottom surface of the strip is cooled ($T_0 < 0.0$) or heated ($T_0 > 0.0$) suddenly. Fig. 5 shows the time dependencies of the normalized stress intensity factor $K_{Ib}/\lambda_{110}|T_0|(\pi c)^{1/2}$ and the value b_0/h indicating the crack contact zone under the cooling process ($T_0 < 0.0$). In this case, because the edge crack is deeper than the tensile zone, the crack tip region ($z = b$) would be subjected to a compressive stress. On the other hand, the part of the crack near the bottom surface $0 \leq z \leq b_0$ would be opened due to the large tensile stress neat the surface, and the stress intensity factor at $z = b$ is equal zero. The unknown crack tip location $z = b_0$ can be obtained from the condition $K_{Ib_0} = 0.0$ at $z = b_0$. Thus the problem may easily be solved by iteration to find b_0 and then compute K_{Ib}. Fig. 6 is the same figures as Fig. 5 under the heating process ($T_0 > 0.0$). Different from the cooling process, the edge crack is deeper than the compressive zone, the crack tip region ($z = b$) would be subjected to a tensile stress and the crack would remain partially open. In other words, the part of the crack, $0 \leq z < a_0$, near the bottom surface would be closed and the stress intensity factor at the crack tip, $z = b$ would remain positive. The unknown crack tip location $z = a_0$ is obtained from the cusp condition $K_{Ia_0} = 0.0$ at $z = a_0$. Thus the problem may easily be solved by iteration to find a_0 and then compute K_{Ib}. The crack contact zone a_0 increases monotonously, and K_{Ib} becomes zero at $F = 0.049$, and the edge crack would be fully closed.

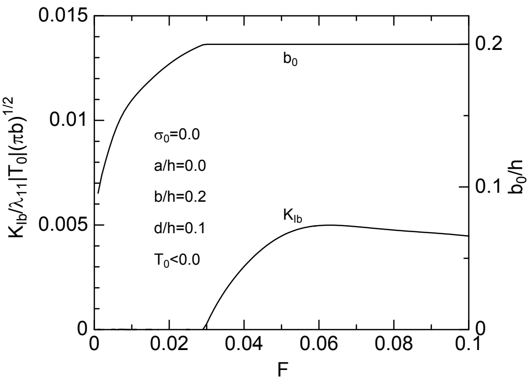


Fig. 5 The transient stress intensity factor K_{Ib} and the crack contact zone b_0 of the edge crack under pure thermal load for $b/h = 0.2$ and $T_0 < 0.0$.

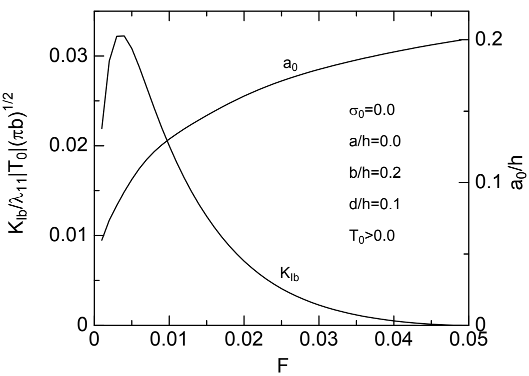


Fig. 6 The transient stress intensity factor K_{Ib} and the crack contact zone a_0 of the edge crack under pure thermal load for $b/h = 0.2$ and $T_0 > 0.0$.

5 Conclusion

The transient fracture problem of an infinite row of parallel cracks in a piezoelectric strip is studied. The effects of the crack spacing, the crack length and the crack location on the fracture behavior are considered. Moreover, taking the crack contact phenomenon into consideration, the transient fracture behavior of a pre-existing edge crack in the strip is considered. The following facts can be found from the numerical results.

1. The stress intensity factors due to both the mechanical and thermal load are lowered by the interaction among cracks, and the influence of it depends on the geometric parameter.
2. In some cases, the stress intensity factors under pure thermal load become negative and the results have no physical meaning. However, when the thermal load is combined with the mechanical load which induces the positive stress intensity factor, those results can be used effectively.
3. Taking the crack contact into consideration, it is found that the edge crack in the strip under the heating process would be fully closed at some time after the thermal shock.

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