Non-Linear Dynamic Deformation of a Piezothermoelastic Laminate

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Abstract This paper presents an analysis on the nonlinear transient behavior of a piezothermoelastic laminate. For the analytical model, a laminated beam is considered to be composed of elastic structural and piezoelectric layers that are subjected to mechanical, thermal, and electrical loads as disturbances or intended control procedures. The deformation of the laminate is analyzed using the classical laminate theory and the von Kármán strain. Equations of motion in terms of the displacements are obtained and analyzed through the Galerkin method. As a result, the dynamic deflection of the laminate is found to be governed by the equation for a polynomial oscillator, and the transient large deformation due to mechanical, thermal, and electrical loads are obtained. Through these results, the characteristics of the transient deformation of the laminate are discussed in detail.

1 Introduction

Piezoelectric materials have been used extensively as sensors and actuators to control structural configuration and to suppress undesired vibration in engineering due to their superior coupling effect between elastic and electric fields. Fiber reinforced plastics (FRP) such as graphite/epoxy are in demand for lightweight structures because they are lighter than general metals and have high specific strength. The structures composed of laminated FRP and piezoelectric materials are called piezothermoelastic laminates and have attracted considerable attention in fields such as aerospace engineering and micro electro mechanical systems. For aerospace applications, structures have to be comparatively large and lightweight. Because of this, they are vulnerable

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to disturbances such as environmental temperature changes and collisions with space debris. As a result, the deformations caused can be relatively large. Therefore, the large deformations of piezothermoelastic laminates were analyzed by several researchers [1-3].

The studies mentioned above [1-3] dealt with the static behavior of piezothermoelastic laminates. However, aerospace applications of these laminates involve dynamic deformation. Therefore, dynamic problems for large deformations of piezothermoelastic laminates have become a focus of study [4-7].

In these analyses [4-7], dynamic deformation deviating arbitrarily from the equilibrium state was not considered, although it is very important from a practical viewpoint for something such as aerospace applications. Therefore, in a previous study Ishihara and Noda [8] analyzed steady vibration deviating from the equilibrium state and obtained the relationship between the deflection of the laminate and its velocity under various loading conditions, excluding the effect of damping.

In practice, it is important to consider the effect of damping on analyzing the dynamic behavior because damping changes such dynamic characteristics as the transient behavior and natural frequencies. Therefore, an analysis of transient dynamic behavior that takes damping into account is important to estimate dynamic characteristics properly. This paper presents an analysis of nonlinear transient dynamic behavior for a piezothermoelastic laminated beam with the damping effect and dynamic deflections that deviate arbitrarily from the equilibrium state considered. For the analytical model, a laminated beam with both ends simply supported is considered, composed of fiber-reinforced laminate and piezoelectric layers subjected to mechanical, thermal, and electrical loads as disturbances or as intended control procedures. Nonlinear large deformation of the laminate is analyzed based on the von Kármán strain [9] and classical laminate theory. Equations of motion for the laminate are derived using the Galerkin method [10]. As a result, the dynamic deflection of the beam is found to be governed by the equation for a polynomial oscillator [11]. According to the equation, the transient large deformation due to mechanical, thermal, and electrical loads are obtained. Moreover, numerical calculation is performed to investigate the nonlinear transient deformation and how to stabilize it.

2 Theoretical analysis

2.1 Problem

The model considered is a simply supported beam with dimensions $a \times b \times h$ and composed of N layers as shown in Fig. 1. Two of the N layers ($z_{k-1} \leq$

$z \leq z_k$, $z_{k'-1} \leq z \leq z_{k'}$) exhibit piezoelectricity while the other layers do not. The beam is laminated in a symmetrical cross-ply manner. The laminate is

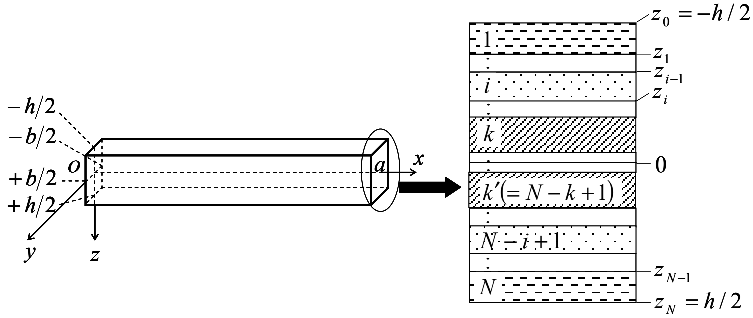


Fig. 1 Analytical model

subjected to the transverse load $Q(t)$ in the z direction and the temperatures $\theta_0(t)$ and $\theta_N(t)$ on the upper ($z = -h/2$) and the lower ($z = h/2$) surfaces of the laminate respectively as mechanical and thermal disturbances. To control the effects of the disturbances, the laminate is also subjected to the electric potentials $V_k(t)$ and $V_{k'}(t)$ on $z = z_{k-1}$ and $z = z_{k'}$ respectively. The surfaces $z = z_k$ and $z = z_{k'-1}$ are both level surfaces of electric potential.

2.2 Governing equations

Based on the classical laminate theory, the von Kármán strain, and the constitutive equation of piezothermoelasticity, the constitutive relations for the laminate are given as follows [8, 12]:

$$N_x = A\left[\frac{\partial u^0}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] - N_x^T - N_x^E \,, \quad M_x = -D\frac{\partial^2 w}{\partial x^2} - M_x^T - M_x^E \quad (1)$$

where N_x and M_x denote the resultant force and moment respectively, u^0 and w denote the displacement components in the x- and z- directions respectively, A and D denote the extensional and bending rigidities respectively. Moreover, N_x^T, M_x^T, N_x^E, and M_x^E are obtained as follows [8]:

$$N_x^T = \frac{1}{2}\left[\Theta_N(t) + \Theta_0(t)\right]\sum_{i=1}^{N} \lambda_i\,(z_i - z_{i-1}),$$

$$M_x^T = \frac{1}{3h}\left[\Theta_N(t) - \Theta_0(t)\right]\sum_{i=1}^{N}\lambda_i\left(z_i^3 - z_{i-1}^3\right),$$

$$N_x^E = e_k\left[V_k(t) - V_{k'}(t)\right] \ , \quad M_x^E = e_k\left[V_k(t) + V_{k'}(t)\right]\frac{z_k + z_{k-1}}{2} \tag{2}$$

where λ_i and e_i denote the stress-temperature coefficient and piezoelectric coefficient respectively for the i-th layer. Equations of motion which integrate the effect of in-plane resultant forces into anti-plane motion are given as follows [8]:

$$\frac{\partial N_x}{\partial x} = 0 \ , \quad \rho h\frac{\partial^2 w}{\partial t^2} + c_d\frac{\partial w}{\partial t} = N_x\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 M_x}{\partial x^2} + Q \tag{3}$$

where ρ and c_d denote the average mass density with respect to z and the damping coefficient. By substituting Eq. (1) into Eq. (3), the equations of motion expressed by the displacements are given as follows:

$$L_1\left(u^0, w\right) = \frac{\partial}{\partial x}\left(N_x^T + N_x^E\right),$$

$$\rho h\frac{\partial^2 w}{\partial t^2} + c_d\frac{\partial w}{\partial t} + L_2(u^0, w) = Q - \frac{\partial^2}{\partial x^2}(M_x^T + M_x^E) - (N_x^T + N_x^E)\frac{\partial^2 w}{\partial x^2} \tag{4}$$

where the definitions of the differentiation operators L_1 and L_2 are given as

$$L_1\left(u^0, w\right) = A\left[\frac{\partial^2 u^0}{\partial x^2} + \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2}\right],$$

$$L_2\left(u^0, w\right) = -A\left[\frac{\partial u^0}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]\frac{\partial^2 w}{\partial x^2} + D\frac{\partial^4 w}{\partial x^4}. \tag{5}$$

2.3 Galerkin method

The Galerkin method [10] is used to solve Eq. (4). Trigonometric functions are chosen as the trial functions and the considered displacements are expressed as series:

$$\left\{u^0, w\right\} = \sum_{m=1}^{\infty}\left\{u_m(t), w_m(t)\right\}\sin\alpha_m x \ : \quad \alpha_m = \frac{m\pi}{a} \tag{6}$$

to satisfy the simple support conditions. Then, the Galerkin method is applied to Eq. (4) to obtain

$$\int_0^a\left[L_1\left(u^0, w\right) - \frac{\partial}{\partial x}\left(N_x^T + N_x^E\right)\right]\sin\alpha_{m'}x\mathrm{d}x = 0,$$

$$\int_0^a \left[\rho h \frac{\partial^2 w}{\partial t^2} + c_d \frac{\partial w}{\partial t} + L_2\left(u^0, w\right) - Q + \frac{\partial^2\left(M_x^T + M_x^E\right)}{\partial x^2} + \left(N_x^T + N_x^E\right)\frac{\partial^2 w}{\partial x^2}\right] \sin\alpha_{m'} x \mathrm{d}x = 0\ , \quad m' = 1,2,3,\ldots,\infty. \tag{7}$$

By substituting Eq. (6) into Eq. (7) and integrating, the simultaneous nonlinear equations with respect to u_m and w_m are obtained. Moreover, by eliminating u_m from the equations, the following simultaneous nonlinear ordinary differential equations with respect to w_m $(m = 1,2,3,\ldots,\infty)$ are obtained

$$\rho h \frac{\mathrm{d}^2 w_m}{\mathrm{d}t^2} + c_d \frac{\mathrm{d}w_m}{\mathrm{d}t} + k_m^L w_m + \sum_{m'=1}^{\infty}\sum_{i=1}^{\infty}\sum_{k=1}^{\infty} k_{m,m'ik}^N w_{m'} w_i w_k = p_m, \quad m = 1,2,3,\ldots,\infty \tag{8}$$

where the definitions of k_m^l, $k_{m,m'ik}^N$, and p_m are given as

$$\begin{aligned} k_m^L &= \alpha_m^2\left[D\alpha_m^2 - \left(N_x^T + N_x^E\right)\right], \\ k_{m,m'ik}^N &= \frac{1}{8} A \alpha_{m'}^2 \alpha_i \alpha_k \left(\Delta_{c,ikm'm} - \sum_{l=1}^{\infty} \frac{2\alpha_i}{\alpha_l}\delta_{ikl}\delta_{m'lm}\right), \\ p_m &= Q_m + \alpha_m^2\left(M_{x,m}^T + M_{x,m}^E\right), \end{aligned} \tag{9}$$

and $Q_m(t)$, $M_{x,m}^T(t)$, and $M_{x,m}^E(t)$ denote the Fourier coefficients of Q, M_x^T, and M_x^E respectively and the definitions of δ_{ij} and $\Delta_{c,ijkl}$ are given in the previous paper [8].

2.4 Polynomial Oscillator

To develop the physical characteristics of the dynamic behavior of the laminate, Eq. (8) is simplified. By truncating the infinite series in Eq. (6), therefore, in Eq.(8), up to one term and considering Eq. (8) for $m = 1$, the following nonlinear equation with respect to w_1 is obtained

$$\rho h \frac{\mathrm{d}^2 w_1}{\mathrm{d}t^2} + c_d \frac{\mathrm{d}w_1}{\mathrm{d}t} + k_1^L w_1 + k_{1,111}^N w_1^3 = p_1. \tag{10}$$

Note that, as in Eq. (6), w_1 denotes the deflection at the center of the laminate $(x = a/2)$. By introducing non-dimensional variables such as

$$U \equiv \frac{w_{11}}{h},\ \tau \equiv \sqrt{\frac{k^N_{1,111}h}{\rho}}t,\ \lambda \equiv -\frac{k^L_1}{k^N_{1,111}h^2},\ \delta \equiv \frac{c_d}{\sqrt{\rho h k^N_{1,111} h}},\ \alpha \equiv \frac{p_1}{k^N_{1,111}h^3}, \tag{11}$$

Eq. (10) is rewritten in non-dimensional form as

$$\frac{\mathrm{d}^2U}{\mathrm{d}\tau^2} + \delta\frac{\mathrm{d}U}{\mathrm{d}\tau} - \lambda U + U^3 = \alpha\ :\quad \delta \geq 0\,,\quad \alpha \geq 0. \tag{12}$$

2.5 Dynamic Behavior

2.5.1 Static large deflection

First, the static large deflection of the laminate and its stability are examined. The static large deflection is obtained as the equilibrium point U_e of the dynamical system described by Eq. (12). From Eq. (12), U_e is obtained for $\alpha = 0$ as:

$$\begin{aligned} &U_e = 0\,(\equiv U_1)\,, \\ &U_e = \pm\sqrt{\lambda}\,(\equiv U_2)\ :\quad \lambda > 0 \end{aligned} \tag{13}$$

and for $\alpha > 0$ as:

$$\begin{aligned} &U_e = U_3\ :\quad \lambda < 0, \\ &U_e = U_4\ :\quad 0 < \lambda < 3\sqrt[3]{(\alpha/2)^2}, \\ &U_e = U_5, U_6, U_7\ :\quad \lambda > 3\sqrt[3]{(\alpha/2)^2} \end{aligned} \tag{14}$$

where the explicit solutions for U_3 through U_7 are given in the previous paper [8]. The stability of the deflection described by Eqs. (13) and (14) can be examined by considering the small deviation of U in the vicinity of U_e as is usual [11]. The variations of the static large deflection of the laminate with parameter λ are shown graphically in Fig. 2, where solid lines denote stable deflections and broken lines denote unstable ones.

2.5.2 Transient large deflection

The transient large deflection that deviates arbitrarily from the equilibrium state is examined. By introducing new non-dimensional variables such as

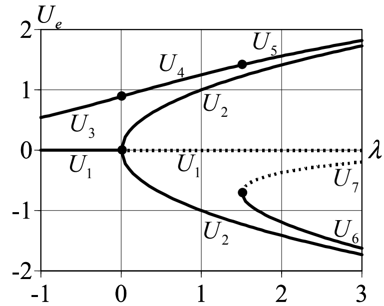


Fig. 2 Variation of static large deflections with parameter λ

$$U' \equiv \frac{U}{\sqrt{|\lambda|}} , \quad \tau' \equiv \tau\sqrt{|\lambda|} , \quad V' \equiv \frac{dU'}{d\tau'} , \quad \alpha' \equiv \frac{\alpha}{|\lambda|\sqrt{|\lambda|}} , \quad \delta' \equiv \frac{\delta}{\sqrt{|\lambda|}}, \tag{15}$$

Eq. (12) is rewritten as

$$\frac{\mathrm{d}^2U'}{\mathrm{d}\tau'^2} + \delta' V' - \mathrm{sgn}\,(\lambda)\cdot U' + U'^3 - \alpha' = 0. \tag{16}$$

By solving Eq. (16) for U' through the Runge-Kutta method, the transient behavior of the laminate is analyzed and the results are presented by orbits, that is, the relationship between the deflection U' and velocity V'. From Eqs. (13) and (14), the final state of the transient behavior is found to be classified into three cases depending on parameters λ and α' as:

$$\left.\begin{array}{lll} \text{case(i)}: & \lambda < 0 & \\ \text{case(ii)}: & \lambda > 0 \quad \text{and} & \alpha' > 2\sqrt{3}/9 \\ \text{case(iii)}: & \lambda > 0 \quad \text{and} & 0 \leqslant \alpha' < 2\sqrt{3}/9 \end{array}\right\} . \tag{17}$$

In order to connect the behavior with engineering sense, the following situation is considered:

$$Q = 0 , \quad \Theta_N = \Theta_0\,(\equiv \Theta) , \quad V_k = V_{k'}\,(\equiv V) \tag{18}$$

where Θ and V are assumed as the thermal disturbance and control procedure respectively. Then, from Eqs. (2), (9), (11), (15), and (18):

$$\alpha' = \frac{4\alpha_1^2\,(z_k + z_{k-1})\,e_k V}{\pi\,|\lambda|\,\sqrt{|\lambda|}h^3 k_{1,111}^N},$$

$$\lambda = \frac{\alpha_1^2}{h^2 k_{1,111}^N}\sum_{i=1}^{N}\lambda_i\,(z_i - z_{i-1})\,(\Theta - \Theta_{cr}) , \quad \Theta_{cr} = \frac{D\alpha_1^2}{\sum\limits_{i=1}^{N}\lambda_i\,(z_i - z_{i-1})} \tag{19}$$

Thus, $\lambda < 0$ and $\lambda > 0$ mean that the laminate is subjected to temperatures that are lower and higher than the buckling temperature respectively, and α'

corresponds to the electric voltage applied to the piezoelectric layers.

Figure 3 shows the transient vibration of the laminate for case (i), where the temperature is lower than the buckling temperature. Figure 4 shows the result for the case (ii), where the temperature is higher than the buckling temperature and the electric voltage is higher than the critical value. From Figs. 3 and 4, the laminate is found to vibrate, decay, and tend to the final deflections U_3' (corresponding to U_3) and U_4' (corresponding to U_4).

Figure 5 shows the transient vibration of the beam for the case (iii), where

Fig. 3 Transient deformation for case (i) ($\lambda < 0$, $\alpha' = 1$, $\delta' = 0.2$)

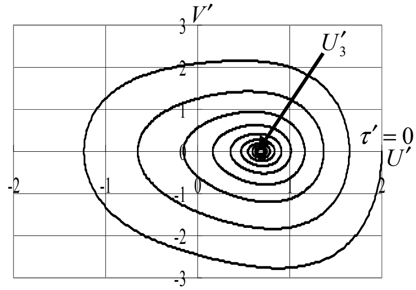
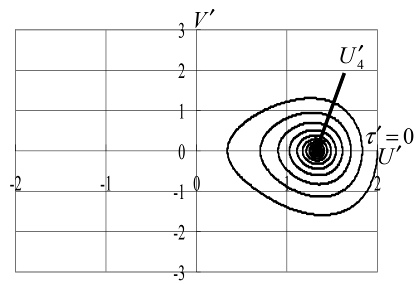
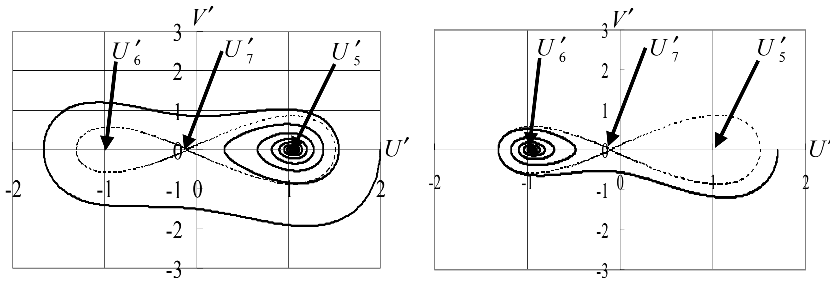


Fig. 4 Transient deformation for case (ii) ($\lambda > 0$, $\alpha' = 1$, $\delta' = 0.2$)



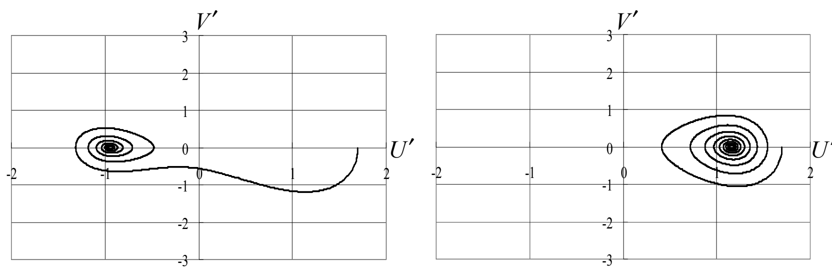
the temperature is higher than the buckling temperature and the electric voltage is smaller than the critical value. In Fig. 5, the points denoted by U_5', U_6' and U_7' correspond to the equilibrium points U_5, U_6 and U_7 respectively and the dotted line shows the separatrix, or the orbit that passes through the unstable equilibrium point U_7' and disregards the effect of damping ($\delta' = 0$). From Fig. 5, it is found that, when the initial deformation is outside the separatrix, deformation reaches out to a branch of the separatrix and tends to the equilibrium point enclosed by the branch.

Finally, Fig. 6 shows the effect of the electric voltage on the transient deformation. The desired final deformation is found to be possible to achieve by applying the appropriate electric voltage, which is of great importance in practical viewpoint.



(*a*) $(U', V')|_{\tau'=0} = (2, 0)$ (*b*) $(U', V')|_{\tau'=0} = (1.7, 0)$

Fig. 5 Transient deformation for case (iii) $(\lambda > 0\ ,\ \alpha' = 0.1\ ,\ \delta' = 0.2)$



(*a*) $\alpha' = 0.1$ (*b*) $\alpha' = 0.4$

Fig. 6 Transient deformation $\left(\lambda > 0\ ,\ \delta' = 0.2\ ,\ (U', V')|_{\tau'=0} = (1.7, 0)\right)$

3 Conclusion

The nonlinear transient behavior of a piezothermoelastic laminate is analyzed with the damping effect and the dynamic deflection that deviates arbitrarily from the equilibrium state considered. For the analytical model, a rectangular laminated beam with both ends simply supported and composed of fiber-reinforced laminate and piezoelectric layers is considered and subjected to mechanical, thermal, and electrical loads as disturbances or as intended control procedures. Nonlinear large deformations of the laminate are analyzed based on the von Kármán strains and classical laminate theory. As a result, the dynamic deflection of the laminate is found to be governed by the equation for a polynomial oscillator. Using the equation, the transient large deformation with the damping effect considered due to mechanical, thermal, and electrical loads is obtained. From the results for the transient large deformation, it is found that appropriate application of the electric voltage to the piezoelectric layers can govern the final deformation of the beam.

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