Mathematical Analysis of Flexural Vibration for a Functionally Graded Material Plate and Vibration Suppression by Flexural Wave Control

Ryuusuke Kawamura, Hiroshi Fujita, Kenichiro Heguri
and Yoshinobu Tanigawa

Abstract In this paper, flexural vibrations of rectangular plates and beams which are consisted of functionally inhomogeneous materials due to cyclic loadings of external force and temperature change are analyzed mathematically. Interference between the flexural vibration due to cyclic loading and that due to cyclic heating is discussed. The amplification effect by loading frequency is also discussed for the deflection and stresses of the beam and the plate. Furthermore, a control problem of the flexural vibration of the FGM beam by the method of wave control is considered. In order to remove progressive wave in flexural waves excited by cyclic loading, intensity and phase lag of control force are derived on the basis of the active sink method. Then, the validity of wave control for the flexural vibration suppression of the FGM beam is discussed.

1 Introduction

Research and development of functionally inhomogeneous materials, such as functionally graded materials, contribute to make performances and functions of structural materials high. The reduction of thermal stresses is one

Ryuusuke Kawamura
Faculty of Engineering, University of Miyazaki, Miyazaki 889-2192, Japan,
e-mail: rkawamura@cc.miyazaki-u.ac.jp

Hiroshi Fujita
Graduate School of Engineering, Osaka Prefecture University, Osaka 599-8531, Japan

Kenichiro Heguri
Osaka Prefecture University, Osaka 599-8531, Japan

Yoshinobu Tanigawa
Professor Emeritus, Osaka Prefecture University, Japan,
e-mail: tanigawayoshinobu20070424@zeus.eonet.ne.jp

of important subjects, in order to secure the material strength and to improve the heat resistance. Assuming applications of FGMs to the field of aerospace, structural members become lightweight and thin-walled. Many thin-walled structural members with high aspect ratio are frequently used. Weight saving in the members often results in decrease of stiffness and natural frequency. Hence, to analyze dynamic responses, such as the vibration of structural members due to cyclic loading and heating, is one of important subjects from a viewpoint of structural strength. In our previous works, effects of material inhomogeneity on flexural vibrations were discussed for the FGM beam [1] and rectangular plate [2,3], which were subjected to the cyclic loading and heating. When the loading frequency is close to a natural frequency of the plate, the vibration due to the cyclic loading and that due to the cyclic heating offset each other. However, flexural responses for the vibrating FGM plate with high aspect ratio due to cyclic loadings are not fully made clear.

A number of low dumping natural modes exist in flexible long and large structures such as large space structures. Approaches to their vibration control can be broadly classified into two categories. One is the vibration modal control method. This is effective to treat the problem of comparatively less vibration modes. However, it is impossible to apply the method to the problem in which structures have a number of vibration modes such as large space structures. Instead, an approach to use the control of propagating waves has been studied for problems of dynamic analysis and applied to the control of vibration for structures. This approach proposed by Tanaka and Kikushima [4,5] is one of wave controls and is referred as the active sink method. This method is regarded as promising, since it enables to make vibration modes asleep in the structures. The study which treats a vibration control problem of the structure composed of FGMs from a viewpoint of waves has not been seen yet.

In this paper, we study an effect of plate aspect ratio on interference between flexural vibrations due to the cyclic loading and heating applying to a FGM plate. We discuss a difference in the deflection amplification between the FGM plate with high aspect ratio and a FGM beam. Then, we attempt to treat a control problem of the flexural vibration of the FGM beam by the method of wave control.

2 Flexural Vibrations for FGM Beam and Rectangular Plate

2.1 Analytical Development

We consider a plate of thickness h as shown in Fig. 1. The plate initially at zero temperature is bounded by planes $z' = 0$ and $z' = h$. The surface $z' = h$ is kept perfectly insulated while the surface $z' = 0$ is exposed, for time $t > 0$, to a prescribed temperature which varies sinusoidally in time with amplitude T_0 and angular frequency ω. The corresponding one-dimensional boundary-value problem for temperature change $T = T(z', t)$ is:

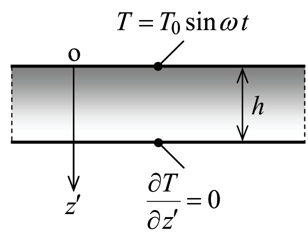


Fig. 1 Inhomogeneous plate.

$$c(z')\rho(z')\frac{\partial T}{\partial t} = \frac{\partial}{\partial z'}\left\{\lambda(z')\frac{\partial T}{\partial z'}\right\}, \tag{1}$$

$$T = 0 \text{ at } t = 0, \tag{2}$$

$$T = T_0 \sin \omega t \text{ at } z' = 0, \tag{3}$$

$$\frac{\partial T}{\partial z'} = 0 \text{ at } z' = h. \tag{4}$$

It is assumed that the specific heat capacity $c\rho$ and the thermal conductivity λ are independently given in a form of power of thickness coordinate z',

$$c(z')\rho(z') = c_0\rho_0\left(1 + \frac{z'}{h}\right)^k, \quad \lambda(z') = \lambda_0\left(1 + \frac{z'}{h}\right)^l, \tag{5}$$

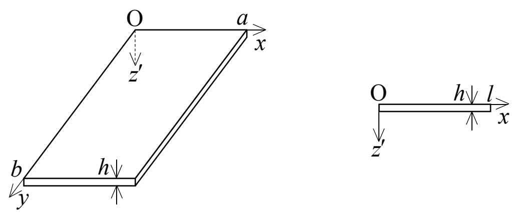
where constants, c_0,$\rho_0$and λ_0, are typical quantities of the specific heat, the mass density, and the thermal conductivity; and exponents k and l are parameters representing the inhomogeneity in the specific heat capacity and the thermal conductivity, respectively.

The solution of Eqs. (1)-(5) may be written in a dimensionless form as follows:

$$\bar{T}(\zeta, \tau) = \sum_{j=1}^{\infty} D_{1j}(\zeta)\, e^{-q_j^2\tau} + D_2(\zeta)\cos\bar{\omega}\tau + D_3(\zeta)\sin\bar{\omega}\tau, \tag{6}$$

where the variables ζ,τ and $\bar{\omega}$ denote the dimensionless quantities of coordinate z', time t, and angular frequency ω, whereas the descriptions in detail of eigenvalue q_j and functions $D_{1j}(\zeta)$,$D_2(\zeta)$,$D_3(\zeta)$ are omitted due to limitations of space.

We assume a beam and a rectangular plate which are subjected to a distributed transverse load p and are exposed to a temperature change T from the stress-free state shown as Fig. 2. If the origin of the coordinate in the



(a) Rectangular plate (b) Beam

Fig. 2 Inhomogeneous rectangular plate and beam.

thickness direction is appropriately chosen in the cross-section of the inhomogeneous beam and plate in which Young's modulus has an arbitrary inhomogeneity in the thickness direction, thermal bending in the inhomogeneous beam and plate can be treated easily. Thus, the coordinate in the thickness direction z whose position of the origin is located at $z' = \eta$ from the top surface $z' = 0$ of the plate is defined as

$$z = z' - \eta. \tag{7}$$

The position η of the origin of the coordinate z is defined as

$$\eta = \int_0^h E(z')z'\,dz' \Bigg/ \int_0^h E(z')\,dz'. \tag{8}$$

The equations of motion for the flexural vibration of the inhomogeneous beam and plate are written as

$$\frac{\partial^4 w}{\partial x^4} + \frac{\mu_b}{c_b}\frac{\partial^2 w}{\partial t^2} = \frac{1}{c_b}\left(p - \frac{\partial^2 M_T}{\partial x^2}\right), \tag{9}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2 w + \frac{\mu_p}{c_p}\frac{\partial^2 w}{\partial t^2} = \frac{1}{c_p}\left[p - \frac{1}{1-\nu}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)M_T\right], \tag{10}$$

where ν is Poisson's ratio; c_b and c_p are flexural rigidities; μ_b and μ_p are mass per unit length and width; M_T is thermal resultant moment.

We assume that the following cyclic transverse load p applies to the plate.

$$p = p_0 + p_1 \sin \omega_1 t, \tag{11}$$

where p_0 is reference load; p_1 is load amplitude; ω_1 is angular frequency.

The thermal resultant moment M_T is defined as

$$M_T = \int_0^h E(z')\alpha(z')T(z',t)(z'-\eta)\,dz'. \tag{12}$$

It is assumed that Young's modulus E, the coefficient of linear thermal expansion α, the mass density ρ are independently given by a power of z' as

$$E(z') = E_0\left(1+\frac{z'}{h}\right)^m, \quad \alpha(z') = \alpha_0\left(1+\frac{z'}{h}\right)^n, \quad \rho(z') = \rho_0\left(1+\frac{z'}{h}\right)^\gamma, \tag{13}$$

where constants E_0 and α_0 are typical quantities of Young's modulus and the coefficient of linear thermal expansion; and exponents, m,n and γ, are parameters representing the inhomogeneity of Young's modulus, the coefficient of linear thermal expansion, and the mass density, respectively.

The thermo-elastic analysis of the flexural vibration for the inhomogeneous plate is outlined below. Because of the linearity of the problem, the solution of Eq. (10) for the out-of-plane deflection w with simple supports may be written in a dimensionless form as the sum of two deflections due to cyclic loading w_1 and to cyclic heating w_2 as

$$\bar{w} = \bar{w}_1 + \bar{w}_2, \tag{14}$$

$$\begin{aligned}\bar{w}_1 = {} & \frac{1}{\bar{\mu}_p}\frac{16}{\pi^2}\sum_{m=odd}^{\infty}\sum_{n=odd}^{\infty}\frac{1}{mn}\left\{\frac{\bar{p}_0}{\Omega_{mn}^2}(1-\cos\Omega_{mn}\tau)\right.\\ & \left. -\frac{\bar{p}_1}{\Omega_{mn}^2-\bar{\omega}_1^2}\left(\sin\bar{\omega}_1\tau-\frac{\bar{\omega}_1}{\Omega_{mn}}\sin\Omega_{mn}\tau\right)\right\}\sin\alpha_m\bar{x}\sin\beta_n\bar{y},\end{aligned} \tag{15}$$

$$\begin{aligned}\bar{w}_2 = {} & \frac{16\bar{h}^2}{(1-\nu)\bar{\mu}_p\pi^2}\sum_{k=odd}^{\infty}\sum_{l=odd}^{\infty}\frac{\alpha_k^2+\beta_l^2}{kl}\left\{\sum_{j=1}^{\infty}\frac{1}{\Omega_{kl}^2+q_j^4}\right.\\ & \times\left(e^{-q_j^2\tau}-\cos\Omega_{kl}\tau+\frac{q_j^2}{\Omega_{kl}}\sin\Omega_{kl}\tau\right)\int_1^2 D_{1j}(\zeta)\zeta^{m+n}(\zeta-1-\bar{\eta})\,d\zeta\\ & +\frac{1}{\Omega_{kl}^2-\bar{\omega}^2}\left(\sin\bar{\omega}\tau-\frac{\bar{\omega}}{\Omega_{kl}}\cos\Omega_{kl}\tau\right)\int_1^2 D_3(\zeta)\zeta^{m+n}(\zeta-1-\bar{\eta})\,d\zeta\\ & \left.+\frac{1}{\Omega_{kl}^2-\bar{\omega}^2}\left(\cos\bar{\omega}\tau-\cos\Omega_{kl}\tau\right)\int_1^2 D_2(\zeta)\zeta^{m+n}(\zeta-1-\bar{\eta})\,d\zeta\right\}\sin\alpha_k\bar{x}\sin\beta_l\bar{y},\end{aligned} \tag{16}$$

where the dimensionless natural angular frequency Ω_{kl} for the flexural vibration of the plate is given by

$$\Omega_{kl} = \sqrt{\frac{\bar{c}_3 \bar{h}^4}{\bar{\mu}_p}} \left(\alpha_k^2 + \beta_l^2\right), \quad \alpha_k = k\pi, \quad \beta_l = \frac{l\pi}{\bar{b}}. \tag{17}$$

The associated stress is also given as the sum of a solution to cyclic loading and that to cyclic heating. It is illustrated that the stress σ_{yy} may be written in a dimensionless form as the sum of the stress due to cyclic loading σ_{yy1} and that due to cyclic heating σ_{yy2},

$$\bar{\sigma}_{yy} = \bar{\sigma}_{yy1} + \bar{\sigma}_{yy2}, \tag{18}$$

$$\bar{\sigma}_{yy\,1} = -\frac{1}{1-\nu^2} \zeta^m (\zeta - 1 - \bar{\eta}) \bar{h}^2 \left(\nu \frac{\partial^2 \bar{w}_1}{\partial \bar{x}^2} + \frac{\partial^2 \bar{w}_1}{\partial \bar{y}^2}\right), \tag{19}$$

$$\bar{\sigma}_{yy2} = -\frac{1}{1-\nu^2} \zeta^m \left\{ (\zeta - 1 - \bar{\eta}) \bar{h}^2 \left(\nu \frac{\partial^2 \bar{w}_2}{\partial \bar{x}^2} + \frac{\partial^2 \bar{w}_2}{\partial \bar{y}^2}\right) + (1+\nu)\zeta^n \bar{T} \right\}. \tag{20}$$

2.2 Numerical Results and Discussion

Interference between the flexural vibration due to cyclic loading and that due to cyclic heating is examined numerically. The amplification effect by the loading frequency is also examined for the deflection and stresses of the beam and the plate.

Typical material properties are chosen from a mild steel. Thickness h, length a, aspect ratio $\bar{b}(= b/a)$ for a plate and thickness h, length l for a beam are given as

$$h = 2 \times 10^{-3}[\mathrm{m}]\,; \quad a = 1.00[\mathrm{m}]\,, \quad \bar{b} = 9.0 \text{ for plate}, \tag{21}$$

$$h = 2 \times 10^{-3}[\mathrm{m}]\,; \quad l = 1.00[\mathrm{m}] \text{ for beam}. \tag{22}$$

So that the maximum deflection amplitude due to cyclic loading is made closely equal to that due to cyclic heating, reference load p_0 and load amplitude p_1 are given as

$$p_0 = 0[\mathrm{kPa}]\,, \quad p_1 = 1[\mathrm{kPa}]. \tag{23}$$

Angular frequencies in cyclic loading and heating $\bar{\omega}_1$, $\bar{\omega}$ are given as

$$\bar{\omega}_1 = \varepsilon_1 \Omega_{11}^{(h)}, \quad \bar{\omega} = \varepsilon\, \Omega_{11}^{(h)}, \tag{24}$$

where ε_1 and ε are parameters, $\Omega_{11}^{(h)}$ is a fundamental natural angular frequency of the flexural vibration of a homogeneous square plate. Setting $\varepsilon_1 = \varepsilon$, the interference in dynamic responses due to cyclic loading and heating is examined, here. Figures 3 and 4 illustrate the time evolution of

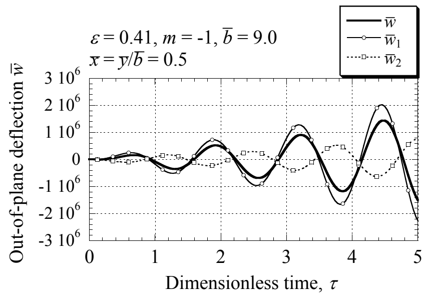


Fig. 3 Time evolution of out-of-plane deflection $\bar{w}$ at the central point $\bar{x} = \bar{y}/\bar{b} = 0.5$ of a FGM plate with aspect ratio $\bar{b} = 9.0$.

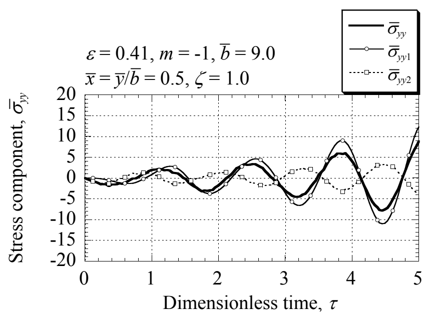


Fig. 4 Time evolution of stress $\bar{\sigma}_{yy}$ at the central point $\bar{x} = \bar{y}/\bar{b} = 0.5$ on the surface $\zeta = 1.0$ in a FGM plate with aspect ratio $\bar{b} = 9.0$.

out-of-plane deflection $\bar{w}$ at the central point $\bar{x} = \bar{y}/\bar{b} = 0.5$ and that of the stress $\bar{\sigma}_{yy}$ in the plate with aspect ratio $\bar{b} = 9$. When the angular frequencies of cyclic loading and heating are closer to the natural angular frequency of the plate, the deflection due to cyclic loading w_1 and that due to cyclic heating w_2 offset each other regardless of aspect ratio. The same result can be observed in the stress σ_{yy} shown in Fig. 4.

Figure 5 shows the time evolutions of dynamic out-of-plane deflections for the plate with aspect ratio $\bar{b} = 9.0$ and for a beam, respectively. The time evolution of the out-of-plane deflection for the plate with high aspect ratio approaches that for a beam gradually.

Figure 6 shows the variation of deflection amplifications with a parameter in cyclic loadings ε. Here, a deflection amplification is defined as the ratio of the maximum dynamic deflection $\bar{w}_{\max}$ to the maximum quasi-static deflection $\bar{w}_{st\ \max}$. The amplification factors for the plate and the beam increase with an increase of the parameter ε. Furthermore, the amplification factor for the plate approaches that for the beam gradually with increase of the parameter ε.

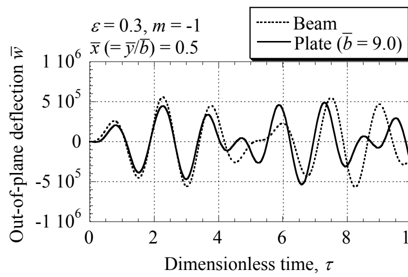


Fig. 5 Time evolution of out-of-plane deflections for a rectangular plate with aspect ratio $\bar{b} = 9.0$ and for a beam.

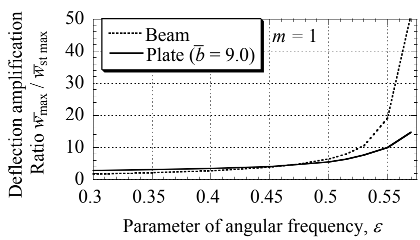


Fig. 6 Variation of deflection amplifications with parameter of angular frequency ε for a rectangular plate and a beam.

3 Vibration Suppression of FGM Beam by Active Sink Method

The concept of wave control for a homogeneous flexible beam has been proposed by Tanaka and Kikushima [4,5]. When energy of an external force is input into structural member, the energy is carried as progressive waves. When the progressive waves arrive at the boundary, reflected waves are exited. Synchronization of interference of these travelling waves causes vibration modes in the structural members. The basic concept of the vibration suppression in the active sink method is to remove reflected waves from structures and to make vibration modes asleep.

We formulate equations for the FGM beam and derive wave vectors by the transfer matrix method. We assume that the beam is subjected to cyclic loading $F_{in} = f_{in} \sin \omega\tau$ at the right boundary edge $\bar{x} = 1.0$ of the beam and control force $F_d = f_d \sin(\omega t + \varphi)$ at the left boundary one $\bar{x} = 0.0$, simultaneously as shown in Fig. 7. The mechanical conditions for the beam are free at both edges. According to the active sink method, we obtain the intensity f_d and the phase lag φ in the control force F_d so as to null the reflected wave in the wave vector. Figures 8 and 9 show an effect of the control force on

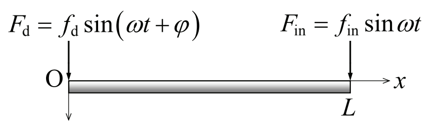


Fig. 7 Vibration suppression of a FGM beam by the active sink method.

the out-of-plane deflection of the beam. The intensity of the cyclic loading is assumed as $f_{in} = 0.05$[N] and the angular frequency ω is set as 0.999 times of third natural angular frequency Ω_3. When the control force is applied to the beam, the maximum amplitude of deflection is decreased about less than 2% of one without the control force.

Figure 10 shows the difference in the time evolution of deflection at the right boundary edge. The amplitude of the flexural vibration of the beam without control force increases with time. On the other hand, the amplitude of the beam vibration with control remains small with elapse of time. Thus, we can confirm the validity of the active sink method for the vibration suppression of the FGM beam.

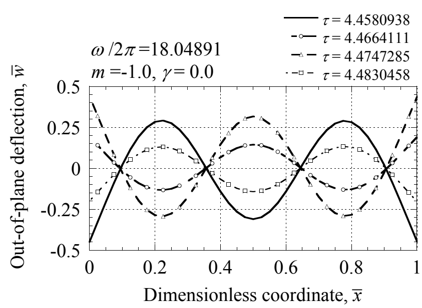


Fig. 8 Longitudinal distribution of out-of-deflection $\bar{w}$ of a FGM beam due to cyclic loading F_{in} without control force.

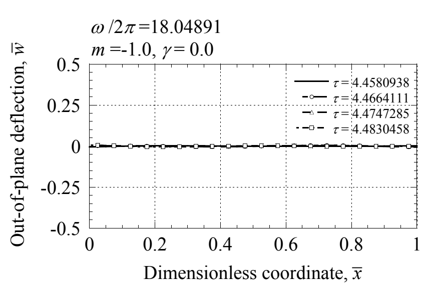


Fig. 9 Longitudinal distribution of out-of-deflection $\bar{w}$ of a FGM beam due to cyclic loading F_{in} with control force F_d.

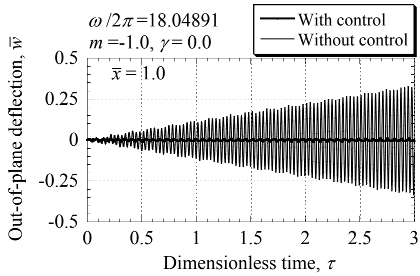


Fig. 10 Effect of addition of control force on the time evolution of out-of-plane deflections at the right boundary edge $\bar{x} = 1.0$ of a FGM beam.

4 Conclusion

Analytical solutions for the FGM beam and plate which are subjected to the cyclic loading and heating are derived. Interference between the flexural vibration due to the cyclic loading and that due to the cyclic heating is discussed numerically. The amplification effect by loading frequency is also discussed for the deflection and stresses of the beam and the plate. A comparison is made for transient flexural responses of the beam and of the plate with high aspect ratio. Furthermore, a vibration suppression of the FGM beam is discussed by applying the active sink method and is illustrated numerically.

Acknowledgements The authors would like to express their gratitude to Professor Hans Irschik and Professor Kazumi Watanabe and organizing committees at the Johannes Kepler University Linz, Austria for their considered efforts to hold this workshop.

References

1. Kawamura R, Tanigawa Y, Hetnarski RB (2003) Thermally induced vibration of an inhomogeneous beam due to a cyclic heating. 177-186 In: Watanabe K, Ziegler F (eds) *IUTAM Symposium on Advanced Materials and Smart Structures*, Solid mechanics and its applications Vol. 106, Kluwer, Dordrecht.
2. Kawamura R, Kusuki S, Matsumoto N (2008) Fundamental thermo-elasticity equations for thermally induced flexural vibration problems for inhomogeneous plates and thermo-elastic dynamical responses to a sinusoidally varying surface temperature. *Journal of Engineering Mathematics*, Vol. 61, No. 2-4, 143-160.
3. Kawamura R, Tanigawa Y, Matsumoto N (2008) Thermally induced vibration of inhomogeneous material rectangular plate due to cyclic variation of thermal and mechanical loads. AIP conference proceedings, Vol. 973, 694-699.
4. Tanaka N, Kikushima Y (1990) Flexural wave control of a flexible beam –Proposition of the active sink method. *Transactions of the Japan Society of Mechanical Engineers* Series C, Vol. 56, No. 522, 351-359 (in Japanese).
5. Tanaka N, Kikushima Y (1990) Flexural wave control of flexible beam –Fundamental characteristics of an active sink system and its verification. *Transactions of the Japan Society of Mechanical Engineers* Series C, Vol. 56, No. 530, 2575-2582 (in Japanese).