Analysis of Weld Induced Plasticity by BFM

Akihide Saimoto

Abstract A method of analysis for the occurrence of localized thermoplastic strain, in a material under plane strain constraint, is studied based on the Body Force Method (BFM). BFM is an indirect boundary type method for elastic stress analysis based on the principle of superposition. Any inelastic strain can be expressed by the embedded force doublets in BFM. That is, in the analysis, a continuously embedded force doublets into the elastic body are used to express the presence of plastic strain. A simplified model of welding-induced plasticity is treated as a numerical example of the present method.

1 Introduction

In order to evaluate a degree of plastic deformation and residual stresses in the body, employment of the commercial finite element code that examines automated elastic-plastic calculation becomes very popular in recent years. The use of commercial code, however, often brings ineffectiveness from the view point of computational efficiency since most of mechanical and structural components are designed for elastic use, and therefore, the size of the plastic zones, even if they may happen due to the localized stress concentration, would be considerably small or restricted. In order to treat problems including limited plasticity efficiently, Blomerus and Hills proposed a dislocation based technique[1]. In their method, edge dislocations which correspond to the occurrence of plastic flow are introduced into the direction of maximum shear. The magnitude of the Burgers vector at the each dislocation point where the plastic flow occurred are determined through the iterative procedure considering the yield criterion. Since the magnitude of Burgers vector

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at each material point can be determined incrementally, the stress redistribution due to the occurrence of yielding can be simulated reasonably. On the other hand, the dislocation approach sometimes exhibits a convergence problem in which the direction of maximum shear stress varied frequently due to the fluctuation of the magnitude of Burgers vector at each material point.

Chen and Nisitani proposed the other approach to treat the limited plasticity based on the BFM. They employed a force doublet embedded in an elastic continuum in order to express the inelastic strain[2, 3]. Although their method is useful for wide range of limited plasticity, it seems difficult to apply the method to special class of plane strain problems in which the plastic strain in the thickness direction becomes the major component. Since its development in 1967, the BFM has been applied for elastic problems of practical importance. The original BFM is a boundary type method for elastic stress analysis, whose base is the principle of superposition. That is, in BFM any elastic problem is expressed in terms of the superposition of fundamental stress fields. As the fundamental solution, stress field due to an isolated point force acting in an infinite elastic body (usually referred as *Kelvin solution*) is preferably employed due to its simplicity. In fact, based on the principle of the BFM, stress components at an arbitrary point P, $\sigma_{ij}(P)$, in an elastic medium can be written as,

$$\sigma_{ij}(P) = \sigma_{ij}^{0}(P) + \int_{\Gamma} \phi^{k}(Q)\sigma_{ij}^{k}(P,Q)d\Gamma(Q), \tag{1}$$

where $P \in R$ is an arbitrary point in the reference region R which is surrounded by the imaginary boundary Γ. $Q \in \Gamma$ is a source point which moves along Γ. $\sigma_{ij}^{k}(P,Q)$ is a fundamental stress solution (stress component σ_{ij} at point P caused by a unit magnitude of point force acting into k_{th}-direction at source point Q) and $\phi^{k}(Q)$ is a density function of the body force which has to be determined so that the given boundary conditions are satisfied.

As discussed in [2, 3], the plastic strain at a point can be replaced by an equivalent force doublet embedded in an perfect elastic solid whose yield stress is infinite. So far, numerical solutions of elastic-plastic problems solved by BFM have been limited to two-dimensional where the plastic strain in the out-of-plain direction can be ignored or almost no influence. However, there exist some important class of problems in which the presence of the out-of-plane plastic strain has to be carefully treated even under the two-dimensional situation. In the present study, the treatment of out-of-plane plastic strain by two-dimensional BFM is discussed in detail. Then the weld-induced plasticity problem is discussed under the assumption that the material is an elastic-perfect-plastic body that obeys Von Mises yield criterion.

2 Solution of Two-Dimensional Elastic Problem by BFM

Before going to further, it would be useful to remind how pure elastic problem is solved by BFM briefly. Consider an infinite sheet with a circular hole of diameter $2a$, subjected to external tensile stresses as illustrated in Fig.1. In this example, the reference region R is an infinite plate excluding the circular disk, therefore, the imaginary boundary Γ is a circular ring of diameter $2a$. The stress component at point P can then be expressed according to Eq.(1) as,

$$\sigma_{xx}(P) = \sigma^0_{xx} + \int_\Gamma \{\phi^x(Q)\sigma^x_{xx}(P,Q) + \phi^y(Q)\sigma^y_{xx}(P,Q)\}d\Gamma(Q), \quad (2)$$

$$\sigma_{yy}(P) = \sigma^0_{yy} + \int_\Gamma \{\phi^x(Q)\sigma^x_{yy}(P,Q) + \phi^y(Q)\sigma^y_{yy}(P,Q)\}d\Gamma(Q), \quad (3)$$

$$\sigma_{xy}(P) = \int_\Gamma \{\phi^x(Q)\sigma^x_{xy}(P,Q) + \phi^y(Q)\sigma^y_{xy}(P,Q)\}d\Gamma(Q) \quad (4)$$

in which $\sigma^x_{ij}(P,Q)$ and $\sigma^y_{ij}(P,Q)$ are stress component at reference point $P(x,y)$ due to a unit magnitude of point force acting in the x and y direction at source point $Q(\xi,\eta)$, in an infinite sheet without any hole. σ^0_{xx} and σ^0_{yy} are the uniform tensile stresses at infinity. $\phi^x(Q)$ and $\phi^y(Q)$ are the unknown densities of body forces which define the magnitude of body forces at point Q per unit length of an imaginary boundary as,

$$dF_x(Q) = \phi^x(Q)d\Gamma,\ dF_y(Q) = \phi^y(Q)d\Gamma. \quad (5)$$

In numerical analysis, the imaginary boundary Γ is divided into several segments and the density of body forces at each segment is assumed to be constant, linear or quadrateral function of the local coordinates as in a same manner in boundary element methods. That is, the unknown densities of body forces are determined through boundary condition defined from the limiting procedure that the reference point $P \in R$ is approached to the boundary point P^Γ from inside of the region R. When the problem is rather simple,

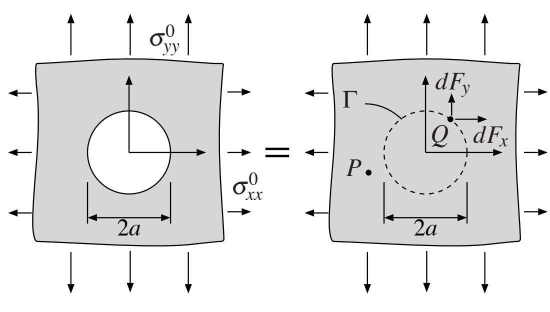

Fig. 1 Analysis of an elastic sheet having a circular hole of diameter $2a$, subjected to external tensile stresses σ^0_{xx} and σ^0_{yy} at infinity

the unknown density of body forces have closed form solution and can be determined theoretically. In fact, the situation illustrated in Fig.1 is one of a such case.

It is well known that two-dimensional elasticity problem can be expressed in terms of two complex potentials $\Omega(z)$ and $\omega(z)$ such that,

$$\sigma_{xx} + \sigma_{yy} = 2\big\{\Omega'(z) + \overline{\Omega'(z)}\big\}, \tag{6}$$

$$\sigma_{yy} - \sigma_{xx} + 2i\sigma_{xy} = 2\big\{\overline{z}\Omega''(z) + \omega'(z)\big\} \tag{7}$$

where z is a complex variable that represents the reference point $z = x + iy$. The Kelvin solution (stress field due to a point force of magnitudes F_x and F_y acting at a point $\zeta = \xi + i\eta$ in an infinite elastic sheet) can be expressed in the form of complex potentials as,

$$\Omega(z) = -\frac{F_x + iF_y}{2\pi(\kappa+1)} \log(z - \zeta), \tag{8}$$

$$\omega(z) = \frac{\kappa(F_x - iF_y)}{2\pi(\kappa+1)} \log(z - \zeta) + \frac{F_x + iF_y}{2\pi(\kappa+1)} \frac{\overline{\zeta}}{z - \zeta}, \tag{9}$$

where κ is a constant relating to Poisson's ratio ν as $\kappa = (3-\nu)/(1+\nu)$ for plane stress and $\kappa = 3 - 4\nu$ for plane strain. i is an imaginary unit and the over-bar denotes the complex conjugate. Using the complex potentials, the elastic fields of Fig.1 can be expressed as,

$$\Omega(z) = \frac{\sigma^0_{xx} + \sigma^0_{yy}}{4} z - \frac{1}{2\pi(\kappa+1)} \oint_\Gamma \log(z - ae^{i\theta})\{\phi^x(\theta) + i\phi^y(\theta)\} a d\theta, \tag{10}$$

$$\begin{aligned}\omega(z) = {} & \frac{\sigma^0_{yy} - \sigma^0_{xx}}{2} z + \frac{\kappa}{2\pi(\kappa+1)} \oint_\Gamma \log(z - ae^{i\theta})\{\phi^x(\theta) - i\phi^y(\theta)\} a d\theta \\ & + \frac{1}{2\pi(\kappa+1)} \oint_\Gamma \frac{ae^{-i\theta}}{z - ae^{i\theta}} \{\phi^x(\theta) + i\phi^y(\theta)\} a d\theta,\end{aligned} \tag{11}$$

since the source point ζ is on the circle of radius a which can be expressed as $\zeta = ae^{i\theta}$. The density functions $\phi^x(\theta)$ and $\phi^y(\theta)$ have closed form solution;

$$\phi^x(\theta) = \underbrace{\frac{\kappa+1}{2(\kappa-1)} \big\{\kappa\sigma^0_{xx} - (\kappa-2)\sigma^0_{yy}\big\}}_{=\rho_x = const.} = \rho_x \cos\theta, \tag{12}$$

$$\phi^y(\theta) = \underbrace{\frac{\kappa+1}{2(\kappa-1)} \big\{\kappa\sigma^0_{yy} - (\kappa-2)\sigma^0_{xx}\big\}}_{=\rho_y = const.} = \rho_y \sin\theta. \tag{13}$$

In fact, substituting Eqs.(12), (13) into Eqs.(10), (11) and by examining the contour integral considering $|z| > a$ using the Cauchy's integral theorem, the exact expressions of complex potentials for Fig.1 are obtained as,

$$\Omega(z) = \frac{\sigma^0_{xx} + \sigma^0_{yy}}{4} z + \frac{\sigma^0_{xx} - \sigma^0_{yy}}{2} \frac{a^2}{z}, \tag{14}$$

$$\omega(z) = \frac{\sigma^0_{yy} - \sigma^0_{xx}}{2} z - \frac{\sigma^0_{xx} + \sigma^0_{yy}}{2} \frac{a^2}{z} + \frac{\sigma^0_{xx} - \sigma^0_{yy}}{2} \frac{a^4}{z^3}. \tag{15}$$

It is readily found that the density functions of the body force in Eqs.(12) and (13) are given by the product of some constant and the components of unit normal $(\cos\theta, \sin\theta)$ at a point Q on the imaginary boundary Γ. Therefore, the expression of boundary integral in Eqs.(10) and (11) can be transformed into a form of area integral by using the Green's theorem as,

$$\Omega(z) = \frac{\sigma^0_{xx} + \sigma^0_{yy}}{4} z + \frac{1}{2\pi(\kappa+1)} \iint_{\overline{R}} \frac{\rho_x - \rho_y}{z - \zeta} d\xi d\eta, \tag{16}$$

$$\omega(z) = \frac{\sigma^0_{yy} - \sigma^0_{xx}}{2} z - \frac{\kappa - 1}{2\pi(\kappa+1)} \iint_{\overline{R}} \frac{\rho_x + \rho_y}{z - \zeta} d\xi d\eta + \frac{1}{2\pi(\kappa+1)} \iint_{\overline{R}} (\rho_x - \rho_y) \frac{\overline{\zeta}}{(z-\zeta)^2} d\xi d\eta, \tag{17}$$

in which $\overline{R}$ is a region inside of the imaginary boundary Γ, usually referred as an *auxiliary region*. Equivalence of Eqs.(10), (11) and Eqs.(16), (17) directly implies that the influence of the body force applied along the imaginary boundary is equivalent to that of due to embedded force doublets into the auxiliary region. The physical meaning of the force doublet is an embedded eigen strain at the point where it is applied. In the problem that includes any inelastic strain as in plastic strain, therefore, the force doublet is used to express its influence.

In the next section, the line weld model and its thermoelastic solution is discussed. Then the procedure for treating a thermoplastic strain is described

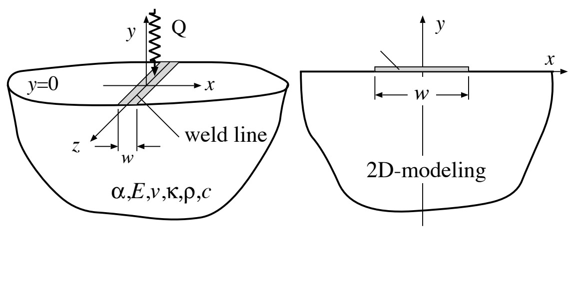

Fig. 2 Simple welding model for stainless steel (Yield stress:σ_Y = 800MPa, Heat flux:Q = 11.2MW/m^2, Linear expansion coefficient:$\alpha = 1.2 \times 10^{-5}$, Young's modulus: E = 200.2GPa, Density:ρ = 7833kg/m^3, Specific heat:c = 586J/kgK, Poisson's ratio:ν = 0.3, Thermal diffusivity: $\kappa = 1.133 \times 10^{-5}$m^2/s and Thermal conductivity:λ = 52W/mK)

under the assumption that the material follows Prandtl-Reuss flow rule for an elastic-perfect-plastic body under plain strain condition.

3 Simplified Model of Line Welding

Fig.2 shows a simplified weld model treated in this monograph. A uniform strength of transient line heater of width "w" is applied to a surface of a semi-infinite medium for a short duration of time with a strength chosen so that the heat flux delivered from the heater resembles to that of expected under actual welding of stainless steel. In a physical sense, the problem is essentially two dimensional which should simplify the analysis, however, an occurrence of plastic flow in the out-of-plane (z) direction make the problem somewhat cumbersome. The resulted thermoelastic field such as temperature rise $\tau(x, y, t)$ and elastic stress components $\sigma_{ij}(x, y, t)$ due to continuous heating of the duration t, can be written under the assumption of plane strain ($\varepsilon_{zz} = 0$) that,

$$\tau(x,y,t) = \frac{Q}{2\pi\kappa\rho c}\int_{-\frac{w}{2}}^{\frac{w}{2}} E_1(S)\,d\xi, \tag{18}$$

$$\frac{\sigma_{xx}(x,y,t)}{\overline{\sigma}} = \int_{-\frac{w}{2}}^{\frac{w}{2}} \left\{ \left(2\frac{y^2}{R^2} - 1 \right) \frac{1-e^{-S}}{S} - E_1(S) \right\} d\xi - \frac{2y}{\pi}\int_{-\infty}^{\infty} \frac{(x-\xi)^2}{R^4} f(\xi,t)d\xi, \tag{19}$$

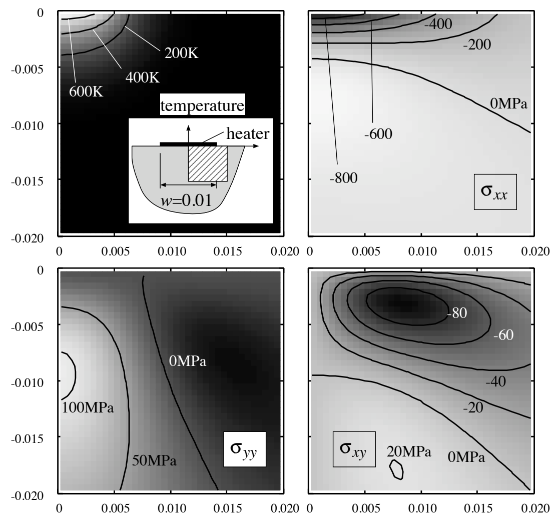

Fig. 3 Temperature and thermoelastic stresses after 1s heating

$$\frac{\sigma_{yy}(x,y,t)}{\overline{\sigma}} = \int_{-\frac{w}{2}}^{\frac{w}{2}} \left\{ \left(1 - 2\frac{y^2}{R^2}\right) \frac{1-e^{-S}}{S} - E_1(S) \right\} d\xi$$
$$- \frac{2y^3}{\pi} \int_{-\infty}^{\infty} \frac{f(\xi,t)}{R^4} d\xi, \quad (20)$$

$$\frac{\sigma_{xy}(x,y,t)}{\overline{\sigma}} = 2y \int_{-\frac{w}{2}}^{\frac{w}{2}} \frac{x-\xi}{R^2} \frac{1-e^{-S}}{S} d\xi - \frac{2y^2}{\pi} \int_{-\infty}^{\infty} \frac{x-\xi}{R^4} f(\xi,t) d\xi, \quad (21)$$

$$\frac{\sigma_{zz}(x,y,t)}{2\overline{\sigma}} = - \int_{-\frac{w}{2}}^{\frac{w}{2}} E_1(S) d\xi - \nu \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(\xi,t)}{R^2} d\xi, \quad (22)$$

where R^2 is a square of distance between reference and source points $R^2 = (x-\xi)^2 + y^2$, (x,y) is a coordinate of reference point, $(\xi, 0)$ is a coordinate of source point, $\overline{\sigma}$ is a constant defined by $\overline{\sigma} = \alpha EQ/4\pi\kappa\rho c(1-\nu)$ in which ρ is a mass density, κ is a thermal diffusivity, c is a specific heat, α is a coefficient of linear expansion, E is a Young's modulus, ν is a Poisson's ratio (the concrete values of those material properties used were shown in the caption of Fig.2). S is a non-dimensional parameter defined by $S = R^2/4\kappa t$, $E_1(x)$ is a integral exponential function defined by $E_1(x) = \int_x^{\infty} \frac{e^{-u}}{u} du$ and $f(\xi,t)$ is a function defined as

$$f(\xi,t) = 2\sqrt{\kappa t} \times \left[\frac{1}{p}(1 - e^{-p^2}) + pE_1(p^2) \right]_{\frac{\xi+w/2}{2\sqrt{\kappa t}}}^{\frac{\xi-w/2}{2\sqrt{\kappa t}}}. \quad (23)$$

4 Expression of Plastic Strain by Force Doublets

As already mentioned, the most fundamental concept for the treatment of plastic strain in BFM is to replace the distribution of plastic strain by force doublets. Consider an elastic-plastic body whose elasticity constants are E for Young's modulus and ν for Poisson's ratio. The plastic part in the region is noted R^p which is surrounded by an elastic foundation R^e. Next, consider an infinitesimally small plastic element $\omega^p \in R^p$ which has stress components $\sigma_{ij}(P)$ and strain components $\varepsilon_{ij}(P) = \varepsilon_{ij}^e(P) + \varepsilon_{ij}^p(P)$ at point $P \in \omega^p$ where $\varepsilon_{ij}^e(P)$ and $\varepsilon_{ij}^p(P)$ are the elastic and plastic components of the strain at point P, respectively. ω^p can be extracted without affecting the stress field if traction $t_i(P) = \sigma_{ij}(P)n_j(P)$ is applied to the outer surface of ω^p, and at the same time, traction $-t_i$ is applied to the inner surface of the cavity which is made by the extraction of ω^p from R^p where $n_j(P)$ is a component of unit normal at P. Then the plastic element ω^p is transposed into an *ideal elastic* element ω^e which has the same elastic properties (E,ν) with region R^e but its yield stress is infinite so that no yielding takes place. Owing to this transposition, stress state is unchanged but the strain state is decreased by the amount of plastic strain $\varepsilon^p_{ij}(P)$. Therefore, if ω^e is embedded into the cavity of the region R^p, some clearances due to shrinkage of the element would be observed. In order to compensate this strain decrease and to embed an ideal element without any gap, an additive stress $T_{ij}(P)$ have to be applied to ω^e. If such procedure is continued until all the plastic element are transposed to an ideal elastic one. After the completion of such transposition, the stress field at an arbitrary point P may be expressed as follows.

$$\sigma_{ij}(P) = \sigma_{ij}^{\text{therm}}(P) - T_{ij}(P) + \iint_{R^p} \frac{\partial \sigma^k_{ij}(P,Q)}{\partial \xi_\ell} T_{k\ell}(Q) dR^p(Q), \qquad (24)$$

where $\sigma_{ij}^{\text{therm}}(P)$ is component of thermoelastic stresses at point P which is shown from Eqs.(19) $\sim$ (22), $T_{ij}(Q)$ is a magnitude of force doublet embedded at point Q, which compensate the strain decrease during the process of transposition from plastic element ω^p to elastic one ω^e. Because of the incremental nature of plasticity, not the total stress but an incremental stress is used to evaluated a present stress state. Then Eq.(24) is replaced by an incremental form as

$$d\sigma_{ij}(P) = d\sigma_{ij}^{\text{therm}}(P) - dT_{ij}(P) + \iint_{R^p} \frac{\partial \sigma^k_{ij}(P,Q)}{\partial \xi_\ell} dT_{k\ell}(Q) dR^p(Q), \qquad (25)$$

in which $dT_{ij}(Q)$ is an increment of the magnitude of force doublet, which is related to the increment of plastic strain at point Q. The total stress can be calculated by a sum of stress increments such that $\sigma_{ij}(P) = \sum d\sigma_{ij}(P)$. When Prandtl-Reuss flow rule is employed, each component of plastic strain increment is assumed to be proportional to the component of deviatoric stress S_{ij} with unknown proportionality constant λ. Therefore, the increment of the magnitude of point force doublet can be expressed as

$$dT_{ij}(Q) = D_{ijk\ell} d\varepsilon^p_{k\ell}(Q) = D_{ijk\ell}\left(\sigma_{k\ell}(Q) - \delta_{k\ell}\frac{\sigma_{mm}(Q)}{3}\right)\lambda(Q), \qquad (26)$$

where $D_{ijk\ell}$ is an elastic modulus tensor and δ_{ij} is Kronecker delta. It should be noted that the term "$-dT_{ij}(P)$" in Eq.(25) is indispensable with no relation to the value of $\partial\sigma^k_{ij}(P,Q)/\partial\xi_\ell$. In fact, stress components due to point force doublet which acts in the z direction $\partial\sigma^z_{ij}(P,Q)/\partial z$ results no influence at any point P under plane strain condition. However, even when $\partial\sigma^k_{ij}(P,Q)/\partial\xi_\ell = 0$, the term $-dT_{ij}(P)$ still gives a non-zero influence at point P. In a practical analysis, the proportional constant $\lambda(Q)$ in Eq.(26) is the unknown parameter to be determined through numerical analysis. Since $\lambda(Q)$ is not only a function of the position Q but also the function of time t, it is required to determine the value of $\lambda(Q)$ step-wisely, considering the yield criterion. For example, when Von Mises criterion for elastic-perfect-plastic body is supposed, the following relation must hold at a point $P \in R^p$ that

$$\sigma_{eq} = \sqrt{(\sigma_{xx} - \sigma_{yy})^2 + (\sigma_{yy} - \sigma_{zz})^2 + (\sigma_{zz} - \sigma_{xx})^2 + 6\sigma_{xy}^2} = \sigma_Y. \quad (27)$$

5 Numerical Procedure and Discussion

For the numerical estimation of residual strain, the time domain is divided into N equally division as $t = n\Delta t, (n = 1, 2, \cdots, N)$ where Δt is a time increment. A space domain is also divided into number of square areas (0.25mm×0.25mm) in which the magnitude of plastic strain (and therefore the magnitude of force doublet) is assumed to be constant over a region and a given time. As a result, the total stress component at the reference time $t = n\Delta t$,$\sigma_{ij}(P)|_n$ can be evaluated as

$$\begin{aligned}\sigma_{ij}(P)|_n = &\sum_{k=1}^{n-1} d\sigma_{ij}(P)|_k + d\sigma_{ij}^{\text{therm}}(P)|_n - dT_{ij}(P)|_n \\ &+ \int_{\Omega} \frac{\partial}{\partial \xi_\ell} \Big\{ \sigma_{ij}^k(P, Q) \Big\} dT_{k\ell}(Q)|_n d\Omega^p(Q), \end{aligned} \quad (28)$$

where $dT_{ij}(P)|_n$ is an increment of the magnitude of force doublet at time $t = n\Delta t$. As seen Eq.(26), $dT_{ij}(P)|_n$ is related to the total stress state at $t = n\Delta t$ but it could be reasonable to evaluate its value from the value of

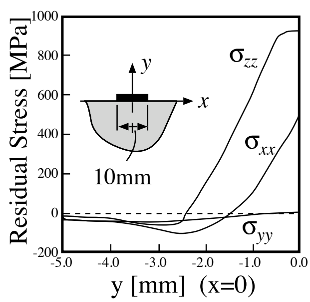

Fig. 4 Residual stress distribution along y axis after complete cool down

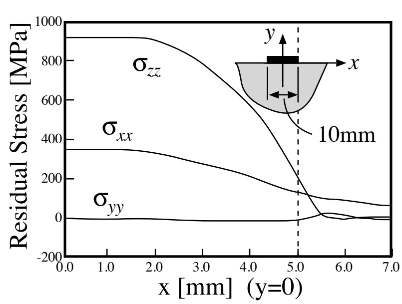

Fig. 5 Residual stress distribution along x axis after complete cool down

total stress at one time step Δt before. That is, $dT_{ij}(P)|_n$ is approximated by

$$dT_{ij}(P)|_n \approx D_{ijk\ell}\left(\sigma_{k\ell}(P)|_{n-1} - \delta_{k\ell}\frac{\sigma_{mm}(P)|_{n-1}}{3}\right)\lambda(P)|_n \tag{29}$$

in which $\lambda(P)|_n$ is unknown parameter yet not determined. Substitution of Eq.(29) into Eq.(28) gives stress components at arbitrary point P at reference time $t = n\Delta t$, if parameter $\lambda(P)|_n$ is provided. In order to determine $\lambda(P)|_n$, the yield criterion is used. However, substitution of Eq.(28) into Eq.(27) leads nonlinear simultaneous equations for the determination of $\lambda(P)|_n$ at each reference point P. These nonlinear simultaneous equations should be solved carefully under the constraint that $\lambda(P)|_n \geq 0$. When $\lambda(P)|_n$ becomes negative, it means the unloading process during plastic deformation so that the value of $\lambda(P)|_n$ should set to be 0. In Figs.4 and 5 the residual stress distribution along the y and x axes after complete cool down are shown. As seen, the out-of-plane residual stress component σ_{zz} exhibits the largest value and the usual plane strain relation $\sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy})$ is violated.

6 Conclusion

A treatment of plastic strain in the direction of out-of-plane based on the principle of the body force method was discussed. The material supposed was elastic-perfect-plastic body that follows Von Mises yield criterion. It was found that the residual stress in the out-of-plane direction σ_{zz} can be estimated independently of the in-plane residual stress components σ_{xx} and σ_{yy}. It was also found that the proposed method provides effective and efficient technique for problems that include limited plasticity.

References

1. P.M. Blomerus and D.A. Hills, An efficient procedure for modeling limited plastic flow, Proc. Inst. Mech. Eng. C, **212**, p.731 (1998)
2. D. -H. Chen and H. Nisitani, Extension of Body Force Method to Elastic-Plastic Problems, Transaction of the Japan Society of Mechanical Engineers, Ser. A, **51**-462, p.571, 1985 (in Japanese)
3. H.Nisitani and D.-H. Chen, Elastic-Plastic Analysis of an Infinite Plate with an Elliptic Hole by Body Force Method, Transaction of the Japan Society of Mechanical Engineers, Ser. A, **51**-465, p.1471, 1985 (in Japanese)