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the strength of nonstandard analysis

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Imme van den Berg
Departamento de Matemática
Universidade de Évora, Évora, Portugal

Vítor Neves
Departamento de Matemática
Universidade de Aveiro, Aveiro, Portugal

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Foreword

Willst du ins Unendliche schreiten?
Geh nur im Endlichen nach alle Seiten!
Willst du dich am Ganzen erquicken,
So must du das Ganze im Kleinsten erblicken.

J. W. Goethe (*Gott, Gemüt und Welt, 1815*)

Forty-five years ago, an article appeared in the Proceedings of the Royal Academy of Sciences of the Netherlands Series A, 64, 432–440 and Indagationes Math. 23 (4), 1961, with the mysterious title "Non-standard Analysis" authored by the eminent mathematician and logician Abraham Robinson (1908–1974).

The title of the paper turned out to be a contraction of the two terms "Non-standard Model" used in model theory and "Analysis". It presents a treatment of classical analysis based on a theory of infinitesimals in the context of a non-standard model of the real number system $\mathbb{R}$.

In the Introduction of the article, Robinson states:

> "It is our main purpose to show that the models provide a natural approach to the age old problem of producing a calculus involving infinitesimal (infinitely small) and infinitely large quantities. As is well-known the use of infinitesimals strongly advocated by Leibniz and unhesitatingly accepted by Euler fell into disrepute after the advent of Cauchy's methods which put Mathematical Analysis on a firm foundation".

To bring out more clearly the importance of Robinson's creation of a rigorous theory of infinitesimals and their reciprocals, the infinitely large quantities, that has changed the landscape of analysis, I will briefly share with the reader a few highlights of the historical facts that are involved.

The invention of the "Infinitesimal Calculus" in the second half of the seventeenth century by Newton and Leibniz can be looked upon as the first funda-

mental new discovery in mathematics of revolutionary nature since the death of Archimedes in 212 BC. The fundamental discovery that the operations of differentiation (flux) and integration (sums of infinitesimal increments) are inverse operations using the intuitive idea that infinitesimals of higher order compared to those of lower order may be neglected became an object of severe criticism. In the "Analyst", section 35, Bishop G. Berkeley states:

> "And what are these fluxions? The velocities of evanescent increments. And what are these same evanescent increments? They are neither finite quantities, nor quantities infinitesimally small, nor yet nothing. May we call them ghosts of departed quantities?"

The unrest and criticism concerning the lack of a rigorous foundation of the infinitesimal calculus led the Academy of Sciences of Berlin, at its public meeting on June 3, 1774, and well on the insistence of the Head of the Mathematics Section, J. L. Lagrange, to call upon the mathematical community to solve this important problem. To this end, it announced a prize contest dealing with the problem of "Infinity" in the broadest sense possible in mathematics.The announcement read:

> "The utility derived from Mathematics, the esteem it is held in and the honorable name of 'exact science' par excellence, that it justly deserves, are all due to the clarity of its principles, the rigor of its proofs and the precision of its theorems. In order to ensure the continuation of these valuable attributes in this important part of our knowledge the prize of a 50 ducat gold medal is for:
>
> A clear and precise theory of what is known as 'Infinity' in Mathematics. It is well-known that higher mathematics regularly makes use of the infinitely large and infinitely small. The geometers of antiquity and even the ancient analysts, however, took great pains to avoid anything approaching the infinity, whereas today's eminent modern analysts admit to the statement 'infinite magnitude' is a contradiction in terms. For this reason the Academy desires an explanation why it is that so many correct theorems have been deduced from a contradictory assumption, together with a formulation of a truly clear mathematical principle that may replace that of infinity without, however, rendering investigations by its use overly difficult and overly lengthy. It is requested that the subject be treated in all possible generality and with all possible rigor, clarity and simplicity."

Twenty-three answers were received before the deadline of January 1, 1786. The prize was awarded to the Swiss mathematician Simon L'Huilier for his essay with motto:

> "The infinite is the abyss in which our thoughts are engulfed."

The members of the "Prize Committee" made the following noteworthy points: None of the submitted essays dealt with the question raised "why so many correct theorems have been derived from a contradictory assumption?" Furthermore, the request for clarity, simplicity and, above all, rigor was not met by the contenders, and almost all of them did not address the request for a newly formulated principle of infinity that would reach beyond the infinitesimal calculus to be meaningful also for algebra and geometry.

For a detailed account of the prize contest we refer the reader to the interesting biography of Lazare Nicolas M. Carnot (1753–1823), the father of the thermodynamicist Sadi Carnot, entitled "Lazare Carnot Savant" by Ch. C. Gillespie (Princeton Univ. Press, 1971), which contains a thorough discussion of Carnot's entry "Dissertation sur la théorie de l'infini mathématique", received by the Academy after the deadline. The above text of the query was adapted from the biography.

In retrospect, the outcome of the contest is not surprising. Nevertheless around that time the understanding of infinitesimals had reached a more sophisticated level as the books of J. L. Lagrange and L. N. Carnot published in Paris in 1797 show.

From our present state of the art, it seems that the natural place to look for a "general principle of infinity" is set theory. Not however for an "intrinsic" definition of infinity. Indeed, as Gian-Carlo Rota expressed not too long ago:

> "God created infinity and man, unable to understand it, had to invent finite sets."

At this point let me digress a little for further clarification about the infinity we are dealing with. During the early development of Cantor's creation of set theory, it was E. Zermelo who realized that the attempts to prove the existence of "infinite" sets, short of assuming there is an "infinite" set or a non-finite set as in Proposition 66 of Dedekind's famous "Was sind und was sollen die Zahlen?" were fallacious. For this reason, Zermelo in his important paper "Sur les ensembles finis et le principe de l'induction complete", Acta Math. 32 (1909), 185–193 (submitted in 1907), introduced an axiom of "infinity" by postulating the existence of a set, say A, non-empty, and that for each of its elements x, the singleton $\{x\}$ is an element of it.

Returning to the request of the Academy: To discover a property that all infinite sets would have in common with the finite sets that would facilitate

their use in all branches of mathematics.What comes to mind is Zermelo's well-ordering principle. Needless to say that this principle and the manifold results and consequences in all branches of mathematics have had an enormous impact on the development of mathematics since its introduction. One may ask what has this to do with the topic at hand? It so happens that the existence of non-standard models depends essentially on it as well and consequently non-standard analysis too.

The construction of the real number system (linear continuum) by Cantor and Dedekind in 1872 and the Weierstrass ε-δ technique gradually replaced the use of infinitesimals. Hilbert's characterization in 1899 of the real number system as a (Dedekind) complete field led to the discovery, in 1907, by H. Hahn, of non-archimedian totally ordered field extensions of the reals. This development brought about a renewed interest in the theory of infinitesimals. The resulting "calculus", certainly of interest by itself, lacked a process of defining extensions of the elementary and special functions, etc., of the objects of classical analysis. It is interesting that Cantor strongly rejected the existence of non-archimedian totally ordered fields. He expressed the view that no actual infinities could exist other than his transfinite cardinal numbers and that, other than 0, infinitesimals did not exist. He also offered a "proof" in which he actually assumed order completeness.

It took one hundred and seventy-five years from the time of the deadline of the Berlin Academy contest to the publication of Robinson's paper "Non-standard Analysis". As Robinson told us, his discovery did not come about as a result of his efforts to solve Leibniz' problem; far from it. Working on a paper on formal languages where the length of the sentences could be countable, it occurred to him to look up again the important paper by T. Skolem "Über die Nichtcharakterisierbarkeit der Zahlenreihe mittels endlich oder abzählbar unendlich vieler Aussagen mit ausschliesslich Zahlenvariablen, Fund. Math. 23 (1934), 150–161*.

Briefly, Skolem showed in his paper the existence of models of Peano arithmetic having "infinitely large numbers". Nevertheless in his models the principle of induction holds only for subsets determined by admissible formulas from the chosen formal language used to describe Peano's axiom system. The non-empty set of the infinitely large numbers has no smallest element and so cannot be determined by a formula of the formal language and is called an external set; those that can were baptized as internal sets of the model.

Robinson, rereading Skolem's paper, wondered what systems of numbers would emerge if he would apply Skolem's method to the axiom system of the real numbers. In doing so, Robinson immediately realized that the real number

*See also: T. Skolem "Peano's Axioms and Models of Arithmetic", in Symposium on the Mathematical Interpretation of Formal Systems, North-Holland, Amsterdam 1955, 1–14.

system was a non-archimedian totally ordered field extension of the reals whose structure satisfies all the properties of the reals, and that, in particular, the set of infinitesimals lacking a least upper bound was an external set.

This is how it all started and the Academy would certainly award Robinson the gold medal.

At the end of the fifties at Caltech (California Institute of Technology) Arthur Erdelyi FRS (1908–1977) conducted a lively seminar entitled "Generalized Functions". It dealt with various areas of current research at that time in such fields as J. Mikusinski's rigorous foundation of the so-called Heaviside operational calculus and L. Schwartz' theory of distributions. In connection with Schwartz' distribution theory, Erdelyi urged us to read the just appeared papers by Laugwitz and Schmieden dealing with the representations of the Dirac-delta functions by sequences of point-functions converging to 0 pointwise except at 0 where they run to infinity. Robinson's paper fully clarified this phenomenon. Reduced powers of $\mathbb{R}$ instead of ultrapowers, as in Robinson's paper, were at play here. In my 1962 Notes on Non-standard Analysis the ultrapower construction was used, but at that time without using explicitly the Transfer Principle.

In 1967 the first International Symposium on Non-standard Analysis took place at Caltech with the support of the U.S. Office of Naval Research. At the time the use of non-standard models in other branches of mathematics started to blossom. This is the reason that the Proceedings of the Symposium carries the title: Applications of Model Theory to Algebra, Analysis and Probability.

A little anecdote about the meeting. When I opened the newspaper one morning during the week of the meeting, I discovered to my surprise that it had attracted the attention of the Managing Editor of the Pasadena Star News; his daily "Conversation Piece" read:

> "A Stanford Professor spoke in Pasadena this week on the subject 'Axiomatizations of Non-standard Analysis which are Conservative Extensions of Formal Systems for Standard Classical Analysis', a fact which I shall tuck away for reassurance on those days when I despair of communicating clearly."

I may add here that from the beginning Robinson was very interested in the formulation of an axiom system catching his non-standard methodology.

Unfortunately he did not live to see the solution of his problem by E. Nelson presented in the 1977 paper entitled "Internal Set Theory". A presentation by Nelson, "The virtue of Simplicity", can be found in this book.

A final observation. During the last sixty years we have all seen come about the solutions of a number of outstanding problems and conjectures,

some centuries old, that have enriched mathematics. The century-old problem to create a rigorous theory of infinitesimals no doubt belongs in this category.

It is somewhat surprising that the appreciation of Robinson's creation was slow in coming. Is it possible that the finding of the solution in model theory, a branch of mathematical logic, had something to do with that?

The answer may perhaps have been given by Augustus de Morgan (1806–1871), who is well-known from De Morgan's Law, and who in collaboration with George Boole (1805–1864) reestablished formal logic as a branch of exact science in the nineteenth century, when he wrote:

> "We know that mathematicians care no more for logic than logicians for mathematics. The two eyes of exact sciences are mathematics and logic: the mathematical sect puts out the logical eye, the logical sect puts out the mathematical eye; each believing that it can see better with one eye than with two."

We owe Abraham Robinson a great deal for having taught us the use of both eyes.

This book shows clearly that we have learned our lesson well.

All the contributors are to be commended for the way they have made an effort to make their contributions that are based on the talks at the meeting "Nonstandard Mathematics 2004" as self-contained as can be expected. For further facilitating the readers, the editors have divided the papers in categories according to the subject. The whole presents a very rich assortment of the nonstandard approach to diverse areas of mathematical analysis.

I wish it many readers.

Wilhelmus A. J. Luxemburg
Pasadena, California
September 2006

Acknowledgements

The wide range of applicability of Mathematical Logic to classical Mathematics, beyond Analysis — as Robinson's terminology *Non-standard Analysis* might imply — is already apparent in the very first symposium on the area, held in Pasadena in 1967, as mentioned in the foreword.

Important indicators of the maturity of this field are the high level of foundational and pure or applied mathematics presented in this book, congresses which take place approximately every two years, as well as the experiences of teaching, which proliferated at graduate, undergraduate and secondary school level all over the world.

This book, made of peer reviewed contributions, grew out of the meeting Non Standard Mathematics 2004, which took place in July 2004 at the Department of Mathematics of the University of Aveiro (Portugal). The articles are organized into five groups, (1) Foundations, (2) Number theory, (3) Statistics, probability and measures, (4) Dynamical systems and equations and (5) Infinitesimals and education. Its cohesion is enhanced by many cross-overs.

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Our special thanks go to Wilhelmus Luxemburg of Caltech, USA, who, being one of the first to recognize the importance of Robinson's discovery of

Nonstandard Analysis, honoured us by writing the foreword. Last but not least, Salvador Sánchez-Pedreño of the University of Murcia, Spain, joined his expert knowledge of Nonstandard Analysis to a high dominion of the LaTeX editing possibilities in order to improve the consistency of the text and to obtain an outstanding typographic result.

Imme van den Berg
Vítor Neves
Editors

Contents

Part I

Foundations

1

The strength of nonstandard analysis

H. Jerome Keisler*

Abstract

A weak theory nonstandard analysis, with types at all finite levels over both the integers and hyperintegers, is developed as a possible framework for reverse mathematics. In this weak theory, we investigate the strength of standard part principles and saturation principles which are often used in practice along with first order reasoning about the hyperintegers to obtain second order conclusions about the integers.

1.1 Introduction

In this paper we revisit the work in [5] and [6], where the strength of nonstandard analysis is studied. In those papers it was shown that weak fragments of set theory become stronger when one adds saturation principles commonly used in nonstandard analysis.

The purpose of this paper is to develop a framework for reverse mathematics in nonstandard analysis. We will introduce a base theory, "weak nonstandard analysis" (*WNA*), which is proof theoretically weak but has types at all finite levels over both the integers and the hyperintegers. In *WNA* we study the strength of two principles that are prominent in nonstandard analysis, the standard part principle in Section 1.6, and the saturation principle in Section 1.9. These principles are often used in practice along with first order reasoning about the hyperintegers to obtain second order conclusions about the integers, and for this reason they can lead to the discovery of new results.

The standard part principle (*STP*) says that a function on the integers exists if and only if it is coded by a hyperinteger. Our main results show that in *WNA*, *STP* implies the axiom of choice for quantifier-free formulas (Theorem 17), *STP*+saturation for quantifier-free formulas implies choice for arithmetical formulas (Theorem 23), and *STP*+saturation for formulas with

*University of Wisconsin, Madison.
keisler@math.wisc.edu

first order quantifiers implies choice for formulas with second order quantifiers (Theorem 25). The last result might be used to identify theorems that are proved using nonstandard analysis but cannot be proved by the methods commonly used in classical mathematics.

The natural models of *WNA* will have a superstructure over the standard integers $\mathbb{N}$, a superstructure over the hyperintegers ${}^*\mathbb{N}$, and an inclusion map $j : \mathbb{N} \to {}^*\mathbb{N}$. With the two superstructures, it makes sense to ask whether a higher order statement over the hyperintegers implies a higher order statement over the integers. As is commonly done in the standard literature on weak theories in higher types, we use functional superstructures with types of functions rather than sets. The base theory *WNA* is neutral between the internal set theory approach and the superstructure approach to nonstandard analysis, and the standard part and saturation principles considered here arise in both approaches. For background in model theory, see [2, Section 4.4].

The theory *WNA* is related to the weak nonstandard theory $NPRA^\omega$ of Avigad [1], and the base theory RCA_0^ω for higher order reverse mathematics proposed by Kohlenbach [7]. The paper [1] shows that the theory $NPRA^\omega$ is weak in the sense that it is conservative over primitive recursive arithmetic (*PRA*) for Π_2 sentences, but is still sufficient for the development of much of analysis. The theory *WNA* is also conservative over *PRA* for Π_2 sentences, but has more expressive power. In Sections 1.11 and 1.12 we will introduce a stronger, second order Standard Part Principle, and give some relationships between this principle and the theories $NPRA^\omega$ and RCA_0^ω.

1.2 The theory PRA^ω

Our starting point is the theory *PRA* of primitive recursive arithmetic, introduced by Skolem. It is a first order theory which has function symbols for each primitive recursive function (in finitely many variables), and the equality relation $=$. The axioms are the rules defining each primitive recursive function, and induction for quantifier-free formulas. This theory is much weaker than Peano arithmetic, which has induction for all first order formulas.

An extension of *PRA* with all finite types was introduced by Gödel [4], and several variations of this extension have been studied in the literature. Here we use the finite type theory PRA^ω as defined in Avigad [1].

There is a rich literature on constructive theories in intuitionistic logic that are very similar to PRA^ω, such as the finite type theory HA^ω over Heyting arithmetic (see, for example, [9]). However, in this paper we work exclusively in classical logic.

We first introduce a formal object N and define a collection of formal objects called **types over** N.

(1) The **base type** over N is N.

(2) If σ, τ are types over N, then $\sigma \to \tau$ is a type over N.

We now build the formal language $L(PRA^\omega)$. $L(PRA^\omega)$ is a many-sorted first order language with countably many variables of each type σ over N, and the equality symbol $=$ at the base type N only. It has the usual rules of many-sorted logic, including the rule $\exists f \forall u\, f(u) = t(u, \ldots)$ where u, f are variables of type $\sigma, \sigma \to N$ and $t(u, \ldots)$ is a term of type N in which f does not occur.

We first describe the symbols and then the corresponding axioms.

$L(PRA^\omega)$ has the following function symbols:

- A function symbol for each primitive recursive function.
- The primitive recursion operator which builds a term $R(m, f, n)$ of type N from terms of type $N, N \to N$, and N.
- The definition by cases operator which builds a term $c(n, u, v)$ of type σ from terms of type N, σ, and σ.
- The λ operator which builds a term $\lambda v.t$ of type $\sigma \to \tau$ from a variable v of type σ and a term t of type τ.
- The application operator which builds a term $t(s)$ of type τ from terms s of type σ and t of type $\sigma \to \tau$.

Given terms r, t and a variable v of the appropriate types, $r(t/v)$ denotes the result of substituting t for v in r. Given two terms s, t of type σ, $s \equiv t$ will denote the infinite scheme of formulas $r(s/v) = r(t/v)$ where v is a variable of type σ and $r(v)$ is an arbitrary term of type N. $\equiv$ is a substitute for the missing equality relations at higher types.

The axioms for PRA^ω are as follows.

- Each axiom of PRA.
- The induction scheme for quantifier-free formulas of $L(PRA^\omega)$.
- Primitive recursion: $R(m, f, 0) = m$, $R(m, f, s(n)) = f(n, R(m, f, n))$.
- Cases: $c(0, u, v) \equiv u$, $c(s(m), u, v) \equiv v$.
- Lambda abstraction: $(\lambda u.t)(s) \equiv t(s/u)$.

The order relations $<, \leq$ on type N can be defined in the usual way by quantifier-free formulas.

In [1] additional types $\sigma \times \tau$, and term-building operations for pairing and projections with corresponding axioms were also included in the language, but

as explained in [1], these symbols are redundant and are often omitted in the literature.

On the other hand, in [1] the symbols for primitive recursive functions are not included in the language. These symbols are redundant because they can be defined from the primitive recursive operator R, but they are included here for convenience.

We state a conservative extension result from [1], which shows that PRA^ω is very weak.

Proposition 1 *PRA^ω is a conservative extension of PRA, that is, PRA^ω and PRA have the same consequences in $L(PRA)$.*

The natural model of PRA^ω is the full functional superstructure $V(\mathbb{N})$, which is defined as follows. $\mathbb{N}$ is the set of natural numbers. Define $V_N(\mathbb{N}) = \mathbb{N}$, and inductively define $V_{\sigma\to\tau}(\mathbb{N})$ to be the set of all mappings from $V_\sigma(\mathbb{N})$ into $V_\tau(\mathbb{N})$. Finally, $V(\mathbb{N}) = \bigcup_\sigma V_\sigma(\mathbb{N})$. The superstructure $V(\mathbb{N})$ becomes a model of PRA^ω when each of the symbols of $L(PRA^\omega)$ is interpreted in the obvious way indicated by the axioms. In fact, $V(\mathbb{N})$ is a model of much stronger theories than PRA^ω, since it satisfies full induction and higher order choice and comprehension principles.

1.3 The theory $NPRA^\omega$

In [1], Avigad introduced a weak nonstandard counterpart of PRA^ω, called $NPRA^\omega$. $NPRA^\omega$ adds to PRA^ω a new predicate symbol $S(\cdot)$ for the standard integers (and S-relativized quantifiers $\forall^S, \exists^S$), and a constant H for an infinite integer, axioms saying that $S(\cdot)$ is an initial segment not containing H and is closed under each primitive recursive function, and a transfer axiom scheme for universal formulas. In the following sections we will use a weakening of $NPRA^\omega$ as a part of our base theory.

In order to make $NPRA^\omega$ fit better with the present paper, we will build the formal language $L(NPRA^\omega)$ with types over a new formal object ${}^*N$ instead of over N. The base type over ${}^*N$ is ${}^*N$, and if σ, τ are types over ${}^*N$ then $\sigma \to \tau$ is a type over ${}^*N$.

For each type σ over N, let ${}^*\sigma$ be the type over ${}^*N$ built in the same way. For each function symbol u in $L(PRA^\omega)$ from types $\vec{\sigma}$ to type τ, $L(NPRA^\omega)$ has a corresponding function symbol ${}^*u$ from types ${}^*\vec{\sigma}$ to type ${}^*\tau$. $L(NPRA^\omega)$ also has the equality relation $=$ for the base type ${}^*N$, and the extra constant symbol H and the standardness predicate symbol S of type ${}^*N$.

We will use the following conventions throughout this paper. When we write a formula $A(\vec{v})$, it is understood that $\vec{v}$ is a tuple of variables that contains

all the free variables of A. If we want to allow additional free variables we write $A(\vec{v}, \ldots)$. We will always let:

- $m, n, \ldots$ be variables of type N,
- $x, y, \ldots$ be variables of type ${}^*N$,
- $f, g, \ldots$ be variables of type $N \to N$.

To describe the axioms of $NPRA^\omega$ we introduce the star of a formula of $L(PRA^\omega)$. Given a formula A of $L(PRA^\omega)$, a **star of** A is a formula ${}^*A$ of $L(NPRA^\omega)$ which is obtained from A by replacing each variable of type σ in A by a variable of type ${}^*\sigma$ in a one to one fashion, and replacing each function symbol in A by its star. The order relations on ${}^*N$ will be written $<$, $\leq$ without stars.

The axioms of $NPRA^\omega$ are as follows:

- The star of each axiom of PRA^ω.
- S is an initial segment: $\neg S(H) \wedge \forall x \forall y\, [S(x) \wedge y \leq x \to S(y)]$.
- S is closed under primitive recursion.
- Transfer: $\forall^S \vec{x}\ {}^*A(\vec{x}) \to \forall \vec{x}\ {}^*A(\vec{x})$, $A(\vec{m})$ quantifier-free in $L(PRA^\omega)$.

It is shown in [1] that if $A(m, n)$ is quantifier-free in $L(PRA)$ and $NPRA^\omega$ proves $\forall^S x \exists y\, {}^*A(x, y)$, then PRA proves $\forall m \exists n\, A(m, n)$. It follows that $NPRA^\omega$ is conservative over PRA for Π_2 sentences.

The natural models of $NPRA^\omega$ are the internal structures ${}^*V(\mathbb{N})$, which are proper elementary extensions of $V(\mathbb{N})$ in the many-sorted sense, with additional symbols S for $\mathbb{N}$ and H for an element of ${}^*\mathbb{N} \setminus \mathbb{N}$.

1.4 The theory *WNA*

We now introduce our base theory *WNA*, weak nonstandard analysis. The idea is to combine the theory PRA^ω with types over N with a weakening of the theory $NPRA^\omega$ with types over ${}^*N$, and form a link between the two by identifying the standardness predicate S of $NPRA^\omega$ with the lowest type N of PRA^ω. In this setting, it will make sense to ask whether a formula with types over ${}^*N$ implies a formula with types over N.

The language $L(WNA)$ of *WNA* has both types over N and types over ${}^*N$. It has all of the symbols of $L(PRA^\omega)$, all the symbols of $L(NPRA^\omega)$ except the primitive recursion operator ${}^*R$, and has one more function symbol j which goes from type N to type ${}^*N$.

We make the axioms of *WNA* as weak as we can so as to serve as a blank screen for viewing the relative strengths of additional statements which arise in nonstandard analysis.

The axioms of *WNA* are as follows:

- The axioms of PRA^ω.
- The star of each axiom of *PRA*.
- The stars of the Cases and Lambda abstraction axioms of PRA^ω.
- S is an initial segment: $\neg S(H) \wedge \forall x \forall y\, [S(x) \wedge y \leq x \rightarrow S(y)]$.
- S is closed under primitive recursion.
- j maps S onto $\mathbb{N}$: $\forall x\, [S(x) \leftrightarrow \exists m\, x = j(m)]$.
- Lifting: $j(\alpha(\vec{m})) = {}^*\alpha(j(\vec{m}))$ for each primitive recursive function α.

The star of a quantifier-free formula of $L(PRA)$, possibly with some variables replaced by H, will be called an **internal quantifier-free formula**. The stars of the axioms of *PRA* include the star of the defining rule for each primitive recursive function, and the induction scheme for internal quantifier-free formulas (which we will call **internal induction**).

The axioms of $NPRA^\omega$ that are left out of *WNA* are the star of the Primitive Recursion scheme, the star of the quantifier-free induction scheme of PRA^ω, and Transfer. These axioms are statements about the hyperintegers which involve terms of higher type.

Note that *WNA* is noncommittal on whether the characteristic function of S exists in type ${}^*N \rightarrow {}^*N$, while the quantifier-free induction scheme of $NPRA^\omega$ precludes this possibility.

In practice, nonstandard analysis uses very strong transfer axioms, and extends the mapping j to higher types. Strong axioms of this type will not be considered here.

Theorem 2 *$WNA + NPRA^\omega$ is a conservative extension of $NPRA^\omega$, that is, $NPRA^\omega$ and $WNA + NPRA^\omega$ have the same consequences in $L(NPRA^\omega)$.*

Proof. Let M be a model of $NPRA^\omega$, and let M^S be the restriction of M to the standardness predicate S. Then M^S is a model of *PRA*. By Proposition 1, the complete theory of M^S is consistent with PRA^ω. Therefore PRA^ω has a model K whose restriction K^N to type N is elementarily equivalent to M^S. By the compactness theorem for many-sorted logic, there is a model M_1 elementarily

equivalent to M and a model K_1 elementarily equivalent to K with an isomorphism $j : M_1^S \cong K_1^N$, such that $\langle K_1, M_1, j\rangle$ is a model of $WNA + NPRA^\omega$. Thus every complete extension of $NPRA^\omega$ is consistent with $WNA + NPRA^\omega$, and the theorem follows. □

Corollary 3 *WNA is a conservative extension of PRA for* Π_2 *formulas. That is, if* $A(m,n)$ *is quantifier-free in* $L(PRA)$ *and* $WNA \vdash \forall m \exists n\, A(m,n)$, *then* $PRA \vdash \forall m \exists n\, A(m,n)$.

Proof. Suppose $WNA \vdash \forall m \exists n\, A(m,n)$. By the Lifting Axiom, $WNA \vdash \forall^S x \exists^S y\ {}^*A(x,y)$. By Theorem 2, $NPRA^\omega \vdash \forall^S x \exists^S y\ {}^*A(x,y)$. Then $PRA \vdash \forall m \exists n\, A(m,n)$ by Corollary 2.3 in [1]. □

Each model of WNA has a $V(\mathbb{N})$ part formed by restricting to the objects with types over N, and a $V(^*\mathbb{N})$ part formed by restricting to the objects with types over $^*N$. Intuitively, the $V(\mathbb{N})$ and $V(^*\mathbb{N})$ parts of WNA are completely independent of each other, except for the inclusion map j at the zeroth level. The standard part principles introduced later in this paper will provide links between types $N \to N$ and $(N \to N) \to N$ in the $V(\mathbb{N})$ part and types $^*N$ and $^*N \to {}^*N$ in the $V(^*\mathbb{N})$ part.

WNA has two natural models, the "internal model" $\langle V(\mathbb{N}), {}^*V(\mathbb{N}), j\rangle$ which contains the natural model ${}^*V(\mathbb{N})$ of $NPRA^\omega$, and the "full model" $\langle V(\mathbb{N}), V(^*\mathbb{N}), j\rangle$ which contains the full superstructure $V(^*\mathbb{N})$ over $^*\mathbb{N}$. In both models, j is the inclusion map from $\mathbb{N}$ into $^*\mathbb{N}$. The full natural model $\langle V(\mathbb{N}), V(^*\mathbb{N}), j\rangle$ of WNA does not satisfy the axioms $NPRA^\omega$. In particular, the star of quantifier-free induction fails in this model, because the characteristic function of S exists as an object of type $^*N \to {}^*N$.

1.5 Bounded minima and overspill

In this section we prove some useful consequences of the WNA axioms.

Given a formula $A(x,\ldots)$ of $L(WNA)$, the **bounded minimum** operator is defined by

$$u = (\mu x < y)\, A(x,\ldots) \leftrightarrow \big[u \le y \wedge (\forall x < u) \neg A(x,\ldots) \wedge [A(u,\ldots) \vee u = y]\big],$$

where u is a new variable. By this we mean that the expression to the left of the $\leftrightarrow$ symbol is an abbreviation for the formula to the right of the $\leftrightarrow$ symbol. In particular, if z does not occur in A, $(\mu z < 1)\, A(\ldots)$ is the (inverted) characteristic function of A, which has the value 0 when A is true and the value 1 when A is false.

In PRA, the bounded minimum operator is defined similarly.

Lemma 4 *Let $A(m, \vec{n})$ be a quantifier-free formula of $L(PRA)$ and let $\alpha(p, \vec{n})$ be the primitive recursive function such that in PRA,*

$$\alpha(p, \vec{n}) = (\mu m < p)\, A(m, \vec{n}).$$

Then

(i) $WNA \vdash {}^*\alpha(y, \vec{z}) = (\mu x < y)\, {}^*A(x, \vec{z})$.

(ii) In WNA, there is a quantifier-free formula $B(p, \ldots)$ such that

$$(\forall m < p)\, A(m, \ldots) \leftrightarrow B(p, \ldots), \qquad (\forall x < y)\, {}^*A(x, \ldots) \leftrightarrow {}^*B(y, \ldots).$$

Similarly for $(\exists x < y)\, {}^*A(x, \ldots)$, and $u = (\mu x < y)\, {}^*A(x, \ldots)$.

Proof. (i) By the axioms of *WNA*, the defining rule for ${}^*\alpha$ is the star of the defining rule for α.

(ii) Apply (i) and observe that in *WNA*,

$$(\forall x < y)\, {}^*A(x, \ldots) \leftrightarrow y = (\mu x < y)\, \neg {}^*A(x, \ldots). \qquad \square$$

Let us write $\forall^\infty x\, A(x, \ldots)$ for $\forall x\, [\neg S(x) \to A(x, \ldots)]$ and $\exists^\infty x\, A(x, \ldots)$ for $\exists x\, [\neg S(x) \wedge A(x, \ldots)]$.

Lemma 5 *(Overspill) Let $A(x, \ldots)$ be an internal quantifier-free formula. In WNA,*

$$\forall^S x\, A(x, \ldots) \to \exists^\infty x\, A(x, \ldots) \quad \textit{and} \quad \forall^\infty x\, A(x, \ldots) \to \exists^S x\, A(x, \ldots).$$

Proof. Work in *WNA*. If $A(H, \ldots)$ we may take $x = H$. Assume $\forall^S x A(x, \ldots)$ and $\neg A(H, \ldots)$. By Lemma 4 (ii) we may take $u = (\mu x < H)\, \neg A(x, \ldots)$. Then $\neg S(u)$. Let $x = u - 1$. We have $x < u$, so $A(x, \ldots)$. Since S is closed under the successor function, $\neg S(x)$. $\square$

We now give a consequence of *WNA* in the language of *PRA* which is similar to Proposition 4.3 in [1] for $NPRA^\omega$. **Σ_1-collection** in $L(PRA)$ is the scheme

$$(\forall m < p) \exists n\, B(m, n, \vec{r}) \to \exists k (\forall m < p)(\exists n < k) B(m, n, \vec{r})$$

where B is a formula of $L(PRA)$ of the form $\exists q\, C$, C quantifier-free.

Proposition 6 *Σ_1-collection in $L(PRA)$ is provable in WNA.*

Proof. We work in *WNA*. By pairing existential quantifiers, we may assume that $B(m,n,\vec{r})$ is quantifier-free. Assume $(\forall m < p)\exists n\, B(m,n,\vec{r})$. Let ${}^*B$ be the formula obtained by starring each function symbol in B and replacing variables of type N by variables of type ${}^*N$.

By the Lifting Axiom and the axiom that S is an initial segment,

$$(\forall x < p)\exists^S y\, {}^*B(x,y,j(\vec{r})).$$

Then

$$\forall^\infty w(\forall x < p)(\exists y < w)\, {}^*B(x,y,j(\vec{r})).$$

By Lemma 4 and Overspill,

$$\exists^S w(\forall x < p)(\exists y < w)\, {}^*B(x,y,j(\vec{r})).$$

By the Lifting Axiom again,

$$\exists k(\forall m < p)(\exists n < k)\, B(m,n,\vec{r}). \qquad \square$$

1.6 Standard parts

This section introduces a standard part notion which formalizes a construction commonly used in nonstandard analysis, and provides a link between the type $N \to N$ and the type ${}^*N$.

In type N let $(n)_k$ be the power of the k-th prime in n, and in type ${}^*N$ let $(x)_y$ be the power of the y-th prime in x. The function $(n,k) \mapsto (n)_k$ is primitive recursive, and its star is the function $(x,y) \mapsto (x)_y$.

Hereafter, when it is clear from the context, we will write t instead of $j(t)$ in formulas of $L(WNA)$.

Intuitively, we identify $j(t)$ with t, but officially, they are different because t has type N while $j(t)$ has type ${}^*N$. This will make formulas easier to read. When a term t of type N appears in a place of type ${}^*N$, it really is $j(t)$.

In the theory *WNA*, we say that x is **near-standard**, in symbols $ns(x)$, if $\forall^S z\, S((x)_z)$. Note that this is equivalent to $\forall n S((x)_n)$. We employ the usual convention for relativized quantifiers, so that $\forall^{ns} x\, B$ means $\forall x\, [ns(x) \to B]$ and $\exists^{ns} x\, B$ means $\exists x\, [ns(x) \wedge B]$. We write

$$x \approx y \text{ if } ns(x) \wedge \forall^S z\, (x)_z = (y)_z.$$

This is equivalent to $ns(x) \wedge \forall n\, (x)_n = (y)_n$. We write $f = {}^o x$, and say f is the **standard part of** x and x is a **lifting of** f, if

$$ns(x) \wedge \forall n f(n) = (x)_n.$$

Note that the operation $x \mapsto {}^o x$ goes from type ${}^*N$ to type $N \to N$. In nonstandard analysis, this often allows one to obtain results about functions of type $N \to N$ by reasoning about hyperintegers of type ${}^*N$.

Lemma 7 *In WNA, suppose that x is near-standard. Then*

(i) If $x \approx y$ then $ns(y)$ and $y \approx x$.

(ii) $(\exists y < H)\, x \approx y$.

Proof. (i) Suppose $x \approx y$. If $S(z)$ then $S((x)_z)$ and $(y)_z = (x)_z$, so $S((y)_z)$. Therefore $ns(y)$, and $y \approx x$ follows trivially.

(ii) Let β be the primitive recursive function $\beta(m,n) = \Pi_{i<m} p_i^{(n)_i}$. By Lifting and defining rules for β and ${}^*\beta$, $\forall x \forall u \forall z\, [z < u \to (x)_z = ({}^*\beta(u,x))_z]$. Therefore $\forall^\infty u\, \forall^S z\, (x)_z = ({}^*\beta(u,x))_z$, and hence $\forall^\infty u\, x \approx {}^*\beta(u,x)$. We have $\forall^S w\, w^w < H$, and by Overspill, there exists w with $\neg S(w) \wedge w^w < H$. Since x is near-standard, $\forall^S u\, [u \leq w \wedge (\forall z < u)\, p_z^{(x)_z} < w]$. By Overspill,

$$\exists^\infty u\, [u \leq w \wedge (\forall z < u)\, p_z^{(x)_z} < w].$$

Let $y = {}^*\beta(u,x)$. Then $x \approx y$. By internal induction,

$$\forall u\, [(\forall z < u)\, p_z^{(x)_z} < w \to {}^*\beta(u,x) < w^u].$$

Then $y \leq w^u \leq w^w < H$. □

We now state the Standard Part Principle, which says that every near-standard x has a standard part and every f has a lifting.

Standard Part Principle (*STP*):

$$\forall^{ns} x \exists f\, f = {}^o x \wedge \forall f \exists x\, f = {}^o x.$$

The following corollary is an easy consequence of Lemma 7.

Corollary 8 *In WNA, STP is equivalent to*

$$(\forall^{ns} x < H) \exists f\, f = {}^o x \wedge \forall f (\exists x < H)\, f = {}^o x.$$

The **Weak Koenig Lemma** is the statement that every infinite binary tree has an infinite branch. The work in reverse mathematics shows that many classical mathematical statements are equivalent to the Weak Koenig Lemma.

Theorem 9 *The Weak Koenig Lemma is provable in WNA + STP.*

Proof. Work in $WNA + STP$. Let $B(n)$ be the formula

$$(\forall m < n)\,\big[(n)_m < 3 \wedge (\forall k < m)\,[(n)_k = 0 \to (n)_m = 0]\big].$$

$B(n)$ says that n codes a finite sequence of 1's and 2's. Write $m \lhd n$ if

$$B(m) \wedge B(n) \wedge m < n \wedge (\forall k < m)\,[(m)_k > 0 \to (m)_k = (n)_k].$$

This says the sequence coded by m is an initial segment of the sequence coded by n. Suppose that $\{n : f(n) = 0\}$ codes an infinite binary tree T, that is,

$$\forall m \exists n\,[m < n \wedge f(n) = 0] \wedge \forall n\,\big[f(n) = 0 \to B(n) \wedge \forall m[m \lhd n \to f(m) = 0]\big].$$

The formulas $B(n)$ and $m \lhd n$ are PRA-equivalent to quantifier-free formulas, which have stars ${}^*B(y)$ and $z\,{}^*\!\lhd y$. By STP, f has a lifting x. By Lemma 4 and Overspill,

$$\exists^\infty y\,\big[{}^*B(y) \wedge (\forall z < y)\,[z\,{}^*\!\lhd y \to (x)_z = 0]\big].$$

Then $ns(y)$, and by the STP there exists $g = {}^o y$. It follows that g codes an infinite branch of T. □

The next proposition gives a necessary and sufficient condition for STP in $WNA + NPRA^\omega$. Let ϕ be a variable of type ${}^*N \to {}^*N$, and write $f \subset \phi$ for $\forall n f(n) = \phi(n)$.

Proposition 10 *In $WNA + NPRA^\omega$, STP is equivalent to*

$$\forall f \exists \phi f \subset \phi \wedge \forall \phi \exists f\,[\forall^S x\, S(\phi(x)) \to f \subset \phi].$$

Proof. Work in $WNA + NPRA^\omega$. Call the displayed sentence STP'.

Assume STP. Take any f. By STP, f has a lifting u. Since $(u, y) \mapsto (u)_y$ is primitive recursive, $\exists \phi \forall y \phi(y) = (u)_y$. Then $\forall n\, f(n) = (u)_n = \phi(n)$, so $f \subset \phi$.

Now take any ϕ and assume that $\forall^S x\, S(\phi(x))$. Using the star of the primitive recursion scheme in $NPRA^\omega$, there exists ψ such that $\forall x(\forall y < x)\,\phi(y) = (\psi(x))_y$. Let $u = \psi(H)$. We then have $(\forall y < H)\,\phi(y) = (u)_y$, so $\forall^S y\,\phi(y) = (u)_y$. It follows that u is near-standard, and by STP there exists f with $f = {}^o u$ and hence $f \subset \phi$.

Now assume STP'. Take any f. By STP' there exist ϕ with $f \subset \phi$. As before there exists ψ such that $\forall x(\forall y < x)\,\phi(y) = (\psi(x))_y$. Let $u = \psi(H)$. Then $\forall n\,(u)_n = \phi(n) = f(n)$, so u is a lifting of f.

Now let u be near-standard. Since $(u, y) \mapsto (u)_y$ is primitive recursive, $\exists \phi \forall y\,\phi(y) = (u)_y$. Then $\forall^S x\, S(\phi(x))$, so by STP' there exists f with $f \subset \phi$. Then $\forall n\, f(n) = (u)_n = \phi(n)$, so $f = {}^o u$. □

1.7 Liftings of formulas

In this section we will define some hierarchies of formulas with variables of type N and $N \to N$, and corresponding hierarchies of formulas with variables of type ${}^*N$. We will then define the lifting of a formula and show that liftings preserve the hierarchy levels and truth values of formulas.

In the following we restrict ourselves to formulas of $L(PRA^\omega)$ with variables of types N and $N \to N$. We now introduce a restricted class of terms, the basic terms, which behave well with respect to liftings.

By a **basic term over** N we mean a term of the form $\alpha(u_1, \ldots, u_k)$ where α is a primitive recursive function of k variables and each u_i is either a variable n of type N or an expression of the form $f(n)$. These basic terms capture all primitive recursive functionals $\beta(\vec{m}, \vec{f})$ in the sense that there is a basic term $t(\vec{m}, \vec{f}, n)$ over N which gives the nth value in the computation of $\beta(\vec{m}, \vec{f})$ for each input $\vec{m}, \vec{f}, n$.

Let QF be the set of Boolean combinations of equations between basic terms over N.

If $A \in QF$, then $(\forall m < n)\, A$, $(\exists m < n)\, A$, and $u = (\mu x < y)\, A$ are PRA^ω-equivalent to formulas in QF.

The set $\Pi^1_0 = \Sigma^1_0$ of **arithmetical formulas** is the set of all formulas which are built from formulas in QF using first order quantifiers $\forall m$, $\exists m$ and propositional connectives.

For each natural number k, Π^1_{k+1} is the set of formulas of the form $\forall f A$ where $A \in \Sigma^1_k$, and Σ^1_{k+1} is the set of formulas of the form $\exists f A$ where $A \in \Pi^1_k$.

We observe that up to PRA^ω-equivalence, $\Pi^1_k \subseteq \Pi^1_{k+1} \cap \Sigma^1_{k+1}$, Π^1_k is closed under finite conjunction and disjunction, and that negations of sentences in Π^1_k belong to Σ^1_k (and vice versa).

In the following we restrict our attention to formulas with variables of type ${}^*N$. We build a hierarchy of formulas of this kind.

By a **basic term over** ${}^*N$ we mean a term of the form ${}^*\alpha(u_1, \ldots, u_k)$ where α is a primitive recursive function of k variables and each u_i is either a variable of type ${}^*N$ or the constant symbol H. NQF is the set of finite Boolean combinations of equations $s = t$ and formulas $S(t)$ where s, t are basic terms over ${}^*N$. Note that the constant symbol H and the predicate symbol S are allowed in formulas of NQF, but the symbol j is not allowed.

The internal quantifier-free formulas are just the formulas $B \in NQF$ in which the symbol S does not occur.

Let $N\Pi^0_0 = N\Sigma^0_0$ be the set of formulas which are built from formulas in NQF using the relativized quantifiers $\forall^S$, $\exists^S$ and propositional connectives. Note that the relations $ns(x)$ and $x \approx y$ are definable by $N\Pi^0_0$ formulas.

For each natural number k, $N\Pi^0_{k+1}$ is the set of formulas of the form $\forall^{ns} x\, A$ where $A \in N\Sigma^0_k$. $N\Sigma^0_{k+1}$ is the set of formulas of the form $\exists^{ns} x\, A$ where $A \in N\Pi^0_k$.

Up to *WNA*-equivalence, $N\Pi^0_k \subseteq N\Pi^0_{k+1} \cap N\Sigma^0_{k+1}$, $N\Pi^0_k$ is closed under finite conjunction and disjunction, and negations of sentences in $N\Pi^0_k$ belong to $N\Sigma^0_k$ (and vice versa).

We now define the lifting mapping on formulas, which sends Π^1_k to $N\Pi^0_k$.

Definition 11 *Let $A(\vec{m}, \vec{f})$ be a formula in Π^1_k, where $\vec{m}, \vec{f}$ contain all the variables of A, both free and bound. The **lifting** $\overline{A}(\vec{z}, \vec{x})$ is defined as follows, where $\vec{z}$ and $\vec{x}$ are tuples of variables of type ${}^*\mathbb{N}$ of the same length as $\vec{m}, \vec{f}$.*

- *Replace each primitive recursive function symbol in A by its star.*
- *Replace each m_i by z_i.*
- *Replace each $f_i(m_k)$ by $(x_i)_{z_k}$.*
- *Replace each quantifier $\forall m_i$ by $\forall^S z_i$, and similarly for $\exists$.*
- *Replace each quantifier $\forall f_i$ by $\forall^{ns} x_i$, and similarly for $\exists$.*

Lemma 12 *(Zeroth Order Lifting) For each formula $A(\vec{m}, \vec{f}) \in \Pi^1_0$, we have $\overline{A}(\vec{z}, \vec{x}) \in N\Pi^0_0$, and*

$$WNA \vdash {}^o\vec{x} = \vec{f} \to [A(\vec{m}, \vec{f}) \leftrightarrow \overline{A}(\vec{m}, \vec{x})].$$

Moreover, if $A \in QF$ then $\overline{A}(\vec{z}, \vec{x})$ is an internal quantifier-free formula.

Proof. It is clear from the definition that $\overline{A}(\vec{z}, \vec{x}) \in N\Pi^0_0$, and if $A \in QF$ then $\overline{A}(\vec{z}, \vec{x})$ is an internal quantifier-free formula. In the case that A is an equation between basic terms, the lemma follows from the Lifting Axiom. The general case is then proved by induction on the complexity of A, using the axiom that j maps $\mathbb{N}$ onto S. □

Lemma 13 *(First Order Lifting) For each formula $A(\vec{m}, \vec{f}) \in \Pi^1_k$, we have $\overline{A}(\vec{z}, \vec{x}) \in N\Pi^0_k$ and*

$$WNA + STP \vdash {}^o\vec{x} = \vec{f} \to [A(\vec{m}, \vec{f}) \leftrightarrow \overline{A}(\vec{m}, \vec{x})].$$

Proof. Zeroth Order Lifting gives the result for $k = 0$. The general case follows by induction on k, using STP. □

1.8 Choice principles in $L(PRA^\omega)$

In this section we state two choice principles in the language $L(PRA^\omega)$, and show that for quantifier-free formulas they are consequences of the Standard Part Principle. Given a function g of type $N \to N$, let $g^{(m)}$ be the function $g^{(m)}(n) = g(2^m 3^n)$.

In each principle, Γ is a class of formulas with variables of types N and $N \to N$, and $A(m, n, \ldots)$ denotes an arbitrary formula in Γ.

$(\Gamma, 0)$-choice $\quad \forall m \exists n\, A(m, n, \ldots) \to \exists g \forall m\, A(m, g(m), \ldots).$

$(\Gamma, 1)$-choice $\quad \forall m \exists f\, A(m, f, \ldots) \to \exists g \forall m\, A(m, g^{(m)}, \ldots).$

When Γ is the set of all quantifier-free formulas of PRA^ω, [7] calls these schemes $QF - AC^{0,0}$ and $QF - AC^{0,1}$ respectively. A related principle is

Γ-comprehension $\quad \exists f \forall m\, f(m) = (\mu z < 1)\, A(m, \ldots).$

Π^1_0-comprehension is called **Arithmetical Comprehension**. The following well-known fact is proved by pairing existential quantifiers.

Proposition 14 *In PRA^ω:*
$(\Pi^0_1, 0)$-choice, $(\Pi^1_0, 0)$-choice, and Arithmetical Comprehension are equivalent;
$(\Pi^0_k, 1)$-choice is equivalent to $(\Sigma^0_{k+1}, 1)$-choice, and implies $(\Sigma^0_{k+1}, 0)$-choice;
$(\Pi^1_k, 1)$-choice is equivalent to $(\Sigma^1_{k+1}, 1)$-choice and implies $(\Sigma^1_{k+1}, 0)$-choice;
$(\Sigma^1_{k+1}, 0)$-choice implies Π^1_k-comprehension.

In PRA^ω, one can define a subset of $\mathbb{N}$ to be a function f such that $\forall n\, f(n) \leq 1$, and define $n \in f$ as $f(n) = 0$. With these definitions, $(\Pi^1_k, 1)$-choice implies Π^1_k-choice and Π^1_k-comprehension in the sense of second order number theory (see [8]).

Lemma 15 *For each internal quantifier-free formula $A(x, y, \vec{z})$,*

$$WNA \vdash \forall^S x \exists^S y\, A(x, y, \vec{z}) \to (\exists^{ns} y < H) \forall^S x\, A(x, (y)_x, \vec{z}).$$

Proof. Work in *WNA*. Assume that $\forall^S x \exists^S y\, A(x, y, \vec{z})$. By Lemma 4, there is a primitive recursive function α such that ${}^*\alpha(u, \vec{z}, w) = (\mu v < w)\, A(u, v, \vec{z})$. By internal induction there exists w such that $w^w < H \wedge \neg S(w)$. Then $\forall^S u\, S({}^*\alpha(u, \vec{z}, w))$ and

$$\forall^S x (\exists y < H)(\forall u < x)\, (y)_u = {}^*\alpha(u, \vec{z}, w).$$

By internal induction, there exists an x such that $\neg S(x)$ and a $y < H$ such that $(\forall u < x)\, (y)_u = {}^*\alpha(u, \vec{z}, w)$. It follows that y is near-standard, and by the definition of α, $(\forall u < x)\, A(u, (y)_u, \vec{z})$. Then

$$(\exists^{ns} y < H) \forall^S x\, A(x, (y)_x, \vec{z}). \qquad \square$$

Theorem 16 *$(QF, 0)$-choice is provable in $WNA + STP$.*

Proof. We work in $WNA + STP$. Let $A(m, n, \vec{r}, \vec{h}) \in QF$ and assume $\forall m \exists n\, A(m, n, \vec{r}, \vec{h})$. Then $\overline{A}$ is an internal quantifier-free formula. By STP, $\vec{h}$ has a lifting $\vec{z}$. By First Order Lifting,

$$A(m, n, \vec{r}, \vec{h}) \leftrightarrow \overline{A}(m, n, \vec{r}, \vec{z}).$$

Then $\forall^S u \exists^S v\, \overline{A}(u, v, \vec{r}, \vec{z})$. By Lemma 15, there is a near-standard y such that $\forall^S u\, \overline{A}(u, (y)_u, \vec{r}, \vec{z})$. By STP, $\exists g\, g = {}^o y$. Then by First Order Lifting, $\forall m\, A(m, g(m), \vec{r}, \vec{h})$. □

Theorem 17 *$(QF, 1)$-choice is provable in $WNA + STP$.*

Proof. We use $(QF, 0)$-choice. Let $A(m, f, \vec{r}, \vec{h}) \in QF$. Assume for simplicity that the tuple $\vec{r}$ is a single variable r. Suppose that $\forall m \exists f\, A(m, f, r, \vec{h})$. By the definition of QF formulas, f occurs in A only in terms of the form $f(m)$ and $f(r)$. Then

$$A(m, f, r, \vec{h}) \leftrightarrow B(m, f(m), f(r), r, \vec{h})$$

where $B \in QF$. Hence $\forall m \exists k\, B(m, (k)_m, (k)_r, r, \vec{h})$. By $(QF, 0)$-choice,

$$\exists f \forall m\, B(m, (f(m))_m, (f(m))_r, r, \vec{h}).$$

Applying $(QF, 0)$-choice to the formula $\forall p\, \exists q\, q = (f((p)_0))_{(p)_1}$, we have

$$\exists g \forall p\, g(p) = (f((p)_0))_{(p)_1},$$

and since $(2^m 3^n)_0 = m$ and $(2^m 3^n)_1 = n$,

$$\exists g \forall m \forall n\, g^{(m)}(n) = g(2^m 3^n) = (f(m))_n.$$

Then $\forall m\, B(m, g^{(m)}(m), g^{(m)}(r), r, \vec{h})$, and $\forall m\, A(m, g^{(m)}, r, \vec{h})$. □

1.9 Saturation principles

We state two saturation principles which formalize methods commonly used in nonstandard analysis. In each principle, Γ is a class of formulas with variables of type ${}^*N$, and $A(x, y, \vec{u})$ denotes an arbitrary formula in the class Γ.

$(\Gamma, 0)$-saturation $\forall^{ns} \vec{u}\, [\forall^S x \exists^S y\, A(x, y, \vec{u}) \rightarrow \exists y \forall^S x\, A(x, (y)_x, \vec{u})]$.
$(\Gamma, 1)$-saturation $\forall^{ns} \vec{u}\, [\forall^S x \exists^{ns} y\, A(x, y, \vec{u}) \rightarrow \exists y \forall^S x\, A(x, (y)_x, \vec{u})]$.

Note that $(\Gamma, 1)$-saturation implies $(\Gamma, 0)$-saturation. $(N\Pi^0_k, 1)$-saturation is weaker than the ${}^*\Pi_k$-saturation principle in the paper [5]. ${}^*\Pi_k$-saturation

is the same as $(N\Pi^0_k, 1)$-saturation except that the quantifiers $\forall^{ns}$, $\exists^{ns}$ are replaced by $\forall$, $\exists$.

In the rest of this section we prove some consequences of $(NQF, 0)$-saturation.

Proposition 18 *Let us write $w = st(v)$ for*

$$S(v) \to w = v \wedge \neg S(v) \to w = 0.$$

In WNA, $(NQF, 0)$-saturation implies that

$$\forall x \exists^{ns} y \forall^S z\, [(y)_z = st((x)_z)].$$

In WNA + STP, $(NQF, 0)$-saturation implies that

$$\forall x \exists f \forall m\, [f(m) = st((x)_m)].$$

Proof. Work in *WNA*. Note that $w = st(v)$ stands for a formula in NQF. Take any x. We have $\forall^S z \exists^S w\, w = st((x)_z)$. Then by $(NQF, 0)$-saturation, $\exists y \forall^S z\, (y)_z = st((x)_z)$, and it follows that $ns(y)$. The second assertion follows by taking $f = {}^o y$. □

Lemma 19

$$WNA \vdash (\forall v > 1)(\exists w < v^{2vH})(\forall x < v)\, (w)_x = (\mu u < H)\, [(y)_{2^x 3^u} = 0].$$

Proof. Use internal induction on v. The result is clear for $v = 2$. Let $\alpha(x) = (\mu u < H)\, [(y)_{2^x 3^u} = 0]$. Assume the result holds for v, that is, $w < v^{2vH} \wedge (\forall x < v)\, (w)_x = \alpha(x)$. Let $z = w * p_v^{\alpha(v)}$. We have $p_v < v^2$ and $\alpha(v) < H$, so $z < v^{2vH} * v^{2H} \leq (v+1)^{2(v+1)H}$ and $(\forall x < v+1)\, (z)_x = \alpha(x)$. This proves the result for $v + 1$ and completes the induction. □

Lemma 20 *In WNA, $(NQF, 0)$-saturation implies that for every formula $A(x, \vec{u}) \in N\Pi^0_0$, $\forall^{ns} \vec{u}\ (\exists y < H) \forall^S x\, (y)_x = (\mu z < 1) A(x, \vec{u})$.*

Proof. Work in *WNA* and assume $(NQF, 0)$-saturation. Let Φ be the set of formulas $A(x, \vec{u})$ such that $\forall^{ns} \vec{u}\, (\exists y < H) \forall^S x\, (y)_x = (\mu z < 1) A(x, \vec{u})$. We prove that $N\Pi^0_0 \subseteq \Phi$ by induction on quantifier rank. Suppose first that $A \in NQF$. Let $C(x, w, \vec{u})$ be the formula $w = (\mu z < 1)\, A(x, \vec{u})$. Then C is a propositional combination of A, $w = 0$, and $w = 1$, so $C \in NQF$. We clearly have $\forall^{ns} \vec{u} \forall^S x\, \exists^S w\, C(x, w, \vec{u})$. By $(NQF, 0)$-saturation and Lemma 7 (ii),

$$\forall^{ns} \vec{u}\, (\exists y < H) \forall^S x\, C(x, (y)_x, \vec{u}),$$

so $A \in \Phi$.

It is clear that the set of formulas Φ is closed under propositional connectives. Suppose all formulas of $N\Pi^0_0$ of quantifier rank at most n belong to Φ, and $A(x,\vec{u}) = \exists^S w B(x,w,\vec{u})$ where $B(x,w,\vec{u}) \in N\Pi^0_0$ has quantifier rank at most n. There is a formula $D(v,\vec{u})$ with the same quantifier rank as B such that in *WNA*, $D(2^x 3^w,\vec{u}) \leftrightarrow B(x,w,\vec{u})$. Then $D \in \Phi$, so

$$\forall^{ns}\vec{u}\,(\exists t < H)\forall^S v\,(t)_v = (\mu z < 1)\,D(v,\vec{u}).$$

Then

$$\forall^{ns}\vec{u}\,(\exists t < H)\forall^S x\forall^S w\,(t)_{2^x 3^w} = (\mu z < 1)\,B(x,w,\vec{u}).$$

Assume that $ns(\vec{u})$ and take t as in the above formula. By Lemma 19 there exists s such that

$$(\forall x < H)\,(s)_x = (\mu w < H)\,[(t)_{2^x 3^w} = 0].$$

It is trivial that $\forall^S x\exists^S y\, y = (\mu z < 1)\,S((s)_x)$. By $(NQF,0)$-saturation and Lemma 7,

$$(\exists y < H)\forall^S x\,(y)_x = (\mu z < 1)\,S((s)_x).$$

Thus whenever $S(x)$,

$$(y)_x = 0 \text{ iff } S((s)_x) \text{ iff } \exists^S w\,[(t)_{2^x 3^w} = 0] \text{ iff } \exists^S w\,B(x,w,\vec{u}).$$

It follows that

$$\forall^{ns}\vec{u}\,(\exists y < H)\forall^S x\,(y)_x = (\mu z < 1)\exists^S w\,B(x,w,\vec{u}),$$

so $A \in \Phi$. □

Theorem 21 *In WNA, $(NQF,0)$-saturation implies $(N\Pi^0_0,0)$-saturation.*

Proof. We continue to work in *WNA* and assume $(NQF,0)$-saturation and $ns(\vec{u})$. Assume that $A(x,y,\vec{u}) \in N\Pi^0_0$ and $\forall^S x\exists^S y\,A(x,y,\vec{u})$. There is a formula $B(v,\vec{u}) \in N\Pi^0_0$ with $B(2^x 3^y,\vec{u})$ *WNA*-equivalent to $A(x,y,\vec{u})$. Applying Lemma 20 to B, we obtain w such that

$$\forall^S v\,(w)_v = (\mu z < 1)\,B(v,\vec{u}),$$

so

$$\forall^S x\forall^S y\,(w)_{2^x 3^y} = (\mu z < 1)\,A(x,y,\vec{u}).$$

By Lemma 19 there exists w' such that

$$(\forall x < H)\,(w')_x = (\mu y < H)\,[(w)_{2^x 3^y} = 0].$$

Then $\forall^S x\,A(x,(w')_x,\vec{u})$, and w is near-standard because $\forall^S x\exists^S y\,A(x,y,\vec{u})$. □

Theorem 22 *In WNA, $(NQF, 0)$-saturation implies that for every formula $A(\vec{u}) \in N\Pi^0_1$, there is a formula $B \in NQF$ such that*

$$\forall^{ns}\vec{u}\,[A(\vec{u}) \leftrightarrow \forall^{ns}x\exists^S y\, B(x, y, \vec{u})].$$

Proof. Work in $WNA + (NQF, 0)$-saturation. Suppose $A \in N\Pi^0_1$. Then there is a least k such that A is equivalent to a formula $\forall^{ns}x\, C$ where C is a prenex formula in $N\Pi^0_0$ of quantifier rank k. If C has the form $\forall^S y\, D$, the quantifier $\forall^S y$ can be absorbed into the quantifier $\forall^{ns}x$, contradicting the assumption that k is minimal. Suppose C has the form $\exists^S y\,\forall^S z\, D(x, y, z, \vec{u})$ and assume that $ns(\vec{u})$. Then $\neg C$ is equivalent to $\forall^S y\,\exists^S z\,\neg D(x, y, z, \vec{u})$. By $(N\Pi^0_0, 0)$-saturation, $\neg C$ is equivalent to $\exists^{ns}z\forall^S y\,\neg D(x, y, (z)_y, \vec{u})$. Then A is equivalent to $\forall^{ns}x\forall^{ns}z\exists^S y\, D(x, y, (z)_y, \vec{u})$. By combining the quantifiers $\forall^{ns}x\forall^{ns}z$, we contradict the assumption that k is minimal. Therefore C must have the form $\exists^S y\, B$ where $B \in NQF$, as required. □

1.10 Saturation and choice

In this section we prove results showing that in $WNA + STP$, saturation principles with quantifiers of type ${}^*N$ imply the corresponding choice principles with quantifiers of type $N \to N$.

Theorem 23 *In WNA+STP, $(NQF, 0)$-saturation implies Arithmetical Comprehension.*

Proof. Work in $WNA + STP$ and assume $(NQF, 0)$-saturation. By Proposition 14, Arithmetical Comprehension is equivalent to $(\Pi^1_0, 0)$-choice. By Theorem 21, $(N\Pi^0_0, 0)$-saturation holds. Let $A(m, n, \vec{r}, \vec{h})$ be an arithmetical formula such that $\forall m\exists n\, A(m, n, \vec{r}, \vec{h})$. By STP, $\vec{h}$ has a lifting $\vec{u}$. By First Order Lifting, we have $\forall^S x\exists^S y\,\overline{A}(x, y, \vec{r}, \vec{u})$, and $\overline{A} \in N\Pi^0_0$. By $(N\Pi^0_0, 0)$-saturation, there exists y such that $\forall^S x\,[S((y)_x) \wedge \overline{A}(x, (y)_x, \vec{r}, \vec{u})]$. Then y is near-standard, and by STP there exists $g = {}^o y$.

By First Order Lifting again, $\forall m A(m, g(m), \vec{r}, \vec{h})$. □

We remark that the axioms of Peano Arithmetic are consequences of Arithmetical Comprehension, so $(NQF, 0)$-saturation implies Peano Arithmetic.

Theorem 24 *In WNA+STP, $(N\Pi^0_k, 0)$-saturation implies $(\Pi^1_k, 0)$-choice, and $(N\Sigma^0_k, 0)$-saturation implies $(\Sigma^1_k, 0)$-choice.*

Proof. Work in $WNA + STP$. For the Π^1_k case, assume $(N\Pi^0_k, 0)$-saturation. Let $A(m, n, \vec{r}, \vec{h}) \in \Pi^1_k$ and suppose that $\forall m\exists n\, A(m, n, \vec{r}, \vec{h})$. Now argue as in the proof of Theorem 23. The Σ^1_k case is similar. □

Theorem 25 *In WNA + STP, $(N\Pi^0_k, 1)$-saturation implies $(\Sigma^1_{k+1}, 1)$-choice.*

Proof. Work in *WNA + STP* and assume $(N\Pi^0_k, 1)$-saturation. It suffices to prove $(\Pi^1_k, 1)$-choice.

Let $A(m, f, \vec{r}, \vec{h}) \in \Pi^1_k$ and suppose that $\forall m \exists f\, A(m, f, \vec{r}, \vec{h})$. By *STP*, $\vec{h}$ has a lifting $\vec{u}$. By First Order Lifting, $\forall^S x \exists^{ns} y\, \overline{A}(x, y, \vec{r}, \vec{u})$ and $\overline{A} \in N\Pi^0_k$. We may rewrite this as $\forall^S x \exists^{ns} y\, [ns(y) \wedge \overline{A}(x, y, \vec{r}, \vec{u})]$ and note that $ns(y) \wedge \overline{A} \in N\Pi^0_k$. By $(N\Pi^0_k, 1)$-saturation, there exists y such that

$$\forall^S x\, [ns((y)_x) \wedge \overline{A}(x, (y)_x, \vec{r}, \vec{u})].$$

Applying $(N\Pi^0_k, 0)$-saturation to the formula $\forall^S x \exists^S z\, z = ((y)_{(x)_0})_{(x)_1}$, we get a near-standard z such that $\forall^S x\, (z)_x = ((y)_{(x)_0})_{(x)_1}$. Then

$$\forall^S x \forall^S w\, (z)_{2^x 3^w} = ((y)_x)_w.$$

By *STP*, there exists $g = {}^o z$. Then for each m, n, $g^{(m)}(n) = g(2^m 3^n) = (z)_{2^m 3^n} = ((y)_m)_n$. Therefore $g^{(m)} = {}^o((y)_m)$ for each m. By First Order Lifting, we get the desired conclusion $\forall m\, A(m, g^{(m)}, \vec{r}, \vec{u})$. □

The literature in reverse mathematics shows that Π^1_1-comprehension is strong enough for almost all of classical mathematics (see [8]).

Let us work in *WNA + STP* and aim for Π^1_1-comprehension. By Theorem 25, $(N\Pi^0_1, 1)$-saturation implies Π^1_1-comprehension. By Theorem 22, $(N\Pi^0_1, 1)$-saturation is equivalent to $(\Gamma, 1)$-saturation where Γ is the set of formulas of the form $\forall^{ns} v \exists^S w\, B$ with $B \in NQF$, so $(\Gamma, 1)$-saturation also implies Π^1_1-comprehension. By Theorem 25 at the next level, $(N\Pi^0_2, 1)$-saturation implies Π^1_2-comprehension, which is stronger than the methods used in most of classical mathematics.

1.11 Second order standard parts

In this section we introduce second order standard parts, which provide a link between the second level of $V(\mathbb{N})$ (type $(N \to N) \to N$), and the first level of $V(^*\mathbb{N})$ (type $^*N \to {}^*N$). We will use $F, G, \ldots$ for variables of type $(N \to N) \to N$, and $\phi, \psi, \ldots$ for variables of type $^*N \to {}^*N$.

ϕ is called **near-standard**, in symbols $ns(\phi)$, if

$$\forall^{ns} x\, S(\phi(x)) \wedge \forall x \forall y\, [x \approx y \to \phi(x) = \phi(y)].$$

We write

$$\phi \approx \psi \text{ if } ns(\phi) \wedge \forall^{ns} x\, \phi(x) = \psi(x).$$

We write $G = {}^o\phi$, and say that G is the **standard part of** ϕ and that ϕ is a **lifting of** G, if

$$ns(\phi) \wedge \forall^{ns} x \forall f\, [{}^o x = f \to \phi(x) = G(f)].$$

Note that the operation $\phi \mapsto {}^o\phi$ goes from type ${}^*N \to {}^*N$ to type $(N \to N) \to N$. The following lemma is straightforward.

Lemma 26 *If* $ns(\phi)$ *and* $\phi \approx \psi$ *then* $ns(\psi)$ *and* $\psi \approx \phi$.

We now state the Second Order Standard Part Principle, which says that every near-standard ϕ has a standard part and every F has a lifting.

Second Order Standard Part Principle:

$$\forall^{ns} \phi\, \exists F\, F = {}^o\phi \wedge \forall F \exists \phi\, F = {}^o\phi.$$

By $WNA{+}STP(2)$ we mean the theory WNA plus both the first and second order standard part principles.

We now take a brief look at the consequences of $STP(2)$ in $WNA{+}NPRA^\omega$. Roughly speaking, in $WNA{+}NPRA^\omega$, the second order standard part principle imposes restrictions of the set of functionals which are reminiscent of constructive analysis. Besides the axioms of WNA, the only axiom of $NPRA^\omega$ that will be used in this section is the star of quantifier-free induction.

A functional G is **continuous** if it is continuous in the Baire topology, that is,

$$\forall f \exists n \forall h\, \big[(\forall m < n)\, h(m) = f(m)] \to G(h) = G(f)\big].$$

Proposition 27 $WNA + NPRA^\omega + STP(2) \vdash \forall G\, G$ *is continuous.*

Proof. Work in $WNA + NPRA^\omega + STP(2)$. Suppose G is not continuous at f. Then

$$\forall n \exists h\, \big[(\forall m < n)\, h(m) = f(m)] \wedge G(f) \neq G(h)\big].$$

By $STP(2)$ there are liftings ϕ of G and x of f. By Lemma 7 and STP,

$$\forall n (\exists y < H)\, \big[(\forall m < n)\, (y)_m = (x)_m] \wedge \phi(x) \neq \phi(y)\big].$$

By the star of QF induction,

$$\exists^\infty w (\exists y < H)\, \big[(\forall u < w)\, (x)_u = (y)_u] \wedge \phi(x) \neq \phi(y)\big].$$

But then $y \approx x$, contradicting the assumption that ϕ is near-standard. □

This result is closely related to Proposition 5.2 in [1], which says that in $NPRA^\omega$, every function $f \in \mathbb{R} \to \mathbb{R}$ is continuous.

The sentence

$$(\exists^2) = \exists G \forall f\, [G(f) = 0 \leftrightarrow \exists n f(n) = 0]$$

played a central role in the paper [7], where many statements are shown to be equivalent to $(\exists^2)$ in RCA_0^ω. Similar sentences are prominent in earlier papers, such as Feferman [3]. It is well-known that

$$PRA^\omega \vdash (\exists^2) \to \exists G\, G \text{ is not continuous.}$$

Corollary 28 $WNA + NPRA^\omega + STP(2) \vdash \neg(\exists^2)$.

1.12 Functional choice and $(\exists^2)$

In this section we obtain connections between WNA and two statements which play a central role in the paper of Kohlenbach [7], the statement $(\exists^2)$ and the functional choice principle $QF - AC^{1,0}$.

In [7], Kohlenbach proposed a base theory RCA_0^ω for higher order reverse mathematics which is somewhat stronger than PRA^ω, and is a conservative extension of the second order base theory RCA_0. Its main axioms are the axioms of PRA^ω and the scheme

$$QF - AC^{1,0}: \qquad \forall f \exists n\, A(f, n, \ldots) \to \exists G \forall f\, A(f, G(f), \ldots)$$

where $A(f, n, \ldots)$ is quantifier-free.

In [7], the formula A in the $QF - AC^{1,0}$ scheme is allowed to be an arbitrary quantifier-free formula in the language $L(PRA^\omega)$. Here we will make the additional restriction that $A(f, n, \ldots)$ is in the class QF as defined in Section 1.7, that is, $A(f, n, \ldots)$ is a Boolean combination of equations and inequalities between basic terms. These formulas only have variables of type N and $N \to N$, and do not have functional variables.

We show now that $QF - AC^{1,0}$ restricted in this way follows from WNA plus the standard part principles.

Theorem 29 $WNA + STP(2) \vdash QF - AC^{1,0}$.

Proof. Work in $WNA + STP(2)$. Assume $\forall f \exists n\, A(f, n, \vec{m}, \vec{h})$. By Zeroth Order Lifting, $\overline{A}(x, u, \vec{v}, \vec{z})$ is an internal quantifier-free formula, and

$$^o x = f \wedge {}^o \vec{z} = \vec{h} \to [A(f, n, \vec{m}, \vec{h}) \leftrightarrow \overline{A}(x, n, \vec{m}, \vec{z})].$$

By Lemma 4 there is a primitive recursive function α such that

$$^*\alpha(x, w, \vec{v}, \vec{z}) = (\mu u < w)\, \overline{A}(x, u, \vec{v}, \vec{z}).$$

By STP, there exists $\vec{z}$ such that $\vec{h} = {}^o\vec{z}$. By the Lambda Abstraction axiom,

$$\exists\phi\forall x\,\phi(x) = {}^*\alpha(x, H, \vec{m}, \vec{z}).$$

Then

$$\forall^{ns}x\,[S(\phi(x)) \wedge \overline{A}(x, \phi(x), \vec{m}, \vec{z})].$$

It follows that ϕ is near-standard. By $STP(2)$, there exists G such that $G = {}^o\phi$. Therefore $\forall f\, A(f, G(f), \vec{m}, \vec{h})$. □

One of the advantages of WNA over $NPRA^\omega$ is that one can add hypotheses which produce external functions and still keep the standard part principles. The simplest hypothesis of this kind is the following statement, which says that the characteristic function of S exists:

$$(1_S \text{ exists}): \qquad \exists\phi\forall y\,\phi(y) = (\mu z < 1)\,S(y).$$

It is clear that

$$NPRA^\omega \vdash \neg(1_S \text{ exists})$$

because by the star of quantifier-free induction, $\forall^S y\,\phi(y) = 0$ implies

$$\exists y\,[\neg S(y) \wedge \phi(y) = 0].$$

However, (1_S exists) is true in the full natural model $\langle V(\mathbb{N}), V({}^*\mathbb{N}), j\rangle$ of WNA. We now connect this principle with the statement $(\exists^2)$.

Theorem 30 *$WNA + STP(2) \vdash (1_S \text{ exists}) \to (\exists^2)$.*

Proof. Work in $WNA + STP(2)$. Let α be the primitive recursive function such that ${}^*\alpha(x, w) = (\mu u < w)\,(x)_u = 0$. Let ϕ be the function 1_S, so that $\forall y\,\phi(y) = (\mu z < 1)\,S(y)$. Then there exists ψ such that $\forall x\,\psi(x) = \phi({}^*\alpha(x, H))$. Observe that

$$\phi({}^*\alpha(x, H)) = 0 \leftrightarrow \exists^S u\,(x)_u = 0,$$

so $\psi(x) = 0 \leftrightarrow \exists^S u\ (x)_u = 0$. Moreover, $\forall x\,\psi(x) < 2$. We show that ψ is near-standard.

Suppose $ns(x)$ and $x \approx y$. We always have $S(\phi(x))$ since $\phi(x) < 2$. If $\psi(x) = 0$ then there exists u such that $S(u)$ and $(x)_u = 0$, so $(y)_u = 0$ and hence $\psi(y) = 0$. This shows that $ns(\psi)$. By $STP(2)$ there exists G such that $G = {}^o\psi$. Consider any f. By STP, f has a lifting x. Then $G(f) = 0$ iff $\psi(x) = 0$ iff $\exists^S u(x)_u = 0$ iff $\exists n\, f(n) = 0$, and thus $(\exists^2)$ holds. □

Let us now go back to Section 1.7 and redefine the set QF of formulas by allowing basic terms of the form $G_i(f_k)$ in addition to the previous basic terms, and redefining the hierarchy Π^1_k by starting with the new QF. Also redefine

the set NQF and the hierarchy $N\Pi^0_k$ by allowing additional basic terms of the form $\phi_i(x_k)$. When $STP(2)$ is assumed, the lifting lemmas from Section 1.7 and the results of Section 1.9 can be extended to the larger classes of formulas just defined. The hierarchies Π^2_k and $N\Pi^1_k$ at the next level can now be defined in the natural way. One can then obtain the following result, with a proof similar to the proofs in Section 1.9.

Theorem 31 *In $WNA + STP(2)$, $(N\Pi^1_k, 0)$-saturation implies $(\Pi^2_k, 0)$-choice, and $(N\Pi^1_k, 1)$-saturation implies $(\Pi^2_k, 1)$-choice.*

1.13 Conclusion

We have proposed weak nonstandard analysis, *WNA*, as a base theory for reverse mathematics in nonstandard analysis.

In $WNA + STP$, one can prove:

The Weak Koenig Lemma,

$(QF, 0)$-choice and $(QF, 1)$-choice,

$(NQF, 0)$-saturation implies $(\Pi^1_0, 0)$-choice.

$(N\Pi^0_k, i)$-saturation implies (Π^1_k, i)-choice, $i = 0, 1$.

In $WNA + STP(2)$ one can prove:

$QF - AC^{1,0}$,

$NPRA^\omega$ implies $\forall G\, G$ is continuous,

1_S exists implies $(\exists^2)$,

$(N\Pi^1_k, i)$-saturation implies (Π^2_k, i)-choice, $i = 0, 1$.

We envision the use of these results to calibrate the strength of particular theorems proved using nonstandard analysis. At the higher levels, this could give a way to show that a theorem cannot be proved with methods commonly used in classical mathematics.

Look again at the natural models of *WNA* discussed at the end of Section 1.4. Let ${}^*V(\mathbb{N})$ be an $\aleph_1$-saturated elementary extension of $V(\mathbb{N})$ in the model-theoretic sense, and consider the internal natural model $\langle V(\mathbb{N}), {}^*V(\mathbb{N}), j\rangle$ and the full natural model $\langle V(\mathbb{N}), V({}^*\mathbb{N}), j\rangle$. Both of these models satisfy the axioms of *WNA*, the *STP*, the statement $(\exists^2)$, and $(N\Pi^1_k, 1)$-saturation. In view of Corollary 28, in the internal natural model the axioms of $NPRA^\omega$ hold and $STP(2)$ fails, while in the full natural model $STP(2)$ holds and the axioms of $NPRA^\omega$ fail.

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2

The virtue of simplicity

Edward Nelson*

Part I. Technical

It is known that IST (internal set theory) is a conservative extension of ZFC (Zermelo-Fraenkel set theory with the axiom of choice); see for example the appendix to [2] for a proof using ultrapowers and ultralimits. But these semantic constructions leave one wondering what actually makes the theory work—what are the inner mechanisms of Abraham Robinson's new logic. Let us examine the question syntactically.

Notational conventions: we use x to stand for a variable and other lowercase letters to stand for a sequence of zero or more variables; variables with a prime $'$ range over finite sets; variables with a tilde $\tilde{}$ range over functions.

We take as the axioms of IST the axioms of ZFC together with the following, in which A is an internal formula:

(T) $\forall^{\mathrm{st}}\mathrm{t}\,[\forall^{\mathrm{st}}\mathrm{x}\mathrm{A} \to \forall\mathrm{x}\mathrm{A}]$, where A has free variables x and the variables of t,

(I) $\forall^{\mathrm{st}}\mathrm{y}'\exists\mathrm{x}\forall\mathrm{y}{\in}\mathrm{y}'\,\mathrm{A} \leftrightarrow \exists\mathrm{x}\forall^{\mathrm{st}}\mathrm{y}\,\mathrm{A}$,

(S) $\forall^{\mathrm{st}}\mathrm{x}\exists^{\mathrm{st}}\mathrm{y}\,\mathrm{A}(\mathrm{x},\mathrm{y}) \to \exists^{\mathrm{st}}\tilde{\mathrm{y}}\forall^{\mathrm{st}}\mathrm{x}\,\mathrm{A}\big(\mathrm{x},\tilde{\mathrm{y}}(\mathrm{x})\big)$.

We have written the standardization principle (S) in functional form and required A to be internal; we call this the *restricted* standardization principle. It can be shown that the general standardization principle is a consequence.

All functions must have a domain. There is a neat way, using the reflection principle of set theory, to ensure that $\tilde{\mathrm{y}}$ has a domain, but let me avoid discussion of this point.

We do not take the predicate symbol *standard* as basic, but introduce it by

$$x \text{ is standard } \leftrightarrow \exists^{\mathrm{st}}y[y = x].$$

In this way $\forall^{\mathrm{st}}$ and $\exists^{\mathrm{st}}$ are new *logical* symbols and (I), (S), (T) are *logical* axioms of Abraham Robinson's new logic.

*Department of Mathematics, Princeton University.
nelson@math.princeton.edu

For any formula A of IST we define a formula A^+, called the *partial reduction* of A. It will always be of the form $\forall^{st}u\exists^{st}vA^\bullet$ where $A^\bullet$ is internal. It is defined recursively as follows:

$$\begin{array}{rcl} \text{if A is internal, } A^+ & \text{is} & A \\ (\neg A)^+ & \text{is} & \forall^{st}\tilde{v}\exists^{st}u\neg A^\bullet\big(u,\tilde{v}(u)\big) \\ (A_1 \vee A_2)^+ & \text{is} & \forall^{st}u_1u_2\exists^{st}v_1v_2[A_1^\bullet \vee A_2^\bullet] \\ (\forall x A)^+ & \text{is} & \forall^{st}u\exists^{st}v'\forall x\exists v{\in}v'A^\bullet \\ (\forall^{st}x A)^+ & \text{is} & \forall^{st}xu\forall^{st}vA^\bullet. \end{array}$$

(We take $\neg$, $\vee$, and $\forall$ as the basic logical operators—the others can be defined in terms of them.) It is understood when forming $(A_1 \vee A_2)^+$ that a variant may be taken (bound variables changed) to avoid colliding variables. If z are the free variables of A, then the *reduction* of A, denoted by A°, is the internal formula

$$\forall u\exists v'\forall z\exists v{\in}v'A^\bullet.$$

This is the same as the partial reduction of the closure of A with $\forall^{st}$ and $\exists^{st}$ replaced by $\forall$ and $\exists$.

We need only show that if A is an axiom of IST, then A° is a theorem, and that for every rule of inference with premise A_1 (or premises A_1 and A_2) and conclusion B, if A_1° is a theorem (or A_1° and A_2° are theorems), then B° is a theorem. This turns out to be quite straightforward in the main, but there is one exception. When I spoke in Aveiro I thought I could present a truly simple syntactical proof of conservativity, but I was mistaken. This remains a desirable goal. So the first part of this paper celebrates the virtue of simplicity by its absence.

The complication lies with the rule of detachment, or modus ponens. First we need a purely internal lemma.

Lemma 1 *(Cross-section) Let A be internal. Then*

$$\exists\widetilde{v}'\forall u'\exists z\forall v{\in}\widetilde{v}'(u)\,A(u,v,z) \leftrightarrow \exists\tilde{v}\forall u'\exists z\forall u{\in}u'\,A\big(u,\tilde{v}(u),z\big).$$

Proof. The backward direction is trivial: let $\widetilde{v}'(u) = \{\tilde{v}(u)\}$. To prove the forward direction, fix $\widetilde{v}'$ and let

$$\Omega = \prod_u \widetilde{v}'(u).$$

Then Ω is the set of all cross-sections of $\widetilde{v}'$. Each $\widetilde{v}'(u)$ is a finite set; give it the discrete topology, so it is compact. Give Ω the product topology, so it is compact by Tychonov's theorem.

By hypothesis, for each u' there exists an element $\tilde{v}_{u'}$ of Ω such that we have $\exists z \forall u{\in}u' A\big(u, \tilde{v}(u), z\big)$ (let $\tilde{v}_{u'}$ be arbitrary outside u'). The u' are a directed set under inclusion, so $u' \mapsto \tilde{v}_{u'}$ is a net in Ω. Since Ω is compact, this net has a limit point $\tilde{v}$, which has the desired property. □

Corollary 2 *(Dual form of cross-section) Again let A be internal. Then*

$$\forall \widetilde{v'} \exists u' \forall z \exists v{\in}\widetilde{v'}(u)\, A(u, v, z) \leftrightarrow \forall \tilde{v} \exists u' \forall z \exists u{\in}u'\, A\big(u, \tilde{v}(u), z\big).$$

Theorem 3 *(Detachment) If* A° *and* $(A \to B)^\circ$ *are theorems, so is* B°.

Proof. Let y be the free variables common to A and B, let w be the remaining free variables of A, and let z be the remaining free variables of B. We shall derive a contradiction from A°, $(A \to B)^\circ$, and $\neg(B^\circ)$. These formulas are

(1) $\forall u_0 \exists v_0' \forall w_0 y_0 \exists v_0{\in}v_0'\, A^\bullet(u_0, v_0, w_0, y_0)$

(2) $\forall \tilde{v}_1 r_1 \exists u_1' s_1' \forall w_1 y_1 z_1 \exists u_1{\in}u_1' s_1{\in}s_1'\, [\neg A^\bullet\big(u_1, \tilde{v}_1(u_1), w_1, y1\big) \vee$
$B^\bullet(r_1, s_1, y_1, z_1)]$

(3) $\exists r_2 \forall s_2' \exists y_2 z_2 \forall s_2{\in}s_2'\, \neg B^\bullet(r_2, s_2, y_2, z_2)$.

Fix r_2 and let $r_1 = r_2$. (That is, delete $\exists r_2$ in (3), replace the variables r_2 by constants also denoted by r_2, delete $\forall r_1$ in (2) and replace each occurrence of r_1 by r_2.) Now apply choice to (1) to pull out v_0' as an existentially quantified function $\widetilde{v_0'}$ of u_0. Notice that (2) has the form of the right hand side of the dual form of the cross-section lemma, so replace it by the left hand side. In this way we obtain

(1′) $\exists \widetilde{v_0'} \forall u_0 z_0 \exists v_0{\in}\widetilde{v_0'}(u_0)\, A^\bullet(u_0, v_0, w_0, y_0)$

(2′) $\forall \widetilde{v_1'} \exists u_1' s_1' \forall z_1 \exists u_i{\in}u_1' s_1{\in}s_1' \forall v_1{\in}\widetilde{v_1'}(u_1, s_1)\, [\neg A^\bullet(u_1, v_1, w_1, y_1) \vee$
$B^\bullet(r_2, s_1, y_1, z_1)]$

(3′) $\forall s_2' \exists z_2 \forall s_2{\in}s_2'\, \neg B^\bullet(r_2, s_2, y_2, z_2)$.

Fix $\widetilde{v_0'}$; let $\widetilde{v_1'}$ be defined by $\widetilde{v_1'}(u, s) = \widetilde{v_0'}(u)$ for all u and s; fix u_1' and s_1'; let $s_2' = s_1'$; fix y_2 and z_2; let $y_1 = y_2$ and $z_1 = z_2$, and let w_1 be arbitrary; let $w_0 = w_1$ and $z_0 = z_2$; fix u_1 and s_1; let $u_0 = u_1$ and $s_2 = s_1$; fix v_0; let $v_1 = v_0$. Then we have

(1″) $A^\bullet(u_1, v_0, w_1, y_2)$

(2″) $\neg A^\bullet(u_1, v_0, w_1, y_2) \vee B^\bullet(r_2, s_1, y_2, z_2)$

(3″) $\neg B^\bullet(r_2, s_1, y_2, z_2)$,

which is a contradiction. □

I have sketched the main step in a syntactical proof of the conservativity of IST over ZFC. But a better argument is needed, one that gives a practical method for converting external proofs into internal proofs. This should be possible. Whenever one uses an ideal object, such as an infinitesimal or a finite set of unlimited cardinal, it depends on the free variables in only a finite way. I expect it to be possible to develop a syntactical procedure that examines the external proof and establishes this dependence in an internal fashion.

Part II. General

Much of mathematics is intrinsically complex, and there is delight to be found in mastering complexity. But there can also be an extrinsic complexity arising from unnecessarily complicated ways of expressing intuitive mathematical ideas. Heretofore nonstandard analysis has been used primarily to simplify proofs of *theorems*. But it can also be used to simplify *theories*. There are several reasons for doing this. First and foremost is the aesthetic impulse, to create beauty. Second and very important is our obligation to the larger scientific community, to make our theories more accessible to those who need to use them. To simplify theories we need to have the courage to leave results in simple, external form—fully to embrace nonstandard analysis as a new paradigm for mathematics.

Much can be done with what may be called *minimal nonstandard analysis*. Introduce a new predicate symbol *standard* applying *only to natural numbers*, with the axioms:

(1) 0 is standard,
(2) if n is standard then $n + 1$ is standard,
(3) there exists a nonstandard number,
(4) if $\mathrm{A}(0)$ and if for all standard n whenever $\mathrm{A}(n)$ then $\mathrm{A}(n + 1)$, then for all standard n, $\mathrm{A}(n)$.

A prime example of unnecessary complication in mathematics is, in my opinion, Kolmogorov's foundational work on probability expressed in terms of Cantor's set theory and Lebesgue's measure theory. A beautiful treatise using these methods is [1], but some probabilists find the alternate treatment in [3] more transparent. Please do not misunderstand what I am saying; these remarks are not polemical. Simplicity is not the only virtue in mathematics and I wish in no way to discount other approaches to the use of nonstandard analysis in probability. I just want to encourage a few others to explore the possibility of using minimal nonstandard analysis in probability theory, functional analysis, differential geometry, or whatever field engages your passion.

In this spirit I shall give a few examples from [3]. A *finite probability space* is a finite set Ω and a strictly positive function pr on Ω such that

$$\sum_{\omega\in\Omega} \mathrm{pr}(\omega) = 1.$$

(The set Ω is finite but we do not require its cardinal to be standard.) An *event* is a subset M of Ω, and its *probability* is

$$\Pr(M) = \sum_{\omega\in M} \mathrm{pr}(\omega).$$

A *random variable* is a function $x : \Omega \to \mathbb{R}$, and its *expectation* is

$$\mathbf{E}x = \sum_{\omega\in\Omega} x(\omega)\mathrm{pr}(\omega).$$

If $a \in \mathbb{R}$, we define

$$x^{(a)}(\omega) = \begin{cases} x(\omega), & |x(\omega)| \le a \\ 0, & \textit{otherwise.} \end{cases}$$

A random variable x *is* L^1 in case

$$\mathbf{E}|x - x^{(a)}| \simeq 0 \text{ for all } a \simeq \infty.$$

Theorem 4 *(Radon-Nikodym) A random variable x is L^1 if and only if we have $\mathbf{E}|x| \ll \infty$ and for all events M with $\Pr(M) \simeq 0$ we have $\mathbf{E}|x|\chi_M \simeq 0$.*

Proof. Suppose that x is L^1. We have $\mathbf{E}|x - x^{(a)}| \le 1$ for all $a \simeq \infty$, so by overspill this is true for some $a \ll \infty$. Then $\mathbf{E}|x| \le \mathbf{E}|x - x^{(a)}| + \mathbf{E}|x^{(a)}| \le 1 + a \ll \infty$. Now let $\Pr(M) \simeq 0$. Let $a \simeq \infty$ be such that $a\Pr(M) \simeq 0$—for example, let $a = 1/\sqrt{\Pr(M)}$. Then

$$\mathbf{E}|x|\chi_M \le \mathbf{E}|x^{(a)}|\chi_M + \mathbf{E}|x - x^{(a)}|\chi_M \le a\Pr(M) + \mathbf{E}|x - x^{(a)}| \simeq 0.$$

Conversely, suppose that $\mathbf{E}|x| \ll \infty$ and that for all M with $\Pr(M) \simeq 0$ we have $\mathbf{E}|x|\chi_M \simeq 0$. Let $a \simeq \infty$ and let $M = \{|x| > a\}$. Then we have $\Pr(M) \le \mathbf{E}|x|/a \simeq 0$, so that $\mathbf{E}|x|\chi_M \simeq 0$; that is, $\mathbf{E}|x - x^{(a)}| \simeq 0$. □

A property holds *almost everywhere* (a.e.) in case for all $\varepsilon \gg 0$ there in an event N with $\Pr(N) \le \varepsilon$ such that the property holds everywhere except possibly on N.

Theorem 5 *(Lebesgue) If x and y are L^1 and $x \simeq y$ a.e., then $\mathbf{E}x \simeq \mathbf{E}y$.*

Proof. Let $z = x - y$. Then $z \simeq 0$ a.e. For all $\lambda \gg 0$ we have $\Pr(\{|z| \geq \lambda\}) \leq \lambda$, so by overspill this holds for some infinitesimal λ. But then

$$|z| \leq |z|\chi_{\{|z| \geq \lambda\}} + \lambda$$

and since z is L^1, $\mathbf{E}|z| \simeq 0$ by the previous theorem. Hence $\mathbf{E}x \simeq \mathbf{E}y$. □

One final example, useful in probability theory but more general. Let I be a finite subset of $[0, 1]$ of the form

$$0 = t_0 < t_1 \cdots < t_{\nu-1} < t_\nu = 1$$

such that $t_\mu \simeq t_{\mu+1}$ for all $0 \leq \mu < \nu$. To the naked eye, I looks just like $[0, 1]$. Although I is finite, it is "uncountable" in the following sense:

Theorem 6 *(Cantor) For any sequence $x : \mathbb{N} \to I$ there exists $t \in I$ such that t is not infinitely close to any x_n with n standard.*

Proof. Construct t_0 by changing the nth decimal digit of x_n, so that $|t_0 - x_n| \geq 10^{-n}$ for all n. Let t be the greatest element of I that is less than t_0; then t is in I and has the desired property. □

References

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[2] Edward Nelson, "Internal set theory: A new approach to nonstandard analysis", Bull. Amer. Math. Soc., **83** (1987) 1165–1198.

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3

Analysis of various practices of referring in classical or non standard mathematics

Yves Péraire*

3.1 Introduction

The thesis underlying this text is that the various approaches of mathematics, both the conventional or the diverse non standard approaches, pure or applied, are characterized primarily by their mode of referring and in particular by the more or less important use of the *reconstructed* reference, the reference to the sets and collections, that I will distinguish from the *direct* reference, the reference to the world of the facts in a broad sense. The direct reference, in traditional mathematics as well as in non standard mathematics (for the main part) is ritually performed in the classical form of modelling, consisting in confronting the facts to a small paradise (a set) correctly structured. So the discourse on the model acts like a metaphor of the modelized reality.

I will defend another approach, the relative approach, which consists in using the mathematical languages like genuine languages of communication, referring directly to the facts, but accepting the usage of the metaphor of sets (revelead as such) too strongly culturally established. My thesis will be illustrated by the presentation of two articles

1. A mathematical framework for Dirac's calculus [14].

2. Heaviside calculus with no Laplace transform [15].

The first one starts with a semantical analysis of Dirac's article introducing the Delta function. Dirac said: "δ is improper, δ' is more improper". If we give it the meaning: "δ is not completely known, and δ' is less known", it works. So it is necessary to translate it in Relative Mathematics' language. We know that the *non relative* attitude consists in the formal affirmation that a model of the delta function is perfectly determined in a paradise, a space of generalized

*Département de Mathématiques, Université Blaise Pascal (Clermont II).
Yves.PERAIRE@math.univ-bpclermont.fr

functions. The continuation of the metaphor requires to provide the paradise with a topology. In the relative approach, we prefer to explore more deeply the concepts of a point, infinity, equality with *words* of relative mathematics. Finally the opposition

NON STANDARD/STANDARD

is replaced by the antagonism

RELATIVE/NON RELATIVE.

3.2 Généralités sur la référentiation

Cette conférence présente sous un autre angle, en les précisant, certaines des idées exposées au congrès PILM 2002 et qui paraîtront dans [13]. Les considérations générales concernant la sémantique des langues mathématiques, qui sous-tendent mon exposé, ne sont pas nouvelles. Toutefois, il est nécessaire de les réactualiser pour tenir compte des pratiques spécifiques des mathématiciens non standard.

On a dit quelquefois que la démarcation entre les diverses pratiques des mathématiques non standard, se ferait à partir du matériel linguistique mis en oeuvre. En gros, les différentes écoles non standard parleraient des même choses mais utiliseraient des dialectes différents.

Nous savons bien pourtant que le langage des mathématiques, classiques ou non standard, est contenu dans le langage du premier ordre du calcul des prédicats. Il est vrai cependant que les différentes écoles non standard n'utilisent pas la même partie de ce langage ; le seul prédicat binaire « $\cdot \in \cdot$ » pour les classiques, le prédicat « $\cdot \in \cdot$ » et le prédicat unaire « $\mathbf{st}\ \cdot$ » pour l'école Nelson-Reeb, j'utilise personnellement le prédicat « $\cdot \in \cdot$ » et un autre prédicat binaire, le prédicat de Wallet « $\cdot\, \mathcal{SR}\, \cdot$ ».

En réalité, ces différents choix induisent (autant qu'ils sont induits par) des pratiques sémantiques différenciées et c'est l'analyse de ce rapport au sens qui permet de comprendre la nature des différentes écoles mathématiques, standard ou non standard qu'elles se veuillent appliquées ou se prétendent pures.

Je distinguerai schématiquement trois mode d'attribution du sens, que je désignerai par les termes

- référence directe,
- référence reconstruite ou
- référence directe élargie.

Ce que j'appelle référence directe, c'est la référence aux faits au sens large, ne se limitant pas à la description des phénomènes, s'autorisant aussi la description des concepts. C'est en gros le mode de référence des physiciens.

La référence reconstruite, c'est la référence aux entités mentales stables engendrées par le discours *en langue mathématique* : les ensembles introduits formellement par le quantificateur $\exists$, les collections.

On pourrait dire que le référent des mathématiques pures consiste essentiellement en ces entités, éjectées à partir des axiomes de la théorie des ensembles. Dans les faits, les choses sont beaucoup plus mélangées. En effet, même si elle n'est pas mise en avant, la référence au monde des "faits élargis" aux concepts naturels, concept d'ensemble, concept naturel de nombre, à une certaine idée du fini et de l'infini, n'est pas absente de la pensée des mathématiciens purs. La pratique des mathématiques appliqués constitue un pas vers le 3ème mode d'attribution du sens ; la modélisation en est la forme la plus classique. Le référent est bien le mode des faits, mais le mode d'expression est systématiquement celui de la métaphore, prise dans l'univers reconstruit.

Les lignes précédentes illustrent ce que j'entend par référent : « ce à quoi le signe linguistique renvoie dans la "réalité extra-linguistique" soit l'univers "réel", soit l'univers imaginaire ». Mais de quels signes parlons-nous ? A quelle langue appartiennent-ils ? Il est tout à fait clair que c'est de la langue de communication, l'anglais ou le français, dont je parle et non pas des languages bien construites, de ZF de IST de RIST ou autre. En réalité ces langues sont utilisées comme matrices pour *produire* des entités structurées et également comme une écriture abrégée de la langue naturelle, une sorte de sténographie.

La pratique que je vais essayer de décrire maintenant, que j'appelle référence directe élargie, prend acte des pratiques sémantiques d'une partie du réseau Georges Reeb, les rend explicites et ouvre la voie à une généralisation. Ce qui caractérise cette approche, ce n'est pas la double référence — au monde des faits d'une part, au monde reconstruit de l'autre — qui est une pratique courante des mathématiques appliquées, mais le fait que ce processus de désignation du sens concerne non plus seulement la langue vernaculaire, mais aussi *la langue mathématique*, que l'on se met a pratiquer alors, comme le français ou l'anglais, en alternant style direct et style métaphorique en quantité variable selon ses préférences. Voici deux exemples de pratiques sémantiques utilisant une métaphore.

Considérons l'énoncé en français : « grand père est mort ». Il possède clairement une référence dans le monde des faits. On peut transcrire cet énoncé par un autre, métaphorique, « grand-père est au paradis ».

Admettons maintenant que ces deux énoncés disent la même chose et que l'on veuille à partir de l'un ou l'autre explorer plus finement le concept de la mort. On peut le faire de plusieurs manières. La méthode la plus directe consiste à utiliser le mot « mort » et à préciser encore le concept par l'introduction d'autre mots. On peut aussi tenter de « pousser la métaphore » un peu plus loin et proposer une description du paradis. Le second exemple provient des mathématiques de la physique.

Le point de départ est l'article de Paul Dirac dans [2], dans lequel est introduite la fonction δ. Si on ne regarde que les formules et les calculs qui figurent dans cet article, alors la définition donnée conduit sans nul doute à une contra-

diction. Il reste à expliquer pourquoi, malgré cela, ils ne conduisent pas à des absurdité physiques et semblent même avoir une certaine puissance explicative.

Le problème de la contradiction tel qu'il a été résolu par Laurent Schwartz [19] a consisté à parler de la fonction de Dirac, qui relie des grandeurs physiques, en termes d'un élément d'un paradis réputé sûr, l'espace des distributions. C'est une métaphore similaire à celle utilisée précédemment pour parler de la mort de grand-père, en effet dans les deux cas

(a) On veut rendre compte d'une relation entrée/sortie : vie/mort d'un côté, [valeur 0]/[valeur 1] de l'autre. Il y a une brève phase de transition de l'état initial à l'état final, que l'on ne peut pas décrire. On pense que l'état des choses pendant cette phase doit expliquer le changement d'état.
(b) Dans chaque exemple *l'indétermination* factuelle de la phase de transition est exprimée par une *détermination* dans un univers mythique, qui prétend expliquer le phénomène. Il faut remarquer toutefois que le mythe utilisé dans la démarche mathématique est d'une bien plus grande cohérence, peut-être parce que la langue mathématique, qui l'a engendré, est exempte d'ambiguïté.
(c) Il y a dans les deux cas une tentation de pousser trop loin la métaphore.

La métaphore de l'ensemble des distributions fonctionne assez bien, elle permet de faire correspondre à certaines formules posées par Dirac des transcriptions acceptables et utilisables mais on sait que certains faits ne sont pas pris en compte par ce modèle, la multiplication n'est pas permise, l'égalité au sens des distributions s'éloigne assez de la réalité de l'égalité physique... D'autres espaces abstraits, fonctions généralisées de Colombeau, ultrapuissances saturées... permettent de tenir un discours formellement cohérent sur la multiplication des fonctions de Dirac.

Cependant, malgré la réelle efficacité mathématique de ces modélisations, on peut souhaiter une approche plus directe, dans laquelle le champ sémantique serait moins encombré par les représentations ensemblistes. Il ne s'agit pas de renoncer complètement à ce type de méthodes, mais plutôt de briser l'automatisme du recours à ces dernières. Ma position sur ce point diverge donc de celle de Gilles Gaston Granger dans [4] pour qui ce qu'il appelle la *sortie de l'irrationnel* ne peut se faire *que par l'introduction* d'un espace fonctionnel. J'ai choisi une approche plus directe du même problème ; elle commence par une analyse sémantique du texte de Paul Dirac. Cette analyse permet de découvrir plusieurs faits, qu'il faut introduire dans la description en langue mathématique. En voici quelques uns.

- le point matériel à une épaisseur non nulle, physiquement infinitésimale,
- la fonction δ est partiellement indéterminée dans le point,

– l'égalité dans les formules de Dirac n' est pas l'égalité classique des ensembles.

Notre travail consistera donc à introduire, dans une langue bien construite, les mots pour exprimer ces faits (par une sorte de traduction de la langue naturelle). En ce qui concerne l'égalité de Dirac, l'analyse du texte fait apparaître que Dirac identifie à δ toute fonction nulle en dehors du point origine et d'intégrale égale à 1, je donnerais donc une définition de l'égalité qui intègre ce fait. Jusqu'à présent, le langage de la théorie relative des ensembles (voir [12]) m'a suffi pour fabriquer tout le lexique nécessaire.

Je propose d'appeller MATHÉMATIQUES RELATIVES la pratique que je viens de décrire. En conséquence les mathématiques classiques devront être qualifiées, pour une large part, de non relatives. En guise d'illustration je vais maintenant présenter les résultats et la philosophie de deux articles concernant les mathématiques de la physique.

1. A mathematical framework for Dirac's calculus [14].
2. Heaviside calculus with no Laplace transform [15].

Une partie importante de chacun de ces articles consiste à mettre en place une définition ad hoc de l'égalité.

3.3 Le calcul de Dirac. L'égalité de Dirac

Une description précise des notions évoquées plus bas, niveaux d'impropriété, dérivées observées, égalité de Dirac est disponible dans [14], chapitres 1 et 2 ainsi que de nombreux exemples de couples de fonctions Dirac-égales. Dans la discussion qui va suivre nous utiliserons des définitions incomplètes pour ne pas cacher l'essentiel, qui est la philosophie de ce travail. En particulier la définition de l'égalité de Dirac, Définition 4 de la section 2.4, est plus complexe.

Comment justifier les calculs de Dirac ? On a compris qu'il fallait reconnaître un sens physique à la notion de point, éclairer aussi la signification du signe « = » et exprimer cela en langue mathématique. L'étape initiale nécessaire pour y parvenir consiste à introduire des prédicats pour *dire l'indétermination.* Cette expression de l'indétermination est cachée dans les formulations de Dirac. En effet Considérons l'énoncé tiré de [2] :

> « *Strictly of course,* $\delta(x)$ *is not a proper function of* x,... $\delta'(x)$, $\delta''(x)$ ... *are even more discontinuous and less proper than* $\delta(x)$ *itself.* »

J'ignore quelle signification précise Dirac donnait au mot « improper », cependant si on l' interprète, en forçant un peu le sens, par « partiellement

indéterminé », la transcription fonctionne. Par contre, la traduction par le mot « anormal » qui a été donnée dans [3], peut provoquer un bloquage.

Voici rapidement comment, dans l'article [14], j'ai exprimé les choses.

1. J'ai utilisé le prédicat de standardicité relative $\cdot SR\cdot$ pour définir les niveaux d'impropriété. On pourra trouver dans [14], première section, la définition précise des niveaux d'impropriété. Rapidement, nous dirons que plus un objet est impropre moins il est standard.
2. A tout nombre p j'associe le point analysé $\wr p \wr$, c'est un halo dont le degré d'infinitésimalité est d'autant plus petit que p est impropre.
3. Je définis la *fonction de Dirac principale* δ de la manière suivante :
   (a) Je fixe un infinitésimal h_1 strictement positif et impropre,
   (b) je pose $\delta = \dfrac{1}{2h_1}\mathrm{Ind}_{[-h_1,h_1]}$.

   On obtient pour δ le graphe suivant, que l'on peut trouver aussi dans les livres de physique. J'ai représenté sur le même graphique la *fonction échelon de Heaviside*, elle vaut 1 pour les valeurs positives de la variable et 0 partout ailleurs.

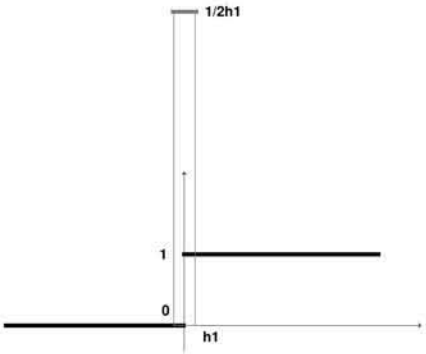

Figure 3.1: La fonction de Heaviside et sa dérivée observée, la fonction de Dirac

   Le réflexe de tout mathématicien sera d'observer que ma définition de δ n'est pas indépendante de la valeur de h_1. La réponse à cette objection nécessite une meilleure investigation de la notion d'égalité.
4. Je définis la Dirac-égalité, « $\overset{D}{=}$ » sur un ensemble de fonctions $\mathcal{C}^\infty$ par morceaux, standard ou pas, de telle sorte que toute fonction ayant toutes ses dérivées infinitésimales pour les valeurs appréciables et d'intégrale infiniment proches de 1 sur tout intervalle $[x, y]$ contenant 0 et ayant des bornes appréciables, soit Dirac-égale à la fonction de Dirac principale.

5. J'ai défini la dérivée observée d'une fonction, standard ou pas, *pouvant présenter des discontinuités* du premier ordre par une formule de la forme

$$f^{]'[}(x) = \frac{f(x+h) - f(x-h)}{2h}$$

qui, à première vue, dépend de h, ou h est infiniment petit d'un ordre de petitesse adapté au niveau d'impropriété de f. Après avoir posé cette définition,

il est préférable de résister à la tentation de chercher une limite quelque-part quand h tend vers 0.

On repousse ainsi l'intrusion dans le champ sémantique d'un objet encombrant. La dérivée observée de la fonction de Heaviside est précisément notre fonction de Dirac principale. On montre facilement que la valeur de $f^{]'[}$ est *Dirac-indépendante* de h :

deux valeurs distinctes de h donnent des valeurs Dirac-égales de la dérivée observée.

Cela répond à l'objection évoquée à l'item 3 et explique aussi pourquoi dans la notation de la dérivée observée je n'ai pas fait apparaître l'accroissement h. On se convaincra facilement que cette approche donne une description plus proche de la réalité physique.

Le chapitre 3 présente les propriétés de base de la dérivation observée. Parmi celles-ci nous avons :

Compatibilité avec la dérivée classique.

$$\text{Si } f \text{ est dérivable } f' \stackrel{D}{=} f^{]'[}.$$

Linéarité au sens de Dirac.

Si f et g ont le même niveau d'impropriété, alors

$$(f+g)^{]'[} \stackrel{D}{=} f^{]'[} + g^{]'[}.$$

Formule de Leibniz au sens de Dirac.

Sous la même condition sur le niveau d'impropriété, alors

$$(f \times g)^{]'[} \stackrel{D}{=} f \times g^{]'[} + f^{]'[} \times g.$$

On trouvera dans [14], section 3 théorème 4, une formulation d'écriture plus complexe mais qui s'applique aux cas ou les fonctions f et g ont des niveaux d'impropriété distincts.

On remarquera que ces égalités ne sont pas testées sur des espaces fonctionnels comme chez Schwartz ou Colombeau et que toutes les multiplications sont autorisées. Cela permet de légitimer des formules de physiques où interviennent des carrés de fonction delta, comme celle-ci dans laquelle δ est la fonction de Dirac principale :

$$\frac{(-1)^{p+1}}{(p+1)!}\, x_+^{p+1}\, \delta^{]p+2[} + \delta^2 \overset{D}{=} \frac{p+2}{2}\, \delta^{]'[}.$$

Il existe une formule analogue dans [1] dans laquelle δ est représentée dans l'espace des fonctions généralisées de Colombeau et où l'égalité est une égalité faible. La formule doit être modifiée si on utilise une autre fonction de Dirac que la fonction de Dirac principale.

Remarques

J'ai qualifié d'égalité la relation $\overset{D}{=}$. On peut préférer voir en cette relation une indiscernabilité ou une approximation de "*quelque chose quelque part*". Je prétends pourtant que la relation de Dirac mérite, autant que l'égalité classique des ensembles, le nom d'égalité. En effet que dit la formule précédente ? En dehors du point origine, supposé d'épaisseur radicalement indétectable, et pour toute valeur accessible de la variable, les fonctions de part et d'autre du signe $\overset{D}{=}$ ainsi que leur dérivées, à tout ordre accessible, prennent des valeurs indistinguables à 10^{-n} près quelle que soit la valeur de n à laquelle on puisse accéder. Les intégrales itérées à tout ordre matériellement possible, avec des bornes d'intégrations de part et d'autre du point $\wr 0 \wr$ sont également indistinguables. On peut dire que deux fonctions Dirac-égales représentent des phénomènes physiquement identiques. En revanche, si on tourne le regard vers le monde des entités, la relation $\overset{D}{=}$ n'est pas une égalité, c'est une relation d'équivalence ; mais on sait bien que ce qui est présenté comme égalité, dans le monde reconstruit, n'est qu'un avatar de l'équivalence logique.

On remarquera que je n'ai pas hésité, dans ma description du calcul de Dirac, à faire usage de l'ensemble des nombres réels, pour parler des grandeurs physiques. Je n'ai commencé à renoncer à la métaphore ensembliste qu'à partir du moment où elle m'a semblé gênante. Kinoshita dans [6] et Grenier dans [5] ont préféré pour leur part utiliser un "*continuum discret*".

Chacun est libre d'utiliser la quantité et le type de représentations ensemblistes qui lui plait, chacun a droit à son style propre.

On remarquera que les formules précédentes parlent *en premier lieu* de faits concernant des grandeurs physiques, pas des ensembles. Il vaut mieux éviter, bien que cela reste possible, et même souhaitable, pour d'autres travaux, d'attribuer un statut ontologique dans la "réalité mathématique" aux

infinitésimaux de différents niveaux dont on fait usage, cela risque de devenir vite désagréable. L'enrichissement du vocabulaire que nous avons réalisé *ici* avait pour but de rendre plus précis le discours en langue mathématique... mais si on tournait les yeux vers les ensembles on pourrait y voir un monde d'entités de plus en plus idéales, un paradis étrange d'idéalités stratifiées...

La fonction de Dirac principale est celle que l'on obtient, prenant au sérieux notre ignorance ce qu'il se passe dans le point $\wr 0 \wr$, quand on applique la définition de la dérivée observée à la fonction échelon de Heaviside.

Dans la section suivante, j'utiliserai des fonctions de Dirac à droite de zéro et de classe $\mathcal{C}^\infty$. Cela peut sembler beaucoup de précisions, pour une fonction mal connue à l'origine. Cela peut se justifier de la manière suivante : dans l'article que je vais présenter le but n'est pas de *décrire* une partie de la réalité physique mais de donner une plus ample "*justification*" pour le concept de Heaviside d'opérateur de dérivation, ayant pour inverse un opérateur d'intégration.

3.4 Calcul de Heaviside sans transformée de Laplace. L'égalité de Laplace

Comme pour la section précédente, j'inviterai le lecteur à se reporter à un article complet, l'article [15]. C'est la philosophique sous-jacente à cet article que je veux présenter ici, je devrai donc passer sous silence les démonstrations, et même donner des définitions incomplètes.

Le calcul opérationnel, quand on l'applique formellement, sans vérifier l'existence d'une transformation de Laplace, donne les bonnes solutions. Ces solutions elles même n'ont le plus souvent pas d'intégrale de Laplace, du moins dans les cas que j'ai pu traiter. D'autre part on a le sentiment que ces méthodes ne devraient pas dépendre de la convergence de l'intégrale de Laplace. Aussi ai-je tenté (avec succès) de mettre en place une notion d'*image généralisée* qui opère même quand les fonctions en jeu ne sont pas Laplace-transformables. Bien sur, ce n'est pas la première tentative dans ce sens. Après le premier travail de Vigneaux publié en 1929 dans [20] d'autres auteurs dont récemment, Komatsu dans [7], G.Lumer et F. Neubrander dans [8, 9] se sont attaqués à ce problème par des méthodes classiques.

L'approche relative permet d'obtenir une solution complète et beaucoup plus directe. Les résultats obtenus sont les suivants.

1. Les difficultés dues à la divergence de l'intégrale de Laplace disparaissent.
2. Les dérivations par rapport aux paramètres sont toujours permises.
3. On peut utiliser des séries divergentes comme images généralisées.
4. Les calculs peuvent faire intervenir des fonctions de Dirac, des peignes de Dirac, etc. sans qu'il soit nécessaire d'introduire des espaces fonctionnels.

5. L'application des méthodes de ce calcul de Heaviside rejustifié, aboutit à une description plus fine des solutions trouvées. En particulier, si il apparaît de l'indétermination dans une équation au dérivées partielles, à cause de la présence d'une fonction delta, alors on retrouve la trace de cette indétermination dans les solutions (Voir l'exemple 1 plus bas, à titre d'illustration).

Indiquons maintenant plus précisément le chemin suivi. Notre calcul va s'appliquer à une classe de bonnes fonctions, par exemple de classe C^∞, mais pas nécessairement intégrables.

Définition de l'égalité de Laplace

Contrairement à ma définition de l'égalité de Dirac, dont la description s'inspirait de l'observation de l'usage que faisait Dirac de l'égalité, ma définition de ce que j'appelle égalité de Laplace n'est pas justifiée par une pratique de Laplace. La définition utilise deux niveaux d'impropriétés définis au moyen du prédicat $\cdot\mathcal{SR}\cdot$. Cette classification permet de définir des ordres de grandeur relatifs dans $\mathbb{R}$.

Nous connaissons les définitions suivantes.

Un nombre x est infinitésimal et on écrit $x \sim 0$ si

$$\forall^{st}\varepsilon > 0 \quad |x| < \varepsilon. \tag{1}$$

Le nombre x est infiniment grand et on écrit $x \sim +\infty$ si

$$\forall^{st} l \quad |x| > l. \tag{1'}$$

Nous dirons maintenant qu'un nombre x est un *infinitésimal relatif*, et nous écrirons $x \approx 0$, si

$$\forall^{Imp}\varepsilon > 0 \quad |x| < \varepsilon. \tag{2}$$

et que x est un *infiniment grand relatif* si

$$\forall^{Imp} l > 0 \quad |x| > l \tag{2'}$$

ce qui s'écrira $x \approx +\infty$. ($\forall^{Imp}x\ F(x)$ est une abréviation pour $\forall x\ (Imp(x) \Rightarrow F(x))$, ce qui s'oralise « pour tout x impropre, $F(x)$ »).

On peut montrer que tout nombre relativement infinitésimal est infinitésimal. Un nombre non relativement infiniment grand est dit *relativement limité*.

On fixe ensuite une fois pour toute :
- un nombre $\bowtie$ relativement infiniment grand,
- une *fonction de Dirac relative* à droite de 0, δ.

Cela signifie que δ reste une fonction de Dirac même pour un observateur idéal capable de "voir" des fonctions de Dirac impropres. Cette fonction δ est très impropre, elle est nulle en dehors d'un intervalle $[0, h]$ avec $h \approx 0$.

J'ai défini ensuite dans [15] une collection $\mathcal{N}$ de fonctions négligeables de classe $\mathcal{C}^\infty$ de deux variables x et t. La généralisation à un nombre plus grand de variables ne pose pas de problème. *On peut même étendre nos résultats aux cas où on a un nombre infiniment grand, impropre, de variables.* Cela peut être utile si on a besoin de faire intervenir des paramètres qui sont des fonctions standard, représentées par un ensemble fini impropre contenant toutes leurs valeurs standard. La classe $\mathcal{N}$ vérifie les propriétés suivantes.

- Si $\alpha(x,t) \in \mathcal{N}$ alors $\alpha(x,t) \approx 0$ pour tout x relativement limité et tout t relativement appréciable.
- $t \cdot \mathcal{N} \subset \mathcal{N}$.

Si $\star$ désigne le produit de convolution des fonctions, alors

- $\mathcal{N} \star \mathcal{N} \subset \mathcal{N}$.
- $\underline{f} \star \mathcal{N} \subset \mathcal{N}$ pour toute bonne fonction f.
- $\dfrac{\partial^{n+m}\mathcal{N}}{\partial x^m \partial t^n} \subset \mathcal{N}$, pour tous $n \in \mathbb{N}$ et $m \in \mathbb{Z}$, limités

On remarque que dans la dernière propriété m peut être négatif. La dérivée d'ordre négatif $-k$ d'une fonction α s'obtient en calculant l'intégrale

$$\underbrace{\int_0^{t_k=t} \cdots \int_0^{t_2} \int_0^{t_1}}_{\text{On intègre } k \text{ fois}} \alpha(s)\, ds\, dt_1 \cdots dt_{k-1}.$$

Remarque. La classe $\mathcal{N}$, "n'est pas" un ensemble pas plus que $t \cdot \mathcal{N}$, $\mathcal{N} \star \mathcal{N}$, $\underline{f} \star \mathcal{N}$ ou $\frac{\partial^{n+m}\mathcal{N}}{\partial x^m \partial t^n}$. Les inclusions précédentes sont des inclusions de collections.

On définit alors l'égalité de Laplace $\overset{\mathcal{L}}{=}$ en posant

$$F \overset{\mathcal{L}}{=} G \overset{Def}{\Leftrightarrow} \exists \alpha \in \mathcal{N}\ F - G = \mathcal{L}\alpha$$

où $\mathcal{L}\alpha$ est la transformée de Laplace classique de α, qui doit donc exister.

On pose ensuite

$$(\mathbf{L}f)(p) \overset{Def}{=} \left[\int_0^{\bowtie} e^{-pt} f(t)\, dt\right] \cdot (\mathcal{L}\delta)(p).$$

La fonction $(\mathcal{L}\delta)(p)$ n'est pas égale à **1**, comme pour la distribution de Dirac. On trouve parfois dans les livres de physique, concernant les *fonctions* de Dirac, l'égalité $(\mathcal{L}\Delta)(p) = 1$. En réalité pour une fonction de Dirac Δ non standard, on a $(\mathcal{L}\delta)(p) \sim 1$ pour les valeurs limitées de p. Notre fonction δ, fixée plus haut, vérifie quant à elle

pour tout $p > 0$ relativement limité, $(\mathcal{L}\delta)(p) \approx 1$.

Démontrons ce dernier point. La deuxième formule de la moyenne donne, pour chaque $p > 0$,

$$(\mathcal{L}\delta)(p) = e^{p\theta h} \int_0^h \delta(t)\, dt, \quad \theta \in [0,1].$$

$h \approx 0$ implique $\theta h \approx 0$, $p\theta h \approx 0$ pour p relativement limité et donc, $e^{p\theta h} \approx 1$. Comme $\int_0^h \delta(t)\, dt \approx 1$, on en déduit que $(\mathcal{L}\delta)(p) \approx 1$ pour tout p relativement limité.

Notation. On écrira $f \sqsupset\!\sqsupset F$ si $F \overset{\mathcal{L}}{=} \mathbf{L}f$ et $f \sqsupset F$ si f admet F comme transformation de Laplace.

Quelques résultats obtenus

Pour toutes bonnes fonctions f et g standard ou impropre

1. $\mathbf{L}f$ est *Laplace-indépendant* de δ. Des choix distincts de la fonction δ donneraient des transformations Laplace-égales.
2. Si $f \sqsupset\!\sqsupset F$ et $g \sqsupset\!\sqsupset F$, alors $f = g$.
3. Si f est de type exponentiel, $f \sqsupset F \Rightarrow f \sqsupset\!\sqsupset F$.
   Ce qui précède implique que toutes les fonctions qui figurent dans les tables de transformations de Laplace sont des images généralisées.
4. $(\mathbf{L}f')(p) \overset{\mathcal{L}}{=} p(\mathbf{L}f)(p) - f(0+) \cdot (\mathcal{L}\delta)(p)$.
5. $\Big(\mathbf{L}\int_0^t f(s)\, ds\Big)(p) \overset{\mathcal{L}}{=} \dfrac{1}{p}(\mathbf{L}f)(p)$.
6. $\mathbf{L}(f \star g) \overset{\mathcal{L}}{=} (\mathbf{L}f) \cdot (\mathbf{L}g)$.
7. Si $f(t) = \sum_{n=0}^{+\infty} a_n t^n$ alors $\mathbf{L}F \overset{\mathcal{L}}{=} \sum_{n=0}^{\bowtie} a_n \dfrac{n!}{p^{n+1}}$, même si la séric $\sum_{n=0}^{+\infty} a_n \dfrac{n!}{p^{n+1}}$ diverge.
8. $\mathbf{L}\dfrac{\partial f}{\partial x} \overset{\mathcal{L}}{=} \dfrac{\partial \mathbf{L}f}{\partial x} \ldots$ etc.

3.5 Exemples

Exemple 1.

Recherche de la solution de l'équation aux dérivées partielles définies pour $x > 0,\ t > 0$.

$$\frac{\partial^2}{\partial x^2} u(x,t) - \frac{\partial}{\partial t} u(x,t) = e^{t^2}$$

$$u(0_+, t) = \Delta(t), \quad u(x, 0_+) = 0 \text{ pour tout } x.$$

Δ est une fonction impropre de Dirac. L'application du calcul formel généralisé donne la solution "exacte"

$$u(x,t) = \left[\Delta(t) - H(t) \star e^{t^2}\right] \star \frac{x}{2\sqrt{\pi t}} e^{\frac{-x^2}{4t}} + H(t) \star e^{t^2}.$$

Cependant LA RÉPONSE PHYSIQUE CORRECTE au problème est la suivante : Pour tous $x > 0$ et $t > 0$,

(a) Si t est appréciable la solution est indiscernable de

$$H(t) \star e^{t^2} - \left[H(t) \star e^{t^2}\right] \star \frac{x}{2\sqrt{\pi t}} e^{\frac{-x^2}{4t}} + \frac{x}{2\sqrt{\pi t}} e^{\frac{-x^2}{4t}}.$$

(b) On ne connaît pas avec précision le comportement de la solution quand t est très petit. Cette imprécision est héritée du celle de la fonction Δ.

Exemple 2.

La fonction e^{-t^2} a une transformation de Laplace et admet un développement en série de rayon infini, cependant sa transformée de Laplace n'est pas égale à la somme — qui n'est pas définie — des transformées de Laplace des termes de la série. On a cependant

$$e^{-t^2} \sqsupset \sum_{2n<\bowtie} (-1)^n \frac{(2n)!}{n! p^{n+1}}.$$

Pour la fonction e^{t^2} il n'y a pas de transformée de Laplace mais on peut écrire une image généralisée :

$$e^{t^2} \sqsupset \sum_{2n<\bowtie} \frac{(2n)!}{n! p^{n+1}}.$$

En conclusion, je dirai que l'examen systématique des questions de sémantique peut accroître considérablement l'efficacité de l'outil mathématique. Nous venons de le constater pour les mathématiques de la physique.

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4

Stratified analysis?

Karel Hrbacek*

It is now over forty years since Abraham Robinson realized that "*the concepts and methods of Mathematical Logic are capable of providing a suitable framework for the development of the Differential and Integral Calculus by means of infinitely small and infinitely large numbers*" (Robinson [29], Introduction, p. 2). The magnitude of Robinson's achievement cannot be overstated. Not only does his framework allow rigorous paraphrases of many arguments of Leibniz, Euler and other mathematicians from the classical period of calculus; it has enabled the development of entirely new, important mathematical techniques and constructs not anticipated by the classics. Researchers working with the methods of nonstandard analysis have discovered new significant results in diverse areas of pure and applied mathematics, from number theory to mathematical physics and economics.

It seems fair to say, however, that acceptance of "nonstandard" methods by the larger mathematical community lags far behind their successes. In particular, the oft-expressed hope that infinitesimals would now replace the notorious ε-δ method in teaching calculus remains unrealized, in spite of notable efforts by Keisler [20], Stroyan [31], Benci and Di Nasso [4], and others. Sociological reasons — the inherent conservativity of the mathematical community, the lack of a concentrated effort at proselytizing — are often mentioned as an explanation. There is also the fact that "nonstandard" methods, at least in the form in which they are usually presented, require heavier reliance on formal logic than is customary in mathematics at large. While acknowledging much truth to all of the above, here I shall concentrate on another contributing difficulty. At the risk of an overstatement, it is this: while it is undoubtedly possible to do calculus by means of infinitesimals in the Robinsonian framework, it does *not* seem possible to do calculus *only* by means of infinitesimals in it. In particular, the promise to replace the ε-δ method by the use of infinitesimals cannot be carried out in full.

*Department of Mathematics, The City College of CUNY, New York, NY 10031.
khrbacek@ccny.cuny.edu

In Section 4.1 I examine this shortcoming in detail, review earlier relevant work, and propose a general plan for extending the Robinsonian framework with the goal of remedying this problem and — possibly — diminishing the need for formal logic as well. Section 4.2 contains a few examples intended to illustrate how mathematical arguments can be conducted in this extended framework. Section 4.3 presents an axiomatic system in which the techniques of Section 4.2 can be formalized, and discusses the motivation and prospects for its further extension.

4.1 The Robinsonian framework

Here and in the rest of the paper, by *Robinsonian framework* I mean any presentation of "nonstandard" methods that postulates a fixed hierarchy of standard, internal and, in most cases, also external sets. Thus the original type-theoretic foundations of Robinson [29], the superstructure method of Robinson and Zakon [30] (also Chang and Keisler [6]), and direct use of ultrafilters à la Luxemburg [21], as well as axiomatic nonstandard set theories like **HST** [13, 18, 19] or Nelson's **IST** [23], are covered by the term, and the discussion in this section applies to all of them. I present the arguments in the "internal picture" employed in **IST** and **HST**; that is, for example, $\mathbf{R}$ denotes the set of all (internal) real numbers, and is referred to as *the standard set of reals*; if needed, ${}^{\circ}\mathbf{R}$ denotes the external set of all standard reals. Superstructure afficionados would use ${}^{*}\mathbf{R}$ for $\mathbf{R}$ and $\mathbf{R}$ for ${}^{\circ}\mathbf{R}$. The same conventions apply to the standard set of natural numbers $\mathbf{N}$ and other standard sets.

The paradigmatic example below, the familiar *nonstandard definition of continuity*, illustrates the difficulty I am concerned about.

Definition 4.1.1 *Let $f : \mathbf{R} \to \mathbf{R}$ be a standard function, and $x \in \mathbf{R}$ a standard real number.*

(i) f is continuous *at x iff for all infinitesimal h, $f(x+h) - f(x)$ is infinitesimal.*

(ii) f is (pointwise) continuous *iff for all standard $x \in \mathbf{R}$, f is continuous at x. Explicitly: $(\forall^{\mathrm{st}} x \in \mathbf{R})(\forall$ infinitesimal $h)$ $(f(x+h)-f(x)$ is infinitesimal$)$.*

It is a basic and useful fact of nonstandard analysis that the notion of continuity of a standard function at a standard point defined above can be extended in a natural way to the notion of continuity at a nonstandard point, and that a standard continuous function $f : \mathbf{R} \to \mathbf{R}$ is continuous at *all* $x \in \mathbf{R}$, even the nonstandard ones. But, what precisely does this mean in the Robinsonian framework, and how do we know that it is true? Certainly not by transfer! In the Robinsonian framework, for a statement about standard

objects to be transferable from the standard to the internal universe, all of its quantifiers have to range over standard sets; formally, it has to be of the form φ^{st} where φ is an $\in$-formula (internal formula). Definition 4.1.1 is not of this form; the quantifier ($\forall$ infinitesimal h) ranges over internal sets. Briefly, **Definition 4.1.1 is not transferable.** A naive attempt to transfer 4.1.1(ii) will likely produce something along the lines of

$$(\forall x \in \mathbf{R})(\forall \text{ infinitesimal } h)\ (f(x+h) - f(x) \text{ is infinitesimal }).$$

As is well known, this statement is equivalent to *uniform* continuity of f (for standard f).

How then do we arrive at our "basic and useful fact"? Every treatment of elementary nonstandard analysis has to answer this question somehow. Moreover, similar difficulties appear with derivatives, integrals —in fact, with every concept defined by nonstandard methods. There seems to be little explicit attention paid to this issue in the literature. Important exceptions are the writings of Péraire [25]-[27], Gordon [11, 12], and Andreev's thesis [1]; discussions of a number of points considered in this paper can be found there. I realized the crucial importance of this issue for teaching of nonstandard analysis during O'Donovan's talk in Aveiro. While describing his experiences with the nonstandard definition of derivative, O'Donovan recounted some questions his students typically ask: "Can we use this formula when x is not standard? When f is not standard?" The answer of course is **NO** —but what then are they supposed to use? After all, a standard function like $\sin x$ does have a derivative at all x![1]

Three implicit responses applicable in Robinsonian framework can be discerned.

Response I (Robinson [29], Goldblatt [10]).

Although Definition 4.1.1 is not expressible by an $\in$-formula, it is *equivalent* to an $\in$-formula, namely, to the *standard definition of continuity.*

Definition 4.1.2 *Let $f : \mathbf{R} \to \mathbf{R}$ be a standard function, and $x \in \mathbf{R}$ standard. f is* continuous at x *iff* $(\forall^{\mathrm{st}}\varepsilon > 0)(\exists^{\mathrm{st}}\delta > 0)(\forall^{\mathrm{st}}y \in \mathbf{R})(\ |y - x| < \delta \Rightarrow |f(y) - f(x)| < \varepsilon)$.

The formula on the right side of Definition 4.1.2 *is* transferable, and yields a natural notion of continuity for all $f : \mathbf{R} \to \mathbf{R}$ and all $x \in \mathbf{R}$ that agrees with Definition 4.1.1 for standard f and x.

[1] I am grateful to R. O'Donovan for many subsequent email exchanges that have been extremely helpful in further clarification of the difficulties with using infinitesimals to teach calculus.

Definition 4.1.3 *Let $f : \mathbf{R} \to \mathbf{R}$ be any (internal) function, and $x \in \mathbf{R}$ any real. f is* continuous at x *iff*

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall y \in \mathbf{R})(|y - x| < \delta \Rightarrow |f(y) - f(x)| < \varepsilon).$$

The problem is resolved, but at the cost of a relapse to the usual ε-δ definition of continuity, at the internal level. This response also illustrates one of the chief reasons why nonstandard methods have to rely so heavily on formal logic. In Definitions 4.1.1(i) and 4.1.2 we have two equivalent formulas, of which one is transferable and the other is not. Transferability is an attribute of formulas; it is a logical, metamathematical concept.

Response II (Nelson [23]).

First, we use Definition 4.1.1(i) to define continuity for standard f, x. Then we let

$$C := {}^s\{\langle f, x\rangle : \ f : \mathbf{R} \to \mathbf{R},\ x \in \mathbf{R},\ f, x \text{ standard, } f \text{ continuous at } x\}$$

(C is a standard set), and finally, for any (internal) f and x, define

Definition 4.1.4 *f is* continuous at x *iff* $\langle f, x\rangle \in C$.

If f is standard and continuous at all standard $x \in \mathbf{R}$ then $(\forall^{\mathrm{st}} x)(\langle f, x\rangle \in C)$, this formula is transferable, and gives $(\forall x)(\langle f, x\rangle \in C)$, i.e., f is continuous at all $x \in \mathbf{R}$, as desired.

The problem here is that this ("somewhat implicit" [23]) definition of continuity is completely divorced from the usual intuition (captured for standard f, x, in different ways, both by the standard ε-δ definition and by the nonstandard definition using infinitesimals): a function f is continuous at x if arguments "near" x yield values "near" $f(x)$. According to 4.1.4, the implicit meaning of the statement "f is continuous at x" for standard f and unlimited x is "$x \in {}^s\{y \in \mathbf{R} : y \text{ standard, } f \text{ continuous at } y\}$"; it is not related to the behavior of f near x in the sense of the order topology on $\mathbf{R}$. Similarly, continuity of $f(x) := x^\nu$, where $\nu \in \mathbf{N}$ is unlimited, translates to "$\nu \in {}^s\{n \in \mathbf{N} : f_n(x) := x^n \text{ is continuous }\}$"; i.e., x^ν is continuous "because" x^n is a continuous function for all finite n! Definition 4.1.4 can be decoded via Nelson's reduction algorithm; it then gives the usual ε-δ definition, at the internal level, as in Response I. We note that here too there are two equivalent formulas, one of which transfers and the other does not.

Response III.

This is a feasible response in an approach based on superstructures, even though I found no discussion of it in the literature. While considering it I switch to the asterisk notation.

We recall that the definition of (pointwise) continuity in terms of infinitesimals can be generalized to standard $f : T_1 \to T_2$, where T_1 and T_2 are arbitrary standard topological spaces. The internal open sets of ${}^*\mathbf{R}$ form a base for an (external) topology on ${}^*\mathbf{R}$, the *Q-topology*. The meaning of "f is continuous at x" for internal f and $x \in {}^*\mathbf{R}$ can be given by "f is continuous at x in the Q-topology on ${}^*\mathbf{R}$." As noted above, this concept has a nonstandard definition, although applying it requires working with ${}^{\circledast}({}^*\mathbf{R})$ in a "second-order" enlargement[2] of the superstructure that contains ${}^*\mathbf{R}$. Yet the difficulty is not resolved. In order to prove that the two definitions of continuity are equivalent for standard f and x, one needs to apply transfer to the (equivalent) standard definitions in terms of open neighborhoods, i.e., fall back on the ε-δ method, in topological disguise. The Robinsonian framework does not provide for direct transfer of nonstandard definitions. This response also begs the question, how do we know that ${}^{\circledast}({}^*f)$ is continuous, and so on. Clearly, an infinite sequence of consecutive enlargements would be needed.

It seems that every attempt to define continuity ultimately has to be grounded on the ε-δ method. As remarked above, the same difficulty appears with derivatives, integrals —in fact, with all standard concepts introduced by nonstandard methods. I see it as a serious problem for the Robinsonian framework, if not as a research tool, surely as a teaching tool and, fundamentally, as a satisfactory answer to the question about the place of infinitesimals, and nonstandard objects in general, in mathematics.

What is to be done?

Contemplation of the three responses suggests some ideas. First, we need to abandon the fixed distinction between standard and nonstandard and be able to treat any (internal) object as if it were standard, and in this capacity subject to application of nonstandard definitions and theorems. This is the idea of **relativization of standardness**. Second, we need to be able to transfer properties described by *arbitrary* (external) formulas, not just $\in$-formulas; for emphasis, I refer to this facility as **general transfer**.

Both ideas have some history in the literature of nonstandard analysis. A definition of relative standardness seems to appear first in Cherlin and Hirschfeld [7], although its model-theoretic roots can be discerned in [6]; but the subsequent development occurred mostly in the axiomatic setting. Gordon [11] defined two notions of relative standardness in **IST** (one of them is essentially the same as in [7]). Wallet [24] proposed to use a binary relative

[2]For some applications of "second-order" enlargements see Molchanov [22]. In the alpha-theory of Benci and DiNasso [4], the $*$-embedding is defined for all sets, but ${}^*({}^*\mathbf{R})$ is only ω_1-saturated, and monads in the Q-topology on ${}^*\mathbf{R}$ are trivial.

standardness predicate as a primitive in an axiomatic treatment, an idea that was developed systematically by Péraire in [25]. The notion of relative standardness *stratifies* the universe into *levels of standardness.* Both Gordon and Péraire give a nonstandard definition of continuity applicable to all f and x (see Definition 4.2.2), and numerous other examples (see also [1, 12, 26, 27, 28]). Gordon's approach does not work as smoothly for concepts whose definitions involve shadows, such as derivative, because standardization does not hold in full. A sufficiently strong (for this purpose) standardization does hold in Péraire's theory **RIST**. Another difference is that, unlike Gordon's, Péraire's relative standardness predicate is a *total* preordering. Yet another stratified nonstandard set theory (not employing a binary relative standardness predicate) was put forward by Fletcher [9].

None of [9, 11, 25] give an explicit formulation of transfer for more than just $\in$-formulas. To my knowledge, the idea of (more) general transfer appears first in the work of Benninghofen and Richter [5]. The main result of [5] is a transfer theorem for a certain (complicated) class of $\in$-st-formulas; Cutland [8] gives some simpler special cases. Although the class of transferable formulas is limited, it has led to interesting applications (see the proof of l'Hôpital rule in Section 4.2, and [5, 32, 33]). However, the idea of relative standardness is not explicit in [5]. Further discussion of the mutual relationship of these various approaches can be found in [16].

In my opinion, the decisive step needed to resolve the difficulty discussed above is to combine relative standardness with general transfer. This step was taken by Péraire in [26] with his proof (in **RIST**) of "polytransfer," essentially, transfer for all formulas that do not quantify over levels of standardness. Many standard concepts have satisfactory nonstandard, transferable definitions in **RIST**. Nevertheless, there are situations (see Example 4 in Section 4.2) where quantification over levels of standardness is both natural and necessary. Moreover, the need to single out the special classes of formulas to which principles of **RIST** apply increases reliance of the framework on formal logic.

It is my belief that for a theory of the "nonstandard" to be fully satisfactory, both foundationally and practically, *all* of its *principles need to apply uniformly at all levels, and to all formulas.* In addition to a complete resolution of the difficulty that is the subject of this discussion, such framework would also diminish the need for appeals to formal logic in practical work: all formulas would be transferable, and equivalent formulas would have equivalent transfers. The axiomatic system **FRIST** presented in Section 4.3 achieves these objectives for internal sets. The examples in the next section show some of the power of internal methods extended by relativized standardness and general transfer.

4.2 Stratified analysis

This section gives several examples intended to illustrate "nonstandard" mathematics in stratified framework. The presentation is informal; an axiomatic system **FRIST** in which all of the arguments in this section can be formalized is described fully in Section 4.3.

Our basic assumption is that the universe of mathematical objects (= sets) is stratified by a binary relation $\sqsubseteq$ into "levels of standardness." The notation $x \sqsubseteq y$ is to be read "*x is standard relative to y*" or "*x is y-standard*", and it is a dense total preordering with a least element 0. The 0-standard sets are called simply *standard*; they form the lowest level of standardness. The class of x-standard sets is denoted $\mathbb{S}_x$.

Let α be a set; *relativization of standardness* to level α consists in regarding $\mathbb{S}_\alpha$, rather than $\mathbb{S}_0$, as the lowest level of standardness. A statement about standard sets is *relativized* to level α by replacing all references to "standard" with "α-standard" (more explicitly, by replacing "*x is y-standard*" with "*x is $\langle y, \alpha\rangle$-standard*").

The key principle that governs the stratified universe is **general transfer**: *All valid statements about standard sets remain valid when relativized to any level α.*

Mathematical practice proceeds by enriching the language with new definitions. We make it a *general rule* that, whenever some standard notion (a new predicate or function) is defined for standard x in terms of some property of x, the definition of the notion is extended to all x by relativizing the defining statement to level x (the definition has to be *fully relativized* —see Section 4.3).

In addition, we make the familiar assumptions that

- all the usual mathematical operations preserve standardness (*the class $\mathbb{S}_0$ of all standard sets satisfies* **ZFC**);
- given any property of x and any standard set A, there is a standard set B, whose standard elements are precisely those standard $x \in A$ having that property (*standardization*);
- for every level $\alpha \sqsupset 0$ there exist α-standard unlimited natural numbers (a much stronger *idealization* is available — see Section 4.3 — but this suffices for calculus).

The examples that follow illustrate the formulation of relativizations and the general rule.

Definition 4.2.1

(i) $r \in \mathbf{R}$ *is* α-limited *iff* $|r| < n$ *for some* α*-standard* $n \in \mathbf{N}$.

(ii) $h \in \mathbf{R}$ *is* α-infinitesimal *iff* $|h| < \frac{1}{n}$ *for all* α*-standard* $n \in \mathbf{N}$, $n \neq 0$.

(iii) $r \approx^{\alpha} s$ *iff* $r - s$ *is* α*-infinitesimal.*

(iv) For α*-limited* $r \in \mathbf{R}$, *the* α-shadow *of* r, $\mathbf{sh}_{\alpha}(r)$, *is the unique* α*-standard* $s \in \mathbf{R}$ *such that* $r \approx^{\alpha} s$.

*(*Remark. *As usual, every limited (i.e.,* 0*-limited) real has a unique* 0*-shadow, by standardization. By general transfer, every* α*-limited real has a unique* α*-shadow.)*

EXAMPLE 1. Continuity (Gordon [11, 12], Péraire [25, 27]).

Definition 4.1.1 relativizes as follows:

Definition 4.2.2 *Let* $f : \mathbf{R} \to \mathbf{R}$ *and* $x \in \mathbf{R}$.

(i) f *is* continuous at x *iff for all* $\langle f, x\rangle$*-infinitesimal* h, $f(x + h) - f(x)$ *is* $\langle f, x\rangle$*-infinitesimal.*

(ii) f *is (pointwise)* continuous *iff it is continuous at all* f*-standard* x. *Explicitly: for all* f*-standard* x *and all* $\langle f, x\rangle$*-infinitesimal* h, $f(x+h)-f(x)$ *is* $\langle f, x\rangle$*-infinitesimal.*

The closure of $\langle f, x\rangle$-standard sets under set-theoretic operations (due to transfer of this property from $\mathbb{S}_0$) implies that f and x are $\langle f, x\rangle$-standard. In particular, for f-standard x, "$\langle f, x\rangle$-standard" is equivalent to "f-standard", and we have:

• f is *continuous* iff for all f-standard x and all f-infinitesimal h, $f(x+h)-f(x)$ is f-infinitesimal.

If f is α-standard, transfer gives:

f is *continuous* iff it is continuous at all α-standard $x \in \mathbf{R}$.

But every $x \in \mathbf{R}$ is α-standard for some $\alpha \sqsupseteq f$, so we have also:

f is *continuous* iff it is continuous at all $x \in \mathbf{R}$.

This reasoning is a special case of a general *global transfer principle* (Section 4.3, Proposition 4.3.1).

In contrast with •, relativized definition of uniform continuity is

Definition 4.2.3 *Let* $f : \mathbf{R} \to \mathbf{R}$. f *is* uniformly continuous *iff for all* x *and all* f*-infinitesimal* h, $f(x + h) - f(x)$ *is* f*-infinitesimal.*

EXAMPLE 2. Derivative (Péraire [27]).

Relativization of the usual nonstandard definition of derivative gives

Definition 4.2.4 *Let $f : \mathbf{R} \to \mathbf{R}$ and $x \in \mathbf{R}$. f is* differentiable at x *iff there is an $\langle f, x\rangle$-standard $L \in \mathbf{R}$ such that $\frac{f(x+h)-f(x)}{h} - L$ is $\langle f, x\rangle$-infinitesimal, for all $\langle f, x\rangle$-infinitesimal $h \neq 0$.*

If this is the case, $f'(x) := L = \mathbf{sh}_{\langle f,x\rangle}\left(\frac{f(x+h)-f(x)}{h}\right)$.

We next give two proofs of an elementary result from calculus, in order to illustrate two styles of work that are supported by the stratified framework.

Proposition 4.2.5 *If f is differentiable at x then f is continuous at x.*

Proof 1. By Definition 4.2.4, for any $\langle f, x\rangle$-infinitesimal h, $f(x + h) - f(x) = Lh + kh$ where k is $\langle f, x\rangle$-infinitesimal. The usual arguments show that the product of an $\langle f, x\rangle$-limited real and an $\langle f, x\rangle$-infinitesimal is an $\langle f, x\rangle$-infinitesimal, and that the sum of two $\langle f, x\rangle$-infinitesimals is $\langle f, x\rangle$-infinitesimal. □

Proof 2. For any standard f and x and any infinitesimal h, $f(x+h)-f(x) = Lh+kh$ and $Lh+kh$ is infinitesimal. Hence a standard function f differentiable at a standard x is continuous at x. By transfer, any function f differentiable at any x is continuous at x. □

The style of the first proof is to give the argument uniformly for all f, x. The advantage is that there is no explicit evocation of transfer; the disadvantage is the need to keep track of the level $\langle f, x\rangle$ throughout the argument.

The second proof is Robinson's, i.e., for standard f, x; followed by transfer of the result to all f, x. The formula being transferred is not an $\in$-formula, but it is easily seen that our general transfer applies to it. This is another instance of the global transfer principle from Proposition 4.3.1 in Section 4.3: any statement that invokes relative standardness only via previously defined standard notions and is valid for all standard sets remains valid for all (internal) sets.

One can, if one so chooses, work in stratified analysis exactly as one would in the Robinsonian framework; that is, give the nonstandard definitions and proofs for standard arguments only. This is what we do in the remaining examples. But in the stratified framework, all such definitions and proofs automatically transfer to definitions and proofs that are meaningful and natural for *all* arguments (as long as no essential use is made of external sets).

Stratified analysis also provides opportunities for proofs and constructions that are not readily available in the Robinsonian framework. They have not been much explored as yet; two examples of what is possible are given below. First we list some general results about infinitesimals.

Lemma 4.2.6

(a) *If $x \in \mathbf{R}$ is α-infinitesimal and $\beta \sqsubseteq \alpha$ then x is β-infinitesimal.*

(*b*) *Every α-limited natural number is α-standard.*

(*c*) *If y is infinitesimal then there is an infinitesimal x such that y is x-infinitesimal.*

Proof. (*a*) is trivial from transitivity of $\sqsubseteq$, and (*b*) is just transfer (to level α) of the well-known fact that every limited natural number is standard.

(*c*) is a consequence of (*b*) and density of $\sqsubseteq$. Let y be a positive infinitesimal and let $\nu \in \mathbf{N}$ be such that $\nu \leq \frac{1}{y} < \nu + 1$. Then $0 \sqsubset \nu$; we fix α such that $0 \sqsubset \alpha \sqsubset \nu$. By (b), ν is α-unlimited and so $y < \frac{1}{n}$ for all α-standard n. So y is α-infinitesimal, hence x-infinitesimal for any α-standard infinitesimal x. □

EXAMPLE 3. l'Hôpital Rule (Benninghofen and Richter [5]; see also [12]).

Proposition 4.2.7 (*l'Hôpital Rule*)
If $\lim_{x\to a} |g(x)| = \infty$ *and* $lim_{x\to a} \frac{f'(x)}{g'(x)} = d \in \mathbf{R}$ *then* $lim_{x\to a} \frac{f(x)}{g(x)} = d$.

Proof. It suffices to prove the proposition for standard f, g, a, d. Also, w.l.o.g. we can let $a = 0$ (replace x by $x - a$). Let x be infinitesimal and y be x-infinitesimal. By Cauchy's Theorem, there is η between x and y (hence, η is infinitesimal) such that $\frac{f(y)-f(x)}{g(y)-g(x)} = \frac{f'(\eta)}{g'(\eta)} \approx d$. Now factor

$$d \approx \frac{f(y)-f(x)}{g(y)-g(x)} = \frac{f(y)-f(x)}{g(y)} \times \frac{g(y)}{g(y)-g(x)} \text{ and observe that } \frac{f(x)}{g(y)} \approx 0,\ \frac{g(x)}{g(y)} \approx 0.$$

($\lim_{x\to 0} |g(x)| = \infty$ implies that for all infinitesimal z, $g(z)$ is unlimited. By transfer to x-level, for all x-infinitesimal z, $g(z)$ is x-unlimited. As y is x-infinitesimal, $\frac{f(x)}{g(y)}$ and $\frac{g(x)}{g(y)}$ are x-infinitesimal.)

It follows that the first factor is infinitely close to $\frac{f(y)}{g(y)}$ and the second to 1. From properties of infinitesimals we conclude that $\frac{f(y)}{g(y)} \approx d$.

By Lemma 4.2.6(c), every infinitesimal y is x-infinitesimal for some infinitesimal x. Hence $\frac{f(y)}{g(y)} \approx d$ holds for *every* infinitesimal y, and we are done.□

Remark. The assumption of density of levels allows for a simpler argument than that of [5, 12], but it is not essential.

EXAMPLE 4. Higher derivatives.

We assume that f, x are standard and $f'(y)$ exists for all $y \approx x$. If $f''(x) = L \in \mathbf{R}$ exists, then $L \approx \frac{f(x+2h)-2f(x+h)+f(x)}{h^2}$ holds for all $h \approx 0$, $h \neq 0$. However, the converse of this statement is false; existence of a standard $L \in \mathbf{R}$ with the above property does not imply that $f''(x)$ exists.

In the stratified framework, we can give a description of $f''(x)$ in terms of values of f, analogous to Definition 4.2.4, using two levels of standardness.

Proposition 4.2.8 *Assume that f and x are standard and $f'(y)$ exists for all $y \approx x$. Then $f''(x)$ exists iff there is a standard $L \in \mathbf{R}$ such that*

$$L \approx \frac{f(x+h_0+h_1) - f(x+h_0) - f(x+h_1) + f(x)}{h_0 h_1}$$

for all $h_0 \approx 0$, $h_1 \approx^{h_0} 0$, $h_0, h_1 \neq 0$.

If this is the case, $f''(x) = L$.

As usual in the stratified framework, the proposition holds for arbitrary f and x provided $\approx$ is replaced by $\approx^{\langle f,x\rangle}$ and "standard" by "$\langle f, x\rangle$-standard."

Proof. We assume that $f'(y)$ exists for all $y \approx x$; in particular, $f'(x)$ and $f'(x+h_0)$ exist. Hence

$$\frac{f(x+h_0+h_1)-f(x+h_0)}{h_1} = f'(x+h_0) + k_1, \quad \frac{f(x+h_1)-f(x)}{h_1} = f'(x) + k_2,$$

where $k_1, k_2 \approx^{h_0} 0$.
From this we get

$$Q := \frac{f(x+h_0+h_1)-f(x+h_0)-f(x+h_1)+f(x)}{h_0 h_1} = \frac{f'(x+h_0)-f'(x)}{h_0} + \frac{(k_1-k_2)}{h_0}.$$

We note that $(k_1 - k_2) \approx^{h_0} 0$ and hence $\frac{(k_1-k_2)}{h_0} \approx^{h_0} 0$, by transfer to the level h_0 of the fact that a quotient of an infinitesimal by a standard real $\neq 0$ is infinitesimal. In particular, $\frac{(k_1-k_2)}{h_0} \approx 0$.

If $L := f''(x)$ exists, we have $\frac{f'(x+h_0)-f'(x)}{h_0} = L + k$ for $k \approx 0$ and hence $Q \approx L$.

Conversely, if $Q \approx L \in {}^\circ\mathbf{R}$ for all h_0, h_1 as above, we have $\frac{f'(x+h_0)-f'(x)}{h_0} \approx L$ for all $h_0 \approx 0, h_0 \neq 0$, and hence $f''(x)$ exists and equals to L. □

By induction we get a characterization of $f^{(n)}(x)$ valid for any *standard* $n \in \mathbf{N}$.

Proposition 4.2.9 *Assume that f and x are standard and $f^{(n-1)}(y)$ exists for all $y \approx x$. Then $f^{(n)}(x)$ exists iff there is a standard $L \in \mathbf{R}$ such that*

$$L \approx \frac{1}{h_0 \dots h_{n-1}} \sum_i (-1)^{i_0+\dots+i_{n-1}} f(x + h^{i_0} + \dots + h^{i_{n-1}})$$

holds for all $\langle h_0, \dots, h_{n-1}\rangle$, where $i = \langle i_0, \dots, i_{n-1}\rangle \in \{0,1\}^n$, $h^{i_k} := h_k$ if $i_k = 0$, $h^{i_k} := 0$ if $i_k = 1$; $h_0 \approx 0$, $h_k \approx^{h_{k-1}} 0$ for $0 < k < n$, and all $h_k \neq 0$.

If this is the case, $f^{(n)}(x) = L$.

General transfer implies that Proposition 4.2.9 holds for **all** n, even the unlimited (hyperfinite) ones, provided $\approx$ is replaced everywhere by $\approx^n$ and

"standard L" by "n-standard L". As there are functions that have derivatives of all orders, *there have to exist "strongly decreasing" sequences of infinitesimals of any hyperfinite length n: $\langle h_0, \ldots, h_{n-1}\rangle$ where each h_k is h_{k-1}-infinitesimal.* Such constructions are not available in **RIST** or in any other framework for nonstandard mathematics.

4.3 An axiomatic system for stratified set theory

The theory **FRIST** (*Fully Relativized Internal Set Theory*) presented here is formalized in first-order logic with equality and two primitive binary predicates, $\in$ (membership) and $\sqsubseteq$ (relative standardness). It postulates the axioms of **ZFC** (the schemata of separation and replacement for $\in$-formulas only), and $(\forall x)(x \sqsubseteq x)$, $(\forall x, y, z)(x \sqsubseteq y \ \wedge \ y \sqsubseteq z \rightarrow x \sqsubseteq z)$, $(\forall x, y)(x \sqsubseteq y \ \vee \ y \sqsubseteq x)$, $(\forall x)(\varnothing \sqsubseteq x)$, $(\exists x)(\neg\, x \sqsubseteq \varnothing)$. Thus $\sqsubseteq$ is a nontrivial total preordering with a least element $0 = \varnothing$; $x \sqsubseteq y$ reads "*x is standard relative to y*" or "*x is y-standard.*" We also write $x \sqsubset y$ for $x \sqsubseteq y \ \wedge \ \neg\, y \sqsubseteq x$, and postulate that $\sqsubseteq$ is *dense*: $(\forall x, y)(x \sqsubset y \rightarrow (\exists z)(x \sqsubset z \sqsubset y))$.

Let α be a set; we let $\mathbb{S}_\alpha := \{x : x \sqsubseteq \alpha\}$ be the class of all α-standard sets; in particular $\mathbb{S} := \mathbb{S}_0 = \{x : x \sqsubseteq 0\}$ is the class of standard sets. If $\bar{\alpha}$ is a list $\alpha_1, \ldots, \alpha_n$, the statement that x is $\bar{\alpha}$-standard is shorthand for "x is $\langle \alpha_1, \ldots, \alpha_n\rangle$-standard"; it is easy to see that (in **FRIST**) this is equivalent to "x is β-standard" for $\beta := \max\{\alpha_1, \ldots, \alpha_n\}$.

For any α let $x \sqsubseteq_\alpha y \equiv (x \sqsubseteq \alpha \wedge y \sqsubseteq \alpha) \vee (x \sqsubseteq y)$. This "relativized" relative standardness predicate treats $\mathbb{S}_\alpha$ as the "standard" (i.e. "level 0") universe, and keeps the higher levels unchanged.

Let Φ be any $\in$-$\sqsubseteq$-formula. Φ^α denotes the *relativization of Φ to level α*, the formula obtained from Φ by replacing each occurrence of $\sqsubseteq$ by $\sqsubseteq_\alpha$. Informally, each occurrence of "x is y-standard" is replaced by "x is $\langle y, \alpha\rangle$-standard." Clearly $x \sqsubseteq_0 y$ is equivalent to $x \sqsubseteq y$, and Φ^0 to Φ.

We now state the principal axioms of **FRIST**.

Transfer: For all α, $(\forall \bar{x} \in \mathbb{S}_0)(\Phi^0(\bar{x}) \ \leftrightarrow \ \Phi^\alpha(\bar{x}))$.

Standardization: For all $\bar{x}$,
$(\forall x \in \mathbb{S}_0)\ (\exists y \in \mathbb{S}_0)\ (\forall z \in \mathbb{S}_0)\ (z \in y \ \leftrightarrow \ z \in x \ \wedge \ \Phi^0(z, x, \bar{x}))$.

Idealization: For all $0 \sqsubset \alpha$, $A, B \in \mathbb{S}_0$ and $\bar{x}$,
$(\forall a \in A^{\mathrm{fin}} \cap \mathbb{S}_0)(\exists x \in B)(\forall y \in a)\ \Phi^\alpha(x, y, \bar{x}) \ \leftrightarrow$
$(\exists x \in B)(\forall y \in A \cap \mathbb{S}_0)\ \Phi^\alpha(x, y, \bar{x})$.

Transfer captures the idea of *full relativization*: whatever is true about the standard universe $\mathbb{S}_0$ is also true about each relativized standard universe $\mathbb{S}_\alpha$. Precisely, for any α and any statement Φ of the $\in$-$\sqsubseteq$-language, if $\mathbf{FRIST} \vdash \Phi$ then also $\mathbf{FRIST} \vdash \Phi^\alpha$. In particular, it follows that we can replace 0 by any α in standardization, and by any $\beta \sqsubset \alpha$ in the other two schemata. It is also easy to show that each $\mathbb{S}_\alpha$ satisfies **ZFC**. Details of these and other results about **FRIST**, as well as a discussion of its relationship to **RIST** and **IST**, can be found in [15].

We prove a version of transfer from the standard universe to the entire internal universe. Let $\Phi(\bar{x})$ be an $\in$-$\sqsubseteq$-formula. We define a new predicate $P(\bar{x})$ by postulating $P(\bar{x}) \equiv \Phi^{\bar{x}}(\bar{x})$, and say that the definition of P is *fully relativized* if it has this form. We note that for any α and any $\bar{x} \in \mathbb{S}_\alpha$, $P(\bar{x}) \leftrightarrow \Phi^\alpha(\bar{x})$; in particular, $P(\bar{x}) \leftrightarrow \Phi(\bar{x})$ for standard $\bar{x}$.

Let the definitions of $P_1, \ldots, P_n$ be fully relativized. Given any formula $\Psi(\bar{x})$ in the $\in$-$\bar{P}$-language, we denote the formula obtained by restricting all quantifiers in Ψ to $\mathbb{S}_\alpha$ by $\mathbb{S}_\alpha \vDash \Psi(\bar{x})$.

Proposition 4.3.1 (*Global Transfer*) $(\forall \bar{x} \in \mathbb{S}_\alpha)[(\mathbb{S}_\alpha \vDash \Psi(\bar{x})) \leftrightarrow \Psi(\bar{x})]$.

Proof. If Ψ is a formula in the $\in$-$\bar{P}$-language, we let $\Psi^{(\alpha)}$ be the formula obtained from Ψ by replacing (first) each occurrence of $(\exists x)$ [$(\forall x)$, resp.] in Ψ by $(\exists x \in \mathbb{S}_\alpha)$ [$(\forall x \in \mathbb{S}_\alpha)$, resp.], and (then) each occurrence of $P_i(\bar{x})$ by $\Phi_i^\alpha(\bar{x})$.

We prove that for all α and all $\bar{x} \in \mathbb{S}_\alpha$, $(\mathbb{S}_\alpha \vDash \Psi(\bar{x})) \leftrightarrow \Psi^{(\alpha)}(\bar{x}) \leftrightarrow \Psi(\bar{x})$, by induction on complexity of Ψ.

If $\Psi(\bar{x})$ is $P_i(\bar{x})$, the claim follows from the definitions and remarks above. If $\Psi(\bar{x})$ is "$x_i \in x_j$" or "$x_i = x_j$" the assertion is trivial, as are the induction steps corresponding to logical connectives.

Assume that, for all β and $\bar{x}, y \in \mathbb{S}_\beta$, $(\mathbb{S}_\beta \vDash \Psi(\bar{x}, y)) \leftrightarrow \Psi^{(\beta)}(\bar{x}, y) \leftrightarrow \Psi(\bar{x}, y)$.

Let $\bar{x} \in \mathbb{S}_\alpha$; we then have $(\mathbb{S}_\alpha \vDash (\forall y)\Psi(\bar{x}, y)) \leftrightarrow (\forall y \in \mathbb{S}_\alpha)(\mathbb{S}_\alpha \vDash \Psi(\bar{x}, y)) \leftrightarrow (\forall y \in \mathbb{S}_\alpha)\Psi^{(\alpha)}(\bar{x}, y) \leftrightarrow [(\forall y)\Psi(\bar{x}, y)]^{(\alpha)}$.

By transfer of the penultimate statement to level β we get its equivalence with $(\forall \beta \sqsupseteq \alpha)(\forall y \in \mathbb{S}_\beta)[\Psi^{(\alpha)}(\bar{x}, y)]^\beta$. It is easily seen that, for $\alpha \sqsubseteq \beta$, $[\Psi^{(\alpha)}]^\beta \leftrightarrow \Psi^{(\beta)}$; hence the last statement is further equivalent to $(\forall \beta \sqsupseteq \alpha)(\forall y \in \mathbb{S}_\beta)\Psi^{(\beta)}(\bar{x}, y) \leftrightarrow (\forall \beta \sqsupseteq \alpha)(\forall y \in \mathbb{S}_\beta)\Psi(\bar{x}, y) \leftrightarrow (\forall y)\Psi(\bar{x}, y)$. (For the last step, note that for every y there is $\beta \sqsupseteq \alpha$ such that $y \in \mathbb{S}_\beta$; e.g. $\beta = \langle \alpha, y \rangle$.) $\square$

Corollary 4.3.2 *For any $A, \bar{p}$, $\{x \in A : \Psi(x, A, \bar{p})\}$ is a set in $\mathbb{S}_{\langle A, \bar{p} \rangle}$.*

Proof. Fix α so that $A, \bar{p} \in \mathbb{S}_\alpha$. Standardization into $\mathbb{S}_\alpha$ yields a set $B \in \mathbb{S}_\alpha$ such that $\mathbb{S}_\alpha \vDash (\forall x)(x \in B \leftrightarrow x \in A \wedge \Psi(x, A, \bar{p}))$. By global transfer, $(\forall x)(x \in B \leftrightarrow x \in A \wedge \Psi(x, A, \bar{p}))$. $\square$

This corollary shows that we can use fully relativized predicates in definitions of sets without fear of encountering external sets. For example, f' is a standard function whenever f is, even though we use infinitesimals to define $f'(x)$.

The consistency of **FRIST** is established by the following theorem, whose proof can be found in [15], Theorems 4.7, 5.1, 5.2.

Theorem 4.3.3 ***FRIST*** *is a conservative extension of* ***ZFC****.*
In fact, ***FRIST*** *has an interpretation in* ***ZFC****, in which the class of standard sets is (definably) isomorphic to the class of all (****ZFC****) sets.*

An interpretation of **FRIST** in **ZFC** involves a new method of iterating ultrapowers, where the stages of the iteration used to obtain the final universe ${}^*\mathbb{V}$ are not indexed by any a priori given linear ordering $\langle \Lambda, \leq \rangle$, but by $\langle {}^*\Lambda, \leq^* \rangle$, a linear ordering constructed from $\langle \Lambda, \leq \rangle$ simultaneously with ${}^*\mathbb{V}$.

FRIST differs from the theory of the same name presented in [15] in two ways.

1. Idealization has been weakened to "bounded idealization" (see **FRBST** of [15]). Foundational reasons for this move are discussed at length in [16] in the context of **IST** vs. **BST**. Without some such weakening of idealization, the second part of Theorem 4.3.3 does not hold.

2. The postulate of density of levels has been added. This property holds in models from [15] based on $\Lambda = \mathbf{Q}$ or any other dense total ordering. There are foundational reasons that seem to favor this assumption, and it also has practical uses (see the proof of l'Hôpital rule).

Like **IST**, **FRIST** is a theory of internal sets. Nelson [23] exhibited a *reduction algorithm* for **IST**; it takes every (bounded) $\in$-st-formula to an $\in$-formula that is equivalent to it for all standard values of the free variables. The question arises, whether such algorithm is also available for **FRIST**. On one level, the answer is *YES*; such algorithm is provided by the interpretation of **FRIST** in **ZFC** from the proof of Theorem 4.3.3. On the other hand, $\in$-formulas yielded by this algorithm are far too complicated to be helpful with practical work. It is possible that a simpler, more natural reduction algorithm can be given, but this matter is still under investigation.

As demonstrated by Nelson and other adherents of **IST**, "internal" methods, cleverly used, suffice for a large area of "nonstandard" applications. Nevertheless, I have always maintained [13, 14, 16] that a truly comprehensive nonstandard set theory has to incorporate external sets as well. There are numerous constructs (nonstandard hulls, Loeb measures,...) where external sets are not just a convenient bookkeeping device but the object of interest per se; a foundational framework that does not allow these constructs can hardly be

universally acceptable to practitioners of "nonstandard" methods. The foundational aspirations of nonstandard set theory also require external sets; "they are there," and have to be accounted for. The guiding "maxim" of this paper, to wit, that all principles should apply uniformly at all levels and to all formulas, gives yet another reason why external sets are necessary. In **FRIST**, transfer, standardization and idealization do indeed satisfy it, but the axioms of **ZFC** do not! In particular, separation applies in **FRIST** only to $\in$-formulas. As soon as we attempt to extend it to all formulas, we introduce external sets (e.g. the set of standard integers $\{n \in \mathbf{N} : n \sqsubseteq 0\}$).

A thorough discussion of various axiomatic nonstandard set theories for external sets and of the difficulties they face can be found in Kanovei and Reeken's monograph [19] and in the survey article [16]. Perhaps the only researcher who considered an extension of the idea of relativization of standardness to external sets in an axiomatic framework was David Ballard. In [3], Ballard proposed an axiomatic system **EST**, where external sets are allowed and any set can be regarded as "standard." **EST** was inspired by Fletcher's [9], and employs neither the binary standardness predicate nor general transfer. After the paper [15] was written, Ballard and I concurrently started to develop an extension of its ideas to external sets. David Ballard died unexpectedly in May 2004 in the midst of his work. My research on this topic is still in progress [17]; results obtained so far indicate that an extension of **FRIST** to a theory that incorporates external sets is both possible and natural.

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5

ERNA at work

C. Impens* and S. Sanders**

Abstract

Elementary Recursive Nonstandard Analysis, in short ERNA, is a constructive system of nonstandard analysis proposed around 1995 by Chuaqui, Suppes and Sommer. It has been shown to be consistent and, without standard part function or continuum, it allows major parts of analysis to be developed in an applicable form. We briefly discuss ERNA's foundations and use them to prove a supremum principle and provide a square root function, both up to infinitesimals.

5.1 Introduction

Hilbert's Program, proposed in 1921, called for an axiomatic formalization of mathematics, together with a proof that this axiomatization is consistent. The consistency proof itself was to be carried out using only what Hilbert called *finitary* methods. The special character of finitary reasoning then would justify classical mathematics. In due time, many characterized Hilbert's informal notion of 'finitary' as that which can be formalized in Primitive Recursive Arithmetic (PRA), proposed in 1923 by Skolem. In PRA one finds (a) an absence of explicit quantification, (b) an ability to define primitive recursive functions, (c) a few rules for handling equality, e.g., substitution of equals for equals, (d) a rule of instantiation, and (e) a simple induction principle.

By Gödel's second incompleteness theorem (1931) it became evident that only *partial* realizations of Hilbert's program are possible. The system proposed by Chuaqui and Suppes is such a partial realization, in that it provides an axiomatic foundation for basic analysis, with a PRA consistency proof ([1], p. 123 and p. 130). Sommer and Suppes's improved system allows definition by recursion (which does away with a lot of explicit axioms)

*Department of Pure Mathematics and Computer Algebra, University of Gent, Belgium. ci@cage.ugent.ac.be

**sasander@cage.ugent.be

and still has a PRA proof of consistency ([2], p. 21). This system is called *Elementary Recursive Nonstandard Analysis*, in short ERNA. Its consistency is proved via Herbrand's Theorem (1930), which is restricted to quantifier-free formulas $Q(x_1, \ldots, x_n)$, usually containing free variables. Alternatively, one might say it is restricted to universal sentences

$$(\forall x_1) \ldots (\forall x_n) Q(x_1, \ldots, x_n),$$

obtained by closing the open quantifier-free formulas by means of universal quantifiers. Herbrand's theorem states that, if a collection of such formulas resp. sentences is consistent, it has a simple 'Herbrand' model and, if it is not, its inconsistency will show up in some finite procedure.

Herbrand's theorem requires that ERNA's axioms be written in a quantifier-free form. As a result, some axioms definitely look artificial; fortunately, theorems don't suffer from the quantifier-free restriction.

Calculus applications of ERNA have been, so far, scarce and sketchy. Thus, [3] contains an outline of an existence theorem for first-order ordinary differential equations, relying on the property, stated without proof, that a continuous function on a compact interval is bounded. As part of a less anecdotical approach we will provide an ERNA version of the supremum principle and deduce from it a square root function. Both results hold *up to infinitesimals*; as ERNA has no standard part function, it is intrinsically impossible to do better.

5.2 The system

The system we are about to describe was first presented in [2], and all our undocumented results are quoted from that paper. The foundations are also exposed, in a more informal manner, in [3].

Notation 5.2.1 $\mathbb{N} = \{0, 1, 2, \ldots\}$ *consists of the (finite) integers.*

Notation 5.2.2 $\vec{x}$ *stands for some finite (possibly empty) sequence* $(x_1, \ldots, x_k)$.

Notation 5.2.3 $\tau(\vec{x})$ *denotes a term in which* $\vec{x} = (x_1, \ldots, x_k)$ *is the list of the distinct free variables.*

5.2.1 The language

- connectives: $\wedge, \neg, \vee, \rightarrow, \leftrightarrow$
- quantifiers: $\forall, \exists$

- an infinite set of variables
- relation symbols:[1]
  - binary $x = y$
  - binary $x \leq y$
  - unary $\mathcal{I}(x)$, read as 'x is infinitesimal', also written '$x \approx 0$'
  - unary $\mathcal{N}(x)$, read as 'x is hypernatural'.
- individual constant symbols:
  - 0
  - 1
  - ε (The Axiom 3 (6) of 5.2.2 shows that ε denotes a positive infinitesimal.)
  - ω (The axioms 3 (7) and 2 (4) of 5.2.2 show that $\omega = 1/\varepsilon$ denotes an infinite hypernatural.)
  - $\uparrow$, read as 'undefined'.

Notation 5.2.4 *'x is defined' stands for '$x \neq \uparrow$'.* (Examples: $1/0$ is undefined, $1/0 = \uparrow$.)

- function symbols:[2]
  - (unary) 'absolute value' $|x|$, 'ceiling' $\lceil x \rceil$, 'weight' $\|x\|$. (For the meaning of $\|x\|$, see Theorem 5.2.3.)
  - (binary) $x+y$, $x-y$, $x \cdot y$, x/y, $x\hat{}y$. (Axiom set 6 and Axiom 12 (4) of 5.2.2 show that $x\hat{}n = x^n$ for hypernatural n, else undefined.)
  - for each $k \in \mathbb{N}$, k k-ary function symbols $\pi_{k,i}$ $(i = 1, \ldots, k)$. (The Axiom schema 7 of 5.2.2 shows that $\pi_{k,i}(\vec{x})$ are the projections of the k-tuple $\vec{x}$.)
  - for each formula φ with $m+1$ free variables, without quantifiers or terms involving min, an m-ary function symbol $\min_\varphi$. (For the meaning of which, see Theorem 5.2.6 and Theorem 5.2.7.)
  - for each triple $(k, \sigma(x_1, \ldots, x_m), \tau(x_1, \ldots, x_{m+2}))$ with $0 < k \in \mathbb{N}$, σ and τ terms not involving min, an $(m+1)$-ary function symbol $\mathrm{rec}^k_{\sigma\tau}$. (Axiom schema 9 of 5.2.2 shows that this is the term obtained from σ and τ by recursion, after the model $f(0, \vec{x}) = \sigma(\vec{x})$, $f(n+1, \vec{x}) = \tau(f(n, \vec{x}), n, \vec{x})$, if terms are defined and don't weigh too much.)

[1] For better readibility we express the relations in x or in (x, y), according to arity.
[2] We denote the values as computed in x or (x, y) according to the arity.

5.2.2 The axioms

Axiom set 1 (Logic). *Axioms of first-order logic.*

Axiom set 2 (Hypernaturals).

1. 0 *is hypernatural;*
2. *if* x *is hypernatural, so is* $x + 1$*;*
3. *if* x *is hypernatural, then* $x \geq 0$*;*
4. ω *is hypernatural.*

Definition 5.2.1 *'x is infinite' stands for '$x \neq 0 \wedge 1/x \approx 0$'; '$x$ is finite' stands for 'x is not infinite'; 'x is natural' stands for 'x is hypernatural and finite'.*

Axiom set 3 (Infinitesimals).

1. *if* x *and* y *are infinitesimal, so is* $x + y$*;*
2. *if* x *is infinitesimal and* y *is finite,* xy *is infinitesimal;*
3. *an infinitesimal is finite;*
4. *if* x *is infinitesimal and* $|y| \leq x$*, then* y *is infinitesimal;*
5. *if* x *and* y *are finite, so is* $x + y$*;*
6. ε *is infinitesimal;*
7. $\varepsilon = 1/\omega$.

Axiom set 4 (Ordered field). *Axioms expressing that the elements, with* $\uparrow$ *excluded, constitute an ordered field of characteristic zero with absolute value function. These include (quantifier-free)*

- *if* x *is defined, then* $x + 0 = 0 + x = x$*;*
- *if* x *is defined, then* $x + (0 - x) = (0 - x) + x = 0$*;*
- *if* x *is defined and* $x \neq 0$*, then* $x \cdot (1/x) = (1/x) \cdot x = 1$.

Axiom set 5 *(Archimedean). If* x *is defined,* $\lceil x \rceil$ *is a hypernatural and* $\lceil x \rceil - 1 < x \leq \lceil x \rceil$.

Theorem 5.2.1 *If* x *is defined, then* $\lceil x \rceil$ *is the least hypernatural* $\geq x$.

Theorem 5.2.2 x *is finite iff there is a natural* n *such that* $|x| \leq n$.

Proof. The statement is trivial for $x = 0$. If $x \neq 0$ is finite, so is $|x|$ because, assuming the opposite, $1/|x|$ would be infinitesimal and so would $1/x$ be by axiom (4) of set 3. By axiom (5) of the same set, the hypernatural $\lceil |x| \rceil < |x|+1$ is then also finite. Conversely, let n be natural and $|x| \leq n$. If $1/|x|$ were infinitesimal, so would $1/n$ be by axiom (4) of set 3, and this contradicts the assumption that n is finite. □

Corollary 5.2.1 *$x \approx 0$ iff $|x| < 1/n$ for all natural $n \geq 1$.*

Axiom set 6 (Power).

1. *if $x \neq \uparrow$, then $x\hat{}0 = 1$;*
2. *if $x \neq \uparrow$ and n is hypernatural, then $x\hat{}(n+1) = (x\hat{}n) \cdot x$.*

Axiom schema 7 (Projection).

If $x_1, \ldots, x_n$ are defined, then $\pi_{n,i}(x_1, \ldots, x_n) = x_i$ for $i = 1, \ldots, n$.

Axiom set 8 (Weight).

1. *If $\|x\|$ is defined, then $\|x\|$ is a nonzero hypernatural.*
2. *If $|x| = m/n \leq 1$ (m and $n \neq 0$ hypernaturals), then $\|x\|$ is defined, $\|x\|.|x|$ is hypernatural and $\|x\| \leq n$.*
3. *If $|x| = m/n \geq 1$ (m and $n \neq 0$ hypernaturals), then $\|x\|$ is defined, $\|x\|/|x|$ is hypernatural and $\|x\| \leq m$.*

Definition 5.2.2 *A* hyperrational *is of the form $\pm p/q$, with p and $q \neq 0$ hypernatural.*

Theorem 5.2.3

1. *If x is not a hyperrational, then $\|x\| = \uparrow$.*
2. *If x is a hyperrational, say $x = \pm p/q$ with p and $q \neq 0$ relatively prime hypernaturals, then*

$$\| \pm p/q\| = \max\{|p|, |q|\}.$$

Remark. In both statements of this theorem, the antecedent can be expressed in a quantifier-free way, but the whole sentence cannot. (This explains why it is a theorem and not part of the axioms.) For instance, $\mathcal{N}(p) \to \neg\mathcal{N}(p|x|)$ expresses 'x is not hyperrational'.

Theorem 5.2.4

1. $\|0\| = 1$;

2. *if* $n \geq 1$ *is hypernatural,* $\|n\| = n$;

3. *if* $\|x\|$ *is defined, then* $\|1/x\| = \|x\|$ *and* $\| \lceil x \rceil \| = | \lceil x \rceil | \leq \|x\|$;

4. *if* $\|x\|$ *and* $\|y\|$ *defined,* $\|x + y\|$, $\|x - y\|$, $\|xy\|$ *and* $\|x/y\|$ *are at most equal to* $(1 + \|x\|)(1 + \|y\|)$, *and* $\|x\hat{}y\|$ *is at most* $(1 + \|x\|)\hat{}(1 + \|y\|)$.

Notation 5.2.5 *For any* $0 < n \in \mathbb{N}$ *we write*

$$\|(x_1, \ldots, x_n)\| = \max\{\|x_1\|, \ldots, \|x_n\|\}.$$

Notation 5.2.6 *For any* $0 < n \in \mathbb{N}$ *we write*

$$2_n^x := \underbrace{2\hat{}(\ldots 2\hat{}(2\hat{}(2\hat{}x)))}_{n\ 2\text{'s}}.$$

Theorem 5.2.5 *If* $\tau(\vec{x})$ *is a term not involving* ω, ε, rec *or* min, *then there exists a* $0 < k \in \mathbb{N}$ *such that*

$$\|\tau(\vec{x})\| \leq 2_k^{\|\vec{x}\|}.$$

Axiom schema 9 (Recursion) *For* $0 < k \in \mathbb{N}$, σ *and* τ *not involving* min*:*

$$\mathrm{rec}_{\sigma\tau}^k(0, \vec{x}) = \begin{cases} \sigma(\vec{x}) & \text{if this is defined, and has weight} \leq 2_k^{\|\vec{x}\|}, \\ \uparrow & \text{if } \sigma(\vec{x}) = \uparrow, \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathrm{rec}_{\sigma\tau}^k(n+1, \vec{x}) = \begin{cases} \tau(\mathrm{rec}_{\sigma\tau}^k(n, \vec{x}), n, \vec{x}) & \text{if defined, with weight} \leq 2_k^{\|\vec{x}, n+1\|}, \\ \uparrow & \text{if } \tau(\mathrm{rec}_{\sigma\tau}^k(n, \vec{x}), n, \vec{x}) = \uparrow, \\ 0 & \text{otherwise.} \end{cases}$$

If σ is constant, the list $\vec{x}$ is empty, and the weight requirements mentioned in this axiom schema are void.

A few words concerning the restrictions included in this axiom schema. One of ERNA's main advantages over the Chuaqui-Suppes system is, that it allows some form of recursion while preserving a finitary consistency proof. In achieving this, a crucial role is played by the weight function, introduced axiomatically but given explicitly in theorem 5.2.3. Recursion is an essential feature of PRA, and it is therefore impossible to prove inside PRA the consistency of a system that has unrestricted recursion. ERNA's axiom schema 9 restricts recursion by truncating objects outgrowing the preset weight standard. In view of the huge bounds allowed, it seems unlikely that access to calculus applications will suffer from this restriction; computing weights is the price to be paid in practice.

Axiom schema 10 (Internal minimum). *For any quantifier-free formula* $\varphi(y, \vec{x})$ *not involving* min *or* $\mathcal{I}$ *we have*

1. $\min_\varphi(\vec{x})$ *is a hypernatural number;*
2. *if* $\min_\varphi(\vec{x}) > 0$, *then* $\varphi(\min_\varphi(\vec{x}), \vec{x})$;
3. *if* n *is a hypernatural and* $\varphi(n, \vec{x})$, *then*

$$\min_\varphi(\vec{x}) \leq n \quad \textit{and} \quad \varphi(\min_\varphi(\vec{x}), \vec{x}).$$

Theorem 5.2.6 *If the quantifier-free formula* $\varphi(y, \vec{x})$ *does not involve* $\mathcal{I}$ *or* min, *and if there are hypernatural* n*'s such that* $\varphi(n, \vec{x})$, *then* $\min_\varphi(\vec{x})$ *is the least of these. If there are none,* $\min_\varphi(\vec{x}) = 0$.

Corollary 5.2.2 *Proofs by hypernatural induction.*

Example 5.2.1 *The sum of two hypernaturals is a hypernatural.*

Proof. Fix any hypernatural x. If the theorem is wrong, there exists at least one y with $\mathcal{N}(y) \wedge \neg\mathcal{N}(x+y)$. By Theorem 5.2.6, there is a least number with these properties, say y_0. Then $y_0 \neq 0$ since $x + 0 = x$ (field axiom) and $\mathcal{N}(x)$ (assumption). From $y_0 \neq 0$, $\mathcal{N}(y_0 - 1)$ (hypernatural axiom). By leastness, $\mathcal{N}(x + (y_0 - 1))$. Hence (field axiom) $\mathcal{N}((x + y_0) - 1)$ and finally $\mathcal{N}(x + y_0)$ (hypernatural axiom). This contradiction proves the theorem. □

Axiom schema 11 (External minimum). *For any quantifier-free formula* $\varphi(y, \vec{x})$ *not involving* min, ω *or* ε *we have*

1. $\min_\varphi(\vec{x})$ *is a hypernatural number;*
2. *if* $\min_\varphi(\vec{x}) > 0$, *then* $\varphi(\min_\varphi(\vec{x}), \vec{x})$;
3. *if* n *is a natural number,* $\|x\|$ *is finite and* $\varphi(n, \vec{x})$, *then* $\min_\varphi(\vec{x}) \leq n$ *and* $\varphi(\min_\varphi(\vec{x}), \vec{x})$.

Remark. $\mathcal{I}$ is allowed in φ.

Theorem 5.2.7 *Let* $\varphi(y, \vec{x})$ *a quantifier-free formula not involving* min, ω *or* ε. *If* $\|\vec{x}\|$ *is finite and if there are natural* n*'s such that* $\varphi(n, \vec{x})$, *then* $\min_\varphi(\vec{x})$ *is the least of these. If there are none,* $\min_\varphi(\vec{x}) = 0$.

Corollary 5.2.3 *Proofs by natural induction.*

Axiom schema 12 ((Un)defined terms).

1. 0, 1, ω, ε *are defined;*

2. $|x|$, $\lceil x \rceil$, $\|x\|$ *are defined iff* x *is;*
3. $x+y$, $x-y$, xy *are defined iff* x *and* y *are;* x/y *is defined iff* x *and* y *are and* $y \neq 0$*;*
4. $x\hat{}y$ *is defined iff* x *and* y *are and* y *is hypernatural;*
5. $\pi_{k,i}(x_1, \ldots, x_k)$ *is defined iff* $x_1, \ldots, x_k$ *are;*
6. *if* x *is not a hypernatural,* $\mathrm{rec}^k_{\sigma\tau}(x, \vec{y})$ *is undefined;*
7. $\min_\varphi(x_1, \ldots, x_k)$ *is defined iff* $x_1, \ldots, x_k$ *are.*

Theorem 5.2.8 (Hypernatural induction) *Let* $\varphi(x)$ *be a quantifier-free formula not involving* min *or* $\mathcal{I}$*, such that*

1. $\varphi(0)$ *holds,*
2. *the implication* $(\mathcal{N}(n) \wedge \varphi(n)) \rightarrow \varphi(n+1)$ *holds.*

Then $\varphi(n)$ *holds for all hypernatural* n*.*

Proof. Suppose, on the contrary, that there is a hypernatural n such that $\neg\varphi(n)$. By Theorem 5.2.6, there is a least hypernatural n_0 such that $\neg\varphi(n_0)$. By our assumption (1), $n_0 > 0$. Consequently, $\varphi(n_0 - 1)$ does hold. But then, by our assumption (2), so would $\varphi(n_0)$. This contradiction proves the theorem. □

Example 5.2.2 *The sum of two hypernaturals is a hypernatural.*

Proof. Fix any hypernatural N and consider the formula $\mathcal{N}(N+x)$. Both $\mathcal{N}(N+0)$ and $\mathcal{N}(N+n) \rightarrow \mathcal{N}(N+n+1)$ are included in axiom set 5.2.2. Hence $\mathcal{N}(N+n)$ for every hypernatural n. □

Example 5.2.3 *Let* $\varphi(n)$ *be a quantifier-free formula not involving* min *or* $\mathcal{I}$*. If* $n_0 < n_1$ *are hypernaturals such that* $n_0 \leq n \leq n_1 - 1 \rightarrow \varphi(n) = \varphi(n+1)$*, then* $\varphi(n_0) = \varphi(n_1)$*.*

Proof. The formula $\varphi(n_0) = \varphi(n_0 + x)$ holds for $x = 0$. If $\varphi(n_0) = \varphi(n_0 + n)$ for any hypernatural $n_0 + n \leq n_1 - 1$, then also $\varphi(n_0) = \varphi(n_0 + n + 1)$ by assumption. Hence $\varphi(n_0) = \varphi(n_0 + n)$ for $0 \leq n \leq n_1 - n_0$. □

For further use we collect here some definable functions, being terms of the language that (provably in ERNA) have the properties of the function.

1. The identity function $id(x) = x$ is $\pi_{1,1}$.

2. For each closed term τ and each arity k, the constant function
$$C_{k,\tau}(x_1, \ldots, x_k) = \tau,$$
is $\pi_{k+1,k+1}(x_1, \ldots, x_k, \tau)$.

3. The hypersequence
$$r(n) = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n \geq 1 \end{cases}$$
is $\mathrm{rec}^k_{\sigma\tau}$ with $k = 1$, $\sigma = 0$, $\tau = C_{2,1}$.

4. The function
$$\zeta(x) = \begin{cases} 1 & \text{if } x = 0 \\ x & \text{otherwise} \end{cases}$$
is $1 + x - r(\lceil |x| \rceil)$.

5. The functions
$$h(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad H(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$
are $\frac{x+|x|}{2\zeta(x)}$ and $\frac{1}{2} + \frac{\zeta(|x|)}{2\zeta(x)}$, respectively.

6. The function
$$1_{(a,b]}(x) = \begin{cases} 1 & \text{if } a < x \leq b \\ 0 & \text{otherwise} \end{cases}$$
is $h(x-a)H(b-x)$. Likewise for the characteristic function of any other interval.

7. For constants a, b and terms ρ, σ, τ, the function
$$d(x) = \begin{cases} \sigma(x) & \text{if } a < x \leq b \text{ and } \rho(x) > 0 \\ \tau(x) & \text{otherwise} \end{cases}$$
is $1_{(a,b]}(x)(h(\rho(x))\,\sigma(x) + (1 - h(\rho(x)))\,\tau(x))$. Likewise for any other type of interval in $a < x \leq b$ and/or any other inequality in $\rho(x) > 0$. Any such construction will be called a *definition by cases*. If no interval is specified, the terms ρ, σ, τ and the resulting function can have more than one free variable.

The next theorem is to be considered as an ERNA version of the supremum principle for a set of type $\{x \mid f(x) < 0\}$.

Notation 5.2.7 *We write $a \ll b$ if $a < b$ and $a \not\approx b$.*

Theorem 5.2.9 *Let $b < c$ be constants such that $d := c - b$ is finite. Further, let $f(x)$ be a term not involving $\mathcal{I}$ or* min*, such that $f(x)$ is never undefined for $b \leq x \leq c$. If*

i. $f(c) \geq 0$,

ii. $f(b) < 0$,

then there is a constant γ with the following properties:

iii. $f(\gamma) \geq 0$,

iv. for every natural number $n \geq 1$ there are $x > \gamma - 1/n$ such that $f(x) < 0$.

If $f(x)$ has the extra property

$$(f(x) < 0 \wedge b < y < x) \rightarrow f(y) < 0 \tag{5.1}$$

then γ is, up to infinitesimals, the only constant $> b$ with the properties (iii) and (iv).

Proof. In order to apply recursion, we choose c as our term σ and use definition by cases to obtain the term

$$\tau(t, n) = \begin{cases} t - d/2^n & \text{if } f(t - d/2^n) \geq 0 \\ t & \text{otherwise.} \end{cases}$$

Note that 'otherwise' is equivalent here to 'if $f(t - d/2^n) < 0$', because we have excluded undefined values for $f(x)$. ERNA's unary function symbol $\mathrm{rec}^1_{\sigma\tau}$ for this particular σ and τ will be shortened to rec. Its properties can be stated simply as

$$\mathrm{rec}(0) = c \quad \text{and} \quad \mathrm{rec}(n+1) = \tau(\mathrm{rec}(n), n)$$

because undefined terms cannot occur, and there are no weight requirements because τ has arity two.

If we prove that for any hypernatural n the two properties

$$f(\mathrm{rec}(n)) \geq 0 \tag{5.2}$$

$$f(\mathrm{rec}(n) - d/2^{n-1}) < 0 \quad (n \geq 1) \tag{5.3}$$

hold, we are done. It suffices to take $\gamma = \mathrm{rec}(\omega)$ and to note that, because $d/2^{\omega-1} \approx 0$,

$$\mathrm{rec}(\omega) - \frac{1}{n} < \mathrm{rec}(\omega) - d/2^{\omega-1}$$

for any natural number $n \geq 1$. We prove (5.2) by hypernatural induction. For $n = 0$ the requirement (5.2) is identical with the assumption (i). Now let n be a hypernatural for which (5.2) holds. If $f(\mathrm{rec}(n) - d/2^n) \geq 0$, the definition of τ implies that $\mathrm{rec}(n+1) = \tau(\mathrm{rec}(n), n) = \mathrm{rec}(n) - d/2^n$, which translates the assumption into $f(\mathrm{rec}(n+1)) \geq 0$. Otherwise, $\mathrm{rec}(n+1) = \tau(\mathrm{rec}(n), n) = \mathrm{rec}(n)$, making the induction hypothesis identical with the requirement $f(\mathrm{rec}(n+1)) \geq 0$.

Next we consider (5.3). Our proof demands that $n = 1$ be treated separately. We have $\mathrm{rec}(1) = \tau(\mathrm{rec}(0), 0) = \tau(c, 0)$, and this is simply b since $f(c - d) = f(b) < 0$. Therefore, the property (5.3) is identical with the assumption (ii). Now the proof for any hypernatural $N \geq 2$. We consider the formula

$$\mathcal{N}(n) \ \wedge \ n \leq N - 2 \ \wedge \ \mathrm{rec}(N - n) \neq \mathrm{rec}(N - n - 1) - d/2^{N-n-1} \qquad (5.4)$$

and consider two possibilities. First possibility: there are no hypernaturals n satisfying (5.4). This means that

$$\mathrm{rec}(N - n) - d/2^{N-n-1} = \mathrm{rec}(N - n - 1) - d/2^{N-n-2}$$

for $0 \leq n \leq N - 2$, and by example 5.2.3 it follows that

$$\mathrm{rec}(N) - d/2^{N-1} = \mathrm{rec}(1) - d = c. \qquad (5.5)$$

As $f(c) < 0$, we conclude that (5.3) holds for our N. Second possibility: there are hypernaturals n satisfying (5.4). If so, let n_0 be the smallest one, as provided by theorem 5.2.6. Then $n_0 \leq N - 2$ and

$$\mathrm{rec}(N - n_0) \neq \mathrm{rec}(N - n_0 - 1) - d/2^{N-n_0-1}$$

i.e.

$$\tau(\mathrm{rec}(N - n_0 - 1), N - n_0 - 1) \neq \mathrm{rec}(N - n_0 - 1) - d/2^{N-n_0-1}.$$

The definition of $\tau(t, n)$ shows that then, inevitably,

$$\tau(\mathrm{rec}(N - n_0 - 1), N - n_0 - 1) = \mathrm{rec}(N - n_0 - 1),$$

meaning that

$$f(\mathrm{rec}(N - n_0 - 1) - d/2^{N-n_0-1}) \equiv f(\mathrm{rec}(N - n_0) - d/2^{N-n_0-1}) < 0. \qquad (5.6)$$

By the leastness of n_0,

$$\mathrm{rec}(N - n) - d/2^{N-n-1} = \mathrm{rec}(N - n - 1) - d/2^{N-n-2}$$

for $0 \leq n \leq n_0 - 1$. Hence

$$\mathrm{rec}(N) - d/2^{N-1} = \mathrm{rec}(N - n_0) - d/2^{N-n_0-1}$$

by example 5.2.3. Substituting in (5.6) gives

$$f(\mathrm{rec}(N) - d/2^{N-1}) < 0,$$

as was to be proved.

Finally, assume the extra property (5.1). If $\gamma' > b$ is another constant with the properties (iii) and (iv), we cannot have $\gamma' \ll \gamma$, as property (iv) for γ would imply that there are $x > \gamma'$ satisfying $f(x) < 0$, which by (5.1) leads to $f(\gamma') < 0$ and contradicts the property (iii) for γ'. Likewise for the possibility $\gamma \ll \gamma'$. Therefore $\gamma' \approx \gamma$. □

This theorem allows us to equip ERNA with a *square root up to infinitesimals* function.

Example 5.2.4 *For every finite constant $p > 0$,* ERNA *provides a constant $\gamma > 0$, unique up to infinitesimals, such that $\gamma^2 \approx p$.*

Proof. It follows from the properties of an ordered field that the term $f(x) = x^2 - p$ and the constants $b = 0, c = 1 + p$ satisfy the requirements of theorem 5.2.9, including the extra requirement (5.1). If γ is the constant resulting from the theorem, then $\gamma^2 \geq p$ and for every natural $n \geq 1$ there are $x > \gamma - 1/n$ with $x^2 < p$. Moreover, $x < 1 + p$ by the properties of the ordered field. Hence $\gamma^2 < x^2 + 2x/n + 1/n^2 < p + 2(1+p)/n + 1/n^2$. By corollary 5.2.1, we conclude that $0 \leq \gamma^2 - p \approx 0$. □

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6

The Sousa Pinto approach to nonstandard generalised functions

R. F. Hoskins*

Abstract

Nonstandard Analysis suggests several ways in which the standard theories of distributions and other generalised functions could be reformulated. This paper reviews the contributions of José Sousa Pinto to this area up to his untimely death four years ago. Following the original presentation of nonstandard models for the Sebastião e Silva axiomatic treatment of distributions and ultradistributions he worked on a nonstandard theory of Sato hyperfunctions, using a simple ultrapower model of the hyperreals. (This in particular allows nonstandard representations for generalised distributions, such as those of Roumieu, Beurling, and so on.) He also considered a nonstandard theory for the generalised functions of Colombeau, and finally turned his attention to the hyperfinite representation of generalised functions, following the work of Kinoshita.

6.1 Introduction

José Sousa Pinto of the University of Aveiro, Portugal, died in August 2000 after a prolonged and debilitating illness. His interest in nonstandard methods, particularly in their application to the study of generalised functions, was of long standing and he will be especially remembered for his part in the organisation of the highly successful **International Colloquium of Nonstandard Mathematics** held at Aveiro [2] in 1994. A most modest and unassuming mathematician, his contributions to NSA are less well known than their value deserves and this paper is concerned to report his work and to stand as some tribute to his memory. From a personal point of view I would also wish to take

*Department of Electronic and Electrical Engineering, Loughborough University, Loughborough, UK.
royhosk@aol.com

this opportunity to acknowledge the value and pleasure I have had in working with him over many years.

6.1.1 Generalised functions and N.S.A.

The theory of generalised functions is a subject of major importance in modern analysis and one that has gone through many changes since the original presentation [18] of the theory of distributions in the form given to it by Laurent Schwartz in the 1950s. Various alternative approaches to the theory have been explored over the years, and the subject has been expanded (and complicated) by the introduction of generalised distributions of several types, ultradistributions, hyperfunctions and so on. In recent years the development of a unifying and simplifying treatment of the whole subject area has become possible through the use of Nonstandard Analysis (N.S.A.) The application of nonstandard methods to distributions and other generalised functions was already considered by Abraham Robinson in his classic text [15] on N.S.A. in 1966, and various workers have extended and developed this approach since then. It was Sousa Pinto who first considered the possibility of developing a nonstandard realisation of the Sebastiao e Silva axioms for Schwartz distributions [6], and later for ultradistributions [7]. His further work on Sato hyperfunctions remained unpublished at the time of his death and an outline of this forms the main part of the present paper.

The first section of the paper briefly reviews standard material on distributions, ultradistributions and Sato hyperfunctions, and summarises the earlier nonstandard re-formulation of that material on which the subsequent development is based.

6.2 Distributions, ultradistributions and hyperfunctions

6.2.1 Schwartz distributions

We recall first some basic facts about distributions. A distribution, in the sense of Schwartz, is a continuous linear **functional** on the space $\mathcal{D} \equiv \mathcal{D}(\mathbb{R})$ of all infinitely differentiable functions of compact support, equipped with an appropriate topology. That is to say, a distribution is simply a member μ of the topological dual $\mathcal{D}'(\mathbb{R})$ of that space. The **distributional derivative** of $\mu \in \mathcal{D}'$ is the distribution $D\mu$ defined by

$$< D\mu, \phi >=< \mu, -\phi' >, \quad \forall \phi \in \mathcal{D}.$$

It follows from this definition that all distributions are infinitely differentiable in this sense. Moreover, it can be shown that every distribution is locally a finite-order derivative of a continuous function. Those distributions which are *globally* representable as finite-order derivatives of continuous functions are called, not unnaturally, **finite-order distributions**. The space of all such finite-order distributions is denoted by $\mathcal{D}'_{fin}(\mathbb{R})$.

Every locally integrable function f defines a so-called **regular** distribution μ_f according to

$$< \mu_f, \phi >= \int_{-\infty}^{+\infty} f(x)\phi(x)dx, \qquad \text{for all } \phi \in \mathcal{D}.$$

$\mathcal{D}'$ contains elements other than such regular distributions, so that $\mathcal{D}'$ is a proper extension of the space $\mathcal{L}_{loc}(\mathbb{R})$ of all locally integrable functions. In this sense distributions may be legitimately described as *generalised functions*. However there is no direct sense in which a distribution can be said to have a value at a point. This becomes particularly clear in the case of those distributions which are not regular. The delta function is the prime example of such a **singular** distribution, being defined simply as that functional δ (obviously linear and continuous) which maps each function $\phi \in \mathcal{D}$ into the number $\phi(0)$. It is a finite-order distribution since it is the second derivative of the continuous function $x_+(t) \equiv tH(t)$, where H denotes the Heaviside unit step.

6.2.2 The Silva axioms

The definition of distributions as equivalence classes of nonstandard internal functions in a nonstandard universe was already made explicit in Abraham Robinson's original text [15] on N.S.A. Several other nonstandard models for $\mathcal{D}'(\mathbb{R})$ have since appeared. In particular such a model was presented by Hoskins and Pinto [6] in 1991, based on the axiomatic treatment of distributions given by Sebastião e Silva [19] in 1956. The Silva axioms for finite-order distributions on an interval $I \subset \mathbb{R}$ can be stated as follows:

Silva axioms for finite order distributions

Distributions are elements of a linear space $\mathcal{E}(I)$ for which two linear maps are defined: $\iota : \mathcal{C}(I) \to \mathcal{E}(I)$ and $D : \mathcal{E}(I) \to \mathcal{E}(I)$, such that

S1 ι is the injective identity, (every $f \in \mathcal{C}$ is a distribution).

S2 To each $\nu \in \mathcal{E}$ there corresponds $D\nu \in \mathcal{E}$ such that, if $\nu = \iota(f) \in \mathcal{C}^1(I)$ then $D\nu = \iota(f')$

S3 For $\nu \in \mathcal{E}$ there exists $f \in \mathcal{C}$ and $r \in \mathbb{N}_0$ such that $\nu = D^r\iota(f)$.

S4 Given $f, g \in \mathcal{C}$ and $r \in \mathbb{N}_0$, the equality $D^r\iota(f) = D^r\iota(g)$ holds if and only if $(f - g)$ is a polynomial of degree $< r$.

Silva gives an abstract model for this set of axioms as follows: define an equivalence relation $\diamond$ on $\mathbb{N}_0 \times \mathcal{C}$ by

$$(r, f) \diamond (s, g) \leftrightarrow \exists m \in \mathbb{N}_0 \left\{ m \geq r, s \wedge (\mathcal{I}_a^{m-r} f - \mathcal{I}_a^{m-s} g) \in \Pi_m \right\}$$

where Π_m is the set of all complex-valued polynomials of degree less than m and $\mathcal{I}_a^k$ is the kth iterated indefinite integral operator with origin at $a \in I$. Now write

$$\mathcal{C}_\infty \equiv \mathcal{C}_\infty(I) = \mathbb{N}_0 \times \mathcal{C}/\diamond$$

Then $\mathcal{C}_\infty$ is a model for the Silva axioms S1-S4, and every model for the Silva axioms is isomorphic to $\mathcal{C}_\infty$. In particular $\mathcal{D}'_{fin}(\mathbb{R})$ is isomorphic to $\mathcal{C}_\infty(\mathbb{R})$. The extension to global distributions of arbitrary order is straightforward. See, for example, the exposition of the Silva approach to distribution theory given in Campos Ferreira, [3].

A nonstandard model for $\mathcal{C}_\infty$

A nonstandard model for these axioms was presented by Hoskins and Pinto [6], using a simple ultrapower model ${}^*\mathbb{R} = \mathbb{R}^{\mathbb{N}}/\sim$ for the hyperreals. It may be summarised as follows:

The internal set ${}^*\mathcal{C}^\infty(\mathbb{R})$ is the nonstandard extension of the standard set $\mathcal{C}^\infty(\mathbb{R})$ of all infinitely differentiable functions on $\mathbb{R}$,

$${}^*\mathcal{C}^\infty(\mathbb{R}) = \{F = [(f_n)_{n\in\mathbb{N}}] : \ f_n \in \mathcal{C}^\infty(\mathbb{R}) \quad \text{for nearly all } n \in \mathbb{N}\}.$$

This set is a differential algebra. We denote by ${}^S\mathcal{C}(\mathbb{R})$ the (external) set of all functions $F \in {}^*\mathcal{C}^\infty(\mathbb{R})$ which are finite-valued and S-continuous at each point of ${}^*\mathbb{R}_b$. An internal function $F \in {}^*\mathcal{C}^\infty(\mathbb{R})$ is then said to be a **predistribution** if it is a finite-order $*$derivative of a function in ${}^S\mathcal{C}(\mathbb{R})$. The set of all such pre-distributions is given by,

$${}^*D^\infty\{{}^S\mathcal{C}(\mathbb{R})\} \equiv \bigcup_{r\geq 0} {}^*D^r\{{}^S\mathcal{C}(\mathbb{R})\}$$

$$= \{F \in {}^*\mathcal{C}^\infty(\mathbb{R}) : \ F = {}^*D^r\Phi \text{ for some } \Phi \in {}^S\mathcal{C}(\mathbb{R}) \text{ and some } r \in \mathbb{N}_0\}.$$

We then have the following (strict) inclusions:

$${}^S\mathcal{C}(\mathbb{R}) \subset {}^*D^\infty\{{}^S\mathcal{C}(\mathbb{R})\} \subset {}^*\mathcal{C}^\infty(\mathbb{R}).$$

The members of ${}^*D^\infty\{{}^S\mathcal{C}(\mathbb{R})\}$ are the nonstandard representatives of finite order distributions on $\mathbb{R}$. Given two such pre-distributions F and G, we say that

they are **distributionally equivalent**, and write $F \Xi G$, if and only if there exists an integer $m \in \mathbb{N}_0$ and a polynomial p_m of degree m (with coefficients in ${}^*\mathbb{R}$) such that

$$ {}^*\mathcal{I}_a^m (F - G) \approx p_m $$

where ${}^*\mathcal{I}_a^m$ denotes the mth-order *indefinite integral operator from $a \in {}^*\mathcal{R}_b$. Then for any $F \in {}^*D^\infty\{{}^S\mathcal{C}(\mathbb{R})\}$ we denote by $\mu_F = {}^\Xi[F]$ the equivalence class containing F and call it a **finite order Ξdistribution**. The set of all such equivalence classes is denoted by

$$ {}^\Xi\mathcal{C}_\infty(\mathbb{R}) \equiv {}^*D^\infty\{{}^S\mathcal{C}(\mathbb{R})\}/\Xi. $$

${}^\Xi\mathcal{C}_\infty(\mathbb{R})$ is a nonstandard model for the Silva axioms and is isomorphic with $\mathcal{D}'_{fin}(\mathbb{R})$.

6.2.3 Fourier transforms and ultradistributions

For the classical Fourier transform of sufficiently well-behaved functions we have

$$ \tilde{f}(x) = \int_{-\infty}^{+\infty} f(y) e^{-ixy} dy \quad ; \quad f(y) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \tilde{f}(x) e^{ixy} dx $$

and the Parseval relation

$$ \int_{-\infty}^{+\infty} f(x) \tilde{g}(x) dx = \int_{-\infty}^{+\infty} \tilde{f}(y) g(y) dy. $$

To extend the definition of Fourier transform to distributions Schwartz used a generalised form of Parseval relation to define $\tilde{\mu}$ as the functional satisfying

$$ < \tilde{\mu}, \phi > = < \mu, \tilde{\phi} > . $$

The difficulty here is that if $\phi \in \mathcal{D}$ then its Fourier transform $\tilde{\phi}$ belongs not to $\mathcal{D}$ but to another space $\mathcal{Z} \equiv \mathcal{Z}(\mathbb{R})$ which comprises all those functions ψ such that $\psi(z)$ is defined on $\mathbb{C}$ as an entire function satisfying an inequality of the form

$$ |z^k \psi(z)| \leq C_k \exp(a|y|), \quad a > 0, \quad k = 0, 1, 2, \ldots . $$

Since $\mathcal{D} \cap \mathcal{Z} = \emptyset$ it follows that although $\tilde{\mu}$ is well defined as a linear continuous functional on the space $\mathcal{Z}$ it is not necessarily defined on $\mathcal{D}$ and may therefore not be a distribution. The members of $\mathcal{Z}'(\mathbb{R})$ are called **ultradistributions**, and constitute another class of generalised functions. Although there are functionals which are both distributions and ultradistributions there exist distributions which are not ultradistributions and ultradistributions which are not distributions.

Nonstandard representation of ultradistributions

In [7] a slight modification of the argument presented in [6] showed that every Ξdistribution may be represented by an internal function in ${}^*\mathcal{D}(\mathbb{R})$. Accordingly we can redefine ${}^{\Xi}\mathcal{C}_\infty(\mathbb{R})$ as follows:

$$ {}^{\Xi}\mathcal{C}_\infty(\mathbb{R}) = {}^*D^\infty({}^S\mathcal{D})/\Xi = \bigcup_{r\geq 0}\{{}^*D^r({}^S\mathcal{D})\}/\Xi $$

where ${}^S\mathcal{D}$ is the ${}^*\mathbb{C}_b$ submodule of ${}^S\mathcal{C}$ comprising all infinitely *differentiable functions of hypercompact support which are finite-valued and S-continuous on ${}^*\mathbb{R}_b$.

If $\tilde{F} \equiv [(\tilde{f}_n)_{n\in\mathbb{N}}]$ is any internal function in ${}^*\mathcal{D}$ then its inverse Fourier transform $F = {}^*\mathcal{F}^{-1}\{\tilde{F}\}$ is defined in the obvious way as $F \equiv [(f_n)_{n\in\mathbb{N}}] = [(\mathcal{F}^{-1}\{f_n\})_{n\in\mathbb{N}}]$ and it follows readily that $\tilde{F} \in {}^*\mathcal{D}$ if and only if $F \equiv \mathcal{F}^{-1}\{\hat{F}\} \in {}^*\mathcal{Z}$. Not every internal function in ${}^*\mathcal{D}$ represents a Ξdistribution and similarly not every internal function in ${}^*\mathcal{Z}$ represents an ultradistribution. However the following result was established in [7].

Let $\mathcal{H}(\mathbb{C})$ denote the space of all standard complex-valued functions which can be extended into the complex plane as entire functions. For each entire function $A(z) = \sum_{n=0}^{\infty} a_n z^n$ in $\mathcal{H}(\mathbb{C})$ define the ∞-*order operator* $\mathbf{A} : \mathcal{Z} \to \mathcal{Z}$ by setting, for each $\phi \in \mathcal{Z}$,

$$ \mathbf{A}[\phi(t)] = \sum_{n=0}^{\infty} a_n(-iD)^n\phi(t) = \sum_{n=0}^{\infty}(-i)^n\phi^{(n)}(t). $$

Then:

Theorem 1 *The inverse Fourier transform of a Ξdistribution in ${}^{\Xi}\mathcal{C}_\infty(\mathbb{R})$ is representable as a finite sum of (standard) ∞-order derivatives of internal functions in ${}^*\mathcal{Z}$ whose standard parts are continuous functions of polynomial growth.*

6.2.4 Sato hyperfunctions

Another approach to the required generalisation of the Fourier transform stems from the work of Carlemann [4]. He observed that if a function f, not necessarily in $\mathcal{L}^1(\mathbb{R})$, satisfies a condition of the form

$$ \int_0^x |f(y)|dy = 0(|x|^\kappa) \qquad \text{for some natural number } \kappa, $$

and if we write

$$ g_1(z) = \int_{-\infty}^{0} f(y)e^{-izy}dy, \quad \text{and} \quad g_2(z) = -\int_0^{+\infty} f(y)e^{-izy}dy $$

then $g_1(z)$ is analytic for all $\Im(z) > 0$ and $g_2(z)$ is analytic for all $\Im(z) < 0$. Moreover, for $\beta > 0$, the function

$$g(x) \equiv g_1(x + i\beta) - g_2(x - i\beta)$$

is the classical Fourier transform of the function $e^{-\beta|t|} f(t)$. The original function f can be recovered by taking the inverse Fourier transform of g and multiplying by $e^{\beta|t|}$. This suggested that a route to a generalisation of the Fourier transform could be found by associating with f a pair of functions $f_1(z)$ and $f_2(z)$ analytic in the upper and lower half-planes respectively. This idea forms the basis of the theory of **hyperfunctions** developed by M. Sato [17] in 1959/60, (although it was anticipated by several other mathematicians). In order to give a brief sketch of this theory it is convenient to introduce the following notation:

$\mathcal{H}(\mathbb{C}\backslash\mathbb{R})$ = the space of all functions analytic outside the real axis.

$\mathcal{H}(\mathbb{C})$ = subspace of all functions in $\mathcal{H}(\mathbb{C}\backslash\mathbb{R})$ which are entire.

$\mathcal{H}^{p,loc}(\mathbb{C}\backslash\mathbb{R})$ = space of all functions θ in $\mathcal{H}(\mathbb{C}\backslash\mathbb{R})$ which are of arbitrary growth to infinity but locally of polynomial growth to the real axis (that is, such that for each compact $K \subset \mathbb{R}$ there exists $C_K > 0$ and $r_K \in \mathbb{N}_0$ such that

$$|\theta(z)| \leq C_K |\Im(z)|^{r_K}$$

for all $z \in \mathbb{C}$ with $\Re(z) \in K$ and sufficiently small $\Im(z) \neq 0$).

Definition 2 *The **hyperfunctions** of Sato are the members of the quotient space $\mathcal{H}_S(\mathbb{R}) = \mathcal{H}(\mathbb{C}\backslash\mathbb{R})/\mathcal{H}(\mathbb{C})$, that is, the set of all equivalence classes $[\theta]$, where $\theta(z)$ is defined and analytic on $\mathbb{C}\backslash\mathbb{R}$ and $\theta_1 \sim \theta_2$ iff $\theta_1 - \theta_2$ is entire.*

Sato hyperfunctions constitute a genuine extension of Schwartz distributions. This is shown by the following crucial result established by Bremermann [1], in 1965:

Theorem 3 (Bremmerman) *If μ is any distribution in $\mathcal{D}'$ then there exists a function $\mu^0(z)$ defined and analytic in $\mathbb{C}\backslash\mathbb{R}$ such that*

$$< \mu, \phi >= \lim_{\varepsilon \to 0} \int_{-\infty}^{+\infty} \{\mu^0(x + i\varepsilon) - \mu^0(x - i\varepsilon)\}\phi(x)dx.$$

Conversely, if $\theta \in \mathcal{H}^{p,loc}(\mathbb{C}\backslash\mathbb{R})$ then there exists $\mu \in \mathcal{D}'$ such that $\theta(z) = \mu^0(z)$. Identifying $\mu \in \mathcal{D}'$ with $[\theta] \in \mathcal{H}_S(\mathbb{R})$ gives an embedding of $\mathcal{D}'$ such that the mapping $S : \mathcal{D}' \to \mathcal{H}^{p,loc}(\mathbb{C}\backslash\mathbb{R})$ is a topological isomorphism.

Remark 4 Note that if μ has compact support then $\mu^0(z)$ is given explicitly by

$$\mu^0(z) = \frac{1}{2\pi i} < \mu, (x-z)^{-1} > .$$

For example, $\delta^0(z) = \frac{1}{2\pi i} < \delta, (x-z) >= -\frac{1}{2\pi i z}$ and we have

$$< \delta, \phi >= \lim_{\varepsilon \to 0} \frac{\varepsilon}{\pi} \int_{-\infty}^{+\infty} \frac{1}{x^2+\varepsilon^2} \phi(x)\, dx = \phi(0).$$

6.2.5 Harmonic representation of hyperfunctions

For each Sato hyperfunction $[\theta]$ in $\mathcal{H}_S(\mathbb{R})$ we can choose some specific function $\theta^0 \in [\theta]$ as the **defining function** of the hyperfunction and write

$$\theta_\pi(x,y) = \theta^0(x+iy) - \theta^0(x-iy).$$

Then θ_π maps the half-plane $\Pi^+ \equiv \mathbb{R} \times \mathbb{R}^+$ into $\mathbb{C}$, and is harmonic on Π^+. Define $\mathbf{H}(\Pi^+)$ to be the linear space of all (real or complex-valued) functions defined and harmonic on Π^+, and let $\Gamma : \mathcal{H}(\mathbb{C}\backslash\mathbb{R}) \to \mathbf{H}(\Pi^+)$ denote the map given by

$$\theta^0 \in \mathcal{H}(\mathbb{C}\backslash\mathbb{R}) \to \Gamma(\theta^0) = \theta_\pi(x,y).$$

Then we have the result

Theorem 5 (Li Bang-He, [13]) *Γ is an onto map and if $\Gamma(\theta^0) = \Gamma(\nu^0)$ then $\theta^0 - \nu^0$ is a complex constant.*

Now suppose that $[\theta]$ is the null hyperfunction, so that θ^0 belongs to $\mathcal{H}(\mathbb{C})$. It is easily shown that $\theta_\pi(x,y)$ is an entire function in both variables (that is, can be extended into $\mathbb{C} \times \mathbb{C}$ as an entire function), is odd in the variable y and such that

$$\lim_{y \to 0} \theta_\pi(x,y) = 0.$$

On the other hand we have immediately from the above theorem,

Corollary 6 *Let $\theta_\pi \in \mathbf{H}(\Pi^+)$ be an harmonic function entire in both variables and odd in the variable y. Then there exists an entire function $\theta \in \mathcal{H}(\mathbb{C})$ such that*

$$\theta_\pi(x,y) = \theta(x+iy) - \theta(x-iy)$$

for all $(x,y) \in \Pi^+$.

Denote by $\mathbf{H}_0(\Pi^+)$ the linear subspace of $\mathbf{H}(\Pi^+)$ comprising all functions which extend into $\mathbb{C} \times \mathbb{C}$ as entire functions in both variables and which are odd in the second variable. Then we have

$$\mathcal{H}_S(\mathbb{R}) \sim \mathbf{H}(\Pi^+)/\mathbf{H}_0(\Pi^+)$$

and every equivalence class $[\theta_\pi(x,y)] \in \mathbf{H}(\Pi^+)/\mathbf{H}_0(\Pi^+)$ is a representation by harmonic functions of the corresponding hyperfunction $[\theta] \in \mathcal{H}_S(\mathbb{R})$. In the sequel we will use analytic or harmonic representation for hyperfunctions as occasion demands.

6.3 Prehyperfunctions and predistributions

Nonstandard representation of hyperfunctions

In the present context $\omega \in {}^*\mathbb{R}$ denotes the infinite hypernatural number defined by $[(n)_{n\in\mathbb{N}}]$ in ${}^*\mathbb{N}_\infty$. Then to each harmonic function $\nu_\pi \in \mathbf{H}(\Pi^+)$ there corresponds an internal function $F_{\{\nu\}} : {}^*\mathbb{R} \to {}^*\mathbb{C}$ defined by

$$F_{\{\nu\}}(x) = {}^*\nu_\pi(x,\varepsilon)$$

where $\varepsilon = \omega^{-1} \in mon(0)$ and ${}^*\nu_\pi \equiv {}^*(\nu_\pi)$ is the nonstandard extension of ν_π. The internal function $F_{\{\nu\}}$ clearly belongs to ${}^*\mathcal{C}^\infty(\mathbb{R})$. If we define a map $\hat{\omega} : \mathbf{H}(\Pi^+) \to {}^*\mathcal{C}^\infty(\mathbb{R})$ by setting $\hat{\omega}(\nu_\pi) = F_{\{\nu\}}$ then we have

$${}^\omega\mathcal{H}_S(\mathbb{R}) \equiv \hat{\omega}(\mathbf{H}(\Pi^+)) \subset {}^*\mathcal{C}^\infty(\mathbb{R})$$

where the inclusion is strict.

The set ${}^\omega\mathcal{H}_S \equiv \hat{\omega}(\mathbf{H}(\Pi^+))$ is an external subset of ${}^*\mathcal{C}^\infty(\mathbb{R})$; however it can be embedded into an internal subset as follows: Consider the nonstandard extension ${}^*\mathbf{H}(\Pi^+)$ of $\mathbf{H}(\Pi^+)$ comprising all *harmonic functions on ${}^*\Pi^+$ and then define,

$${}^\omega\mathbf{H}(\mathbb{R}) \equiv \{F \in {}^*\mathcal{C}^\infty(\mathbb{R}) : [\exists \Theta \in {}^*\mathbf{H}(\Pi^+) : F(x) = \Theta(x,\varepsilon), \forall_{x\in{}^*R}]\}.$$

Then we have

$${}^\omega\mathcal{H}_S \equiv \hat{\omega}(\mathbf{H}(\Pi^+)) \subset {}^\omega\mathbf{H}(\mathbb{R}) \subset {}^*\mathcal{C}^\infty(\mathbb{R})$$

where ${}^\omega\mathbf{H}(\mathbb{R})$ contains elements which are infinitely close to members of ${}^\omega\mathcal{H}_S \equiv \hat{\omega}(\mathbf{H}(\Pi^+))$ and also elements which are far from any internal function in that space. The members of ${}^\omega\mathbf{H}(\mathbb{R})$ will generally be called **prehyperfunctions**. Prehyperfunctions which are near some element of ${}^\omega\mathcal{H}_S(\mathbb{R})$ are said to be **nearstandard prehyperfunctions**, and the others may be called **remote prehyperfunctions**. The set of all such nearstandard prehyperfunctions will be denoted by

$${}^\omega\mathbf{H}_{ns}(\mathbb{R}) \supset {}^\omega\mathcal{H}_S \equiv \hat{\omega}(\mathbf{H}(\Pi^+)).$$

The elements of ${}^\omega\mathbf{H}(\mathbb{R})$ enjoy an important property which is not shared by all internal functions in ${}^*\mathcal{C}^\infty(\mathbb{R})$, namely:

Theorem 7 *Every prehyperfunction in* ${}^{\omega}\mathbf{H}(\mathbb{R})$ *on the line may be extended into the hypercomplex plane as a* *analytic function in the infinitesimal strip*

$$\Omega_{\varepsilon} = \{z \in {}^{*}\mathbb{C} : \ |Im(z)| < \varepsilon\}.$$

Proof. Any function is analytic at the centre of an open disc on which it is harmonic. Hence any internal function $F(x)$, $x \in {}^{*}R$ in ${}^{\omega}\mathbf{H}(\mathbb{R})$ extends into the hypercomplex plane $z = \xi + i\eta$ and is *analytic on the disc

$$(\xi - x)^2 + \eta^2 < \varepsilon^2.$$

For $\xi = x$ we have $-\varepsilon < \eta < \varepsilon$ and since x may be any hyperreal it follows that the internal function $F_{\{\nu\}}(x)$ extends as a *analytic function into the infinitesimal strip in ${}^{*}\mathbb{C}$ defined by $|Im(z)| < \varepsilon$. □

The converse does not hold: not every *analytic internal function in the infinitesimal strip Ω_{ε} is a prehyperfunction on the line. The product of two prehyperfunctions, for example, is a *analytic function on the strip Ω_{ε} but need not itself be a prehyperfunction: ${}^{\omega}\mathbf{H}(\mathbb{R})$ is a linear space over ${}^{*}\mathbb{C}$ but not an algebra. Now let $\mathcal{A}(\Omega_{\varepsilon})$ denote the set of all internal functions in ${}^{*}\mathcal{C}^{\infty}(\mathbb{R})$ which may be extended *analytically into the strip Ω_{ε}. $\mathcal{A}(\Omega_{\varepsilon})$ is a differential subalgebra of ${}^{*}\mathcal{C}^{\infty}(\mathbb{R})$ with respect to the usual operations of addition, multiplication and *differentiation. Moreover we have,

$${}^{\omega}\mathcal{H}_S(\mathbb{R}) \subset {}^{\omega}\mathbf{H}_{ns}(\mathbb{R}) \subset {}^{\omega}\mathbf{H}(\mathbb{R}) \subset \mathcal{A}(\Omega_{\varepsilon}) \subset {}^{*}\mathcal{C}^{\infty}(\mathbb{R}).$$

The product of any two prehyperfunctions makes sense within the algebra $\mathcal{A}(\Omega_{\varepsilon})$ although such a product will not generally be a prehyperfunction. In view of the above inclusions it seems appropriate to call the members of $\mathcal{A}(\Omega_{\varepsilon})$ **generalised prehyperfunctions**.

Finally let $\mathcal{O}(\Omega_{\varepsilon})$ be the subset of all internal functions $\Theta \in \mathcal{A}(\Omega_{\varepsilon})$ such that ${}^{*}D^{k}\Theta(x) \approx 0$ for all $x \in {}^{*}\mathbb{R}_b$ and for all $k \in \mathbb{N}_0$. $\mathcal{O}(\Omega_{\varepsilon})$ is a linear ${}^{*}\mathbb{C}_b$-submodule (but not an ideal) of $\mathcal{A}(\Omega_{\varepsilon})$ and therefore

$$\mathcal{A}(\Omega_{\varepsilon})/\mathcal{O}(\Omega_{\varepsilon})$$

is a module over ${}^{*}\mathbb{C}_b$: its elements might conveniently be called **generalised ${}^{\Xi}$-hyperfunctions**. Every such generalised ${}^{\Xi}$-hyperfunction which may be represented by an internal function in ${}^{\omega}\mathbf{H}_{ns}(\mathbb{R})$ is called a **standard ${}^{\Xi}$-hyperfunction** or simply a **${}^{\Xi}$-hyperfunction**.

The set of all ${}^{\Xi}$-hyperfunctions on the line is defined by

$${}^{\Xi}\mathcal{H}_S(\mathbb{R}) \equiv {}^{\omega}\mathcal{H}_S(\mathbb{R})/\mathcal{O}(\Omega_{\varepsilon})$$

and we have the isomorphism

$$\mathcal{H}_S(\mathbb{R}) \sim {}^{\Xi}\mathcal{H}_S(\mathbb{R}).$$

6.4 The differential algebra $\mathcal{A}(\Omega_\varepsilon)$

6.4.1 Predistributions of finite order

Although not every continuous function f on $\mathbb{R}$ may be continued analytically into the complex plane, every such function does admit an analytic representation in the sense that there exists a unique hyperfunction $[f_\pi(x,y)]$ in $\mathcal{H}_S(\mathbb{R})$ such that $f_\pi(x,y) \to f(x)$ as $y \downarrow 0$ uniformly on compacts. Then the internal function $F(x) = {}^*f_\pi(x,\varepsilon)$ belongs to ${}^\omega\mathcal{H}_S(\mathbb{R}) \subset {}^\omega\mathbf{H}_{ns}(\mathbb{R}) \subset \mathcal{A}(\Omega_\varepsilon)$ and is such that

$$F(x) \approx {}^*f(x), \qquad \text{for all } x \in {}^*\mathbb{R}_b.$$

$F(x)$ is S-continuous at every point (standard and nonstandard) of ${}^*\mathbb{R}_b$; that is,

$$\forall x, y \in {}^*\mathbb{R}_b \, [x \approx y \Rightarrow F(x) \approx F(y)].$$

Reciprocally, every internal function $F \in \mathcal{A}(\Omega_\varepsilon)$ which is finite and S-continuous at every point $x \in {}^*\mathbb{R}_b$ is infinitely close to a (standard) continuous function f defined on $\mathbb{R}$. We denote by ${}^S\mathcal{C}(\Omega_\varepsilon)$ the subalgebra of all functions in $\mathcal{A}(\Omega_\varepsilon)$ which are finite and S-continuous on ${}^*\mathbb{R}_b$.

An internal function $F : {}^*\mathbb{R} \to \mathbb{C}$ is said to be **finitely *differentiable** at $x \in {}^*\mathbb{R}$ if it is *differentiable at x and, in addition, ${}^*DF(x)$ is a bounded number.

Theorem 8 *An internal function $F \in \mathcal{A}(\Omega_\varepsilon)$ which is finitely *differentiable on ${}^*\mathbb{R}_b$ belongs to ${}^S\mathcal{C}(\Omega_\varepsilon)$.*

Definition 9 *For any internal function F in ${}^S\mathcal{C}(\Omega_\varepsilon)$ the standard function $f = st(F)$ will be called the **shadow** of F while the regular distribution $\nu_F = st_{\mathcal{D}}(F) \in \mathcal{D}'(\mathbb{R})$ generated by F will be called the **$\mathcal{D}$-shadow** of F.*

Now let Φ be an arbitrary function in ${}^S\mathcal{C}(\Omega_\varepsilon)$. Since ${}^S\mathcal{C}(\Omega_\varepsilon)$ is not a differential algebra the derivative, ${}^*D\Phi$, will not necessarily belong to ${}^S\mathcal{C}(\Omega_\varepsilon)$. However it is easy to see that the functional $\mu_{{}^*D\Phi} : \mathcal{D}(\mathbb{R}) \to \mathbb{C}$ defined by

$$< \mu_{{}^*D\Phi}, \phi >= st(< \Phi, -{}^*\phi' >)$$

is a well defined distribution. Hence ${}^*D\Phi$ has a well-defined $\mathcal{D}$-shadow. If, in addition, ${}^*D\Phi$ itself belongs to ${}^S\mathcal{C}(\Omega_\varepsilon)$ then $\nu_{{}^*D\Phi}$ is the (regular) distribution generated by the standard function $f' = st({}^*D\Phi)$ which is the standard derivative of the function $f = st(\Phi)$. Let ${}^*D^0\{{}^S\mathcal{C}(\Omega_\varepsilon)\}$ denote the set of derivatives of functions in ${}^S\mathcal{C}(\Omega_\varepsilon)$ and for each $k \in \mathbb{N}_0$ define

$${}^*D^k\{{}^S\mathcal{C}(\Omega_\varepsilon)\} = {}^*D\left\{{}^*D^{k-1}\{{}^S\mathcal{C}(\Omega_\varepsilon)\}\right\}$$

where ${}^*D^0\{{}^S\mathcal{C}(\Omega_\varepsilon)\} \equiv {}^S\mathcal{C}(\Omega_\varepsilon)$. We can now define the following (external) subset of ${}^*\mathcal{H}(\Omega_\varepsilon)$:

$$ {}^S\mathcal{C}_\infty(\Omega_\varepsilon) = \bigcup_{k=0}^{\infty} {}^*D^k\{{}^S\mathcal{C}(\Omega_\varepsilon)\}. $$

Then for each $F \in {}^S\mathcal{C}_\infty(\Omega_\varepsilon)$ there exists an S-continuous function $\Phi \in {}^S\mathcal{C}(\Omega_\varepsilon)$ and an integer $r \in \mathbb{N}_0$ such that $F = {}^*D^r\Phi$. The functional $\mu_F : \mathcal{D}(\mathbb{R})$, defined by

$$ < \mu_F, \phi >= st\left((-1)^r < \Phi, {}^*\phi^{(r)} >\right) = st\left({}^*\int_{{}^*K_\phi} {}^*\phi(x)F(x)dx\right) $$

is a distribution. We call μ_F the **D-shadow** of the internal function F and write $\mu_F = st_{\mathcal{D}}(F)$. That is to say, we extend $st_{\mathcal{D}}$ into ${}^S\mathcal{C}_\infty(\Omega_\varepsilon)$ as a mapping with values in $\mathcal{D}'(\mathbb{R})$. An internal function $F \in {}^*\mathcal{H}(\Omega_\varepsilon)$ is said to be **${}^*S_{\mathcal{D}}$differentiable** if, for every $\phi \in \mathcal{D}$ there exists a standard number b_ϕ such that

$$ < \tau^{-1}\{F(x+\tau) - F(x)\}, {}^*\phi >\approx b_\phi $$

for all $\tau \approx 0$, $\tau \neq 0$. As is easily confirmed, every function F in ${}^S\mathcal{C}_\infty(\Omega_\varepsilon)$ is ${}^*S_{\mathcal{D}}$differentiable, and we define the **^{S}differential order** of F to be the number ${}^So(F)$ defined by

$$ {}^So(F) = \min\{j \in \mathbb{N}_0 : \ F = {}^*D^j\Phi, \quad \text{for some } \Phi \in {}^S\mathcal{C}(\Omega_\varepsilon)\}. $$

Accordingly we call ${}^S\mathcal{C}_\infty(\Omega_\varepsilon)$ the set of all **predistributions of finite ^{S}differential order**.

Replacing the (standard) concept of **distribution of finite order** by the (nonstandard) concept of **predistribution of finite ^{S}differential order** it is clear that ${}^S\mathcal{C}_\infty(\Omega_\varepsilon)$ with the ${}^*D$ operator constitutes a natural (nonstandard) model for the axiomatic definition of distributions of finite order given by J.S. Silva.

Distributional equivalence

We may now glue together all internal functions in ${}^S\mathcal{C}_\infty(\Omega_\varepsilon)$ which have the same distributional shadow. That is to say, we define the following equivalence relation on ${}^S\mathcal{C}_\infty(\Omega_\varepsilon)$:

$F, G \in {}^S\mathcal{C}_\infty(\Omega_\varepsilon)$ are **distributionally equivalent**, written $F \Xi G$, if and only if they have the same distributional shadow. The quotient space

$$ {}^\Xi\mathcal{C}_\infty(\Omega_\varepsilon) = {}^S\mathcal{C}_\infty(\Omega_\varepsilon)/\Xi $$

is a ${}^*\mathbb{C}_b$-module which is isomorphic to the space $\mathcal{C}_\infty(\mathbb{R})$ of J.S. Silva distributions of finite order.

6.4.2 Predistributions of local finite order

Finite order predistributions in ${}^{S}\mathcal{C}_{\infty}(\mathbb{R})$ are not the only prehyperfunctions which have a distributional shadow. Let $F \in {}^{\omega}\mathcal{H}_{S}(\mathbb{R})\backslash{}^{S}\mathcal{C}_{\infty}(\Omega_{\varepsilon})$ be a prehyperfunction such that for every compact $K \subset \mathbb{R}$ there exists an integer $r_K \in \mathbb{N}_0$ and an internal function $\Psi_K \in {}^{\omega}\mathcal{H}_{S}(\mathbb{R})$ which is S-continuous on some *-neighbourhood of ${}^*K$ so that

$$F(x) = {}^*D^{r_K}\Phi_K(x)$$

for all $x \in {}^*K \subset \mathbb{R}_b$. The smallest such r_K will be called the ^{S}differential order of F on ${}^*K$, and denoted ${}^{S}o_K(F)$. If $\phi \in \mathcal{D}(\mathcal{R})$ has support contained in K then

$$< F, {}^*\pi >=< \Phi_K, (-1)^{r_K}\, {}^*\phi^{(r_K)} >$$

is a bounded number and so F has a distributional shadow in $\mathcal{D}_K(\mathbb{R}) \subset \mathcal{D}(\mathbb{R})$. Since K may be any compact in $\mathcal{R}$ it follows that F has a shadow in $\mathcal{D}(\mathcal{R})$ and so $\mu = st_D(F)$ will be a well defined (standard) distribution in $\mathcal{D}'(\mathcal{R})$.

Denote by ${}^{S}\mathcal{C}_{\pi}(\mathbb{R})$ the subset of all prehyperfunctions which have a distributional shadow. Then we have the inclusion

$${}^{S}\mathcal{C}_{\infty}(\mathbb{R}) \subset {}^{S}\mathcal{C}_{\pi}(\mathbb{R})$$

Moreover ${}^{S}\mathcal{C}_{\pi}(\mathbb{R})\backslash{}^{S}\mathcal{C}_{\infty}(\mathbb{R})$ is not empty since it contains, for example, the internal function $F : {}^*\mathbb{R} \to {}^*\mathbb{C}$ defined by

$$F(x) = \sum_{i=0}^{+\infty} {}^*D^i \left\{ \frac{1}{\pi} \frac{\omega}{1+\omega^2(x-i)^2} \right\}$$

The members of ${}^{S}\mathcal{C}_{\pi}(\mathbb{R})$ will be called **predistributions of local finite order** or, more simply, **predistributions**.

6.4.3 Predistributions of infinite order

Let Φ be any internal function in ${}^{S}\mathcal{C}(\Omega_{\varepsilon})$ and suppose that there exists an harmonic function $g \in \mathbf{H}(\Pi^{+})$ such that $\Phi(x) = {}^*g(x,\varepsilon)$. Let also $r \equiv [r_n]$ be an arbitrary infinite hypernatural number. For every $n \in \mathbb{N}$ the function

$$\frac{\partial^{r_n} g}{\partial x^{r_n}}(x,y), \qquad (x,y) \in \Pi^{+}$$

is again harmonic on Π^{+}, and so ${}^*D^r\Phi$ is a generalised prehyperfunction in ${}^*\mathcal{H}(\Omega_{\varepsilon})$. The internal function ${}^*D^r\Phi$ is locally bounded by $r!\,\omega^r$; that is to say,

for each compact $K \subset \mathbb{R}$ there exists a bounded constant C_K such that for all $x \in {}^*K$ we have

$$|{}^*D^r\Phi(x)| \leq C_K r!\,\omega^r.$$

In general the infinite order derivative ${}^*D^r\Phi$ may have neither an ordinary shadow nor a distributional shadow, and $st_{\mathcal{D}}({}^*D^r\Phi)$ may have no meaning. On the other hand, for any $\phi \in \mathcal{D}(\mathbb{R})$, we have

$$< {}^*D^r\Phi, {}^*\phi >= (-1)^r < \Phi, {}^*\phi^r > .$$

But, since there exist test functions in $\mathcal{D}(\mathbb{R})$ whose derivatives may grow arbitrarily with the order, then

$$< {}^*D^r\Phi, {}^*\phi > \equiv \left[\left(< g(x,1/n), (-1)^{r_n}\phi^{(r_n)}(x) >\right)_{n\in N}\right]$$

will not in general be a bounded hypercomplex number. Hence $st_{\mathcal{D}}({}^*D^r\Phi)$ may have no meaning. However we can define a family of standard part maps which allow us to attach a type of shadow to derivatives of the form ${}^*D^r\Phi$ for infinite $r \in {}^*\mathbb{N}_\infty$ and internal functions Φ in ${}^S\mathcal{C}(\Omega_\varepsilon)$. These standard part maps are defined on certain subspaces of $\mathcal{D}(\mathbb{R})$, for example on those of so-called **Roumieu type** which we recall very briefly as follows.

Spaces of Roumieu type [16]

Let $\mathcal{M}$ denote the set of all positive real sequences $(M_p)_{p\in\mathbb{N}_0}$ such that

(a) $(M_p)^2 \leq M_{p-1}M_{p+1}, \quad p = 0, 1, \ldots,$

(b) $M_p \leq Ah^p \min_{0\leq q\leq p}\{M_pM_{p-q}\}$, $p = 0, 1, \ldots,$ for some positive constants A and h,

(c) $\sum_{p=0}^{+\infty}(M_p)^{-1/p} < +\infty$. Further, let $\mathcal{D}^{(M_p)}(\mathbb{R})$ be the subset of $\mathcal{D}(\mathbb{R})$ comprising all functions ϕ whose derivatives satisfy

$$|\phi^{(p)}(x)| \leq Ah^pM_p, \qquad p = 0, 1, \ldots,$$

for some sequence $(M_p) \in \mathcal{M}$, where A and h are positive constants (generally dependent on ϕ).

It can be shown that $\mathcal{D}^{(M_p)}(\mathbb{R})$ is not empty for every sequence $\{M_p\}_{p\in\mathbb{N}_0} \in \mathcal{M}$. In particular, if there exists $p \in \mathbb{N}_0$ such that $M_p = +\infty$ for all $p \geq p_0$ then $\{M_p\}_{p\in\mathbb{N}_0}$ belongs to $\mathcal{M}$ and $\mathcal{D}^{(M_p)}(\mathbb{R}) \equiv \mathcal{D}(\mathbb{R})$. The space $\mathcal{D}^{(M_p)}(\mathbb{R})$ is the union of the family of spaces

$$\left\{\mathcal{D}^{(M_p)}_{K,h}(\mathbb{R})\right\}_{K\subset\mathbb{R},\,h>0}$$

where K runs over the set of all compact subsets of $\mathbb{R}$ and h runs over all positive numbers. For each real number $h > 0$ and compact $K \subset \mathbb{R}$, $\mathcal{D}_{K,h}^{(M_p)}(\mathbb{R})$ contains all functions $\phi \in \mathcal{D}^{(M_p)}(\mathbb{R})$ with support contained in K and satisfying the above inequality for that particular value of $h > 0$. Each space $\mathcal{D}_{K,h}^{(M_p)}$ is a Banach space with respect to the norm

$$\|\phi\|_{\mathcal{D}_{K,h}^{(M_p)}} = \sup_{p \geq 0} \left\{ \frac{1}{h^p M_p} \sup_{x \in K} |\phi^{(p)}(x)| \right\},$$

and $\mathcal{D}^{(M_p)}(\mathbb{R})$ is provided with the inductive limit topology.

$\mathcal{D}'^{(M_p)}$ denotes the topological dual of $\mathcal{D}^{(M_p)}(\mathbb{R})$ and its elements are sometimes called **generalised distributions in the sense of Roumieu**. A linear functional is in $\mathcal{D}^{(M_p)}(\mathbb{R})$ if and only if it is continuous on that space for every $h > 0$ and compact $K \subset \mathbb{R}$.

6.5 Conclusion

The above outline of Sousa Pinto's nonstandard treatment of hyperfunctions is reported in greater detail in [10]. Sousa Pinto's later work, developing the hyperfinite approach to distributions initiated by Kinoshita, is given in [8] and [9], but most comprehensively in his last publication, the book [20], which is now available in an English translation.

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7

Neutrices in more dimensions

Imme van den Berg*

Abstract

Neutrices are convex subgroups of the nonstandard real number system, most of them are external sets. They may also be viewed as modules over the external set of all limited numbers, as such non-noetherian. Because of the convexity and the invariance under some translations and multiplications, the external neutrices are appropriate models of orders of magnitude of numbers. Using their strong algebraic structure a calculus of *external numbers* has been developped, which includes solving of equations, and even an analysis, for the structure of external numbers has a property of completeness. This paper contains a further step, towards linear algebra and geometry. We show that in $\mathbb{R}^2$ every neutrix is the direct sum of two neutrices of $\mathbb{R}$. The components may be chosen orthogonal.

7.1 Introduction

7.1.1 Motivation and objective

Consider the problem to specify a mathematical model for the intuitive notion of "order of magnitude". Orders of magnitude have some intrinsic vagueness, haziness or superficiality. They are bounded and invariant under at least some additions, or alternatively, translations. Classically, orders of magnitude are modelled with the O- or o-notation, which can be seen as additive groups of real functions in one variable [19]. One may argue that there is some friction between the intuitive notion of order of magnitude, which is about numbers, and its model, which concerns functions. If one wishes to preserve the properties of boundedness and invariance under some additions in the real number system, one enters into conflict with the archimedian property, a conflict also

*University of Évora, Portugal.
ivdb@uevora.pt

known as the *Sorites* paradox [28, 35]. However, this difficulty can be circumvented, if one models within the real number system of nonstandard analysis.

Nonstandard analysis disposes of so-called external sets, which do not correspond to sets of classical analysis. External sets of real numbers may be convex and bounded without having an infimum and supremum, and they are invariant under at least some additions. It may be shown that such an external set E has a group property: There exists $\varepsilon > 0$ such that whenever $x \in E$ one has $x + l\varepsilon \in E$ for limited (i.e., bounded by a standard integer) real numbers l. So it is natural to consider convex (external) additive subgroups of $\mathbb{R}$, which have been called *neutrices* in [29] and [30]. The term is borrowed from Van der Corput [20], who uses it to designate groups of functions, which may be more general than Oh's and oh's. Within the nonstandard real number system there exists a rich variety of neutrices; simple examples are $\varnothing$, the set of all infinitesimals, and $\pounds$, the set of all limited real numbers, for more intricate examples see [29, 30] and also section 7.3.2. *External numbers* are the sum of a (nonstandard) real number and a neutrix. One could define an *order of magnitude* simply to be a convex (external) subset of the nonstandard reals; then it is in fact an *external interval*, bounded by two external numbers (see [5] for a proof).

The external numbers satisfy a calculus, which is rich enough to include addition, subtraction, multiplication, division, order, solution of equations and calculation of integrals. All in all, this calculus of orders of magnitude resembles closely the calculus of the reals. We refer to [29, 30] for definitions, results and notations. We recall here a notation for two orders of magnitudes which are not neutrices. The set of positive *appreciable* numbers @ corresponds to the external interval $(\varnothing, \pounds]$ and the set ∞ of positive unlimited numbers corresponds to the external interval $(\pounds, \mathbb{R}]$.

The neutrices and external numbers have been applied in various settings (the terminology not being explicit in earlier papers): singular perturbation theory ([2, this paper describes the discovery of the "canard" phenomenon, the set of parameters for which it occurs is an external interval)], [22], [10], [11, papers on exponentially small thicknesses of boundary layers], [12, thickness of transitions of boundary layers]), asymptotics [5, sets of numbers having the same (nonstandard) asymptotic expansion, and domains of validity of asymptotic approximations], probability theory [6, modelling of mass and queue of probability distributions] and psychology [7, inperfect knowledge of maximal utility]. Special mention has to be made of the work of Bosgiraud, who applies the external numbers nontrivially to problems of modelling and calculation of insecure statistical events in a series of papers [13, 14, 15, 16, 17, 18].

This paper makes a step towards an external calculus in more variables. The main result (7.2.2, decomposition theorem) states that every neutrix in

the two-dimensional real space is the direct sum of two neutrices in the one-dimensional nonstandard real line. The neutrices are uniquely determined, and correspond to orthogonal directions. So in a sense the neutrices in the plane possess a dimension, too, which may be interpreted as "length" times "width". In a second paper [8] we extend the decomposition theorem to $\mathbb{R}^k$ for arbitrary standard k, and also show that the decomposition fails if k is unlimited. One reason for dividing the publication of the result into two separate papers is that the proofs of the two-dimensional case and the k-dimensional case are both rather lengthy. It is to be noted that the proof of the k-dimensional case uses in an essential way the result in two dimensions, but is by no means an extension of its proof by some form of external induction.

For standard dimension, the result answers in part a conjecture by Georges Reeb, who suggested that one should be able to recognize the dimension of a space on its external subsets. He also conjectured that there should be a relatively easy nonstandard proof of the topological dimension theorem, or the invariance of domain theorem, but this remains unsettled (some progress has been made by Reveilles [34]).

One may define an *external point* to be the sum of a (nonstandard) vector and a neutrix. On the basis of the decomposition theorem it could be interesting to develop fragments of linear algebra, matrix-calculus, geometry and multivariate analysis and statistics, in order to model, in the case of more variables, approximate qualifications ("small", "enormous", "somewhat", "good"), approximate phenomena ("mistbanks", "stains", "spheres of influence", "superficiality"), and approximate calculus and reasoning.

7.1.2 Setting

Roughly spoken, an internal set is a set defined by a formula of classical analysis (which may contain parameters), and an external set cannot be defined this way.

In k-dimensional space, except for its linear subspaces, every neutrix is an external set. The class of external sets differs from one nonstandard model to another, or alternatively, from one nonstandard axiomatics to another. However, the particular external sets met in applications are to a large extent the same. Indeed, usually they reduce either to $\varnothing$ or to $\pounds$; these two sets have essentially the same properties through all common nonstandard models and axiomatics.

The setting of this paper is the axiomatic system Internal Set Theory (IST) of Nelson [32, 33], and we refer to [24] and [23] for up-to-date presentations and terminology. The language of this system contains two primitive symbols $\in$ and *st* (standard). It differs from model-theoretic approaches in the way that

nonstandard elements are already contained in infinite standard sets, instead of extensions of such sets. It has the advantage of simplicity, of saturation in all (standard) cardinals, and of full validity of the "Fehrele principle" [5]: no *galactic* formula, i.e. a Σ_1-formula, starting with the "external quantifier" $\exists x\,(st\,x \wedge \cdots)$ is equivalent to a *halic* formula, i.e. a Π_1-formula, starting with the "external quantifier" $\forall x\,(st\,x \longrightarrow \cdots)$ unless they are equivalent to an internal formula. These are exactly the two characteristics that enable to prove the classification theorem of halflines (theorem 7.3.23), which has as a direct consequence that every order of magnitude is an external interval, and constitutes a crucial step in the proof of the decomposition theorem for neutrices in two dimensions (theorem 7.3.42).

Formally, Nelson's axiomatics does not regard external sets, but there are no major problems if we consider "external sets" which have only internal elements, and are defined by an external formula in which all "external quantifiers" range over standard sets. This is supposedly the case for neutrices, and for sets reduced to such. Axiomatics which enable to deal with this kind of external sets are given by [31, 27, 1, 25, 26].

7.1.3 Structure of this article

In section 7.2 we define formally the notion of neutrix and state the decomposition theorem for neutrices in 2-dimensional space.

The decomposition theorem is proved in section 7.3. The actual proof, which is contained in section 7.3.3, needs some elementary external geometry (section 7.3.1) and algebra (section 7.3.2). In particular we study two kinds of division for neutrices in one dimension, and their relation to certain geometric properties of neutrices in two dimensions. We give special attention to practical aspects, like the calculation of the divisions, and present many examples.

In general, the proof is neither fully algebraic, nor fully analytic. Instead, it tends to be a mixture of algebraic and analytic arguments, where typically algebraic operations are adapted to the order of magnitude of the quantities involved.

7.2 The decomposition theorem

Definition 7.2.1 *Let $k \in \mathbb{N}$, $k \geq 1$ be standard. A* neutrix *is a convex additive subgroup of $\mathbb{R}^k$.*

A convex (external) subset N of $\mathbb{R}$ is a subgroup if and only if it is symmetric with respect to 0 and whenever $x \in N$ one has $2x \in N$. Then by external induction $nx \in N$ for all standard $n \in \mathbb{N}$; by convexity, one has $lx \in$

$\mathbb{N}$ for all limited $l \in \mathbb{R}$. This indicates an alternative way to define neutrices: they should be *modules* over $\pounds$, the external ring of all limited real numbers: a subset N of $\mathbb{R}$ is a neutrix if and only if $\pounds \cdot N = N$.

The only internal neutrices of $\mathbb{R}^k$ are its linear subspaces. Thus the only internal neutrices of $\mathbb{R}$ are $\{0\}$ and $\mathbb{R}$ itself. Two obvious external neutrices in $\mathbb{R}$ are $\pounds$ itself and $\varnothing$, the external set of all infinitesimals. The neutrices of the form $\varepsilon\pounds$ for some positive $\varepsilon \in \mathbb{R}$ ("ε-galaxies") are isomorphic to $\pounds$ and the neutrices of the form $\varepsilon\varnothing$ for some positive $\varepsilon \in \mathbb{R}$ ("ε-halos") are isomorphic to $\varnothing$. Every neutrix $N \neq \{0\}$ is non-noetherian in an external sense: there exists always a strictly ascending chain of subneutrices $(N_n)_{n\in\mathbb{N}}$ with $N_1 \subsetneqq N_2 \subsetneqq \cdots \subsetneqq N_n \subsetneqq N_{n+1} \subsetneqq \cdots$ for standard indices n. Indeed, let $\omega \in \mathbb{R}$ be positive unlimited, and put $N_n = \omega^n\pounds$. Then $N_n \nsubseteq N_{n+1}$ for all indices n. Consider ω^ω which is not an element of N_n for all standard indices n. Let $\varepsilon \in N$ be sufficiently small such that also $\varepsilon\omega^\omega \in N$. Then $(\varepsilon N_n)_{st\,n}$ is a strictly ascending chain of $\pounds$-submodules of N. It may be proved [5] that for any strictly ascending chain $(N_n)_{st\,n}$ of neutrices, the union $\cup_{st\,n} N_n$ is neither isomorphic (for internal homomorphisms) to $\pounds$ nor to $\varnothing$. This suggests that there is a rich variety of external neutrices in $\mathbb{R}$, non-isomorphic with respect to internal homomorphisms. Still, as it is tacitly understood that a neutrix is defined by a bounded formula ϕ of Internal Set Theory, a neutrix has a simple logical form, for ϕ may be supposed to be internal, galactic or halic [5].

The main theorem of this paper asserts that in a sense augmenting the dimension to two does not generate entirely new types of neutrices, for any neutrix in $\mathbb{R}^2$ may be decomposed into two neutrices of $\mathbb{R}$. We adopt the notation $Nx \equiv \{nx \mid n \in N\}$ for the neutrix of all multiples of some vector x with coefficients in some neutrix $N \subset \mathbb{R}$.

Theorem 7.2.2 (Decomposition theorem) *Let $N \subset \mathbb{R}^2$ be a neutrix. Then there are neutrices $N_1 \supset N_2$ in $\mathbb{R}$ and orthonormal vectors u_1, u_2 such that*

$$N = N_1u_1 \oplus N_2u_2$$

Moreover, if $M_1 \supset M_2$ in $\mathbb{R}$ are neutrices and v_1, v_2 are orthonormal vectors with $N = M_1v_1 \oplus M_2v_2$, it holds that

$$M_1 = N_1 \ , \ M_2 = N_2.$$

7.3 Geometry of neutrices in $\mathbb{R}^2$ and proof of the decomposition theorem

The present section is divided into various subsections, conform the various stages of the proof of the decomposition theorem.

The first subsection contains the definitions of the notions of thickness, width (smallest thickness) and length (largest thickness) of neutrices, which are the basic ingredients of the proof. We prove an important theorem, called the "sector-theorem", expressing convexity of the thicknesses of neutrices in two dimensions on not too large sectors. One of the consequences is that the thicknesses in most directions are minimal. This implies that the width of a neutrix is realized in some direction, so in the decomposition represented in the main theorem we obtained already the neutrix N_2. Other important notions used in the proof are near-orthogonality and near-parallelness.

The most important step in the proof of the decomposition theorem in $\mathbb{R}^2$ consists in establishing a direction, which realizes the length of the neutrix; then up to a rotation, the neutrix will simply be "length times width".

This part of the proof uses a form of euclidean geometry in which the points and lines may have non-zero thickness, in fact such a thickness takes the form of a neutrix. We have to adapt some definitions, operations and theorems of ordinary, exact euclidean geometry to this geometry of "clouds" or "mistbanks". This is done in the first subsection, while the second subsection contains an algebraic tool: the "division" of a neutrix by another, the result of which may be calculated through a "subtraction", after taking logarithms.

The final subsection establishes the existence of a direction which realizes the length. It uses an argument of "external analysis": every (external) lower halfline has a supremum, which is an *external number*, i.e., the sum of an ordinary real number and a neutrix (theorem 7.3.23). In a sense, the external set of directions which realize the length is the supremum of an external set of directions which do *not* realize the length.

7.3.1 Thickness, width and length of neutrices

Definition 7.3.1 *Let $N \subset \mathbb{R}^2$ be a neutrix. We call N* square *in case there exists a neutrix $M \subset \mathbb{R}$ such that $N = M \times M$.*

Definition 7.3.2 *Let $N \subset \mathbb{R}^2$ be a neutrix and $r \in \mathbb{R}^2$ be a non-zero vector. The* thickness *of N in the direction of r is the neutrix T_r of $\mathbb{R}$ defined by*

$$T_r = \left\{ x \in \mathbb{R} \;\middle|\; x \cdot \frac{r}{\|r\|} \in N \right\}.$$

The width *W of N is defined by*

$$W = \bigcap_{r \in S^1} T_r,$$

and its length *L by*

$$L = \bigcup_{r \in S^1} T_r.$$

In an obvious way, if X is a subset of $\mathbb{R}^2$ one may define $\|X\| = \{\|x\| \mid x \in X\}$. So, if r is unitary, we may write the thickness in the direction of r alternatively as $T_r = \pm\|N \cap \mathbb{R}r\|$. As an example, consider the neutrix $\mathcal{L} \times \varnothing \subset \mathbb{R}^2$. Let $r = \binom{\cos\phi}{\sin\phi}$. Then $T_r = \mathcal{L}$ if $\phi \simeq 0 \pmod{\pi}$, otherwise $T_r = \varnothing$. Hence $W = \varnothing$ and $L = \mathcal{L}$. It will be shown later on that for every neutrix the width is assumed in some direction (in fact most of the directions) and the same holds for the length (in only few directions). For square neutrices $N = M^2$ all thicknesses are equal to M, hence their width and length are also equal to M.

Definition 7.3.3 *Let $N \subset \mathbb{R}^2$ be a neutrix and $x, y \in \mathbb{R}^2$. We call x and y* nearly orthonormal *if $\|x\| = \|y\| = 1$ and $< x, y > \simeq 0$. We call x and y* nearly orthogonal *if $\frac{x}{\|x\|}$ and $\frac{y}{\|y\|}$ are nearly orthonormal.*

Let $\varepsilon \simeq 0$. Then the vectors $\binom{1}{0}$ and $\binom{\varepsilon}{\sqrt{1-\varepsilon^2}}$ are nearly orthonormal. The vectors $\binom{1/\varepsilon}{0}$ and $\binom{\varepsilon}{1}$ are nearly orthogonal. We introduce now a notion of near-orthogonality with respect to neutrices.

Definition 7.3.4 *Let $N \subset \mathbb{R}^2$ be a non-square neutrix and $x, y \in \mathbb{R}^2$ be non-zero vectors. We call a line $\mathbb{R}y$* nearly parallel *to N if $T_y > W$. We call x a* near-normal *vector of N if x is nearly orthogonal to all $y \in S^1$ such that y is nearly parallel to N.*

As an example, consider the neutrix $N \equiv \mathcal{L} \times \varnothing \subset \mathbb{R}^2$. The vector $\binom{0}{1}$ is nearly orthogonal to N. Indeed unitary vectors y_ε such that y is nearly parallel, i.e. $T_{y_\varepsilon} = \mathcal{L} > \varnothing$, are all of the form $\binom{\varepsilon}{\sqrt{1-\varepsilon^2}}$ with $\varepsilon \simeq 0$. Then $\left\langle \binom{0}{1}, \binom{\sqrt{1-\varepsilon^2}}{\varepsilon} \right\rangle \simeq 0$. Note that all vectors of the form $\binom{\alpha}{1+\beta}$ with $\alpha, \beta \simeq 0$ are also nearly orthogonal to N.

The notions of near-parallelness and near-normality will be justified for neutrices N in two dimensions by theorem 7.3.6. In fact it is a consequence of the next simple, but important theorem, based on the convexity of N, that most directions have the same thickness. This proves a fortiori the existence of a direction which realizes the width.

Theorem 7.3.5 (Sector-theorem) *Let $N \subset \mathbb{R}^2$ be a neutrix and a and b two unitary vectors which make an angle θ with $0 \leq \theta \lessapprox \pi$. Let c be a unitary vector, which makes an angle γ with a such that $0 \leq \gamma \leq \theta$. Then*

$$T_c \geq \min(T_a, T_b).$$

Proof. Without restriction of generality, we may assume that $T_b \geq T_a > 0$ and that $a = \binom{1}{0}$. Let $x \in \mathbb{R}a \cap N$. Consider the change of scale $M = N/\|x\|$.

The image of x is $a \in M$. Notice that also $b \in M$, for $T_b \geq T_a$. Let ζ be the intersection of the lines ab and $\mathbb{R}c$. Then $\zeta \in M$ by convexity and $\|\zeta\|$ is appreciable. Because $\pounds\zeta \in M$ it holds that $\zeta/\ \|\zeta\| \in M$. Put $z = \|x\| \cdot \zeta/\ \|\zeta\|$. Then $z \in N$. Because $\|z\| = \|x\|$ we conclude that $T_c \geq T_a$. □

Theorem 7.3.6 *Let $N \subset \mathbb{R}^2$ be a neutrix. Then there exists an unitary vector u such that $T_u = W$. In fact the set of directions corresponding to an unitary vector r with $T_r = W$ contains an interval of the form $(\alpha + \varnothing, \alpha + \pi + \varnothing)$.*

Proof. Let $M = \min\left(T_{\binom{1}{0}}, T_{\binom{0}{1}}\right)$. By the sector-theorem $T_u \geq M$ for all unitary u in the first quadrant. But $T_{\binom{1}{0}} = T_{\binom{-1}{0}}$ and $T_{\binom{0}{1}} = T_{\binom{0}{-1}}$ hence $T_u \geq M$ for all unitary u in every quadrant. Hence $M = W$.

If N is square, one has $T_r = W$ for all unitary vectors r. If not, we may rotate N in order to obtain that $T_{\binom{1}{0}} = T_{\binom{-1}{0}} > W$. Let r be an unitary vector which makes an angle θ with the horizontal axis such that $0 \lessapprox \theta \lessapprox \pi$. If $T_r > W$, one should have $T_{\binom{0}{1}} > W$, by the sector-theorem applied to $[\theta, \pi]$ or $[0, \theta]$, depending to whether $\theta \leq \frac{\pi}{2}$ or $\theta \geq \frac{\pi}{2}$. This implies a contradiction, hence $T_r = W$. Thus we proved the second assertion,with $\alpha = 0$. □

Corollary 7.3.7 *Let $N \subset \mathbb{R}^2$ be a neutrix with width W. Let u, v be two orthonormal vectors. Then $T_u = W$ or $T_v = W$.*

The proof of the decomposition theorem for a neutrix of $\mathbb{R}^2$ is easy, once we know that its length is realized in some direction.

Theorem 7.3.8 *Let $N \subset \mathbb{R}^2$ be a neutrix with length L and width W. Assume there exists a unitary vector u such that $N \cap \mathbb{R}u = Lu$. Let v be unitary such that $u \perp v$. Then $N = Lu \oplus Wv$. Moreover, if u', v' are orthonormal and $N_1, N_2 \subset \mathbb{R}$ are neutrices with $N_1 \supset N_2$ such that $N = N_1u' \oplus N_2v'$, one has $N_1 = L$ and $N_2 = W$.*

Proof. By corollary 7.3.7 it holds that $T_v = W$. Because N is a neutrix, we have $Lu \oplus Wv \subset N$.

Conversely, let $n \in N$. Let p be the orthogonal projection of n on $\mathbb{R}u$. Because $\|n\| \in L$ and $\|p\| \leq \|n\|$ one has $\|p\| \in L$, so $p \in N$. Then $n - p \in N$, and because $T_v = W$, it follows that $n - p \in Wv$. Hence $n = p + (n - p) \in Lu \oplus Wv$ and $N \subset Lu \oplus Wv$. We conclude that $N = Lu \oplus Wv$.

We prove now the uniqueness part. The neutrix N cannot contain vectors larger than its length, so $N_1 \subset L$. If there exists $\lambda \in L \setminus N_1$, all vectors n in N satisfy $\|n\| < |\lambda|$, so L cannot be the length of N. So $L \supset N_1$, from which we conclude that $L = N_1$. By corollary 7.3.7, one has $N_2 = W$. □

It is now almost straightforward to prove the decomposition theorem for neutrices in $\mathbb{R}^2$ if their length is of the form $\lambda\pounds$.

Proposition 7.3.9 *Let $\lambda \in \mathbb{R}, \lambda > 0$. Let N be a neutrix with length $L = \lambda\pounds$ and width W. Then there are orthonormal vectors u and v such that $N = Lu \oplus Wv$.*

Proof. By rescaling if necessary we may assume that $\lambda = 1$. Let $u \in N$ be unitary. Then $\pounds u \in N$ because N is a neutrix, and $N \cap \mathbb{R}u \subset \pounds u$ because the length of N is $\pounds$. Hence $N \cap \mathbb{R}u = \pounds u = Lu$.

Let v be unitary such that $u \perp v$. Then $N = Lu \oplus Wv$ by theorem 7.3.8.□

The decomposition theorem is also easily proved for subneutrices of a given neutrix with length less than the length of this neutrix. Indeed, we have the following definition and proposition.

Definition 7.3.10 *Let $N \subset \mathbb{R}^2$ be a neutrix with length L and width W. Let $M \subset \mathbb{R}$ be a neutrix with $W \subset M \subset L$. We define*

$$N_M = \{n \in N \mid \|n\| \in M\}.$$

Clearly N_M is a neutrix with length M. Its length is realized in any direction for which N contains a vector u with $T_u \geq M$. Then the next proposition is a direct consequence of theorem 7.3.8.

Proposition 7.3.11 *Let $N \subset \mathbb{R}^2$ be a neutrix with length L and width W. Let $M \subset \mathbb{R}$ be a neutrix with $W \subset M \subset L$. Assume there is a unitary vector u such that $T_u \geq M$. Let v be a unitary vector v such that (u, v) is orthonormal. Then $N_M = Mu \oplus Wv$.*

Definition 7.3.12 *Let $N \subset \mathbb{R}^2$ be a neutrix. We call N* lengthy *if it is not square, and if its length is not of the form $L = \varepsilon\pounds$ for some $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$.*

So the two-dimensional decomposition theorem will follow once we have proved that lengthy neutrices assume there length. Note that the proof of proposition 7.3.9 does not work, because for every $\lambda \in L$ there exist $n \in N$ such that $\|n\| = \infty \cdot \lambda$. In order to prove that a lengthy neutrix assumes its length too, we work "from outside in". We consider lines with unitary directions u such that $T_u < L$, with u in an appropriate segment of the unit circle, in such a way that all lines "leave to the right". We divide the segment into two (external) classes: directory vectors for lines which are "leaving upside" and directory vectors for lines which are "leaving downside". It follows from the completion argument mentioned earlier that the two classes do not entirely fill

up the segment, just as two disjoint open intervals within some closed interval omit at least one point. The unitary vectors left out are then exactly those directions v such that $T_v = L$.

We introduce an appropriate scaling and orientation for lengthy neutrices, and make also precise what we understand by "leaving to the right upside" and "leaving to the right downside".

Definition 7.3.13 *Let $N \subset \mathbb{R}^2$ be a lengthy neutrix. We call N* appropriately scaled *if $W \subsetneqq \varnothing$ and $L \supsetneqq \pounds$.*

Proposition 7.3.14 *A lengthy neutrix $N \subset \mathbb{R}^2$ is homothetic to an appropriately scaled neutrix.*

Proof. Let $\lambda, \omega \in L \setminus W$ be positive such that $\lambda/\omega \simeq 0$. Consider $M = N/\omega$. Its width is $W/\omega \subset W/\lambda \subset \varnothing$ and its length is L/ω which contains at least some unlimited elements. If $W/\lambda \subsetneqq \varnothing$ we are done. If $W/\lambda = \varnothing$ we have $W/\omega \subsetneqq W/\lambda$, so clearly $W/\omega \subsetneqq \varnothing$. Hence M is appropriately scaled. □

Definition 7.3.15 *Let $N \subset \mathbb{R}^2$ be a lengthy appropriately scaled neutrix with length L. We call N* appropriately oriented *if*

$$N_{\pounds} = \pounds \begin{pmatrix} 1 \\ 0 \end{pmatrix} \oplus W \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Lemma 7.3.16 *Let $N \subset \mathbb{R}^2$ be a lengthy appropriately scaled neutrix. There exists a rotation ρ of the plane such that $\rho(N)$ is appropriately oriented.*

Proof. Let $W \subsetneqq \varnothing$ be the width of N. Then by proposition 7.3.11 there are orthonormal vectors u and v such that $N_{\pounds} = \pounds u \oplus Wv$. So let ρ be a rotation such that $\rho(u) = \binom{1}{0}$. Then $\rho(N)_{\pounds} = \pounds \binom{1}{0} \oplus W \binom{0}{1}$. □

Lemma 7.3.17 *Let $N \subset \mathbb{R}^2$ be a lengthy appropriately scaled and oriented neutrix with length L and width W. Let $\lambda \in L, \lambda \gtrapprox 0$. Then there exist orthonormal vectors $u \simeq \binom{1}{0}$ and $v \simeq \binom{0}{1}$ such that $N_{\pounds\lambda} = \pounds\lambda u \oplus Wv$.*

Proof. If λ is limited, the property follows from definition 7.3.15. If λ is unlimited, any element $n \in N$ with $\|n\| = \lambda$ is of the form $n = \binom{\xi}{\varepsilon\xi}$ with $\xi \simeq \infty$ and $\varepsilon \simeq 0$. Take $u = \frac{1}{\sqrt{1+\varepsilon^2}}\binom{1}{\varepsilon}$ and $v = \frac{1}{\sqrt{1+\varepsilon^2}}\binom{-\varepsilon}{1}$. Then the proposition follows from lemma 7.3.16. □

In the final part of this section we consider points and lines close to a lengthy appropriately scaled and oriented neutrix.

Definition 7.3.18 *Let $N \subset \mathbb{R}^2$ be a lengthy appropriately scaled and oriented neutrix with and length L and width W. Let $q \in \mathbb{R}^2$ be such that $\|q\| \in L$. Then q is said to be* infinitely close *to N if $q \simeq n$ for some element $n \in N$. Let $u \simeq \binom{1}{0}$, $v \simeq \binom{0}{1}$ be orthonormal vectors such that $N_{\pounds\lambda} = \pounds\lambda u \oplus Wv$ for some unlimited λ with $\lambda \geq \|q\|$. Then $q = \xi u + \eta v$ with $\eta \simeq 0$. It is called a* lower *point if $\eta < W$ and an* upper *point if $\eta > W$. Let x be a unitary vector, with $x \simeq \binom{1}{0}$. We say that the nearly parallel line $\mathbb{R}x$ is* downward *if it contains an infinitely close lower point, and* upward *if it contains an infinitely close upper point.*

As an example, consider $N = \omega\varnothing \times \frac{1}{\omega^2}\pounds$. The line containing the vector $\binom{1}{1/\omega}$ is nearly parallel upward, and the line containing the vector $\binom{1}{-1/\omega}$ is nearly parallel downward. By convexity, a nearly parallel line cannot be both downward and upward with respect to a neutrix N. By the next proposition, if its intersection with N is not maximal, it should be either one.

Proposition 7.3.19 *Let $N \subset \mathbb{R}^2$ be a lengthy appropriately scaled and oriented neutrix with length L. Let $x \simeq \binom{1}{0}$ be a unitary vector. Assume that $T_x < L$. Then the nearly parallel line x is either downward or upward with respect to N.*

Proof. Let W be the width of N. Let λ be unlimited such that $T_x < \lambda < L$. By proposition 7.3.9 there exist orthonormal vectors $u \simeq \binom{1}{0}$ and $v \simeq \binom{0}{1}$ such that $N_{\pounds l} = \pounds\lambda \oplus Wv$. By continuity $x \cap \pounds\lambda \oplus \varnothing v$ contains some point $y = \xi u + \eta v$ with $\xi \in L^+$ and $|\eta| > W$. If $\eta < W$ the line x is downward, and $\eta > W$ the line x is upward. □

Definition 7.3.20 *Let $N \subset \mathbb{R}^2$ be a lengthy appropriately scaled and oriented neutrix with length L and width W. Let $x \notin N$ be an infinitely close lower point and $y \notin N$ be an infinitely close upper point, with $\|x\| = \|y\|$. Let $u \simeq \binom{1}{0}$ and $v \simeq \binom{0}{1}$ be orthonormal vectors such that $u \perp xy$. If for some unlimited λ with $|\lambda| \geq \|x\|$ it holds that $N_{\pounds\lambda} = \pounds\lambda u \ominus Wv$, the points x and y are called* opposite *with respect to N. Also, the lines $\mathbb{R}x$ and $\mathbb{R}y$ are called* opposite *with respect to N.*

The final proposition of this section states that opposite lines generate the same thicknesses.

Proposition 7.3.21 *Let $N \subset \mathbb{R}^2$ be a lengthy appropriately scaled and oriented neutrix. Let $a, b \simeq \binom{1}{0}$ be unitary such that $\mathbb{R}a$ is a nearly parallel downward line, and $\mathbb{R}b$ an opposite nearly parallel upward line. Then $T_a = T_b$.*

Proof. Let L be the length of N and W be its width. Let $x \in \mathbb{R}a$ be an infinitely close lower point and y be its opposite infinitely close upper point on $\mathbb{R}b$. Let $\lambda \in L^+, |\lambda| \geq \|x\|$ be such that $N_{\pounds\lambda} = \pounds\lambda u \oplus Wv$, where $u \simeq \binom{1}{0}$ and $v \simeq \binom{0}{1}$ are orthonormal vectors such that $u \perp xy$. Let $\alpha > 0$ be such that $T_a < \alpha < \|x\|$. Then $\alpha a \notin N$, so there exist $\xi \leq \|x\|$, $\eta > W$ such that $\alpha a = \xi u - \eta v$. Because $N_{\pounds\lambda} = \pounds\lambda u \oplus Wv$, it holds that $\alpha b = \xi u + \eta v \notin N$, so $\alpha \notin T_b$. Hence $T_b \subset T_a$. In a symmetric manner we prove that $T_a \subset T_b$. We conclude that $T_a = T_b$. □

7.3.2 On the division of neutrices

Let $N \subset \mathbb{R}^2$ be a neutrix with width W and length L. We assume that N is of the form $N = L\binom{1}{0} \oplus W\binom{0}{1}$. Consider all lines $\mathbb{R}x$ with x of the form $x = \binom{1}{y}$, $y \in \mathbb{R}$.

The following sets appear to be of interest:

1. $R = \left\{ y \,\middle|\, \mathbb{R}\binom{1}{y} \cap L\binom{1}{0} \oplus W^C\binom{0}{1} = \emptyset \right\}$
2. $S = \left\{ y \,\middle|\, \mathbb{R}\binom{1}{y} \cap L^C\binom{1}{0} \oplus W\binom{0}{1} \neq \emptyset \right\} \qquad (L \subsetneqq \mathbb{R}).$

Clearly, if $y \in R$ or $y \in S$ the neutrix realizes its length in the direction x, i.e. one has $T_x = L$. If $y \in S$, the line $\mathbb{R}\binom{1}{y}$ leaves N on its "small" side. On the other hand, if $y \in R^C$, the line $\mathbb{R}\binom{1}{y}$ leaves N on its "large" side. But N may have "corners", i.e., there may exist lines $\mathbb{R}x$ which after leaving N enter into the set $L^C\binom{1}{0} \oplus W^C\binom{0}{1}$. Then $S \subsetneqq R$. An example of such a neutrix is $N = \pounds\binom{1}{0} \oplus \varepsilon\pounds\binom{0}{1}$, with $\varepsilon \simeq 0$, $\varepsilon > 0$. An example of a neutrix without such "corners" is given by $N = \pounds\binom{1}{0} \oplus \varnothing\binom{0}{1}$, see also [5].

If $W \subsetneqq L$, the sets R and S are neutrices, in fact they result from an algebraic operation applied to W and L. The first one is the well-known division operator on ideals or modules, commonly written ":" [36]. We recall here its definition, in the context of neutrices.

Definition 7.3.22 *Let $M, N \subset \mathbb{R}$ two neutrices. We write*

$$M : N = \{x \in \mathbb{R} \mid (\forall n \in N)(nx \in M)\} .$$

One can also abbreviate by

$$M : N = \{x \in \mathbb{R} \mid Nx \subset M)\} .$$

Since $N(\pounds x) = (\pounds N)x = Nx \subset M$, the set $M : N$ is a neutrix. Notice that

$$N(M : N) \subset M$$

and that $M : N$ is the maximal set X which satisfies the property

$$N \cdot X \subset M. \tag{7.1}$$

As such, we call $M : N$ the *solution* of the equation (7.1).

The second one is also a sort of division, that we note M/N. This division is based on a sort of inverse. Its definition needs more knowledge on neutrices, and will be postponed.

The study of divisions is highly related, but distinct from earlier work on the division of neutrices by Koudjeti [29] (see also [30]), mainly in the sense that our definitions are of algebraic or analytic nature, instead of set-theoretic. We use many of his tools and results, sometimes in a slightly modified form.

Following Koudjeti, the argumentation becomes more simple and intrinsic, if instead of multiplications of neutrices we study additions of lower halflines.

The transformation from neutrices N to lower halflines is done by the *symmetrical logarithm* $\log_s(N) = \log(N^+ \setminus \{0\})$; formally we define $\log_s\{0\} = \emptyset$. We transform halflines G back by the *symmetrical exponential* $\exp_s(G) = [-\exp(G), \exp(G)]$; formally we define $\exp_s \emptyset = \{0\}$.

Below we recall some fundamental properties of halflines and neutrices. A neutrix I is *idempotent* if $I \cdot I = I$. A lower halfline H is *idempotent* if $H + H = H$. A lower halfline is idempotent if and only if it is of the form $(-\infty, N)$ or $(-\infty, N]$, where N is a neutrix. A lower halfline H is idempotent if and only if $\exp_s H$ is an idempotent neutrix. A neutrix N is idempotent if and only if $\log_s N$ is an idempotent lower halfline.

Let $\varepsilon \simeq 0$, $\varepsilon > 0$. Examples of idempotent neutrices are, in increasing order $\{0\}$, $\pounds \cdot e^{-@/\varepsilon}$, $\pounds \cdot \varepsilon^{\infty}$, $\varnothing$, $\pounds$, $\pounds e^{(1/\varepsilon)^@}$ and $\mathbb{R}$. The neutrices $\pounds \cdot e^{-(1/\varepsilon)^2 - @/\varepsilon}$, $(1/\varepsilon)^{1/\varepsilon} \cdot \pounds \cdot \varepsilon^{\infty}$, $(1/\varepsilon) \cdot \varnothing$, $\varepsilon\pounds$ and $\pounds e^{\Gamma(1/\varepsilon) + (1/\varepsilon)^@}$ are not idempotent. They are all of the form $a \cdot I$ where I is an idempotent neutrix, and a is a real number. In fact every neutrix can be written is this form. This is a consequence of the following classification theorem of lower half-lines, which with successive generalizations has been proved in [9, 3, 5]:

Theorem 7.3.23 *Every lower half-line $H \subset \mathbb{R}$ has a representation either of the form $H = (-\infty, r + N)$ or $H = (-\infty, r + N]$, where N is a neutrix, which is unique, and r is a real number, determined up to the neutrix N.*

We see that a lower half-line H may be written in the form $H = r + K$, where K is of the form $(-\infty, N)$ or $(-\infty, N]$, i.e. the lower halfline K is idempotent. Then $\exp_s H = e^r \exp_s K$, and we obtained as a consequence that every neutrix is the product of a real number and an idempotent neutrix.

With respect to the above theorem we recall some notation. Halflines of the form $(-\infty, r + N)$ may be called *open*, and halflines of the form $(-\infty, r + N]$

closed. The external set $r + N$ is called an *external number*; it has been shown [29], [30] that many algebraic laws valid for the real number system continue to be valid for the external numbers. Certain analytic laws, too, on behalf of the above theorem. For instance, it may be justified to call the external number $r + N$ the *supremum* $\sup H$ of H. We call $\overline{H} \equiv (-\infty, \sup H]$ the *closure* of H, and $\underline{H} \equiv (-\infty, \sup H)$ the *interior* of H.

The pointwise addition of lower halflines H and K, satisfies

$$H + K = \begin{cases} K & \sup H \subsetneqq \sup K, \text{ or } \sup H = \sup K \text{ and } K \text{ is open} \\ H & \sup H \supsetneqq \sup K, \text{ or } \sup H = \sup K \text{ and } K \text{ is closed.} \end{cases} \tag{7.2}$$

Similar definitions and rules hold for upper halflines, working with infimums instead of supremums. Since lower halflines may be translated into idempotent lower halflines, and neutrices may be rescaled to idempotent neutrices, in defining algebraic operations we may restrict ourselves to the idempotent cases. The extension to the general case is straightforward and will be briefly addressed to at the end of this section.

We turn first to the problem of defining subtractions. The first operation $\div$ will correspond to the division :, and the second operation $\cdot\cdot$, which will correspond to the division /, is defined through inverses.

Definition 7.3.24 *Let H, K be idempotent lower halflines. We define $H \div K$ by*

$$H \div K = \{x \mid (\forall k \in K)(k + x \in H)\}.$$

Notice that $K + (H \div K) \subset H$ and that we have a maximality property similar to the division :, i.e. $H \div K$ may be called the (maximal) *solution* of the equation $K + X = H$.

Definition 7.3.25 *Let H be an idempotent lower halfline. We define the* symmetrical inverse $(-H)_s$ *of H by*

$$(-H)_s = \begin{cases} \overline{H} & H \text{ open} \\ \underline{H} & H \text{ closed} \end{cases}$$

If H is open ("boundary to the left of zero"), its symmetric reciprocal is a idempotent halfline, which is closed ("boundary to the right of zero"); in a sense the "distance" of the "boundaries" of H and $(-H_s)$ to zero is equal. So there is some geometric justification in calling the reciprocal symmetric. Formally it holds that $(-\emptyset)_s = \mathbb{R}$ and $(-\mathbb{R})_s = \emptyset$.

Definition 7.3.26 *Let H, K be idempotent lower halflines. We define $H \cdot\cdot K$ by*

$$H \cdot\cdot K = H + (-K)_s.$$

Notice that $H \cdot\cdot K$ is an idempotent lower halfline, too.

Proposition 7.3.27 *Let H, K be lower halflines. Let $S = \sup H$ and $T = \sup K$. Then*

1. $(-H)_s = -H^C$.
2. $H \cdot\cdot K = H - K^C$.
3. $H \cdot\cdot K = \begin{cases} (-K)_S & S \subsetneqq T, \text{ or } S = T \text{ and } H \text{ is closed} \\ H & S \supsetneqq T, \text{ or } S = T \text{ and } H \text{ is open.} \end{cases}$

The proofs are straightforward, using formula (7.2) in 3.

Proposition 7.3.28 *Let H, K be two idempotent lower half-lines of $\mathbb{R}$. Let $S = \sup H$ and $T = \sup K$. Then*

1. $(H \div K) = (H^C - K)^C$.
2. $H \div K = \begin{cases} (-K)_s & S \subsetneqq T, \text{ or } S = T \text{ and } K \text{ is open} \\ H & S \supsetneqq T, \text{ or } S = T \text{ and } K \text{ is closed.} \end{cases}$

Proof.

1. Let $x \in H \div K$. Suppose $x \in H^C - K$. Then there exist $y > H$, $k \in K$ such that $x = y - k$. Thus $x + k > H$, so $x \notin H \div K$, a contradiction. Then $x \in \left(H^C - K\right)^C$, hence $H \div K \subset \left(H^C - K\right)^C$. Conversely, let $x \in \left(H^C - K\right)^C$. Suppose $x \notin H \div K$. Then there exists $k \in K$, $z \in H^C$ such that $k + x = z$. Thus $x \in H^C - K$, a contradiction. So $x \in \left(H^C - K\right)^C$ and $\left(H^C - K\right)^C \subset H \div K$. We conclude that $H \div K = \left(H^C - K\right)^C$.
2. Straightforward, from 1 and formula (7.2). □

The following theorem is a direct consequence of propositions 7.3.27.3 and 7.3.28.2.

Theorem 7.3.29 *Let H, K be two idempotent lower half-lines of $\mathbb{R}$. Whenever $H \neq K$, it holds that*

$$H \cdot\cdot K = H \div K,$$

but

$$H \cdot\cdot H = \underline{H} \subsetneqq \overline{H} = H \div H.$$

So, generically, the subtraction $\cdot\cdot$, defined through reciprocals, and the subtraction $\div$, defined through solutions, yields the same result; thus the equation $K + X = H$ can be solved through reciprocals. In the exceptional case of the subtraction of two identical halflines the outcomes are strongly interrelated, for $H \cdot\cdot H$ is the interior of $H \div H$, and $H \div H$ the closure of $H \cdot\cdot H$.

We will now consider divisions. We start by defining the symmetric inverse. We relate this notion to the symmetric reciprocal.

Definition 7.3.30 *Let $N \subset \mathbb{R}$ be an idempotent neutrix. The* symmetric inverse $(N^{-1})_s$ *is defined by*

$$(N^{-1})_s = \exp_s (-\log_s N)_s .$$

Notice that $(N^{-1})_s$ is an idempotent neutrix, for it is the exponential of an idempotent lower halfline.

Below we calculate the symmetric inverse for some familiar neutrices. We have always $\varepsilon \simeq 0$, $\varepsilon > 0$.

$\mathbf{N}$	$\log_s \mathbf{N}$	$(-\log_s \mathbf{N})_s$	$(\mathbf{N}^{-1})_s$
$\{0\}$	$\emptyset$	$\mathbb{R}$	$\mathbb{R}$
$\pounds e^{-@/\varepsilon}$	$(-\infty, \varnothing/\varepsilon)$	$(-\infty, \varnothing/\varepsilon]$	$\pounds e^{\varnothing/\varepsilon}$
$\pounds \varepsilon^{\infty}$	$(-\infty, \pounds \log \varepsilon)$	$(-\infty, \pounds \log 1/\varepsilon]$	$\pounds (1/\varepsilon)^{@}$
$\varnothing$	$(-\infty, \pounds)$	$(-\infty, \pounds]$	$\pounds$
$\pounds$	$(-\infty, \pounds]$	$(-\infty, \pounds)$	$\varnothing$
$\pounds e^{(1/\varepsilon)^{@}}$	$(-\infty, \pounds/\varepsilon]$	$(-\infty, \pounds/\varepsilon)$	$\pounds e^{-\infty/\varepsilon}$
$\mathbb{R}$	$\mathbb{R}$	$\emptyset$	$\{0\}$

Table 7.1: Inverses of some idempotent neutrices.

In [29] the symmetric inverse of a (convex) set N is defined to be $\frac{1}{N^C} \cup \{0\}$.

The next theorem states that, if N is a neutrix, the two definitions are equivalent.

Proposition 7.3.31 *Let $N \subset \mathbb{R}$ be a neutrix. Then*

$$(\mathbf{N}^{-1})_s = \frac{1}{N^C} \cup \{0\}.$$

Proof. The equality holds formally for $N = \{0\}$ and $N = \mathbb{R}$. Let $0 \subsetneqq N \subsetneqq \mathbb{R}$. We prove only that $(\mathbf{N}^{-1})_s \subset \frac{1}{N^C} \cup \{0\}$. Clearly $0 \in \frac{1}{N^C} \cup \{0\}$. Now assume

that x is a nonzero element of $(\mathbf{N}^{-1})_s$, that we may suppose to be positive by reasons of symmetry. Then

$$\begin{aligned}\log x &\in (-\log_s N)_s \\ &= -(\log_s N)^C \\ &= -(\log(|N| \setminus \{0\})^C \\ &= -\log(\mathbb{R}^+ \setminus N) \\ &= \log \tfrac{1}{\mathbb{R}^+ \setminus N}.\end{aligned}$$

So $x \in 1/N^C$. Hence $(\mathbf{N}^{-1})_s \subset \frac{1}{N^C} \cup \{0\}$. □

We define the symmetric division through multiplication by the symmetric inverse, and relate it to the symmetric subtraction.

Definition 7.3.32 *Let $M, N \subset \mathbb{R}$ be two idempotent neutrices. The* symmetric division *M/N of M and N is defined by*

$$M/N = M \cdot (N^{-1})_s.$$

Proposition 7.3.33 *Let M, N be idempotent neutrices. Then*

$$M/N = \exp_s(\log_s M \cdot\cdot \log_s N).$$

Proof. We have, using the algebraic relations $M = \exp_s \log_s M$ and $\exp_s H \cdot \exp_s K = \exp_s(H + K)$,

$$\begin{aligned}M/N &= M \cdot (N^{-1})_s \\ &= \exp_s \log_s M \cdot \exp_s(-\log_s N)_s \\ &= \exp_s (\log_s M + (-\log_s N)_s) \\ &= \exp_s (\log_s M \cdot\cdot \log_s N)\,.\end{aligned}$$ □

We refrain from giving a general formula for M/N and point out that it can be calculated with the help of the propositions 7.3.27 and 7.3.33. However in some special cases M/N may be readily calculated.

1. $M \subsetneqq N \subset \varnothing$: $M/N = M$, $N/M = (M^{-1})_s$.
2. $\pounds \subset M \subsetneqq N$: $M/N = (N^{-1})_s$, $N/M = N$.
3. $M \subset \varnothing$: $M/M = M$.
4. $M \supset \pounds$: $M/M = (M^{-1})_s$.

Observe that always $M/M \subset \varnothing$.

We consider now the division $M : N$, of definition 7.3.22. It bears the following relation to the subtraction $\div$:

Proposition 7.3.34 *Let M, N be idempotent neutrices. Then*

$$M : N = \exp_s(\log_s M \div \log_s N).$$

Proof. We prove only that $M : N \subset \exp_s(\log_s M \div \log_s N)$. Formally, one has $0 \in \exp_s(\log_s M \div \log_s N)$. Let $x \in M : N, x \neq 0$. For reasons of symmetry, we may suppose that $x > 0$. Because $xN \subset M$ one has $\log x + \log_s N \subset \log_s M$. So $\log x \in \log_s M \div \log_s N$, hence $x \in \exp_s(\log_s M \div \log_s N)$. We conclude that $M : N \subset \exp_s(\log_s M \div \log_s N)$. □

As a consequence of theorem 7.3.29 and propositions 7.3.33 and 7.3.34 we obtain that $M/N = M : N$ whenever $M \neq N$. If $M = N$ we have

$$M : M = \exp_s(\log_s M \div \log_s M) = \exp_s \overline{\log_s M} = \begin{cases} (M^{-1})_s & M \subset \varnothing \\ M & M \supset \pounds. \end{cases}$$

Notice that always $M : M \supset \pounds$. Because $M/M \subset \varnothing$, we obtain

$$M/M \subsetneqq M : M.$$

The division has the following set-theoretic characterization.

Proposition 7.3.35 *Let $M, N \subset \mathbb{R}$ be two neutrices. Then*

$$M : N = \left(\frac{M^C}{N \setminus \{0\}}\right)^C.$$

Proof. The proof is very similar to the proof of proposition 7.3.28.1. □

It follows from the results above that, analogously to the subtraction $\div$, the division $M : N$ can in practice be calculated through inverses. See also the table and the special cases we presented earlier.

We indicate briefly how subtractions of non-idempotent halflines can be reduced to subtractions of idempotentent halflines, and consider also the analogous reduction for divisions of neutrices.

Let F, G be two lower halflines. By theorem 7.3.23 there exist real numbers f and g and idempotent halflines H and K such that

$$F = f + H \quad , \quad G = g + K.$$

We define

$$F \cdot\cdot G = f - g + H \cdot\cdot K \quad , \quad F \div G = f - g + H \div K.$$

In the same manner, let M, N be neutrices. Let $m, n \in \mathbb{R}$ and I, J be idempotent neutrices such that

$$M = mI \quad , \quad N = nJ.$$

We define

$$M/N = \frac{m}{n} I/J.$$

As regards to the operation $M : N$, the relation

$$M : N = \frac{m}{n}(I : J) \tag{7.3}$$

readily follows from definition 7.3.22. It is a matter of straightforward verification to show that the above formulae do not depend on the choice of f, g, m and n, and that, mutatis mutandis, the properties considered earlier in this section continue to hold.

We state three useful properties of the division $:$. We recall that a neutrix N is *linear* if there exists $\varepsilon > 0$ such that $N = \varnothing\varepsilon$ or $N = \pounds\varepsilon$, else N is *nonlinear*. Nonlinear neutrices have the property that $N = \omega \cdot N$ for at least some $\omega \simeq +\infty$ (see [5]).

Proposition 7.3.36 *Let $M, M_1, M_2, N, N_1, N_2 \subset \mathbb{R}$ be neutrices.*

1. *If $M_1 \subset M_2$, it holds that $M_1 : N \subset M_2 : N$.*
2. *If $N_1 \subset N_2$, it holds that $M : N_1 \supset M : N_2$.*
3. *If $M \subsetneqq N$, one has $M : N = \varnothing$ if and only if there exists $\varepsilon > 0$ such that $M = \varnothing\varepsilon$ and $N = \pounds\varepsilon$, otherwise $M : N \subsetneqq \varnothing$.*

Proof. We only prove 3. Let $\varepsilon, \eta > 0$. As regards to linear neutrices we have the following table

$:$	$\varnothing\varepsilon$	$\pounds\varepsilon$
$\varnothing\eta$	$\pounds\varepsilon/\eta$	$\varnothing\varepsilon/\eta$
$\pounds\eta$	$\pounds\eta/\varepsilon$	$\pounds\eta/\varepsilon$

So the only possibility for such neutrices M, N to obtain $M : N = \varnothing$, is when $M = \varnothing\varepsilon$ and $N = \pounds\eta$, with ε/η appreciable, i.e., when $N = \pounds\varepsilon$. Else the condition $M \subsetneqq N$ ensures that $\varepsilon/\eta \simeq 0$, respectively $\eta/\varepsilon \simeq 0$, which implies that $M : N \subsetneqq \varnothing$.

Assume M is nonlinear. Let $\omega \simeq +\infty$ be such that $M/\omega = M$. Then

$$M : N = \frac{M}{\omega} : N = \frac{1}{\omega}(M : N) \subset \frac{1}{\omega}\varnothing \subsetneqq \varnothing.$$

The case that N is nonlinear is similar. This concludes the proof. □

Finally we establish the relation between the two divisions and the families of directions in the plane R and S.

Theorem 7.3.37 *Let $N \subset \mathbb{R}^2$ be a neutrix with width W and length L, of the form $N = Lu \oplus Wv$, where u and v are orthonormal vectors. Let*

$$R = \left\{ y \,\middle|\, \mathbb{R}\binom{1}{y} \cap Lu \oplus W^C v = \emptyset \right\}$$

and, if $L \subsetneqq \mathbb{R}$, let

$$S = \left\{ y \,\middle|\, \mathbb{R}\binom{1}{y} \cap L^C u \oplus Wv \neq \emptyset \right\}.$$

Then $R = W : L$ and $S = W/L$.

Proof. Without restriction of generality, we may assume that $u = \binom{1}{0}$ and $v = \binom{0}{1}$. First, let $y \in R$. If $y = 0$, clearly, $y \in W : L$. Assume $y \neq 0$. Then there exist $\lambda \in L, \lambda \neq 0$ such that $\binom{\lambda}{\lambda y} \notin L\binom{1}{0} \oplus W^C\binom{0}{1}$, so

$$y = \frac{\lambda y}{\lambda} \in \left(\frac{W^C}{L \setminus \{0\}} \right)^C = W : L.$$

Hence $R \subset W : L$. Conversely, let $y \in W : L$. If $y = 0$, clearly $y \in R$. Assume $y \neq 0$. Suppose there is $\lambda \in L$ such that $\binom{\lambda}{\lambda y} \in L\binom{1}{0} \oplus W^C\binom{0}{1}$. Then $y \in \frac{W^C}{L\setminus\{0\}}$, which means that $y \notin W : L$, a contradiction. So $y \in R$, hence $W : L \subset R$. We conclude that $R = W : L$.

Second, let $y \in S$. Then there exists $\mu \in L^C$ such that $\binom{\mu}{\mu y} \in L^C\binom{1}{0} \oplus W\binom{0}{1}$. So $\mu y \in W$ and $y \in \frac{W}{L^C} = W/L$. Hence $S \subset W/L$. Conversely, let $y \in W/L$. Then there is $\mu \in L^C$ and $\eta \in W$ such that $y = \frac{\eta}{\mu}$. So $\binom{\mu}{\mu y} \in L^C\binom{1}{0} \oplus W\binom{0}{1}$, which means that $y \in S$. Hence $W/L \subset S$. We conclude that $S = W/L$. □

7.3.3 Proof of the decomposition theorem

Let us consider a neutrix in $\mathbb{R}^2$. We know already that, if it is square or not lengthy, it can be decomposed into two neutrices of $\mathbb{R}$. For lengthy neutrices, we sketch here the remaining part of the proof of the decomposition theorem.

A lengthy neutrix may be assumed to be appropriately scaled and oriented. By theorem 7.3.8 it suffices to look for a direction in the plane with maximal thickness. The set of directions with maximal thickness will be obtained as the complement of the directions with nonmaximal thickness, where by the sector-theorem we may restrain ourselves to directions nearly parallel to the neutrix. By proposition 7.3.19 and 7.3.21 they are divided into two "equal" opposite parts. The set of directions with maximal thicknesses will then be the supremum, in the sense of theorem 7.3.23, of the set of downward nearly parallel directions, or alternatively, the infimum of the set of upward nearly parallel directions. In fact, the "gap" between the two opposite families of directions is of the form $\binom{1}{x+W:L}$, where W is the width of the neutrix, and L its length.

We turn now to the proper proof, and start with some terminology.

Definition 7.3.38 *Let $N \subset \mathbb{R}^2$ be a lengthy appropriately scaled and oriented neutrix. We write*

$$\begin{aligned} D &= \left\{ y \simeq 0 \,\middle|\, \tbinom{1}{y} \text{ is nearly parallel downward} \right\} \\ U &= \left\{ y \simeq 0 \,\middle|\, \tbinom{1}{y} \text{ is nearly parallel upward} \right\} \\ S_D &= \left\{ x \in \mathbb{R} \middle|\, D + x = D \right\} \\ S_U &= \left\{ x \in \mathbb{R} \middle|\, U + x = U \right\}. \end{aligned}$$

For $y \simeq 0$ we write

$$I_y = \left\{ x \in \mathbb{R} \,\middle|\, T_{\binom{1}{y+x}} = T_{\binom{1}{y}} \right\},$$

and we define

$$I = \bigcap_{y \simeq 0} I_y.$$

Theorem 7.3.39 *Let $N \subset \mathbb{R}^2$ be a lengthy appropriately scaled and oriented neutrix with length L and width W. Then*

$$I = W : L \subsetneqq \varnothing.$$

Proof. It follows from proposition 7.3.36.3 that $W : L \subsetneqq \varnothing$. Let $x \in I$. Suppose $x \notin W : L$. By proposition 7.3.35 there are $\eta \in W^C$ and $\lambda \in L$ such that $x = \eta/\lambda$. By proposition 7.3.9 there are orthonormal vectors $u \simeq \binom{1}{0}$ and $v \simeq \binom{0}{1}$ such that $N_{\pounds\lambda} = \pounds\lambda u \oplus Wv$; up to a rotation we may assume that $u = \binom{1}{0}$ and $v = \binom{0}{1}$. So $\binom{\lambda}{\eta} \notin N$. Hence $T_{\binom{1}{x}} \subset T_{\binom{1}{\eta/\lambda}} \subsetneqq \pounds\lambda \subset T_{\binom{1}{0}}$. Hence $x \notin I_u \supset I$, a contradiction. This implies that $x \in W : L$, which means that $I \subset W : L$.

Conversely, let $x \in W : L$. Let $y \simeq 0$. By proposition 7.3.11 there are orthonormal vectors u, v such that $N_{T\binom{1}{y}} = T\binom{1}{y}u \oplus Wv$. It follows from

theorems 7.3.37 and 7.3.8 that $T_{\binom{1}{y+x}} = T_{\binom{1}{y}}$ for all $x \in W : T_{\binom{1}{y}} \supset W : L$, so $x \in I_y$. Because y is arbitrary, it holds that $x \in I$. Hence $W : L \subset I$. We conclude that $I = W : L$. □

Theorem 7.3.40 *Let $N \subset \mathbb{R}^2$ be a lengthy appropriately scaled and oriented neutrix with length L and width W. Then*

1. *For all $y, z \in D$ such that $y \leq z$ one has $T_{\binom{1}{y}} \leq T_{\binom{1}{z}}$.*
2. *For all $y, z \in U$ such that $y \leq z$ one has $T_{\binom{1}{y}} \geq T_{\binom{1}{z}}$.*
3. *For all $\varepsilon > W : L$ there is $y \in D, z \in U$ such that $z - y \leq \varepsilon$.*
4. *$S_D = S_U = W : L$.*
5. *There exists $x \simeq 0$ such that $D = [\varnothing, x + W : L)$ and $U = (x + W : L, \varnothing]$.*

Proof.

1. Suppose $T_{\binom{1}{y}} > T_{\binom{1}{z}}$. By proposition 7.3.21 there is $x \in U$ such that $T_{\binom{1}{x}} = T_{\binom{1}{y}}$. So $y < z < x$, $y \simeq x$, while $T_{\binom{1}{y}} > T_{\binom{1}{z}} < T_{\binom{1}{x}}$. This contradicts the sector-theorem. Hence $T_{\binom{1}{y}} \leq T_{\binom{1}{z}}$.
2. Analogous to 1.
3. Because $W : L$ is a neutrix, one has $\varepsilon/2 > W : L$. By proposition 7.3.35 there exist $\lambda \in L, \eta > W$ such that $\eta/\lambda \leq \varepsilon/2$. Let u, v be orthonormal such that $N_{\mathcal{L}\lambda} = \mathcal{L}\lambda u \oplus Wv$. We see that $\lambda u + \eta v, \lambda u - \eta v \notin N$, so $-\eta/\lambda \in D$ and $\eta/\lambda \in U$, while $\eta/\lambda - (-\eta/\lambda) \leq \varepsilon$.
4. From 3 we derive that $S_D \subset W : L$. Conversely, let $y \in D$. Then it follows from theorem 7.3.39 that $T_{\binom{1}{y+W:L}} = T_{\binom{1}{y}} < L$. This implies that $y + x \in D$ for all $x \in W : L$. Hence $W : L \subset S_D$. We conclude that $S_D = W : L$. The proof that $S_U = W : L$ is analogous.
5. By theorem 7.3.23 and 4 the set D is either of the form $D = [\varnothing, x + W : L)$ or $D = [\varnothing, x + W : L]$. We show that the second possibility is absurd. Then the only way to satisfy 3 is when $U = (x + W : L, \varnothing]$. By 7.3.39 and 1 one has $T_{\binom{1}{y}} \leq T_{\binom{1}{x}}$ for all $y \in D$. Then also $T_{\binom{1}{z}} \leq T_{\binom{1}{x}}$ for all $z \in U$, by proposition 7.3.21. Because $D \cup U = \varnothing$, one has $T_{\binom{1}{y}} \leq T_{\binom{1}{x}}$ for all $y \simeq 0$. By theorem 7.3.6 all other thicknesses are equal to W. Hence $T_r \leq T_{\binom{1}{x}} < L$ for all unitary vectors r. Then L cannot be the length of N, a contradiction. Hence $D = [\varnothing, x + W : L)$. The proof that $U = (x + W : L, \varnothing]$ is analogous. □

Theorem 7.3.41 *Let $N \subset \mathbb{R}^2$ be a lengthy appropriately scaled and oriented neutrix with length L and width W. Then there exists $x \simeq 0$ such that $\left\{y \middle| T_{\binom{1}{y}} = L\right\} = x + W : L$.*

Proof. By theorem 7.3.40.5 there exists $x \simeq 0$ such that $\varnothing \setminus (D \cup U) = x + W : L$. By theorem 7.3.39 we have $T_{\binom{1}{y}} = T_{\binom{1}{x}}$ for all $y \in x + W : L$. Suppose $T_{\binom{1}{x}} < L$. By proposition 7.3.19 either $x \in D$ or $x \in U$, a contradiction. Hence $T_{\binom{1}{x}} \geq L$, in fact $T_{\binom{1}{y}} = L$ for all $y \in x + W : L$. □

Theorem 7.3.42 (Two-dimensional decomposition theorem) *Let $N \subset \mathbb{R}^2$ be a neutrix. Then there are neutrices L, W with $W \subset L \subset \mathbb{R}$, and orthonormal vectors u, v such that*

$$N = Lu \oplus Wv.$$

Moreover, the neutrix L is the length of N, and W is its width.

Proof. Let L be the length of N and W its width. If $L = W$ one has $N = L\binom{1}{0} \oplus W\binom{0}{1}$. If $L = \pounds\lambda$ for some $\lambda \in \mathbb{R}$, the theorem follows from proposition 7.3.9. In the remaining cases we may assume that N is appropriately scaled and oriented. By theorem 7.3.41 there exist a unitary vector u such that $T_u = L$. Let v be unitary such that u, v are orthonormal. By theorem 7.3.8 we have $N = Lu \oplus Wv$. The last part of the theorem also follows from theorem 7.3.8. □

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Part II

Number theory

8

Nonstandard methods for additive and combinatorial number theory. A survey

Renling Jin*

8.1 The beginning

In this article my research on the subject described in the title is summarized. I am not the only person who has worked on this subject. For example, several interesting articles by Steve Leth [21, 22, 23] were published around 1988. I would like to apologize to the reader that no efforts have been made by the author to include other people's research.

My research on nonstandard analysis started when I was a graduate student in the University of Wisconsin. A large part of my thesis was devoted towards solving the problems posed in [19]. By the time when my thesis was finished, many of the problems had been solved. However, some of them were still open including [19, Problem 9.13]. It took me another three years to find a solution to [19, Problem 9.13]. Before this my research on nonstandard analysis was mainly focused on foundational issues concerning the structures of nonstandard universes. After I told Steve about my solution to [19, Problem 9.13], he immediately informed me how it could be applied to obtain interesting results in combinatorial number theory. This opened a stargate in front of me and lead me into a new and interesting field.

For nonstandard analysis we use a superstructure approach. We fix an $\aleph_1$-saturated nonstandard universe ${}^*\mathbb{V}$. For each standard set A we write ${}^*A$ for the nonstandard version of A in ${}^*\mathbb{V}$.

*Department of Mathematics, College of Charleston, Charleston, SC 29424.
jinr@cofc.edu

8.2 Duality between null ideal and meager ideal

Given an ordered measure space Ω such as the Lebesgue measure space on the real line with the natural order, one can discuss the relationship between measurable sets and open sets[1]. The null ideal on Ω is the collection of null sets, i.e. the sets with measure zero, and the meager ideal is the collection of all meager sets[2]. The sets in an ideal are often considered to be small. The duality between null ideal and meager ideal means that there exists a meager set with positive measure, i.e. the smallness in terms of null ideal is incomparable with the smallness in terms of meager ideal. However, it is usually true that if the space also has an additive structure, then the sum of two set with positive measure may not be meager. See Corollary 8.2.2 for example. What can we say about a Loeb space?

Let H be a hyperfinite integer and let $[0, H]$ be an interval of integers. The term $[a, b]$ in this article always means the interval of integers between a and b including a and b if they are also integers. On $[0, H]$ one can construct a Loeb measure generated by the normalized counting measure. By a Loeb space we always mean the hyperfinite set $[0, H]$ with the Loeb measure generated by the normalized counting measure[3]. On $[0, H]$ there is also a natural order and an additive structure. However, the order topology on $[0, H]$ is discrete, therefore uninteresting. In [19] a U-topology is introduced for each cut $U \subseteq [0, H]$ that gives meaningful analogy of the order topology on the real line. An infinite initial segment U of non-negative integers is called a cut if it is closed under addition. For example $\mathbb{N}$, the set of all standard non-negative integers, is the smallest cut. We often write $x > U$ for a positive integer x and a cut U if $x \notin U$. Let $U \subseteq [0, H]$ be a cut. A set $A \subseteq [0, H]$ is called U-open if for every $x \in A$, there exists a positive integer $y > U$ such that $[x - y, x + y] \cap [0, H] \subseteq A$. A U-topology is the collection of all U-open sets and a U-meager set is a meager set in terms of U-topology. In [19] it was proven that for any cut $U \subseteq [0, H]$ there is always a U-meager set of Loeb measure one in $[0, H]$. The question 9.13 in [19] asked whether the sum (modulo $H + 1$) of two sets in $[0, H]$ with positive Loeb measure can be U-meager for some cut $U \subseteq [0, H]$. For two sets A and B and a binary operation $\circ$ between A and B, we write $A \circ B$ for the set $\{a \circ b : a \in A \text{ and } b \in B\}$. For a number k, we write kA for the set $\{ka : a \in A\}$. In [11] we prove the following theorem.

[1] An order on a space can generate a topology called order topology on the space so that a set is open if it is the union of open intervals.

[2] A set A in a topological space is nowhere dense if every non-empty open set O contains another non-empty open set R disjoint from A. A meager set is the union of finitely many or countably many nowhere dense sets. A meager set is also called a set of the first category.

[3] The Loeb space here is often called the hyperfinite uniform Loeb space.

Theorem 8.2.1 *Let H be a hyperfinite integer and $U \subseteq [0, H]$ be any cut. If $A, B \subseteq [0, H]$ are two internal sets with positive Loeb measure, then $A \ominus_H B$ is not U-nowhere dense, where $\oplus_H$ is the usual addition modulo $H + 1$.*

Note that Theorem 8.2.1 yields a negative answer to [19, Problem 9.13]. Also note that the theorem is still true if $\oplus_H$ is replaced by the usual addition $+$ and the sumset $A+B$ is considered to live in $[0, 2H]$. Theorem 8.2.1 has several corollaries in the standard world. If one lets U be the cut $\bigcap_{n\in\mathbb{N}}[0, \frac{H}{n}]$, then Theorem 8.2.1 implies the following well known non-trivial fact.

Corollary 8.2.2 *If A and B are two sets of reals with positive Lebesgue measure, then $A + B$ must contain a non-empty open interval of reals.*

Corollary 8.2.2 was credited to Steinhaus in [21].

Let $A \subseteq \mathbb{N}$ be infinite. The upper Banach density $BD(A)$ of A is defined by

$$BD(A) = \lim_{k\to\infty} \sup_{n\in\mathbb{N}} \frac{|A \cap [n, n + k]|}{k + 1}.$$

A set $C \subseteq \mathbb{N}$ is called piecewise syndetic if there is a positive integer k such that $C + [0, k]$ contains arbitrarily long sequence of consecutive numbers. The definition of upper Banach density, syndeticity, and piecewise syndeticity can be found in [1, 7]. If one let $U = \mathbb{N}$, then Theorem 8.2.1 implies the following result.

Corollary 8.2.3 *Let $A, B \subseteq \mathbb{N}$. If $BD(A) > 0$ and $BD(B) > 0$, then $A + B$ is piecewise syndetic.*

Corollary 8.2.3 was pointed out to me by Steve Leth. By choosing other cuts U one can have more corollaries. These corollaries also have their own corollaries. The reader can find more of them in [11, 18].

Corollary 8.2.2 and Corollary 8.2.3 can also be proven using standard methods [13]. However, Theorem 8.2.1 does not have a standard version. The generality of Theorem 8.2.1 shows the advantages of the nonstandard methods. Theorem 8.2.1 reveals a universal phenomenon, which says that if two sets are large in terms of "measure", then $A + B$ must not be small in terms of "order-topology".

8.3 Buy-one-get-one-free scheme

Excited by the results such as Corollary 8.2.3, I was eager to let people know what I had obtained. After a talk I gave at a meeting in 1997, I was informed

by a member in the audience that Corollary 8.2.3 had probably already been proven in [1] or in [7]. This made me rush to the library to check out the book and the paper; I was anxious to see whether my efforts were a waste of time. Fortunately, they weren't; in fact, Corollary 8.2.3 complemented a theorem in [7] which says that if a set $A \subseteq \mathbb{N}$ has positive upper Banach density, then $A - A$ is syndetic. From [1, 7] I also learned of terms such as *upper Banach density, syndeticity, piecewise syndeticity,* etc. the first time.

One thing which caught my eye when I read [1, 7] was the use of Birkhoff Ergodic Theorem. It is natural for a nonstandard analyst to think what one can achieve if Birkhoff Ergodic Theorem is applied to some problems in a Loeb measure space setting. With that in mind, I derived Theorem 8.3.1 and Theorem 8.3.2 as lemmas in [12].

Given a set $A \subseteq \mathbb{N}$, the lower asymptotic density $\underline{d}(A)$ of A is defined by

$$\underline{d}(A) = \liminf_{n\to\infty} \frac{|A \cap [1, n]|}{n},$$

where $|X|$ means the cardinality of X when X is finite. Later, we will use $|X|$ representing the internal cardinality of X when X is a hyperfinite set. For a set A and a number x we often write $A \pm x$ for $A \pm \{x\}$ and write $x \pm A$ for $\{x\} \pm A$.

Theorem 8.3.1 *Suppose $A \subseteq \mathbb{N}$ with $BD(A) = \alpha$. Then there is an interval of hyperfinite length $[H, K]$ such that for almost all $x \in [H, K]$ in terms of the Loeb measure on $[H, K]$, we have $\underline{d}((^*A - x) \cap \mathbb{N}) = \alpha$. On the other hand, if $A \subseteq \mathbb{N}$ and there is a positive integer x such that $\underline{d}((^*A - x) \cap \mathbb{N}) \geqslant \alpha$, then $BD(A) \geqslant \alpha$.*

Given a set $A \subseteq \mathbb{N}$, the Shnirel'man density $\sigma(A)$ of A is defined by

$$\sigma(A) = \inf_{n\geqslant 1} \frac{|A \cap [1, n]|}{n}.$$

Theorem 8.3.2 *Suppose $A \subseteq \mathbb{N}$ with $BD(A) = \alpha$. Then there is a positive integer x such that $\sigma((^*A - x) \cap \mathbb{N}) = \alpha$.*

Theorem 8.3.1 is [12, Lemma 2] and Theorem 8.3.2 is the combination of [12, Lemma 3, Lemma 4 and Lemma 5]. It is often the case that a result involving Shnirel'man density is obtained first. Then people explore possible generalizations to some results involving lower asymptotic density. The behaviors of these two densities are quite similar. From Theorem 8.3.1 and Theorem 8.3.2 we can see that the behavior of upper Banach density is also similar to the behavior of lower asymptotic density or Shnirel'man density. We

can now claim that there is a theorem involving upper Banach density parallel to each existing theorem involving lower asymptotic density or Shnirel'man density. This is the scheme that can be called *buy-one-get-one-free* because we can get a parallel theorem involving upper Banach density for free as soon as a theorem involving lower asymptotic density or Shnirel'man density is obtained. I would now like to briefly describe how this works.

Given a set $A \subseteq \mathbb{N}$ with $BD(A) = \alpha$, there is a positive integer x (may be nonstandard) such that $\underline{d}((^*A - x) \cap \mathbb{N}) = \alpha$ (or $\sigma((^*A - x) \cap \mathbb{N}) = \alpha$). This means that in $x + \mathbb{N}$, a copy of $\mathbb{N}$ above x, the set $^*A$ has lower asymptotic density (or Shnirel'man density) α. Now apply the existing theorem involving $\underline{d}$ (or σ) to the set $^*A \cap (x + \mathbb{N})$ to obtain a result about $^*A$. Finally, pushing down the result to the standard world, one can obtain a parallel theorem involving upper Banach density. Corollary 8.3.3 and Corollary 8.3.4 below are the results obtained using this scheme.

The first one is a corollary parallel to Mann's Theorem. Mann's Theorem says that if two sets $A, B \subseteq \mathbb{N}$ both contain 0, then $\sigma(A + B) \geqslant \min\{\sigma(A) + \sigma(B), 1\}$. Mann's Theorem is an important theorem; in [20] it is referred as one of three pearls in number theory. We can now easily prove the following corollary of Theorem 8.3.2.

Corollary 8.3.3 *$BD(A + B + \{0, 1\}) \geqslant \min\{BD(A) + BD(B), 1\}$ for all $A, B \subseteq \mathbb{N}$.*

The addition of $\{0, 1\}$, which substitutes the condition $0 \in A \cap B$ in Mann's Theorem, is necessary because without it, Corollary 8.3.3 is no longer true. For example if A and B both are the set of all even numbers, then $BD(A) = BD(B) = BD(A + B) = \frac{1}{2}$.

The second corollary is parallel to Plünnecke's Theorem [24, p. 225] which says that if $B \subseteq \mathbb{N}$ is a basis of order h, then $\sigma(A + B) \geqslant \sigma(A)^{1-\frac{1}{h}}$ for every set $A \subseteq \mathbb{N}$. A set $B \subseteq \mathbb{N}$ is called a basis of order h if every non-negative integer n is the sum of at most h non-negative integers (repetition is allowed) from B. In the upper Banach density setting we can define piecewise basis of order h. A set $B \subseteq \mathbb{N}$ is called a piecewise basis of order h if there is a sequence of intervals $[a_k, b_k] \subseteq \mathbb{N}$ with $\lim_{k\to\infty}(b_k - a_k) = \infty$ such that every integer $n \in [0, b_k - a_k]$ is the sum of at most h integers (repetition allowed) from $(B - a_k) \cap [0, b_k - a_k]$. If B is a basis of order h, then it must also be a piecewise basis of order h because one can take $b_k = k$ and $a_k = 0$.

Corollary 8.3.4 *If B is a piecewise basis of order h, then $BD(A + B) \geqslant BD(A)^{1-\frac{1}{h}}$ for every set $A \subseteq \mathbb{N}$.*

Corollary 8.3.3 and Corollary 8.3.4 can also be proven using the standard methods [13] from Ergodic Theory. For more results similar to Corollary 8.3.3 and Corollary 8.3.4, see [12].

8.4 From Kneser to Banach

In the last section we didn't use the full power of Theorem 8.3.1. Suppose $A \subseteq \mathbb{N}$ has upper Banach density α. We use only one x such that $\underline{d}((^*A - x) \cap \mathbb{N}) = \alpha$ while there are almost all x in an interval $[H, K]$ of hyperfinite length such that $\underline{d}((^*A - x) \cap \mathbb{N}) = \alpha$. We can take this advantage and prove a theorem involving upper Banach density parallel to Kneser's Theorem [9]. For two sets $A, B \subseteq \mathbb{N}$ we write $A \sim B$ if $(A \setminus B) \cup (B \setminus A)$ is a finite set. Kneser's Theorem[4] says that for all $A, B \subseteq \mathbb{N}$, if $\underline{d}(A + B) < \underline{d}(A) + \underline{d}(B)$, then there is a positive integer g and a set $G \subseteq [0, g-1]$ such that $A + B \subseteq G + g\mathbb{N}$, $A + B \sim G + g\mathbb{N}$, and $\underline{d}(A + B) = \frac{|G|}{g} \geqslant \underline{d}(A) + \underline{d}(B) - \frac{1}{g}$.

Kneser's Theorem was motivated by Mann's Theorem. Can Mann's Theorem be true if one replaces Shnirel'man density with lower asymptotic density? There are obvious counterexamples. Let d, k, and k' be positive integers. Suppose $G = \{0, d, 2d, \ldots, (k-1)d\}$ and $G' = \{0, d, 2d, \ldots, (k'-1)d\}$. Suppose also that $g > (k + k' - 2)d$, $A = G + g\mathbb{N}$, and $B = G' + g\mathbb{N}$. Then $\underline{d}(A + B) = \frac{k+k'-1}{g} = \underline{d}(A) + \underline{d}(B) - \frac{1}{g}$. Roughly speaking, Kneser's Theorem says that the only kind of counterexamples which make the inequality false in Mann's Theorem with σ replaced by $\underline{d}$ are similar to the one just described.

In [13] a parallel theorem [13, Theorem 3.8] was obtained. Let $A, B \subseteq \mathbb{N}$ with $BD(A) = \alpha$ and $BD(B) = \beta$. Then there are intervals $[a_n, b_n]$ and $[c_n, d_n]$ such that $\lim_{n\to\infty}(b_n - a_n) = \infty$, $\lim_{n\to\infty}(d_n - c_n) = \infty$, $\lim_{n\to\infty} \frac{|A \cap [a_n, b_n]|}{b_n - a_n + 1} = \alpha$, and $\lim_{n\to\infty} \frac{|B \cap [c_n, d_n]|}{d_n - c_n + 1} = \beta$. We hope to characterize the structure of $A + B$ inside the intervals $[a_n + c_n, b_n + d_n]$. However, [13, Theorem 3.8] when restricted to the addition of two sets, only characterized the structure of $A + B$ on a very small part of $\mathbb{N}$. The reason for this is because we used only one x and one y with $\underline{d}((^*A - x) \cap \mathbb{N}) = \alpha$ and $\underline{d}((^*B - y) \cap \mathbb{N}) = \beta$ in the proof.

During the summer of 2003 my undergraduate research partner Prerna Bihani and I conducted an undergraduate research project funded by the College of Charleston to work on theorems parallel to Kneser's Theorem. The work done during the summer and the following year produced the paper [2], which contains Theorem 8.4.1. To avoid some technical difficulties we considered only the sum of two copies of the same set in [2].

[4] The version of Kneser's Theorem in [9] is about the addition of multiple sets. We stated the version here for the addition of two sets just for simplicity.

Theorem 8.4.1 *Let A be a set of non-negative integers such that $BD(A) = \alpha$ and $BD(A+A) < 2\alpha$. Let $\{[a_n, b_n] : n \in \mathbb{N}\}$ be a sequence of intervals such that $\lim_{n\to\infty}(b_n - a_n) = \infty$ and $\lim_{n\to\infty} \frac{|A\cap[a_n,b_n]|}{b_n-a_n+1} = \alpha$. Then there are $g \in \mathbb{N}$, $G \subseteq [0, g-1]$, and $[c_n, d_n] \subseteq [a_n, b_n]$ for each $n \in \mathbb{N}$ such that*

(1) $\lim_{n\to\infty} \frac{d_n-c_n}{b_n-a_n} = 1$,

(2) $A + A \subseteq G + g\mathbb{N}$,

(3) $(A + A) \cap [2c_n, 2d_n] = (G + g\mathbb{N}) \cap [2c_n, 2d_n]$ *for all* $n \in \mathbb{N}$,

(4) $BD(A + A) = \frac{|G|}{g} \geqslant 2\alpha - \frac{1}{g}$.

Note that (1) above shows that the structure of $A + A$ is characterized on a large portion of $[2a_n, 2b_n]$. Note also that we cannot replace $[2c_n, 2d_n]$ with $[2a_n, 2b_n]$ in (3) because all conditions for A still hold if we delete any elements from $A \cap ([a_n, b_n] \setminus [c_n, d_n])$.

The proof of Theorem 8.4.1 can be described in several steps. Given a hyperfinite integer N, we know that for almost all $x, y \in [a_N, b_N]$ we have $\underline{d}((^*A-x)\cap\mathbb{N}) = \underline{d}((^*A-y)\cap\mathbb{N}) = \alpha$. We can also assume that $\underline{d}((x-^*A)\cap\mathbb{N}) = \underline{d}((y-^*A)\cap\mathbb{N}) = \alpha$. Step one: characterize the structure of $^*(A+A)$ in $x+y+\mathbb{Z}$ using Kneser's Theorem, where $\mathbb{Z}$ is the set of all standard integers. Step two: show that the structures of $^*(A+A)\cap(x+y+\mathbb{Z})$ for almost all $x, y \in [a_N, b_N]$ are consistent with one another so that these structures can be combined into one structure. Hence we can characterize the structure of $^*(A + A)$ in $[2c_N, 2d_N]$, where $\frac{d_N-c_N}{b_N-a_N} \approx 1$. Step three: prove that for different hyperfinite integers N and N', the structure of $^*(A + A)$ in $[2a_N, 2b_N]$ and the structure of $^*(A + A)$ in $[2a_{N'}, 2b_{N'}]$ are consistent so that these structures of $^*(A + A)$ in $[2a_N, 2b_N]$ for all hyperfinite integers N can be combined into one structure of $^*(A + A)$ in $\bigcup\{[a_N, b_N] : N \text{ is hyperfinite}\}$. Step five: pushing down the structure of $^*(A + A)$ to the standard world results Theorem 8.4.1.

The methods developed in [13] do not seem to be enough for proving Theorem 8.4.1. So it is interesting to see whether one can produce a reasonably nice and short standard proof of the theorem.

In [3] the structure of A was characterized when $\underline{d}(A)$ is very small and $\underline{d}(A + A) \leqslant c\underline{d}(A)$ for some constant $c \geqslant 2$. It is also interesting to see how one can characterize the structure of $A + A$ when $BD(A + A) \leqslant cBD(A)$ for some constant $c \geqslant 2$.

8.5 Inverse problem for upper asymptotic density

In January of 2000, I was invited to give a talk at the DIMACS workshop "Unusual Applications of Number Theory". One of the workshop organizers

was Melvyn Nathanson to whom I am grateful for being the first number theorist to express an interest in my research on number theory not to mention his continued encouragement. During the workshop I had a chance to meet another number theorist G. A. Freiman who is well-known for his work on inverse problems in additive and combinatorial number theory. He gave me a preprint of his list of open problems [5]. This list and the book [24] have since gotten me interested in the inverse problems.

Inverse problems study the properties of A when $A+A$ satisfies certain conditions. Freiman discovered a phenomenon that if $A+A$ is small, then A must have some arithmetic structure. In fact Kneser's Theorem and Theorem 8.4.1 can be viewed as two examples of the phenomenon. One can characterize the arithmetic structure of A from the structure of $A+A$ in Theorem 8.4.1 and characterize the structure of A and the structure of B from the structure of $A+B$ in Kneser's Theorem (see [2] for details). In this section we characterize the structure of A when the upper asymptotic density of $A+A$ is small. Given $A \subseteq \mathbb{N}$, the upper asymptotic density $\bar{d}(A)$ of A is defined by

$$\bar{d}(A) = \limsup_{n\to\infty} \frac{|A \cap [1,n]|}{n}.$$

Without loss of generality we always assume $0 \in A$ in this section. We can also assume that $\gcd(A) = 1$ because if $\gcd(A) = d > 1$, then we can recover the structure of A from the structure of A', where $A' = \{a/d : a \in A\}$. When $0 \in A$ and $\gcd(A) = 1$, one can easily prove, using Freiman's result (1) at the beginning of the next section, that $\bar{d}(A+A) \geqslant \frac{3}{2}\bar{d}(A)$ if $\bar{d}(A) \leqslant \frac{1}{2}$ and $\bar{d}(A+A) \geqslant \frac{1+\bar{d}(A)}{2}$ if $\bar{d}(A) \geqslant \frac{1}{2}$. The following two examples show that the lower bounds above are optimal.

Example 8.5.1 *For every real number $0 \leqslant \alpha \leqslant 1$, let*

$$A = \{0\} \cup \bigcup_{n=1}^{\infty} [\lceil (1-\alpha)2^{2^n} \rceil, 2^{2^n}].$$

Then $\bar{d}(A) = \alpha$, $\bar{d}(A+A) = \frac{1+\alpha}{2}$ if $\alpha \geqslant \frac{1}{2}$, and $\bar{d}(A+A) = \frac{3}{2}\alpha$ if $\alpha \leqslant \frac{1}{2}$.

Example 8.5.2 *Let $k, m \in \mathbb{N}$ be such that $k \geqslant 4$ and 0, m, $2m$ are pairwise distinct modulo k. Let $A = k\mathbb{N} \cup (m + k\mathbb{N})$. Then $\bar{d}(A) = \frac{2}{k} = \alpha \leqslant \frac{1}{2}$ and $\bar{d}(A+A) = \frac{3}{k} = \frac{3}{2}\alpha$. It is easy to choose k, m such that $\gcd(A) = 1$.*

So we can say that $\bar{d}(A+A)$ is small when $\bar{d}(A+A) = \min\{\frac{3}{2}\bar{d}(A), \frac{1+\bar{d}(A)}{2}\}$ and we need to characterize the structure of A when $\bar{d}(A+A)$ is small.

Clearly the characterization of the structure of A should cover the cases in both Example 8.5.1 and Example 8.5.2. We hope to show that A must have

the structure described in one of the examples above when $\bar{d}(A+A)$ is small. However, some variations of the examples are unavoidable. If the set A is replaced by $A' \subseteq A$ with $\bar{d}(A') = \alpha$ in both Example 8.5.1 and Example 8.5.2, then $\bar{d}(A'+A')$ is also small. Furthermore, if $A = A' \cup A''$ where

$$A' = \{0\} \cup \bigcup_{n=1}^{\infty} [\lceil (1-\alpha)2^{2^{2n}} \rceil, 2^{2^{2n}}]$$

and A'' is an arbitrary subset of

$$\bigcup_{n=1}^{\infty} [\lceil (1-\alpha)2^{2^{2n+1}} \rceil, 2^{2^{2n+1}}],$$

then again $\bar{d}(A+A)$ is small. This example shows that we can only hope to characterize the structure of A along the increasing sequence h_n such that $\lim_{n\to\infty} \frac{|A\cap[1,h_n]|}{h_n} = \alpha$.

It was a long journey for me to arrive at the most recent result in [16] due to the technical difficulties of the proof. First the structure of A was characterized in [14] when $\bar{d}(A+A+\{0,1\})$ is small. Later the structure of A was characterized in [15] when $\bar{d}(A+A)$ is small and A contains two consecutive numbers. Finally in [16] the following theorem was proven.

Theorem 8.5.3 *Let $\bar{d}(A) = \alpha > 0$.*

Part I: Assume $\alpha > \frac{1}{2}$. Then $\bar{d}(A+A) = \frac{1+\alpha}{2}$ implies that for every increasing sequence $\{h_n : n \in \mathbb{N}\}$ with $\lim_{n\to\infty} \frac{|A\cap[0,h_n]|}{h_n+1} = \alpha$, we have

$$\lim_{n\to\infty} \frac{|(A+A)\cap[0,h_n]|}{h_n+1} = \alpha.$$

Part II: Assume $\alpha < \frac{1}{2}$ and $\gcd(A) = 1$. Then $\bar{d}(A+A) = \frac{3}{2}\alpha$ implies that either (a) there exist $k > 4$ and $c \in [1, k-1]$ such that $\alpha = \frac{2}{k}$ and $A \subseteq k\mathbb{N} \cup (c + k\mathbb{N})$ or (b) for every increasing sequence $\{h_n : n \in \mathbb{N}\}$ with $\lim_{n\to\infty} \frac{|A\cap[0,h_n]|}{h_n+1} = \alpha$, there exist two sequences $0 \leqslant c_n \leqslant b_n \leqslant h_n$ such that

$$\lim_{n\to\infty} \frac{|A\cap[b_n,h_n]|}{h_n - b_n + 1} = 1,$$

$$\lim_{n\to\infty} \frac{c_n}{h_n} = 0,$$

and $[c_n+1, b_n-1]\cap A = \emptyset$ for every $n \in \mathbb{N}$.

Part III: Assume $\alpha = \frac{1}{2}$ *and* $\gcd(A) = 1$. *Then* $\bar{d}(A+A) = \frac{3}{2}\alpha$ *implies that either (a) there exists* $c \in \{1, 3\}$ *such that* $A \subseteq 4\mathbb{N} \cup (c+4\mathbb{N})$ *or (b) for every increasing sequence* $\{h_n : n \in \mathbb{N}\}$ *with* $\lim_{n\to\infty} \frac{|A\cap[0,h_n]|}{h_n+1} = \alpha$, *we have*

$$\lim_{n\to\infty} \frac{|(A+A)\cap[0,h_n]|}{h_n+1} = \alpha.$$

I would like to make some remarks here on Theorem 8.5.3. First, the proof of Part I is easy; the most difficult part is Part II. Second, Part I and (b) of Part III cannot be improved so that set A has the structure similar to the structure described in (b) of Part II. For example, if one lets

$$A = \{0\} \cup \bigcup_{n=1}^{\infty} \left([3\cdot 2^{2^n-3}, 4\cdot 2^{2^n-3}] \cup [5\cdot 2^{2^n-3}, 2^{2^n}]\right),$$

then $\bar{d}(A) = \frac{1}{2}$ and $\bar{d}(A+A) = \frac{1+\bar{d}(A)}{2}$. Clearly A does not have the structure described in (b) of Part II.

The main ingredient of the proof of Theorem 8.5.3 is the following lemma in nonstandard analysis. For an internal set $A \subseteq [0, H]$ and a cut $U \subseteq [0, H]$ we define the lower U-density $\underline{d}_U(A)$ by

$$\underline{d}_U(A) = \sup\left\{\inf\left\{st\left(\frac{|A\cap[0,n]|}{n+1}\right) : n \in U \smallsetminus [0,m]\right\} : m \in U\right\},$$

where st means the standard part map. Note that if $U = \mathbb{N}$ and $A \subseteq \mathbb{N}$, then $\underline{d}(A) = \underline{d}_U(^*A)$. A set $I = \{a, a+d, a+2d, \ldots\}$ is called an arithmetic progression with difference d. An arithmetic progression can be finite (hyperfinite) or infinite. If an arithmetic progression is finite (hyperfinite), then its cardinality (internal cardinality) is its length. A set $I \cup J$ is called a bi-arithmetic progression if both I and J are arithmetic progressions with the same difference d and $I + I$, $I + J$, and $J + J$ are pairwise disjoint. A finite (hyperfinite) bi-arithmetic progression $I \cup J$ has its length $|I| + |J|$. Let U be a cut. A bi-arithmetic progression $B \subseteq U$ is called U-unbounded if both I and J are upper unbounded in U.

Lemma 8.5.4 *Let* H *be hyperfinite and* $U = \bigcap_{n\in\mathbb{N}}[0, \frac{H}{n}]$. *Suppose* $A \subseteq [0, H]$ *be such that* $0 < \underline{d}_U(A) = \alpha < \frac{2}{3}$. *If* $A \cap U$ *is neither a subset of an arithmetic progression of difference greater than* 1 *nor a subset of a* U*-unbounded bi-arithmetic progression, then there is a standard positive real number* $\gamma > 0$ *such that for every* $N > U$, *there is a* $K \in A$, $U < K < N$, *such that*

$$\frac{|(A+A)\cap[0,2K]|}{2K+1} \geqslant 3\frac{|A\cap[0,K]|}{2K+1} + \gamma.$$

Lemma 8.5.4 is motivated by Kneser's Theorem. It basically says that either $A+A$ is large in an interval $[0,2K]$ with $K > \frac{H}{n}$ for some standard n or A has desired arithmetic structure in an interval $[0,K]$ with $K > \frac{H}{n}$ for some standard n. The proof uses the fact that U is an additive semi-group. This can be done only in a nonstandard setting. It is interesting to see whether this lemma can be replaced by a standard argument with a reasonable length.

Recently G. Bordes [4] generalized Part II of Theorem 8.5.3 for sets A with small upper asymptotic density. He characterized the structure of A when $\bar{d}(A) \leqslant \alpha_0$ for some small positive number α_0 and $\bar{d}(A+A) < \frac{5}{3}\bar{d}(A)$. It is interesting to see whether one can replace α_0 by a relatively large value, say $\frac{2}{5}$, in Bordes' Theorem.

8.6 Freiman's $3k-3+b$ conjecture

After Theorem 8.5.3 was proven, I realized that the same methods used there could also be used to advance the existing results towards the solution of Freiman's $3k-3+b$ conjecture [5]. This is important because the conjecture is about the inverse problem for the addition of *finite sets*. Let A be a finite set of integers with cardinality $k>0$. It is easy to see that $|A+A| \geqslant 2k-1$. On the other hand, if $|A+A| = 2k-1$, then A must be an arithmetic progression. In the early 1960s, Freiman obtained the following generalizations [6].

(1) Let $A \subseteq \mathbb{N}$. Suppose $k = |A|$, $0 = \min A$, and $n = \max A$. Suppose also $\gcd(A) = 1$. Then $|A+A| \geqslant 3k-3$ if $n \geqslant 2k-3$ and $|A+A| \geqslant k+n$ if $n \leqslant 2k-3$.

(2) If $k > 3$ and $|2A| = 2k-1+b < 3k-3$, then A is a subset of an arithmetic progression of length at most $k+b$

(3) If $k > 6$ and $|2A| = 3k-3$, then either A is a subset of an arithmetic progression of length at most $2k-1$ or A is a bi-arithmetic progression.

In [6] a result was also mentioned without proofs for characterizing the structure of A when $k > 10$ and $|A+A| = 3k-2$. In [10] an interesting generalization of (3) above was obtained by Hamidoune and Plagne, where the condition $|2A| = 3k-3$ is replaced by $|A+tA| = 3k-3$ for every integer t. However, no further progress of this kind had been made for a larger value of $|A+A|$ before my recent work. In fact, Freiman made the following conjecture in [5] five years ago.

Conjecture 8.6.1 *There exists a natural number K such that for any finite set of integers A with $|A| = k > K$ and $|A+A| = 3k-3+b < \frac{10}{3}k-5$ for*

some $b \geqslant 0$, A is either a subset of an arithmetic progression of length at most $2k-1+2b$ or a subset of a bi-arithmetic progression of length at most $k+b$.

Note that the conclusion of Conjecture 8.6.1 could be false if one allows $|A+A| = \frac{10}{3}k - 5$. Simply let A be the union of three intervals $[0, a-1]$, $[b, b+a-1]$, and $[2b, 2b+a-1]$, where $k = 3a$ and b is a sufficiently large integer. Clearly $|A+A| = \frac{10}{3}k - 5$. Since b can be as large as we want, we can choose a b so that set A is neither a subset of an arithmetic progression of a restricted length nor a subset of a bi-arithmetic progression of a restricted length.

Using nonstandard methods such as Lemma 8.5.4, I was able to prove the following theorem in [17].

Theorem 8.6.2 *Suppose $f : \mathbb{N} \mapsto \mathbb{N}$ is a function with $\lim_{n\to\infty} \frac{f(n)}{n} = 0$. There exists a natural number K such that for any finite set of integers A with $|A| = k$, if $k > K$ and $|A+A| = 3k - 3 + b$ for some $0 \leqslant b \leqslant f(k)$, then A is either a subset of an arithmetic progression of length at most $2k-1+2b$ or a subset of a bi-arithmetic progression of length at most $k+b$.*

Theorem 8.6.2 gives a new result even for $f(x) \equiv 2$. However, we still have a long way before solving Conjecture 8.6.1. It is already interesting to see whether we can obtain the same result with $f(x) = \alpha x$ for some positive real number α.

The ideas for proving Theorem 8.6.2 are similar to the proof of Theorem 8.5.3, but much more technical. Suppose Theorem 8.6.2 is not true; then one can find a sequence of counterexamples A_n such that $|A_n| \to \infty$. Given a hyperfinite integer N, let $A = A_N$. Without loss of generality, we can assume that $0 = \min A$, $H = \max A$, $\gcd(A) = 1$, and $\alpha \approx \frac{|A|}{H+1} \gg 0$. Note that $\frac{|A+A|-3|A|+3}{H} \approx 0$. Hence $|A+A|$ is almost the same as $3|A|-3$ from the nonstandard point of view. Using the case-by-case argument, we can show that if $\frac{|A|}{H+1} \ll \frac{1}{2}$, then A is a subset of a bi-arithmetic progression. If $\frac{|A|}{H+1} \approx \frac{1}{2}$ and $b = |A+A| - 3|A| + 3$, then we can show that $H+1 \leqslant 2|A| - 1 + 2b$ when A is not a subset of a bi-arithmetic progression. The proof for the case $\frac{|A|}{H+1} \approx \frac{1}{2}$ is much harder than the proof for the case $\frac{|A|}{H+1} \ll \frac{1}{2}$ although the former depends on the latter. In both cases Lemma 8.5.4 was used to get structural information of A on an interval with length longer than $\frac{H}{n}$ for some standard positive integer n.

There are some similarities between our methods and analytic methods. In order to detect some structural properties of $A \subseteq [0, n]$, one may need to show that either A is uniformly distributed on $[0, n]$ or A has a greater density on a well formed subset of $[0, n]$. The analytic methods usually look for a large Fourier coefficient $\bar{A}(r)$ (cf. [8, Corollary 2.5]) or a large exponential sum $\sum_{i=0}^{n-1} A(i)e^{\frac{2\pi i}{n}}$ (cf. [24, Theorem 2.9]) to detect the greater density on a well

formed subset of $[0, n]$ when n is a prime number. When n is not a prime number then one needs to replace it with a prime number $p > n$ and consider A in $[0, p]$ instead. This replacement may not work well for Conjecture 8.6.1 as the structure of A needs to be very precise. In our methods we look for the greater density of $A \subseteq [0, H]$ on an interval $[0, K]$ for some $K > U$ by checking the value of $\underline{d}_U(A)$, where $U = \bigcap_{n\in\mathbb{N}}[0, \frac{H}{n}]$. If $\underline{d}_U(A) \geqslant \frac{2}{3}$, then the density of A on $[0, K]$ for some $K > U$ is significantly greater than $|A|/H$, which will lead to a contradiction that $|A + A|$ is almost the same as $3k - 3$. If $\underline{d}_U(A) = 0$, then the density of A on $[K, H]$ is significantly greater than $|A|/H$, which will again lead to a contradiction. Otherwise either $|(A + A) \cap [0, 2K]|$ is large, which is impossible by the fact that $\frac{|A+A|}{2H+1} \lessapprox \frac{3}{2}\alpha$, or A has very nice structural properties on $[0, K]$ following Lemma 8.5.4, which will force A to have the structure we hope for.

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9

Nonstandard methods and the Erdős-Turán conjecture

Steven C. Leth*

9.1 Introduction

A major open question in combinatorial number theory is the Erdős-Turán conjecture which states that if $A = \langle a_n \rangle$ is a sequence of natural numbers with the property that $\sum_{n=1}^{\infty} 1/a_n$ diverges then A contains arbitrarily long arithmetic progressions [1]. The difficulty of this problem is underscored by the fact that a positive answer would generalize Szemerédi's theorem which says that if a sequence $A \subset \mathbb{N}$ has positive upper Banach Density then A contains arbitrarily long arithmetic progressions. Szemerédi's theorem itself has been the object of intense interest since first conjectured, also by Erdős and Turán, in 1936. First proved by Szemerédi in 1974 [9], the theorem has been re-proved using completely different approaches by Furstenberg in 1977 [2, 3] and Gowers in 1999 [4], with each proof introducing powerful new methods.

The Erdős-Turán conjecture immediately implies that the primes contain arbitrarily long arithmetic progressions, and it was thought by many that a successful proof for the primes would be the result of either a proof of the conjecture itself or significant progress toward the conjecture. However, very recently Green and Tao were able to solve the question for the primes without generalizing Szemerédi's result in terms of providing weaker density conditions on a sequence guaranteeing that it contain arithmetic progressions.

In this paper we outline some possible ways in which nonstandard methods might be able to provide new approaches to attacking the Erdős-Turán conjecture, or at least other questions about the existence of arithmetic progressions. Heavy reference will be made to results in [7] and [8], and the proofs for all results quoted but not proved here appear in those two sources.

*Department of Mathematical Sciences, University of Northern Colorado, Greeley, CO 80639.
steven.leth@unco.edu

9.2 Near arithmetic progressions

We begin with some definitions that first appear in [8].

Definition 9.2.1 *Let $A \subset \mathbb{N}$, and let $I = [a, b]$ be an interval in $\mathbb{N}$. We will write* $\boldsymbol{l(I)}$ *for the length of I (i.e. $l(I) = b-a+1$) and we will write* $\boldsymbol{\delta(A, I)}$ *or* $\boldsymbol{\delta(A, [a, b])}$ *for the density of the set A on the interval I. Thus $\delta(A, I) = \frac{|A \cap I|}{l(I)}$.*

Definition 9.2.2 *Let t, d and w be in $\mathbb{N}$, and let $\alpha \in \mathbb{R}$ with $0 < \alpha < 1$. For $A \subset \mathbb{N}$ and I an interval in $\mathbb{N}$ of length $l(I)$ we say that A contains a t-termed α-homogeneous cell of distance d and width w in I or simply a* $\boldsymbol{< t, \alpha, d, w >}$ *cell in I iff there exists $b \in I$ with $b + (t-1)d + w$ also in I such that for each $\nu, \xi = 0, 1, 2, \ldots, t-1$:*

$$\delta\big(A, [b+\xi \cdot d, b+\xi \cdot d+w]\big) \geq (1-\alpha)\delta\big(A, [b+\nu \cdot d, b+\nu \cdot d+w]\big) \geq (1-\alpha)^2\delta(A, I).$$

If each $\delta(A, [b + \xi \cdot d, b + \xi \cdot d + w])$ is simply nonzero, i.e. the intervals are nonempty, then we say that A contains a $\boldsymbol{< t, d, w >}$ *cell.*

For $\beta > 0$ and $0 \leq u \leq w$ we will say that a $< t, \alpha, d, w >$ cell is $\boldsymbol{u, \beta}$ ***uniform*** *if for each $\nu = 0, 1, 2, ..., t-1$, and all x such that $u \leq x \leq w$:*

$$(1-\beta)\delta(A, J_\nu) \leq \delta(A, [b + \nu \cdot d, b + \nu \cdot d + x]) \leq (1+\beta)\delta(A, J_\nu),$$

where J_ν denotes the interval $[b + \nu \cdot d, b + \nu \cdot d + w]$.

It is clear that an actual arithmetic progression of length t and distance d is an example of a $< t, \alpha, w, d >$ cell with $w = 0$ and α any non-negative number. Furthermore, this cell is u, β uniform for $u = 0$ and any non-negative β. We could view the existence of a $< t, \alpha, d, w >$ cell inside a sequence A as a weak form of an actual arithmetic progression inside A. These cells are "near" arithmetic progressions in some (perhaps rather weak) sense, and intuitively are "nearer" to arithmetic progressions as the size of w decreases. In some of the results that we look at w will be "small" in the sense that the ratio of w to d will be small compared to the ratio of d to the length of the interval I. In other results w will be "small" by actually being bounded by a finite number while d gets arbitrarily large.

Definition 9.2.3 *Let I be an interval in $\mathbb{N}$, and $A \subset I$, with $r > 1 \in \mathbb{R}$ and $m \in \mathbb{N}$. We say that A has the* $\boldsymbol{m, r}$ ***density property*** *on I iff for any interval $J \subset I$, if $l(J) \geq \frac{l(I)}{m}$, then $\delta(A, J) \leq r\delta(A, I)$.*

Theorem 9.2.1 below gives a condition for the existence of "near" arithmetic progressions for any sequence on any interval I in which the density does not

drastically increase as the size of the subinterval decreases. More specifically, it provides an absolute constant such that whenever the density of a sequence does not increase beyond a fixed ratio for any subinterval of size greater than the length of I divided by that fixed constant, then the sequence will contain a $<t, \alpha, w, d>$ cell with some relative "smallness" conditions for w.

A complete proof of this theorem appears in [8], but we will outline the proof here, as it provides the clearest illustration of how the use of the nonstandard model provides us with a new set of tools for questions of this type.

Theorem 9.2.1 *Let $h(x)$ be any increasing real valued function such that $h(x) > 0$ whenever $x > 0$, and let $g(x)$ be any real valued function which approaches infinity as x approaches infinity. For all real $\alpha > 0$, $r > 1$ and $j, t \in \mathbb{N}$ there exists a standard natural number m such that for all $n > m$, whenever I is an interval of length n and any nonempty set $A \subset I$ has the m, r density property on I then A contains a u, β uniform $< t, \alpha, d, w >$ cell with $\frac{u}{w} < h(\frac{w}{d})$, $\frac{w}{d} < h(\frac{d}{n})$, $\beta < h(\frac{d}{n})$ and $\frac{n}{g(m)} < d < \frac{n}{j}$. Furthermore, we may take w and d to be powers of 2.*

Proof. (Sketch only). Suppose $h(x), g(x), \alpha, j, r, t$ are given as in the statement above and that no such m exists. By "overspill" there exists an M, N in ${}^*\mathbb{N} - \mathbb{N}$ with $M < N$ and a hyperfinite internal set A such that A has the M, r density property on an interval of length N but A contains no $< t, \alpha, d, w >$ cell on this interval with the required properties. Since the conditions are translation invariant we may assume that the interval is $[0, N-1]$. We now define a standard function $f : [0,1] \longrightarrow [0,1]$ by:

$$f(x) = st\left(\frac{|A \cap [0, xN]|}{|A \cap [0, N]||}\right).$$

Using the fact that A has the M, r density property on $[0, N]$ it is not difficult to show that $f(x)$ satisfies a Lipschitz condition with constant r. Thus, the function f is absolutely continuous, differentiable almost everywhere and equal to the integral of its derivative. Since $f(1) = 1$, $f(0) = 0$ and f is the integral of its derivative, it must be that the Lebesgue measure of $\{x : f'(x) \geq (1 - \frac{\alpha}{4})\}$ is nonzero. Thus, there exists a real number $c \geq 1$ such that the Lebesgue measure of the set

$$E = \left\{x : \ c - \frac{\alpha}{4} \leq f'(x) \leq c\right\}$$

is nonzero.

By using the Lebesgue density theorem it is straightforward to show that any set of positive measure contains arbitrarily long arithmetic progressions, and that, in fact, these progressions may have arbitrarily small differences

between elements. This allows us to obtain a $<t, \alpha, D, W>$ cell, with $D, W \in {}^*\mathbb{N} - \mathbb{N}$ with the property that there exists $B \in {}^*\mathbb{N} - \mathbb{N}$ such that

$$st\left(\frac{B}{N}\right), st\left(\frac{B+D}{N}\right), st\left(\frac{B+2D}{N}\right), \ldots, st\left(\frac{B+(t-1)D}{N}\right)$$

forms an arithmetic progression in E. The α homogeneity follows from the definition of E. The fact that f is differentiable at each point in E allows us to obtain the uniformity condition, and allows us to take U, D and W arbitrarily small but not infinitesimal to N. This, in turn, allows us the freedom to make those quantities powers of 2.

We are thus able to obtain a U, β uniform $< t, \alpha, D, W >$ cell for A in $[0, N-1]$ with all the properties required in the theorem, contradicting our assumption. □

Definition 9.2.4 *For A a sequence of positive integers we define the* ***upper Banach Density*** *of A or* ***$BD(A)$*** *by:*

$$BD(A) = \inf_{x \in \mathbb{N} - \{0\}} \max_{a \in \mathbb{N}} \frac{|A \cap [a+1, a+x]|}{x}.$$

Upper Banach density is often simply called Banach density, and is sometimes referred to in the literature as strong upper density, with notation $d^*(A)$ in place of $BD(A)$. That notation is used in [7].

The theorem allows us to obtain some results about the existence of uniform $<t, \alpha, d, w>$ cells in sequences that are relatively sparse (certainly too sparse to necessarily contain actual arithmetic progressions). The theorem below, also proved in [8], is of this type.

Theorem 9.2.2 *Let $\alpha > 0$ and $t > 2 \in \mathbb{N}$ be given, $h(x)$ be any continuous real valued function such that $h(x) > 0$ whenever $x > 0$ and let A be a sequence in $\mathbb{N}$ with the property that for all $\varepsilon > 0$, $|A \cap [0, n-1]| > n^{1-\varepsilon}$ for sufficiently large n. Then for sufficiently large n, A contains a u, β uniform $<t, \alpha, d, w>$ cell on $[0, n-1]$ with w and d powers of 2 and such that:*

$$\frac{u}{w} < h\left(\frac{w}{d}\right), \ \frac{w}{d} < h\left(\frac{\log d}{\log n}\right), \ \text{and} \ \ \beta < h\left(\frac{\log d}{\log n}\right).$$

The condition that $\frac{w}{d} < h\left(\frac{\log d}{\log n}\right)$ is not very strong, and it would be desirable to improve this weak "smallness" condition. The theorem below shows that, even for $t = 3$, if we keep the density condition on A as above then we cannot improve this smallness condition to $\frac{w}{d} < \left(\frac{d}{n}\right)^{\alpha}$ for any $\alpha > 0$, even if we do not insist on any homogeneity or uniformity conditions.

Theorem 9.2.3 *Let $\alpha > 0$. There exist constants $r > 0$ such that for arbitrarily large n there are subsets A of $[0, n-1]$ such that*

$$\left|A \cap [0, n-1]\right| > \frac{n}{2^{r \log\log n \sqrt{\log n}}},$$

and yet A contains no $<3, d, w>$ cell in $[0, n-1]$ satisfying $\frac{w}{d} < \left(\frac{d}{n}\right)^{\alpha}$, with w and d a power of 2.

Here we recall that a $<t, d, w>$ cell is merely a collection of t intervals in arithmetic progression on which A is nonempty.

Proof. Since the statement is strictly stronger as α decreases, we will assume that $\alpha \leq 1$. In [5, p. 98] it is shown that there exists a constant $c > 0$ and a sequence A satisfying

$$\left|A \cap [0, n-1]\right| > \frac{n}{e^{c\sqrt{\log n}}} \text{ for sufficiently large } n$$

that contains no 3-term arithmetic progression. This result is due to Behrend. For convenience we will adjust the constant and use log base 2 here, and also replace e with 2. By adjusting the constant if necessary (and using $2A$) we may assume that A contains no two consecutive numbers. We may also translate so that $0 \in A$ without changing the density condition. Thus, we begin with a sequence A which contains 0 and no two consecutive numbers and satisfies

$$\left|A \cap [0, n-1]\right| > \frac{n}{2^{c\sqrt{\log n}}} \text{ for sufficiently large } n.$$

Let $N \in {}^*\mathbb{N} - \mathbb{N}$ and let

$$\beta = (1/2)^{2/\alpha}; \quad m_0 = N; \quad m_1 = \text{the largest power of 2 less than } \beta^{(1+\alpha/2)} N$$

$$m_{k+1} = \text{the smallest power of 2 greater than } \left(\frac{m_k}{N}\right)^{\alpha} m_k$$

$$L = \text{the smallest number such that } m_{L+1} \leq 1.$$

We now wish to show by induction that

$$m_k \leq \beta^{(1+\alpha/2)^k} N. \tag{9.2.1}$$

To see this we note that for $k = 1$ the definition of m_1 guarantees this. By the construction we have

$$\left(\frac{m_k}{N}\right)^{\alpha} m_k < m_{k+1} \leq 2 \left(\frac{m_k}{N}\right)^{\alpha} m_k.$$

so that, assuming the induction hypothesis,

$$\begin{aligned} m_{k+1} &\leq 2\Big(\frac{m_k}{N}\Big)^{\alpha} m_k \\ &\leq 2\Big(\beta^{(1+\alpha/2)^k}\Big)^{\alpha} \beta^{(1+\alpha/2)^k} N \\ &= 2\Big(\beta^{\alpha(1+\alpha/2)^k}\Big)\beta^{(1+\alpha/2)^k} N \\ &= 2\Big(\beta^{\alpha/2(1+\alpha/2)^k}\Big)\Big(\beta^{\alpha/2(1+\alpha/2)^k}\Big)\beta^{(1+\alpha/2)^k} N \\ &\leq 2\beta^{\alpha/2}\beta^{\alpha/2(1+\alpha/2)^k+(1+\alpha/2)^k} N \\ &\leq \beta^{\alpha/2(1+\alpha/2)^k+(1+\alpha/2)^k} N \\ &= \beta^{(1+\alpha/2)^{k+1}} N, \end{aligned}$$

completing the induction step, and establishing 9.2.1 above.

We will define a subset B of $[0, N-1]$ with the property that it contains no $<3, d, w>$ cell in $[0, N-1]$ satisfying $\frac{w}{d} < \big(\frac{d}{N}\big)^{\alpha}$, with w and d a power of 2, and such that

$$\big|B \cap [0, N-1]\big| > \frac{N}{2^{cL\sqrt{\log N}}}$$

by essentially using block copies of initial segments of ${}^*A$. More specifically we let

$$B_1 = [0, m_1 - 1]$$

and, for $1 \leq k < L$

$$i \in B_{k+1} \text{ iff } \left\lfloor \frac{i'}{m_{k+1}} \right\rfloor \in {}^*A, \text{ where } i' \text{ is the remainder of } i \text{ mod } m_k.$$

When k is finite, the fact that $0 \in {}^*A$ means that B_k intersected with $B_1 \cap \ldots \cap B_{k-1}$ has cardinality at least a noninfinitesimal multiple of that of $B_1 \cap \ldots \cap B_{k-1}$. When k is in ${}^*\mathbb{N} - \mathbb{N}$ the density condition on A guarantees that at least a $\frac{1}{2^{c\sqrt{\log n}}}$ portion of B_k intersects with $B_1 \cap \ldots \cap B_{k-1}$, where $n < N$. Thus

$$B_1 \cap \ldots \cap B_k \text{ has cardinality at least } \frac{N}{2^{ck\sqrt{\log N}}}. \tag{9.2.2}$$

We now let

$$B = B_1 \cap \ldots \cap B_L.$$

Now suppose that B contains a $<3, d, w>$ cell on $[0, N-1]$ with $\frac{w}{d} < \big(\frac{d}{N}\big)^{\alpha}$ where both w and d are powers of 2. We show that this forces an actual arithmetic progression of length 3 in ${}^*A$ and thus in A (by transfer), contradicting our assumption about A.

To see this, we let i be such that $m_{i+1} \leq d < m_i$. Then since ${}^*A$ contains no two consecutive numbers, the $<3, d, w>$ cell on $[0, N-1]$ must be completely contained inside one of the blocks of length m_i, i.e. inside some $[\nu m_i, (\nu+1)m_i]$. But

$$w \leq \left(\frac{d}{N}\right)^{\alpha} d < \left(\frac{m_i}{N}\right)^{\alpha} m_i \leq m_{i+1},$$

and since w, d and the m_k's are powers of 2

$$w | m_{i+1} \quad \text{and} \quad m_i | d,$$

so that there exist 3 intervals of length m_{i+1} inside $[\nu m_i, (\nu+1)m_i]$ which contain elements of B and are in arithmetic progression. By the construction this means that ${}^*A$ contains an arithmetic progression of length 3, and then by transfer, so does A.

It remains to estimate L in terms of N. From 9.2.1 we see that

$$m_k \leq 1 \text{ when } \beta^{(1+\alpha/2)^k} N \leq 1 \quad \text{i.e. } N \leq (1/\beta)^{(1+\alpha/2)^k}$$

so that

$$\log N \leq (1+\alpha/2)^k \log(1/\beta)$$

or

$$\log\log N \leq k \log(1+\alpha/2) + \log\log(1/\beta).$$

Thus

$$m_k \leq 1 \text{ whenever } k \geq \frac{\log\log N - \log\log(1/\beta)}{\log(1+\alpha/2)}.$$

This and the definition of L now yield

$$L+1 \leq \frac{\log\log N - \log\log(1/\beta)}{\log(1+\alpha/2)}.$$

The above inequality, the definition of B and 9.2.2 imply that for any $r > c/\log(1+\alpha/2)$

$$\left|B \cap [0, N-1]\right| > \frac{N}{2^{r \log\log N \sqrt{\log N}}}.$$

For such an r the result now follows by transfer. □

We note that the density condition given in the theorem above is stronger than simply being greater than $n^{1-\varepsilon}$ for sufficiently large n, since $n^{1-\varepsilon}$ is of the form

$$\frac{n}{2^{c \log N}} < \frac{n}{2^{\log\log n \sqrt{\log n}}} \quad \text{for large } n.$$

Thus, weak as theorem 9.2.2 is in its smallness conditions, there are clear limits to how much it can be strengthened for sets of this relative sparseness.

It appears to be more promising to look at denser sequences in the hope of maintaining a stronger "smallness" condition. The theorem below is proved in [8], and provides just one possible example of conditions like this that might provide a means for approaching deep questions about arithmetic progressions. The proof of theorem 9.2.4 below is similar to that of the proof of theorem 3 given above. In particular these proofs illustrate how "smallness" conditions that may not seem very strong can be used to show that somewhat denser sets contain actual arithmetic progressions.

Theorem 9.2.4 *The Erdős-Turán conjecture follows if we can show that for fixed t and constant $c > 0$, there exists n_0 such that for all $n > n_0$, whenever the sequence A satisfies*

$$\left|A \cap [0, n-1]\right| > \frac{n}{(c \log n)^{2 \log \log n}}$$

then A contains a $<t, d, w>$ cell on $[0, n-1]$ with $\frac{w}{d} < \frac{d}{n}$ where both w and d are powers of 2.

9.3 The interval-measure property

The conditions given below are natural from the nonstandard perspective and not at all so from the standard perspective. They might provide another means of attacking questions about arithmetic progressions. These definitions first appear in [7].

Definition 9.3.1 *Let A be an internal subset of ${}^*\mathbb{N}$, $y, z \in {}^*\mathbb{N}$ with $z - y \in {}^*\mathbb{N} - \mathbb{N}$. Then we say that A has the **IM (interval-measure)** property on $[y, z]$ iff for every real, standard $\beta > 0$ there is a real, standard $\alpha > 0$ such that whenever $[u, v] \subset [y, z]$ with $v - u \in {}^*\mathbb{N} - \mathbb{N}$ and the largest gap of A on $[u, v]$ is $\leq \alpha(v - u)$ then*

$$\lambda\left(st\left\{\left(\frac{a-y}{z-y}\right) : a \in A \cap [y, z]\right\}\right) \geq 1 - \beta.$$

*If A is a standard subset of $\mathbb{N}$, we say that A has the **SIM (standard interval-measure)** property iff ${}^*A$ has the IM property on every interval $[y, z] \subset {}^*\mathbb{N}$ with $z - y \in {}^*\mathbb{N} - \mathbb{N}$, and*

$$\lambda\left(st\left\{\left(\frac{a-y}{z-y}\right) : a \in {}^*A \cap [y, z]\right\}\right) > 0 \text{ on some such interval } [y, z].$$

Here λ is used to denote Lebesgue measure.

A somewhat cumbersome but standard equivalent definition for the SIM property is given in [7]. Through the use of the standard equivalent it is easy to see that if A has the SIM property then given $\beta > 0$ there exists a fixed $\alpha > 0$ that works for every infinite interval. The theorem below follows immediately from theorem 3.2 in that same work.

Theorem 9.3.1 *Let $t \in \mathbb{N}$, $0 < \beta < 1/t$, and suppose that $A \subset \mathbb{N}$ has the SIM property, with α corresponding to the given β. Then there exists a constant $j \in \mathbb{N}$ such that whenever A contains a $<\boldsymbol{t, d, w}>$ cell consisting of the intervals $[b + \xi \cdot d, b + \xi \cdot d + w]$ for $0 \leq \xi \leq t - 1$ in which the largest gap of A on each of these intervals is $\leq \alpha w$ then A contains $<\boldsymbol{t, d, j}>$ subcell, i.e. consisting of intervals of the form $[b' + \xi \cdot d, b' + \xi \cdot d + j] \subset [b + \xi \cdot d, b + \xi \cdot d + w]$.*

This result is significant in that a fixed constant size to the intervals is a much stronger "smallness" condition than was achieved in previous results. However, the assumption that A is a SIM set is a strong condition. It is shown in [7] that any sequence $A = \langle a_n \rangle$ in which $\lim_{n\to\infty}(a_{n+1} - a_n) = \infty$ does not have the SIM property, so no pure density condition weaker than positive Banach density can imply that A contains a SIM set. On the other hand, sets may certainly have the SIM property without having positive Banach density. Even the question of whether or not positive Banach density is sufficient for a sequence to contain a SIM set is still open. A positive solution to either of the conjectures below would be a major step toward establishing the SIM condition as a useful tool for questions of this type. Since the two conjectures together imply Szemerédi's theorem, at least one of them is certain to be quite difficult.

Conjecture 9.3.1 *Let $A \subset \mathbb{N}$ have positive Banach density. Then A contains a subset B with the SIM property.*

Conjecture 9.3.2 *Every set $A \subset \mathbb{N}$ with the SIM property contains arbitrarily long arithmetic progressions.*

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Part III

Statistics, probability and measures

10

Nonstandard likelihood ratio test in exponential families

Jacques Bosgiraud*

Abstract

Let $(p_\theta)_{\theta\in\Theta}$ be an exponential family in $\mathbb{R}^k$. After establishing nonstandard results about large deviations of the sample mean $\overline{X}$, this paper defines the nonstandard likelihood ratio test of the null hypothesis $H_0 : \theta \in \mathrm{hal}(\widetilde{\Theta}_0)$, where $\widetilde{\Theta}_0$ is a standard subset of Θ and $\mathrm{hal}(\widetilde{\Theta}_0)$ its halo. If α is the level of the test, depending on whether $\frac{\ln\alpha}{n}$ is infinitesimal or not we obtain different rejection criteria. We calculate risks of the first and second kinds (external probabilities) and prove that this test is more powerful than any "regular" nonstandard test based on $\overline{X}$.

10.1 Introduction

10.1.1 A most powerful nonstandard test

In a preceding paper [9], we proved that the nonstandard likelihood ratio test (NSLRT) is more powerful than the nonstandard chi-squared test. Our purpose now is to generalize this result to classical exponential families in $\mathbb{R}^k$ (where k is standard): the NSLRT is more powerful than any nonstandard test issuing from a (family of) standard test(s) with rejection criterion $\overline{X} \in R$, where $\overline{X}$ is the sample mean and R a sufficiently regular set. The standard likelihood ratio test is not so powerful (see §10.5). We hope that viewing the problem from a general perspective will lead to a clearer understanding of its structure and simpler and better proofs. In §10.3, we establish some results about large deviations of the mean, more or less similar to classical results, before studying the NSLRT in §10.4 and comparing it to nonstandard "regular" tests based on $\overline{X}$ in §10.5.

*Université Paris VIII, Paris.
jacques.bosgiraud@univ-paris8.fr

This paper follows also papers about nonstandard tests for monotone likelihood ratio families ([5], [6], [8]) and we shall maintain the same definition for a nonstandard test in this paper. For notation and definitions of nonstandard analysis, external calculus, and external probabilities we refer to [15], [11], [14], [4].

10.2 Some basic concepts of statistics

10.2.1 Main definitions

A *statistical family* is a triplet $(\Omega, \mathcal{F}, \mathcal{P})$ where Ω is a set, $\mathcal{F}$ a σ-field of subsets of Ω, $\mathcal{P}$ a family of probabilities on $(\Omega, \mathcal{F})$: $\mathcal{P} := \{P_\theta : \theta \in \Theta\}$. In the following, we suppose that Ω is a subset of $\mathbb{R}^d$ (where d is a standard integer), that Θ is a subset of $\mathbb{R}^k$ (where k is a standard integer) and that there exists a positive mesure μ defined on $(\Omega, \mathcal{F})$ such that for each $\theta \in \Theta$, P_θ is absolutely continuous with respect to μ: we note $P_\theta := p_\theta \mu$ where p_θ is a $\mathcal{F}$-measurable function defined on Ω.

So, $X : \Omega \to \mathbb{R}^d$, $x \to x$ is a random variable with distribution P_θ.

Let n be a (standard or nonstandard) integer; denoting $\mathcal{X} := \Omega^n$, $P_\theta^n := P_\theta^{\otimes n}$ is a probability defined on $(\mathcal{X}, \mathcal{F}^{\otimes n})$ and we can write $P_\theta^n = p_\theta^n \mu^n$, where $\mu^n := \mu^{\otimes n}$ and $p_\theta^n(x_1, \ldots, x_n) = \prod_{i=1}^n p_\theta(x_i)$. A *n-sample* $(x_1, \ldots, x_n)$ is an element of $\mathcal{X}$. For $i = 1, \ldots, n$ we define $X_i : \mathcal{X} \to \mathbb{R}^d$ by $X_i(x_1, \ldots, x_n) = x_i$; then $X_1, \ldots, X_n$ are independent identically distributed (i.i.d.) random variables with distribution P_θ. $(X_1, \ldots, X_n)$ is a *n-sampling* of X. Note that some authors do not distinguish X_i and x_i, sample and sampling.

A *statistic* is a measurable function

$$T : \mathcal{X} \to \mathbb{R}^m, \quad (x_1, \ldots, x_n) \to T(x_1, \ldots, x_n).$$

For example, the sample mean $\overline{X} := \frac{1}{n} \sum_{i=1}^n X_i$ is a statistic (here $m = d$). We note $E_\theta T$ the expectation of T for the probability P_θ^n.

10.2.2 Tests

Let Θ_0 a nonempty proper subset of Θ, $\Theta_1 := \Theta_0^{\mathbf{C}}$ its complement and $\alpha \in]0, 1[$. A *level-α test* of $H_0 : \theta \in \Theta_0$ against $H_1 : \theta \in \Theta_1$ is a statistic $\varphi : \mathcal{X} \to [0, 1]$ such that $\forall \theta \in \Theta_0$, $E_\theta \varphi \leq \alpha$.

$H_0 : \theta \in \Theta_0$ is called the null hypothesis; $1 - \varphi$ is the probability of accepting H_0.

$H_1 : \theta \in \Theta_1$ is the alternative hypothesis and φ is the probability of rejecting H_0.

Also, $\sup_{\theta\in\Theta_0} E_\theta\varphi$ is called the size of the test. $\{(x_1,\dots,x_n) \in \mathcal{X}, \varphi(x_1,\dots,x_n) = 1\}$ is called the *rejection set* and $\{(x_1,\dots,x_n) \in \mathcal{X}, \varphi(x_1,\dots,x_n) = 0\}$ is called the *acceptation set*; if $\varphi(x_1,\dots,x_n) \in]0,1[$ the test is said to be randomized.

Very often, φ is defined through a statistic T and $\varphi(x_1,\dots,x_n) = 1 \Leftrightarrow T(x_1,\dots,x_n) \in R$ (in fact, we shall write $\varphi = 1 \Leftrightarrow T \in R$) where R is a subset of $\mathbb{R}^m$. R is also called the *rejection set* and "$T \in R$" is the *rejection criterium* (e.g. if $m = 1$, $T > t_0$ is a rejection criterium; t_0 is a constant depending on Θ_0 and α).

For $\theta \in \Theta_0$, $E_\theta\varphi$ is the risk of the first kind (at θ); for $\theta \in \Theta_1$, $E_\theta(1-\varphi)$ is the risk of the second kind (at θ); for $\theta \in \Theta$, $E_\theta\varphi$ is the power of φ (at θ).

For testing a given null hypothesis, there are generally a lot of level-α tests (for example the constant test $\varphi := \alpha$). A level-α test ϕ is said *uniformly the most powerful* (U.M.P.) if for any level-α test φ, ϕ is uniformly more powerful than ψ, i.e. $\forall\theta \in \Theta_1, E_\theta\phi \geq E_\theta\varphi$ (in fact "more powerful" means "at least as powerful"). U.M.P. tests only exist in particular cases: for example for 1-dimensional exponential families if $H_0 : \theta \leq \theta_0$ (where $\theta_0 \in \Theta$, an interval of $\mathbb{R}$). So, some more sophisticated notions are used to compare the power of two tests: for exemple the relative efficiency (cf. §10.5). It is not possible to summarize this notion in some words; so we suggest to refer to [1], [2] or [12].

10.3 Exponential families

10.3.1 Basic concepts

The following classical results and more information about exponential families can be found in [10] or [13]. Let k be a standard integer. We denote by $(x \mid y) = \sum_{j=1}^k x_j y_j$ the scalar product of x and y, vectors of $\mathbb{R}^k$. Let μ be a probability mesure on $\Omega := \mathbb{R}^k$ (the σ-field is the field of borelian sets) and let

$$\Theta := \left\{\theta \in \mathbb{R}^k : \int \exp(\theta \mid x)\mu(dx) < \infty\right\}.$$

The set Θ is convex and for $\theta \in \Theta$, let

$$\psi(\theta) = \ln \int \exp(\theta \mid x)\mu(dx).$$

The function ψ is convex and continuous on Θ^0 (the interior of Θ).

The statistical family $\{P_\theta : \theta \in \Theta\}$ defined by $P_\theta := p_\theta\mu$ where

$$p_\theta(x) = \exp\big((\theta \mid x) - \psi(\theta)\big)$$

is the (full) *exponential family associated to* μ. A lot of classical statistical families (e.g. multinomial distributions, multidimensional normal distributions,...) are exponential families, generally after reparametrization. This reparametrization can be chosen such that Θ^0 is nonempty. Let

$$\Theta' := \{\theta \in \Theta : \ E_\theta \|X\| < \infty\}$$

where $\|\|$ denotes the Euclidean norm.

For $\theta \in \Theta'$ we define $\lambda(\theta) := E_\theta X$; this mapping is 1-1 from Θ' onto $\Lambda := \lambda(\Theta')$: Θ' contains Θ^0 and λ is a 1-1 diffeomorphism from Θ^0 onto Λ^0 (cf. [10, pp.74,75]). To see this, notice that for $\theta \in \Theta^0$, $E_\theta X = \nabla\psi(\theta)$ and all derivatives of ψ exist at θ. For $\theta_1, \theta_2 \in \Theta^0$, $\theta_1 \neq \theta_2$, ψ is strictly convex on the line joining θ_1 and θ_2 and then $\big(\theta_1 - \theta_2 \mid \lambda(\theta_1) - \lambda(\theta_2)\big) > 0$.

10.3.2 Kullback-Leibler information number

For $\theta_0 \in \Theta$ and $\theta \in \Theta'$, the Kullback-Leibler information number is given by

$$I(\theta, \theta_0) = \psi(\theta_0) - \psi(\theta) + \big(\theta - \theta_0 \mid \lambda(\theta)\big).$$

If Θ_0 is a proper subset of Θ, let

$$I(\theta, \Theta_0) := \inf\{I(\theta, \theta_0) : \ \theta_0 \in \Theta_0\}.$$

For $(\xi, \xi_0) \in \Lambda^2$, we set

$$J(\xi, \xi_0) := I\big(\lambda^{-1}(\xi), \lambda^{-1}(\xi_0)\big),$$

and for $A \subset \Lambda$ and $\xi \in \Lambda$, let

$$J(\xi, A) := \inf\{J(\xi, a) : \ a \in A\},$$

$$J(A, \xi) := \inf\{J(a, \xi) : \ a \in A\}.$$

The classical likelihood ratio test of $H_{\Theta_0} : \theta \in \Theta_0$ against $\theta \in \Theta \setminus \Theta_0$ is based on the statistic

$$R_{\Theta_0} := \frac{1}{n}\left(\ln \sup_{\theta\in\Theta} \prod_{i=1}^n p_\theta(X_i) - \ln \sup_{\theta_0\in\Theta_0} \prod_{i=1}^n p_{\theta_0}(X_i)\right)$$

where $(X_i)_{1\le i\le n}$ is a n-sampling of X (see § 10.2.1). Here, denoting $\overline{X} = \frac{1}{n}\sum_{i=1}^n X_i$

$$\begin{aligned} R_{\Theta_0} &= \sup_{\theta\in\Theta}\big\{(\theta \mid \overline{X}) - \psi(\theta)\big\} - \sup_{\theta_0\in\Theta_0}\big\{(\theta_0 \mid \overline{X}) - \psi(\theta_0)\big\} \\ &= \inf_{\theta_0\in\Theta_0} \sup_{\theta\in\Theta}\big\{(\theta - \theta_0 \mid \overline{X}) - \psi(\theta) + \psi(\theta_0)\big\}. \end{aligned}$$

If $\overline{X} \in \Lambda$, then $\lambda^{-1}(\overline{X})$ is the maximum likelihood estimator of θ and so

$$R_{\Theta_0} = \inf_{\theta_0 \in \Theta_0} I(\lambda^{-1}(\overline{X}), \theta_0) = I(\lambda^{-1}(\overline{X}), \Theta_0) = J(\overline{X}, \Lambda_0)$$

where $\Lambda_0 := \lambda(\Theta_0)$.

$\overline{\Lambda}$ (the closure of Λ) is the closure of the convex hull of the support of μ (cf. [10]); so, in any case, $\overline{X} \in \overline{\Lambda}$. As $\sup\{(\theta \mid \xi) - \psi(\theta) : \theta \in \Theta\}$ is lower semi-continuous with respect to ξ, we define (as in [10]), for $\xi \in \partial\Lambda$ (the boundary of Λ) and $\xi_0 \in \Lambda$, $J(\xi, \xi_0)$ by

$$J(\xi, \xi_0) := \liminf_{\xi' \to \xi, \xi' \in \Lambda} J(\xi', \xi_0).$$

So, if $\overline{X} \notin \Lambda$ (then $\overline{X} \in \partial\Lambda$) it remains possible to write

$$R_{\Theta_0} = J(\overline{X}, \Lambda_0).$$

In the following, we shall suppose that, for each $\xi_0 \in \Lambda$, $J(\cdot, \xi_0)$ is continuous on $\overline{\Lambda}$. This assumption is verified by classical exponential families, but it is possible to build counter-examples (cf. exercise 7.5.6 in [10]). Note that if $\xi_0 \in \Lambda \setminus \mathrm{hal}(\partial\Lambda)$ is limited, this assumption implies the S-continuity of $J(\cdot, \xi_0)$ on any limited subset of $\overline{\Lambda}$. Indeed, if ξ_0 is standard, it is obvious. If ξ_0 is non-standard, and if ξ is limited, let $\theta_0 := \lambda^{-1}(\xi_0)$; as λ is a diffeomorphism from Θ^0 onto Λ^0, then ${}^o\theta_0 = \lambda^{-1}({}^o\xi_0)$ and so, as ψ is continuous on Θ^0, we can write

$$J(\xi, \xi_0) - J(\xi, {}^o\xi_0) = \psi(\theta_0) - \psi({}^o\theta_0) + ({}^o\theta_0 - \theta_0 \mid \xi) \simeq 0.$$

The S-continuity of $J(\cdot, \xi_0)$ is then deduced from the S-continuity of $J(\cdot, {}^o\xi_0)$.

In the following, if A is a subset of $\mathbb{R}^k$, A^0 will denote its interior, $\overline{A}$ its closure, ∂A its boundary for the euclidian topology of $\mathbb{R}^k$; its shadow ${}^oA$ is a closed standard set (cf. [11, p. 63]). If A is a subset of $\overline{\Lambda}$, $\widehat{\partial} A$ will denote its boundary for the induced topology of $\overline{\Lambda}$.

10.3.3 The nonstandard test

In this paper n will be supposed unlimited, so we shall use classical asymptotic properties of the likelihood ratio test (cf. §10.4). Let $\widetilde{\Theta}_0$ be a standard subset of Θ, such that $\mathrm{hal}(\widetilde{\Theta}_0)$ — its halo — is included in a standard compact K included in Θ^0.

Let $\mathcal{F} := \left\{\Theta_0 \text{ internal} : \widetilde{\Theta}_0 \subset \Theta_0 \subset \mathrm{hal}(\widetilde{\Theta}_0)\right\}$ and let Φ_{Θ_0} be the likelihood ratio test of $H_{\Theta_0} : \theta \in \Theta_0$ against $\theta \notin \Theta_0$ of size α. Recall that Φ_{Θ_0} is defined in the following way: there exists a number $d := d(\alpha, \Theta_0)$ such that

$$\sup_{\theta \in \Theta_0} P_\theta^n(R_{\Theta_0} > d) \leq \alpha \leq \sup_{\theta \in \Theta_0} P_\theta^n(R_{\Theta_0} \geq d).$$

Consequently, if $R_{\Theta_0} < d$ then $\Phi_{\Theta_0} = 0$, if $R_{\Theta_0} > d$ then $\Phi_{\Theta_0} = 1$ and one randomizes if $R_{\Theta_0} = d$.

Definition 10.3.1 *The nonstandard likelihood ratio test (NSLRT) of the nonstandard null hypothesis* $(H_0) : \theta \in hal(\widetilde{\Theta}_0)$ *of level* α *is defined by* $\Phi_0 \equiv \inf\{\Phi_{\Theta_0} : \Theta_0 \in \mathcal{F}\}$.

We prove in this paper that, except for a case studied in §10.4.3 where Φ_0 is randomized, Φ_0 is equal to 0 or 1:

- either $\forall \Theta_0 \in \mathcal{F}, \Phi_{\Theta_0} = 1$ and then $\Phi_0 = 1$ (if H_{Θ_0} is rejected for each $\Theta_0 \in \mathcal{F}$ then H_0 is rejected),
- or $\exists \Theta_0 \in \mathcal{F}, \Phi_{\Theta_0} = 0$ and then $\Phi_0 = 0$ (if H_{Θ_0} is accepted for at least one $\Theta_0 \in \mathcal{F}$ then H_0 is accepted).

At the beginning of §10.4 we shall explain that α has to be infinitesimal. We shall prove in §10.5 that this NSLRT is uniformly more powerful than any "regular" (this term will be defined later) nonstandard test based on $\overline{X}$.

Convention: if E is an external event, and $PR(E)$ its external probability and if η is an external number, "the probability of E is equal to η" means $PR(E) = \eta$ and "the probability of E is exactly equal to η" means $PR(E) \equiv \eta$.

10.3.4 Large deviations for $\overline{X}$

In exponential families, $J(\cdot, \lambda(\theta_0))$ can be regarded as the Cramér transform of P_{θ_0}. More generally, classical literature establishes that

$$\lim_{n\to\infty} \frac{1}{n} \ln P^n(\overline{X}_n \in A) = -\inf_{x\in A} C(x)$$

where A is a Borel set of $\mathbb{R}^k$ such that $A \subset \overline{A^0}$ and $\overline{X}_n$ the mean of n i.i.d. random variables taking values in $\mathbb{R}^k$, C the Cramér transform of their distribution (see [16], section 4: the proof is based on the minimax theorem for compact convex sets). Even more generally, if the i.i.d. random variables take values in some topological vector space E, classical literature (cf. [3]) establishes the same result if A is a finite union of convex Borel sets of E. For exponential families, it is possible to establish specific proofs by using half-spaces (cf. [10], chap. 7 and exercises 7.5.1 to 7.5.6). In a more general setting, half-spaces are also used in [3], section 2.

Our propositions 10.3.2 and 10.3.3 give nonstandard results which imply (by transfer) the classical results. If the first part of the proofs is based on the law of large numbers as in the classical literature, the second part, based on infinitesimal pavings, seems to be original.

10.3.5 *n*-regular sets

Definition 10.3.2 *Let A be a subset of $\overline{\Lambda}$ and $a \in \partial A$. We shall say that A is n-regular in a iff $\exists b \in hal(a)$, $\exists \rho \in \left]\frac{\mathcal{L}}{\sqrt{n}}, \emptyset\right]$ such that $\rho\left\|\lambda^{-1}(b)\right\| \simeq 0$ and $B(b,\rho) \subset A$ (where $B(b,\rho)$ is the open ball of centrum b and radius ρ).*

Proposition 10.3.1 *Let A be a limited subset of $\overline{\Lambda}$ such that ${}^oA = \overline{({}^oA)^0}$ and $hal(\partial A) = hal(\partial\, {}^oA)$; then A is n-regular in any point $a \in \partial A$.*

Proof. We first claim that $A \setminus \mathrm{hal}(\partial A) = {}^oA \setminus \mathrm{hal}(\partial\, {}^oA)$. Indeed, if $x \in A \setminus \mathrm{hal}(\partial A)$ then ${}^ox \in {}^oA$ and ${}^ox \notin \mathrm{hal}(\partial A) = \mathrm{hal}(\partial\, {}^oA)$ and so ${}^ox \in {}^oA \setminus \mathrm{hal}(\partial\, {}^oA)$; consequently, ${}^ox \in ({}^oA)^0$ and so $x \in ({}^oA)^0$; finally, as $x \notin \mathrm{hal}(\partial A) = \mathrm{hal}(\partial\, {}^oA)$, $x \in {}^oA \setminus \mathrm{hal}(\partial\, {}^oA)$. Conversely, if $x \in {}^oA \setminus \mathrm{hal}(\partial\, {}^oA)$, there exists $y \in A$ such that $x \simeq y$; then $x \in A$ for if not $x \in \mathrm{hal}(\partial A) = \mathrm{hal}(\partial\, {}^oA)$.

If $a \notin \mathrm{hal}(\partial\Lambda)$, choose an infinitesimal ρ such that $\rho > \frac{\mathcal{L}}{\sqrt{n}}$. Then for each $b \in A \setminus \mathrm{hal}(\partial A)$, $B(b,\rho) \subset A$: it is obvious if b is standard and if b is nonstandard, with ${}^ob$ its shadow, there exists a standard ρ_0 such that $B({}^ob,\rho_0) \subset A$ and then $B(b,\rho) \subset B({}^ob,\rho_0) \subset A$. So

$$\forall \varepsilon \gtrapprox 0,\ \exists b \in A,\ d(a,b) < \varepsilon \text{ and } B(b,\rho) \subset A.$$

Then the internal set $\{\varepsilon \in \mathbb{R} : \exists b \in A, d(a,b) < \varepsilon \wedge B(b,\rho) \subset A\}$ contains all appreciable ε. Cauchy's principle yields that

$$\exists \varepsilon \simeq 0,\ \exists b \in A,\ d(a,b) < \varepsilon \text{ and } B(b,\rho) \subset A;$$

as $a \notin \mathrm{hal}(\partial\Lambda)$ then $b \notin \mathrm{hal}(\partial\Lambda)$ and so $\lambda^{-1}(b)$ is limited (because λ is a standard diffeomorphism from Θ^0 onto Λ^0). Finally $\rho\left\|\lambda^{-1}(b)\right\| \simeq 0$.

If $a \in \mathrm{hal}(\partial\Lambda)$, fix $\rho = n^{-1/4}$; if $b \in A \setminus (\mathrm{hal}(\partial\Lambda) \cup \mathrm{hal}(\partial A))$ then $B(b,\rho) \subset A$ and, as $\lambda^{-1}(b)$ is limited $\rho\left\|\lambda^{-1}(b)\right\| < n^{-1/8}$ (for example). So the internal set

$$\left\{\varepsilon \in \mathbb{R} :\ \exists b \in A,\ d(a,b) < \varepsilon \ \wedge\ B(b,\rho) \subset A \ \wedge\ \rho\left\|\lambda^{-1}(b)\right\| < n^{-1/8}\right\}$$

contains all appreciable ε. Cauchy's principle yields that

$$\exists \varepsilon \simeq 0,\ \exists b \in A,\ d(a,b) < \varepsilon \wedge B(b,\rho) \subset A \wedge \rho\left\|\lambda^{-1}(b)\right\| < n^{-1/8}. \qquad \square$$

For example, a standard set A such that $A \subset \overline{A^0}$ (e.g. an open standard set) is n-regular in any boundary point. It is possible to prove that a limited convex set A such that ${}^oA$ has a nonempty interior is also n-regular in any boundary point (we shall not use this result in the following).

Lemma 10.3.1

(i) Let $(\xi_0, \xi_1) \in \Lambda^2$ *and let* $\xi \in]\xi_0, \xi_1[$ *(the segment between* ξ_0 *and* ξ_1*). Then* $J(\xi, \xi_1) < J(\xi_0, \xi_1)$.

(ii) Let A *be a nonempty subset of* Λ *and* $\xi_1 \in \Lambda^0 \setminus A$. *Then* $J(A, \xi_1) = J(\overline{A}, \xi_1) = J(\hat{\partial} A, \xi_1)$.

(iii) Let E *and* F *be subsets of* Λ *such that* $\overline{E}$ *is a compact set included in* F^0*; let* $\xi \in \Lambda^0 \setminus \overline{E}$. *Then* $J(F, \xi) < J(E, \xi)$ *and* $J(\xi, F) < J(\xi, E)$.

Proof. *(i)* As Λ is convex, $\xi \in \Lambda$. We set $\xi_0 - \xi =: \alpha(\xi - \xi_1)$ where $\alpha > 0$; denoting $\theta = \lambda^{-1}(\xi)$ and $\theta_1 := \lambda^{-1}(\xi_1)$, we have (using corollary 2.5 in [10])

$$J(\xi_0, \xi_1) - J(\xi, \xi_1) = (\theta - \theta_1 \mid \xi_0 - \xi) + J(\xi_0, \xi) = \alpha(\theta - \theta_1 \mid \xi - \xi_1) + J(\xi_0, \xi) > 0.$$

(ii) Let $\xi_0 \in A$. If $\xi_0 \notin \hat{\partial} A$, let $\xi \in \hat{\partial} A \cap]\xi_0, \xi_1[$. Using *(i)*, we can write $J(\xi, \xi_1) \leq J(\xi_0, \xi_1)$. So $J(\hat{\partial} A, \xi_1) \leq J(A, \xi_1)$. Conversely, using the continuity of $J(\cdot, \xi_1)$, we have

$$J(\hat{\partial} A, \xi_1) \geq J(\overline{A}, \xi_1) = J(A, \xi_1).$$

(iii) If $\xi \in \overline{F}$, then $J(F, \xi) = J(\xi, F) = 0$ and $J(E, \xi) \neq 0$, $J(\xi, E) \neq 0$ because $\xi \notin \overline{E}$.

We suppose now that $\xi \notin F$. Let $\xi_E \in \hat{\partial} E$ be such that $J(E, \xi) = J(\xi_E, \xi)$; there exists at least one $\xi_F \in \hat{\partial} F \cap]\xi_E, \xi[$ and then, using *(i)*, $J(\xi_F, \xi) < J(\xi_E, \xi)$ and so $J(F, \xi) < J(E, \xi)$.

Let now $\xi'_E \in \hat{\partial} E$ be such that $J(\xi, E) = J(\xi, \xi'_E)$; there exists at least one $\xi'_F \in \hat{\partial} F \cap]\xi'_E, \xi[$. We set $\theta_E := \lambda^{-1}(\xi'_E)$ and $\theta_F := \lambda^{-1}(\xi'_F)$. Then

$$\begin{aligned} J(\xi, \xi'_E) - J(\xi, \xi'_F) &= \\ &= \psi(\theta_E) - \psi(\theta) + (\theta - \theta_E \mid \xi) - \psi(\theta_F) + \psi(\theta) - (\theta - \theta_F \mid \xi) \\ &= \psi(\theta_E) - \psi(\theta_F) + (\theta_F - \theta_E \mid \xi) \\ &= \psi(\theta_E) - \psi(\theta_F) + (\theta_F - \theta_E \mid \xi'_F) + (\theta_F - \theta_E \mid \xi - \xi'_F) \\ &= I(\theta_F, \theta_E) + (\theta_F - \theta_E \mid \xi - \xi'_F). \end{aligned}$$

Then setting $\xi - \xi'_F =: \beta(\xi'_F - \xi'_E)$ where $\beta > 0$, we can write

$$J(\xi, \xi'_E) - J(\xi, \xi'_F) = J(\xi'_F, \xi'_E) + \beta(\theta_F - \theta_E \mid \xi'_E - \xi'_F) > 0,$$

according to corollary 2.5 in [10]. □

Proposition 10.3.2 *Let A be a limited subset of $\Lambda \setminus hal(\partial\Lambda)$, $\xi_1 = \lambda(\theta_1)$ be a limited point of $\Lambda \setminus hal(\partial\Lambda)$ and $\xi_0 \in \hat{\partial} A$ such that $J(\xi_0, \xi_1) \simeq J(A, \xi_1)$. If A is n-regular in ξ_0, then*

$$\frac{1}{n} \ln P_{\theta_1}^n(\overline{X} \in A) \simeq -J(A, \xi_1),$$

and if furthermore $d(\xi_1, A) \not\simeq 0$, then

$$-\frac{1}{n} \ln P_{\theta_1}^n(\overline{X} \in A) = J(A, \xi_1)(1 + \emptyset) = @.$$

Proof. For all $(\theta_1, \theta_2) \in \Theta^2$, denoting $x = (x_1, \ldots, x_n)$ and $\overline{x} = \frac{1}{n}\sum_{j=1}^n x_j$ (where $x_j = (x_{j,i})_{1 \le i \le k} \in \mathbb{R}^k$) we can write

$$P_{\theta_1}^n(dx) = \exp\Big(n\big((\theta_1 - \theta_2 \mid \overline{x}) - \psi(\theta_1) + \psi(\theta_2)\big)\Big) P_{\theta_2}^n(dx).$$

Let $\theta_0 := \lambda^{-1}(\xi_0)$ and let $\xi_2 =: \lambda(\theta_2)$ and $\rho > \frac{\mathcal{L}}{\sqrt{n}}$ be such that $\xi_2 \in \mathrm{hal}(\xi_0)$ and $B(\xi_2, \rho) \subset A$. As λ is a standard diffeomorphism from Θ^0 onto Λ^0, $\theta_0, \theta_1, \theta_2$ are limited. Then

$$\begin{aligned} P_{\theta_1}^n\big(\overline{X} \in A\big) &\ge P_{\theta_1}^n\big(\overline{X} \in B(\xi_2, \rho)\big) \\ &\ge \int_{\overline{X} \in B(\xi_2, \rho)} \exp\Big(n\big((\theta_1 - \theta_2 \mid \overline{X}) - \psi(\theta_1) + \psi(\theta_2)\big)\Big)\, dP_{\theta_2}^n \\ &\ge \exp\Big(n \int_{\overline{X} \in B(\xi_2, \rho)} \big((\theta_1 - \theta_2 \mid \overline{X}) - \psi(\theta_1) + \psi(\theta_2)\big)\, dP_{\theta_2}^n\Big) \end{aligned}$$

by Jensen's inequality.

According to (8.4) in [4], $P_{\theta_2}^n(\overline{X}_i \in [\xi_{2,i} - \rho/2, \xi_{2,i} + \rho/2]) = 1 + \emptyset$ for each $i = 1 \ldots k$. Then the nonstandard law of large numbers yields $P_{\theta_2}^n(\overline{X} \in B(\xi_2, \rho)) = 1 + \emptyset$ and we can write (denoting by $\mathcal{L}^k$ the external set of limited vectors of $\mathbb{R}^k$)

$$\begin{aligned} \frac{1}{n} \ln P_{\theta_1}^n(\overline{X} \in A) &\ge (\theta_1 - \theta_2 \mid \xi_2 + \mathcal{L}^k \rho) - \psi(\theta_1) + \psi(\theta_2) + \frac{1}{n} \ln(1 + \emptyset) \\ &= (\theta_1 - \theta_2 \mid \xi_2) + \mathcal{L}\rho - \psi(\theta_1) + \psi(\theta_2) + \frac{\emptyset}{n} \\ &= -I(\theta_2, \theta_1) + \emptyset. \end{aligned}$$

As

$$I(\theta_2, \theta_1) - I(\theta_0, \theta_1) = (\theta_0 - \theta_1 \mid \xi_2 - \xi_0) + I(\theta_2, \theta_0),$$

with $\|\theta_0 - \theta_1\|$ limited, $\|\xi_2 - \xi_0\| \simeq 0$ and $I(\theta_2, \theta_0) = @\|\theta_2 - \theta_0\| = @\|\xi_2 - \xi_0\|$ (cf. lemma 3.2.2 in [13]), we have

$$-I(\theta_2, \theta_1) = -I(\theta_0, \theta_1) + \emptyset = -J(A, \xi_1) + \emptyset,$$

and so

$$\frac{1}{n}\ln P_{\theta_1}^n(\overline{X}\in A)\geq -J(A,\xi_1)+\emptyset.$$

Conversely, the limited set A is included in a hypercube $[-p,p]^k$ where p is a standard integer; pave it with $n^k(2p)^k$ hypercubes (T_l) with side $\delta=\frac{1}{n}$. Among these hypercubes, eliminate the ones which do not intersect A and for the others choose $t_l\in A\cap T_l$. In the aim of simplicity, we shall denote again the selected hypercubes by $(T_l)_{1\leq l\leq N}$. Let $\theta_l:=\lambda^{-1}(t_l)$, θ_l is limited because t_l is limited and $t_l\notin \mathrm{hal}(\partial\Lambda)$. From the relation $N<n^k(2p)^k$ we deduce $\frac{\ln N}{n}=\frac{\ln n}{n}\mathcal{L}$. Then, we can write

$$\begin{aligned}
P_{\theta_1}^n(\overline{X}\in A) &\leq \sum_{l=1}^{N}\int_{\{\overline{X}\in T_l\}} dP_{\theta_1}^n\\
&= \sum_{l=1}^{N}\int_{\{\overline{X}\in T_l\}} \exp\Big(n\big((\theta_1-\theta_l\mid\overline{X})-\psi(\theta_1)+\psi(\theta_l)\big)\Big)\,dP_{\theta_l}^n\\
&\leq \sum_{l=1}^{N}\exp\left(n\left(\left(\theta_1-\theta_l\mid t_l+\frac{\mathcal{L}^k}{n}\right)-\psi(\theta_1)+\psi(\theta_l)\right)\right)\\
&\leq N\max_{l=1\ldots N}\exp\left\{n\left(\left(\theta_1-\theta_l\mid t_l+\frac{\mathcal{L}^k}{n}\right)-\psi(\theta_1)+\psi(\theta_l)\right)\right\}.
\end{aligned}$$

Thus

$$\begin{aligned}
\frac{1}{n}\ln P_{\theta_1}^n(\overline{X}\in A) &\leq \frac{1}{n}\ln N+\max_{l=1\ldots N}\left\{\left(\theta_1-\theta_l\mid t_l+\frac{\mathcal{L}^k}{n}\right)-\psi(\theta_1)+\psi(\theta_l)\right\}\\
&\leq \frac{1}{n}\ln N+\max_{l=1\ldots N}\left\{(\theta_1-\theta_l\mid t_l)-\psi(\theta_1)+\psi(\theta_l)\right\}+\frac{\mathcal{L}}{n}\\
&\leq \frac{1}{n}\ln N+\max_{l=1\ldots N}-I(\theta_l,\theta_1)+\frac{\mathcal{L}}{n}\\
&\leq -J(A,\xi_1)+\emptyset.
\end{aligned}$$

Finally,

$$\frac{1}{n}\ln P_{\theta_1}^n(\overline{X}\in A)=-J(A,\xi_1)+\emptyset.$$

As we said previously, according to lemma 3.2.2 in [13], if a and ξ_1 are limited elements of $\Lambda\setminus\mathrm{hal}(\Lambda)$, then $J(a,\xi_1)=@d(a,\xi_1)$. Consequently, if $d(A,\xi_1)=@$ then $J(A,\xi_1)=@$. Thus

$$-\frac{1}{n}\ln P_{\theta_1}^n(\overline{X}\in A)=J(A,\xi_1)(1+\emptyset)=@.\qquad\square$$

In fact, the continuity of $J(\cdot,\xi_1)$ allows us to generalize this result to limited subsets of $\overline{\Lambda}$:

Proposition 10.3.3 *Let A be a limited subset of $\overline{\Lambda}$, ξ_1 be a limited point of $\Lambda \setminus hal(\partial\Lambda)$ and let $\xi_0 \in \hat{\partial}A$ such that $J(\xi_0, \xi_1) \simeq J(A, \xi_1)$. If A is n-regular in ξ_0 then*

$$\frac{1}{n} \ln P^n_{\theta_1}(\overline{X} \in A) \simeq -J(A, \xi_1).$$

Proof. Let $\xi_2 =: \lambda(\theta_2) \in \text{hal}(\xi_0)$ and $\rho > \frac{\mathcal{L}}{\sqrt{n}}$ be such that $B(\xi_2, \rho) \subset A$ and $\rho \|\theta_2\| \simeq 0$. As in the proof of proposition 10.3.2, we can write

$$\begin{aligned} \frac{1}{n} \ln P^n_{\theta_1}(\overline{X} \in A) &\geq (\theta_1 - \theta_2 \mid \xi_2 + \mathcal{L}^k\rho) - \psi(\theta_1) + \psi(\theta_2) + \frac{1}{n}\ln(1+\emptyset) \\ &= (\theta_1 - \theta_2 \mid \xi_2) + \mathcal{L}\rho + \emptyset - \psi(\theta_1) + \psi(\theta_2) + \emptyset \\ &= -J(\xi_2, \xi_1) + \emptyset = -J(\xi_0, \xi_1) + \emptyset = -J(A, \xi_1) + \emptyset \end{aligned}$$

because $\|\xi_2 - \xi_0\| \simeq 0$ and $J(\cdot, \xi_1)$ is S-continuous.

Conversely, as in the proof of proposition 10.3.2, use a paving $(T_l)_{1 \leq l \leq N}$ with side $\delta = \frac{1}{n}$ where each T_l is closed and such that $T_l \cap A \neq \varnothing$, but choose t_l in T_l (maybe outside A) such that:

$$\begin{array}{l} \text{if } \theta_{1,i} - \theta_{l,i} > 0 \text{ then } t_{l,i} \text{ is maximal in } T_l \\ \text{if } \theta_{1,i} - \theta_{l,i} \leq 0 \text{ then } t_{l,i} \text{ is minimal in } T_l \end{array}$$

So, for $\overline{X} \in T_l$ we can write $(\theta_1 - \theta_l \mid \overline{X}) \leq (\theta_1 - \theta_l \mid t_l)$ and then

$$\begin{aligned} P^n_{\theta_1}(\overline{X} \in A) &\leq \sum_{l=1}^{N} \int_{\{\overline{X} \in T_l\}} dP^n_{\theta_1} \\ &\leq \sum_{l=1}^{N} \int_{\{\overline{X} \in T_l\}} \exp\Big(n\big((\theta_1 - \theta_l \mid \overline{X}) - \psi(\theta_1) + \psi(\theta_l)\big)\Big)\, dP^n_{\theta_l} \\ &\leq \sum_{l=1}^{N} \exp\Big(n\big((\theta_1 - \theta_l \mid t_l) - \psi(\theta_1) + \psi(\theta_l)\big)\Big). \end{aligned}$$

Thus

$$\begin{aligned} \frac{1}{n} \ln P^n_{\theta_1}(\overline{X} \in A) &\leq \frac{1}{n} \ln N + \max_{l=1\ldots N} \{(\theta_1 - \theta_l \mid t_l) - \psi(\theta_1) + \psi(\theta_l)\} \\ &\leq \frac{1}{n} \ln N + \max_{l=1\ldots N} \{-J(t_l, \xi_1)\}. \end{aligned}$$

As $J(\cdot, \xi_1)$ is S-continuous, $J(t_l, \xi_1) = J({}^o t_l, \xi_1) + \emptyset$ and then, as ${}^o t_l \in {}^o A$

$$\frac{1}{n} \ln P^n_{\theta_1}(\overline{X} \in A) \leq -J({}^o A, \xi_1) + \emptyset.$$

We now claim that $J({}^oA, \xi_1) \simeq J(A, \xi_1)$. Indeed, if $a \in A$, then ${}^oa \in {}^oA$ and $J(a, \xi_1) \simeq J({}^oa, \xi_1)$; so $J({}^oA, \xi_1) \lesssim J(A, \xi_1)$. Conversely, if $a \in {}^oA$, there exists $a' \in A$ such that $a \simeq a'$ and thus such that $J(a, \xi_1) \simeq J(a', \xi_1)$; so $J(A, \xi_1) \lesssim J({}^oA, \xi_1)$.

Finally $\frac{1}{n} \ln P_{\theta_1}^n(\overline{X} \in A) \leq -J(A, \xi_1) + \emptyset$. □

10.3.6 *n*-regular sets defined by Kullback-Leibler information

Let Θ_0 be an internal subset of Θ, included in a standard compact K included in the interior of Θ.

Let $c_0 := c_0(\Theta_0) := \sup\{I(\theta, \Theta_0) : \theta \in \Theta\} \leq \infty$. For $c \in]0, c_0[$, let $A_c := \left\{\xi \in \overline{\Lambda} : J(\xi, \Lambda_0) \leq c\right\}$ where $\Lambda_0 := \lambda(\Theta_0)$. We shall see in lemma 10.3.2 that $J(\cdot, \Lambda_0)$ is continuous; then if $c < c' < c_0$, A_c is a proper subset of $A_{c'}$ (for if not, $\left\{\xi \in \overline{\Lambda} : J(\xi, \Lambda_0) \leq \frac{c+c'}{2}\right\} = \left\{\xi \in \overline{\Lambda} : J(\xi, \Lambda_0) < \frac{c+c'}{2}\right\}$ is a connected component of $\overline{\Lambda}$ which is a connected set.)

Lemma 10.3.2 *If $c \leq c_0$ is limited, then $\hat{\partial} A_c$ is a limited compact set.*

Proof. $\hat{\partial} A_c = \left\{\xi \in \overline{\Lambda} : J(\xi, \Lambda_0) = c\right\}$ is closed in $\overline{\Lambda}$ since the function $\xi \to J(\xi, \Lambda_0)$ is continuous on $\overline{\Lambda}$ as we prove now. By transfer, we just have to prove this continuity for a standard Λ_0: if ξ_1 and ξ_2 are such that $\|\xi_1 - \xi_2\| \simeq 0$, if $\xi_0 \in \Lambda_0$ is such that $J(\xi_1, \xi_0) \simeq J(\xi_1, \Lambda_0)$ then, as $J(\cdot, \xi_0)$ is S-continuous, $J(\xi_2, \xi_0) \simeq J(\xi_1, \xi_0)$ and so $J(\xi_2, \Lambda_0) \lesssim J(\xi_1, \Lambda_0)$; similarly $J(\xi_1, \Lambda_0) \lesssim J(\xi_2, \Lambda_0)$.

We prove now that if ξ is unlimited, then $J(\xi, \Lambda_0)$ is unlimited. For each $\xi_0 \in \Lambda_0$, ξ_0 belongs to the standard compact K; then $\|\xi - \xi_0\| \simeq \infty$ and so $J(\xi, \xi_0) \simeq \infty$ (cf. [10, p. 177]). Thus $J(\xi, \Lambda_0)$ is unlimited.

Therefore $\left\{\xi \in \overline{\Lambda} : J(\xi, \Lambda_0) = c\right\}$ is limited. □

Proposition 10.3.4 *Let $c \leq c_0$ be limited and θ_1 be a limited point of Θ' such that $\xi_1 := \lambda(\theta_1) \notin hal(\partial\Lambda)$.*

(i) If $\xi_1 \notin A_c$, then $\frac{1}{n} \ln P_{\theta_1}^n(\overline{X} \in A_c) = \frac{1}{n} \ln P_{\theta_1}^n(\overline{X} \in A_c^0) \simeq -J(\hat{\partial} A_c, \xi_1)$.

(ii) If $\xi_1 \in A_c$, then $\frac{1}{n} \ln P_{\theta_1}^n(\overline{X} \in A_c) \simeq 0$.

(iii) If $\xi_1 \notin A_c^{\mathbf{C}}$, then $\frac{1}{n} \ln P_{\theta_1}^n(\overline{X} \in A_c^{\mathbf{C}}) = \frac{1}{n} \ln P_{\theta_1}^n(\overline{X} \in \overline{A_c^{\mathbf{C}}}) \simeq -J(\hat{\partial} A_c, \xi_1)$.

(iv) If $\xi_1 \in A_c^{\mathbf{C}}$, then $\frac{1}{n} \ln P_{\theta_1}^n(\overline{X} \in A_c^{\mathbf{C}}) \simeq 0$.

Proof. We first prove the both similar results *(i)* and *(iii)*, and then the both similar results *(ii)* and *(iv)* which are obtained in a same way.

(i) If Θ_0 and c are standard, A_c is a standard compact set such that $A_c = \overline{A_c^0}$. Indeed, $A_c^0 = \{\xi \in \Lambda : J(\xi, \Lambda_0) < c\}$. If ξ is such that $J(\xi, \Lambda_0) = c$, let $\lambda_0 \in \overline{\Lambda}_0$ (a compact set) be such that $J(\xi, \Lambda_0) = J(\xi, \overline{\Lambda}_0) = J(\xi, \lambda_0) = c$. According to lemma 10.3.1 (i), for any $\xi' \in]\xi, \lambda_0[$, $J(\xi', \lambda_0) < J(\xi, \lambda_0)$ and so $J(\xi', \Lambda_0) < c$. Thus $]\xi, \lambda_0[\in A_c^0$ and then $\xi \in \overline{A_c^0}$. So, according to proposition 10.3.1, A_c and A_c^0 are n-regular in any point of ∂A_c and then, according to proposition 10.3.3, we can write $\frac{1}{n} \ln P_{\theta_1}^n(\overline{X} \in A_c) \simeq -J(A_c, \xi_1)$ and $\frac{1}{n} \ln P_{\theta_1}^n(\overline{X} \in A_c^0) \simeq -J(A_c^0, \xi_1)$. Finally, lemma 10.3.1 (ii) yields $J(A_c, \xi_1) = J(A_c^0, \xi_1) = J(\widehat{\partial} A_c, \xi_1)$.

Now, if c or Θ_0 are not standard, the continuity of $J(\xi, \cdot)$ on $\lambda(K)$ implies ${}^oA_c({}^o\Theta_0) = A_{\circ c}({}^o\Theta_0)$ and the hypothesis of proposition 10.3.1 is verified since

$$\mathrm{hal}(\partial A_c) = \mathrm{hal}(\widehat{\partial} A_c) \cup \left\{\xi \in \partial\Lambda : \ J(\xi, \Lambda_0) \lessapprox c\right\}$$

and

$$\mathrm{hal}(\partial\, {}^oA_c) = \mathrm{hal}(\widehat{\partial}\, {}^oA_c) \cup \left\{\xi \in \partial\Lambda : \ J(\xi, {}^o\Lambda_0) \lessapprox {}^oc\right\}.$$

Indeed, on one hand, the continuity of $J(\xi, \cdot)$ implies

$$\left\{\xi \in \partial\Lambda : J(\xi, \Lambda_0) \lessapprox c\right\} = \left\{\xi \in \partial\Lambda : J(\xi, {}^o\Lambda_0) \lessapprox {}^oc\right\},$$

and on the other hand, this continuity also implies

$$\begin{aligned} \mathrm{hal}(\hat{\partial} A_c) &= \left\{\xi \in \overline{\Lambda} : J(\xi, \lambda(\Theta_0)) \simeq c\right\} \\ &= \left\{\xi \in \overline{\Lambda} : J(\xi, \lambda({}^o\Theta_0)) \simeq {}^oc\right\} = \mathrm{hal}(\hat{\partial}\, {}^oA_c). \end{aligned}$$

Then we can conclude as before, using proposition 10.3.1, proposition 10.3.3 and lemma 10.3.1.

(iii) In the same way, $A_c^{\mathbf{C}}$ is n-regular in any point of $\widehat{\partial} A_c^{\mathbf{C}} = \widehat{\partial} A_c$, but $A_c^{\mathbf{C}}$ is not always limited. Meanwhile, $\widehat{\partial} A_c$ is limited and so the proof (in proposition 10.3.3) of $\frac{1}{n} \ln P_{\theta_1}^n(\overline{X} \in A_c^{\mathbf{C}}) \geq -J(\hat{\partial} A_c, \xi_1) + \emptyset$ remains valid.

Suppose now that $\xi_1 \in A_c$. A_c is included in a standard hypercube $U_p := [-p, p]^k$ (where p is a standard integer). If we set $S_{p,i} := \left\{x \in \mathbb{R}^k : x_i > p\right\}$ and $S'_{p,i} := \left\{x \in \mathbb{R}^k : x_i < -p\right\}$ then $U_p^{\mathbf{C}} = \bigcup_{i=1}^k (S_{p,i} \cup S'_{p,i})$. It is clear that

$$P_{\theta_1}^n(\overline{X} \in A_c^{\mathbf{C}}) = P_{\theta_1}^n(\overline{X} \in U_p \setminus A_c) + P_{\theta_1}^n(\overline{X} \in U_p^{\mathbf{C}}).$$

On one hand, as $U_p \setminus A_c$ is limited and n-regular in any boundary point, we can write

$$\frac{1}{n} \ln P_{\theta_1}^n(\overline{X} \in U_p \setminus A_c) \simeq -J(U_p \setminus A_c, \xi_1) = -J(\widehat{\partial} U_p \cup \widehat{\partial} A_c, \xi_1).$$

Using lemma 10.3.1(iii) with $E = \widehat{\partial} U_p$ and $F = A_c^{\mathbf{C}}$, we have $J(\widehat{\partial} U_p \cup \widehat{\partial} A_c, \xi_1) = J(\widehat{\partial} A_c, \xi_1)$ and so $\frac{1}{n} \ln P_{\theta_1}^n(\overline{X} \in U_p \setminus A_c) \simeq -J(\widehat{\partial} A_c, \xi_1)$.

On the other hand,

$$P_{\theta_1}^n(\overline{X} \in U_p^{\mathbf{C}}) \leq \sum_{i=1}^{k} (P_{\theta_1}^n(\overline{X} \in S_{p,i}) + P_{\theta_1}^n(\overline{X} \in S'_{p,i})).$$

We know that $\frac{1}{n} \ln P_{\theta_1}^n(\overline{X} \in S_{p,i}) \simeq -J(S_{p,i}, \xi_1)$ and $\frac{1}{n} \ln P_{\theta_1}^n(\overline{X} \in S'_{p,i}) \simeq -J(S'_{p,i}, \xi_1)$ (it is a classical result concerning half-spaces: cf. chap. 7 in [10]); so

$$\frac{1}{n} \ln P_{\theta_1}^n(\overline{X} \in U_p^{\mathbf{C}}) \leq \max_{i=1\ldots k} \max(-J(S_{p,i}, \xi_1), -J(S'_{p,i}, \xi_1)) + \emptyset$$

which is less than $-J(\widehat{\partial} A_c, \xi_1)$ according to lemma 10.3.1(i). Then $\frac{1}{n} \ln P_{\theta_1}^n(\overline{X} \in A_c^{\mathbf{C}}) \leq -J(\hat{\partial} A_c, \xi_1) + \emptyset$.

Finally, $\frac{1}{n} \ln P_{\theta_1}^n(\overline{X} \in A_c^{\mathbf{C}}) \simeq -J(\widehat{\partial} A_c, \xi_1)$.

(ii) If $\xi_1 \in \mathrm{hal}(\widehat{\partial} A_c)$, then exists $\xi_0 \in \widehat{\partial} A_c$ such that $J(\xi_0, \xi_1) \simeq J(A, \xi_1) \simeq 0$ and it remains possible to use proposition 10.3.1, proposition 10.3.3 and lemma 10.3.1 as for the proof of *(i)*.

If $\xi_1 \notin \mathrm{hal}(\widehat{\partial} A_c)$, then $\xi_1 \in {}^o A_c \setminus \mathrm{hal}(\widehat{\partial} A_c)$ according to the proof of *(i)* and so there is a standard number ρ such that the ball $B(\xi, \rho)$ is included in ${}^o A_c \setminus \mathrm{hal}(\widehat{\partial} A_c)$ and consequently included in A_c. Then the nonstandard law of large numbers yields $P_{\theta_1}^n(\overline{X} \in A_c) = 1 + \emptyset$ and finally $\ln P_{\theta_1}^n(\overline{X} \in A_c) \simeq 0$.

(iv) is obtained in a similar way. □

Definition 10.3.3 *Let $\theta \in \Theta'$. Let h_θ be the function defined on $[0, c_0[$ by $h_\theta(c) := J(A_c^{\mathbf{C}}, \lambda(\theta))$ and g_θ the function defined on $[0, c_0[$ by $g_\theta(c) := J(A_c, \lambda(\theta))$.*

As $c \to A_c$ is increasing, g_θ is a non-increasing function and h_θ is a non-decreasing function. Let $\gamma(\theta) := I(\theta, \Theta_0)$; $\lambda(\theta) \in A_c$ iff $c \geq \gamma(\theta)$ and so $g_\theta(c) = 0$ if and only if $c \in [\gamma(\theta), c_0[$ and $h_\theta(c) = 0$ if and only if $c \in]0, \gamma(\theta)]$.

Proposition 10.3.5

(i) The functions $(\theta, c) \to g_\theta(c)$ and $(\theta, c) \to h_\theta(c)$ are S-continuous on $\{\theta \in \Theta \setminus \mathrm{hal}(\partial\Theta) : \theta$ limited$\} \times ([0, c_0[\cap \mathcal{L})$.

(ii) The function g_θ is decreasing on $[0, \gamma(\theta)]$ and h_θ is increasing on $[\gamma(\theta), c_0]$.

Proof. Let θ be standard and $\theta' \simeq \theta$, then $\lambda(\theta) \simeq \lambda(\theta')$; let c and c' be such that $c' \simeq c$. Then ${}^oA_{c'} = {}^oA_c$ and

$$J(A_c, \lambda(\theta')) \simeq J(A_c, \lambda(\theta)) \simeq J({}^oA_c, \lambda(\theta)).$$

Similarly, $J(A_{c'}, \lambda(\theta')) \simeq J({}^oA_{c'}, \lambda(\theta))$.

The monotonicity of g_θ is deduced from lemma 10.3.1 (iii) which shows that if $c' < c$ and $\lambda(\theta) \notin A_{c'}$ then $J(A_{c'}, \lambda(\theta)) > J(A_c, \lambda(\theta))$.

The continuity and monotonicity of h_θ are proved in the same way. □

10.4 The nonstandard likelihood ratio test

Recall that $\mathcal{F} := \{\Theta_0 \text{ internal} : \widetilde{\Theta}_0 \subset \Theta_0 \subset \text{hal}(\widetilde{\Theta}_0)\}$, that the NSLRT is defined by $\Phi_0 :\equiv \inf\{\Phi_{\Theta_0} : \Theta_0 \in \mathcal{F}\}$, that Φ_{Θ_0} is defined through R_{Θ_0} and that, for exponential families, $R_{\Theta_0} = J(\overline{X}, \Lambda_0)$ where $\Lambda_0 := \lambda(\Theta_0)$. So, the relation

$$\sup_{\theta \in \Theta_0} P_\theta^n(R_{\Theta_0} > d) \leq \alpha \leq \sup_{\theta \in \Theta_0} P_\theta^n(R_{\Theta_0} \geq d)$$

(see §10.2.1) can be written

$$\sup_{\theta \in \Theta_0} P_\theta^n(\overline{X} \in A_d^{\mathbf{C}}(\Theta_0)) \leq \alpha \leq \sup_{\theta \in \Theta_0} P_\theta^n(\overline{X} \in \overline{A_d^{\mathbf{C}}(\Theta_0)}).$$

Consequently, $A_d^{\mathbf{C}}$ is the rejection set (relatively to $\overline{X}$), A_d^0 the acceptation set and the test Φ_{Θ_0} is randomized if $\overline{X} \in \widehat{\partial} A_c$.

As n is illimited, if $d < c_0(\Theta_0)$ and if d is limited then, for $\theta \in \Theta_0$, proposition 10.3.4 yields

$$\frac{1}{n} \ln P_\theta^n(\overline{X} \in A_d^{\mathbf{C}}) \simeq \frac{1}{n} \ln P_\theta^n(\overline{X} \in \overline{A_d^{\mathbf{C}}}) \simeq -J(\hat{\partial} A_d, \lambda(\theta)).$$

As $\sup_{\theta \in \Theta_0} -J(\hat{\partial} A_d, \lambda(\theta)) = -d$ by definition of A_d, we have

$$d(\alpha, \Theta_0) \simeq -\frac{\ln \alpha}{n}.$$

This is the nonstandard form of a classical result of Bahadur ([1], [2]). We shall study the NSLRT according as these $d(\alpha, \Theta_0)$ are infinitesimal or appreciable. In this latter case, we shall have to distinguish the cases $d(\alpha, \Theta_0) < c_0(\Theta_0)$ and $d(\alpha, \Theta_0) \geq c_0(\Theta_0)$.

In the following, $\widetilde{c}_0$ stands for $c_0(\widetilde{\Theta}_0)$, $\widetilde{\Lambda}_0$ for $\lambda(\widetilde{\Theta}_0)$ and $R_0 = J(\overline{X}, \widetilde{\Lambda}_0)$ for $R_{\widetilde{\Theta}_0}$; A_c will denote $A_c(\widetilde{\Lambda}_0)$; the notations $h_\theta(c)$ and $g_\theta(c)$ will also refer

to these $A_c := A_c(\widetilde{\Lambda}_0)$. Note that if $\Theta_0 \in \mathcal{F}$, then the S-continuity of λ on K implies

$$R_{\Theta_0} \simeq R_0.$$

The following lemmas will be used for the calculation of the risks of the first and the second kind. They are valid not only for exponential families, but also for any statistical family. Here Θ'' is a standard subset of Θ.

Lemma 10.4.1 *Let T be a statistic, $\mathcal{S}$ a (perhaps external) subset of Θ'' and $F : D \to \mathbb{R}_+, (\theta, c) \to F_\theta(c)$ a standard continuous function defined on a domain $D = \{(\theta, c) \in \Theta'' \times \mathbb{R}_+ : d_\theta \leq c \leq d'_\theta\}$ such that*

- *the functions $\theta \to d_\theta$ and $\theta \to d'_\theta$ are continuous,*
- *for each $\theta \in \mathcal{S}$, F_θ is decreasing on $[d_\theta, d'_\theta]$,*
- *for each θ in $\mathcal{S}$ and c nearly standard in $]d_\theta, d'_\theta[$,*

$$\frac{1}{n} \ln P_\theta^n(T \geq c) \simeq F_\theta(c).$$

Then, for each θ nearly standard in $\mathcal{S}$ and C_0 standard in $[d_\theta, d'_\theta[$

$$PR_\theta^n(T \gtrapprox C_0) \equiv \mathcal{L}e^{-n\mathbb{@}+nF_\theta(C_0)}.$$

Proof. $\underline{PR}_\theta^n(T \overset{>}{\not\approx} C_0) \equiv \sup_{w=\mathbb{@}} P_\theta^n(T > C_0 + w)$. Now write

$$\frac{1}{n} \ln P_\theta^n(T \geq C_0 + w) \simeq F_\theta(C_0 + w) \simeq {}^oF_\theta(C_0 + w).$$

So, as ${}^oF_\theta = F_{\theta_0}$ (where $\theta_0 := {}^o\theta$) is a standard continuous decreasing function, we have

$$\frac{1}{n} \ln \underline{PR}_\theta^n(T \gtrapprox C_0) \equiv \left]-\infty, \sup_{w=\mathbb{@}} F_\theta(C_0 + w)\right[\equiv]-\infty, F_\theta(C_0) + \emptyset[\,.$$

Similarly,

$$\frac{1}{n} \ln \overline{PR}_\theta^n(T \gtrapprox C_0) \equiv]-\infty,\ {}^oF_\theta(C_0) + \emptyset[\equiv]-\infty, F_\theta(C_0) + \emptyset[\,,$$

for if not, there exists $\eta < 0$, $\eta \simeq 0$ such that

$$\forall \varepsilon \in \mathrm{hal}^+(0),\ \frac{1}{n} \ln P_\theta^n(T \geq C_0 + \varepsilon) > {}^oF_\theta(C_0) + \eta$$

where $\mathrm{hal}^+(0) := \{\varepsilon \in \mathrm{hal}(0) : \varepsilon \geq 0\}$.

Then $\left\{\varepsilon \in \mathbb{R}_+ : \frac{1}{n} \ln P_\theta^n(T \geq C_0 + \varepsilon) > {}^oF_\theta(C_0) + \eta\right\}$ is an internal set including $\mathrm{hal}^+(0)$ and consequently (by Cauchy's principle) including an interval $[0, w_0]$ where w_0 is standard. But

$$\frac{1}{n} \ln P_\theta(T \geq C_0 + w_0) \simeq F_\theta(C_0 + w_0) \simeq {}^oF_\theta(C_0 + w_0)$$

with ${}^oF_\theta(C_0 + w_0) < {}^oF_\theta(C_0)$ which is a contradiction, because both these numbers are standard. Finally

$$\underline{PR}_\theta^n(T \gtrapprox C_0) \equiv \overline{PR}_\theta^n(T \gtrapprox C_0) \equiv \mathcal{L}e^{-n@+nF_\theta(C_0)}. \qquad \square$$

Lemma 10.4.2 *Let T be a statistic, $\mathcal{S}$ a (perhaps external) subset of Θ'' and $G : E \to \mathbb{R}_+, (\theta, c) \to G_\theta(c)$ a standard continuous function defined on a domain $E = \{(\theta, c) \in \Theta'' \times \mathbb{R}_+ : e_\theta \leq c \leq e'_\theta\}$ such that*

- *the functions $\theta \to e_\theta$ and $\theta \to e'_\theta$ are continuous,*
- *for each $\theta \in \mathcal{S}$, G_θ is increasing on $[e_\theta, e'_\theta]$,*
- *for each θ in $\mathcal{S}$ and c nearly standard in $]e_\theta, e'_\theta[$,*

$$\frac{1}{n} \ln P_\theta^n(T \leq c) \simeq G_\theta(c).$$

Then, for each θ nearly standard in $\mathcal{S}$ and C_0 standard in $[e_\theta, e'_\theta[$

$$PR_\theta^n(T \lessapprox C_0) \equiv \mathcal{L}e^{nG_\theta(C_0)+n\emptyset}.$$

Proof. $\overline{PR}_\theta^n(T \lessapprox C_0) \equiv \inf_{t=@} P_\theta^n(T \leq C_0 + t)$. Now write

$$\frac{1}{n} \ln P_\theta^n(T \leq C_0 + t) \simeq G_\theta(C_0 + t) \simeq {}^oG_\theta(C_0 + t).$$

So, as ${}^oG_\theta$ is a standard continuous increasing function, we have

$$\frac{1}{n} \ln \overline{PR}_\theta^n(T \lessapprox C_0) \equiv \left]-\infty, \inf_{t=@} G_\theta(C_0 + t)\right] \equiv]-\infty, G_\theta(C_0) + \emptyset].$$

As in lemma 10.4.1, one establishes that $\overline{PR}_\theta^n(T \lessapprox C_0) \equiv \underline{PR}_\theta^n(T \lessapprox C_0)$ and finally

$$PR_\theta^n(T \lessapprox C_0) \equiv \mathcal{L}e^{nG_\theta(C_0)+n\emptyset}. \qquad \square$$

10.4.1 $\dfrac{\ln \alpha}{n}$ infinitesimal

Proposition 10.4.1 *For* $\frac{\ln \alpha}{n} \simeq 0$,

(i) if $R_0 \simeq 0$ *then* $\Phi_0 \equiv 0$ *otherwise* $\Phi_0 \equiv 1$;

(ii) for $\theta \in hal(\widetilde{\Theta}_0)$, *the risk of the first kind is exactly equal to* $\mathcal{L}e^{-n@-nh_\theta(0)}$;

(iii) for a limited $\theta \in \Theta \setminus (hal(\partial\Theta) \cup hal(\widetilde{\Theta}_0))$, *the risk of the second kind is exactly equal to* $\mathcal{L}e^{-ng_\theta(0)+n\emptyset}$.

Proof. *(i)* If $R_0 \not\simeq 0$ then $\forall \Theta_0 \in \mathcal{F}$, $R_{\Theta_0} \not\simeq 0$ and so $\forall \Theta_0 \in \mathcal{F}$, $\Phi_{\Theta_0} = 1$ because $d(\alpha, \Theta_0) \simeq -\frac{\ln \alpha}{n} \simeq 0$ and finally $\Phi_0 \equiv 1$.

If $R_0 \simeq 0$, $\overline{X} \in \mathrm{hal}(\widetilde{\Lambda}_0)$ (for if not, $R_0 = J(\overline{X}, \widetilde{\Lambda}_0)$ is appreciable) and so $\lambda^{-1}(\overline{X}) \in \widetilde{\Theta}_0 \cup \{\lambda^{-1}(\overline{X})\} \in \mathcal{F}$. Then, $\Phi_{\Theta_0} = 0$ for this $\Theta_0 := \widetilde{\Theta}_0 \cup \{\lambda^{-1}(\overline{X})\}$. Finally, $\Phi_0 \equiv 0$.

(ii) According to proposition 10.3.4 (iii) and (iv), for a limited $c \leq \widetilde{c}_0$, we have $\frac{1}{n} \ln P_\theta^n(R_0 \geq c) \simeq -h_\theta(c)$. Let Θ'' a standard compact set such that $\mathrm{hal}(\widetilde{\Theta}_0) \subset \Theta'' \subset \Theta \setminus \mathrm{hal}(\partial\Theta)$ (for example $\Theta'' = K$). Then according to proposition 10.3.5, $(\theta, c) \to -h_\theta(c)$ is continuous on $\Theta'' \times [0, c_0(\Theta'')]$ and for $\theta \in \Theta''$, $-h_\theta$ is decreasing on $[\gamma(\theta), c_0(\Theta'')]$.

We can now apply 10.4.1 with $D = \{(\theta, c) \in \Theta'' \times \mathbb{R}_+ : \gamma(\theta) \leq c \leq c_0(\Theta'')\}$, $T = R_0$, $\mathcal{S} = \mathrm{hal}(\widetilde{\Theta}_0)$, $F_\theta = -h_\theta$, $C_0 = 0$ to obtain

$$PR_\theta^n\big(R_0 \overset{>}{\not\simeq} 0\big) \equiv \mathcal{L}e^{-n@-nh_\theta(0)}.$$

(iii) According to proposition 10.3.4 (i) and (ii), for a limited $c \leq \widetilde{c}_0$, we have $\frac{1}{n} \ln P_\theta^n(R_0 \leq c) \simeq -g_\theta(c)$. For any limited $\theta \in \Theta \setminus (\mathrm{hal}(\partial\Theta) \cup \mathrm{hal}(\widetilde{\Theta}_0))$, there exists a standard compact set $\Theta'' \subset \Theta \setminus \mathrm{hal}(\partial\Theta)$, such that $\theta \in \Theta''$. According to proposition 10.3.5, $(\theta, c) \to -g_\theta(c)$ is continuous on $\Theta'' \times [0, c_0(\Theta'')]$ and for $\theta \in \Theta''$, $-g_\theta$ is increasing on $[0, \gamma(\theta)]$.

We can now apply lemma 10.4.2 with $D = \{(\theta, c) \in \Theta'' \times \mathbb{R}_+ : 0 \leq c \leq \gamma(\theta)\}$, $\mathcal{S} = \Theta'' \setminus \mathrm{hal}(\widetilde{\Theta}_0)$, $T = R_0$, $G_\theta = -g_\theta$, $C_0 = 0$, to obtain $PR_\theta^n(R_0 \lesssim 0) \equiv \mathcal{L}e^{-ng_\theta(0)+n\emptyset}$. □

Remark 10.4.1 *Let* $\theta \in hal(\widetilde{\Theta}_0)$; *the nonstandard version of the law of large numbers yields*

$$PR_\theta^n(\overline{X} \in hal(\lambda(\theta))) = 1 + \emptyset \Rightarrow PR_\theta^n(\overline{X} \in hal(\widetilde{\Lambda}_0)) = 1 + \emptyset.$$

As the reject criterion is $\overline{X} \notin hal(\widetilde{\Lambda}_0)$, *it is not logical to take* α *appreciable, but infinitesimal.*

10.4.2 $0 \lessapprox \dfrac{1}{n}|\ln \alpha| \lessapprox \widetilde{c}_0$

If $\widetilde{c}_0 = \infty$, we shall suppose that $\frac{|\ln \alpha|}{n}$ is limited because some preliminary results (in §10.3) are no longer available (for example A_c is no longer limited if $c \simeq \infty$).

Proposition 10.4.2 *For* $0 \lessapprox \frac{|\ln \alpha|}{n} \lessapprox \widetilde{c}_0$ *and* $\frac{|\ln \alpha|}{n}$ *limited, put* $C := {}^o\!\left(-\frac{\ln \alpha}{n}\right)$.

(i) If $R_0 \gtrapprox C$ *then* $\Phi_0 \equiv 1$ *otherwise* $\Phi_0 \equiv 0$.

(ii) For $\theta \in hal(\widetilde{\Theta}_0)$, *the risk of the first kind is exactly equal to* $\mathcal{L}e^{-n@-nh_\theta(C)}$.

(iii) For a limited $\theta \in \Theta \setminus ((hal(\partial\Theta) \cup hal(\widetilde{\Theta}_0))$, *if* $\gamma(\theta) \gtrapprox C$ *then the risk of the second kind is exactly equal to* $\mathcal{L}e^{-ng_\theta(C)+n\emptyset}$; *and if* $\gamma(\theta) \lessapprox C$ *then the risk of the second kind is exactly equal to* $1 + \mathcal{L}e^{-n@-nh_\theta(C)}$.

Proof. *(i)* If $R_0 \gtrapprox C$ then $\forall \Theta_0 \in \mathcal{F}, R_{\Theta_0} \gtrapprox -\frac{\ln \alpha}{n}$, so $R_{\Theta_0} > d(\alpha, \Theta_0)$ and $\Phi_{\Theta_0} = 1$. Finally, $\Phi_0 \equiv 1$.

If $R_0 \simeq 0, \Phi_0 \equiv 0$, as in proposition 10.4.1 (i).

If $0 \lessapprox R_0 \lessapprox C$, then $\overline{X} \notin \text{hal}(\widetilde{\Lambda}_0)$; we prove that $\exists \Theta_0 \in \mathcal{F}$, $\Phi_{\Theta_0} = 0$. If not, the external set $\mathcal{F}$ is included in the internal set $\left\{\Theta_0 : \widetilde{\Theta}_0 \subset \Theta_0 \subset K \wedge \Phi_{\Theta_0} \neq 0\right\}$, and using the Cauchy's principle, there exists an internal set $\Theta_0 \supset \text{hal}(\widetilde{\Theta}_0)$ such that $\Phi_{\Theta_0} \neq 0$, i.e. $R_{\Theta_0} \geq d(\alpha, \Theta_0) \simeq C$. Denoting by $\widehat{\Theta}_0$ the shadow of this Θ_0 we have $R_{\widehat{\Theta}_0} \simeq R_{\Theta_0} \gtrsim C$.

$\widehat{\Theta}_0$ and $\widetilde{\Theta}_0$ are two distinct relatively compact standard sets such that $\partial\widetilde{\Theta}_0 \subset \widehat{\Theta}_0^0$ and then such that $\partial\widetilde{\Lambda}_0 \subset \widehat{\Lambda}_0^0$ (where $\widehat{\Lambda}_0 := \lambda(\widehat{\Theta}_0)$). According to lemma 10.3.1 (iii), $J(\xi, \widehat{\Lambda}_0) < J(\xi, \widetilde{\Lambda}_0)$ for each $\xi \notin \widetilde{\Lambda}_0$ and then

$$R_{\widehat{\Theta}_0} = J(\overline{X}, \widehat{\Lambda}_0) \simeq J({}^o\overline{X}, \widehat{\Lambda}_0) < J({}^o\overline{X}, \widetilde{\Lambda}_0) \simeq J(\overline{X}, \widetilde{\Lambda}_0) = R_0.$$

As $J({}^o\overline{X}, \widehat{\Lambda}_0)$ and $J({}^o\overline{X}, \widetilde{\Lambda}_0)$ are both standard numbers, this contradicts $R_{\widehat{\Theta}_0} \gtrsim C$ and $R_0 \lesssim C$.

(ii) The proof is similar to the proof of proposition 10.4.1 (ii), with $C_0 = C$ instead of $C_0 = 0$. We note that the risk of the first kind calculated in §10.4.1 is a particular case of this result.

(iii) For the first result, the proof is similar to the proof of proposition 10.4.1 (iii), with $C_0 = C$ instead of $C_0 = 0$ and $\mathcal{S} = \left\{\theta \in \Theta'' \setminus \text{hal}(\widetilde{\Theta}_0) : \gamma(\theta) \gtrapprox C\right\}$ instead of $\mathcal{S} = \Theta'' \setminus \text{hal}(\widetilde{\Theta}_0)$. For the second, we choose a standard compact set $\Theta'' \subset \Theta \setminus \text{hal}(\partial\Theta)$, such that $\theta \in \Theta''$. In fact, the proof of (ii) is valid with $\mathcal{S} = \left\{\theta \in \Theta'' : \gamma(\theta) \lesssim C\right\}$ and one obtains

$$PR_\theta^n(R_0 \gtrsim C) \equiv \mathcal{L}e^{-n@-nh_\theta(C)} \qquad \square$$

10.4.3 $\frac{1}{n}|\ln\alpha| \gtrsim\!\!\!\approx \widetilde{c}_0$

The number $\widetilde{c}_0$ is standard because $\widetilde{\Theta}_0$ is standard and $\widetilde{c}_0 \neq \infty$; Λ is a limited set for if not $J(\xi, \widetilde{\Lambda}_0)$ is unlimited because ξ is unlimited (see the proof of lemma 10.3.2) and then $c_0(\widetilde{\Theta}_0) \geq \sup_{\xi\in\Lambda} J(\xi, \widetilde{\Lambda}_0)$ is unlimited. We shall suppose here that for each subset Θ_0 of Θ, the function $\theta \to I(\theta, \Theta_0)$ is continuous on Θ. This assumption is verified by classical exponential families. Then for any Θ_0, $c_0(\Theta_0) = \sup_{\xi\in\Lambda} J(\xi, \lambda(\Theta_0))$.

For any $\Theta_0 \subset K$, if $c_0(\Theta_0) \lesssim\!\!\!\not\approx \frac{1}{n}|\ln\alpha|$ then $d(\alpha, \Theta_0) = c_0(\Theta_0)$ and so $\Phi_{\Theta_0} < 1$ because $A_d = \overline{\Lambda}$. If $\overline{X} \in \Lambda^0$, then $\Phi_{\Theta_0} = 0$ and if $\overline{X} \in \partial\Lambda$, either $\Phi_{\Theta_0} = 0$, or $0 < \Phi_{\Theta_0} < 1$ (there is a randomization); the problem is not simple, as the example of multinomial distributions shows (cf. [9]). We note that Kallenberg avoids this case in theorem 3.3.2 of [13].

Proposition 10.4.3 *For* $\frac{1}{n}|\ln\alpha| \gtrsim\!\!\!\approx \widetilde{c}_0$,

(i) $\Phi_0 < 1$,

(ii) if $\overline{X} \in \Lambda^0$ *then* $\Phi_0 = 0$.

Proof. As Φ_0 is a non-increasing function of α, we just have to give proofs for $\frac{1}{n}|\ln\alpha| \simeq c_0$.

(i) We prove that $\exists\Theta_0 \in \mathcal{F}$, $\Phi_{\Theta_0} < 1$. If not, $\forall\Theta_0 \in \mathcal{F}$, $\max\Phi_{\Theta_0} = 1$ (which means that Φ_{Θ_0} takes value 1), and by Cauchy's principle $\exists\widehat{\Theta}_0 \supset \mathrm{hal}(\widetilde{\Theta}_0)$, such that $\max\Phi_{\widehat{\Theta}_0} = 1$. According to lemma 10.3.1 (iii), for any $\xi \in \Lambda$, $J({}^o\xi, {}^o\widehat{\Lambda}_0) < J({}^o\xi, {}^o\widetilde{\Lambda}_0)$ because $\mathrm{hal}(\widetilde{\Lambda}_0) \subset \widehat{\Lambda}_0 := \lambda(\widehat{\Theta}_0)$. Both these numbers being standard, the relation

$$J(\xi, \widehat{\Lambda}_0) \simeq J({}^o\xi, {}^o\widehat{\Lambda}_0) < J({}^o\xi, {}^o\widetilde{\Lambda}_0) \simeq J(\xi, \widetilde{\Lambda}_0)$$

implies $J(\xi, \widehat{\Lambda}_0) \lesssim\!\!\!\approx J(\xi, \widetilde{\Lambda}_0)$ and then $c_0(\widehat{\Theta}_0) \lesssim\!\!\!\not\approx c_0(\widetilde{\Theta}_0)$. So $c_0(\widehat{\Theta}_0) \lesssim\!\!\!\not\approx \frac{|\ln\alpha|}{n}$ which contradicts $\max\Phi_{\widehat{\Theta}_0} = 1$.

(ii) Let $\overline{X} \in \Lambda^0$; if $\forall\Theta_0 \in \mathcal{F}, \Phi_{\Theta_0} \neq 0$, by Cauchy's principle $\exists\widehat{\Theta}_0 \supset \mathrm{hal}(\widetilde{\Theta}_0)$ such that $\Phi_{\widehat{\Theta}_0} \neq 0$. But $c_0(\widehat{\Theta}_0) \lesssim\!\!\!\approx \widetilde{c}_0 \simeq \frac{|\ln\alpha|}{n}$ and so, as $\overline{X} \in \Lambda^0$, $\Phi_{\widehat{\Theta}_0} = 0$, which contradicts $\Phi_{\widehat{\Theta}_0} \neq 0$. □

Remark 10.4.2 *Let* θ *be a limited element of* $\Theta \setminus (hal(\partial\Theta) \cup (\widetilde{\Theta}_0))$*; the nonstandard version of the law of large numbers yields* $PR^n_\theta(\overline{X} \in hal(\lambda(\theta))) = 1+\emptyset$ *and thus* $PR^n_\theta(\overline{X} \in \Lambda^0) = 1 + \emptyset$*. So, for this* θ*, the risk of the second kind is equal to* $1 + \emptyset$*: if* $\frac{1}{n}|\ln\alpha| \gtrsim\!\!\!\approx \widetilde{c}_0$*, the NSLRT is not consistent (in the classical*

theory, a sequence of tests $(\varphi_n)_{n\geq n_0}$ [where n is the size of the sample] is consistent if for any $\theta \in \Theta_1$, $\lim_{n\to\infty} E_\theta(1-\varphi_n) = 0$: then, for $n \simeq \infty$, the risk of the second kind is infinitesimal).

Remark 10.4.3 *As the example of multinomial distributions shows (cf. [9]), it seems impossible to give general results about the value of Φ_0 if $\overline{X} \in \partial\Lambda$. We can only say that Φ_0 is randomized if $\overline{X}$ belongs to a subset of $\partial\Lambda$ and takes the value 0 if $\overline{X}$ belongs to the complement of this subset. This subset can be empty (as shown in [9]), and then $\Phi_0 \equiv 0$ in any case.*

10.5 Comparison with nonstandard tests based on $\overline{X}$

In the classical theory, for testing $\theta \in \Theta_0$ against $\theta \in \Theta_1$, one fixes the level α and one tries to find a test that has maximum power in any $\theta \in \Theta_1$. However, such uniformly most powerful (U.M.P.) tests exist only in a few exceptional cases: for example for 1-dimensional exponential families, if $\Theta_0 := \{\theta \in \Theta : \theta \leq \theta_0\}$. If no U.M.P. exists, one studies an asymptotic approach, for sample sizes tending to infinity. One of the tools to compare asymptotically the power of two tests of the same hypothesis is the *Bahadur relative efficiency.* In a more general framework (not only for exponential families) Bahadur ([1] or [2], see also [12]) has proved that the likelihood ratio test is at last as "efficient" as any test based on a rejection criterion of the form $T > c$ (with randomization if $T = c$ if necessary), where T is a statistic built with the n-sampling. However, it may be that to get the same power in a point $\theta \in \Theta \setminus \Theta_0$ the size of the sampling for the likelihood ratio test has to be larger than for the test based on T: thus was introduced the notion of "Bahadur deficiency" (cf. [13]).

We shall prove that the NSLRT is uniformly the most powerful in a large family of nonstandard tests if $\alpha \lneqq 1$ and $\frac{\ln \alpha}{n} \lneqq \widetilde{c}_0$. The standard likelihood ratio test is not as powerful in the corresponding standard family.

Let (φ_{Θ_0}) be a family of tests of level α (where φ_{Θ_0} tests $H_{\Theta_0} : \theta \in \Theta_0$ against $\theta \notin \Theta_0$). The corresponding level-α nonstandard test φ_0 is defined by

$$\varphi_0 :\equiv \inf\{\varphi_{\Theta_0} : \Theta_0 \in \mathcal{F}\}$$

and when we shall refer to a "nonstandard test", it will be defined thusly.

10.5.1 Regular nonstandard tests

Suppose that for each Θ_0, φ_{Θ_0} is a test of size α defined by a rejection set $R := R(\alpha, \Theta_0)$ such that

$$\sup_{\theta\in\Theta_0} P_\theta^n(\overline{X} \in R) \leq \alpha \leq \sup_{\theta\in\Theta_0} P_\theta^n(\overline{X} \in R \cup \widehat{\partial} R).$$

If R is n-regular in any point of $\widehat{\partial}R$ and if $\widehat{\partial}R$ is limited, φ_{Θ_0} will be called "regular". The most famous example of such a test is the chi-squared test since the relative frequency F is in fact the sample mean for the multivariate distribution. In order to get $R \neq \emptyset$ we shall suppose that $\frac{1}{n}|\ln\alpha| \lneqq \widetilde{c}_0$ and so $\frac{1}{n}|\ln\alpha| \lneqq c_0(\Theta_0)$ for $\Theta_0 \in \mathcal{F}$. If φ_{Θ_0} is regular for any $\Theta_0 \in \mathcal{F}$, φ_0 will be said to be "regular". We prove now that Φ_0 is more powerful than any regular level-α nonstandard test φ_0 in the following sense: $\Phi_0 \geq \varphi_0$. (For standard tests Φ and φ, it is clear that if $\Phi \geq \varphi$, then Φ is more powerful than φ —in the classical sense— because $E_\theta\Phi \geq E_\theta\varphi$ for any θ.)

Proposition 10.5.1 *For $\alpha \lneqq 1$ and $\frac{1}{n}|\ln\alpha| \lneqq \widetilde{c}_0$, if φ_0 is a regular level-α nonstandard test, then $\Phi_0 \geq \varphi_0$.*

Proof. As n is illimited and as R is n-regular at any point of $\widehat{\partial}R$ which is limited, we can write

$$\frac{1}{n}\ln P_\theta^n(\overline{X} \in R) \simeq \frac{1}{n}\ln P_\theta^n(\overline{X} \in \overline{R}) \simeq -J(\widehat{\partial}R, \lambda(\theta)),$$

so $\inf_{\theta\in\Theta_0} J(\hat{\partial}R, \lambda(\theta)) =: J(\hat{\partial}R, \lambda(\Theta_0))$ verifies $J(\hat{\partial}R, \lambda(\Theta_0)) \simeq -\frac{\ln\alpha}{n}$.

Put $c := {}^o\big(\frac{|\ln\alpha|}{n}\big)$. The region where $\varphi_0 \neq 0$ is included in the external subset $R_{\varphi_0} := \bigcap_{\Theta_0\in\mathcal{F}} R(\alpha, \Theta_0)$. We prove that $R_{\varphi_0} \subset \{\xi \in \overline{\Lambda}, J(\xi, \widetilde{\Lambda}_0) \gtrapprox c\} \equiv \{\Phi_0 \equiv 1\}$ which will prove that $\Phi_0 \geq \varphi_0$ (we remember that Φ_0 takes only values 0 and 1).

Let $\xi_0 \in \overline{\Lambda}$ such that $J(\xi_0, \widetilde{\Lambda}_0) \lneqq c$; we now claim that $\exists\Theta_0 \in \mathcal{F}$, $\xi_0 \notin R(\alpha, \Theta_0)$. Indeed, suppose that $\forall\Theta_0 \in \mathcal{F}$, $\xi_0 \in R(\alpha, \Theta_0)$. Then, by Cauchy's principle $\exists\widehat{\Theta}_0 \supset \mathrm{hal}(\widetilde{\Theta}_0)$ such that $\xi_0 \in R(\alpha, \widehat{\Theta}_0)$. Denote $\theta_0 := \lambda^{-1}(\xi_0)$.

• If $\xi_0 \in \mathrm{hal}(\widetilde{\Lambda}_0)$, then $\theta_0 \in \widehat{\Theta}_0$ since λ is a standard diffeomorphism from Θ^0 onto Λ^0.

Suppose $\rho_0 := d(\xi_0, \hat{\partial}R(\alpha, \widehat{\Theta}_0)) > \frac{\mathcal{L}}{\sqrt{n}}$ and so $B(\xi_0, \rho_0) \cap \overline{\Lambda} \subset R(\alpha, \widehat{\Theta}_0)$. As $\overline{\Lambda}$ contains the support of P_{θ_0}, the nonstandard law of large numbers yields $P_{\theta_0}^n(\overline{X} \in R(\alpha, \hat{\Theta}_0)) = 1 + \emptyset$ which is impossible because $P_{\theta_0}^n(\overline{X} \in R(\alpha, \hat{\Theta}_0)) \leq \alpha$.

Suppose then $\rho_0 \leq \frac{\mathcal{L}}{\sqrt{n}}$. Let $\xi_1 \in \hat{\partial}R(\alpha, \widehat{\Theta}_0) \cap \mathrm{hal}(\xi_0)$; the n-regularity at ξ_1 implies that $\exists\ \xi_2 \in \mathrm{hal}(\xi_1)$, $\exists\rho \in \big]\frac{\mathcal{L}}{\sqrt{n}}, \emptyset\big]$ such that $B(\xi_2, \rho) \subset R(\alpha, \widehat{\Theta}_0)$. As $d(\xi_0, \xi_2) \simeq 0$, $\xi_2 \in \lambda(\widehat{\Theta}_0)$ and $\theta_2 := \lambda^{-1}(\xi_2) \in \widehat{\Theta}_0$. As previously, $P_{\theta_2}^n(\overline{X} \in R(\alpha, \hat{\Theta}_0)) = 1 + \emptyset$ which is impossible because $P_{\theta_2}^n(\overline{X} \in R(\alpha, \hat{\Theta}_0)) \leq \alpha$.

• If $\xi_0 \notin \mathrm{hal}(\widetilde{\Lambda}_0)$, we can write

$$\begin{aligned} J(\xi_0, \lambda(\widehat{\Theta}_0)) &\simeq J(\xi_0, \lambda({}^o\widehat{\Theta}_0)) \simeq J({}^o\xi_0, \lambda({}^o\widehat{\Theta}_0)) \\ &< J({}^o\xi_0, \lambda(\widetilde{\Theta}_0)) \simeq J(\xi_0, \lambda(\widetilde{\Theta}_0)) \\ &\lneqq c. \end{aligned}$$

But $J({}^o\xi_0, \lambda({}^o\widehat{\Theta}_0))$ and $J({}^o\xi_0, \lambda(\widetilde{\Theta}_0)) \neq 0$ are both standard numbers; as $\lambda(\widetilde{\Theta}_0) \subset (\lambda({}^o\widehat{\Theta}_0))^0$ we have $J({}^o\xi_0, \lambda({}^o\widehat{\Theta}_0)) < J({}^o\xi_0, \lambda(\widetilde{\Theta}_0))$ and so $J(\xi_0, \lambda(\hat{\Theta}_0)) \lesssim c$.

On the other hand $J(R(\alpha, \hat{\Theta}_0), \lambda(\hat{\Theta}_0)) \simeq -\frac{\ln \alpha}{n} \simeq c$: this contradicts $\xi_0 \in R(\alpha, \hat{\Theta}_0)$. □

10.5.2 Case when $\widetilde{\Theta}_0$ is convex

For any $\Theta_0 \in \mathcal{F}$ and any $\theta_1 \in \Theta \setminus \mathrm{hal}(\widetilde{\Theta}_0)$, there is a level-$\alpha$ test $\Psi^{\theta_1}_{\Theta_0}$ which is the most powerful at θ_1 (for any level-α test φ testing $\theta \in \Theta_0$ against $\theta \in \Theta_1$, $E_{\theta_1}\varphi \leq E_{\theta_1}\Psi^{\theta_1}_{\Theta_0}$). This test is defined in the following way (see [13, p. 49]) let

$$f(x) := \int_{\overline{\Theta}_0} \exp\Big(n\big((\theta_0 - \theta_1 \mid x) - \psi(\theta_0) + \psi(\theta_1)\big)\Big)\tau(d\theta_0)$$

where τ is the least favorable distribution. Then $\Psi^{\theta_1}_{\Theta_0} = 1$ if $f(\overline{X}) < C$ and $\Psi^{\theta_1}_{\Theta_0} = 0$ if $f(\overline{X}) > C$ where C is a constant depending on α, Θ_0, θ_1. So, the rejection set (relatively to $\overline{X}$) is $\big\{\xi \in \overline{\Lambda} : f(\xi) < C\big\}$.

Definition 10.5.1 *For $\theta_1 \in \Theta \setminus \mathrm{hal}(\widetilde{\Theta}_0)$, let* $\Psi^{\theta_1}_0 :\equiv \inf\big\{\Psi^{\theta_1}_{\Theta_0} : \Theta_0 \in \mathcal{F}\big\}$.

We might think that $\Psi^{\theta_1}_0$ is "the most powerful at θ_1 nonstandard level-α test of H_0"; but in fact, in some cases, Φ_0 is more powerful than $\Psi^{\theta_1}_{\Theta_0}$:

Proposition 10.5.2 *If $\widetilde{\Theta}_0$ is convex, then for any $\theta_1 \in \Theta \setminus \mathrm{hal}(\widetilde{\Theta}_0)$, $\Phi_0 \geq \Psi^{\theta_1}_{\Theta_0}$.*

Proof. Let $\theta_1 \in \Theta \setminus \mathrm{hal}(\widetilde{\Theta}_0)$. We prove that the rejection sets

$$A := \big\{\xi \in \overline{\Lambda} : \ f(\xi) < C\big\}$$

are n-regular and then the proposition 10.5.1 yields the result.

The set $\widetilde{\Theta}_0$ is convex and $d(\theta_1, \widetilde{\Theta}_0) \not\simeq 0$ and so exists a standard cone $\Delta \subset \mathbb{R}^k$ and a standard positive number $\varepsilon > 0$ such that if $u \in \Delta$ and $\|u\| = 1$ then $(\theta_0 - \theta_1 \mid u) < -\varepsilon$ for any $\theta_0 \in \widetilde{\Theta}_0$. Therefore, for any $\Theta_0 \in \mathcal{F}$, for any $\theta \in \Theta_0$ and for any $u \in \Delta \setminus \{0\}$, $(\theta_0 - \theta_1 \mid u) < 0$.

Choose $\Theta_0 \in \mathcal{F}$. If $u \in \Delta \setminus \{0\}$,

$$f(x+u) := \int_{\overline{\Theta}_0} \exp\big(n(\theta_0 - \theta_1 \mid u)\big) \cdot$$
$$\cdot \exp\Big(n\big((\theta_0 - \theta_1 \mid x) - \psi(\theta_0) + \psi(\theta_1)\big)\Big)\tau(d\theta_0) < f(x).$$

Let now $a \in \partial A$, then $f(a) = C$. For any $u \in \Delta \setminus \{0\}$, $f(a+u) < C$, so $a + (\Delta \setminus \{0\}) \subset A$. It is now possible to choose a point $b \in a + \Delta$ such that $d(a,b) = \frac{1}{n}$ (for example) and such that $B(b, \frac{1}{n^{3/2}}) \subset a + \Delta$. If $a \notin \text{hal}(\partial\Lambda)$, this proves that A is n-regular in a. If $a \in \text{hal}(\partial\Lambda)$, we have to use Cauchy's principle like at the end of the proof of proposition 10.3.1 □

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11

A finitary approach for the representation of the infinitesimal generator of a markovian semigroup

Schérazade Benhabib*

Abstract

This work is based on Nelson's paper [1], where the central question was: under suitable regularity conditions, what is the form of the infinitesimal generator of a Markov semigroup?

In the elementary approach using IST [2], the idea is to replace the continuous state space, such as $\mathbb{R}$ with a finite state space X possibly containing an unlimited number of points. The topology on X arises naturally from the probability theory. For $x \in X$, let $\mathcal{I}_x$ be the set of all $h \in \mathcal{M}$ vanishing at x where $\mathcal{M}$ is the multiplier algebra of the domain $\mathcal{D}$ of the infinitesimal generator. To describe the structure of the semigroup generator A, we want to split $Ah(x) = \sum_{y \in X \setminus \{x\}} a(x,y)\ h(y)$ so that the contribution of the external set F_x of the points far from x appears separately. A definition of the quantity $\alpha_{ah}(x) = \sum_{y \in F} a(x,y)\ h(y)$ is given using the least upper bound of the sums on all internal sets W included in the external set F. This leads to the characterization of the global part of the infinitesimal generator.

11.1 Introduction

Let X be a finite state space possibly containing an unlimited number N of points. For all x in X, let $a(x,y)$ be a real number for each y on X such that $a(x,y)$ is positive if $y \neq x$ and $\sum_y a(x,y) = 0$. Then $A = (a(x,y))$ is a finite $N \times N$ matrix, with positive off-diagonal elements and row sums 0. Thus $P^t = \sum_n \frac{t^n A^n}{n!}$ exists and is a markovian semigroup with infinitesimal generator A.

*EIGSI, 17041 La Rochelle, France.
scherazade.benhabib@eigsi.fr

Then we can define externally the domain of definition of A and its multiplier algebra as follows.

Definition 11.1.1 *The domain $\mathcal{D}$ of definition of the infinitesimal generator A is the external set*

$$\mathcal{D} = \left\{ f \in \mathbb{R}^X / \ f \text{ limited and } Af \text{ limited} \right\} \tag{11.1.1}$$

And its multiplier algebra is the external set

$$\mathcal{M} = \{ f \in \mathcal{D} / \ \forall g \in \mathcal{D} \ fg \in \mathcal{D} \} \tag{11.1.2}$$

The topology on X arises naturally from the probability theory and a proximity relation is defined on X, more particulary from the way the functions in the multiplier algebra $\mathcal{M}$ of the domain $\mathcal{D}$ of the infinitesimal generator act on X.

Definition 11.1.2 *Let u and v be in X, we say that u is close to v ($u \simeq v$) if and only if*

$$\forall h \in \mathcal{M} \ \ h(u) \simeq h(v)$$

Thus, the elements of $\mathcal{M}$ have a macroscopic property of continuity formally analogous to the S-continuity.

With this relation, we define the external sets

$$C_x = \{ y \in X : \ y \simeq x \} \tag{11.1.3}$$

of points close to x and

$$F_x = \{ y \in X : \ y \not\simeq x \} \tag{11.1.4}$$

of those far from x.

For each x in X, $(a(x,y))$ are positive real numbers if $x \neq y$ and can take unlimited values. Let $\mathcal{I}_x$ be the set of all h in $\mathcal{M}$ vanishing at x, and for h in $\mathcal{I}_x$ let us define

$$Ah(x) = \sum_{y \in X \setminus \{x\}} a(x,y) \ h(y) \tag{11.1.5}$$

To describe the structure of the semigroup generator, we want to split $Ah(x)$ so that the contribution of the points far from x in X, appears separately.

One is tempted to define directly the quantity

$$\alpha_{ah}(x) = \sum_{y \in F_x} a(x,y) h(y) \tag{11.1.6}$$

which alas has no meaning because of the external character of the set of indices F_x

Nevertheless, we will show in Section 11.2 how we can attach a definite meaning to it. That leads in Section 11.3 to a definition for an α_x-integrable function. Theorem 11.3.1 establishes the representation of $Ah(x)$ for functions in $\mathcal{M}$ which have a zero at x of the third order. This is the global or integral part of A. Theorem 11.3.2 and its corollary extend the notion of α_x-integrable to functions which have a zero of the first or the second order.

11.2 Construction of the least upper bound of sums in IST

Lemma 11.2.1 *Let E be an external subset of X and f a positive function defined on X. If there exists a standard natural number n_0 such that for all internal sets W contained in E we have*

$$\sum_{y\in W} f(y) \leq n_0 \tag{11.2.1}$$

then there exists a unique up to an infinitesimal least number $\alpha(f)$ such that

$$\sum_{y\in W} f(y) \lesssim \alpha(f) \tag{11.2.2}$$

for all internal sets W contained in E.

Proof. Let n_0 be a standard natural number such that for all internal sets W contained in E we have (11.2.1).

By external induction, there is a least n_0 such that (11.2.1) holds. Now for each standard natural k there is a largest natural number j with $0 \leq j \leq 2^k$ such that for all such W

$$\sum_{y\in W} f(y) \leq n_0 - \frac{j}{2^k} \tag{11.2.3}$$

Let $\alpha(k) = n_0 - \frac{j}{2^k}$. Then for each standard k we have a standard $\alpha(k)$, so there is a standard sequence $k \to \alpha(k)$. This sequence is decreasing, therefore it has a standard limit $\alpha(f)$. Moreover, for all standard k and all such W, $\left(\alpha(f) - \sum_{y\in W} f(y)\right) \leq \frac{1}{2^k}$. □

This proof uses very elementary tools. It can easily be formulated in the minimal non-standard analysis introduced by Nelson in [3] in which "standard"

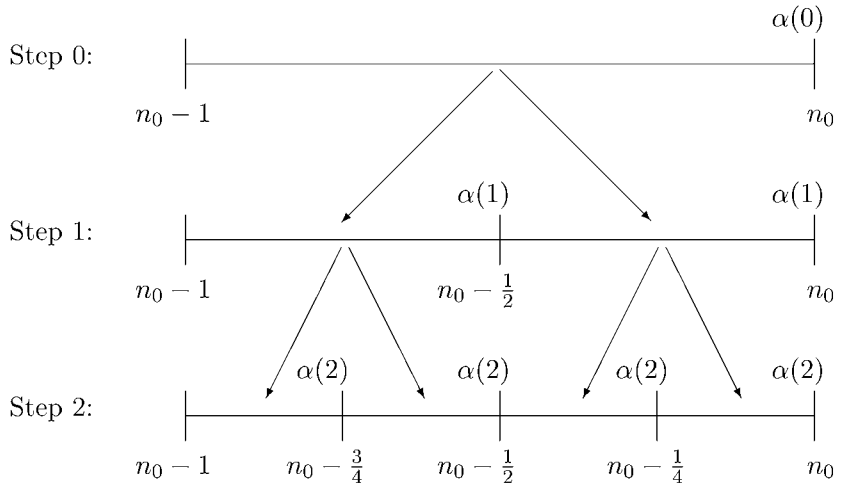

Figure 11.1

applies only to natural numbers. On the contrary, if all the force of IST is used, one could check that

$$\alpha(f) = \sup{}^{s}\left\{x \in \mathbb{R}^+;\ \exists^{int} W \subset E\ \sum_{y \in W} f(y) \lesssim x\right\}.$$

Figure 11.1 exhibits the way the sequence is built.

Definition 11.2.1

1. *If f is a positive function defined on X, and if the conditions required for Lemma 11.2.1 are verified the quantity*

$$\alpha_f = \sum_{y \in E} f(y) \tag{11.2.4}$$

is defined to be the standard limit $\alpha(f)$.

2. *If the conditions for Lemma 11.2.1 are not verified, let α_f be the formal symbol ∞ and say that (11.2.2) is not defined.*

3. *In the general case, let $f = f^+ - f^-$ be the decomposition of f into the difference of two positive functions. Say that α_f is defined in case both α_{f^+} and α_{f^-} are defined, and let α_f be their difference:*

$$\alpha_f = \alpha_{f^+} - \alpha_{f^-}$$

Notation 11.2.1 *For each x in X, if f depends on x, write $\alpha_f(x)$ instead of α_f.*

11.3 The global part of the infinitesimal generator

Definition 11.3.1 *For $x \in X$ and $h \in \mathcal{M}$, let $f = a(x,\cdot)h$, then h is said to be α_x-integrable if and only if $\alpha_f(x)$ is defined.*

Notation 11.3.1 *In that case write $\alpha_{ah}(x)$ instead of $\alpha_f(x)$.*

Definition 11.3.2

1. *h is a function in $\mathcal{I}_x$ if and only if $h \in \mathcal{M}$ and $h(x) = 0$.*
2. *h is a function in $\mathcal{I}_x^2$ if and only if $h \in \mathcal{M}$ and $h = \sum_{k=1}^{p} e_k\ g_k$ with p limited and e_k, g_k in $\mathcal{I}_x$.*
3. *h is a function in $\mathcal{I}_x^3$ if and only if $h \in \mathcal{M}$ and is of the form $h = \sum_1^p e_k f_k g_k$ with $e_k, f_k, g_k \in \mathcal{I}_x$ and p limited.*
4. *We say that h has a zero at x of the first order when $h \in \mathcal{I}_x$, of the second order when $h \in \mathcal{I}_x^2$, of the third order when $h \in \mathcal{I}_x^3$.*

Theorem 11.3.1 *If h has a zero at x of the third order, then h is α_x-integrable and $Ah(x)$ is infinitely close to $\alpha_{ah}(x)$.*

Proof. The proof goes through the following three steps:

1. Let h be in $\mathcal{I}_x^3$, therefore it is of the form $h = \sum_1^q f_k^2\ g_k$ with q limited and f_k, g_k in $\mathcal{I}_x$.
2. If $f, g \in \mathcal{I}_x$ then $f^2 g$ is α_x-integrable and so is h.
3. $A(f^2 g)(x) \simeq \alpha_{a(f^2 g)}(x)$ thus $Ah(x) \simeq \alpha_{ah}(x)$.

These steps are proved as follows:

1. Let $h \in \mathcal{I}_x^3$, as
$$efg = \frac{1}{4}(e+f)^2 g - \frac{1}{2}e^2 g - \frac{1}{2}f^2 g,$$
therefore h is of the mentioned form.
2. Let $f^2 g = \left(f^2 g^+\right) - \left(f^2 g^-\right)$ and W be an internal subset of F_x
$$\left|\sum_W a(x,y) f^2(y) g^+(y)\right| \leq ||g|| A(f^2)(x)$$

As $||g||$ and $A(f^2)$ are limited, they are both infinitely close to a standard number. Taking the integer part of $^\circ\big(||g||A(f^2)(x)\big)$ plus 1, we get the existence of the natural number n_0 required by Lemma 11.2.1.

The same argument applies to f^2g^-. Thus $\alpha_{a(f^2g^+)}(x)$ and $\alpha_{a(f^2g^-)}(x)$ exist and as $\alpha_{a(f^2g)}(x) = \alpha_{a(f^2g^+)}(x) - \alpha_{a(f^2g^-)}(x)$, then f^2g is α_x-integrable, and so is h.

3. Let $\beta \gg 0$. Since $g \in \mathcal{I}_x$ there is an internal set W of F_x such that $g^+(y) \leq \beta$ for $y \in W^C$. We have

$$\begin{aligned}
\big|A(f^2g^+)(x) - \alpha_{a(f^2g^+)}\big| &= \Big|\sum_{y\in X\setminus\{x\}} a(x,y)f^2(y)g^+(y) - \alpha_{a(f^2g^+)}\Big| \\
&\leq \sum_{y\in X\setminus\{x\}} a(x,y)f^2(y)g^+(y) - \sum_{y\in W} a(x,y)f^2(y)g^+(y) \\
&= \sum_{y\in W^C\setminus\{x\}} a(x,y)f^2(y)g^+(y) \\
&\leq \beta \sum_{y\in W^C\setminus\{x\}} a(x,y)f^2(y) \\
&\leq \beta \sum_{y\in X\setminus\{x\}} a(x,y)f^2(y) \\
&= \beta A(f^2)(x)
\end{aligned}$$

Since $A(f^2)$ is limited and $\beta \gg 0$ is arbitrary, $A(f^2g^+) \simeq \alpha_{a(f^2g^+)}(x)$. The same applies to f^2g^-.

Then $A(f^2g^-)(x) \simeq \alpha_{a(f^2g^-)}(x)$ and $\alpha_{a(f^2g)} = \alpha_{a(f^2g^+)}(x) + \alpha_{a(f^2g^-)}(x)$. Thus $A(f^2g)(x) \simeq \alpha_{a(f^2g)}(x)$. Therefore $A(h)(x) \simeq \alpha_{(ah)}(x)$. □

Theorem 11.3.2 *If h is positive and has a zero at x of the first order, then h is α_x-integrable and α_h is less or infinitely close to $Ah(x)$.*

Proof. Let W be any internal subset of F_x and h a positive function in $\mathcal{I}_x^+$, so

$$0 \leq \sum_W a(x,y)h(y) \leq A(h)(x)$$

As $A(h)(x)$ is limited, we argue as in Theorem 11.3.1 and apply Lemma 11.2.1. One concludes that h is α_x-integrable.

As the inequality above holds for all internal subset W of F_x, we get that $\alpha_{ah}(x)$ is less or infinitely close to $A(h)(x)$. □

Corollary 11.3.1 *If h has a zero at x of the second order, then h is α_x-integrable.*

Proof. Let h be a function in $\mathcal{I}_x^2$, as

$$fg = \frac{1}{4}(e+g)^2 - \frac{1}{4}(e-g)^2$$

h is also of the form $h = \sum_{k=1}^{q} \beta_k f_k^2$ with $\beta_k = \pm 1$, q limited and f_k^2 in $\mathcal{I}_x^+$. Consequently we can apply Theorem 11.3.2 and h is α_x-integrable. □

11.4 Remarks

The next goal will be to find an IST construction of the pure diffusion part of the generator of the semi-group and to complete a work already in progress concerning the rescaling of the time parameter for the markovian semigroup P^t.

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12

On two recent applications of nonstandard analysis to the theory of financial markets

Frederik S. Herzberg*

Abstract

Suitable notions of "unfairness" that measure how far an empirical discounted asset price process is from being a martingale are introduced for complete and incomplete-market settings. Several limit processes are involved each time, prompting a nonstandard approach to the analysis of this concept. This leads to an existence result for a "fairest price measure" (rather than a martingale measure) for an asset that is simultaneously traded on several stock exchanges. This approach also proves useful when describing the impact of a currency transaction tax.

12.1 Introduction

Nonstandard analysis is most successful when it concerns itself with mathematical concepts that involve several limit processes in a non-trivial way. One such example is the quantification of a stochastic process's distance from being a martingale.

Given the well-known correspondence between the existence of a martingale measure for a discounted asset price process and the non-existence of arbitrage opportunities, there is a natural economic interpretation to the comparison of two positive stochastic processes in regard to "how far" they actually are from being a martingale. We will render two economic questions related to such-conceived "unfairness" mathematical: (1) Is there a "fairest price measure" (rather than a martingale measure) for an asset that is simultaneously traded at multiple stock exchanges? (2) Does a currency transaction tax imposed on a currency that is subject to herd behaviour among traders have a fairness-

*Mathematical Institute, University of Oxford, Oxford OX1 3LB, United Kingdom.
herzberg@maths.ox.ac.uk / herzberg@wiener.iam.uni-bonn.de

enhancing impact on the currency exchange rates preceding a financial crash? Both questions will be shown to allow for an affirmative answer.

Techniques from nonstandard analysis have already been successfully applied to mathematical finance in the work of Cutland, Kopp and Willinger [5]. This contribution is based on two more recent papers by the author [8, 9].

12.2 A fair price for a multiply traded asset

In this Section, we will *en passant* motivate notions of "unfairness" for both complete-market as well as incomplete-market settings.

Theorem 12.2.1 *Let $n \in \mathbb{N}$, $p > 0$ and $c, N > 0$. Consider an adapted probability space $(\Gamma, \mathcal{G}, \mathbb{P})$ and $\mathcal{B}[0,1] \otimes \mathcal{G}_1$-measurable geometric Brownian motions with constant multiplicative drift $\tilde{g}_i : \Gamma \times [0,1] \to \mathbb{R}^{d \cdot n}$, $i \in \{1, \ldots, n\}$. Then there exist processes g_i, $i \in \{1, \ldots, n\}$ on an adapted probability space $(\Omega, \mathcal{F}, \mathbb{Q})$, equivalent to the processes $\tilde{g}_i$ on Γ (in the sense of adapted equivalence [7]), such that there is a probability measure $\mathbb{M}_m$ on Ω in the class of measures*

$$\mathcal{C}(\Omega, g) := \left\{ Q : \mathcal{F}_1 \to [0,1] : \begin{array}{c} Q \text{ probability measure,} \\ \forall A \in \mathcal{F}_1 \quad \frac{1}{N} \cdot \mathbb{Q}(A) \leq Q(A) \leq N \cdot \mathbb{Q}(A), \\ \forall i \neq j \in \{1, \ldots, n\} \int_0^1 \frac{\mathrm{Cov}_Q\left((g_i)_s, (g_j)_s\right)}{\mathbb{E}_Q\left|(g_i g_j)_s\right|} ds \geq c \end{array} \right\}$$

minimising

$$Q \mapsto m_\Omega(Q, g) := \sum_{i=1}^{n} \int_0^1 \int_s^1 \int_\Omega \left| (g_i)_s - \mathbb{E}_Q\left[(g_i)_t \,|\mathcal{F}_s\right] \right|^p dQ \; dt \, ds$$

in the class of Loeb extensions of finitely additive measures from $({}^*\mathcal{C})(\Omega, G)$ (G being an arbitrary lifting of g).

Analogously, there is a probability measure $\mathbb{M}_n$ minimising

$$Q \mapsto n_\Omega(Q, g) = \int_0^1 \mathbb{E}_\mathbb{Q} \left| \frac{1}{g_t} \frac{d}{du}\Big|_{u=0} \mathbb{E}\left[g_{t+u} \mid \mathcal{G}_t\right] \right| dt$$

(where $n_\Omega(Q, g)$ is defined to be $+\infty$ if the derivative in 0 in this definition does not a.s. exist as a continuous function in t) in the class of Loeb extensions of finitely additive measures from $({}^*\mathcal{C})(\Omega, G)$. Moreover,

$$\inf_{\mathcal{C}(\Omega, g)} m_\Omega(\cdot, g) \leq \inf_{\mathcal{C}(\Gamma, \tilde{g})} m_\Gamma(\cdot, \tilde{g})$$

as well as

$$\inf_{\mathcal{C}(\Omega, g)} n_\Omega(\cdot, g) \leq \inf_{\mathcal{C}(\Gamma, \tilde{g})} n_\Gamma(\cdot, \tilde{g}).$$

Remark 12.2.1 *The original paper [8] does not state the Theorem and the subsequent Lemmas as precisely as it is done here, although the exact result becomes clear from the proofs in that paper.*

The proof for this Theorem can be split into the following Lemmas which might also be interesting in their own right.

Lemma 12.2.1 *Using the notation of the previous Theorem 12.2.1, for any hyperfinite adapted space Ω [12],*

$$\inf_{\mathcal{C}(\Omega,g)} m_\Omega(\cdot,g) \leq \inf_{\mathcal{C}(\Gamma,\tilde{g})} m_\Gamma(\cdot,\tilde{g}).$$

Lemma 12.2.2 *Under the assumptions of Theorem 12.2.1 and choosing Ω to be any hyperfinite adapted space, the infimum of $m_\Omega(\cdot,g)$ in the class of Loeb extensions of finitely additive measures from $({}^*\mathcal{C})(\Omega,G)$ is attained by some measure $\mathbb{M}_m \in \mathcal{C}(\Omega,g)$.*

Proofs for both of these Lemmas can be found in a recent paper by the author [8].

Although they look very similar, it is technically slightly more demanding to prove the following two Lemmas (which in turn obviously entail the second half of the Theorem, i.e. the assertions concerned with the map n).

Lemma 12.2.3 *Using the notation of the previous Theorem 12.2.1, for any hyperfinite adapted space Ω [12],*

$$\inf_{\mathcal{C}(\Omega,g)} n_\Omega(\cdot,g) \leq \inf_{\mathcal{C}(\Gamma,\tilde{g})} n_\Gamma(\cdot,\tilde{g}).$$

Lemma 12.2.4 *Under the assumptions of Theorem 12.2.1 and choosing Ω to be any hyperfinite adapted space, the infimum of $n_\Omega(\cdot,g)$ on $\mathcal{C}(\Omega,g)$ is attained by some measure $\mathbb{M}_n \in \mathcal{C}(\Omega,g)$.*

Easy results are

Lemma 12.2.5 *A semimartingale x is a P-martingale on Ω if and only if $m_\Omega(P,x)=0$. The function $m_\Omega(P,\cdot)^{\frac{1}{p}}$ on the space of measurable processes of Ω satisfies the triangle inequality and is 1-homogeneous. For $p=2$, it defines an inner product on the space*

$$\mathcal{E} := \left\{ x : \Omega \times [0,1] \to \mathbb{R}^d \ : \ x \text{ measurable}, \ m(P,x) < +\infty \right\}$$

which becomes a Hilbert space by this construction.

Lemma 12.2.6 *A semimartingale x is a P-martingale on Ω if and only if $n_\Omega(P,x) = 0$. The function $n_\Omega(P,\cdot)$ on the space of measurable processes of Ω remains unchanged when multiplying the argument by constants* (scaling invariance).

We can generalise this to the following

Definition 12.2.1 *Let Ω be an adapted probability space. Define*

$$\mathcal{L}(\Omega,\mathbb{R}^d) := \left\{x : \Omega \times [0,1] \to \mathbb{R}^d \,:\, x \text{ measurable}\right\}.$$

A function $\Upsilon : \mathcal{L}(\Omega,\mathbb{R}^d) \to [0,+\infty]$ is an incomplete-market notion of unfairness *if and only if it satisfies the triangle inequality, is 1-homogeneous, and assigns 0 to a semimartingale y if and only if y is a martingale. Υ is said to be a* complete-market notion of unfairness *if and only if it remains unchanged under multiplication by constants and Υ vanishes exactly for those semimartingales that are in fact martingales.*

By Lemmas 12.2.5 and 12.2.6 respectively, the function $m_\Omega(Q,\cdot)^{\frac{1}{p}}$ (for a fixed probability measure Q) is an example for an incomplete-market notion of unfairness, whereas similarly $n_\Omega(Q,\cdot)$, again for a fixed probability measure Q, is an example for a complete-market notion of unfairness.

Remark 12.2.2 *The distinction between notions of unfairness for complete and incomplete markets can be justified by the following reasoning: If it is, under assumption of completeness, possible to buy as much of an asset as one intends to, multiplication of the discounted price process by a constant does not enhance the arising arbitrage opportunities at all; therefore, for complete markets, a suitable notion of unfairness should be scaling invariant in that it does not change under multiplication of the argument — which is conceived as being a discounted price process — by constants.*

12.3 Fairness-enhancing effects of a currency transaction tax

In this Section, we shall analyse the empirical price of an asset that is subject to herd-behaviour preceding a financial crash and how the resulting distortion can be mitigated through imposing a transaction tax. This is of particular relevance if the crashing asset in question is a currency, given the massive (usually destabilising and therefore adverse) macroeconomic consequences of such financial crises.

We shall restrict our attention to a model of currency prices where extrapolation from the observed behaviour of other traders, up to white noise and constant inflation, completely accounts for the evolution of the currency price. This is to say that, in analogy to a Nash equilibrium, we assume that every agent is acting in such a manner that he gains most if all other agents follow his pattern. In our model this pattern will consist in using some average value of past currency prices as a "sunspot", that is a proxy for a general perception that depreciation or appreciation of the particular currency in question is due.

In order to introduce the model for the pre-crash discounted currency price, consider $\alpha : \mathbb{R} \to \mathbb{R}$, a piecewise constant function, such that

$$\alpha = \alpha(1) > 0 \text{ on } (0, +\infty) \quad , \quad \alpha = \alpha(-1) < 0 \text{ on } (-\infty, 0).$$

We assume that the logarithmic discounted price process $x^{(\upsilon)}$ is governed, given some initial condition that without loss of generality can be taken to be $x^{(\upsilon)} = 0$, by the stochastic differential equation

$$\begin{aligned} dx_t^{(\upsilon)} = \alpha \left(x_t^{(\upsilon)} - \sum_{u \in I} p_u \cdot x_{(t-u)\vee 0}^{(\upsilon)} \right) \chi_{\left\{ \left| x_t^{(\upsilon)} - \sum_{u \in I} p_u \cdot x_{(t-u)\vee 0}^{(\upsilon)} \right| \geq \upsilon \right\}} dt \\ + \sigma \cdot db_t - \frac{\sigma^2}{2} dt, \end{aligned} \tag{12.3.1}$$

where $r > 0$ is the logarithmic discount rate (assumed to be constant), b is the one-dimensional Wiener process, $(p_u)_{u \in I}$ is a convex combination — i.e. $I \subset (0, +\infty)$ is finite, $\forall u \in I \quad p_u > 0$ and $\sum_{u \in I} p_u = 1$. The parameter υ depends very much on the tax rate we assume. If ρ is the logarithmic tax rate and T the expected time during which one will hold the asset (that is the expected duration of the upward or downward tendency of the stock price), one can compute υ as follows:

$$\upsilon = T \cdot \rho.$$

The equation (12.3.1), is of course first of all only a formal equation that — thanks to the boundedness of α — can be made rigorous using the theory of stochastic differential equations developed by Hoover and Perkins [11] or Albeverio et al. [2] for instance.

We will introduce the following abbreviation:

$$\psi^{(\upsilon)} := \chi_{\{|\cdot| \geq \upsilon\}} \cdot \alpha.$$

12.4 How to minimize "unfairness"

Intuitively, the map $n_\Omega(P, \cdot)$ (for a fixed probability measure P on a probability space Ω) when applied to a discounted asset price process measures how

often (in terms of time and probability) and how much it will be the case that one may expect to obtain a multiple (or a fraction) of one's portfolio simply by selling or buying the stock under consideration.

For the following, we will drop the first component of n (indicating the probability measure) if no ambiguity can arise; for internal hyperfinite adapted spaces and Loeb hyperfinite adapted spaces, we will assume that the canonical measure (the internal uniform counting measure and its Loeb measure, respectively) on the space is referred to.

First of all we derive a formula that will make explicitly computing n easier in our specific setting:

Lemma 12.4.1 *If $x^{(\upsilon)}$ satisfies (12.3.1) for some $\upsilon > 0$ on an adapted probability space $(\Gamma, \mathcal{G}, \mathbb{Q})$, then the discounted price process $\left(\exp\left(x_t^{(\upsilon)}\right) : t \geq 0\right)$ is of finite unfairness. More specifically,*

$$\begin{aligned} n_\Gamma\left(\exp\left(x^{(\upsilon)}\right)\right) &= \int_0^1 \mathbb{E}\left[\left|\psi^{(\upsilon)}\left(x_t^{(\upsilon)} - \sum_{i\in I} p_i x^{(\upsilon)}_{(t-i)\vee 0}\right)\right|\right] dt \\ &= \int_0^1 \mathbb{E}\left[\left|\left.\frac{d}{du}\right|_{u=0} \mathbb{E}\left[x^{(\upsilon)}_{t+u}\middle|\mathcal{G}_t\right] + \frac{\sigma^2}{2}\right|\right] dt \end{aligned}$$

Proof. The proof is more or less a formal calculation, provided one is aware of the path-continuity of our process and the fact that the filtrations generated by b and $x^{(\upsilon)}$ are identical. For this implies that, given $t > 0$, the value

$$\psi^{(\upsilon)}\left(x^{(\upsilon)}_{t+u} - \sum_{i\in I} p_i x^{(\upsilon)}_{(t+u-i)\vee 0}\right)(\omega)$$

does not change within sufficiently small times u — almost surely for all those paths ω where $x_t^{(\upsilon)}(\omega) - \sum_{i\in I} p_i x^{(\upsilon)}_{(t-i)\vee 0}(\omega) \notin \{\pm\upsilon\}$, this condition itself being satisfied with probability 1. Now, using this result and the martingale property of the quotient of the exponential Brownian motion and its exponential bracket, we can deduce that for all $t > 0$ almost surely:

$$\begin{aligned} &\frac{1}{\exp\left(x_t^{(\upsilon)}\right)} \left.\frac{d}{du}\right|_{u=0} \mathbb{E}\left[\exp\left(x^{(\upsilon)}_{t+u}\right)\middle|\mathcal{G}_t\right] \\ &\qquad = \left.\frac{d}{du}\right|_{u=0} \mathbb{E}\left[\exp\left(x^{(\upsilon)}_{t+u} - x_t^{(\upsilon)}\right)\middle|\mathcal{G}_t\right] \\ &\qquad = \left.\frac{d}{du}\right|_{u=0} \exp\left(\psi^{(\upsilon)}\left(x_t^{(\upsilon)} - \sum_{i\in I} p_i x^{(\upsilon)}_{(t-i)\vee 0}\right) u\right) \cdot \\ &\qquad\qquad \cdot \mathbb{E}\left[\exp\left(\sigma b_{t+u} - \frac{\sigma^2}{2}(t+u) - \left(\sigma b_t - \frac{\sigma^2}{2} t\right)\right)\middle|\mathcal{G}_t\right] \end{aligned}$$

$$
\begin{aligned}
&= \frac{d}{du}\Big|_{u=0} \exp\left(\psi^{(\upsilon)}\left(x_t^{(\upsilon)} - \sum_{i\in I} p_i x_{(t-i)\vee 0}^{(\upsilon)}\right) u\right) \cdot \\
&\qquad \cdot \mathbb{E}\left[\exp\left(\sigma b_{t+u} - \frac{\sigma^2}{2}(t+u)\right)\Big|\, \mathcal{G}_t\right] \exp\left(-\sigma b_t + \frac{\sigma^2}{2} t\right) \\
&= \frac{d}{du}\Big|_{u=0} \exp\left(\psi^{(\upsilon)}\left(x_t^{(\upsilon)} - \sum_{i\in I} p_i x_{(t-i)\vee 0}^{(\upsilon)}\right) u\right) \cdot 1 \\
&= \psi^{(\upsilon)}\left(x_t^{(\upsilon)} - \sum_{i\in I} p_i x_{(t-i)\vee 0}^{(\upsilon)}\right)
\end{aligned}
$$

Analogously, one may prove the second equation in the Lemma: for, one readily has almost surely

$$
\begin{aligned}
&\frac{d}{du}\Big|_{u=0} \mathbb{E}\left[x_{t+u}^{(\upsilon)}\Big|\, \mathcal{G}_t\right] \\
&\qquad = \frac{d}{du}\Big|_{u=0} \left(\mathbb{E}\left[\psi^{(\upsilon)}\left(x_t^{(\upsilon)} - \sum_{i\in I} p_i x_{(t-i)\vee 0}^{(\upsilon)}\right) u\Big|\, \mathcal{G}_t\right] - \frac{\sigma^2}{2} u + x_t^{(\upsilon)}\right) \\
&\qquad = \psi^{(\upsilon)}\left(x_t^{(\upsilon)} - \sum_{i\in I} p_i x_{(t-i)\vee 0}^{(\upsilon)}\right) - \frac{\sigma^2}{2}
\end{aligned}
$$

In order to proceed from these pointwise almost sure equations to the assertion of the Theorem, one will apply Lebesgue's Dominated Convergence Theorem, yielding

$$
n_\Gamma\left(\exp\left(x^{(\upsilon)}\right)\right) = \int_0^1 \mathbb{E}\left[\left|\frac{d}{du}\Big|_{u=0} \mathbb{E}\left[x_{t+u}^{(\upsilon)}\Big|\, \mathcal{G}_t\right] + \frac{\sigma^2}{2}\right|\right] dt. \qquad \square
$$

For finite hyperfinite adapted probability spaces, an elementary proof for the main Theorem of this Section can be contrived:

Lemma 12.4.2 *For any hyperfinite number H we will let $X^{(\upsilon)}$ for all $\upsilon > 0$ denote the solution to the hyperfinite initial value problem*

$$
\begin{aligned}
&X_0^{(\upsilon)} = 0, \\
&\forall t \in \left\{0, \ldots, 1 - \frac{1}{H!}\right\} \\
&X_{t+\frac{1}{H!}}^{(\upsilon)} - X_t^{(\upsilon)} = \\
&\quad = \alpha\left(X_t^{(\upsilon)} - \sum_{u\in I} p_u \cdot X_{(t-u)\vee 0}^{(\upsilon)}\right) \chi_{\left\{\left|X_t^{(\upsilon)} - \sum_{u\in I} p_u \cdot X_{(t-u)\vee 0}^{(\upsilon)}\right| \geq \upsilon\right\}} \cdot \frac{1}{H!} \\
&\qquad + \sigma \cdot \pi_{t+\frac{1}{H!}} \cdot \frac{1}{(2H!)^{1/2}} - \frac{\sigma^2}{2}\frac{1}{H!}
\end{aligned}
\tag{12.4.1}
$$

(where $\pi_{\ell/H!} : \Omega = \{\pm 1\}^{H!} \to \{\pm 1\}$ is for all hyperfinite $\ell \leq H!$ the projection to the ℓ-th coordinate) which is just the hyperfinite analogue to (12.3.1). Using this notation, and considering a hyperfinite adapted probability space of mesh size $H!$, one has for all $k < H!$,

$$\sum_{k<H!} \mathbb{E}\left| X^{(\tau)}_{k/H!} - \mathbb{E}\left[X^{(\tau)}_{\frac{k+1}{H!}} \middle| \mathcal{F}_{k/H!} \right] - \frac{\sigma^2}{2}\frac{1}{H!} \right| \leq \sum_{k<H!} \mathbb{E}\left| X^{(\upsilon)}_{k/H!} - \mathbb{E}\left[X^{(\upsilon)}_{\frac{k+1}{H!}} \middle| \mathcal{F}_{k/H!} \right] - \frac{\sigma^2}{2}\frac{1}{H!} \right|$$

for all $\tau \geq \upsilon$. As a consequence, $m\left(X^{(\cdot)}, \Omega\right)$ is monotonely decreasing for all hyperfinite adapted probability spaces $\Omega = \{\pm 1\}^{H!}$.

Proof of Lemma 12.4.2. It suffices to prove the result for finite (rather than merely hyperfinite) adapted spaces. By transfer to the nonstandard universe, we will obtain the same result for infinite hyperfinite H as well. The proof of this Lemma for finite H relies on exploiting the assumption that α is piecewise constant, since $\alpha = \alpha(1)\chi_{(0,+\infty)} + \alpha(-1)\chi_{(-\infty,0)} + \alpha(0)\chi_{\{0\}}$ yields

$$\begin{aligned}
&\forall k < H! \quad \forall \upsilon > 0 \\
&\mathbb{E}\left[X^{(\upsilon)}_{\frac{k+1}{H!}} \middle| \mathcal{F}_{k/H!} \right] - X^{(\upsilon)}_{\frac{k}{H!}} + \frac{\sigma^2}{2}\frac{1}{H!} \\
&\quad = \frac{1}{H!}\alpha\left(X^{(\upsilon)}_{k/H!} - \sum_{u\in I} p_u \cdot X^{(\upsilon)}_{\left(\frac{k}{H!}-u\right)\vee 0} \right) \chi_{\left\{ \left| X^{(\upsilon)}_{k/H!} - \sum_{u\in I} p_u \cdot X^{(\upsilon)}_{\left(\frac{k}{H!}-u\right)\vee 0} \right| \geq \upsilon \right\}} \\
&\quad = \frac{1}{H!}\begin{pmatrix} \alpha(1)\chi_{\left\{ X^{(\upsilon)}_{k/H!} - \sum_{u\in I} p_u \cdot X^{(\upsilon)}_{\left(\frac{k}{H!}-u\right)\vee 0} \geq \upsilon \right\}} \\ + \alpha(-1)\chi_{\left\{ X^{(\upsilon)}_{k/H!} - \sum_{u\in I} p_u \cdot X^{(\upsilon)}_{\left(\frac{k}{H!}-u\right)\vee 0} \leq -\upsilon \right\}} \end{pmatrix},
\end{aligned}$$

which immediately follows from the construction of Anderson's random walk [3] $B_\cdot = \frac{1}{\sqrt{2H!}}\pi_\cdot$ and the recursive difference equation defining the process $X^{(\upsilon)}$. However, this last equation implies

$$\begin{aligned}
&\forall k < H! \quad \forall \upsilon > 0 \\
&\mathbb{E}\left| \mathbb{E}\left[X^{(\upsilon)}_{\frac{k+1}{H!}} \middle| \mathcal{F}_{k/H!} \right] - X^{(\upsilon)}_{\frac{k}{H!}} + \frac{\sigma^2}{2}\frac{1}{H!} \right| \\
&\qquad = \frac{1}{H!}\mathbb{E}\left| \begin{matrix} \alpha(1)\chi_{\left\{ X^{(\upsilon)}_{k/H!} - \sum_{u\in I} p_u \cdot X^{(\upsilon)}_{\left(\frac{k}{H!}-u\right)\vee 0} \geq \upsilon \right\}} \\ + \alpha(-1)\chi_{\left\{ X^{(\upsilon)}_{k/H!} - \sum_{u\in I} p_u \cdot X^{(\upsilon)}_{\left(\frac{k}{H!}-u\right)\vee 0} \leq -\upsilon \right\}} \end{matrix} \right|
\end{aligned}$$

$$= \frac{1}{H!} \left(\begin{array}{l} |\alpha(1)|\, \mathbb{P}\left\{ X^{(\upsilon)}_{k/H!} - \sum\limits_{u\in I} p_u \cdot X^{(\upsilon)}_{(\frac{k}{H!}-u)\vee 0} \geq \upsilon \right\} \\ + |\alpha(-1)|\, \mathbb{P}\left\{ X^{(\upsilon)}_{k/H!} - \sum\limits_{u\in I} p_u \cdot X^{(\upsilon)}_{(\frac{k}{H!}-u)\vee 0} \leq -\upsilon \right\} \end{array} \right).$$

Now all that remains to be shown is that

$$\sum_{k<H!} \mathbb{P}\left\{ X^{(\upsilon)}_{k/H!} - \sum_{u\in I} p_u \cdot X^{(\upsilon)}_{(\frac{k}{H!}-u)\vee 0} \geq \upsilon \right\}$$

and

$$\sum_{k<H!} \mathbb{P}\left\{ X^{(\upsilon)}_{k/H!} - \sum_{u\in I} p_u \cdot X^{(\upsilon)}_{(\frac{k}{H!}-u)\vee 0} \leq -\upsilon \right\}$$

are monotonely decreasing in υ. The former statement is a consequence of the following assertion:

$$\begin{aligned} \forall \upsilon' \leq \upsilon\ \forall \ell < H! \quad & \sum_{k\leq \ell} \mathbb{P}\left\{ X^{(\upsilon)}_{k/H!} - \sum_{u\in I} p_u \cdot X^{(\upsilon)}_{(\frac{k}{H!}-u)\vee 0} \geq \upsilon \right\} \\ & \leq \sum_{k\leq \ell} \mathbb{P}\left\{ X^{(\upsilon')}_{k/H!} - \sum_{u\in I} p_u \cdot X^{(\upsilon')}_{(\frac{k}{H!}-u)\vee 0} \geq \upsilon' \right\} \end{aligned} \tag{12.4.2}$$

One can prove this estimate by considering the minimal ℓ such that the pointwise inequality

$$\sum_{k\leq \ell} \chi_{\left\{ X^{(\upsilon)}_{k/H!} - \sum_{u\in I} p_u \cdot X^{(\upsilon)}_{(\frac{k}{H!}-u)\vee 0} \geq \upsilon \right\}} \leq \sum_{k\leq \ell} \chi_{\left\{ X^{(\upsilon')}_{k/H!} - \sum_{u\in I} p_u \cdot X^{(\upsilon')}_{(\frac{k}{H!}-u)\vee 0} \geq \upsilon' \right\}} \tag{12.4.3}$$

fails to hold. Then one has an $\omega \in \Omega$ such that

$$\begin{aligned} \forall k < \ell \quad & \chi_{\left\{ X^{(\upsilon)}_{k/H!} - \sum_{u\in I} p_u \cdot X^{(\upsilon)}_{(\frac{k}{H!}-u)\vee 0} \geq \upsilon \right\}}(\omega) \\ & = \chi_{\left\{ X^{(\upsilon')}_{k/H!} - \sum_{u\in I} p_u \cdot X^{(\upsilon')}_{(\frac{k}{H!}-u)\vee 0} \geq \upsilon' \right\}}(\omega), \end{aligned} \tag{12.4.4}$$

$$1 = \chi_{\left\{ X^{(\upsilon)}_{\ell/H!} - \sum_{u\in I} p_u \cdot X^{(\upsilon)}_{(\frac{\ell}{H!}-u)\vee 0} \geq \upsilon \right\}}(\omega), \tag{12.4.5}$$

$$0 = \chi_{\left\{ X^{(\upsilon')}_{\ell/H!} - \sum_{u\in I} p_u \cdot X^{(\upsilon')}_{(\frac{\ell}{H!}-u)\vee 0} \geq \upsilon' \right\}}(\omega). \tag{12.4.6}$$

But equation (12.4.4) implies, via the difference equation for $X^{(\cdot)}$ (12.4.1), inductively in k the relation

$$\forall k < \ell \quad X^{\upsilon}_k(\omega) = X^{\upsilon'}_k(\omega).$$

If one combines this with

$$1 = \chi_{\left\{ X^{(v)}_{\ell/H!} - \sum_{u\in I} p_u \cdot X^{(v)}_{\left(\frac{\ell}{H!}-u\right)\vee 0} \geq v \right\}}(\omega)$$

(which is equation (12.4.5)) and $v \leq v'$, one can derive — again via the recursive difference equation (12.4.1) — that $X^{(v')}_{\ell/H!} \geq X^{(v)}_{\ell/H!}$ applied in $t = \frac{\ell}{H!}$ as well as the estimates

$$\begin{aligned}
\chi_{\left\{ X^{(v')}_{\ell/H!} - \sum_{u\in I} p_u \cdot X^{(v')}_{\left(\frac{k}{H!}-u\right)\vee 0} \geq v' \right\}}(\omega) &= \chi_{\left\{ X^{(v')}_{\ell/H!} - \sum_{u\in I} p_u \cdot X^{(v)}_{\left(\frac{\ell}{H!}-u\right)\vee 0} \geq v' \right\}}(\omega) \\
&\geq \chi_{\left\{ X^{(v)}_{\ell/H!} - \sum_{u\in I} p_u \cdot X^{(v)}_{\left(\frac{\ell}{H!}-u\right)\vee 0} \geq v' \right\}}(\omega) \\
&\geq \chi_{\left\{ X^{(v)}_{\ell/H!} - \sum_{u\in I} p_u \cdot X^{(v)}_{\left(\frac{\ell}{H!}-u\right)\vee 0} \geq v \right\}}(\omega) \\
&= 1.
\end{aligned}$$

This contradicts equation (12.4.6). Hence, the estimate (12.4.3) has been established for all $k < H!$, leading to (12.4.2).

Similarly, one can prove

$$\begin{aligned}
\forall v' \leq v \; \forall \ell < H! \quad & \sum_{k\leq \ell} \mathbb{P}\left\{ X^{(v)}_{k/H!} - \sum_{u\in I} p_u \cdot X^{(v)}_{\left(\frac{k}{H!}-u\right)\vee 0} \leq -v \right\} \\
& \leq \sum_{k\leq \ell} \mathbb{P}\left\{ X^{(v')}_{k/H!} - \sum_{u\in I} p_u \cdot X^{(v')}_{\left(\frac{k}{H!}-u\right)\vee 0} \leq -v' \right\}
\end{aligned}$$

which entails that $\sum_{k<H!} \mathbb{P}\left\{ X^{(v)}_{k/H!} - \sum_{u\in I} p_u \cdot X^{(v)}_{\left(\frac{k}{H!}-u\right)\vee 0} \leq -v \right\}$ must be monotonely decreasing in v. □

Using nonstandard analysis and the model theory of stochastic processes as developed by Keisler and others [10, 12, 7], we can prove the following result:

Theorem 12.4.1 *Suppose* $\left(y^{(v)} : v > 0\right)$ *is a family of stochastic processes on an adapted probability space* Γ *such that* $y^{(v)}$ *solves the stochastic differential equation (12.3.1) formulated above for all* $v > 0$. *Then the function* $\sigma \mapsto n_\Gamma\left(y^{(\sigma)}\right)$ *attains its minimum on* $[0, S]$ *in* S.

Proof. By the previous Lemmas 12.4.2 and the formula for n from Lemma 12.4.1, the assertion of the Theorem holds true internally for hyperfinite adapted spaces Γ, if we replace $y^{(\sigma)}$ by a lifting $Y^{(\sigma)}$ and if we let n_Ω when applied to internal processes denote the hyperfinite analogue of the standard n_Ω. Now, according to results by Hoover and Perkins [11] as well as Albeverio

et al. [2], the solution $X^{(\upsilon)}$ of the hyperfinite initial value problem (12.4.1) is a lifting for the solution $x^{(\upsilon)}$ of (12.3.1) on a hyperfinite adapted space for any $\upsilon \geq 0$. Now, $y \mapsto n_\Omega(y)$ is the expectation of a conditional process in the sense of Fajardo and Keisler [7]. Therefore, due to the Adapted Lifting Theorem [7], we must have

$$\forall \upsilon \geq 0 \quad {}^\circ n_\Omega\left(X^{(\upsilon)}\right) = n_\Omega\left(x^{(\upsilon)}\right)$$

(where we identify n with its internal analogue when applied to internal processes). Since the internal equivalent of the Theorem's assertion holds for internal hyperfinite adapted space, the previous equation implies that it is also true for Loeb hyperfinite adapted spaces.

Now let $\left(y^{(\upsilon)} : \upsilon > 0\right)$ be a family of processes on some (not necessarily hyperfinite) adapted probability space Γ with the properties as in the Theorem. Because of the universality of hyperfinite adapted spaces [7, 12], we will find a process $x^{(\upsilon)}$ on any hyperfinite adapted space Ω such that x and y are automorphic to each other. This implies [2] that $x^{(\upsilon)}$ satisfies (12.3.1) as well. Furthermore, as one can easily see using Lebesgue's Dominated Convergence Theorem,

$$\forall \upsilon > 0 \quad n_\Omega\left(x^{(\upsilon)}\right) = n_\Gamma\left(y^{(\upsilon)}\right).$$

Due to our previous remarks on the solutions of (12.3.1) on hyperfinite adapted spaces, this suffices to prove the Theorem. □

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13

Quantum Bernoulli experiments and quantum stochastic processes

Manfred Wolff*

Abstract

Based on a W^*-algebraic approach to quantum probability theory we construct basic discrete internal quantum stochastic processes with independent increments. We obtain a one-parameter family of (classical) Bernoulli experiments as linear combinations of these basic processes.

Then we use the nonstandard hull of the internal GNS-Hilbert space $\mathcal{H}_\tau$ corresponding to the chosen state τ (the underlying quantum probability measure) in order to derive nonstandard hulls of our internal processes. Finally continuity requirements lead to the specification of a certain subspace $\mathcal{L}$ of $\widehat{\mathcal{H}}_\tau$ to which the nonstandard hulls of our internal processes can be restricted and which turns out to be isomorphic to the Loeb-Guichardet space introduced by Leitz-Martini [10]. A subspace of $\mathcal{L}$ then is shown to be isomorphic to the symmetric Fock space $\mathcal{F}_+\left(L^2([0,1],\lambda)\right)$ and our basic processes agree with the processes of Hudson and Parthasarathy on this subspace.

13.1 Introduction

In the early 1980's Hudson and Parthasarathy [5], Hudson and Lindsay [4], and Hudson and Streater [6] started the theory of quantum stochastic processes, in particular of quantum Brownian motion. In [12] Parthasarathy showed one way how to make the passage from quantum random walk to diffusion. All these topics are extensively treated in P.A. Meyer's compendium [11]. For another general introduction to this field see [13] where an extensive motivation from quantum physics may be found. The Hudson-Parthasarathy approach is based on the classical symmetric Fock space over $L^2(\mathbb{R}_+\,,\,\lambda)$ where

*Mathematisches Institut, Eberhard-Karls-Univ. Tübingen.
manfred.wolff@uni-tuebingen.de

λ denotes the Lebesgue measure (see section 13.6). Guichardet [3] gave an interesting representation of this Fock space as an L^2-direct sum of spaces $L^2_n = L^2(X_n)$ where X_n is the space of all subsets of the unit interval $[0,1]$ having exactly n elements. The measure on X_n is defined via the corresponding Lebesgue measure on $\mathbb{R}^n_+$. This representation was extensively used by Maassen developing his kernel approach to general quantum stochastic processes.

Journé [7] invented the so called toy Fock spaces (bébé Fock in French) as discrete approximations of the symmetric Fock space. Let $\Omega_0 = \{0,1\}$ be equipped with the uniform distribution μ_0. Then the toy Fock space of order n is the space $L^2(\Omega_0^n, \mu_0^{\otimes n})$. A rigorous discrete approximation of the Guichardet space and the basic quantum stochastic processes defined on it which uses the space $L^2(\Omega_0^{\mathbb{N}}, \mu_0^{\otimes \mathbb{N}})$ as a toy Fock space is given by Attal [2].

There is another approach based also on the Guichardet space to prove such approximation theorems using nonstandard analysis which was developed by Leitz-Martini [10]. He discretized the Guichardet space in the following manner: let $T = \{\frac{k}{N} : k \in {}^*\mathbb{Z},\ 0 \leq k \leq N\}$ be the hyperfinite time line where $N \in {}^*\mathbb{N}$ is infinitely large. Let Γ_n be the set of all internal subsets of T of internal cardinality n and set $\Gamma = \bigcup_{k=0}^{N} \Gamma_k$. Now let M be an internal subset of Γ and set

$$m(M) = \sum_{n \leq N} \frac{|M \cap \Gamma_n|}{N^n}$$

where $|A|$ denotes the internal cardinality of A. Then m is an internal measure on Γ with $m(\Gamma) \approx e$. The space $L^2(\Gamma, L_m)$, where L_m denotes the Loeb measure associated to m, contains the Guichardet space as a Banach sublattice. In fact $L^2(\Gamma, L_m)$ is the nonstandard version of the Guichardet space, so to speak the *Loeb–Guichardet space* over the time interval $[0,1]$. Among other things Leitz-Martini constructed all the relevant discrete quantum stochastic processes and showed that their nonstandard hulls exist in a precise manner and form the basic quantum stochastic processes: the time process, the creation and annihilation processes, and the number process.

All approaches mentioned so far start with some kind of discrete approximation of the *symmetric Fock space* or the isomorphic Guichardet space over $\mathbb{R}_+$, $[0,1]$ respectively.

In contrast to these approaches we present here a new one based on the more natural *$W^\star$-algebraic* foundation of quantum stochastic as described e.g. in [8]. In [13] the author sketched a different though similar approach in the finite-dimensional setting but he was not able to make the transition from the finite-dimensional to the continuous infinite-dimensional case. Our access is exactly the noncommutative version of Anderson's way [1] to Brownian motion. In particular we are able to *justify* the use of the symmetric Fock space (or equivalently the Guichardet space) in the context of quantum stochastic

processes. We believe that by our approach the applications of nonstandard analysis as given by Leitz-Martini [10] are also better understandable.

We begin with the discrete quantum Bernoulli experiment modelled in the von Neumann algebra of all $2^n \times 2^n$-matrices over $\mathbb{C}$. Then we prove a Moivre-Laplace type theorem for quantum Bernoulli experiments, i.e. we show that these discrete stochastic processes converge to quantum Brownian motion in the same sense in which the classical approximation of the usual Brownian motion is treated by Anderson [1]. Furthermore we construct the basic quantum stochastic processes out of the corresponding discrete versions in the case of a vacuum state. Finally we show how the symmetric Fock space approach fits into our setting.

Applications to the theory of stochastic processes are under preparation.

13.2 Abstract quantum probability spaces

Recall that almost all information about a given probability space (Ω, Σ, μ) is to be found in the pair $(\mathcal{A}, \mu)$ with $\mathcal{A} = L^\infty(\Omega, \Sigma, \mu)$. Let us denote by $[f]$ the equivalence class modulo negligible functions in which the essentially bounded measurable function f is contained. Then $[f] \to \int_\Omega f(\omega) d\mu(\omega) =: \mu(f)$ defines a positive, linear, order continuous normalized functional on $\mathcal{A}$.

Analogously a **quantum probability space** is a pair $(\mathcal{A}, \tau)$ where $\mathcal{A}$ denotes a $W^\star$-algebra and τ is a positive, linear order continuous normalized functional on $\mathcal{A}$, or a **normal state** for short. The reader not familiar with the abstract theory of $W^\star$-algebras may look at them as subalgebras of the algebra of all bounded operators on an appropriate Hilbert space which are closed under involution $^\star : a \to a^\star$ and which are also closed with respect to the strong operator topology. In this context order continuity means the following: whenever a downward directed net (a_α) of nonnegative selfadjoint operators from $\mathcal{A}$ converges to the operator 0 with respect to the strong operator topology then $\lim_\alpha \tau(a_\alpha) = 0$ holds. In particular one may interpret $L^\infty(\Omega, \Sigma\mu)$ as the $W^\star$–algebra of all multiplication operators $M_f : g \to M_f(g) = fg$ ($f \in L^\infty(\Omega, \Sigma, \mu)$) on the Hilbert space $L^2(\Omega, \Sigma, \mu)$.

Let $(\mathcal{A}, \tau)$ be a quantum probability space. Let $\mathcal{A}_1, \mathcal{A}_2$ be $W^\star$-subalgebras generating the $W^\star$-subalgebra $\mathcal{A}_3$; moreover let τ_k be the restriction of τ to $\mathcal{A}_k$ ($k = 1, 2, 3$). Then $\mathcal{A}_1$ and $\mathcal{A}_2$ are called **stochastically independent** if $(\mathcal{A}_3, \tau_3)$ is isomorphic to $(\mathcal{A}_1 \overline{\otimes} \mathcal{A}_2, \tau_1 \overline{\otimes} \tau_2)$ where $\overline{\otimes}$ denotes the $W^\star$-tensor product.

Remark: Let us point out that this notion of independence agrees with the usual one in the classical (commutative) case $\mathcal{A} = L^\infty(\Omega, \Sigma, \mu)$ but that

there are other notions of independence in the quantum stochastic setting also generalizing the classical one (e.g. free independence).

Two probability spaces $(\mathcal{A}, \tau)$ and $(\mathcal{A}', \tau')$ are called **equivalent** if there exists an order continuous algebraic isomorphism $\varphi : \mathcal{A} \to \mathcal{A}'$ with $\tau' = \tau \circ \varphi$.

Now let us construct the L^2-space corresponding to the probability space $(\mathcal{A}, \tau)$. It is nothing else than the so called GNS-space (after Gelfand, Naimark, and Segal). To this end we introduce the $\mathbb{C}$-valued mapping $(a|b) = \tau(a^\star b)$ on $\mathcal{A} \times \mathcal{A}$. It is sesquilinear, that means, it is linear in the second argument, antilinear in the first argument, and it satisfies $(a|a) \geq 0$ for all $a \in \mathcal{A}$. The space $L_\tau = \{a : (a|a) = 0\}$ is a left ideal, and on $\mathcal{A}/L_\tau$ there is well-defined a scalar product by $(\hat{a}|\hat{b}) = (a|b)$ where $\hat{a} = a + L_\tau$ denotes the equivalence class in which a is contained. $\mathcal{H}_\tau$ is the completion of $\mathcal{A}/L_\tau$ with respect to the associated scalar product norm $\|\hat{a}\| = \sqrt{(\hat{a}|\hat{a})} = \sqrt{\tau(a^\star a)}$. The representation $\pi_\tau : \mathcal{A} \to \mathcal{L}(\mathcal{H}_\tau)$ is given by $\pi_\tau(a)(b + L_\tau) = ab + L_\tau$ which is well-defined since L_τ is a left ideal. The representation is injective whenever $\mathcal{A}$ is simple (i.e. has no nontrivial closed ideals), e.g. $\mathcal{A} = M_n(\mathbb{C})$, the algebra of all $n \times n$-matrices.

In case $\mathcal{A} = L^\infty(\Omega, \mu)$ the space $\mathcal{H}_\tau$ turns out indeed to be the space $L^2(\Omega, \mu)$ and in this way $L^\infty(\Omega, \mu)$ is represented as the algebra of operators of multiplication by L^∞-functions (see above).

13.3 Quantum Bernoulli experiments

Let us begin with the easiest example of a quantum probability space: Let $\mathcal{A} = M_2(\mathbb{C})$ be the $W^\star$-algebra of all complex 2×2 matrices $a = (a_{ik})$ and choose $\lambda \in]1/2, 1]$. Then $\tau_\lambda(a) = \lambda a_{11} + (1 - \lambda)a_{22}$ defines a normal state on $\mathcal{A}$. As in the case of classical probability we will often write $\mathbb{E}_\lambda(a)$ in place of $\tau_\lambda(a)$, and we call it the **expectation of** a.

Now consider the following model of tossing a fair coin: choose $\mathcal{A}_0$ as the commutative $W^\star$-algebra $\mathbb{C}^{\{0,1\}}$ and let $\mu_0 : \mathcal{A}_0 \to \mathbb{C}$ be given by $\mu_0(f) = \frac{1}{2}(f(0) + f(1))$. The underlying classical probability space is to be rediscovered in $\mathcal{A}_0$ as the set of two idempotents $1_{\{0\}}$ and $1_{\{1\}}$ where 1_A denotes the indicator function of a subset A of a given set. $\mathcal{A}_0$ is spanned also by the elements

$$X = -1_{\{0\}} + 1_{\{1\}} \quad \text{and} \quad 1 = 1_{\{0\}} + 1_{\{1\}} = 1_{\{0,1\}}.$$

We now construct all injective $\star$-algebra homomorphisms $\pi : \mathcal{A}_0 \to \mathcal{A}$ satisfying $\tau_\lambda \circ \pi = \mu_0$. In the sense of section 13.2 this means that we will find all abstract probability spaces $(\mathcal{B}, \tau_{\lambda|_\mathcal{B}})$ which are equivalent to $(\mathcal{A}_0, \mu_0)$. An easy calculation shows that they are given by $\pi := \pi_z$ with

$$\pi(1_{\{1\}}) = \frac{1}{2}\begin{pmatrix} 1 & z \\ \overline{z} & 1 \end{pmatrix} =: P_z \quad , \quad \pi(1_{\{0\}}) = \frac{1}{2}\begin{pmatrix} 1 & -z \\ -\overline{z} & 1 \end{pmatrix} =: Q_z$$

and linear extension, where $z \in \mathbb{C}$ is a parameter with $|z| = 1$. $\pi_z(X) = P_z - Q_z = \left(\begin{smallmatrix} 0 & z \\ \bar{z} & 0 \end{smallmatrix}\right) =: X_z$.

In summary we have seen that the simplest quantum probability space $(\mathcal{A}, \tau_\lambda)$ contains a one-parameter family $(\mathcal{A}_z, \tau_\lambda|_{\mathcal{A}_z})$ of models of classical coin tossing.

For $w = e^{is}$, $z = e^{it}$ $(-\pi < s, t \leq \pi)$ the commutator $[X_w, X_z]_- = X_w X_z - X_z X_w$ is easily determined:

$$[X_w, X_z] = 2\,i\,\sin(s-t) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

So X_w and X_z commute if $w = z$ or $w = -z$.

We obtain $\mathbb{E}_\lambda([X_w, X_z]) = 2\,i\,(2\lambda - 1)\sin(s-t)$. The absolute value of it attains its maximum at $s - t = \pm\pi/2$. The pair (X_w, X_z) with $z = -iw$ is called a **quantum coin tossing**. It is unique up to automorphisms. More precisely this means the following: choose $w = e^{it}$ $(t \in]-\pi, \pi])$, and consider the automorphism φ on $\mathcal{A}$ given by $\varphi(a) = u^\star a u$ where

$$u = \begin{pmatrix} e^{-it/2} & 0 \\ 0 & e^{it/2} \end{pmatrix}.$$

Then $\varphi(X_1) = X_w$, $\varphi(X_{-i}) = X_{-iw}$ and $\tau_\lambda \circ \varphi = \tau_\lambda$.

So in the following we need only to consider the quantum coin tossing (X_1, X_{-i}). Notice that $X_1 = \sigma_x$, $X_{-i} = \sigma_y$ and $[X_w, X_{-iw}]x = 2i\sigma_z$, where $\sigma_x, \sigma_y, \sigma_z$ denote the Pauli matrices. The following important matrices can be constructed by (σ_x, σ_y):

$$b^- = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \frac{1}{2}(\sigma_x + i\sigma_y), \tag{13.3.1}$$

$$b^+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \frac{1}{2}(\sigma_x - i\sigma_y), \tag{13.3.2}$$

$$b^\circ = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{1} - \frac{1}{2i}[\sigma_x, \sigma_y]. \tag{13.3.3}$$

Obviously the following formulae hold:

$$\sigma_x = b^+ + b^-,\ \sigma_y = i \cdot (b^+ - b^-),\ [b^-, b^+] = \sigma_z = \mathbf{1} - 2b^\circ,\ X_w = \overline{w} b^+ + w b^-.$$

These formulae show that the *random variables* b^+ and b^- are in a certain sense more basic than X_w though they are not selfadjoint.

As in the classical case we describe the experiment of n $(\in \mathbb{N})$ independent coin tossings by the n-fold tensor product $(\mathcal{A}^{\otimes n}, \tau_\lambda^{\otimes n})$. The algebra corresponding to the k-th trial is described by the elements

$$\underbrace{\mathbf{1} \otimes \cdots \otimes \mathbf{1}}_{(k-1)\text{ times}} \otimes x \otimes \underbrace{\mathbf{1} \otimes \cdots \otimes \mathbf{1}}_{(n-k)\text{ times}} =: x_k$$

where $x \in \mathcal{A}$ is arbitrary. Describing k times repetition of our quantum coin tossing we obtain $S_{w,k} := \sum_{j=1}^{k} X_{w\,j}$. We have

$$[S_{w,k}, S_{-iw,k}] = 2i \sum_{j=1}^{k} \sigma_{z,\,k}.$$

The pair $(S_{w,k}, S_{-iw,k})$ is called a **quantum Bernoulli experiment**.

Setting

$$\mathcal{A}_k := \mathcal{A}^{\otimes k} \otimes \underbrace{\mathbf{1} \otimes \cdots \otimes \mathbf{1}}_{n-k \text{ times}} = \{a \otimes \underbrace{\mathbf{1} \otimes \cdots \otimes \mathbf{1}}_{n-k \text{ times}} : \ a \in \mathcal{A}^{\otimes k}\}$$

we obtain a natural filtering $\mathbb{C} \cdot \mathbf{1} \subset \mathcal{A}_1 \subset \cdots \subset \mathcal{A}_n = \mathcal{A}^{\otimes n}$. Moreover there also exists a conditional expectation $\mathbb{E}_k$ from $\mathcal{A}_n$ onto $\mathcal{A}_k$ which is given by $\mathbb{E}_k(a \otimes b) = \tau_\lambda^{\otimes(n-k)}(b) \cdot a$, where $b \in \mathcal{A}^{\otimes(n-k)}$, and linear extension.

13.4 The internal quantum processes

Now we consider a polysaturated model ${}^*V(\mathbb{R})$ of the full structure $V(\mathbb{R})$ over $\mathbb{R}$. As usual we replace the standard integer n above by an infinitely large integer N. Then all considerations in the previous section remain true for internal objects.

We use the discrete time interval $T = \{\frac{k}{N} : 0 \leq k < N\}$ denoting its elements by $\underline{s}, \underline{t}, \underline{u}, \ldots$ Moreover we rescale also the family $(\mathcal{A}_k)$ by setting $\mathcal{A}_{\underline{t}} := \mathcal{A}_{N\underline{t}}$.

Set $\rho = \frac{1}{\sqrt{N}}$. Then we introduce

$$a_{\underline{t}}^{\pm} = \rho b_{\underline{t}}^{\pm}, \tag{13.4.1}$$

$$a_{\underline{t}}^{\circ} = b_{\underline{t}}^{\circ}, \tag{13.4.2}$$

$$a_{\underline{t}}^{\bullet} = \rho^2 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{13.4.3}$$

These elements are the internal increments by which we construct the basic internal stochastic processes

$$A_{\underline{t}}^{\sharp} = \Sigma_{0 \leq \underline{s} < \underline{t}}\, a_{\underline{s}}^{\sharp}, \ \sharp \in \{+, -, 0, \bullet\} =: I.$$

$(A_{\underline{t}}^{+})_{\underline{t}}$ is called the internal **creation process**, $(A_{\underline{t}}^{-})_{\underline{t}}$ the **annihilation process**, $(A_{\underline{t}}^{\bullet})_{\underline{t}}$ the **time process** and finally $(A_{\underline{t}}^{\circ})_{\underline{t}}$ the **number process**.

The internal Brownian motion corresponding to the parameter $w \in \mathbb{C}$ then is given by $B_{w,\underline{t}} = \bar{w} A_{\underline{t}}^{+} + w A_{\underline{t}}^{-}$ $(|w| = 1)$. It satisfies the internal difference equation

$$dB_{w,\underline{t}} = \overline{w} a_{\underline{t}}^{+} + w a_{\underline{t}}^{-}.$$

Like in the previous section we consider the state $\tau_\lambda^{\otimes N} =: \tau_{\lambda,N}$ and we obtain

$$\mathbb{E}_{\lambda,N}(B_{w,\underline{t}}) := \tau_{\lambda,N}(B_{w,\underline{t}}) = 0, \quad \mathbb{E}_{\lambda,N}(B_{w,\underline{t}}^2) = \underline{t}. \tag{13.4.4}$$

The second equation follows from the facts that

$$\tau_{\lambda,N}\left((\bar{w}a_{\underline{s}}^+ + wa_{\underline{s}}^-)(\bar{w}a_{\underline{u}}^+ + wa_{\underline{u}}^-)\right) = 0$$

for $\underline{s} \neq \underline{u}$, but $\tau_{\lambda,N}\left((\bar{w}a_{\underline{s}}^+ + wa_{\underline{s}}^-)^2\right) = \tau_{\lambda,N}(1/N) = 1/N$.

The commutator is

$$\left[B_{w,\underline{t}}, B_{-w,\underline{t}}\right] = 2i\rho\Sigma_{0\leq\underline{s}<\underline{t}}\,\sigma_{z,\underline{s}}.$$

The so-called internal Itô-table for the increments is easily computed:

	$a_{\underline{s}}^\bullet$	$a_{\underline{s}}^+$	$a_{\underline{s}}^\circ$	$a_{\underline{s}}^-$
$a_{\underline{s}}^\bullet$	$\rho^2 a_{\underline{s}}^\bullet$	0	0	$\rho^2 a_{\underline{s}}^-$
$a_{\underline{s}}^+$	$\rho^2 a_{\underline{s}}^+$	0	0	$\rho^2 a_{\underline{s}}^\circ$
$a_{\underline{s}}^\circ$	0	$a_{\underline{s}}^+$	$a_{\underline{s}}^\circ$	0
$a_{\underline{s}}^-$	0	$a_{\underline{s}}^\bullet$	$a_{\underline{s}}^-$	0

Let $0 \neq v \in {}^*\mathbb{C}$ be arbitrary. Then the process $(P_{\underline{t}})$ given by

$$P_{\underline{t}}^v = A_{\underline{t}}^\circ + |v|B_{\frac{v}{|v|},\underline{t}} + |v|^2 A_{\underline{t}}^\bullet$$

is called the **Poisson Process with parameter** v.

Now we compute the **characteristic operator functions** of these processes. Whenever $V = (V_{\underline{t}})_{\underline{t}}$ is a stochastic process its characteristic operator function is defined by $f_V(u,\underline{t}) = \exp(iuV_{\underline{t}})$, where $u \in {}^*\mathbb{R}$. Obviously $\dot{f}_V(u,\underline{t})\,|_{u=0} = iV_{\underline{t}}$ and $f_V(u,\underline{t})$ is unitary iff $V_{\underline{t}}$ is selfadjoint.

The processes $V = (V_{\underline{t}})$ we considered so far are all of the form

$$V_{\underline{t}} = \sum_{0\leq\underline{s}<\underline{t}} d_{\underline{s}},$$

where

$$d_{\underline{s}} = \underbrace{1\otimes\cdots\otimes 1}_{N\underline{s}-1 \text{ times}} \otimes d \otimes \underbrace{1\otimes\cdots\otimes 1}_{N-N\underline{s} \text{ times}},$$

and $d \in \mathcal{A}$ is appropriately chosen. Since $d_{\underline{u}}$, $d_{\underline{v}}$ commute for $\underline{u} \neq \underline{v}$ we obtain

$$f_V(u,\underline{t}) = \bigotimes_{0\leq\underline{s}<\underline{t}} (\exp(iud))_{\underline{s}}.$$

The most important formulas are the following ones:

$$\exp(iuB_{w,\underline{t}}) = \bigotimes_{0\le\underline{s}<\underline{t}} (\cos(\rho u)\underline{\mathbf{1}} + i\sin(u)X_w)_{\underline{s}}$$

$$\exp(iuP_{\underline{t}}) = \bigotimes_{0\le\underline{s}<\underline{t}} Q_{\underline{s}}(u)$$

where $Q(u) = \frac{1}{\xi}\begin{pmatrix} 1+\rho^2|v|^2e^{iu\xi} & \rho v(e^{iu\xi}-1) \\ \rho\bar{v}(e^{iu\xi}-1) & \rho^2|v|^2+e^{iu\xi} \end{pmatrix}$ with $\xi = 1+\rho^2|v|^2$.

13.5 From the internal to the standard world

The operator norm topology on $\mathcal{A}$ is too fine in order to obtain reasonable nonstandard hulls. Moreover the processes discussed so far are internally bounded but not necessarily S-bounded. For example $B_{w,k/N}$ is selfadjoint, hence its operator norm is equal to its spectral radius. This number is easily computed as $\frac{k}{\sqrt{N}}$, or in other words $\|B_{w,\underline{t}}\| = \sqrt{N}\cdot\underline{t}$.

In the following we only consider τ_λ for $\lambda = 1$. The other case will be treated elsewhere. We set $\tau_1 =: \tau$ and we construct the Hilbert space $\mathcal{H}_\tau$ corresponding to $(\mathcal{A},\tau)$ ($\mathcal{A}$ the algebra of 2×2-matrices, see section 13.2). To this end we denote the columns of the 2×2-matrix a by $a = (a_1^\downarrow, a_2^\downarrow)$. Then $\tau(a^*b) = (a_1^\downarrow \mid b_1^\downarrow)$ where the latter is the canonical scalar product on $\mathbb{C}^2$. Hence $L_\tau = \{a : a_2^\downarrow = 0\}$ and $\pi_\tau(a)(b+L_\tau) = ab_1^\downarrow + L_\tau$. In particular $\mathcal{H}_\tau \cong \mathbb{C}^2$, where the isometric isomorphism is given by $a + L_\tau \to a_1^\downarrow$.

We have previously introduced the random variable $X_1 = \sigma_x$. Its equivalence class in $\mathcal{H}_\tau$ is denoted by $\widetilde{X_1} = X_1 + L_\tau$ We set $e_0 = b_1^{\circ\downarrow} = \binom{1}{0} = \widetilde{X_1^0}$ and $e_1 = b_1^{-\downarrow} = \binom{0}{1} = \widetilde{X_1}$.

Now we consider $\tau_N = \tau^{\otimes N}$. Obviously $\mathcal{H}_{\tau_N} \cong ({}^*\mathbb{C}^2)^{\otimes N}$ holds. Let $\omega \in \{0,1\}^T$ be an arbitrary internal function. Then we define e_ω by $e_\omega = \otimes_{k=0}^{N-1} e_{\omega(k/N)}$. The set $\{e_\omega : \omega\in\{0,1\}^T\}$ forms an orthonormal basis in $\mathcal{H}_{\tau_N}$.

$\mathcal{H}_{\tau_N}$ is an internal Hilbert space, so its nonstandard hull $\widehat{\mathcal{H}}_{\tau_N}$ is a Hilbert space.

The formulae (13.3.1) up to (13.4.3) above lead to the following equations, which result from the fact that $\mathcal{H}_{\tau_N}$ is the GNS-Hilbert space of $(\mathcal{A}_N,\tau_N)$. In order to make the formulae as simple as possible we denote the addition mod 2 by $\oplus$, i.e. we consider $\Omega = \{0,1\}^T$ as the cartesian product of the group $(\mathbb{Z}_2,\oplus)$. Then we set $\omega^+ = 1_T \oplus \omega$ and $\omega^- = \omega$ for $\omega\in\{0,1\}^T$. Moreover we write $1_{\underline{t}}$ in place of $1_{\{\underline{t}\}}$.

With all theses conventions we obtain

$$a_{\underline{t}}^{\pm} e_\omega = \frac{\omega^{\pm}(\underline{t})}{\sqrt{N}} e_{\omega \oplus 1_{\underline{t}}}, \tag{13.5.1}$$

$$a_{\underline{t}}^{\bullet} e_\omega = \frac{\omega^{+}(\underline{t})}{N} e_\omega = a_{\underline{t}}^{-} a_{\underline{t}}^{+} e_\omega, \tag{13.5.2}$$

$$a_{\underline{t}}^{\circ} e_\omega = \omega^{-}(\underline{t}) e_\omega = N \cdot a_{\underline{t}}^{+} a_{\underline{t}}^{-} e_\omega. \tag{13.5.3}$$

13.5.1 Brownian motion

First of all we show that our internal Brownian motion $(B_{w,\underline{t}})$ leads to the standard Brownian motion on a subspace of $\widehat{\mathcal{H}}_{\tau_N}$. To this end we recall that the standard Brownian motion (β_t) on $L^2(\Omega, L_\mu)$ (with $\Omega = \{0,1\}^T$, $\mu = \mu_N = \mu_0^{\otimes N}$, L_μ the corresponding Loeb measure) can be viewed as a family of multiplication operators $f \mapsto \beta_t f$ on $L^2(\Omega, L_\mu)$ which are selfadjoint but unbounded (cf. the end of section 13.2). From equation (13.4.4) it follows that $B_{w,\underline{t}}$ is S-bounded in $\mathcal{H}_N$. We claim that classical Brownian motion (or better to say: Anderson's Brownian motion) is equivalent (in the sense of section 13.2) to our $(\widehat{B}_{w,\circ\underline{t}})$ where $(\widehat{B}_{w,\circ\underline{t}})$ denotes the family of selfadjoint operators on a subspace of $\widehat{\mathcal{H}}_{\tau_N}$ coming from the family $(B_{w,\underline{t}})$. To this end let $|w| = 1$ and

$$P_1 = \frac{1}{2}\begin{pmatrix} \underline{1} & w \\ w & 1 \end{pmatrix} \quad , \quad P_0 = \frac{1}{2}\begin{pmatrix} \underline{1} & -w \\ -w & 1 \end{pmatrix}.$$

For $\omega \in \Omega$ we set $P_\omega = \otimes_{0 \le k < N} P_{\omega(k/N)}$. It follows $\tau_N(P_\omega) = 2^{-N} = \mu(\{\omega\})$. We denote the internal $W^\star$-algebra generated by $\{P_\omega : \omega \in \{0,1\}^T\}$ by $\mathcal{B}(w)$. The formula

$$b_{w,t} := \overline{w} a_{\underline{t}}^{+} + w a_{\underline{t}}^{-} = \frac{\rho}{2}(P_{1_{\underline{t}}^{-}} - P_{1_{\underline{t}}^{+}})$$

shows that $(B_{w,t}) = \sum_{0 \le \underline{s} < \underline{t}} b_{w,\underline{t}}$ is contained in $\mathcal{B}(w)$.

The internal L^∞-space $L^\infty(\Omega, \mu)$ is nothing else than $L^\infty(\{0,1\}, \mu_0)^{\otimes N}$. Thus it is not very hard to see that $\pi_N := \pi_w^{\otimes N}$ (see section 13.2) maps $L^\infty(\Omega, \mu)$ onto $\mathcal{B}(w)$ with $\tau_N \circ \pi_N|_{\mathcal{B}(w)} = \mu$. Now let $D = \{f \in L^\infty(\Omega, \mu) : f \text{ is } S\text{-bounded}\}$. Then

$$\begin{aligned} \pi_N(D) &\subset \{a \in \mathcal{A}_N : \|a\| \text{ is } S\text{-bounded}\} \\ &\subset \{x \in \mathcal{H}_{\tau_N} : \|x\|_2 = \sqrt{\tau(x^* x)} \text{ is } S\text{-bounded}\}. \end{aligned}$$

Moreover for f, $g \in D$ we have

$$\int_\Omega {}^\circ\bar{f}\, {}^\circ g \, dL_\mu \approx \mu(\bar{f} g) = \tau_N\left(\pi_N(\bar{f} g)\right) = (\pi_N(f) | \pi_N(g))$$

because of $\pi_N(\bar{f}g) = \pi_N(f)^\star \pi_N(g)$. Since $L^2(\Omega, L_\mu)$ is the completion of $\{{}^\circ f : f \in D\}$ the mapping ${}^\circ f \to \widehat{\pi_N(f)}$ can be extended to a linear isometry U from $L^2(\Omega, L_\mu)$ onto a subspace $\mathcal{M}$, say, of $\mathcal{H}_{\tau_N}$.

For $t \in [0,1]$ set $\tilde{t} = \min\{\underline{s} \in T : \underline{s} > t\}$. We show that $U(\beta_t) = \widehat{B_{w,\tilde{t}}}$. Define

$$y_{\underline{t}}(\omega) = \begin{cases} 1 & \omega(\underline{t}) = 1 \\ -1 & \omega(\underline{t}) = 0 \end{cases}$$

and $Y_{\underline{t}} = \frac{1}{\sqrt{N}} \sum_{0 \le \underline{s} < \underline{t}} y_{\underline{s}}$. Then $\beta_t(\omega) = {}^\circ Y_{\tilde{t}}(\omega)$. Moreover $\pi_N(Y_{\underline{t}}) = B_{w,\underline{t}}$. Set $M_n = \{\omega : |Y_{\tilde{t}}(\omega)| \le n\}$. Then for $n \in \mathbb{N}$ we have $M_n = \{\omega : |\beta_t(\omega)| \le n\}$. This implies that $\lim_{n\to\infty} \beta_t 1_{M_n} = \beta_t$ holds in $L^2(\Omega, L_\mu)$. Therefore for every standard $\varepsilon > 0$ there exists a standard n_0 with $\|\beta_t(1_\Omega - 1_{M_n})\|_2 < \varepsilon$ for $n \ge n_0$ standard. But this in turn implies $\|Y_{\tilde{t}}(1_\Omega - 1_{M_n})\|_2 < \varepsilon$ for n standard, $n \ge n_0$. Since π_N is an isometry also for the internal L^2-norm on $\mathcal{H}_{\tau_N}$ we obtain $\|B_{w,\tilde{t}}\pi_N(1_\Omega - 1_{M_n})\|_2 < \varepsilon$ for these $n \ge n_0$. $\|B_{w,\tilde{t}}\,\pi_N(1_{M_n})\|_2 = \|Y_{\tilde{t}} 1_{M_n}\|_2 \le n$ implies that $B_{w,\tilde{t}}\,\pi_N(1_{M_n}) \in \mathcal{H}_{\tau_N,\mathrm{fin}}$ and $U(\beta_t 1_{M_n}) = \widehat{B_{w,\tilde{t}}\,\pi_N(1_{M_n})}$. But this in turn gives $U(\beta_t) = \widehat{B_{w,\tilde{t}}}$. That the operator of multiplication by β_t is mapped onto the operator of multiplication by $\widehat{B_{w,\tilde{t}}}$ follows from an easy additional argument.

13.5.2 The nonstandard hulls of the basic internal processes

The deeper problem is to find a joint subspace $\mathcal{K}$ of $\widehat{\mathcal{H}}_{\tau_N}$ such that all processes have standard parts as closable operators densely defined in $\mathcal{K}$. We begin the construction with some more notations: For $\omega \in \{0,1\}^T =: \Omega$ we set $M_\omega = \{\underline{t} : \omega(\underline{t}) = 1\}$, and $|\omega| := |M_\omega|$, that is the (internal) cardinality of M_ω. Using these notations we obtain the following formulae by means of (13.4.4)–(13.5.2). Let $y = \sum_{\omega\in\Omega} y_\omega e_\omega$ be arbitrary. Then

$$A^\bullet_{\underline{t}} y = \sum_{\omega\in\Omega} \left(\frac{1}{N} \sum_{0 \le \underline{s} < \underline{t}} \omega^+(\underline{s}) \right) y_\omega e_\omega \tag{13.5.4}$$

$$A^\circ_{\underline{t}} y = \sum_{\omega\in\Omega} \left(\sum_{0 \le \underline{s} < \underline{t}} \omega(\underline{s}) \right) y_\omega e_\omega \tag{13.5.5}$$

$$A^+_{\underline{t}} y = \sum_{\varphi\in\Omega} \left(\frac{1}{\sqrt{N}} \sum_{0 \le \underline{s} < \underline{t}} y_{\varphi \oplus 1_{\underline{s}}} \varphi^-(\underline{s}) \right) e_\varphi \tag{13.5.6}$$

$$A^-_{\underline{t}} y = \sum_{\varphi\in\Omega} \left(\frac{1}{\sqrt{N}} \sum_{0 \le \underline{s} < \underline{t}} y_{\varphi \oplus 1_{\underline{s}}} \varphi^+(\underline{s}) \right) e_\varphi \tag{13.5.7}$$

Consider now the projection $\mathbb{E}_{\underline{t}}$ from $\mathcal{H}$ onto $\mathcal{H}_{\underline{t}}$ which is the Hilbert space coming from $\mathcal{A}_{\underline{t}}$ (see section 13.3). $\mathbb{E}_{\underline{t}}$ is given by

$$\mathbb{E}_{\underline{t}}(e_\omega) = \begin{cases} e_\omega & \omega(\underline{s}) = 0 \text{ for all } \underline{s} \geq t \\ 0 & \text{else} \end{cases}$$

Then we obtain $\mathbb{E}_{\underline{t}} A^\sharp_{\underline{t}} = A^\sharp_{\underline{t}} \mathbb{E}_{\underline{t}}$ for each $\sharp \in \{\bullet, \circ, +, -\}$ or in other words *the family* $(A^\sharp_{\underline{t}})_{\underline{t}<1}$ *is adapted to* $(\mathcal{H}_{\underline{t}})_{\underline{t}<1}$.

We prove now some continuity properties. To this end we introduce

$$\mathcal{H}^{(n)} := \left\{ \sum_{|\omega|=n} y_\omega e_\omega : y_\omega \in {}^*\mathbb{C} \right\}$$

and we set $\mathcal{H} := \mathcal{H}_{\tau_N}$ for short.

Then $\mathcal{H} = \perp_{n=0}^{N} \mathcal{H}^{(n)}$ where $\perp$ denotes the orthogonal direct sum. Moreover $A^\bullet_{\underline{t}}(\mathcal{H}^{(n)}) \subset \mathcal{H}^{(n)}$, $A^\circ_{\underline{t}}(\mathcal{H}^{(n)}) \subset \mathcal{H}^{(n)}$, $A^+_{\underline{t}}(\mathcal{H}^{(n)}) \subset \mathcal{H}^{n+1}$, and finally $A^-_{\underline{t}}(\mathcal{H}^{(n)}) \subset \mathcal{H}^{(n-1)}$ where $\mathcal{H}^{(-1)} := \{0\} =: \mathcal{H}^{(N+1)}$.

For $y = \sum_{|\omega|=n} y_\omega e_\omega$ and $0 \leq \underline{s} < \underline{t}$ we obtain

$$\begin{aligned} \|A^\bullet_{\underline{t}} y - A^\bullet_{\underline{s}} y\|^2 &= \sum_{|\omega|=n} \left| \frac{1}{N} \sum_{\underline{s} \leq \underline{u} < t} \omega^+(s) \right|^2 |y_\omega|^2 \\ &\leq \sum_{|\omega|=n} \left| \frac{1}{N} \sum_{\underline{s} \leq \underline{u} < \underline{t}} 1 \right|^2 |y_\omega|^2 = (\underline{t} - \underline{s})^2 \|y\|^2. \end{aligned} \tag{13.5.8}$$

It follows that $\|A^\bullet_{\underline{t}} - A^\bullet_{\underline{s}}\| \leq (\underline{t} - \underline{s})$, in particular the mapping $\underline{t} \mapsto A^\bullet_{\underline{t}}$ is S-continuous from T to $\mathcal{L}(\mathcal{H})$ with respect to the operator norm.

Let $y = \sum_{|\omega|=n} y_\omega \in \mathcal{H}^{(n)}$, and $0 \leq \underline{s} < \underline{t}$

$$\|A^\circ_{\underline{t}} y - A^\circ_{\underline{s}} y\|^2 = \sum_{|\omega|=n} \Big(\sum_{\underline{s} \leq \underline{u} < \underline{t}} \omega(\underline{s}) \Big)^2 |y_\omega|^2.$$

It follows that $\|A^\circ_{\underline{t}}|_{\mathcal{H}^{(n)}}\| \leq n$, but the mapping $\underline{t} \mapsto A^\circ_{\underline{t}} y$ is not S-continuous in general. So finally we have to choose an appropriate subspace in order to get S-continuity of $\underline{t} \to A^\circ_{\underline{t}} y$.

Next we show the following inequality:

Proposition 1 *For* $0 \leq \underline{s} < \underline{t}$

$$\|(A^+_{\underline{t}} - A^+_{\underline{s}})|_{\mathcal{H}^{(n)}}\|^2 \leq (n+1)(\underline{t} - \underline{s}) \tag{13.5.9}$$

holds.

Proof. Let $y = \sum_{|\omega|=n} y_\omega e_\omega \in \mathcal{H}^{(n)}$ be arbitrary. Then

$$(A_{\underline{t}}^+ - A_{\underline{s}}^+)y = \frac{1}{\sqrt{N}} \sum_{|\varphi|=n+1} \left(\sum_{\underline{s} \le \underline{u} < \underline{t}} y_{\varphi \oplus 1_{\underline{u}}} \varphi(\underline{u}) \right) e_\varphi$$

holds. If $\varphi(u) = 0$ then $|\varphi \oplus 1_{\underline{u}}| = n+1$ hence $y_{\varphi \oplus 1_{\underline{u}}} = 0$. It follows

$$\begin{aligned} \|(A_{\underline{t}}^+ - A_{\underline{s}}^+)y\|^2 &= \frac{1}{N} \sum_{|\varphi|=n+1} \left(\left| \sum_{\underline{s} \le \underline{u} < \underline{t}} y_{\varphi \oplus 1_{\underline{u}}} \right| \right)^2 \\ &\le \frac{n+1}{N} \sum_{|\varphi|=n+1} \sum_{\underline{s} \le \underline{u} < \underline{t}} |y_{\varphi \oplus 1_{\underline{u}}}|^2 \end{aligned}$$

Let $\omega \in \Omega$ be arbitrary with $|\omega| = n$. Then there exist at most $N(\underline{t} - \underline{s}) - |\omega \cdot 1_{[\underline{s},\underline{t}[}|$ different φ with $\varphi \oplus 1_{\underline{u}} = \omega$ and $\underline{s} \le \underline{u} < \underline{t}$. This gives $\|(A_{\underline{t}}^+ - A_{\underline{s}}^+)y\|^2 \le \frac{N(\underline{t}-\underline{s})}{N}(n+1)\|y\|^2$, and the asserted inequality follows. □

Corollary 1

$$\|(A_{\underline{t}}^- - A_{\underline{s}}^-)|_{\mathcal{H}^{(n)}}\|^2 \le (\underline{t} - \underline{s})n. \tag{13.5.10}$$

Proof. $A_{\underline{t}}^-$ is the adjoint of $A_{\underline{t}}^+$.

By construction we have $\mathcal{H}^{(n)} \perp \mathcal{H}^{(m)}$ for $n \neq m$. So the subspace $\mathcal{K}^\infty$ of $\hat{\mathcal{H}}$, generated by all the spaces $\hat{\mathcal{H}}^{(n)}$ $(n \in \mathbb{N})$ is nothing else than the orthogonal direct sum $\mathcal{K}^\infty = \perp_{n \in \mathbb{N}_0} \hat{\mathcal{H}}^{(n)}$. Recall the definition $\tilde{t} = \min\{\underline{u} \in T : \underline{u} > \underline{t}\}$. Then from the inequality (13.5.8) it follows that by

$$\hat{A}_{t,n}^\bullet \hat{y} = (A_{\tilde{t},n}^\bullet y)^\wedge$$

there is uniquely defined an operator on $\hat{\mathcal{H}}^{(n)}$ bounded by t. So we obtain the operator $\hat{A}_t^\bullet = \perp_{n \in \mathbb{N}_0} \hat{A}_{t,n}$ on $\mathcal{K}^\infty$ which is selfadjoint and bounded by t. Moreover $\|\hat{A}_t^\bullet - \hat{A}_s^\bullet\| \le t - s$ holds. □

In order to define the nonstandard hull of one of the other operators $A_{\underline{t}}^\sharp$ $(\sharp \in \{+, -, 0\})$ we choose the following joint domain of definition:

$$\mathcal{K}_0^\infty := \left\{ y \in \mathcal{K}^\infty : y = \perp_{n \in \mathbb{N}_0} \hat{y}_n, \hat{y}_n \in \hat{\mathcal{H}}^{(n)}, \sum_{n=0}^\infty n \|\hat{y}_n\|_2^2 < \infty \right\}$$

which is dense in $\mathcal{K}^\infty$. From the inequalities (13.5.9) to (13.5.10) we obtain that by

$$\begin{aligned} \hat{A}_t^\circ(\perp_{n \in \mathbb{N}_0} \hat{y}_n) &= \perp_{n \in \mathbb{N}_0} (A_{\tilde{t}}^\circ y_n)^\wedge \\ \hat{A}_t^\pm(\perp_{n \in \mathbb{N}_0} \hat{y}_n) &= \perp_{n \in \mathbb{N}_0} (A_{\tilde{t}}^\pm y_n)^\wedge \end{aligned}$$

the operators $\hat{A}_t^\sharp : \mathcal{K}_0^\infty \to \mathcal{K}^\infty$ are uniquely defined. Moreover the following proposition holds:

Proposition 2 *$\hat{A}_t^\circ$ is selfadjoint and $\hat{A}_t^+$ and $\hat{A}_t^-$ are adjoint to each other.*

Proof. Set $\hat{A}_{t,n}^\sharp = \hat{A}_t^\sharp \mid \hat{\mathcal{H}}_n$. These operators are bounded and $\hat{A}_{t,n}^\circ$ is selfadjoint, whereas $\hat{A}_{t,n}^+$ and $\hat{A}_{t,(n+1)}^-$ are adjoint to each other. Now the assertion is obvious. □

The final problem we have to solve is to find an appropriate filtration with the necessary conditions of continuity. To this end we calculate the difference $\mathbb{E}_{\underline{t}} - \mathbb{E}_{\underline{s}}$ for $0 \leq \underline{s} < \underline{t}$. For $y \in \mathcal{H}^{(n)}$ we obtain

$$(\mathbb{E}_{\underline{t}} - \mathbb{E}_{\underline{s}})y = \sum_{|\omega|=n,\, \omega \leq 1_{[\underline{s},\underline{t}]}} y_\omega e_\omega.$$

Let $\underline{t} = k/N$, $\underline{s} = l/N$. It follows

$$\begin{aligned} \|\mathbb{E}_{\underline{t}} y - \mathbb{E}_{\underline{s}} y\|^2 &= \sum_{|\omega|=n,\, \omega \leq 1_{[\underline{s},\underline{t}[}} |y_\omega|^2 \\ &\leq \max_{|\omega|=n} |y_\omega|^2 \binom{k-l}{n} \\ &= \max_{|\omega|=n} |y_\omega|^2 N^n (\underline{t} - \underline{s})^n \cdot \Big(1 - \frac{1}{k-l}\Big) \cdots \Big(1 - \frac{n-1}{k-l}\Big). \end{aligned}$$

So the mapping $y \to \mathbb{E}_{\underline{t}} y$ will be S-continuous if $\max_{|\omega|=n} |y_\omega|^2 N^n$ will be finite.

This consideration leads to the internal measure m on $\Omega = \{0,1\}^T$ given by $m(\{\omega\}) = \frac{1}{N^{|\omega|}}$. Then (Ω, m) is the direct sum of $(\Omega_n, m_n)_{n \leq N}$ with $\Omega_n = \{\omega \in \Omega : |\omega| = n\}$ and $m_n = m|_{\Omega_n}$. Notice that $m(\Omega_n) = \binom{N}{n} \cdot \frac{1}{N^n}$, so that $m(\Omega) = \sum_{k=0}^N \binom{N}{k} \cdot \frac{1}{N^k} = \big(1 + \frac{1}{N}\big)^N \approx e$. Moreover $\sum_{k=L}^N m(\Omega_k) \approx 0$ for all $L \approx \infty$.

Let (Ω, L_m) be the Loeb measure space associated to (Ω, m). Then $L_m(\bigcup_{k\in\mathbb{N}} \Omega_k) = e$ and $\Omega \setminus (\bigcup_{k\in\mathbb{N}} \Omega_k)$ is an L_m- null set. Finally $L^2(\Omega, L_m) \cong \perp_{n\in\mathbb{N}} L^2(\Omega_n, L_{m_n})$ where L_{m_n} is the Loeb measure associated to m_n.

Consider the external subspace X of all S-bounded internal *complex valued functions on Ω. Then for $f \in X$ the complex-valued function $^\circ f : \omega \to {}^\circ(f(\omega))$ is in $L^2(\Omega, L_m)$ and the space $\{^\circ f : f \in X\}$ is dense in $L^2(\Omega, L_m)$. Moreover $^\circ f = \sum_{k\in\mathbb{N}_0} {}^\circ f_k$ in the L^2-sense, where $f_k = f\,|_{\Omega_k}$.

Now we map X onto a subspace Y of $\mathcal{H}$ by the mapping V defined as follows:

$$V(f) = \sum_{\omega\in\Omega} f(\omega) N^{-|\omega|/2}\, e_\omega,$$

and set

$$W(f) = \widehat{V(f)}.$$

Then $\int_\Omega |f|^2 dm = \|V(f)\|^2$. Let $\mathcal{L}$ be the closure of $\{W(f) : f \in X\} =: \mathcal{L}^{(b)}$ in $\mathcal{K}^\infty$. Then the last equation shows $L^2(\Omega, L_m) \cong \mathcal{L}$. Moreover $X_k = \{f \in X : f(\Omega_\ell) = 0 \text{ for } \ell \neq k\}$ is mapped onto a subspace of $\mathcal{H}^{(k)}$ (k standard). Let $\mathcal{L}_k$ be the closure in $\mathcal{K}^\infty$ of $\{W(f) : f \in X_k\} =: \mathcal{L}_k^{(b)}$. Then $\mathcal{L} = \perp_{k\in\mathbb{N}} \mathcal{L}_k$ and $\mathcal{L}_k \cong L^2(\Omega_k, L_{m_k})$.

The following facts are easily seen to hold:

Theorem 1

$$\begin{aligned}
\mathcal{L}_k^{(b)} &\subset \mathcal{K}_0^\infty \text{ for } k \in \mathbb{N}_0 \\
\hat{A}_{\underline{t}}^\bullet \mathcal{L}_k^{(b)} &\subset \mathcal{L}_k \\
\hat{A}_{\underline{t}}^\circ \mathcal{L}_k^{(b)} &\subset \mathcal{L}_k \\
\hat{A}_{\underline{t}}^+ \mathcal{L}_k^{(b)} &\subset \mathcal{L}_{k+1}^{(b)} \\
\hat{A}_{\underline{t}}^- \mathcal{L}_k^{(b)} &\subset \mathcal{L}_{k-1}^{(b)}.
\end{aligned}$$

Moreover $\mathcal{L} \cap \mathcal{K}_0^\infty$ is dense in $\mathcal{L}$ and it is the joint domain of definition of the operators $\hat{A}_t^\sharp$, $\sharp \in \{\bullet, \circ, +, -\}$ which are closable and essentially selfadjoint in case $\sharp = \bullet, \circ$, and adjoint to each other in case $\sharp = +, -$.

Remark: We map an internal set M of T onto its indicator function $1_M \in \{0,1\}^T = \Omega$. This mapping ψ is bijective and therefore we obtain an isometric isomorphism U of $L^2(\Omega, m)$ onto the internal Guichardet space of Leitz-Martini. This shows that our considerations concerning the continuity of the filtering leads in a very natural manner to the Loeb-Guichardet space for the basic quantum stochastic processes (cf. section 13.1).

13.6 The symmetric Fock space and its embedding into $\mathcal{L}$

Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{H}^{\otimes n}$ be the nth tensor product of $\mathcal{H}$. We denote the full symmetric group of n elements by $\S_n$. Then by

$$P_n(x_1 \otimes \cdots \otimes x_n) = \frac{1}{n!} \sum_{\pi\in\S_n} x_{\pi(1)} \otimes \cdots \otimes x_{\pi(n)}$$

there is uniquely defined an orthogonal projection, the range of which is called the **nth symmetric tensor product** $\mathcal{H}^{\odot n}$. The **symmetric Fock space**

$\mathcal{F}_+(\mathcal{H})$ **over** $\mathcal{H}$ is the orthogonal sum

$$\mathcal{F}_+(\mathcal{H}) = \perp_{n=0}^{\infty} \mathcal{H}^{\odot n}$$

where as usual $\mathcal{H}^{\odot 0} = \mathbb{C}$.

For the special case $\mathcal{H} = L^2([0,1],\lambda)$ (λ: Lebesgue measure) the tensor product $\mathcal{H}^{\otimes n}$ is nothing else than $L^2([0,1]^n,\lambda^n)$ where λ^n denotes the n-dimensional Lebesgue measure, and the projection P_n is given by

$$P_n f(t_1,\dots t_n) = \frac{1}{n!}\sum_{\pi\in S_n} f(t_{\pi(1)},\dots t_{\pi(n)}).$$

Now we give the more or less classical definition of the basis quantum processes. To this end we set

$$f_1 \circ \cdots \circ f_n := P_n(f_1 \otimes \cdots \otimes f_n).$$

For $0 \le t < 1$ the definition of the creation operator is given by

$$\mathcal{A}_t^+((f_1 \circ \cdots \circ f_n) := 1_{[0,t[} \circ f_1 \circ \cdots \circ f_n \tag{13.6.1}$$

and linear extension to the largest possible domain.

Similarly the annihilation operator is given by

$$\mathcal{A}_t^-(f_1 \circ \cdots \circ f_n) = \sum_{j=1}^{n} \left(1_{[0,t[}|f_j\right) f_1 \circ \cdots \circ \hat{f}_j \circ \cdots \circ f_n, \tag{13.6.2}$$

where the expression $\hat{f}_j$ indicates that this factor is omitted.

The number operator is given by

$$\mathcal{A}_t^\circ(f_1 \circ \cdots \circ f_n) = 1_{[0,t[}\cdot f_1 \circ \cdots \circ f_n + \cdots + f_1 \circ f_2 \circ \cdots \circ f_{n-1} \circ 1_{[0,t[}\cdot f_n, \tag{13.6.3}$$

whereas the time-operator is

$$\mathcal{A}_t^\bullet(f_1 \circ \cdots \circ f_n) = t f_1 \circ \cdots \circ f_n. \tag{13.6.4}$$

Remark: These operators are those ones introduced by Hudson and Parthasarathy [5] as the basic quantum processes.

Consider now the set $X_n = \{(t_1,t_2,\dots,t_n) : 0 \le t_1 < \cdots < t_n \le 1\}$. It is a measurable subset of $[0,1]^n$. Let f be an element of $L^2(X_n,\lambda^n)$. Setting

$$\tilde{f}(t_1,\dots,t_n) = \begin{cases} f(t_1,\dots,t_n) & (t_1,\dots,t_n) \in X_n \\ 0 & \text{otherwise} \end{cases}$$

we obtain an element $\tilde{f}$ of $L^2([0,1]^n, \lambda^n)$. Then $P_n(\tilde{f}) \in L^2([0,1],\lambda)^{\odot n}$ and $\|P_n(\tilde{f})\|^2 = \int_{X_n} |f|^2 d\lambda^n$, i.e. $\|f\| = \|P_n(\tilde{f})\|$ holds. On the other hand let $f \in L^2([0,1],\lambda)^{\odot n}$ be arbitrary. Then $f = n!\, P_n(\widetilde{f_{|X_n}})$ (equality in the L^2-sense). It follows that the mapping $f \to P_n(\tilde{f})$ is a linear isometry from $L^2(X_n, \lambda^n)$ onto $L^2([0,1],\lambda)^{\odot n}$. Its inverse is given by $g \to n!\, g_{|X_n} =: U_n(g)$.

Now we embed $L^2(X_n, \lambda^n)$ isometrically into the space $\mathcal{L}_n$ constructed at the end of the previous section. To this end let $\omega \in \Omega_n$ be arbitrary. Set $t_1(\omega) = \min(\underline{s} : \omega(\underline{s}) = 1)$ and by induction $\underline{t}_{k+1}(\omega) = \min(\underline{s} > \underline{t}_k : \omega(\underline{s}) = 1)$. Then $\vec{\underline{t}}(\omega) := (\underline{t}_1(\omega), \ldots, \underline{t}_n(\omega)) \in {}^*X_n \cap T^n$. Moreover if $\vec{\underline{t}} := (k_1/N, \ldots, k_n/N) \in {}^*X_n \cap T^n$ then $\vec{\underline{t}} = \vec{\underline{t}}(\omega)$ for $\omega = 1_M$ where $M = \{\underline{t}_1(\omega), \ldots, \underline{t}_n(\omega)\}$.

We consider the dense subspace C_n of $L^2(X_n, \lambda^n)$ consisting of the restrictions to X_n of continuous functions on the closure $\overline{X_n}$. For $f \in C_n$ we set $S(f)(\omega) = {}^*f(\vec{\underline{t}}(\omega))$. This is an element of $L^2(\Omega_n, m_n)$ satisfying $\|f\|_2 \approx \|S(f)\|_2$. It follows that C_n is linearly and isometrically embedded into $L^2(\Omega_n, L_{m_n})$, and this embedding can obviously be extended to the whole of $L^2(X_n, \lambda^n)$. Since $L^2(\Omega_n, L_{m_n})$ is linearly and isometrically embedded in $\mathcal{L}_n$, we obtain that the symmetric tensor product $L^2([0,1],\lambda)^{\odot n}$ can be embedded in $\mathcal{L}_n$. An obvious extension then yields an embedding denoted by U of the symmetric Fock space $\mathcal{F}_+(L^2([0,1],\lambda)$ into $\mathcal{L}$. Call $\mathcal{G}$ the image of this embedding and set $\mathcal{G}_0 = G \cap \mathcal{K}_0^\infty$.

Theorem 2 *$\mathcal{G}_0$ is dense in $\mathcal{G}$. Moreover the restrictions of $A^\sharp$ to $\mathcal{G}_0$ yield closed densely defined operators which are selfadjoint for $\sharp \in \{\circ, \bullet\}$ and adjoint to each other in the other cases and which moreover satisfy*

$$\mathcal{A}_t^\sharp = U^{-1} A_t^\sharp U$$

on $U^{-1}(\mathcal{G}_0)$ where $\mathcal{A}^\sharp$ are the operators given by equations (13.6.1) up to (13.6.4).

The proof is obvious.

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14

Applications of rich measure spaces formed from nonstandard models

Peter Loeb*

Abstract

We review some recent work by Yeneng Sun and the author. Sun's work shows that there are results, some used for decades without a rigourous foundation, that are only true for spaces with the rich structure of Loeb measure spaces. His joint work with the author uses that structure to extend an important result on the purification of measure valued maps.

14.1 Introduction

In 1975 [11], the author constructed a class of standard measure spaces formed on nonstandard models. These spaces, now called "Loeb spaces" in the literature, are very close to the underlying internal spaces, and are rich in structure. (See the author's three chapters and Osswald's two chapters in [12] for background.) We will briefly review here some recent work by Yeneng Sun that uses these measure spaces. Sun has shown that there are results, some used in applications for decades without a rigorous foundation, that are only true for spaces with the rich structure of Loeb measure spaces. We will follow the description of Sun's work with a detailed proof of a special case of the recent result in [14] by Yeneng Sun and the author on the purification of measure-valued maps. A counterexample shows that the result we give is false if the Loeb space we use is replaced by the unit interval with Lebesgue measure. We note in passing here that the result in [13] by Osswald, Sun, Zhang and the author presents yet another example needing rich measure spaces.

*Department of Mathematics, University of Illinois, Urbana, IL 61801.
loeb@math.uiuc.edu

14.2 Recent work of Yeneng Sun

We begin with some work in the late 1990's of Yeneng Sun ([16], [17], [18], and Chapter 7 of [12]). That work, Sun's alone, is published elsewhere, so what is said here is an invitation to read, and not a substitute for, the original articles. To set the background we consider the following question: Does it make sense to speak of an infinite number of independent individuals or random variables indexed by points in a uniform probability space? Can one, for example, reasonably consider an infinite number of independent tosses of a fair coin with the tosses indexed by a uniform probability space, and if so, does it make sense to say that half the tosses should be heads? Of course, there is no problem in speaking of independent random variables indexed by the first n integers with each integer having probability $1/n$. An infinite index set with a uniform probability measure, however, must be uncountable; there is no uniform probability measure on the full set of natural numbers. The problem thus evolves to finding the possible meaning of an uncountable family of independent random variables. Whatever way one approaches this problem, the usual measure-theoretic tools fail.

A natural attempt to generalize coin tossing replaces the natural numbers with points in the interval $[0,1]$, and thus replaces sequences of 1's and -1's with functions from $[0,1]$ to the two-point set $\{-1,1\}$. This is a reasonable model for an uncountable family of independent coin tosses. In 1937, Doob [2] exhibited a problem with this and similar spaces of functions when standard techniques are applied. To see the problem, let Ω denote the set of $\{-1,1\}$-valued functions on $[0,1]$, and let P be the product measure on Ω constructed from the measure taking the values $1/2$ at 1 and $1/2$ at -1. In the usual construction of an appropriate σ-algebra on Ω, measurable sets are formed from the algebra of cylinder sets, with functions in each cylinder set restricted at only a finite number of elements of $[0,1]$ to take either the value 1 or -1. It follows that each measurable set in Ω is determined in a way described below by a countable subset of $[0,1]$, and so the following result holds.

Proposition 14.2.1 *Fix any $h \in \Omega$. Set*

$$M_h := \{\omega \in \Omega : \ \omega(t) = h(t) \text{ except for countably many } t \in [0,1]\}.$$

Then M_h has P-outer measure 1.

Proof. For each measurable $B \subseteq \Omega$, there is a countable set $C \subset [0,1]$ such that for all α, β in Ω, if $\alpha(t) = \beta(t)$ for all $t \in C$, then $\alpha \in B$ if and only if $\beta \in B$. Suppose $M_h \subseteq B$. Given $\omega \in \Omega$, Let ω' agree with ω on C and agree with h on $[0,1] \setminus C$. Then $\omega' \in M_h$, so $\omega' \in B$. It then follows that $\omega \in B$. Thus $B = \Omega$, so the outer measure $P^*(M_h) = 1$. □

Remark 14.2.2 Note that if $\omega \in M_h$, then since Lebesgue measure λ of a countable set is 0, $\omega = h$ λ-a.e. on $[0,1]$. This is true if h is nonmeasurable, or if $h \equiv 1$, or if $h \equiv -1$. Now outer measure when applied to the intersection of measurable sets with M_h is finitely additive, hence countably additive. Since the outer measure of M_h is 1, one can trivially extend P to a measure $\overline{P}$ with $\overline{P}(M_h) = 1$. Thus, no matter what h might be, one can claim that $\overline{P}$ almost every function is equal to h at λ-almost very point of $[0,1]$. This, and other highly questionable arguments have been used for decades to work around the measure theoretic problem indicated by Doob's example.

Another approach to representing a continuum of independent random variables is to consider a function $f(i,\omega)$, called a process, where i is an index from an uncountable probability space called the parameter space (the probability measure need not be uniform) and ω is taken from a second probability space called the sample space. The question then is whether it makes sense to work with the usual product of these two probability spaces. Sun has shown in Proposition 7.33 of [12] that no matter what kind of measure spaces, even Loeb measure spaces, one might take as the parameter space and sample space of a process, independence and joint measurability with respect to the classical measure-theoretic product, i.e., formed using measurable rectangles as in [15], are never compatible with each other except for a trivial case. Here is the exact statement of that proposition.

Proposition 14.2.3 (Sun) *Let $(I,\mathcal{I},\mu)$ and $(X,\mathcal{X},\nu)$ be any two probability spaces. Form the classical, complete product probability space $(I \times X, \mathcal{I} \otimes \mathcal{X}, \mu \otimes \nu)$. Let f be a function from $I \times X$ to a separable metric space. If f is jointly measurable on the product probability space, and for $\mu \otimes \mu$-almost all $(i_1,i_2) \in I \times I$, f_{i_1} and f_{i_2} are independent (call this **almost sure pairwise independence**), then, for μ-almost all $i \in I$, $f(i,\cdot)$ is a constant function on X.*

Sun notes that Proposition 14.2.3 is still valid when μ has an atom A. The almost sure pairwise independence condition implies the essential constancy of the random variables f_i for almost all $i \in A$.

In the articles cited at the beginning of this section, Sun has shown that a construction overcoming these measure-theoretic problems is obtained by forming the internal product of internal factors, and then taking not just the Loeb space of each factor, but also the Loeb space of the internal product. This yields a rich extension of the usual product σ-algebra formed from the Loeb factors, one that still has the Fubini Property equating the integral over the product space to the iterated integrals over the factor spaces forming that product. Here in more detail is that construction.

Sun's starts with internal spaces $(T, \mathcal{T}, \lambda)$ and $(\Omega, \mathcal{A}, P)$. The space T may be a hyperfinite set with λ given by uniform weights. He then forms the Loeb spaces $\left(T, L_\lambda(\mathcal{T}), \widehat{\lambda}\right)$ and $\left(\Omega, L_\mu(\mathcal{A}), \widehat{P}\right)$. He lets $\lambda \otimes P$ denote the internal product measure, while $\mathcal{T} \otimes \mathcal{A}$ denotes the internal product σ-algebra, and $L_\lambda(\mathcal{T}) \otimes L_P(\mathcal{A})$ denotes the classical product σ-algebra formed from $L_\lambda(\mathcal{T})$ and $L_P(\mathcal{A})$ as in [15]. On the other hand, forming the Loeb space from the internal product $\mathcal{T} \otimes \mathcal{A}$ produces a larger σ-algebra $L_{\lambda\otimes P}(\mathcal{T} \otimes \mathcal{A})$ on $T \times \Omega$. (See the following propositions.) Here are some properties of what we shall call the big product space $\left(T \times \Omega, L_{\lambda\otimes P}(\mathcal{T} \otimes \mathcal{A}), \widehat{\lambda \otimes P}\right)$.

Proposition 14.2.4 (Keisler-Sun [7]) *The big product space depends only on the Loeb factor spaces* $\left(T, L_\lambda(\mathcal{T}), \widehat{\lambda}\right)$ *and* $\left(\Omega, L_P(\mathcal{A}), \widehat{P}\right)$.

Proposition 14.2.5 (Anderson [1]) *If* $E \in L_\lambda(\mathcal{T}) \otimes L_P(\mathcal{A})$, *then* $E \in L_{\lambda\otimes P}(\mathcal{T} \otimes \mathcal{A})$ *and* $\widehat{\lambda \otimes P}(E) = \widehat{\lambda} \otimes \widehat{P}(E)$.

Proposition 14.2.6 (Hoover (first example), Sun [17]) *The inclusion of* $L_\lambda(\mathcal{T}) \otimes L_P(\mathcal{A})$ *in* $L_{\lambda\otimes P}(\mathcal{T} \otimes \mathcal{A})$ *is strict if and only if both* $\widehat{\lambda}$ *and* $\widehat{P}$ *have non-atomic parts.*

Proposition 14.2.7 (Keisler [6]) *A Fubini Theorem holds for the big product space.*

In his articles, cited above, Sun shows that to work with independence in a continuum setting, as has been attempted in informal mathematical applications without rigor, and for decades, one needs a rich product σ-algebra such as the σ-algebra of a big product space. Here is that general result as stated in Proposition 7.4.1 of [12] and proved in Theorem 6.2 of [17].

Proposition 14.2.8 (Sun) *Let* X *be a complete, separable and metrizable topological space. Let* $\mathcal{M}(X)$ *be the space of Borel probability measures on* X, *where* $\mathcal{M}(X)$ *is endowed with the topology of weak convergence of measures. Let* μ *be any Borel probability measure on the space* $\mathcal{M}(X)$. *If both* $\widehat{\lambda}$ *and* $\widehat{P}$ *are atomless, then there is a process* f *from* $(T\times\Omega, L_{\lambda\otimes P}(\mathcal{T}\otimes\mathcal{A}), \widehat{\lambda \otimes P})$ *to* X *such that the random variables* $f_t = f(t, \cdot)$ *are almost surely pairwise independent (i.e., for* $\widehat{\lambda \otimes \lambda}$*-almost all* $(t_1, t_2) \in T\times T$, f_{t_1} *and* f_{t_2} *are independent), and the probability measure on* $\mathcal{M}(X)$ *induced by the function* $\widehat{P} f_t^{-1}$ *from* T *to* $\mathcal{M}(X)$ *is the given measure* μ.

Remark 14.2.9 Here, for any given $t \in T$, $\widehat{P} f_t^{-1}$ is the probability measure on X induced by the random variable $f_t : \Omega \to X$. It is the measurable mapping from T to $\mathcal{M}(X)$ taking the value $\widehat{P} f_t^{-1}$ at each $t \in T$ that induces the measure μ on $\mathcal{M}(X)$.

Remark 14.2.10 A consequence of the proposition is that when the space $\left(T, L_\lambda(\mathcal{T}), \widehat{\lambda}\right)$ is a hyperfinite Loeb counting probability space and $\widehat{\mu}$ is a non-atomic Loeb probability measure, it makes sense to use the big product space as the underlying product space for an infinite number of equally weighted, independent random variables or agents.

As part of his work with large product spaces, Sun has extended the law of large numbers. The usual strong law states that if random variables X_i, $i \in \mathbb{N}$, are independent with the same distribution and finite mean m, then $\frac{1}{n}\Sigma_{i=1}^{n} X_i$ tends almost surely to the constant random variable m. That is, for almost all samples ω, the value of the sequence at ω tends to the constant m.

Theorem 14.2.11 (Sun) *Let f be a real-valued integrable process on the big product space. If the random variables $f_t := f(t, \cdot)$ are almost surely pairwise independent, then for almost all samples $\omega \in \Omega$, the mean of the sample function $f_\omega := f(\cdot, \omega)$ on the parameter space T is the mean of f viewed as a random variable on the big product space. There is no requirement of identical distributions.*

Another facet of this same work deals with independence. It is well known that for a finite collection of random variables, pairwise independence is strictly weaker than mutual independence. Sun has shown that for processes on big product spaces, pairwise independent and mutual independent coalesce, and they coalesce with other notions of independence that are distinct for a finite number of random variables. This implies asymptotic results for finite families of random variables where the families are ordered by containment and have increasing cardinality.

14.3 Purification of measure-valued maps

We now turn to a special case of the joint work of Yeneng Sun with the author in [14]. That special case generalizes the following celebrated theorem of Dvoretzky, Wald and Wolfowitz (see [3], [4], [5]).

Theorem 14.3.1 *Let A be a finite set and $\mathcal{M}(A)$ the space of probability measures on A. Let $(T, \mathcal{T})$ be a measurable space, and μ_k, $k = 1, \cdots, m$, finite, atomless signed measures on $(T, \mathcal{T})$. Given $f : T \to \mathcal{M}(A)$ so that for each $a \in A$, $f(\cdot)(\{a\})$ is $\mathcal{T}$-measurable, there is a $\mathcal{T}$-measurable map $g : T \to A$ so that for each $a \in A$, and each $k \leq m$,*

$$\int_T f(t)(\{a\}) d\mu_k(t) = \mu_k(\{t \in T : g(t) = a\}).$$

This theorem justifies the elimination, i.e., purification, of randomness in some settings. For example, in games, T represents information available to the players, and A represents the actions players may choose, given $t \in T$. Every player's objective is to maximize her own expected payoff, but that payoff depends on the actions chosen by all the players.

For each player, a mapping from T to A is called a pure strategy. A mapping from T to $\mathcal{M}(A)$ is called a mixed strategy; in this case, each player chooses a "lottery on A". A Nash equilibrium is achieved if every player is satisfied with her own choice of strategy given the choices of the other players.

In quite general settings, such an equilibrium can be achieved using mixed strategies. When results such as Theorem 14.3.1 apply, those mixed strategies can be purified yielding an equilibrium with the same expected payoffs. We refer the reader to [14] for more information on the game-theoretic consequences of Theorem 14.3.1 and its extension. We will concentrate here on the proof of that extension, a proof simpler than that of the more general result in [14].

We extend the DWW Theorem from a finite set A to a complete, separable metric space. There always exists a Borel bijection from such a space to a compact metric space. That is, if A is uncountable, then it follows from Kuratowski's theorem (see [15], p. 406) that there is a Borel bijection from A to $[0,1]$. On the other hand, for a countable set A, one can use a bijection from A to $\{0, 1, 1/2, \ldots, 1/n, \ldots\}$. Therefore, it suffices to let A be a compact metric space. For simplicity we will work with a finite set of measures, but may extend to a countable collection.

Theorem 14.3.2 *Let K be a finite set, and let A be a compact metric space. For each $k \in K$, let μ_k be a nonatomic, finite, signed Loeb measure on a Loeb measurable space $(T, \mathcal{T})$. If f is a $\mathcal{T}$-measurable mapping from T to $\mathcal{M}(A)$, then there is a $\mathcal{T}$-measurable mapping g from T to A such that for each $k \in K$ and for all Borel sets B in A, $\int_T f(t)(B)\mu_k(dt) = \mu_k\left(g^{-1}[B]\right)$. This is equivalent to the condition that for each $k \in K$ and for any bounded Borel measurable function θ on A,*

$$\int_T \int_A \theta(a) f(t)(da)\mu_k(dt) = \int_T \theta(g(t))\mu_k(dt).$$

Example 14.3.3 Sun and the author show by a counter example in [14] that the Loeb measures of the theorem cannot be replaced with measures absolutely continuous with respect to Lebesgue measure on $[0,1]$. In their example, the compact set A is the interval $[-1,1]$, and the measure-valued map is given by $f(t) := (\delta_t + \delta_{-t})/2$ for each $t \in [0,1]$. The example uses just two measures, $\mu_1 = \lambda$ on $[0,1]$, and $\mu_2 = 2t\lambda$ on $[0,1]$. If there is a function g that works, then it is not hard to see using the functions on A given by $\theta(a) = |a|$ and $\theta(a) = a^2$

that $\int_0^1 (t - |g(t)|)^2 \lambda(dt) = 0$. That is, $g(t)$ must take the value t or $-t$ λ-a.e. on $[0,1]$. It is also not hard to see that this is impossible. The moral is that a Lebesgue measurable g can't switch values on $[0,1]$ fast enough to work.

To set up the proof of the theorem, we let $\widetilde{\mu}_k$ be the signed internal measure generating μ_k for each $k \in K$. Also, we set $P_0 = \frac{1}{c}\Sigma_{k\in K} |\widetilde{\mu}_k|$, where $|\widetilde{\mu}_k|$ is the internal total variation of $\widetilde{\mu}_k$ and $c = \Sigma_{k\in K} |\widetilde{\mu}_k| (T)$. Then P_0 is an internal probability measure on T, and for each $k \in K$, $\widetilde{\mu}_k << P_0$. Let $P = \frac{1}{\mathrm{st}(c)}\Sigma_{k\in K} |\mu_k|$. The probability measure P is the Loeb measure generated by P_0, and for each $k \in K$, $\mu_k << P$. Let $\widetilde{\beta}_k$ be the internal Radon-Nikodym derivative of $\widetilde{\mu}_k$ with respect to P_0. Since $\mu_k << P$, $\widetilde{\beta}_k$ is S-integrable with respect to P_0 and $\beta_k := \mathrm{st}\,\widetilde{\beta}_k$ is the Radon-Nikodym derivative of μ_k with respect to P. We let $\mathcal{B}$ denote the collection of Borel subsets of A.

The following lemma is the only part of the proof that needs nonstandard analysis.

Lemma 14.3.4 *Let $\{\phi_i : i \in \mathbb{N}\}$ be a countable, dense (with respect to the sup-norm topology) subcollection of the continuous real-valued functions on A. Assume that there is a sequence of $\mathcal{T}$-measurable mappings $\{g_n,\ n \in \mathbb{N}\}$ from T to A such that for each $i \in \mathbb{N}$ and $k \in K$, the sequence $\int_T \phi_i(g_n(t))\beta_k(t)P(dt)$ converges; let $c_{i,k} \in \mathbb{R}$ denote the limit. Then, there is a $\mathcal{T}$-measurable mapping g from T to A such that for each $i \in \mathbb{N}$ and $k \in K$,*

$$\int_T \phi_i(g(t))\beta_k(t)P(dt) = c_{i,k}.$$

Proof. For each $n \in \mathbb{N}$, let $h_n : T \to {}^*A$ be a $\mathcal{T}_0$-measurable lifting of g_n with respect to the internal measure P_0. Then for each n and $i \in \mathbb{N}$ and each $k \in K$,

$$\int_T \phi_i(g_n(t))\beta_k(t)P(dt) \simeq \int_T {}^*\phi_i(h_n(t))\widetilde{\beta}_k(t)P_0(dt),$$

and so

$$\lim_{\substack{n\in\mathbb{N}\\ n\to\infty}} \left(\mathrm{st}\left|\int_T {}^*\phi_i(h_n(t))\widetilde{\beta}_k(t)P_0(dt) - c_{i,k}\right|\right) = 0.$$

Using $\aleph_1$-saturation, we may extend the sequence h_n to an internal sequence and choose an unlimited integer $H \in {}^*\mathbb{N}$ so that for every $i \in \mathbb{N}$ and each $k \in K$,

$$\int_T {}^*\phi_i(h_H(t))\widetilde{\beta}_k(t)P_0(dt) \simeq c_{i,k}.$$

The desired function g is obtained by setting $g(t) := \mathrm{st}(h_H(t)) \in A$ at each $t \in T$. □

Now for the proof of Theorem 14.3.2, we note first that since $\mathcal{M}(A)$ is a compact metric space under the Prohorov metric ρ (which induces the topology of weak convergence of measures), there is a sequence of simple functions $\{f_n\}_{n=1}^{\infty}$ from $(T, \mathcal{T})$ to $\mathcal{M}(A)$ such that

$$\forall t \in T, \lim_{n\to\infty} \rho(f_n(t), f(t)) = 0.$$

Assume we know that for each $n \in \mathbb{N}$, there is a $\mathcal{T}$-measurable mapping g_n from T to A such that for each $k \in K$, and any bounded Borel measurable function θ on A,

$$\int_T \int_A \theta(a) f_n(t)(da)\mu_k(dt) = \int_T \theta(g_n(t))\mu_k(dt).$$

Let $\mathcal{G} = \{\phi_i : i \in \mathbb{N}\}$ be a countable dense subset of the continuous real-valued functions on A with the sup-norm topology. For a given $\phi_i \in \mathcal{G}$ and each $t \in T$, $\int_A \phi_i(a) f_n(t)(da) \to \int_A \phi_i(a) f(t)(da)$. Moreover, each integral is bounded by the maximum value of $|\phi_i|$ on A, so by the bounded convergence theorem, for each $k \in K$,

$$\begin{aligned}\int_T \phi_i(g_n(t))\beta_k(t)P(dt) &= \int_T \phi_i(g_n(t))\mu_k(dt) \\ &= \int_T \int_A \phi_i(a) f_n(t)(da)\mu_k(dt) \\ &\to \int_T \int_A \phi_i(a) f(t)(da)\mu_k(dt)\end{aligned}$$

By the lemma, there is a $\mathcal{T}$-measurable mapping g from T to A such that for each $k \in K$,

$$\begin{aligned}\int_T \phi_i(g(t))\mu_k(dt) &= \int_T \phi_i(g(t))\beta_k(t)P(dt) \\ &= \int_T \int_A \phi_i(a) f(t)(da)\mu_k(dt).\end{aligned}$$

Since this is true for each ϕ_i in $\mathcal{G}$, it is true with ϕ_i replaced by an arbitrary bounded Borel measurable function θ on A. Therefore, without loss of generality, we may assume that $f : T \to \mathcal{M}(A)$ is simple.

Next, we fix a sequence of Borel measurable, finite partitions $\mathcal{P}^m = \{A_1^m, \ldots, A_{l_m}^m\}$ of A such that the diameter of each set in $\mathcal{P}_m$ is at most $1/2^m$, and $\mathcal{P}_{m+1}$ is a refinement of $\mathcal{P}_m$. Also for each l, $1 \le l \le l_m$, we pick a point $a_l^m \in A_l^m$.

Since f is a simple function from T to $\mathcal{M}(A)$, there is a $\mathcal{T}$-measurable partition $\{S_j\}_{j=1}^{N}$ of T such that $f \equiv \gamma_j \in \mathcal{M}(A)$ on S_j. It follows that for

each $B \in \mathcal{B}$, and each $k \in K$, $\int_T f(t)(B)\mu_k(dt) = \sum_{j=1}^N \gamma_j(B)\mu_k(S_j)$. By Lyapunov's Theorem, each S_j can be decomposed by a finite, $\mathcal{T}$-measurable partition $\{T_1^{j,m}, \ldots, T_{l_m}^{j,m}\}$ so that for every $l \leq l_m$ and $k \in K$, $\mu_k\left(T_l^{j,m}\right) = \gamma_j(A_l^m)\mu_k(S_j)$, whence

$$\begin{aligned}\int_T f(t)(A_l^m)\mu_k(dt) &= \sum_{j=1}^N \gamma_j(A_l^m)\mu_k(S_j) \\ &= \sum_{j=1}^N \mu_k\left(T_l^{j,m}\right).\end{aligned}$$

(As an alternative to the above use of the Lyapunov theorem, one can with small modifications of the proof here use the author's Lyapunov theorem [10], but then all of the simple functions must be modified on a P-null set T_0 so that for each of them, the corresponding partition sets S_j of T are internal.)

Now for each $m \geq 1$, we define a $\mathcal{T}$-measurable mapping $g_m : T \to A$ so that for each $l \leq l_m$ and each $j \leq N$, $g_m(t) \equiv a_l^m$ on $T_l^{j,m}$. For each continuous, real-valued ψ on A, for each $m \geq 1$ and each $k \in K$,

$$\begin{aligned}\sum_{l=1}^{l_m} \psi(a_l^m)\left(\int_T f(t)(A_l^m)\mu_k(dt)\right) &= \sum_{l=1}^{l_m} \psi(a_l^m)\left(\sum_{j=1}^N \gamma_j(A_l^m)\mu_k(S_j)\right) \\ &= \sum_{l=1}^{l_m}\sum_{j=1}^N \left(\psi(a_l^m)\gamma_j(A_l^m)\right)\mu_k(S_j) \\ &= \sum_{l=1}^{l_m}\sum_{j=1}^N \psi(a_l^m)\mu_k\left(T_l^{j,m}\right)\end{aligned}$$

approximates as $m \to \infty$ the integral

$$\begin{aligned}\int_T \left(\int_A \psi(a)f(t)(da)\right)\mu_k(dt) &= \sum_{l=1}^{l_m} \int_T \left(\int_{A_l^m} \psi(a)f(t)(da)\right)\mu_k(dt) \\ &= \sum_{l=1}^{l_m}\sum_{j=1}^N \int_{s_j}\left(\int_{A_l^m} \psi(a)\gamma_j(da)\right)\mu_k(dt) \\ &= \sum_{l=1}^{l_m}\sum_{j=1}^N \left(\int_{A_l^m} \psi(a)\gamma_j(da)\right)\mu_k(S_j)\end{aligned}$$

so

$$\begin{aligned}\int_T \psi(g_m(t))\beta_k(t)P(dt) &= \int_T \psi(g_m(t))\mu_k(dt) \\ &= \sum_{l=1}^{l_m}\sum_{j=1}^{N} \psi(a_l^m)\mu_k\left(T_l^{j,m}\right) \\ &\to \int_T \left(\int_A \psi(a)f(t)(da)\right)\mu_k(dt)\end{aligned}$$

It follows from the lemma that there is a $\mathcal{T}$-measurable mapping g from T to A such that for each $k \in K$ and for each continuous real-valued function θ on A from a countable dense set of such functions, and therefore for each bounded Borel measurable function θ on A,

$$\int_T \theta(g(t))\mu_k(dt) = \int_T \int_A \theta(a)f(t)(da)\mu_k(dt). \qquad \square$$

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15

More on S-measures

David A. Ross*

15.1 Introduction

In their important (but often overlooked) paper [1], C. Ward Henson and Frank Wattenberg introduced the notion of *S-measurability*, and showed that S-measurable functions are "approximately standard" (in a sense made precise in the next section).

In a recent paper ([3]), the author used this machinery to transform a well-known nonstandard plausibility argument for the Radon-Nikodým theorem into a correct and complete nonstandard proof of the theorem. The problem of finding an "essentially nonstandard" proof for Radon-Nikodým had been one of long standing. Although Luxemburg gave a nonstandard proof as long ago as 1972 ([2]), he obtained the result as a consequence of another equally-deep theorem in analysis due to Riesz. (Beate Zimmer [6] has recently proved vector-valued extensions of Radon-Nikodym starting from the same plausibility argument, though using different standardizing machinery than that in [3] or the present paper.)

In this paper I use an S-measure argument very like the one in [3] to give an intuitive nonstandard proof of the Riesz result used by Luxemburg. Along the way I give a new proof for the main technical result from [1] on S-measures, a new nonstandard proof for Egoroff's Theorem, and a nonstandard proof for the existence of the conditional expectation operator which is more elementary than that in [3].

15.2 Loeb measures and S-measures

I will assume that we work in a nonstandard model in the sense of Robinson, and that this model is as saturated as it needs to be to carry out all constructions; in particular, it is an enlargement.

*Department of Mathematics, University of Hawaii, Honolulu, HI 96822.
ross@math.hawaii.edu

Suppose X a set and $\mathcal{A}$ is an algebra on X. There are two natural algebras on ${}^*X$: $\mathcal{A}_0 = \{{}^*A : A \in \mathcal{A}\}$ (the algebra of *standard* subsets of ${}^*X$), and ${}^*\mathcal{A}$. Note that except in the simplest cases, (i) neither $\mathcal{A}_0$ nor ${}^*\mathcal{A}$ are σ-algebras; (ii) $\mathcal{A}_0$ is external; and (iii) ${}^*\mathcal{A}$ is vastly larger than $\mathcal{A}_0$.

These leads to two distinct σ-algebras:

1. $\mathcal{A}_S$ = the smallest σ-algebra containing $\mathcal{A}_0$
2. $\mathcal{A}_L$ = the smallest σ-algebra containing ${}^*\mathcal{A}$

Recall that if μ is a (finitely- or countably-additive) finite measure on $(X, \mathcal{A})$ then ${}^*\mu$ maps ${}^*\mathcal{A}$ to ${}^*[0, \infty)$, and in particular is not normally a measure, unless the range of μ is finite. However, ${}^\circ{}^*\mu$ takes its values in $[0, \infty)$, and so $({}^*X, {}^*\mathcal{A}, {}^\circ{}^*\mu)$ is an external, standard, finitely-additive finite measure space.

The following was first noticed by Loeb. It is an immediate consequence of $\aleph_1$-saturation and the Carathéodory Extension Theorem, or can be proved directly; see [4] for details.

Theorem 1 *${}^\circ{}^*\mu$ extends to a σ-additive measure μ_L on $({}^*X, \mathcal{A}_L)$. Moreover, $\forall E \in \mathcal{A}_L$*

$$\mu_L(E) = \inf\{\mu_L(A) \ : \ E \subseteq A \in {}^*\mathcal{A}\} \tag{15.2.1}$$

$$= \sup\{\mu_L(A) \ : \ A \subseteq E, \ A \in {}^*\mathcal{A}\} \tag{15.2.2}$$

Of course, the measure μ_L remains σ-additive when restricted to any σ-algebra contained in $\mathcal{A}_L$, in particular $\mathcal{A}_S$; this corresponds to replacing ${}^*\mathcal{A}$ by $\mathcal{A}_0$ as the generating algebra. However, it is not so obvious that the approximation properties (15.2.1) and (15.2.2) hold with this replacement as well. The next lemma, which asserts that they do, guarantees that a set $E \subseteq {}^*X$ has "*S-measure* 0" in the sense of Henson and Wattenberg [1] precisely when $E \subseteq D$ for some $D \in \mathcal{A}_S$ with $\mu_L(D) = 0$. For this lemma, and the duration of the paper, it will be convenient to assume that $\mathcal{A}$ is a σ-algebra.

Lemma 1 *$\forall E \in \mathcal{A}_S$,*

$$\mu_L(E) = \inf\{\mu(A) : \ E \subseteq {}^*A, \ A \in \mathcal{A}\} \tag{15.2.3}$$

$$= \sup\{\mu(A) : \ {}^*A \subseteq E, \ A \in \mathcal{A}\} \tag{15.2.4}$$

Proof. Let

$$\begin{aligned} \mathcal{B} &= \{E \in \mathcal{A}_S : \ \forall \varepsilon > 0 \ \exists A, B \in \mathcal{A} \text{ such that } {}^*A \subseteq E \subseteq {}^*B \\ &\quad \text{and } \mu(B \setminus A) < \varepsilon\} \\ &= \{E \in \mathcal{A}_S : \ \forall \varepsilon > 0 \ \exists A, B \in \mathcal{A} \text{ such that } {}^*A \subseteq E \subseteq {}^*B \\ &\quad \text{and } \mu_L({}^*B) < \mu_L(E) + \varepsilon \text{ and } \mu_L({}^*A) > \mu_L(E) - \varepsilon\}. \end{aligned}$$

Evidently ${}^*A \in \mathcal{B}$ for every $A \in \mathcal{A}$. It suffices to show that $\mathcal{B}$ is a σ-algebra, so that $\mathcal{A}_S \subseteq \mathcal{B}$. If $A \subseteq E \subseteq B$ then $B^{\complement} \subseteq E^{\complement} \subseteq A^{\complement}$ and $\mu_L(B^{\complement} \setminus A^{\complement}) = \mu_L(A \setminus B)$, so $\mathcal{B}$ is closed under complements. It remains to show that $\mathcal{B}$ is closed under countable unions. Let $E = \bigcup_n E_n$ where $E_n \in \mathcal{B}$ $(n \in \mathbb{N})$ increases to E, and let $\varepsilon > 0$. For some N, $\mu_L(E_N) > \mu_L(E) - \varepsilon/2$, and for some $A \in \mathcal{A}$, ${}^*A \subseteq E_N$ and $\mu_L({}^*A) > \mu_L(E_N) - \varepsilon/2$; it follows that ${}^*A \subseteq E$ and $\mu_L({}^*A) > \mu_L(E) - \varepsilon$. For the exterior approximation, there exists $B_n \in \mathcal{A}$ with $E_n \subseteq {}^*B_n$ and $\mu_L({}^*B_n \setminus E_n) < \varepsilon 2^{-(n+1)}$. Put $B = \bigcup_n B_n$, then $E = \bigcup_n E_n \subseteq \bigcup_n {}^*B_n \subseteq {}^*B$, and

$$\begin{aligned}\mu(B) = \mu_L({}^*B) &\leq \mu_L\Big({}^*B \setminus \bigcup_n {}^*B_n\Big) + \mu_L\Big(\Big(\bigcup_n {}^*B_n\Big) \setminus E\Big) + \mu_L(E) \\ &< 0 + \sum_n \varepsilon 2^{-(n+1)} + \mu_L(E) \leq \mu_L(E) + \varepsilon. \qquad \square\end{aligned}$$

Call a function $f : {}^*X \to \mathbb{R}$ *approximately standard* provided:

1. f is $\mathcal{A}_S$-measurable;
2. the restriction $g = f|_X$ of f to X is an $\mathcal{A}$-measurable function from X to $\mathbb{R}$; and
3. $f \approx {}^*g$ almost everywhere (with respect to μ_L).

For example, suppose $h : X \to \mathbb{R}$ is a bounded $\mathcal{A}$-measurable function. For $r \in \mathbb{R}$, ${}^\circ{}^*h^{-1}(-\infty, r] = \bigcap_{n\in\mathbb{N}} {}^*h^{-1}(-\infty, r + 1/n)$, so ${}^\circ{}^*h$ is $\mathcal{A}_S$-measurable. For $x \in X$, $h(x) = {}^*h(x) = {}^\circ{}^*h(x)$, so $h = ({}^\circ{}^*h)|_X$. It follows that ${}^\circ{}^*h$ is approximately standard.

The main result of this section is the theorem of Henson and Wattenberg, that *all* $\mathcal{A}_S$-measurable functions are approximately standard. The proof here differs from theirs, and makes it possible to prove some new results about integrability.

Denote by **APS** the set of all approximately standard functions. The following enumerates some useful properties of **APS**.

Lemma 2

1. **APS** *is closed under finite linear combinations.*
2. *If* $f, g \in$ **APS** *then* $\min\{f, g\} \in$ **APS** *and* $\max\{f, g\} \in$ **APS**.
3. *If* $f_n \in$ **APS**, $n \in \mathbb{N}$, *and* f_n *monotonically increases pointwise to* f, *then* $f \in$ **APS**.

4. *Let* $h : X \to \mathbb{R}$ *be an* $\mathcal{A}$*-measurable function, and put* $E = \{x \in {}^*X : |{}^*h(x)| < \infty\}$. *Then (i)* $E \in \mathcal{A}_S$, *(ii)* $\mu_L({}^*X \setminus E) = 0$, *and (iii)* ${}^\circ({}^*h\chi_E) \in$ **APS** *(where* χ_E *is the characteristic function of* E*).*

Proof.

1. Follows immediately from the observation that if $f, g : {}^*X \to \mathbb{R}$ and $\alpha, \beta \in \mathbb{R}$ then $(\alpha f + \beta f)|_X = (\alpha f|_X + \beta f|_X)$.

2. As in (1), note that $\max\{f, g\}|_X = \max\{f|_X, g|_X\}$ and $\min\{f, g\}|_X = \min\{f|_X, g|_X\}$.

3. Let $g_n = f_n|_X$, $A_n = \{x \in {}^*X : {}^*g_n(x) \not\approx f_n(x)\}$, and $A = \bigcup_{n\in\mathbb{N}} A_n$. Since $\mu_L(A_n) = 0$ for each n, $\mu_L(A) = 0$. Let $g = f|_X$ and note $g = \sup_n g_n$. Fix a standard $\varepsilon > 0$, and for $n \in \mathbb{N}$ let $E_n = \{x \in X : g_n(x) > g(x) - \varepsilon\}$. Since $X = \bigcup_{n\in\mathbb{N}} E_n$, $\mu_L({}^*E_n) = \mu(E_n)$ increases to $\mu(X) = \mu_L({}^*X)$, so $\mu_L(A \cup ({}^*X \setminus \bigcup_{n\in\mathbb{N}} {}^*E_n)) = 0$. Suppose $x \in ({}^*X \setminus A) \cap \bigcup_{n\in\mathbb{N}} {}^*E_n$. For some $n, m \in \mathbb{N}$, $x \in {}^*E_n$ and $f_m(x) > f(x) - \varepsilon$. It follows that ${}^*g(x) < {}^*g_n(x) + \varepsilon \approx f_n(x) + \varepsilon \le f(x) + \varepsilon < f_m(x) + 2\varepsilon \approx {}^*g_m(x) + 2\varepsilon \le {}^*g(x) + 2\varepsilon$. Since ε was arbitrary, ${}^*g(x) \approx f(x)$.

4. Put $E_n = \{x \in X : |h(x)| < n\}$, so $E = \bigcup_{n\in\mathbb{N}} {}^*E_n \in \mathcal{A}_S$. $X = \bigcup_{n\in\mathbb{N}} E_n$, so $\mu_L({}^*E_n) = \mu(E_n)$ increases to $\mu(X) = \mu_L({}^*X)$, and $\mu_L({}^*X \setminus E) = 0$. The proof of (iii) is now like the example preceding the statement of this lemma. □

Theorem 2 *Every* $\mathcal{A}_S$*-measurable function is approximately standard.*

Proof. Note that if $A \in \mathcal{A}$ then $\chi_{{}^*A} = {}^*\chi_A = {}^\circ{}^*\chi_A$ is in **APS** by Lemma 2 (4). By (1) and (3) of Lemma 2 and a suitable version of the Monotone Class Theorem (for example, Theorem 3.14 of [5]), every bounded $\mathcal{A}_S$-measurable function is in **APS**. If $f \ge 0$ is $\mathcal{A}_S$-measurable then $f = \sup_n \max\{f, n\}$, so is in **APS** by (2) and (3) of Lemma 2. Any general $\mathcal{A}_S$-measurable f can be written $f = \max\{f, 0\} - \max\{-f, 0\}$ so is in APS by (1) and (2) of Lemma 2. □

For example, suppose $E \in \mathcal{A}_S$; denote by $S(E)$ the set $X \cap E$ of standard elements of E. If f is the characteristic function of E then $g = f|_X$ is the characteristic function of $S(E)$. By the theorem, $S(E)$ is $\mathcal{A}$-measurable and $\mu(S(E)) = \mu_L({}^*S(E)) = \int {}^*g \, d\mu_L = \int f \, d\mu_L = \mu_L(E)$.

Corollary 1 *Let* $f : {}^*X \to \mathbb{R}$ *be approximately standard, and* $p > 0$. *Put* $g = f|_X$. *Then* $f \in \mathcal{L}^p(\mu_L)$ *if and only if* $g \in \mathcal{L}^p(\mu)$, *in which case* $\int f^p d\mu_L = \int g^p d\mu$ *and* $\|f\|_p = \|g\|_p$.

Proof. Without loss of generality, $f \geq 0$. Let $s_n : {}^*X \to \mathbb{R}$, $n \in \mathbb{N}$, be simple functions increasing pointwise to f^p. Put $t_n = s_n|_X$, then t_n is a sequence of simple functions on X increasing pointwise to g^p. Moreover, if $s_n = \sum_{k=1}^m \alpha_k \chi_{E_k}$ then $t_n = \sum_{k=1}^m \alpha_k \chi_{S(E_k)}$, and $\int s_n d\mu_L = \sum_{k=1}^m \alpha_k \mu_L(E_k) = \sum_{k=1}^m \alpha_k \mu(S(E_k)) = \int t_n d\mu$. Let $n \to \infty$, and it follows that $\int f^p d\mu_L$ and $\int g^p d\mu$ are either both infinite or are both finite and equal. □

The next result lets us extend the notion of approximate standardness to completion-measurable functions.

Lemma 3 *Let $f : {}^*X \to \mathbb{R}$; the following are equivalent:*

(i) f is measurable with respect to the μ_L-completion of $\mathcal{A}_S$;

(ii) $f|_X$ is measurable with respect to the μ-completion of $\mathcal{A}$, and $f \approx {}^*(f|_X)$ off a μ_L-nullset in $\mathcal{A}_S$.

Proof. Recall (without proof) the standard fact that a function is completion measurable if and only if it agrees with a measurable function off a nullset.

$(i \Rightarrow ii)$ There is an $\mathcal{A}_S$-measurable f' and a μ_L-nullset $A \in \mathcal{A}_S$ such that $f = f'$ off A. Let $g = f|_X$ and $g' = f'|_X$, then g' is $\mathcal{A}$-measurable, ${}^*g' \approx f'$ off a μ_L-nullset $B \in \mathcal{A}_S$, and $\{x \in X : g(x) \neq g'(x)\} \subseteq S(A)$. Since $\mu_L(A) = 0$, $\mu(S(A)) = 0$ by the example above, and so g' is measurable with respect to the μ-completion of $\mathcal{A}$. Note also that ${}^*\mu(S(A)) = 0$, so $\mu_L(A \cup B \cup {}^*S(A)) = 0$. For $x \in {}^*X, x \notin A \cup B \cup {}^*S(A), f(x) = f'(x) \approx g'(x) = g(x)$, proving (ii).

$(ii \Rightarrow i)$ Let $A \in \mathcal{A}_S$ be a μ_L-nullset with $f \approx {}^*(f|_X)$ off A. Let $g = f|_X$, g' $\mathcal{A}$-measurable such that $g = g'$ off a μ-nullset $B \in \mathcal{A}$, $E = \{x \in {}^*X : |{}^*g'(x)| < \infty\}$, and $f' = {}^\circ({}^*g'\chi_E)$. By Lemma 2, f' is in **APS**. For $x \notin E \cup {}^*B \cup A$, $f'(x) \approx {}^*g'(x) = {}^*g(x) \approx f(x)$, so in fact $f'(x) = f(x)$. Since f' is $\mathcal{A}_S$-measurable, f is completion measurable. □

15.3 Egoroff's Theorem

The most striking application [1] made of the S-measure construction was a proof of Egoroff's Theorem. Their proof relied on a nonstandard condition shown by Robinson to be equivalent to approximate uniform convergence.

Here I use the machinery above to give an alternate proof of the theorem not depending on Robinson's condition.

Theorem 3 *Let $(X, \mathcal{A}, \mu)$ be a finite measure space, $f, f_n : X \to \mathbb{R}$ measurable, $f_n \to f$ a.e. Then*

$$\forall \varepsilon > 0 \; \exists A \in \mathcal{A} \; \mu(A) > \mu(X) - \varepsilon \; \& \; f_n \to f \text{ uniformly on } A.$$

Proof. For $n \in \mathbb{N}$ put $g_n = \inf_{m>n} f_m$ and $h_n = \sup_{m>n} f_m$. Note that for almost all x, $h_n(x) - g_n(x)$ is nonnegative and nondecreasing in n with $\lim_{n\to\infty} h_n(x) - g_n(x) = 0$. Put $\hat{g}_n = {}^\circ({}^*g_n)$, $\hat{h}_n = {}^\circ({}^*h_n)$. Let $E = \{x \in {}^*X : \lim_{n\to\infty} \hat{h}_n(x) - \hat{g}_n(x) \neq 0\}$. Note $E \in \mathcal{A}_S$, so $\mu_L(E) = \mu(S(E)) = \mu(\emptyset) = 0$. It follows that for some $A \in \mathcal{A}$, ${}^*A \subseteq E^{\complement}$ and $\mu(A) > \mu(X) - \varepsilon$. It remains to show that f_n converges to f uniformly on A; equivalently, that $h_n - g_n$ converges to 0 uniformly on A.

Fix $\delta > 0$. For $x \in {}^*A$ there is a $k \in \mathbb{N}$ such that $\hat{h}_k(x) - \hat{g}_k(x) < \delta$. Note ${}^*h_k(x) - {}^*g_k(x) < \delta$ as well. Let $\phi(x)$ be the least k such that ${}^*h_k(x) - {}^*g_k(x) < \delta$. The function $\phi : {}^*A \to \mathbb{N}$ is internal and finite-valued, so has a standard upper bound $N \in \mathbb{N}$. Then ${}^*h_k - {}^*g_k < \delta$ on ${}^*A$ for every $k \geq N$, so $h_k - g_k < \delta$ on A; this completes the proof. □

15.4 A Theorem of Riesz

Theorem 4 *Let T be a continuous linear real functional on $\mathcal{L}^2(X, \mathcal{A}, \mu)$; then there is a g such that for every $f \in \mathcal{L}^2(X, \mathcal{A}, \mu)$, $T(f) = \int fg\, d\mu$.*

After the proof we show that the function g is actually in $\mathcal{L}^2(X, \mathcal{A}, \mu)$.

Proof. By saturation let $\widehat{\mathcal{A}}$ be a *-finite algebra with $\mathcal{A}_0 \subseteq \widehat{\mathcal{A}} \subseteq {}^*\mathcal{A}$. There is an internal *-partition Π of ${}^*X$ which corresponds to $\widehat{\mathcal{A}}$ in the sense that the latter is the internal closure of the former under hyperfinite unions.

Now, define

$$\hat{\gamma}(x) := \begin{cases} \dfrac{{}^*T(\chi_p)}{{}^*\mu(p)}, & x \in p \in \Pi \text{ and } {}^*\mu(p) > 0; \\ 0, & \text{otherwise.} \end{cases}$$

Let $a : \Pi \to {}^*X$ be an internal choice function, that is, $a_p \in p$ for $p \in \Pi$. For any bounded, measurable $f : X \to \mathbb{R}$,

$$T(f) = \sum_p {}^*T({}^*f\chi_p) \tag{15.4.1}$$

$$\approx \sum_p {}^*f(a_p)\,{}^*T(\chi_p) \tag{15.4.2}$$

$$= \sum_p {}^*f(a_p)\hat{\gamma}(a_p)\,{}^*\mu(p) \tag{15.4.3}$$

$$= \sum_p \int_p {}^*f(a_p)\hat{\gamma}(a_p)\, {}^*d\mu \tag{15.4.4}$$

$$= \sum_p \int_p {}^*f(a_p)\hat{\gamma}(x)\, {}^*d\mu \tag{15.4.5}$$

$$\approx \sum_p \int_p {}^*f(x)\hat{\gamma}(x)\, {}^*d\mu \tag{15.4.6}$$

$$= \int {}^*f\hat{\gamma}\, {}^*d\mu \tag{15.4.7}$$

The steps from (15.4.1) to (15.4.2) and (15.4.5) to (15.4.6) follow from boundedness of f and the definition of $\widehat{\mathcal{A}}$. For (15.4.2) to (15.4.3) note that by continuity of T, if $p \in \Pi$ and ${}^*\mu(p) = 0$ then ${}^*T(\chi_p) = 0$.

Moreover, if ${}^*f$ is replaced in the above by any internal function $h : {}^*X \to {}^*\mathbb{R}$ with the property that h is constant on each $p \in \Pi$, then all but the first equality in the above still holds, and we obtain ${}^*T(h) = \int h\hat{\gamma}\, d^*\mu$. In particular, if $h = \chi_A$ for some $A \in \widehat{\mathcal{A}}$ then ${}^*T(\chi_A) = \int_A \hat{\gamma}\, d^*\mu$.

The proof now proceeds as follows: (i) Show $\hat{\gamma}$ is finite almost everywhere (and therefore $\gamma =^\circ \hat{\gamma}$ exists almost everywhere). (ii) Show that $\hat{\gamma}$ is S-integrable (and therefore $\gamma =^\circ \hat{\gamma}$ is integrable). (iii) Put $G = \mathbb{E}[\gamma|\mathcal{A}_S]$ (the conditional expectation of γ). (iv) Let g be the restriction of G to X; by Theorem 2 g is $\mathcal{A}$-measurable. (v) Show that g works.

For (i), write $[\hat{\gamma} < n] = \{x \in {}^*X : \hat{\gamma}(x) < n\}$ $(n \in {}^*\mathbb{N})$, and $[\hat{\gamma} < \infty] = \bigcup_{n\in\mathbb{N}}[\hat{\gamma} < n]$. Suppose (for a contradiction) that $\mu_L([\hat{\gamma} < \infty]) < 1 - r$ for some standard $r > 0$. Then $\mu([\hat{\gamma} < n]) < 1 - r$ for each standard $n \in \mathbb{N}$, so $\mu([\hat{\gamma} < H]) < 1 - r$ for some infinite H. But then $\infty > T(1) \approx {}^*\int 1\hat{\gamma} d^*\mu$ (by the note above) $\geq {}^*\int_{[\hat{\gamma}\geq H]} \hat{\gamma} d^*\mu \geq H {}^*\mu([\hat{\gamma} \geq H]) > rH$, which is infinite, a contradiction.

For (ii), the reader is referred to [4] for a discussion of S-integrability. In particular, to show that $\hat{\gamma}$ is S-integrable, it suffices to show that $\forall H$ infinite, ${}^*\int_{[\hat{\gamma}>H]} \hat{\gamma} d^*\mu \approx 0$. (Indeed, this can be adopted as the *definition* of S-integrability.)

So, fix such an H. Note $[\hat{\gamma} > H] \in \widehat{\mathcal{A}}$; it follows that ${}^*\int_{[\hat{\gamma}>H]} \hat{\gamma} d^*\mu = {}^*T(\chi_{[\hat{\gamma}>H]})$. Suppose (for a contradiction) that ${}^*T(\chi_{[\hat{\gamma}>H]}) > r > 0$, r standard. By (i), ${}^*\mu([\hat{\gamma} > H]) \approx 0$. It follows from transfer that for every standard $n \in \mathbb{N}$ there is a $B_n \in \mathcal{A}$ with $T(\chi_{B_n}) > r$ and $\mu(B_n) < 1/2^{-n}$. Put $B^n := \bigcup_{m\geq n} B_m$. Then $\mu(B^n) < \sum_{m\geq n} 2^{-m} = 2^{-n+1}$, but $T(\chi_{B^n}) > r$. As $n \to \infty$, $\chi_{B^n} \to 0$ in $\mathcal{L}^2$ but $T(\chi_{B^n}) > r$, contradicting continuity of T.

It follows from S-integrability of $\hat{\gamma}$ that $\gamma =^\circ \hat{\gamma}$ is integrable, so its conditional expectation with respect to $\mathcal{A}_S$, $\mathbb{E}[\gamma|\mathcal{A}_S]$, exists; put $G = \mathbb{E}[\gamma|\mathcal{A}_S]$. (Conditional expectation is defined in the next section, where a simple nonstandard proof for its existence is given. The reader is referred to chapter 9 of [5] for properties of the conditional expectation.)

Finally, let $g = G|_X$, which (as noted above) is $\mathcal{A}$-measurable by Theorem 2. It remains to show that g satisfies the conclusion of Theorem 4.

Let $f \in \mathcal{L}^2(X, \mathcal{A}, \mu)$, and suppose first that f is bounded. This ensures that $({}^*f)\hat{\gamma}$ is S-integrable, and $({}^\circ{}^*f)\gamma \in \mathcal{L}^1({}^*X, \mathcal{A}_L, \mu_L)$.

Then

$$T(f) \approx \int {}^*f\hat{\gamma}\, d^*\mu \tag{15.4.8}$$

$$\approx \int ({}^\circ{}^*f)\gamma\, d\mu_L \tag{15.4.9}$$

$$= \int \mathbb{E}[({}^\circ{}^*f)\gamma|\mathcal{A}_S]\, d\mu_L \tag{15.4.10}$$

$$= \int ({}^\circ{}^*f)\mathbb{E}[\gamma|\mathcal{A}_S]\, d\mu_L \tag{15.4.11}$$

$$= \int ({}^\circ{}^*f)G\, d\mu_L \tag{15.4.12}$$

$$= \int fg\, d\mu \tag{15.4.13}$$

The step from (15.4.8) to (15.4.9) is by S-integrability, (15.4.10) to (15.4.11) is a standard property of conditional expectation (using the fact that ${}^\circ{}^*f$ is $\mathcal{A}_S$-measurable), and (15.4.12) to (15.4.13) is Corollary 1 with $p = 1$.

For unbounded $f \in \mathcal{L}^2(X, \mathcal{A}, \mu)$ and $n \in \mathbb{N}$, let $f_n = \max\{-n, \min\{f, n\}\}$. By the result above, $T(f_n) = \int f_n g\, d\mu$. By continuity of T and Lebesgue's Dominated Convergence Theorem ([5], Theorem 5.9), $T(f) = \int fg\, d\mu$. □

The Riesz Theorem is usually stated in the following nominally stronger form.

Corollary 2 *Let T be a continuous linear real functional on $\mathcal{L}^2(X, \mathcal{A}, \mu)$; then there is a $g \in \mathcal{L}^2(X, \mathcal{A}, \mu)$ such that for every $f \in \mathcal{L}^2(X, \mathcal{A}, \mu)$, $T(f) = \int fg\, d\mu$.*

Proof. It suffices to show that any g satisfying the conclusion of Theorem 4 is already in $\mathcal{L}^2(X, \mathcal{A}, \mu)$.

Since T is continuous, it is bounded, so there is a constant C such that for any $f \in \mathcal{L}^2(X, \mathcal{A}, \mu)$, $T(f) \leq C\|f\|_2$.

Put $g_n = \min\{g, n\}$. Then $\|g_n\|_2^2 = \int g_n^2\, d\mu \leq \int g_n g\, d\mu = T(g_n) \leq C\|g_n\|_2$. Divide both sides of this inequality by $\|g_n\|_2$ and square, obtain $\int g_n^2\, d\mu = \|g_n\|_2^2 \leq C^2$. The result now follows by Fatou's Lemma ([5], Lemma 5.4). □

15.4.1 Conditional expectation

Theorem 5 *Suppose $(X, \mathcal{A}, \mu)$ is a probability measure, that $\mathcal{B} \subseteq \mathcal{A}$ is another σ-algebra, and that $f \in \mathcal{L}^1(X, \mathcal{A}, \mu)$. There is a $\mathcal{B}$-measurable function $g : X \to \mathbb{R}$ such that for any $B \in \mathcal{B}$, $\int_B f\, d\mu = \int_B g\, d\mu$.*

The function g is called the *conditional expectation* of f on $\mathcal{B}$, and denoted by $\mathbb{E}[f|\mathcal{B}]$.

To avoid circularity, it is necessary to show that $\mathbb{E}[f|\mathcal{B}]$ exists without use of the Riesz Theorem (or related results, such as the Radon-Nikodým Theorem). Such a proof appears in [3]. This section presents a modification of that proof which is even more elementary, in that it does not require the Hahn decomposition.

Without loss of generality $f \geq 0$. Put $\mathcal{G} = \{g \in \mathcal{L}^1(X, \mathcal{B}, \mu) : \forall B \in \mathcal{B}, \int_B g\, d\mu \leq \int_B f\, d\mu\}$. Note if $g_1, g_2 \in \mathcal{G}$ then $\max\{g_1, g_2, 0\} \in \mathcal{G}$. Let $r = \sup\{\int_X g d\mu : g \in \mathcal{G}\}$, and for $n \in \mathbb{N}$ let $g_n \in \mathcal{G}$ with $\int_X g_n d\mu > r - 1/2^n$; we may assume that $0 \leq g_1 \leq g_2 \leq \cdots$. Put $g = \sup_n g_n$. Note that if $B \in \mathcal{B}$, $\int_B g\, d\mu = \sup_n \int_B g_n\, d\mu \leq \int_B f\, d\mu$, so $g \in \mathcal{G}$. Moreover, $\int_X g\, d\mu = r$. It remains to show that for any $B \in \mathcal{B}$, $\int_B f\, d\mu = \int_B g\, d\mu$.

Suppose not; then there is a $B \in \mathcal{B}$ and $\varepsilon > 0$ with $\int_B f\, d\mu - \int_B g\, d\mu = \varepsilon$.

For some $\delta > 0$, $\varepsilon' = \int_B (f - \hat{g}) d\mu > 0$ where $\hat{g} = g + \delta\chi_B$. $\hat{g}$ *almost* witnesses a contradiction; the perturbation by δ needs to be localized to a slightly smaller set.

As in the proof of Theorem 4 let $\widehat{\mathcal{B}}$ be a *-finite algebra with $\{{}^*B : B \in \mathcal{B}\} \subseteq \widehat{\mathcal{B}} \subseteq {}^*\mathcal{B}$, and let Π be the internal hyperfinite *-partition of ${}^*X$ corresponding to $\widehat{\mathcal{B}}$. Let $B^+ = \bigcup\{b \in \Pi : \int_b {}^*(f - \hat{g}) d^*\mu > 0\}$. Put $s = {}^\circ \int_{B^+} {}^*(f - \hat{g}) d^*\mu$.

Note that for every $C \in \mathcal{B}$, if $C \subseteq B$ then $\int_C (f - \hat{g}) d\mu = \int_{{}^*C} {}^*(f - \hat{g}) d^*\mu \leq s$; in particular, $\varepsilon' \leq s$.

For $n \in \mathbb{N}$ consider the statement,

$$\exists B_n \in \mathcal{B},\ B_n \subseteq B,\ \int_{B_n} (f - \hat{g}) d\mu > s - 2^{-n}$$

As this holds in the nonstandard model (with B^+ for B_n), it holds by transfer in the standard model. For $m \in \mathbb{N}$ put $B^m = \bigcup_{n>m} B_n$, and put $B_\infty = \bigcap_m B^m$. Note $B_\infty \in \mathcal{B}$ and $B_\infty \subseteq B$, so $\int_{B_\infty} (f - \hat{g}) d\mu \leq s$. On the other hand, since always $\int_{B_n} (f - \hat{g}) d\mu > s - 2^{-n}$, $\int_{B^m} (f - \hat{g}) d\mu > s - \sum_{n>m} 2^{-n} = s - 2^{-m}$, therefore $\int_{B_\infty} (f - \hat{g}) d\mu \geq s$.

Put $g' = g + \delta\chi_{B_\infty}$. If $C \in \mathcal{B}$,

$$\int_C (f - g')\, d\mu = \int_{C \setminus B_\infty} (f - g)\, d\mu + \int_{C \cap B_\infty} (f - \hat{g})\, d\mu\,.$$

The first term in this sum is nonnegative since $g \in \mathcal{G}$. The second is nonnegative since otherwise $\int_{B_\infty \setminus C}(f - \hat{g})d\mu > s$. It follows that $\int_C (f - g')d\mu \geq 0$, so $g' \in \mathcal{G}$. Since $s > 0$, $\mu(B_\infty) > 0$, so $\int g'\, d\mu = \int g\, d\mu + \delta\mu(B_\infty) > r$, a contradiction.

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16

A Radon-Nikodým theorem for a vector-valued reference measure

G. Beate Zimmer*

Abstract

The conclusion of a Radon-Nikodým theorem is that a measure μ can be represented as an integral with respect to a reference measure such that for all measurable sets A, $\mu(A) = \int_A f_\mu(x)\, d\lambda$ with a (Bochner or Lebesgue) integrable derivative or density f_μ. The measure λ is usually a countably additive σ-finite measure on the given measure space and the measure μ is absolutely continuous with respect to λ. Different theorems have different range spaces for μ, which could be the real numbers, or Banach spaces with or without the Radon-Nikodým property. In this paper we generalize to derivatives of vector valued measures with respect a vector-valued reference measure. We present a Radon-Nikodým theorem for vector measures of bounded variation that are absolutely continuous with respect to another vector measure of bounded variation. While it is easy in settings such as $\mu << \lambda$, where λ is Lebesgue measure on the interval $[0,1]$ and μ is vector-valued to write down a nonstandard Radon-Nikodým derivative of the form $\varphi : {}^*[0,1] \to \text{fin}({}^*E)$ by $\varphi_\mu(x) = \sum_{i=1}^{H} \frac{{}^*\mu(A_i)}{{}^*\lambda(A_i)}\, 1_{A_i}(x)$, a vector valued reference measure does not allow this approach, as the quotient of two vectors in different Banach spaces is undefined. Furthermore, generalizing to a vector valued control measure necessitates the use of a generalization of the Bartle integral, a bilinear vector integral.

16.1 Introduction and notation

For nonstandard notions and notations not defined here we refer for example to the book by Albeverio, Fenstad, Hoegh-Krøhn and Lindstrøm [1] and the survey on nonstandard hulls by Henson and Moore [6] or the introduction

*Department of Mathematics and Statistics, Texas A&M University - Corpus Christi, Corpus Christi, TX 78412.

to nonstandard analysis in [7]. Plenty of information about vector measures can be found in [5].

Throughout, Ω is a set and Σ is a σ-algebra of subsets of Ω. E, F and G denote Banach spaces. $\mu : \Sigma \to E$ and $\nu : \Sigma \to F$ are countably additive vector measures of bounded variation. The variation of ν is defined as $|\nu|(A) = \sup \sum_{\pi \in \Pi} \|\nu(A_\pi)\|$ where $A \in \Sigma$ and Π is a finite partition of A into sets in Σ. By Proposition I.1.9 in [5] the variation $|\nu|$ of a countably additive measure ν is also a countably additive measure on Σ.

We think of the nonstandard model in terms of superstructures $V(X)$ and $V(^*X)$ connected by the monomorphism $* : V(X) \to V(^*X)$ and call an element $b \in V(^*X)$ internal, if it is an element of a standard entity, i.e. if there is an $a \in V(X)$ with $b \in {}^*a$. We assume that the nonstandard model is at least $\aleph$-saturated, where $\aleph$ is an uncountable cardinal number such that the cardinality of the σ-algebra Σ of $|\nu|$-measurable subsets of Ω is less than $\aleph$.

Let $(^*\Omega, L_{|\nu|}(^*\Sigma), \widehat{|\nu|})$ denote the Loeb space constructed from the nonstandard extension of the measure space $(\Omega, \Sigma, |\nu|)$. This measure space is obtained by extending the measure $^\circ{}^*|\nu|$ from $^*\Sigma$ to the σ-algebra generated by $^*\Sigma$. The completion of this σ-algebra is denoted by $L_{|\nu|}(^*\Sigma)$. The Loeb measure $\widehat{|\nu|}$ is a standard countably additive measure.

The functions we work with take their values in a Banach space E or its nonstandard hull $\widehat{E}$. The nonstandard hull of a Banach space E is defined as $\widehat{E} = \text{fin}(^*E)/\approx$, the quotient of the elements of bounded norm by elements of infinitesimal norm. The nonstandard hull is a standard Banach space and contains E as a subspace. We denote the quotient map from $\text{fin}(^*E)$ onto $\widehat{E}$ by π or π_E. Whenever we encounter products of Banach spaces, we equip them with the ℓ_∞ norm: $\|e \times f\| = \max(\|e\|, \|f\|)$.

By a result of Loeb in [8], there exists an internal *-finite partition of $^*\Omega$ consisting of sets $A_1, \ldots, A_H \in {}^*\Sigma$ $(H \in {}^*\mathbb{N})$ such that the partition is finer than the image under $*$ of any finite partition of Ω into sets in Σ. The proof uses a concurrent relation argument. From this fine partition we discard all partition sets of $^*|\nu|$-measure zero. The remaining sets still form an internal collection, which we also denote by $A_1, \ldots, A_H$.

16.2 The existing literature

Ross gives a nice detailed nonstandard proof of the Radon-Nikodým theorem for two real-valued σ-finite measures $\mu << \nu$ on a measurable space in [9]. Previously we have studied nonstandard Radon-Nikodým derivatives of vector measures $\mu : \Sigma \to E$ with respect to Lebesgue measure on $[0, 1]$. We have shown that through nonstandard analysis a unifying approach to

Radon-Nikodým derivatives independent of the Radon-Nikodým property or lack thereof of E can be found. The generalized derivatives we constructed are not necessarily essentially separably valued and therefore not always Bochner integrable. However, a generalization of the Bochner integral in [10] for functions with values in the nonstandard hull of a Banach space allows one to integrate the generalized derivatives. In [11] with methods similar to those used by Ross [9] or Bliedtner and Loeb in [3], and with the use of local reflexivity we "standardize" the generalized derivatives from maps ${}^*\Omega \to \widehat{E}$ to maps $\Omega \to E''$ with values in the second dual of the original Banach space E.

The vector measures book [5] by Diestel and Uhl is still considered as the authoritative work on Radon-Nikodým theorems. The standard literature yields very little on vector-vector derivatives; only Bogdan [4] together with his student Kritt has made an initial foray into this field. In their article they assume that they have two vector measures $\mu : \Sigma \to E$ and $\nu : \Sigma \to F$ with $\mu << |\nu|$, and they assume that the range space of μ has the Radon-Nikodým property and the range space of ν is uniformly convex. They then define a derivative and integral in the form $\mu(A) = \int_A \frac{d\mu}{d|\nu|}(\omega) \cdot \frac{d|\nu|}{d\nu}(\omega)\, d\nu$, where $\frac{d|\nu|}{d\nu}$ is a function from Ω into F', the dual space of the range of ν. The integrand is then of the form $\frac{d\mu}{d\nu}(\omega) = e \cdot f'$, a function from Ω into $E \times F'$. A trilinear product on $E \times F \times F'$ with values in E can be defined as $e \cdot f' \cdot f = f'(f) \cdot e$. This is needed to integrate simple functions of the form $\sum_{i=1}^n e_i \cdot f_i' \cdot 1_{A_i} : \Omega \to E \times F'$ with respect to the F-valued measure ν as follows:

$$\int_\Omega \sum_{i=1}^n e_i \cdot f_i' \cdot 1_{A_i}\, d\nu = \sum_{i=1}^n e_i \cdot f_i' \cdot \nu(A_i) = \sum_{i=1}^n e_i \cdot f_i'(\nu(A_i)) \in E.$$

The main difficulty is the definition of $\frac{d|\nu|}{d\nu}$, the derivative of a real-valued measure with respect to a vector-valued measure, the opposite setting from the ordinary Radon-Nikodým theorems. Bogdan uses his assumption that F is uniformly convex to find this derivative. Since $\nu << |\nu|$ and since uniform convex spaces are reflexive and hence have the Radon-Nikodým property, there is a Bochner integrable function $\frac{d\nu}{d|\nu|} : \Omega \to F$ such that for all measurable sets A, the vector measure can be written as a Bochner integral $\nu(A) = \int_A \frac{d\nu}{d|\nu|}\, d|\nu|$. A standard theorem (e.g. Theorem II.2.4 in [5]) about vector measures states that the norm of the Radon-Nikodým derivative is a Radon-Nikodým derivative for the total variation of the vector measure, i.e. for all $A \in \Sigma$

$$|\nu|(A) = \int_A \left\| \frac{d\nu}{d|\nu|} \right\| d|\nu|.$$

This implies that $|\nu|$-almost everywhere on Ω, $\left\| \frac{d\nu}{d|\nu|} \right\| = 1$. For uniformly convex spaces F there is a continuous map $g : S_F \to S_{F'}$, where S_X denotes

the unit sphere of X, such that $\langle f, g(f)\rangle = 1$ for all $f \in S_F$. In this case, the composition $g \circ \frac{d\nu}{d|\nu|}$ is a map from Ω into the unit sphere of F'. The derivative of μ with respect to ν is defined as $\frac{d\mu}{d\nu} = \frac{d\mu}{d|\nu|} \cdot \left(g \circ \frac{d\nu}{d|\nu|}\right) : \Omega \to E \times F'$, or, if one regards the product $e \times f'$ as a rank one continuous linear operator from F to E, the derivative can be considered a map: $\frac{d\mu}{d\nu} : \Omega \to L(F, E)$. Since $\frac{d\mu}{d|\nu|}$ and $\frac{d\nu}{d|\nu|}$ are both Bochner integrable and g is continuous and $g \circ \frac{d\nu}{d|\nu|}$ is almost everywhere of norm one, it is easy to show that the result is integrable with respect to the vector measure ν with an integral that uses approximation by simple functions.

16.3 The nonstandard approach

With nonstandard analysis we can significantly weaken the assumptions on the Banach spaces E and F. We use Bogdan's idea of writing the derivative as $\frac{d\mu}{d|\nu|} \cdot \frac{d|\nu|}{d\nu} = \frac{d\mu}{d|\nu|} \cdot \left(g \circ \frac{d\nu}{d|\nu|}\right)$.

The derivatives $\frac{d\mu}{d|\nu|}$ and $\frac{d\nu}{d|\nu|}$ are derivatives of a vector measure with respect to a real valued measure. They can be found and integrated with the methods described in [10], [11] or [12].

In [10] we defined a Banach space $M(\widehat{|\nu|}, \widehat{E})$ of extended integrable functions as the set of equivalence classes under equality $\widehat{|\nu|}$-almost everywhere of functions $f : {}^*\Omega \to \widehat{E}$ for which there is an internal, $*$-simple, S-integrable $\varphi : {}^*\Omega \to {}^*E$ such that $\pi_E \circ \varphi = f$ $\widehat{|\nu|}$-almost everywhere on ${}^*\Omega$. Such a φ is called a lifting of f. On $M(\widehat{|\nu|}, \widehat{E})$ we defined an integral by setting $\int_A f \, d\widehat{|\nu|} = \pi\left(\int_A \varphi \, d^*|\nu|\right)$, for all $A \in {}^*\Sigma$, where the integral of the internal simple function φ is defined in the obvious way. This integral can be extended to sets in the Loeb σ-algebra and generalizes the Bochner integral in the sense that $M(\widehat{|\nu|}, \widehat{E})$ contains $L_1(\widehat{|\nu|}, \widehat{E})$ and the integrals agree on that subspace. However, $M(\widehat{|\nu|}, \widehat{E})$ also contains functions which fail to be essentially separably valued and hence fail to be measurable. Lemma 1 in [11] translates to this situation as:

Lemma 16.3.1 *Let E be any Banach space and let $\mu : \Sigma \to E$ be a $|\nu|$-absolutely continuous countably additive vector measure of bounded variation. Then the internal $*$-simple function $\varphi_\mu : {}^*\Omega \to {}^*E$ defined by*

$$\varphi_\mu(\omega) = \sum_{i=1}^{H} \frac{{}^*\mu(A_i)}{{}^*|\nu|(A_i)} \, 1_{A_i}(\omega)$$

is S-integrable, where $A_1, \ldots, A_H$ is the fine partition of ${}^*\Omega$ introduced above.

S-integrability implies that the function is $\widehat{|\nu|}$-almost everywhere $\text{fin}({}^*E)$-valued. This allows us compose an S-integrable internal function with the quotient map $\pi_E : \text{fin}({}^*E) \to \widehat{E}$ to make it a nonstandard hull valued function defined on the Loeb space $\left({}^*\Omega, L_{|\nu|}({}^*\Sigma), \widehat{|\nu|}\right)$. Define $f_\mu \in M(\widehat{|\nu|}, \widehat{E})$ by

$$f_\mu = \pi_E \circ \varphi_\mu.$$

Theorem 4 in [11] asserts that if the vector measure μ has a Bochner integrable Radon-Nikodým derivative, then the generalized derivative "is" the Radon-Nikodým derivative in the following sense:

Theorem 16.3.2 *Let $\mu : \Sigma \to E$ be a countably additive vector measure with a Bochner integrable Radon-Nikodým derivative $f : \Omega \to E$. Define the generalized Radon-Nikodým derivative $f_\mu : {}^*\Omega \to \widehat{E}$ as above. Then*

$$\pi_E \circ {}^*f = f_\mu \quad \textit{on each set } A_i \textit{ in the fine partition of } {}^*\Omega.$$

Of course, the same arguments also work for finding a $\widehat{F}$-valued generalized derivative $f_\nu \in M(\widehat{|\nu|}, \widehat{F})$ of the vector measure $\nu : \Sigma \to F$ with respect to its total variation $|\nu|$. Differentiating ν with respect to its total variation gives one extra feature of the derivative:

Lemma 16.3.3 *Let F be a Banach space and let $\nu : \Sigma \to F$ be a countably additive vector measure of bounded variation. Define*

$$\varphi_\nu(\omega) = \sum_{i=1}^{H} \frac{{}^*\nu(A_i)}{{}^*|\nu|(A_i)}\ 1_{A_i}(\omega).$$

Then $f_\nu = \pi_F \circ \varphi_\nu$ is $\widehat{|\nu|}$-almost everywhere of norm one.

Proof. Since $\nu << |\nu|$, the S-integrability of f_ν follows from Lemma 16.3.1. The norm condition follows from an adaptation of a basic standard result (Theorem II.2.4 in [5]): if f is Bochner integrable and a vector measure F is defined by $F(A) = \int_A f\, d|\nu|$, then $|F|(A) = \int_A \|f\| d|\nu|$. By construction of the generalized derivative,

$${}^*\nu(A) \approx \int_A \varphi_\nu = \int_A \sum_{i=1}^{H} \frac{{}^*\nu(A_i)}{{}^*|\nu|(A_i)}\ 1_{A_i}(\omega)\, d^*|\nu|$$

for all $A \in {}^*\Sigma$. In this situation it implies that

$${}^*|\nu|(A) = \int_A {}^*\left\| \frac{d\nu}{d|\nu|} \right\| d^*|\nu|$$

and hence ${}^*\|\varphi_\nu(\omega)\| = 1$ for $\widehat{|\nu|}$-almost every $\omega \in {}^*\Omega$. □

16.4 A nonstandard vector-vector integral

The next theorem allows us to generalize the generalized Bochner integral to a vector-vector integral which for any simple function obviously agrees with the Bartle integral developed in [2]. For the definition of the Bartle integral, E, F and G are Banach spaces equipped with a continuous bilinear multiplication $E \times F \to G$ and $\nu : \Sigma \to F$ is a countably additive vector measure of bounded variation. Such a product can arise by taking $E = F = G$ to be a C^*-algebra, or by taking $E = F'$ and $G = \mathbb{R}$, or by taking $E = L(F, G)$. A function $f : \Omega \to E$ is Bartle integrable if there is a sequence of simple functions s_n that converges to f almost everywhere and for which the sequence $\lambda_n(A) = \int_A s_n \, d\nu$ converges in the norm of G for all $A \in \Sigma$. Then $\int_A f \, d\nu = \lim_{n\to\infty} \lambda_n(A)$ exists in norm uniformly for all $A \in \Sigma$. Essentially bounded measurable functions are Bartle-integrable. The Bartle integral is a countably additive set function.

Theorem 16.4.1 *Let E, F and G be Banach spaces equipped with a continuous bilinear multiplication $E \times F \to G$. Assume that $\nu : \Sigma \to F$ is a countably additive vector measure of bounded variation and $f \in M(\widehat{|\nu|}, \widehat{E})$. The integrand f has an internal lifting φ_f and there is an internal $*$-simple lifting φ_ν of the generalized derivative of ν with respect to $|\nu|$. Then for all $A \in {}^*\Sigma$, the integral $\int_A f \, d\widehat{\nu}$ defined by*

$$\int_A f \, d\widehat{\nu} = \pi_G \left(\int_A \varphi_f \cdot \varphi_\nu \, d^*|\nu| \right)$$

is a well-defined countably additive $\widehat{G}$-valued integral.

Proof. The function $f \in M(\widehat{|\nu|}, \widehat{E})$ has a internal, S-integrable and $*$-simple lifting $\varphi_f : {}^*\Omega \to \text{fin}({}^*E)$. By Lemma 16.3.3 $\varphi_\nu : {}^*\Omega \to \text{fin}({}^*F)$ is internal, $*$-simple, S-integrable and of norm one almost everywhere. Continuity of the multiplication $E \times F \to G$ implies that the product $\varphi_f \cdot \varphi_\nu$ is an S-integrable and internal $*$-simple function from ${}^*\Omega \to \text{fin}({}^*G)$. This means that it is a lifting of an extended Bochner integrable function in $M(\widehat{|\nu|}, \widehat{G})$ and hence the $\widehat{G}$-valued generalized integral with respect to $\widehat{|\nu|}$ applies and gives a well-defined vector-vector integral. Since the extended Bochner integral is countably additive, so is this integral. □

The use of nonstandard analysis simplifies standard arguments here, since in the standard setting this argument would only work if the Banach space F had the Radon-Nikodým property.

16.5 Uniform convexity

Bogdan needed the assumption that F is uniformly convex to convert the F-valued derivative $\frac{d\nu}{d|\nu|}$ into the F'-valued derivative $\frac{d|\nu|}{d\nu}$. He composed the derivative $\frac{d\nu}{d|\nu|}$ with a continuous map g from the unit sphere of F to the unit sphere of its dual space F' such that for all $f \in S_F$, $\langle f, g(f)\rangle = 1$. The resulting map $g \circ \frac{d\nu}{d|\nu|} : \Omega \to F'$ is $|\nu|$ measurable and inherits the integrability of $\frac{d\nu}{d|\nu|}$.

If the Banach space F happens to be uniformly convex, we can compose the nonstandard extension of g with φ_ν. In this case, $\langle f, g(f)\rangle = 1$ for all $f \in S_F$ translates into $\left\langle \frac{{}^*\nu(A_i)}{{}^*|\nu|(A_i)}, {}^*g\left(\frac{{}^*\nu(A_i)}{{}^*|\nu|(A_i)}\right)\right\rangle = 1$ for $i = 1, \ldots, H$ i.e.

$$\left\langle {}^*\nu(A_i), {}^*g\left(\frac{{}^*\nu(A_i)}{{}^*|\nu|(A_i)}\right)\right\rangle = {}^*|\nu|(A_i).$$

Theorem 16.5.1 *Let E be any Banach space and let F be a uniformly convex vector space. If the vector measure $\mu : \Sigma \to E$ is absolutely continuous with respect to the total variation of the vector measure $\nu : \Sigma \to F$, then there is an extended Bochner integrable function $f : {}^*\Omega \to \widehat{E \times F'}$ such that for all $A \in \Sigma$*

$$\mu(A) = \int_{{}^*A} f \, d\widehat{\nu}.$$

Proof. Define $\varphi_f = \varphi_\mu \cdot ({}^*g \circ \varphi_\nu)$, where φ_μ is a lifting of the generalized derivative $\frac{d\mu}{d|\nu|}$, φ_ν is a lifting of the generalized derivative $\frac{d\nu}{d|\nu|}$ and g is the continuous map from S_F to $S_{F'}$. This lifting is internal, S-integrable, $*$-simple and takes its values in $\mathrm{fin}({}^*E) \times {}^*S_{F'}$. Define $f = \pi_{E\times F'} \circ \varphi_f$. Then

$$\begin{aligned}
\int_{{}^*\Omega} f \, d\widehat{\nu} &= \pi_E\left(\int_{{}^*\Omega} \varphi_\mu(\omega) \cdot {}^*g\left(\varphi_\nu(\omega)\right) \, d^*\nu\right) \\
&= \pi_E\left(\sum_{i=1}^{H} \frac{{}^*\mu(A_i)}{{}^*|\nu|(A_i)} \cdot {}^*g\left(\frac{{}^*\nu(A_i)}{{}^*|\nu|(A_i)}\right) {}^*\nu(A_i)\right) \\
&= \pi_E\left(\sum_{i=1}^{H} \frac{{}^*\mu(A_i)}{{}^*|\nu|(A_i)} \cdot {}^*|\nu|(A_i)\right) \\
&= \pi_E\left({}^*\mu({}^*\Omega)\right) \\
&= \mu(\Omega),
\end{aligned}$$

and similarly for $\mu(A)$, where (A_i) is replaced by $(A_i \cap {}^*A)$ throughout. Since the fine partition refines the standard partition of Ω into A and its complement, $A_i \cap {}^*A$ is either the empty set or A_i. □

If E has the Radon-Nikodým property, then this lifting coincides with the image of the derivative found by Bogdan in the nonstandard hull of the space of vector-vector integrable functions on Ω.

16.6 Vector-vector derivatives without uniform convexity

We can weaken the assumption on F: it follows from the Hahn-Banach theorem that for each nonzero $f \in F$ there is an $f' \in F'$ with $\|f'\| = 1$ and $f'(f) = \|f\|$. Without uniform convexity, the choice of f' is not necessarily unique, and the map $f \mapsto f'$ need not be continuous on the unit sphere of F. We can forego the use of a function g defined on all of S_F. All we need is a derivative on each set A_i in the fine partition of ${}^*\Omega$.

Theorem 16.6.1 *Let F be any Banach space and let $\nu : \Sigma \to F$ be a countably additive vector measure of bounded variation. Then there is a function $g_{|\nu|} \in M(\widehat{|\nu|}, \widehat{F'})$ such that*

$$\int_{{}^*A} g_{|\nu|}\, d\widehat{\nu} = {}^\circ\big({}^*|\nu|({}^*A)\big) = |\nu|(A)$$

for all $A \in \Sigma$.

Proof. As the collection of sets A_i is internal, we can use the axiom of choice to get an internal collection of norm one functionals $g_i \in {}^*S_{F'}$ with

$$g_i\,({}^*\nu(A_i)) = {}^*|\nu|(A_i) \quad \text{for } i = 1, \ldots, H.$$

Define an internal $*$-simple S-integrable function. $\psi_{|\nu|} : {}^*\Omega \to {}^*S_{F'}$ by

$$\psi_{|\nu|} = \sum_{i=1}^{H} g_i \cdot 1_{A_i}.$$

This function is a lifting of an extended integrable function $g_{|\nu|} \in M(\widehat{|\nu|}, \widehat{F'})$. We can define a continuous bilinear multiplication $F' \times F \to \mathbb{R}$ by $f' \cdot f = f'(f)$ and then use the integral from theorem 16.4.1. In this case the quotient map π_G becomes the standard part map $^\circ : \mathrm{fin}({}^*\mathbb{R}) \to \mathbb{R}$. By the choice of the fine partition, $A_i \cap {}^*A$ is either the empty set or equals A_i.

$$\begin{aligned} \int_{{}^*A} g_{|\nu|}\, d\widehat{\nu} &= {}^\circ\left(\int_{{}^*A} \psi_{|\nu|}\, d{}^*\nu\right) \\ &= {}^\circ\left(\int_{{}^*A} \sum_{i=1}^{H} g_i \cdot 1_{A_i}\, d{}^*\nu\right) \end{aligned}$$

$$\begin{aligned} &= {}^\circ\left(\sum_{i=1}^{H} g_i \cdot {}^*\nu(A_i \cap {}^*A)\right) \\ &= {}^\circ\left(\sum_{i=1}^{H} {}^*|\nu|(A_i \cap {}^*A)\right) \\ &= {}^\circ({}^*|\nu|({}^*A)) \end{aligned}$$

for any set $A \in \Sigma$. This finishes the proof, as for all $A \in \Sigma$, $|\nu|(A) = {}^\circ({}^*|\nu|({}^*A))$. □

The last theorem constructed a derivative for any countably additive vector measure of bounded variation with respect to its total variation as an element of $M(\widehat{|\nu|}, \widehat{F'})$. This was the main difficulty in differentiation with respect to a vector measure. Now we can differentiate one vector measure with respect to another vector measure.

Theorem 16.6.2 *Let E and F be any Banach spaces, let $\mu : \Sigma \to E$ and $\nu : \Sigma \to F$ be a countably additive vector measures of bounded variation such that $\mu << |\nu|$. Then there is a function $h \in M(\widehat{|\nu|}, \widehat{E \times F'})$ such that for all $A \in \Sigma$*

$$\mu(A) = \int_{{}^*A} h \, d\widehat{\nu}.$$

Proof. The continuous trilinear multiplication $E \times F' \times F \to E$ defined by $e \times f' \times f = f'(f) \cdot e$ extends to a continuous bilinear multiplication $\widehat{E \times F'} \times \widehat{F} \to \widehat{E}$ defined by $\pi(e \times f') \times \pi(f) = \pi_E(f'(f) \cdot e)$. The extended integrable function $h : {}^*\Omega \to \widehat{E \times F'}$ is defined through its lifting $\varphi = \varphi_\mu \cdot \psi_{|\nu|}$, the product of the lifting φ_μ of $\frac{d\mu}{d|\nu|} \in M(\widehat{|\nu|}, \widehat{E})$ and the lifting $\psi_{|\nu|}$ of $\frac{d|\nu|}{d\nu} \in M(\widehat{|\nu|}, \widehat{F'})$ found in theorem 16.6.1. Setting

$$\varphi = \varphi_\mu \cdot \psi_{|\nu|} = \sum_{i=1}^{H} \frac{{}^*\mu(A_i)}{{}^*|\nu|(A_i)} \cdot g_i \cdot 1_{A_i},$$

we obtain a $*$ simple internal function with values in $\mathrm{fin}({}^*E) \times {}^*S_{F'}$. The S-integrability of φ follows from the S-integrability of φ_μ and Lemma 16.3.3 which asserts that ${}^*\|\psi_{|\nu|}\| \approx 1$ almost everywhere. Then for all $A \in \Sigma$

$$\begin{aligned} \mu(A) &= \pi_E\left({}^*\mu({}^*A)\right) \\ &= \pi_E\left(\sum_{i=1}^{H} {}^*\mu(A_i \cap {}^*A)\right) \\ &= \pi_E\left(\sum_{i=1}^{H} \frac{{}^*\mu(A_i \cap {}^*A)}{{}^*|\nu|(A_i \cap {}^*A)} \cdot {}^*|\nu|(A_i \cap {}^*A)\right) \end{aligned}$$

$$
\begin{aligned}
&= \pi_E \left(\sum_{i=1}^{H} \frac{{}^*\mu(A_i \cap {}^*A)}{{}^*|\nu|(A_i \cap {}^*A)} \cdot (g_i \cdot 1_{A_i \cap {}^*A}) \cdot {}^*\nu(A_i \cap {}^*A) \right) \\
&= \pi_E \left(\int_{{}^*A} \varphi_\mu \cdot \psi_{|\nu|} \, d^*\nu \right) \\
&= \int_{{}^*A} h \, d\widehat{\nu},
\end{aligned}
$$

and $h : {}^*\Omega \to \widehat{E \times F'}$ is an extended integrable generalized vector-vector derivative $\frac{d\mu}{d\nu}$. □

16.7 Remarks

All the constructions above depend on the choice of a fine partition $A_1, \ldots, A_H$ of ${}^*\Omega$ that refines all finite standard partitions of Ω into measurable sets. A different choice of the partition would yield different derivatives, especially in the case of a non-uniformly convex Banach space F. However, for the standard analyst there is no noticeable difference between different choices of fine partitions, as the integrals over any standard set will be the same, no matter which fine partition is used.

Representing a vector measure $\nu : \Sigma \to F$ as an integral of its generalized derivative $\frac{d\nu}{d|\nu|}$ simplifies some of the results on Loeb completions of internal measures [13] by Živaljević.

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17

Differentiability of Loeb measures

Eva Aigner*

Abstract

We introduce a general definition of S-differentiability of an internal measure and compare different special cases. It will be shown how S-differentiability of an internal measure yields differentiability of the associated Loeb measure. We give some examples.

17.1 Introduction

In this paper we present some new results about differential properties of Loeb measures. The theory of differentiable measures was suggested by Fomin [9] as an infinite dimensional substitute for the Sobolev-Schwartz theory of distributions and has extended rapidly to a strong field of research. In particular during the last ten years it has become the foundation for many applications in different fields such as quantum field theory (see e.g. [10] or [14]) or stochastic analysis (see e.g. [4], [5], [6] and [15]).

There are many different notions of measure differentiability. The following two are most common (see e.g. [3], [6], [9], [13] and [15]). Let ν be a Borel measure on a locally convex space E and y an element of E.

1) The measure ν is called *Fomin-differentiable along* y if for all Borel subsets $B \subset E$ the limit

$$\lim_{r \to 0,\, r \in \mathbb{R}} \frac{\nu(B + ry) - \nu(B)}{r}$$

exists.

2) The measure ν is called *Skorohod-differentiable with respect to* y, if there exists another Borel measure ν' on E, such that for all continuous real-

*Ludwig-Maximilians-Universität,München, Germany.

valued bounded functions g on E

$$\lim_{r\to 0,\, r\in\mathbb{R}} \frac{\int g(x-ry)d\nu(x) - \int g(x)d\nu(x)}{r} = \int g(x)d\nu'(x).$$

The relationships between these two approaches were studied by Averbukh, Smolyanov and Fomin [3] and Bogachev [6].

A very general definition of differentiability of a curve of measures is given by Weizsäcker in [16].

In this paper we define measure differentiability as follows.

Definition 17.1.1 *Let $(\Omega, \mathcal{F}, \nu)$ be a measure space, $(\nu_r)_{-\varepsilon<r<\varepsilon}$, $\varepsilon \in \mathbb{R}^+$, a curve of nonnegative finite σ-additive measures on Ω such that $\nu = \nu_0$ and $\tilde{\mathcal{C}}$ a set of $\mathcal{F}$-measurable real-valued bounded functions on Ω. We say that ν is* differentiable with respect to the set $\tilde{\mathcal{C}}$ *if there exists a signed finite σ-additive measure ν' on Ω, such that for all functions g of $\tilde{\mathcal{C}}$*

$$\lim_{r\to 0} \frac{\int g(\omega)d\nu_r(\omega) - \int g(\omega)d\nu(\omega)}{r} = \int g(\omega)d\nu'(\omega).$$

The measure ν' is called a derivative of ν.

Note that Definition 17.1.1 covers the cases mentioned above, since for a locally convex space Ω and a fixed vector $y \in \Omega$ a curve $(\nu_r)_{-\varepsilon<r<\varepsilon}$ can be defined by $\nu_r(B) = \nu(B + ry)$ for all Borel subsets $B \subset \Omega$. When choosing $\tilde{\mathcal{C}} = \{1_B : B \in \mathcal{F}\}$, where 1_B is the indicator function, we obtain Fomin-differentiability. When choosing $\tilde{\mathcal{C}}$ as the set of all continuous bounded functions we obtain Skorohod-differentiability.

Differentiability (in the above sense) for Loeb measures has not been studied previously as far as we are aware. Hence the aim of this paper is to present the foundation and basic results. Since we want to obtain results for Loeb measures, in the first part we provide and discuss natural and very general assumptions for the underlying internal measures. The arising results for the Loeb measures — in particular a powerful theorem for the case of Fomin differentiability — are presented and discussed in the second part. Short and simple examples will illustrate the results.

A more complex and detailed description of differential properties of Loeb measures will be given in the author's thesis ([1]). A main topic of that thesis is also the application of the basic results to nonstandard representations of abstract Wiener spaces (see [7], [8] and [12]), which yields new insights also in standard mathematics.

The reader should be familiar with the basic results on nonstandard analysis and the Loeb measure construction, presented e.g. in [2], [7] and [12].

17.2 S-differentiability of internal measures

Standing Assumption

Throughout this paper let Ω be an internal set, $\mathcal{A}$ an internal ${}^*\sigma$-field on Ω and $\mu \geq 0$ an internal ${}^*\sigma$-additive S-bounded measure on $\mathcal{A}$. Let $(\mu_t)_{t\in J}$ be an internal curve of nonnegative ${}^*\sigma$-additive S-bounded measures on $\mathcal{A}$. Since we want to obtain an external curve of Loeb measures, we assume that for some $\varepsilon \in \mathbb{R}^+$ the internal parameter set J is either an interval of ${}^*\mathbb{R}$ containing the standard interval $I =]-\varepsilon, \varepsilon[$ or J is a discrete interval $\{\frac{-k}{H}, \frac{-k+1}{H}, \ldots, \frac{k-1}{H}, \frac{k}{H}\}$ with $H \in {}^*\mathbf{N} \setminus \mathbf{N}$, $k \in \{1, \ldots, H\}$ and $\varepsilon \leq \frac{k}{H}$. Moreover, the internal curve $(\mu_t)_{t\in J}$ shall be S-continuous in the following sense. If $t, s \in J$ with $t \approx s$, then $\mu_t(A) \approx \mu_s(A)$ for each $A \in \mathcal{A}$. Finally we assume that $\mu = \mu_0$.

We now introduce S-differentiability for internal measures.

Definition 17.2.1 *Suppose we have a (not necessarily internal) set $\mathcal{C}$ of internal ${}^*\mathbb{R}$-valued functions on Ω, each being $\mathcal{A}$-measurable and S-bounded. We say that the internal measure μ is S-differentiable with respect to the set $\mathcal{C}$ if there exists an internal S-bounded signed measure μ' on $\mathcal{A}$ so that for all $f \in \mathcal{C}$ and for all infinitesimals $t \in J$, $t \neq 0$*

$$\frac{\int f(\omega)\, d\mu_t(\omega) - \int f(\omega)\, d\mu(\omega)}{t} \approx \int f(\omega)\, d\mu'(\omega).$$

We call μ' an (internal) derivative of μ.

Note that a derivative is not uniquely determined by the above definition. If μ is S-differentiable with respect to a set $\mathcal{C}$ and if for some infinitesimal $t \in J$ the internal measure $\frac{\mu_t - \mu}{t}$ has limited values, then $\frac{\mu_t - \mu}{t}$ is a derivative of μ.

In this section we will regard and compare S-differentiability for different sets $\mathcal{C}$ of functions. Following the standard literature we define S-Fomin-differentiability.

Definition 17.2.2 *The internal measure μ is called S-Fomin-differentiable if the differentiability is with respect to $\mathcal{C} = \{1_A : A \in \mathcal{A}\}$ where 1_A is the indicator function.*

Note that in the case of S-Fomin-differentiability each measure $\frac{\mu_t - \mu}{t}$ with $t \approx 0$, $t \in J \setminus \{0\}$, is a derivative of μ. The following proposition shows the power of S-Fomin-differentiability.

Proposition 17.2.1 *If μ is S-Fomin-differentiable and μ' is a derivative of μ, then μ is S-differentiable with respect to the set $\mathcal{C}$ of all S-bounded ${}^*\mathbb{R}$-valued $\mathcal{A}$-measurable functions. The Fomin-derivative μ' is also a derivative with respect to $\mathcal{C}$.*

Proof. Let μ be S-Fomin-differentiable and let μ' be a derivative of μ. Let $t \neq 0$ be an infinitesimal of J and set $\tilde{\mu} := \frac{\mu_t - \mu}{t}$. Since $\mu' \approx \tilde{\mu}$ on $\mathcal{A}$ we obtain $\int f(\omega) d\mu'(\omega) \approx \int f(\omega) d\tilde{\mu}(\omega)$ for all S-bounded $\mathcal{A}$-measurable functions $f : \Omega \to {}^\star\mathbb{R}$. □

When considering measures with Lebesgue densities, then the following lemma is useful.

Lemma 17.2.1 *Let $\Omega = {}^\star\mathbb{R}$, $\mathcal{A}$ the internal field of Borel subsets, y a fixed element of ${}^\star\mathbb{R}$, μ an internal S-bounded measure on $\mathcal{A}$. Take $J = {}^\star\mathbb{R}$ and for all $t \in J$ define $\mu_t(A) = \mu(A + ty)$ for all $A \in \mathcal{A}$. Assume that μ has an internal Lebesgue density f satisfying the following three conditions:*

1) $f \geq 0$, $f = 0$ only on a set of internal Lebesgue measure 0.

2) f is *differentiable in the direction of y (with derivative $f'_y(x) := f'(x) \cdot y$).

3) If $t \approx 0$, then for all $x \in {}^\star\mathbb{R}$ with $f(x) \neq 0$

$$\frac{1}{t}\left(\frac{f(x+ty)}{f(x)} - 1\right) \approx \frac{f'_y(x)}{f(x)}.$$

Now if the internal function $\beta^y_\mu : {}^\star\mathbb{R} \to {}^\star\mathbb{R}$, defined by

$$\beta^y_\mu(x) = \begin{cases} \dfrac{f'_y(x)}{f(x)} & \text{if } f(x) \neq 0 \\ 0 & \text{if } f(x) = 0, \end{cases}$$

is S_μ-integrable, then μ is S-Fomin-differentiable and if μ' is a derivative, then for all $A \in \mathcal{A}$

$$\mu'(A) \approx \int_A \beta^y_\mu(x)\, d\mu(x).$$

Proof. Since β^y_μ is S_μ-integrable, $A \mapsto \int_A \beta^y_\mu(x) d\mu(x)$ defines an S-bounded measure. Now let $N = \{x \in {}^\star\mathbb{R} : f(x) = 0\}$. Then $\lambda(N) = 0$, where λ is the internal Lebesgue measure. If $t \approx 0$, $t \neq 0$, then

$$\begin{aligned} \left|\frac{\mu_t(A) - \mu(A)}{t} - \int_A \beta^y_\mu(x) d\mu(x)\right| &\leq \int_A \left|\frac{f(x+ty) - f(x)}{t} - \beta^y_\mu(x) \cdot f(x)\right| d\lambda(x) \\ &= \int_{A \setminus N} \left|\frac{1}{t}\left(\frac{f(x+ty)}{f(x)} - 1\right) - \frac{f'_y(x)}{f(x)}\right| d\mu(x) \\ &\approx 0. \end{aligned}$$

Here we have used condition 3) and the fact that μ is S-bounded. □

Example 17.2.1 Let $\Omega = {}^\star\mathbb{R}$, $\mathcal{A}$ be the internal field of Borel subsets and μ defined by $\mu(A) = \int_A \frac{1}{1+x^2}\, d\lambda(x)$. Fix an element $y \in {}^\star\mathbb{R}$ and define the curve by $\mu_t(A) = \int_{A+ty} \frac{1}{1+x^2}\, d\lambda(x)$, $t \in {}^\star\mathbb{R}$. If y is limited, then it's easy to see that the assumptions of Lemma 17.2.1 are satisfied. Hence the measure μ is S-Fomin-differentiable and if μ' is a derivative, then $\mu'(A) \approx \int_A \frac{-2xy}{1+x^2}\, d\mu(x)$ for each $A \in \mathcal{A}$. If y is an unlimited element of ${}^\star\mathbb{R}$, then μ is not S-Fomin-differentiable, because for $t = \frac{1}{y}$ and for the internal interval $A = {}^\star[0,1] \subset {}^\star\mathbb{R}$ the value $\frac{\mu_t(A)-\mu(A)}{t}$ is unlimited.

Remark Note that Lemma 17.2.1 is also true for $\Omega = {}^\star\mathbb{R}^L$, $L \in {}^\star\mathbf{N}$. In the author's thesis [1] this is used to show the S-Fomin-differentiability of a nonstandard representation of the Wiener measure introduced by Cutland and Ng in [8].

We now turn to other forms of S-differentiability. Again, following the standard literature, we define S-Skorohod-differentiability.

Definition 17.2.3 *Let Ω be a subset of ${}^\star M$ where M is a metric space and let $\mathcal{A}$ be an internal ${}^\star\sigma$-field on Ω. The measure μ is called S-Skorohod-differentiable if it is S-differentiable with respect to the set of all S-bounded, $\mathcal{A}$-measurable functions $f : \Omega \to {}^\star\mathbb{R}$ that are S-continuous. This means, $f(\omega) \approx f(\tilde{\omega})$ for $\omega, \tilde{\omega} \in \Omega$, $\omega \approx \tilde{\omega}$.*

If Ω is as described in Definition 17.2.3, then, as a consequence of Proposition 17.2.1, S-Fomin-differentiability implies S-Skorohod-differentiability. The following example shows that the converse is not true.

Example 17.2.2 Fix a natural number $H \in {}^\star\mathbf{N} \setminus \mathbf{N}$ and let $\Omega \subset {}^\star\mathbb{R}$, $\Omega = \{\frac{1}{H} \cdot z : z \in {}^\star\mathbf{Z}\}$. The measure μ is the counting measure defined on the field of internal subsets of Ω, i.e.

$$\mu(A) = \frac{\left|A \cap \left[0, \frac{H-1}{H}\right]\right|}{H}.$$

Here $\left[0, \frac{H-1}{H}\right] = \{0, \frac{1}{H}, \ldots, \frac{H-1}{H}\}$ and $|A|$ is the internal number of elements. Now let $J \subset \Omega, J = [-\frac{l}{H}, \frac{l}{H}]$ with $l \in {}^\star\mathbf{N}$, $\frac{l}{H} \approx \frac{1}{2}$ and let $(\mu_t)_{t\in J}$ be defined by $\mu_t(A) = \mu(A+t)$. Note that for any $k \in {}^\star\mathbf{N}$ with $\frac{k}{H} \in J$ we obtain:

$$\frac{\mu_{\frac{k}{H}} - \mu}{\frac{k}{H}}(A) = \frac{1}{k}\left(\left|A \cap \left[\frac{-k}{H}, \frac{-1}{H}\right]\right| - \left|A \cap \left[\frac{H-k}{H}, \frac{H-1}{H}\right]\right|\right)$$

and

$$\frac{\mu_{\frac{-k}{H}} - \mu}{\frac{-k}{H}}(A) = \frac{1}{k}\left(\left|A \cap \left[0, \frac{k-1}{H}\right]\right| - \left|A \cap \left[\frac{H}{H}, \frac{H+k-1}{H}\right]\right|\right).$$

Now choose $k \in {}^*\mathbf{N}$ with $\frac{k}{H} \in J$ and $\frac{k}{H} \approx 0$ and set $A = [0, \frac{k-1}{H}]$. Then

$$\frac{\mu_{\frac{k}{H}} - \mu}{\frac{k}{H}}(A) = 0 \quad \text{and} \quad \frac{\mu_{\frac{-k}{H}} - \mu}{\frac{-k}{H}}(A) = 1.$$

Hence μ is not S-Fomin-differentiable. It is not hard to check that μ is S-Skorohod-differentiable with internal derivatives $\frac{\mu_t - \mu}{t}$ for all infinitesimals $t \in J \setminus \{0\}$. We will give a standard application of this example in Example 17.3.1.

Finally we consider S-differentiability with respect to *continuous functions. We will see that this is equivalent to S-Fomin-differentiability. The following proposition can be easily shown by transfer of Urysohn's Lemma.

Proposition 17.2.2 *Let Ω be an internal *normal space, $\mathcal{A}$ the internal field of Borel subsets. If $(\mu_t)_{t\in J}$ is a curve of *regular measures, then μ is S-differentiable with respect to the set $\mathcal{C}$ of all internal *continuous S-bounded functions if and only if μ is S-Fomin-differentiable.*

17.3 Differentiability of Loeb measures

In this section we show how S-differentiability of an internal measure yields differentiability (in the sense of Definition 17.1.1) of the corresponding Loeb measure. Recall the Standing Assumption on page 240.

Now we can define a curve of Loeb measures in a unique way. Let $\varepsilon \in \mathbb{R}^+$ as described in the Standing Assumption (page 240), $I =]-\varepsilon, \varepsilon[$. For each $r \in I$ choose $t \in J$ such that $t \approx r$ and set $\mu_r := \mu_t$. Let us denote the associated Loeb spaces by $(\Omega, L_{\mu_r}(\mathcal{A}), (\mu_r)_L)$. Since the Loeb σ-fields $L_{\mu_r}(\mathcal{A})$ are not necessarily identical we choose a joint σ-field $\mathcal{F} \subset \bigcap_{r\in I} L_{\mu_r}(\mathcal{A})$. We now define the curve $((\mu_L)_r)_{r\in I}$ of measures on $\mathcal{F}$ by $(\mu_L)_r := (\mu_r)_L$ restricted to $\mathcal{F}$.

Let us mention some obvious connections between S-differentiability and differentiability.

Lemma 17.3.1 *Let μ be S-differentiable with respect to a set $\mathcal{C}$ of internal *$\mathbb{R}$-valued, $\mathcal{A}$-measurable and S-bounded functions on Ω and let μ' be an internal derivative of μ.*

1) Then for all $f \in \mathcal{C}$

$$\lim_{r\to 0} \frac{{}^\circ\left(\int f(\omega)d\mu_r(\omega)\right) - {}^\circ\left(\int f(\omega)d\mu(\omega)\right)}{r} = {}^\circ\left(\int f(\omega)d\mu'(\omega)\right).$$

The convergence is uniform, if $\mathcal{C}$ is internal.

2) Suppose

$$\tilde{\mathcal{C}} := \{g : \Omega \to \mathbb{R} : \text{ there is an } f \in \mathcal{C} \text{ with } {}^\circ(f(\omega)) = g(\omega) \text{ for all } \omega \in \Omega\}$$

and

$$\mathcal{F}_{\mu'} := \Big(\bigcap_{r\in I} L_{\mu_r}(\mathcal{A})\Big) \cap L_{\mu'}(\mathcal{A}).$$

Then the Loeb measure μ_L, restricted to $\mathcal{F}_{\mu'}$, is differentiable with respect to the set $\tilde{\mathcal{C}}$ and the Loeb extension $(\mu')_L$ of ${}^\circ(\mu')$, restricted to $\mathcal{F}_{\mu'}$, is a derivative $(\mu_L)'$ of μ_L. This means that for all $g \in \tilde{\mathcal{C}}$:

$$\lim_{r\to 0} \frac{\int g(\omega)d(\mu_L)_r(\omega) - \int g(\omega)d\mu_L(\omega)}{r} = \int g(\omega)d(\mu')_L(\omega).$$

Note that, if μ' and γ' are two internal derivatives of μ, then $\int g(\omega)d(\mu')_L(\omega) = \int g(\omega)d(\gamma')_L$ for all $g \in \tilde{\mathcal{C}}$, but the Loeb measures $(\mu')_L$ and $(\gamma')_L$ on $\mathcal{A}$ and hence also the σ-fields $L_{\mu'}(\mathcal{A})$ and $L_{\gamma'}(\mathcal{A})$ may be different.

Example 17.3.1 In Example 17.2.2 we defined an internal counting measure μ. Since μ is S-Skorohod-differentiable with internal derivatives $\frac{\mu_t-\mu}{t}$, $t \in J \setminus \{0\}$, $t \approx 0$, we can apply Lemma 17.3.1. But as we have seen in Example 17.2.2, for $k \in {}^*\mathbf{N}$ with $\frac{k}{H} \in J$ and $\frac{k}{H} \approx 0$ and $A = [0, \frac{k-1}{H}]$

$$\left(\frac{\mu_{\frac{k}{H}} - \mu}{\frac{k}{H}}\right)_L (A) = 0 \quad \text{and} \quad \left(\frac{\mu_{\frac{-k}{H}} - \mu}{\frac{-k}{H}}\right)_L (A) = 1.$$

Nevertheless, there exists a σ-field, on which the Loeb measures $\left(\frac{\mu_t-\mu}{t}\right)_L$ coincide for all infinitesimals $t \in J \setminus \{0\}$. Let $\mathcal{B}(\mathbb{R})$ be the (standard) σ-field of all Borel subsets of $\mathbb{R}$. For $B \in \mathcal{B}(\mathbb{R})$ let $st^{-1}[B] = \{\omega \in \Omega : {}^\circ\omega \in B\}$. According to the usual approach to standard Lebesgue measure by a nonstandard counting measure (see [7] or [12]), the external set $st^{-1}[B]$ is an element of $L_{\mu_r}(\mathcal{A})$ for all $r \in I$ and $(\mu_r)_L(st^{-1}[B]) = \nu_r(B)$, where $\nu_r(B) = \int_{B+r} 1_{[0,1]}(x)d\lambda(x)$ with standard Lebesgue measure λ. But $st^{-1}[B]$ is also an element of $L_{\frac{\mu_t-\mu}{t}}(\mathcal{A})$ for all infinitesimals $t \in J \setminus \{0\}$ and

$$\left(\frac{\mu_t - \mu}{t}\right)_L (st^{-1}[B]) = 1_B(1) - 1_B(0).$$

Hence the Loeb measures $\left(\frac{\mu_t-\mu}{t}\right)_L$, restricted to the σ-field $\{st^{-1}[B] : B \in \mathcal{B}(\mathbb{R})\}$, coincide for all infinitesimals $t \in J \setminus \{0\}$. If we define a measure ν' on $\mathcal{B}(\mathbb{R})$ by $\nu'(B) = 1_B(1) - 1_B(0)$, then the S-differentiability of the internal counting measure μ yields the Skorohod-differentiability (with respect to $1 \in \mathbb{R}$) of the standard measure ν with derivative ν'.

We will now see that in the case of S-Fomin-differentiability the σ-field $\mathcal{F}$ doesn't depend on the chosen internal derivative and the derivative $(\mu_L)'$ of μ_L is uniquely determined. Moreover, the differentiability of μ_L is true not only with respect to standard parts of internal functions, but also with respect to all $\mathcal{F}$-measurable S-bounded real-valued functions on Ω.

We gather these facts together into the following, which is the main theorem of this paper.

Theorem 17.3.1 *Let μ be S-Fomin-differentiable and $\mathcal{F} = \bigcap_{r\in I} L_{\mu_r}(\mathcal{A})$. Then μ_L is differentiable on $\mathcal{F}$ with respect to the set $\tilde{\mathcal{C}} = \{1_B : B \in \mathcal{F}\}$ and the differentiability is uniform on $\tilde{\mathcal{C}}$. The derivative $(\mu_L)'$ is uniquely determined and is absolutely continuous with respect to μ_L. If μ' is an internal derivative of μ, then the Loeb extension $(\mu')_L$ is defined on $\mathcal{F}$ and coincides with $(\mu_L)'$. In particular this is true for all internal measures $\frac{\mu_t-\mu}{t}$, where $t \in J \setminus \{0\}$ is infinitesimal.*

Proof. Let μ' be a derivative of μ. We will show the following statements:

(A) For all internal sets $A \in \mathcal{A}$ the limit

$$\lim_{r\to 0} \frac{(\mu_L)_r(A) - \mu_L(A)}{r}$$

exists and is equal to $(\mu')_L(A)$. The convergence is uniform on the internal field $\mathcal{A}$.

(B) If $N \in \mathcal{F}$ is a μ_L-nullset, then

$$\lim_{r\to 0} \frac{(\mu_L)_r(N) - \mu_L(N)}{r} = 0.$$

The convergence is uniform for all μ_L-nullsets of $\mathcal{F}$. Any μ_L-nullset of $\mathcal{F}$ is also a $(\mu')_L$-nullset of $\mathcal{F}$.

(C) If $B \in \mathcal{F}$ and if $A \in \mathcal{A}$ is μ_L-equivalent to B, then

$$\lim_{r\to 0} \frac{(\mu_L)_r(B) - \mu_L(B)}{r} = \lim_{r\to 0} \frac{(\mu_L)_r(A) - \mu_L(A)}{r},$$

in particular the left limit exists. The convergence is uniform on $\mathcal{F}$.

(D) The Loeb extension $(\mu')_L$ is defined on $\mathcal{F}$ and for all $B \in \mathcal{F}$

$$(\mu')_L(B) = \lim_{r\to 0} \frac{(\mu_L)_r(B) - \mu_L(B)}{r}.$$

(A) follows from Lemma 17.3.1.

To prove *(B)* let N be a μ_L-nullset of $\mathcal{F}$, i.e. there exists a sequence $(N_{\frac{1}{n}})_{n\in\mathbf{N}} \subset \mathcal{A}$ so that for all $n \in \mathbf{N}$ we have $N \subseteq N_{\frac{1}{n}}$, $N_{\frac{1}{n+1}} \subseteq N_{\frac{1}{n}}$ and

$\mu(N_{\frac{1}{n}}) \leq \frac{1}{n}$. Let $\tilde{N} := \bigcap_{n=1}^{\infty} N_{\frac{1}{n}}$. Then $\tilde{N} \in \mathcal{F}$, $\mu_L(\tilde{N}) = 0$ and we obtain for all $r \in I$

$$(\mu_L)_r(N) \leq (\mu_L)_r(\tilde{N}).$$

Since $\mu_L(N) = 0$ we get

$$\left|\frac{(\mu_L)_r(N) - \mu_L(N)}{r}\right| = \left|\frac{(\mu_L)_r(N)}{r}\right| \leq \left|\frac{(\mu_L)_r(\tilde{N})}{r}\right|.$$

Therefore it is sufficient to show that $\lim_{r\to 0} \frac{(\mu_L)_r(\tilde{N})}{r} = 0$. Now since $\mu_L(\tilde{N}) = 0$,

$$\begin{aligned}\frac{(\mu_L)_r(\tilde{N})}{r} &= \frac{(\mu_L)_r(\tilde{N}) - \mu_L(\tilde{N})}{r} \\ &= \frac{\lim_{n\to\infty}(\mu_L)_r(N_{\frac{1}{n}}) - \lim_{n\to\infty}\mu_L(N_{\frac{1}{n}})}{r} \\ &= \lim_{n\to\infty}\frac{(\mu_L)_r(N_{\frac{1}{n}}) - \mu_L(N_{\frac{1}{n}})}{r}.\end{aligned}$$

It follows from *(A)*, that for each $n \in \mathbf{N}$ the limit $\lim_{r\to 0}\frac{(\mu_L)_r(N_{\frac{1}{n}}) - \mu_L(N_{\frac{1}{n}})}{r}$ exists and is equal to $(\mu')_L(N_{\frac{1}{n}})$. Since $(\mu')_L$ is defined on the smallest σ-field containing $\mathcal{A}$, the limit $\lim_{n\to\infty}(\mu')_L(N_{\frac{1}{n}}) = (\mu')_L(\tilde{N})$ also exists. Because of the uniform convergence stated in *(A)* and since $(N_{\frac{1}{n}})_{n\in\mathbf{N}} \subset \mathcal{A}$ we can exchange the limits as follows:

$$\begin{aligned}\lim_{r\to 0}\frac{(\mu_L)_r(\tilde{N})}{r} &= \lim_{r\to 0}\lim_{n\to\infty}\frac{(\mu_L)_r(N_{\frac{1}{n}}) - \mu_L(N_{\frac{1}{n}})}{r} = \\ \lim_{n\to\infty}\lim_{r\to 0}\frac{(\mu_L)_r(N_{\frac{1}{n}}) - \mu_L(N_{\frac{1}{n}})}{r} &= \lim_{n\to\infty}(\mu')_L(N_{\frac{1}{n}}) = (\mu')_L(\tilde{N}).\end{aligned}$$

So the measure μ_L is differentiable at $\tilde{N}$ and the value of the derivative is $(\mu')_L(\tilde{N})$. It remains to show that $(\mu')_L(\tilde{N}) = 0$. To this end we use an argument due to Smolyanov and Weizsäcker [15]. Let us consider the function

$$f : I \longrightarrow \mathbb{R}, r \mapsto (\mu_L)_r(\tilde{N})$$

Then f is differentiable at $t = 0$ and $f'(0) = (\mu')_L(\tilde{N})$. Since f is nonnegative and $f(0) = 0$, the first derivative of f in 0 must be 0. Hence $(\mu')_L(\tilde{N}) = 0$. The uniformity of the convergence can be seen by using again the uniform convergence on $\mathcal{A}$. Of course $(\mu')_L(N) = 0$ since $N \subset \tilde{N}$. Hence the μ_L-nullsets of $\mathcal{F}$ are also $(\mu')_L$-nullsets of $\mathcal{F}$.

(C) Here we show the differentiability for an arbitrary element of $\mathcal{F}$. So let $B \in \mathcal{F}$, $A \in \mathcal{A}$ μ_L-equivalent to B and $r \in I$. Since $\mu_L(B) = \mu_L(A)$, it is sufficient to show that

$$\lim_{r \to 0} \frac{(\mu_L)_r(B) - (\mu_L)_r(A)}{r} \;=\; 0.$$

Since μ_L is nonnegative the following estimate is easy to verify.

$$|(\mu_L)_r(B) - (\mu_L)_r(A)| \;\leq\; (\mu_L)_r(A \triangle B),$$

where $A \triangle B$ denotes the symmetric difference $(A \setminus B) \cup (B \setminus A)$. Now *(B)* yields

$$\lim_{r \to 0} \left| \frac{(\mu_L)_r(B) - (\mu_L)_r(A)}{r} \right| \;\leq\; \lim_{r \to 0} \frac{(\mu_L)_r(A \triangle B)}{|\, r \,|} \;=\; 0.$$

The uniform convergence follows from *(A)* and *(B)*. Hence *(C)* is proved.

(D) follows from *(A)*, *(B)* and *(C)*. □

Example 17.3.2 Let μ be a measure with internal Lebesgue density satisfying the assumptions of Lemma 17.2.1 for a fixed $y \in {}^\star\mathbb{R}$. Recall that the internal curve is given by $\mu_t(A) = \mu(A + ty)$ for all Borel subsets $A \subset {}^\star\mathbb{R}$. Let $\varepsilon \in \mathbb{R}^+$, $I =]-\varepsilon, \varepsilon[$. Since μ is S-Fomin-differentiable, we can apply Theorem 17.3.1. Hence μ_L is differentiable on $\mathcal{F} = \bigcap_{r \in I} L_{\mu_r}(\mathcal{A})$ with respect to $\tilde{\mathcal{C}} = \{1_B : B \in \mathcal{F}\}$. Moreover, for the uniquely determined derivative $(\mu_L)'$ we obtain:

$$(\mu_L)'(B) \;=\; \int_B {}^\circ\beta_\mu^y(x)\, d\mu_L(x)$$

for all $B \in \mathcal{F}$, where β_μ^y is defined in Lemma 17.2.1

The power of Fomin-differentiability is also shown in the last result.

Corollary 17.3.1 *If μ is S-Fomin-differentiable with an internal derivative μ' and $\mathcal{F}$ is defined as in Theorem 17.3.1, then the Loeb measure μ_L, restricted to $\mathcal{F}$, is differentiable with respect to the set $\tilde{\mathcal{C}}$ of all $\mathcal{F}$-measurable real-valued bounded functions on Ω. The Loeb measure $(\mu')_L$, restricted to $\mathcal{F}$, is the derivative of $(\mu)_L$.*

Proof. This is routine integration theory using the uniform differentiability on $\mathcal{F}$ and the approximation of measurable functions by simple functions. □

Remark An application of Fomin-differentiability is given in [15] by Smolyanov and Weizsäcker. There, a nonnegative measure ν on a locally convex space

E is considered which is Fomin-differentiable along all elements of a Hilbert subspace of E. Using this measure differentiability, Smolyanov and Weizsäcker introduce an operator on a subspace of $\mathcal{L}^2(E,\nu)$. In the Gaussian case this operator is the derivative operator of the Malliavin calculus (see [11]).

The question of the assumptions that are needed for differential properties of Loeb measures to yield such a derivative operator is the subject of ongoing investigation.

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Part IV

Differential systems and equations

18

The power of Gâteaux differentiability

Vítor Neves*

Abstract

The search for useful non standard minimization conditions on C^1 functionals defined on Banach spaces lead us to a very simple argument which shows that if a C^1 function $f : E \to F$ between Banach spaces is actually Gâteaux differentiable on finite points along finite vectors, then it is uniformly continuous on bounded sets if and only if it is lipschitzian on bounded sets. The following is a development of these ideas starting from locally convex spaces.

18.1 Preliminaries

This section consists of an informal description of tools to frame the discourse and therefore contains only a small number of theorems and few proofs. Careful foundational treatments may be found in [3], [9], [14] or [1]; we shall also use Nelson's quantifiers $\forall^{st}$ and $\exists^{st}$ as in [8].

We assume the existence of two set-theoretical structures $\mathcal{A}$ and $\mathcal{B}$ and a function ${}^*(\cdot) : \mathcal{A} \to \mathcal{B}$ satisfying the properties we proceed to describe. Elements of either of the structures which are sets shall be called **entities**[1].

(P1) $\mathcal{A}$ *is a model of the relevant Analysis* in the sense that any object of Classical Mathematical Analysis as well as the mathematical structures under study have an interpretation in $\mathcal{A}$ and theorems about them are true in $\mathcal{A}$. In particular the complete ordered field $\mathbb{R}$ of real numbers or any other classical space are elements of $\mathcal{A}$.

*Departamento de Matemática, Universidade de Aveiro.
vneves@mat.ua.pt
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[1]We shall also admit the existence of *atoms* i.e. objects without elements which are not the empty set; for instance numbers are atoms.

The image by ${}^*(\cdot)$ of any given element $a \in \mathcal{A}$ will be denoted ${}^*a$ and be called **standard** as well as a itself; elements of standard sets in $\mathcal{B}$ shall be called **internal**; non internal sets in $\mathcal{B}$ are called **external**.

A set of sets verifies the **Finite Intersection Property**, abbreviated f.i.p., if *all its finite subsets have non-empty intersection.* Denote the cardinal (number of elements) of a set S by $card(S)$ and its power set by $\mathcal{P}(S)$.

(P2) Polysaturation[2]

Given $E \in \mathcal{A}$ and $\mathcal{C} \subseteq {}^*\mathcal{P}(E)$, if $\mathcal{C}$ verifies the f.i.p. and $card(\mathcal{C}) < card(\mathcal{A})$ then $\bigcap \mathcal{C} \neq \emptyset$

There is an encompassing formal first order language $\mathcal{L}$ with equality $=$ and $\in$ (to be read as the membership relation), with the usual connectives $\neg$ for negation, $\wedge$ for conjunction, $\vee$ for disjunction, $\Rightarrow$ for implication, $\Leftrightarrow$ for equivalence, universal $\forall$ and existential $\exists$ quantifiers and enough constants, predicate and function symbols to denote any elements, relations and functions under consideration either in $\mathcal{A}$ or in $\mathcal{B}$.

Say that a formula of $\mathcal{L}$ is **bounded** if its quantified subformulae, hereby including the formula itself, are of the form

$$\forall x\ [x \in a \Rightarrow \phi] \qquad \text{or} \qquad \exists x\ [x \in a \wedge \phi]$$

for some constant a and formula ϕ; a **sentence** is a formula without free variables. $\mathcal{B}$ contains a *formal* copy of $\mathcal{A}$ in the following sense.

(P3) Transfer

If $\phi(a_1, \cdots, a_n)$ is a bounded sentence with occurrences of constants a_i $(1 \leq i \leq n)$ and no more constants, then $\phi(a_1, \cdots, a_n)$ is true in $\mathcal{A}$ iff $\phi({}^*a_1, \cdots, {}^*a_n)$ is true in $\mathcal{B}$[3].

It is always useful to keep in mind theorems 18.1.1 and 18.1.2 below. Say that a formula of $\mathcal{L}$ is **standard** (resp. **internal**) if it is bounded and all its constants denote standard (resp. internal) elements of $\mathcal{B}$.

Theorem 18.1.1 (Principles of Standard and of Internal Definition) *A set $B \in \mathcal{B}$ is standard (resp. internal) iff there exists $a \in \mathcal{A}$ and a standard (resp. internal) formula ϕ such that $B = \{x \in {}^*a |\ \phi(x)\}$ or, more informally, a set in $\mathcal{B}$ is standard or internal iff it is definable by a respectively standard or internal formula.*

[2] Actually this is stronger than needed for our purposes, but it is powerful and easy to formulate.

[3] $\mathcal{B}$ is a kind of elementary extension of $\mathcal{A}$ with embedding ${}^*(\cdot)$.

Call **monad** any intersection $\bigcap_{C\in\mathcal{C}} {}^*C$ with $\emptyset \neq \mathcal{C} \in \mathcal{A}$.

Theorem 18.1.2 (Cauchy's Principle) *Any internal set which contains a monad* $\bigcap_{C\in\mathcal{C}} {}^*C$ *also contains one of the standard sets* ${}^*C$.

Hyper-real numbers $\lambda \in {}^*\mathbb{R}$ are classified the following way

$$\begin{aligned} \lambda \textit{ is } \textbf{finite} &:\equiv \exists r \in \mathbb{R}\ |x| \leq {}^*r \\ \lambda \textit{ is } \textbf{infinitesimal} &:\equiv \forall r \in \mathbb{R}\ [r > 0 \Rightarrow |x| \leq {}^*r] \\ \lambda \textit{ is } \textbf{infinite} &:\equiv \lambda \textit{ is not finite.} \end{aligned}$$

$\mathcal{O}$ and μ denote respectively the set of finite and infinitesimal hyper-real numbers.

For any given (real) locally convex space S, let Γ_S denote a gauge, i.e. a *directed set*, of semi-norms defining the topology of S; $\mathbf{fin}({}^*S)$ and $\mu({}^*S)$ shall denote respectively the sets of finite and infinitesimal elements in ${}^*S$, i.e., for any given $x \in {}^*S$,

$$x \in \mathbf{fin}({}^*S) :\equiv \forall \gamma \in \Gamma_S\ \gamma(x) \in \mathcal{O} \tag{18.1.1}$$

$$x \in \mu({}^*S) :\equiv \forall \gamma \in \Gamma_S\ \gamma(x) \in \mu \tag{18.1.2}$$

When Γ_S is unbounded, in the sense that

$$\forall x \in S \quad \Gamma_S(x) := \{\gamma(x) |\ \gamma \in \Gamma_S\} \quad \text{is unbounded,} \tag{18.1.3}$$

it so happens that

$$\forall x \in {}^*S \quad \left[x \in \mu({}^*S) \Leftrightarrow \forall^{st}\gamma \in \Gamma_S\ \gamma(x) \leq 1\right]. \tag{18.1.4}$$

Condition (18.1.4) has the syntactical advantage of dispensing with a quantifier; as we will present strongly syntactical reasoning,

we assume from now on that condition (18.1.3) *is always verified.*

Denote ${}^\sigma C$ the set of **standard** elements of ${}^*C$, i.e.,

$${}^\sigma C := \{{}^*x |\ x \in C\}.^4$$

If E and F are locally convex spaces, $L^{(k)}(E,F)$ denotes the space of continuous k-linear maps $l : E^k \to F$; a map $l \in {}^*L^{(k)}(E,F)$ is said to be **finite** (resp. **infinitesimal**) if $l(\mathbf{fin}({}^*E)^k) \subseteq \mathbf{fin}({}^*F)$ (resp. $l(\mathbf{fin}({}^*E)^k) \subseteq \mu({}^*F)$).

It is useful to keep in mind the following results ([4] is a most complete source; also see [14, chap. 10]).

[4] We sometimes identify C with ${}^\sigma C$ for the sake of simplifying notation.

Theorem 18.1.3 *In a locally convex space S,*

1. $x \in \mu(^*S)$ *if and only if* $\forall \lambda \in \mathcal{O}\ \lambda x \in \mu(^*S) \qquad (x \in {}^*S)$.
2. $x \in \mathbf{fin}(^*S)$ *if and only if* $\forall \lambda \in \mu\ \lambda x \in \mu(^*S) \qquad (x \in {}^*S)$.
3. $\forall x \in \mu(^*S)\ \exists \lambda \in {}^*\mathbb{R} \setminus \mathcal{O}\ \lambda x \in \mu(^*S)$, *therefore it is also true that*

$$\forall x \in {}^*S\ [x \in \mu(^*S) \ \Leftrightarrow\ \exists \lambda \in {}^*\mathbb{R} \setminus \mathcal{O}\ \lambda x \in \mu(^*S)]$$

Let $x \approx y$ mean that x is **infinitely near** y, i.e., $x - y \in \mu(^*S)$; also let **st** denote **standard part** whenever appropriate, i.e.,

$$y\ =\ \mathbf{st}(x)\ :\equiv\ y \in {}^\sigma S\ \ \&\ \ x \approx y;$$

$\mathbf{ns}(^*S)$ will denote the set of **near-standard vectors** of $^*S$, i.e.,

$$\mathbf{ns}(^*S) := {}^\sigma S + \mu(^*S).$$

A function $f : {}^*E \to {}^*F$ is **S-continuous on** $A \subseteq {}^*E$ if for all $x, y \in A$, $f(x) \approx f(y)$, whenever $x \approx y$; by definition, any standard function f is S-continuous if $^*f$ is. Theorem 18.1.2 implies that S-continuity of internal functions is nothing else than continuity measured with standard tolerances; this and condition (18.1.3) imply:

Lemma 18.1.1 *Suppose $A \in {}^*\mathcal{P}(E)$. An internal function $f : A \to {}^*F$ is S-continuous at $x \in A$ iff*

$$\forall \nu \in {}^\sigma\Gamma_F\, \forall \varepsilon \in {}^\sigma\mathbb{R}\ \exists \gamma \in {}^\sigma\Gamma_E\ \exists \delta \in {}^\sigma\mathbb{R}\ \forall y \in A \\ [\gamma(x - y) < \delta \Rightarrow \nu(f(y) - f(x)) < \varepsilon].$$

In particular

Theorem 18.1.4 *A function $f : A \subseteq E \to F$ between locally convex spaces E and F is*

1. *continuous (on A) iff it is S-continuous on ${}^*A \cap \mathbf{ns}(^*E)$;*
2. *uniformly continuous (on A) iff it is S-continuous on $^*A$.*

And

Theorem 18.1.5 *Given locally convex spaces E and F and an internal $^*k$-linear function $l : {}^*E^k \to {}^*F$, the following conditions are equivalent*

1. *l is S-continuous.*

2. $l(\mathbf{fin}({}^*E)) \subseteq \mathbf{fin}({}^*F)$ *(i.e. the internal S-continuous maps are the internal finite maps).*

3. $l(\mu({}^*E)) \subseteq \mu({}^*F)$.

It might also be interesting to notice the following

Theorem 18.1.6 *Let E and F be locally convex spaces, A be a subset of ${}^*E$ and $\phi : (A \times \mathbf{fin}({}^*E)) \to {}^*F$ be an internal function which is linear in the second coordinate. If ϕ is S-continuous in the sense that*

$$\forall x, y \in A \ \forall u, v \in \mathbf{fin}({}^*E) \ \ [x \approx y \ \& \ u \approx v \ \Rightarrow \ \phi(x,u) \approx \phi(y,v)],$$

then, for all $x \in A$, $\phi(x, \cdot)$ is a finite map.

Proof. Assume $\phi : (A \times \mathbf{fin}({}^*E)) \to {}^*F$ is S-continuous, $x \in A \subseteq {}^*E$ and that $u \in \mathbf{fin}({}^*E)$; for all $t \in \mu$, $tu \approx 0$, therefore $t\phi(x,u) = \phi(x,tu) \approx \phi(x,0) = 0$, hence, by theorem 18.1.3, $\phi(x,u) \in \mathbf{fin}({}^*F)$ as required. □

Observe that, when S is normed, an element $x \in {}^*S$ is infinitesimal (resp. finite) iff $\|x\| \in \mu$ (resp. $\|x\| \in \mathcal{O}$) and the following holds.

Corollary 18.1.1 *Given normed spaces E and F and a k-linear function $l : E^k \to F$, the following conditions are equivalent*

1. *l is continuous*

2. $\forall x \in {}^*S \ [\, \|x\| \approx 0 \ \Rightarrow \ \|l(x)\| \approx 0 \,]$

3. $\forall x \in {}^*S \ [\, \|x\| \in \mathcal{O} \ \Rightarrow \ \|l(x)\| \in \mathcal{O} \,]$.

Of course theorem 18.1.5 and corollary 18.1.1 essentially say that, for linear maps *continuity is equivalent to continuity at zero.*

18.2 Smoothness

Let E and F be complete locally convex spaces, A be an internal subset of ${}^*E$, $f : A \to {}^*F$ and $l_{(\cdot)} : A \to {}^*L(E,F)$ be internal and write

$$f(x+tv) = f(x) + tl_x(v) + t\iota \quad (x \in A, \ v \in {}^*E, \ t \in {}^*\mathbb{R}, \ \iota \in {}^*F). \qquad (18.2.1)$$

f is **GS-differentiable** at x with **GS-derivative** l_x, if $\iota \approx 0$ whenever $v \in \mathbf{fin}({}^*E)$ and $t \approx 0$; f is **uniformly differentiable** at $a \in {}^*E$ with **uniform derivative** $l_{(\cdot)}$ if f is GS-differentiable at x and l_x is finite, for all $x \approx a$. We shall also call $l : A \times \mathbf{fin}({}^*E) \to {}^*F$ the derivative of f and define

$$df(x,v) := l(x,v) := l_x(v) := df_x(v) \qquad (x, v \in {}^*E).$$

Observe that, when one takes x and v in E within equation (18.2.1), the *GS*-derivative $df(x,v)$ is just the Gâteaux derivative (**G-derivative** for short) of f at x along v. The following theorem is well known too (see [13]).

Theorem 18.2.1 *If E and F are Banach spaces, the function $f : E \to F$ has G-derivative $df(x,y)$, for all $x \in E$, and $x \mapsto df(x,\cdot) : E \to L(E,F)$ is continuous for the uniform topology on $L(E,F)$, then $x \mapsto df(x,\cdot)$ is actually a Fréchet derivative.*

The following is a simple generalization of proposition 2.4 in [15].

Theorem 18.2.2 *If the internal map $f : {}^*E \to {}^*F$ is uniformly differentiable at all points of an internal set $A \in {}^*\mathcal{P}(E)$ with derivative df, then*

1. *f is S-continuous on A*
2. *$df : A \times \mathbf{fin}({}^*E) \to {}^*F$ verifies*
   (a) *$df(A \times \mathbf{fin}({}^*E)) \subseteq \mathbf{fin}({}^*F)$*
   (b) *df is S-continuous in the sense that*

$$\forall x, y \in A \ \forall u, v \in \mathbf{fin}({}^*E) \ [x \approx y \ \& \ u \approx v \ \Rightarrow \ df_x(u) \approx df_y(v)].$$

Proof. Borrowing from [15]: suppose that $f : {}^*E \to {}^*F$ is internal and uniformly differentiable at all points of the internal set A, that $x, y \in A$ and that $x \approx y$; take $\lambda \in {}^*\mathbb{R} \setminus \mathcal{O}$ such that $\lambda(x-y) \in \mu({}^*E)$ (theorem 18.1.3) and define $t := \frac{1}{\lambda}$; t is infinitesimal and, for some $\iota \in \mu({}^*F)$,

$$\begin{aligned} f(x) - f(y) &= f(y + t\lambda(x-y)) - f(y) \\ &= t df_y(\lambda(x-y)) \ + \ t\iota \\ &= df_y(x-y) \ + \ t\iota \in \mu({}^*F) \end{aligned}$$

by theorem 18.1.5, and 1 is proven.

2 (a) is an immediate consequence of the fact that the maps df_x are finite and this is useful in proving S-continuity of df in (b): take $x, y \in A$ and $u, v \in \mathbf{fin}({}^*E)$ such that $x \approx y \ \& \ u \approx v$; observe that

$$df(x,u) - df(y,v) = df_x(u-v) + df_x(v) - df_y(v).$$

As the maps df_x are finite, by theorem 18.1.5, $df(x, u-v) \in \mu({}^*F)$ and it is enough to show that, under the present assumptions, $df_x(v) - df_y(v) \approx 0$.

Now, taking λ and t as above and $v \in \mathbf{fin}(^*E)$, there exist infinitesimal vectors $\iota, \eta, \delta \in {}^*F$ such that

$$\begin{aligned} f(x) - f(y) &= df_y(x-y) + t\iota && (18.2.2)\\ f(x+tv) - f(x) &= tdf_x(v) + t\eta && (18.2.3)\\ f\big(y + t(v + \lambda(x-y))\big) - f(y) &= tdf_y(v) + df_y(x-y) + t\delta. && (18.2.4) \end{aligned}$$

Summing (18.2.2)+(18.2.3)-(18.2.4), we obtain

$$t(df_x(v) - df_y(v)) = t(\iota + \eta - \delta)$$

and $df_x(v) - df_y(v) \approx 0$ as required. □

Moreover Stroyan also showed in [15] that uniform differentiability is a very good generalization of G-differentiability in that, among other properties, it overcomes the impossibility of topologizing $L(E,F)$ in such a way that the evaluation map $(x,y) \mapsto l(x) : L(E,F) \times E \to F$ be continuous when E and F are not normable locally convex spaces[5]; we established the equivalence between uniform and C_{qb} differentiability of standard functions on complete locally convex spaces, and discuss relations with a weaker kind of differentiability when $\mathbf{fin}(^*E) = \mathbf{ns}(^*E)$, in [12]. The main results in this context read as follows.

Assume that U is a non-empty open subset of E, $f : U \to F$, $a \in {}^\sigma U + \mu(^*E)$ and $k \in \mathbb{N}$.

Definition 18.2.1 (Stroyan [15]) *The function f is k-uniformly differentiable at a if there are finite maps $d^{(i)}f_{(\cdot)} : U \to L^{(i)}(E,F)$, $(1 \le i \le k)$ — the **derivatives** of f — such that, with $d := d^{(1)}$, for all $x \in \mathbf{fin}(^*E)$ and all $t \in \mu(^*\mathbb{R})$, there exist $\eta \in \mu(^*F)$ and infinitesimal maps η_i $(2 \le i \le k)$, so that*

$$\begin{aligned} f(a+tx) &= f(a) + tdf_a(x) + t\eta && (18.2.5)\\ d^{(i)}f_{a+tx}(\cdot) &= d^{(i)}f_a(\cdot) + td^{(i+1)}f_a(x,\cdot) + t\eta_i(\cdot)\ (1 \le i < k). && (18.2.6) \end{aligned}$$

*f is k-uniformly differentiable in U if it is k-uniformly differentiable at all $a \in {}^\sigma U + \mu(^*E)$. When f is 1-uniformly differentiable we just say that it is **uniformly** differentiable.*

The following may also be found in [15].

Theorem 18.2.3 *When f is k-uniformly differentiable, not only f itself is S-continuous but also the derivatives $d^{(i)}f : \mathbf{ns}(^*E) \times \mathbf{fin}(^*E) \to {}^*F$ are S-continuous:*

$$\forall x, y \in \mathbf{ns}(^*E)\ \forall u, v \in \mathbf{fin}(^*E) \quad [x \approx u\ \&\ y \approx v \ \Rightarrow\ d^{(i)}f_x(u) \approx d^{(i)}f_y(v)] \quad (1 \le i \le k).$$

[5] A quite simple and clear explanation of this can be found in [7, page 2].

Theorem 18.2.4 (Taylor's formula [15]) *f is k-uniformly differentiable in U if there are maps $df^{(i)} : U \to L^{(i)}_{(\cdot)}(E,F)$ $(1 \leq i \leq k)$, such that, whenever $a \in {}^{\sigma}U + \mu({}^*E)$, $x \in \mathbf{fin}({}^*E)$, $t \in \mu$, there exists $\eta \in \mu({}^*F)$ such that*

$$f(a+tx) = f(a) + \sum_{i=1}^{k} \frac{t^i}{i!} df^{(i)}_a(x, \cdots, x) + t^k \eta.$$

Relations with F-differentiability on Banach spaces are actually studied in [14, chap. 5]. On what regards standard definitions of smoothness we take from [6], whereto we refer the reader for details, namely on **convergence structures**.

Definition 18.2.2 *Let $\mathcal{N}$ denote the filter of neighborhoods of zero in $\mathbb{R}$, $\mathcal{B}$ be a filter on E and, for any filter $\mathcal{F}$ in a (real) vector space, $\mathcal{NF}$ denote the filter generated by $\big\{\{rb|\ r \in N\ \&\ b \in F\}|\ N \in \mathcal{N}\ \&\ F \in \mathcal{F}\big\}$; also, for $k \in \mathbb{N}$, let $\mathcal{B}^k$ be the filter in E^k generated by $\{B_1 \times \cdots \times B_k|\ B_1, \cdots B_k \in \mathcal{B}\}$ and, for any filter $\mathcal{F}$ in $L^{(k)}(E,F)$, $F(\mathcal{B}^k)$ be generated by $\{\bigcup_{l \in F} l(B)|\ F \in \mathcal{F}\ \&\ B \in \mathcal{B}\}$; finally, if ϕ is a function and $\mathcal{F}$ is a filter on the domain of ϕ, $\phi(\mathcal{F})$ is generated by $\{\phi(F)|\ F \in \mathcal{F}\}$.*

1. *$\mathcal{B}$ is **quasi-bounded** if $\mathcal{NB}$ converges to zero; Λ_{qb} denotes the convergence structure of quasi-bounded convergence; when a filter $\mathcal{F}$ converges to $x \in E$ with respect to Λ_{qb}, we write $\mathcal{F} \in \Lambda_{qb}(x)$.*

2. *A filter $\mathcal{F}$ in $L^{(k)}(E,F)$ qb-converges to zero if $\mathcal{F}(\mathcal{B}^k)$ converges to zero in F whenever $\mathcal{B}$ is quasi-bounded in E.*

3. *A function $l : U \to L^{(k)}(E,F)$ is **qb-continuous** if $l(\mathcal{F}) \in \Lambda_{qb}(l(a))$ whenever $a \in U$ and $\mathcal{F}$ is a filter in E converging to a.*

4. *Recall that U is an open subset of E and $f : U \to F$. We say that f is **of class** C^k_{qb} if*

   (a) *f is qb-continuous.*

   (b) *There exist maps $d^i f_{(\cdot)} : U \to L^{(i)}(E,F)$ $(1 \leq i \leq k)$ such that*

      i. *$d^i f_{(\cdot)}$ is qb-continuous, for all $i = 1, \cdots, k$*

      ii. *For all $a \in U$, all $x \in E$ and all $i = 0, \cdots, k-1$*

$$\lim_{t \to 0} \frac{1}{t} \left(d^i f_{a+tx} - d^i f_a \right) = d^{i+1} f_a(x, \cdot)$$

for the topology of pointwise convergence on $L^i(E,F)$.

Theorem 18.2.5 ([12]) *The function f is of class C^k_{qb}, with derivatives $d^i f_{(\cdot)}$, $(1 \leq i \leq k)$ iff it is k-uniformly differentiable with the same derivatives.*

The following is also a consequence (for example see [6]) either of this theorem (18.2.5) or of theorem 18.2.4.

Theorem 18.2.6 *The function $f : E \to F$, between Banach spaces E and F, is of Fréchet class C^k, with derivatives $d^i f_{(\cdot)}$, $(1 \leq i \leq k)$ iff it is k-uniformly differentiable with the same derivatives.*

18.3 Smoothness and finite points

Recall that E and F denote real locally convex spaces.

Lemma 18.3.1 *Suppose $A \in {}^*\mathcal{P}(E)$. The following conditions are equivalent for any internal map $f : A \to {}^*F$ which is GS-differentiable at all $x \in A \cap \mathbf{fin}({}^*E)$, with GS-derivative $df : (A \cap \mathbf{fin}({}^*E)) \times \mathbf{fin}({}^*E) \to {}^*F$.*

1. *f is uniformly differentiable at all $x \in A \cap \mathbf{fin}({}^*E)$*
2. *f is S-continuous on $A \cap \mathbf{fin}({}^*E)$*
3. *$df((A \cap \mathbf{fin}({}^*E)) \times \mathbf{fin}({}^*E)) \subseteq \mathbf{fin}({}^*F)$*

Proof. For the sake of simplicity, we assume $A = {}^*E$; the proof of the general case is an easy adaptation thereof.

Suppose that $df : \mathbf{fin}({}^*E) \times \mathbf{fin}({}^*E) \to {}^*F$ is the GS-derivative of the internal map $f : {}^*E \to {}^*F$.

$(1 \Rightarrow 2)$ This is 1 in theorem 18.2.2 above.

$(2 \Rightarrow 3)$ Assume that f is S-continuous on $\mathbf{fin}({}^*E)$, let $df_{(\cdot)} : \mathbf{fin}({}^*E) \to {}^*L(E, F)$ be the GS-derivative of f. Pick x and v in $\mathbf{fin}({}^*E)$; we must show that $df_x(v)$ is finite; suppose this is not the case, take a semi-norm $\gamma \in \Gamma_F$ such that $\gamma(df_x(v)) \notin \mathcal{O}$ and let $t := \frac{1}{\gamma(df_x(v))}$; t is infinitesimal and, by hypothesis, there exists $\iota \in \mu({}^*F)$ such that $f(x + tv) = f(x) + tdf_x(v) + t\iota$; but then

$$1 = \gamma(tdf_x(v)) = \gamma\left(f(x + tv) - f(x) - t\iota\right) \approx 0$$

which is impossible; it follows that $df_x(v)$ must be finite as required.

$(3 \Rightarrow 1)$ In the presence of GS-differentiability, 3 completes the definition of uniform differentiability. □

Say that a standard function $f : E \to F$ is GS, or uniformly, differentiable at $x \in {}^*E$ with derivative df_x if respectively ${}^*f$ is GS, or uniformly differentiable, at x with derivative ${}^*df$.

Theorem 18.3.1 *If a standard function $f : E \to F$ is GS-differentiable with standard derivative df_x at all $x \in \mathbf{fin}(^*E)$, the following conditions are equivalent*

1. *f is uniformly differentiable on bounded sets with derivative df.*
2. *f is uniformly continuous on bounded sets.*
3. *For any bounded subset B of E^2, $df(B)$ is bounded in F.*
4. *df is uniformly continuous on bounded subsets of E^2.*

As we said before, the proof is easy. Start by keeping in mind that

Theorem 18.3.2 *A subset B of the locally convex space S is bounded iff $^*B \subseteq \mathbf{fin}(^*S)$; in particular, if E and F are normed and $L(E;F)$ is endowed with the usual uniform norm, a subset B of $L(E,F)$ is bounded iff*

$$\forall \phi \in {}^*B \ \forall x \in \mathbf{fin}(^*E) \quad \phi(x) \in \mathbf{fin}(^*F). \tag{18.3.1}$$

Proof of thm. 18.3.1. Equivalences (1 $\Leftrightarrow$ 2 $\Leftrightarrow$ 3) are immediate consequences of lemma 18.3.1 and theorem 18.3.2 (note that, by theorem 18.3.2 above, $^*B \cap \mathbf{fin}(^*E) = {}^*B$ if and only if B is bounded).

(3 $\Rightarrow$ 4) Follows from theorems 18.3.2 and 18.2.2.

(4 $\Rightarrow$ 2) If we show that condition 4 implies that df_x is a finite map whenever B is a bounded subset of E and $x \in {}^*B$, we are done; but this follows from theorem 18.1.6. □

As an application to Banach spaces:

Corollary 18.3.1 *Let E and F be Banach spaces and $f : E \to F$ be a S-continuous and GS-differentiable function at all $x \in \mathbf{fin}(^*E)$ with standard derivative df, then*

1. *df is actually the Fréchet derivative of f and f is of class C^1.*
2. ***(Mean Value)** Denoting $[x,y]$ the line segment from x to y,*

$$\forall x, y \in E \ \exists z \in [x,y] \quad \|f(x) - f(y)\| \leq \|df_z\| \cdot \|x - y\|; \tag{18.3.2}$$

*therefore f is **Lipschitzian** on bounded sets, i.e., for each bounded subset B of E, there exists $K \in \mathbb{R}$ such that*

$$\forall x, y \in B \quad \|f(x) - f(y)\| \leq K\|x - y\|. \tag{18.3.3}$$

3. *For all infinitely near finite vectors $x, y, z \in {}^*E$, there exists an infinitesimal $\iota \in {}^*F$ such that*

$$f(x) - f(y) \;=\; df_z(x-y) + \|x-y\|\iota \tag{18.3.4}$$

Proof. 1. As f is S-continuous on $\mathbf{fin}({}^*E)$, it is uniformly continuous on bounded sets and hence, by theorem 18.3.1, of Fréchet class C^1.

2. Condition (18.3.2) is true in view of part 1 above; therefore condition (18.3.3) is true (by theorems 18.3.1 and 18.3.2) because $\mathbf{fin}({}^*E)$ is *convex.

3. We borrow from [14, page 97]: the Fréchet derivative $x \mapsto df_x$ is S-continuous at finite points by 2b in theorem 18.2.2 and we may apply Transfer of the Mean Value condition (18.3.2) to the function

$$g \;\equiv\; v \mapsto f(v) - df_z(v-z),$$

whose derivative $v \mapsto dg_v = df_v - df_z$ is infinitesimal whenever $v \approx z$, and therefore verifies, for some $c \in [x, y]$, hence $c \approx z$,

$$\frac{\|f(x) - f(y) - df_z(x-y)\|}{\|x-y\|} = \|g(x) - g(y)\| \le \left\| dg_c\left(\frac{x-y}{\|x-y\|}\right)\right\| \approx 0. \quad \square$$

18.4 Smoothness and the nonstandard hull

This section evolves along main ideas due to Manfred Wolff.

Recall that $\cdot \approx \cdot$ is an equivalence relation on ${}^*S$, with equivalence classes $\hat{x} := x + \mu({}^*S)$ and let $\hat{S}$ be the **nonstandard hull** of ${}^*S$ with semi-norms $\hat{\gamma}$ or

$$\hat{S} := \mathbf{fin}({}^*S)_{/\approx} \quad \& \quad \hat{\gamma}(\hat{x}) := \mathbf{st}(\gamma(x)) \quad (x \in \mathbf{fin}({}^*S)).$$

Nonstandard hulls are complete spaces, but may vary with the chosen model of analysis $\mathcal{A}$; actually Banach spaces with invariant nonstandard hulls are finite dimensional, but this is not the case with locally convex non-normable ones; spaces with invariant non-standard hulls were characterized in [5] and were named **HM-spaces** by Keith Stroyan (relevant data is summarized in detail in [14, chap. 10]; also see [10, sec. 3.9] for nonstandard hulls of metric spaces).

Any internal function between locally convex spaces, $f : A \subseteq {}^*E \to {}^*F$ that is S-continuous on $\mathbf{fin}({}^*E) \cap A$ and such that $f(\mathbf{fin}({}^*E) \cap A) \subseteq \mathbf{fin}({}^*F)$ has a natural **non-standard hull** $\hat{f} : \hat{A} \subseteq \hat{E} \to \hat{F}$ too defined by

$$\begin{aligned} \hat{A} \;&:=\; \{\hat{x} |\; x \in \mathbf{fin}({}^*E) \cap A\} \\ \hat{f}(\hat{x}) \;&:=\; \widehat{f(x)} \quad (x \in A \cap \mathbf{fin}({}^*E)); \end{aligned}$$

note that $\hat{f}$ *is not* a standard function in the sense we have been considering, for it certainly is not *a priori* an element of the model of analysis $\mathcal{A}$, nevertheless an application of Nelson's Algorithm ([8], [11]) will provide us with definitions of differentiability for functions without reference to nonstandard extensions.

From now on $f : {}^*E \to {}^*F$ denotes an internal GS-differentiable function with derivative df such that

$$f(\mathbf{fin}({}^*E)) \subseteq \mathbf{fin}({}^*F) \tag{18.4.1}$$

and

$$d\hat{f}(\hat{x}, \hat{v}) := \widehat{df(x, v)} \tag{18.4.2}$$

whenever this is a good definition, namely when df is S-continuous on $\mathbf{fin}({}^*E)$ and $x, v \in \mathbf{fin}({}^*E)$. For any GS-differentiable function $h : E \to F$

$$\Delta(h, x, v, t) := \left(\frac{1}{t}\Big(h(x + tv) - h(x) - dh(x, v)\Big)\right) \quad (x, v \in {}^*E;\ t \in {}^*\mathbb{R} \setminus \{0\}).$$

18.4.1 Strong uniform differentiability

Definition 18.4.1 *An* ***internal*** *function $f : {}^*E \to {}^*F$ is* ***Strongly Uniformly Differentiable (SUD*** *for short) if it is uniformly differentiable on* $\mathbf{fin}({}^*E)$ *and*

$$f(\mathbf{fin}({}^*E)) \subseteq \mathbf{fin}({}^*F).$$

Let $\mathcal{P}$ and $\mathcal{Q}$ denote respectively gauges of semi-norms for E and F. In these terms, f is SUD when the two following conditions are simultaneously verified

$$\forall (x, v) \in {}^*E^2 \quad \big[(x, v) \in \mathbf{fin}({}^*E)^2 \Rightarrow df(x, v) \in \mathbf{fin}({}^*F)\big]$$
$$\forall (x, v) \in {}^*E^2\ \forall t \in {}^*\mathbb{R} \quad \big[(x, v) \in \mathbf{fin}({}^*E)^2 \wedge 0 \neq t \in \mu \Rightarrow \Delta(f, x, v, t) \in \mu({}^*F)\big];$$

these expand respectively to

$$\forall (x, v) \in {}^*E^2 \ \Big[\forall^{st} p \in {}^*P\, \exists^{st} m \in {}^*\mathbb{N}\ p(x) + p(v) \leq m \Rightarrow \forall^{st} q \in {}^*Q \exists^{st} n \in {}^*\mathbb{N}\ q\big(df(x, v)\big) \leq n\Big]$$
$$\forall (x, v) \in {}^*E^2\ \forall t \in {}^*\mathbb{R} \ \Big[\forall^{st} p \in {}^*P\, \exists^{st} m \in {}^*\mathbb{N}\ p(x) + p(v) \leq m \wedge \forall^{st} n \in {}^*\mathbb{N}\ 0 < |t| \leq \frac{1}{n} \Rightarrow \forall^{st} q \in {}^*Q\ q(\Delta(f, x, v, t)) \leq 1\Big];$$

which reduce respectively to the following, where we leave the domains implicit,

$$\forall (x, v)\ \forall^{st} q\ \exists^{st}(p, n)\ \forall^{st} m\ \big[p(x) + p(v) \leq m \Rightarrow\ q\big(df(x, v)\big) \leq n\big]$$
$$\forall (x, v, t)\, \forall^{st} q\, \exists^{st}(p, n)\, \forall^{st} m \Big[p(x) + p(v) \leq m \wedge 0 < |t| \leq \frac{1}{n} \Rightarrow q(\Delta(f, x, v, t)) \leq 1\Big];$$

Nelson's algorithm then gives

$$\forall^{st} q\ \forall^{st} \widetilde{m}\ \exists^{st\ fin} P \times N\ \forall (x,v)\ \exists (p,n) \in P \times N \\ \big[p(x) + p(v) \leq \widetilde{m}(p,n)\ \Rightarrow\ q\big(df(x,v)\big) \leq n\big] \tag{18.4.3}$$

$$\forall^{st} q\ \forall^{st} \widetilde{m}\ \exists^{st\ fin} P \times N\ \forall (x,v,t)\ \exists (p,n) \in P \times N \\ \Big[p(x) + p(v) \leq \widetilde{m}(p,n)\ \wedge\ 0 < |t| \leq \frac{1}{n} \Rightarrow q(\Delta(f,x,v,t)) \leq 1\Big]. \tag{18.4.4}$$

Formulas (18.4.3) and (18.4.4) together define strong uniform differentiability of an internal function and we actually proved the following

Theorem 18.4.1 *An internal GS-differentiable function $f : {}^*E \to {}^*F$ is **SUD** iff*

$$\forall^{st} q\ \forall^{st} \widetilde{m}\ \exists^{st\ fin} P \times N\ \forall (x,v,t)\ \exists (p,n) \in P \times N \\ \Big[\Big[p(x) + p(v) \leq \widetilde{m}(p,n)\ \wedge\ 0 < |t| \leq \frac{1}{n}\Big] \Rightarrow \\ \big[q\big(df(x,v)\big) \leq n\ \wedge\ q(\Delta(f,x,v,t)) \leq 1\big]\Big]; \tag{18.4.5}$$

When f is standard, transfer of (18.4.5) provides the definition, 18.4.2 below, of SU differentiability for functions between locally convex spaces in *any* "universe" $\mathcal{A}$:

Definition 18.4.2 *A **standard** function $h : S \to T$ between locally convex spaces S and T, with unbounded gauges of semi-norms respectively Σ and Θ, is **Strongly Uniformly Differentiable**, with derivative $dh_{(.)} : S \to L(S,T)$, (**SUD** for short) when, leaving implicit that $\sigma \in \Sigma$, $\tau \in \Theta$, $\widetilde{m} : \Sigma \times \mathbb{N} \to \mathbb{N}$, $x \in S$, $v \in S$ and $t \in \mathbb{R} \setminus \{0\}$,*

$$\forall \tau\ \forall \widetilde{m}\ \exists^{fin} C \times N \subseteq \Sigma \times \Theta\ \forall (x,v,t)\ \exists (p,n) \in C \times N \\ \Big[\Big[\sigma(x) + \sigma(v) \leq \widetilde{m}(\sigma,n)\ \wedge\ 0 < |t| \leq \frac{1}{n}\Big] \Rightarrow \\ \big[\tau\big(df(x,v)\big) \leq n\ \wedge\ \tau(\Delta(h,x,v,t)) \leq 1\big]\Big]. \tag{18.4.6}$$

***NB:** variants of this formula where any inequality $\leq$ is taken to be strict, i.e., to be $<$, are equivalent.*

18.4.2 The non-standard hull

Theorem 18.4.2 *Let E and F be locally convex spaces and $f : {}^*E \to {}^*F$ be an internal GS-differentiable function with derivative df such that*

$$f(\mathbf{fin}({}^*E)) \subseteq \mathbf{fin}({}^*F). \tag{18.4.7}$$

f is SUD iff $\hat{f} : \hat{E} \to \hat{F}$ is uniformly differentiable with derivative $d\hat{f} : \hat{E}^2 \to \hat{F}$ given by

$$d\hat{f}(\hat{x}, \hat{v}) := \widehat{df(x, v)}.$$

Proof. Let us first assume that the internal function $f : {}^*E \to {}^*F$ is SUD with derivative df, and takes finite vectors of ${}^*E$ into finite vectors of ${}^*F$.

Define

$$\widehat{\mathcal{P}} := \{\hat{p} |\ p \in \mathcal{P}\} \quad , \quad \widehat{\mathcal{Q}} := \{\hat{q} |\ q \in \mathcal{Q}\};$$

although there might exist continuous semi-norms not of the type $\hat{\gamma}$, $\hat{\mathcal{P}}$ and $\hat{\mathcal{Q}}$ are unbounded directed families of semi-norms, so that condition (18.1.4) still holds with obvious adaptations. Also recall that

$$\hat{f}(\hat{x}) = \widehat{f(x)}$$

and observe that equation

$$d\hat{f}(\hat{x}, \hat{v}) := \widehat{df(x, v)}$$

defines $d\hat{f}$ well, in view of theorems 18.1.5 and 18.1.6 and lemma 18.3.1.

Take $\hat{q} \in \hat{\mathcal{Q}}$ and $\widetilde{m} : \hat{\mathcal{Q}} \times \mathbb{N} \to \mathbb{N}$. Define

$$\widetilde{M}(p, n) := \widetilde{m}(\hat{p}, n) \quad (p \in \mathcal{P},\ n \in \mathbb{N});$$

$\widetilde{M}$ is a standard function, therefore we may deduce from (18.4.5) that there exists a standard finite set $P \times N \subseteq {}^*\mathcal{P} \times {}^*\mathbb{N}$ such that

$$\begin{aligned}
&\forall (x, v, t) \in {}^*E^2 \times {}^*\mathbb{R}\ \exists (p, n) \in P \times N \\
&\Big[p(x) + p(v) < \widetilde{M}(p, n)\ \wedge\ 0 < |t| \leq \frac{1}{n}\ \Rightarrow \\
&\qquad\qquad q(df(x, v)) \leq n\ \wedge\ q(\Delta(f, x, v, t)) < 1 \Big].
\end{aligned}$$

P and N being standard and finite, define

$$\hat{N} := N \quad , \quad \hat{P} := \{\hat{p} |\ p \in P\},$$

and take $(\hat{x}, \hat{v}, t)$; whatever the representatives x and v, there is a corresponding pair $(p, n) \in P \times N$, which may be vary with x and v, but always verifies (18.4.3); suppose that

$$\hat{p}(\hat{x}) + \hat{p}(\hat{v}) < \widetilde{M}(\hat{p}, n) \ \wedge \ 0 < |t| \leq \frac{1}{n}; \tag{18.4.8}$$

all the numbers involved here are standard, $\hat{p}(\hat{x}) \approx p(x)$ & $\hat{p}(\hat{v}) \approx p(v)$ as well as $\hat{q}(\widehat{df(x,v)}) \approx q(df(x,v))$; in particular, it follows

$$\begin{array}{rcl} p(x) + p(v) & < & \widetilde{m}(\hat{p}, n) \ = \ \widetilde{M}(p, n) \\ & \text{so that} & \\ q(df(x,v)) & < & n \\ & \text{an thus} & \\ \hat{q}(\widehat{df(x,v)}) & \leq & n \\ & \text{that is} & \\ \hat{q}(d\hat{f}(\hat{x}, \hat{v})) & \leq & n \end{array}$$

and the first factor in the consequent of (18.4.5) follows for $\hat{f}$ as required; for the remaining factor of the consequent, observe that, actually

$$p(x) + p(v) < \widetilde{M}(p, n) \ \wedge \ 0 < |t| \leq \frac{1}{n};$$

implies

$$q\left(\frac{1}{t}\Big(f(x+tv) - f(x) - df(x,v)\Big)\right) \ = \ q(\Delta(f, x, v, t)) \ < \ 1$$

too; t and n are standard, $0 \neq t$ and f as well as df are S-continuous by lemma 18.3.1, thus

$$\frac{1}{t}\Big(f(x+tv) - f(x) - df(x,v)\Big) \ \approx \ \frac{1}{t}\Big(\hat{f}(\hat{x}+t\hat{v}) - \hat{f}(\hat{x}) - \widehat{df(x,v)}\Big);$$

finally

$$\hat{q}(\Delta(\hat{f}, , x, \hat{v}, t)) \leq 1.$$

The proof that $\hat{f}$ is SUD with derivative $d\hat{f}$ is finished.

Now suppose that $\hat{f}$ is itself uniformly differentiable with derivative $d\hat{f} : \hat{E}^2 \to \hat{F}$. Take $q \in {}^{\sigma}\mathcal{Q}$ and a standard function $\widetilde{m} : {}^*(\mathcal{P} \times \mathbb{N}) \to {}^*\mathbb{N}$; since $\widehat{(\cdot)}$ is 1-1

$$\widetilde{M}(\hat{p}, n) := \widetilde{m}(p, n)$$

defines a function from $\widehat{\mathcal{P}} \times \mathbb{N}$ into $\mathbb{N}$ and we may apply (18.4.6) appropriately written in terms of $\hat{f}$, and obtain the corresponding finite set $\hat{P} \times N \subseteq \widehat{\mathcal{P}} \times \mathbb{N}$; pick $x, v \in E$, $t \in \mathbb{R} \setminus \{0\}$ and use (18.4.6), with $<$ for $\leq$, in order to obtain $(\hat{p}, n) \in \hat{P} \times N$ so that

$$\begin{aligned}\Big[\hat{p}(\hat{x}) + \hat{p}(\hat{v}) \leq \widetilde{M}(\hat{p}, n) \ \wedge \ 0 < |t| \leq \frac{1}{n}\Big] \ \Rightarrow \\ \big[\hat{q}\big(d\hat{f}(\hat{x}, \hat{v})\big) < n \ \wedge \ \hat{q}(\Delta(\hat{f}, \hat{x}, \hat{v}, t)) < 1\big];\end{aligned} \tag{18.4.9}$$

therefore, if

$$p(x) + p(v) < \widetilde{m}(p, n) \ \wedge \ 0 < |t| \leq \frac{1}{n} \tag{18.4.10}$$

then

$$\hat{p}(\hat{x}) + \hat{p}(\hat{v}) \leq \widetilde{M}(\hat{p}, n) \ \wedge \ 0 < |t| \leq \frac{1}{n},$$

hence

$$\hat{q}\big(d\hat{f}(\hat{x}, \hat{v})\big) < n \ \wedge \ \hat{q}(\Delta(\hat{f}, \hat{x}, \hat{v}, t)) < 1$$

or

$$\hat{q}\big(\widehat{df(x, v)}\big) < n \ \wedge \ \hat{q}(\Delta(\hat{f}, \hat{x}, \hat{v}, t)) < 1.$$

It immediately follows that

$$q\big(df(x, v)\big) \ \leq \ n,$$

because both $\hat{q}\big(\widehat{df(x, v)}\big)$ and the n above are standard; moreover $\widehat{(\cdot)}$ is linear and when $t \not\approx 0$

$$q(\Delta(f, x, v, t)) \ \leq \ 1; \tag{18.4.11}$$

it so happens that $t \overset{\tau}{\mapsto} q(\Delta(f, x, v, t))$ is always an internal function and we actually have shown that

$$\forall q \in {}^{\sigma}\mathcal{Q} \ \exists n \in {}^{\sigma}\mathbb{N} \ \left[0 < |t| < \frac{1}{n} \Rightarrow q(\Delta(f, x, v, t)) \ \leq \ 1\right]$$

which is the same as

$$\forall q \in {}^{\sigma}\mathcal{Q} \ \big[0 \neq t \approx 0 \ \Rightarrow \ q(\Delta(f, x, v, t)) \approx 0\big]$$

and thus (18.4.11) also holds when $t \approx 0$. Again because $\widehat{(\cdot)}$ is 1-1, we may define

$$P \ := \ \{p |\ \hat{p} \in \hat{P}\}$$

and we have shown that (18.4.5) holds. □

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19

Nonstandard Palais-Smale conditions

Natália Martins* and Vítor Neves**

Abstract

We present nonstandard versions of the Palais-Smale condition (**PS**) below, some of them generalizations, but still sufficient to prove Mountain Pass Theorems, which are quite important in Critical Point Theory.

19.1 Preliminaries

For notation, other preliminaries and basic references on Nonstandard Analysis we suggest section 1 in article [13] in this volume, namely we assume that we have two set-theoretical structures $\mathcal{A}$ and $\mathcal{B}$ — the former a model of "the relevant" Analysis — and a 1-1 function ${}^*(\cdot) : \mathcal{A} \to \mathcal{B}$ which satisfies the Transfer and Polysaturation Principles. Specific notation and results follow. Recall that ${}^*\mathbb{R}$ denotes the set of ***hyperreal numbers*** and $\mathcal{O}$ the set of finite hyperreal numbers; if $x \in {}^*\mathbb{R}$ is infinitesimal we write $x \approx 0$. If $x, y \in {}^*\mathbb{R}$ are such that $x - y \approx 0$, we say that x is ***infinitely close*** to y and write $x \approx y$. Also remember that finite hyperreal numbers have a standard part:

Theorem 19.1.1 (Standard Part Theorem) *If $x \in \mathcal{O}$ there exists a unique $r \in \mathbb{R}$ such that $x \approx r$; r is called the* ***standard part*** *of x and is denoted by* $\mathbf{st(x)}$ *or* $\mathbf{{}^\circ x}$*. Moreover, for all $x, y \in \mathcal{O}$, $st(x + y) = st(x) + st(y)$, $st(xy) = st(x)st(y)$ and if $x \leq y$ then $st(x) \leq st(y)$.*

*Departamento de Matemática, Universidade de Aveiro.
nataliam@mat.ua.pt
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**vneves@mat.ua.pt

Next we restrict some definitions to normed spaces; just recall that a normed space is a locally convex space whose topology is defined by a single norm and also recall that, for any set $A \in \mathcal{A}$,

$$^{\sigma}A := \{^{*}a : \ a \in A\},$$

and often A and $^{\sigma}A$ are identified for simplicity.

Definition 19.1.1 *Let $(E, \|\cdot\|)$ be a real normed space and $x \in {}^{*}E$.*

1. $x \in \mathbf{fin}(^{*}E)$, *i.e., x is* ***finite*** *if $\|x\| \in \mathcal{O}$;*
2. $x \in \mu(^{*}E)$, *i.e., x is* ***infinitesimal*** *if $\|x\| \approx 0$ in $^{*}\mathbb{R}$; $x \approx 0$ means $x \in \mu(^{*}E)$;*
3. $x \in \mathbf{ns}(^{*}E)$, *i.e., x is* ***near-standard*** *if there exists $a \in {}^{\sigma}E$ such that $x \approx a$, which means $x - a \approx 0$;*
4. $x \in \mathbf{pns}(^{*}E)$, *i.e., x is* ***pre-near-standard*** *if for all positive $\varepsilon \in \mathbb{R}$ there exists $a \in {}^{\sigma}E$ such that $\|x - a\| < \varepsilon$.*

It follows from the above definitions that $\mathbf{ns}(^{*}E) \subseteq \mathbf{pns}(^{*}E)$ and $\mathbf{ns}(^{*}E) \subseteq \mathbf{fin}(^{*}E)$. In general, $\mathbf{ns}(^{*}E) \neq \mathbf{pns}(^{*}E)$ and $\mathbf{ns}(^{*}E) \neq \mathbf{fin}(^{*}E)$.

Theorem 19.1.2 *Let $(E, \|\cdot\|)$ be a real normed space.*

1. *E is complete if and only if $\mathbf{pns}(^{*}E) = \mathbf{ns}(^{*}E)$;*
2. *E is finite dimensional if and only if $\mathbf{fin}(^{*}E) = \mathbf{ns}(^{*}E)$;*
3. *$A \subseteq E$ is compact if and only if $^{*}A \subseteq \mathbf{ns}(^{*}E)$ and $\mathbf{st}(^{*}A) = A$.*

The nonstandard extension of $\mathbb{N}$, $^{*}\mathbb{N}$, is the set of ***hypernatural numbers*** and $^{*}\mathbb{N}_{\infty}$ denotes the set of infinite hypernatural numbers. It is also useful to keep in mind the following properties of sequences and functions in a real normed space $(E, \|\cdot\|)$.

Proposition 19.1.1 *Suppose $(x_n)_{n\in\mathbb{N}}$ is a sequence in E. Then*

1. *$(x_n)_{n\in\mathbb{N}}$ is bounded if and only if $\forall n \in {}^{*}\mathbb{N}_{\infty}\ x_n \in \mathbf{fin}(^{*}E)$;*
2. *$(x_n)_{n\in\mathbb{N}}$ converges to $a \in E$ if and only if $\forall n \in {}^{*}\mathbb{N}_{\infty}\ x_n \approx a$;*
3. *$(x_n)_{n\in\mathbb{N}}$ has a convergent subsequence if and only if $\exists m \in {}^{*}\mathbb{N}_{\infty}$, $x_m \in \mathbf{ns}(^{*}E)$.*

Theorem 19.1.3 *Let E and F be two real normed spaces and $f : E \to F$. The function f is continuous on $a \in E$ if and only if*

$$\forall x \in {}^{*}E\ [x \approx a \Rightarrow f(x) \approx f(a)].$$

19.2 The Palais-Smale condition

Many results in Critical Point Theory involve the following condition, originally introduced in 1964 [10] by Palais and Smale:

Definition 19.2.1 *If E is a real Banach space, the C^1 functional $f : E \to \mathbb{R}$ satisfies the* ***Palais-Smale condition*** *if for all $(x_n)_{n\in\mathbb{N}} \in E^{\mathbb{N}}$,*

$$\textbf{(PS)} \qquad (f(x_n))_{n\in\mathbb{N}} \text{ is bounded and } \lim_{n\to\infty} f'(x_n) = 0 \;\Rightarrow\; (x_n)_{n\in\mathbb{N}} \text{ has a convergent subsequence.}$$

Suppose $(E, \|\cdot\|)$ is a real Banach space with continuous dual E' and duality pairing $\langle\cdot,\cdot\rangle$. Let $C^1(E,\mathbb{R})$ denote the set of continuously Fréchet differentiable functionals defined on E; a is ***critical point*** (resp. ***almost critical point***) of f if $f'(a) = 0$ (resp. $f'(a) \approx 0$), i.e., if $\langle f'(a), v\rangle = 0$ (resp. $\langle f'(a), v\rangle \approx 0$) for all $v \in E$ (resp. $v \in \mathbf{fin}(^*E)$); $f(a)$ is a ***critical value*** of f if $f^{-1}(f(a))$ contains critical points.

If $f : E \to \mathbb{R}$ is Fréchet differentiable we will denote by K the set of all critical points of f, that is,

$$K = \{x \in E : \; f'(x) = 0\}$$

and, for each $c \in \mathbb{R}$, K_c will denote the set of critical points with value c, that is,

$$K_c = \{x \in E : \; f'(x) = 0 \;\wedge\; f(x) = c\} = K \cap f^{-1}(c).$$

The following is easy to prove.

Proposition 19.2.1 *Suppose that $f \in C^1(E,\mathbb{R})$ satisfies* **(PS)**. *Then*

1. *If f is bounded, K is a compact set.*
2. *For each $a, b \in \mathbb{R}$ such that $a \le b$,*

$$\{u \in E : \; a \le f(u) \le b \;\wedge\; f'(u) = 0\} = f^{-1}([a,b]) \cap K$$

is a compact set.

Note that

$$f \text{ satisfies } \textbf{(PS)} \text{ and } K \text{ is compact } \not\Rightarrow \; f \text{ is bounded}$$

and

$$K \text{ is compact and } f \text{ is bounded } \not\Rightarrow \; f \text{ satisfies } \textbf{(PS)},$$

as can be seen with the following two examples.

Example 19.2.1 The real function $f(x) = x^3$ $(x \in \mathbb{R})$ satisfies (**PS**), $K = \{0\}$ is compact and f is not bounded.

Example 19.2.2 Let

$$f(x) = \begin{cases} 2 - x^2 & \text{if } x \in [-1, 1] \\ \frac{1}{x^2} & \text{if } x \notin [-1, 1] \end{cases}$$

f is bounded, $K = \{0\}$ but f does not satisfies (**PS**).

Example 19.2.3 The real functions $\exp(x)$, $\cos(x)$, $\sin(x)$ and all the constant functions defined on $\mathbb{R}$ do not satisfy (**PS**).

Now we will present an important class of functionals that satisfies (**PS**).

Proposition 19.2.2 *If E is a finite dimensional real Banach space and $f \in C^1(E, \mathbb{R})$ is coercive (i.e. $f(x) \to +\infty$ whenever $\|x\| \to +\infty$), then f satisfies* (**PS**).

Proof. Let $(x_n)_{n\in\mathbb{N}} \subseteq E$ be such that $(f(x_n))_{n\in\mathbb{N}}$ is bounded and $\lim_{n\to\infty} f'(x_n) = 0$. Then, for all $n \in {}^*\mathbb{N}_\infty$, $f(x_n) \in \mathcal{O}$ (Proposition 19.1.1). Since f is coercive, for all $n \in {}^*\mathbb{N}_\infty$, $x_n \in \mathbf{fin}({}^*E)$. E is finite dimensional then, by Theorem 19.1.2, $\mathbf{fin}({}^*E) = \mathbf{ns}({}^*E)$ and therefore, for all $n \in {}^*\mathbb{N}_\infty$, $x_n \in \mathbf{ns}({}^*E)$. Finally, Proposition 19.1.1 implies that $(x_n)_{n\in\mathbb{N}}$ has a convergent subsequence. □

19.3 Nonstandard Palais-Smale conditions

The following definition often shortens statements

Definition 19.3.1 *Suppose $f \in C^1(E, \mathbb{R})$. We say that a sequence $(u_n)_{n\in\mathbb{N}}$ is a **Palais-Smale sequence for** f if*

$$(f(u_n))_{n\in\mathbb{N}} \text{ is bounded and } \lim_{n\to\infty} f'(u_n) = 0.$$

Therefore,

*f satisfies the **Palais-Smale condition** (**PS**) if every Palais-Smale sequence for f has a convergent subsequence.*

Suppose E is a real Banach space and $f \in C^1(E, \mathbb{R})$. Next we present some nonstandard variations of (**PS**):

$$(PS0) \qquad \begin{array}{c} (u_n)_{n\in\mathbb{N}} \text{ is a Palais-Smale sequence} \\ \Downarrow \\ \exists m \in {}^*\mathbb{N}_\infty \; u_m \in \mathbf{ns}({}^*E) \end{array}$$

$$(PS1) \qquad f(u) \in \mathcal{O} \;\wedge\; f'(u) \approx 0 \Rightarrow u \in \mathbf{ns}({}^*E)$$

$$(PS2) \qquad \begin{array}{c} f(u) \in \mathcal{O} \;\wedge\; f'(u) \approx 0 \\ \Downarrow \\ u \in \mathbf{fin}({}^*E) \;\wedge\; \mathbf{st}(f(u)) \text{ is a critical value of } f \end{array}$$

$$(PS3) \qquad \begin{array}{c} (u_n)_{n\in\mathbb{N}} \text{ is a Palais-Smale sequence} \\ \Downarrow \\ (u_n)_{n\in\mathbb{N}} \text{ is bounded} \;\wedge\; \forall n \in {}^*\mathbb{N}_\infty \; \mathbf{st}(f(u_n)) \text{ is a critical value of } f \end{array}$$

$$(PS4) \qquad \begin{array}{c} (u_n)_{n\in\mathbb{N}} \text{ is a Palais-Smale sequence} \\ \Downarrow \\ (u_n)_{n\in\mathbb{N}} \text{ is bounded} \;\wedge\; \exists n \in {}^*\mathbb{N}_\infty \; \mathbf{st}(f(u_n)) \text{ is a critical value of } f \end{array}$$

Proposition 19.3.1 *If $f \in C^1(E, \mathbb{R})$ then*

$$(PS1) \Leftrightarrow \Big[f(u) \in \mathcal{O} \wedge f'(u) \approx 0 \Rightarrow u \in \mathbf{ns}({}^*E) \;\wedge\; \mathbf{st}(f(u)) \textit{ is a critical value of } f \Big].$$

Proof. The implication $\Leftarrow$ is trivial. For the proof of the other implication, suppose that $u \in {}^*E$ is such that $f(u) \in \mathcal{O}$ and $f'(u) \approx 0$. By $(PS1)$, there exists $a \in {}^\sigma E$ such that $u \approx a$ and, therefore, from the continuity of f and f' it follows that (Theorem 19.1.3)

$$f(a) \approx f(u) \quad \text{and} \quad f'(a) \approx f'(u) \approx 0$$

too, so that $f(a) = \mathbf{st}(f(u))$ and $f(a)$ is a critical value of f. □

Nonstandard versions of (**PS**) and (**PS**) itself are related as follows.

Theorem 19.3.1 *For any real Banach space E we have*

$$(PS1) \Rightarrow (\mathbf{PS}) \Leftrightarrow (PS0) \Rightarrow (PS2) \Leftrightarrow (PS3) \Rightarrow (PS4).$$

Proof. It is clear that $(\mathbf{PS}) \Leftrightarrow (PS0)$, $(PS1) \Rightarrow (PS0)$ and $(PS3) \Rightarrow (PS4)$, therefore

$$(PS1) \Rightarrow (\mathbf{PS}) \Leftrightarrow (PS0) \quad \text{and} \quad (PS3) \Rightarrow (PS4)$$

are true.

First we prove that $(PS0) \Rightarrow (PS2)$. Let $u \in {}^*E$ such that $f(u) \in \mathcal{O}$ and $f'(u) \approx 0$. Fix $M \in \mathbb{R}^+$ such that $|f(u)| < M$. Suppose $u \notin \mathbf{fin}({}^*E)$. For each $n \in \mathbb{N}$, define the standard set

$$H_n := \Big\{x \in E : \ |f(x)| < M \ \wedge \ \|f'(x)\| < \frac{1}{n} \ \wedge \ \|x\| > n\Big\}.$$

Since, for each $n \in \mathbb{N}$,

$$u \in {}^*H_n = \Big\{x \in {}^*E : \ |f(x)| < M \ \wedge \ \|f'(x)\| < \frac{1}{n} \ \wedge \ \|x\| > n\Big\},$$

we conclude that ${}^*H_n \neq \emptyset$ and the Transfer Principle says that $H_n \neq \emptyset$.

For each $n \in \mathbb{N}$, take $x_n \in H_n$. Then $(x_n)_{n\in\mathbb{N}}$ is a Palais-Smale sequence but, for all $n \in {}^*\mathbb{N}_\infty$, $x_n \notin \mathbf{fin}({}^*E)$, and hence, for all $n \in {}^*\mathbb{N}_\infty$, $x_n \notin \mathbf{ns}({}^*E)$. This is a contradiction with $(PS0)$ and therefore $u \in \mathbf{fin}({}^*E)$.

Now we will prove that if $f(u) \in \mathcal{O}$ and $f'(u) \approx 0$, then $\mathbf{st}(f(u))$ is a critical value of f.

Let $\alpha = \mathbf{st}(f(u))$ and, for each $n \in \mathbb{N}$, define

$$F_n := \Big\{x \in E : \ |f(x)| < M \ \wedge \ \|f'(x)\| < \frac{1}{n} \ \wedge \ |\alpha - f(x)| < \frac{1}{n}\Big\}.$$

Since, for each $n \in \mathbb{N}$,

$$u \in {}^*F_n = \Big\{x \in {}^*E \ : \ |f(x)| < M \ \wedge \ \|f'(x)\| < \frac{1}{n} \ \wedge \ |\alpha - f(x)| < \frac{1}{n}\Big\}$$

then, ${}^*F_n \neq \emptyset$ and therefore $F_n \neq \emptyset$.

For each $n \in \mathbb{N}$ take $x_n \in F_n$. Then $(x_n)_{n\in\mathbb{N}}$ is a Palais-Smale sequence and from $(PS0)$ we conclude

$$\exists a \in {}^\sigma E \ \exists m \in {}^*\mathbb{N}_\infty \ \ x_m \approx a.$$

Since $f \in C^1(E, \mathbb{R})$,

$$\alpha \approx f(x_m) \approx f(a) \ \text{ and } \ 0 \approx f'(x_m) \approx f'(a).$$

Therefore $\alpha = f(a)$, $f'(a) = 0$, α is a critical value of f and $(PS0) \Rightarrow (PS2)$ is proven.

Lets prove that $(PS2) \Rightarrow (PS3)$. Let $(u_n)_{n\in\mathbb{N}}$ be a Palais-Smale sequence. Then,

$$\forall n \in {}^*\mathbb{N}_\infty \ [\, f(u_n) \in \mathcal{O} \ \wedge \ f'(u_n) \approx 0\,].$$

From $(PS2)$ we conclude that

$$\forall n \in {}^*\mathbb{N}_\infty \ [\, u_n \in \mathbf{fin}({}^*E) \ \wedge \ \mathbf{st}(f(u_n)) \text{ is a critical value of } f\,]$$

so we may conclude that $(u_n)_{n\in\mathbb{N}}$ is bounded and

$$\forall n \in {}^*\mathbb{N}_\infty \ \mathbf{st}(f(u_n)) \text{ is a critical value of } f.$$

Finally we will prove that $(PS3) \Rightarrow (PS2)$. Let $u \in {}^*E$ such that $f(u) \in \mathcal{O}$ and $f'(u) \approx 0$. Let $M \in \mathbb{R}^+$ such that $|f(u)| < M$. Suppose $u \notin \mathbf{fin}({}^*E)$; we construct a Palais-Smale sequence $(x_n)_{n\in\mathbb{N}}$ using the sets H_n, as in the proof of the first part of the proof that $(PS0) \Rightarrow (PS2)$, in such a way that $(x_n)_{n\in\mathbb{N}}$ is not bounded, which contradicts $(PS3)$.

Suppose now that $f(u) \in \mathcal{O}$ and $f'(u) \approx 0$. We need to prove that $\alpha = \mathbf{st}(f(u))$ is a critical value of f. Just as in the second part of the proof of $(PS0) \Rightarrow (PS2)$, we can build a Palais-Smale sequence $(x_n)_{n\in\mathbb{N}}$ such that for all $n \in {}^*\mathbb{N}_\infty$, $f(x_n) \approx \alpha$. From $(PS3)$ we conclude that for all $n \in {}^*\mathbb{N}_\infty$, $\mathbf{st}(f(x_n)) = \alpha$ is a critical value of f. □

More can be said when E is separable:

Theorem 19.3.2 *When E is a separable Banach space,* (**PS**) *and* $(PS1)$ *are equivalent; in other words: if E is a separable Banach space, a C^1 function $f : E \to \mathbb{R}$ verifies the Palais-Smale condition if and only if almost critical points where f is finite are near-standard.*

Proof. Suppose f satisfies (**PS**) and $u \in {}^*E$ is such that $f(u) \in \mathcal{O}$ and $f'(u) \approx 0$. If $u \notin \mathbf{ns}({}^*E)$, it follows from Theorem 19.1.2 that $u \notin \mathbf{pns}({}^*E)$, that is,

$$\exists \varepsilon \in \mathbb{R}^+ \ \forall y \in {}^\sigma E \ \ \|u - y\| > \varepsilon.$$

Let $V := \{v_p : p \in \mathbb{N}\}$ be dense in E. We will construct a Palais-Smale sequence $(x_n)_{n\in\mathbb{N}}$ in E such that for all $N \in {}^*\mathbb{N}_\infty$, $x_N \notin \mathbf{ns}({}^*E)$ which contradicts (**PS**).

Let $M \in \mathbb{R}^+$ be such that $|f(u)| < M$. For each $n \in \mathbb{N}$, define

$$C_n := \left\{ x \in E : \ |f(x)| < M \wedge \|f'(x)\| < \frac{1}{n} \wedge \forall p \in \mathbb{N}\,[\, p \le n \Rightarrow \|x - v_p\| > \varepsilon\,] \right\}.$$

Since, for each $n \in \mathbb{N}$, $u \in {}^*C_n$, we conclude that ${}^*C_n \neq \emptyset$ and therefore $C_n \neq \emptyset$.

For each $n \in \mathbb{N}$, take $x_n \in C_n$. Let $N \in {}^*\mathbb{N}_\infty$ and $v \in {}^\sigma E$. Since $\overline{V} = E$, there exists $p_0 \in \mathbb{N}$ such that $\|v - v_{p_0}\| < \frac{\varepsilon}{2}$. Since

$$\|x_N - v\| \geq \|x_N - v_{p_0}\| - \|v_{p_0} - v\| > \frac{\varepsilon}{2}$$

we conclude that $x_N \notin \mathbf{ns}({}^*E)$ which contradicts (**PS**). □

It is not clear whether this equivalence is true in general.

The following result is consequence of the fact that if E is finite dimensional, then E is separable and $\mathbf{fin}({}^*E) = \mathbf{ns}({}^*E)$.

Theorem 19.3.3 *If E is finite dimensional*

$$(PS1) \Leftrightarrow (\mathbf{PS}) \Leftrightarrow (PS0) \Leftrightarrow (PS2) \Leftrightarrow (PS3) \Leftrightarrow (PS4).$$

Another easy observation:

Proposition 19.3.2 *Any C^1 functional in a Banach space which verifies $(PS4)$ and admits a Palais-Smale sequence, has at least one critical point.*

Proof. Suppose that $(u_n)_{n\in\mathbb{N}}$ is a Palais-Smale sequence for the functional $f \in C^1(E, \mathbb{R})$. Then there exist $m \in {}^*\mathbb{N}_\infty$ such that $\mathbf{st}(f(u_m))$ is a critical value of f. Hence, there exists $a \in E$ such that $f(a) = \mathbf{st}(f(u_m))$ and $f'(a) = 0$. □

Next we present an example which shows that

$$(PS2) \not\Rightarrow (PS1).$$

Example 19.3.1 Let H be an infinite dimensional Hilbert space and define

$$\begin{array}{rcl} f : H & \to & \mathbb{R} \\ x & \mapsto & f(x) = g(\|x\|^2 - 1) \end{array}$$

where $g : \mathbb{R} \to \mathbb{R}$ is given by

$$g(t) = \begin{cases} 0 & \text{if} \quad t \leq 0 \\ t^2 \exp^{-\frac{1}{t^2}} & \text{if} \quad t > 0 \end{cases}.$$

Observe that g is a C^1 function and

$$g'(t) \approx 0 \;\Leftrightarrow\; [t \leq 0 \;\vee\; t \approx 0]. \tag{19.3.1}$$

Also,

$$\begin{array}{rcl} h : H & \to & \mathbb{R} \\ x & \mapsto & \|x\|^2 - 1 \end{array}$$

is a C^1 functional and

$$\forall a \in H \ \forall x \in H \ \langle h'(a), x\rangle = 2a \bullet x$$

where $\cdot \bullet \cdot$ denotes the inner product in H. Therefore, f is a C^1 functional. We will prove that f does not satisfy $(PS1)$ but satisfies $(PS2)$.

By Theorem 19.1.2 we can take $u \in {}^*H$ such that $u \in \mathbf{fin}({}^*H) \setminus \mathbf{ns}({}^*H)$ and $\|u\| = 1$. Hence, $f(u) = 0$ and $f'(u) = 0$, which shows that f does not satisfy $(PS1)$.

Note that

$$v \notin \mathbf{fin}({}^*H) \Rightarrow f(v) \notin \mathcal{O}$$

and

$$\begin{aligned} f'(v) \approx 0 \ &\Leftrightarrow \ \|f'(v)\| \approx 0 \\ &\Leftrightarrow \ \forall x \in \mathbf{fin}({}^*H) \ \langle f'(v), x\rangle = (2v \bullet x) g'(\|v\|^2 - 1) \approx 0. \end{aligned} \tag{19.3.2}$$

Next we will prove that

$$f'(v) \approx 0 \ \Rightarrow \ [\, \|v\| \leq 1 \ \vee \ \|v\| \approx 1\,]. \tag{19.3.3}$$

If $f'(v) \approx 0$ and $v \neq 0$, by (19.3.2) either $2v \bullet \frac{v}{\|v\|} \approx 0$, and thus $\|v\| \approx 0$, or $g'(\|v\|^2 - 1) \approx 0$, so that, by (19.3.1), $\|v\| \leq 1$ or $\|v\| \approx 1$; in all possible cases (19.3.3) holds.

Since

$$[\, \|v\| \leq 1 \ \vee \ \|v\| \approx 1\,] \ \Rightarrow \ f(v) \approx 0$$

and 0 is a critical value of f, we may conclude that

$f(v) \in \mathcal{O} \ \wedge \ f'(v) \approx 0 \ \Rightarrow \ v \in \mathbf{fin}({}^*H) \ \wedge \ \mathbf{st}(f(v)) = 0$ is a critical value of f

proving that f does satisfy $(PS2)$.

Remark 19.3.1 Other consequences of Example 19.3.1*:*

1. If we assume H to be separable, Theorem 19.3.2 shows that

$$(PS2) \not\Rightarrow (\mathbf{PS}).$$

2. Moreover, the fact that $f \in C^1(E, \mathbb{R})$ and satisfies $(PS2)$, does not imply that the sets

$$f^{-1}([a, b]) \cap K \qquad (a \leq b)$$

are compact (see Proposition 19.2.1).

It follows easily from Proposition 19.1.1 that condition $(PS3)$ is equivalent to

$$\begin{gathered}(u_n)_{n\in\mathbb{N}} \text{ is a Palais-Smale sequence} \\ \Downarrow \\ (u_n)_{n\in\mathbb{N}} \text{ is bounded } \wedge \text{ all convergent subsequences of } (f(u_n))_{n\in\mathbb{N}} \\ \text{converge to a critical value of } f\end{gathered}$$

and $(PS4)$ is equivalent to

$$\begin{gathered}(u_n)_{n\in\mathbb{N}} \text{ is a Palais-Smale sequence} \\ \Downarrow \\ (u_n)_{n\in\mathbb{N}} \text{ is bounded } \wedge \text{ there is a subsequence of } (f(u_n))_{n\in\mathbb{N}} \\ \text{which converges to a critical value of } f.\end{gathered}$$

Moreover, in the finite dimensional case, these two standard conditions are equivalent.

19.4 Palais-Smale conditions per level

In this section we present a weaker *compactness condition* for C^1 functionals introduced in 1980 [3] by Brézis, Coron and Nirenberg. In the survey book [7] the reader can find more variants of the (**PS**) condition.

Definition 19.4.1 *Suppose $f \in C^1(E,\mathbb{R})$ and $c \in \mathbb{R}$. We say that $(u_n)_{n\in\mathbb{N}}$ is a **Palais-Smale sequence of level** c (for f) if*

$$\lim_{n\to\infty} f(u_n) = c \quad \text{and} \quad \lim_{n\to\infty} f'(u_n) = 0.$$

*f satisfies the **Palais-Smale condition of level** c, $(\mathbf{PS})_c$, if every Palais-Smale sequence of level c has a convergent subsequence.*

Remark 19.4.1 Suppose that $f \in C^1(E,\mathbb{R})$. Then

1. If f satisfies (**PS**), then f satisfies $(\mathbf{PS})_c$ for all $c \in \mathbb{R}$.
2. If f satisfies $(\mathbf{PS})_c$, then the set of critical points of value c, K_c, is compact.

Example 19.4.1 The function $\exp(x) : \mathbb{R} \to \mathbb{R}$ satisfies $(\mathbf{PS})_c$ for all c except for $c = 0$. The real functions $\sin(x)$ and $\cos(x)$ defined in $\mathbb{R}$ satisfy $(\mathbf{PS})_c$ for all c except for $c = 1$ and $c = -1$.

19.5 Nonstandard variants of Palais-Smale conditions per level

As above, E is a real Banach space, $f \in C^1(E, \mathbb{R})$ and $c \in \mathbb{R}$.

Obvious adaptations of conditions $(PS0)$, $(PS1)$ and $(PS2)$ provide the following conditions *per* level:

$$(PS0)_c \qquad \begin{array}{c} (u_n)_{n\in\mathbb{N}} \text{ is a Palais-Smale sequence of level } c \\ \Downarrow \\ \exists m \in {}^*\mathbb{N}_\infty \; u_m \in \mathbf{ns}({}^*E) \end{array}$$

$$(PS1)_c \qquad f(u) \approx c \;\wedge\; f'(u) \approx 0 \Rightarrow u \in \mathbf{ns}({}^*E)$$

$$(PS2)_c \qquad \begin{array}{c} f(u) \approx c \;\wedge\; f'(u) \approx 0 \\ \Downarrow \\ u \in \mathbf{fin}({}^*E) \;\wedge\; c = \mathbf{st}(f(u)) \text{ is a critical value of } f \end{array}$$

Note that if $(u_n)_{n\in\mathbb{N}}$ is a Palais-Smale sequence of level c, then

$$\forall n \in {}^*\mathbb{N}_\infty \; \mathbf{st}(f(u_n)) = c$$

hence, the adaptation of conditions $(PS3)$ and $(PS4)$ to this context are equivalent to the standard condition:

$$\begin{array}{c} (u_n)_{n\in\mathbb{N}} \text{ is a Palais-Smale sequence of level } c \\ \Downarrow \\ (u_n)_{n\in\mathbb{N}} \text{ is bounded } \wedge\; c \text{ is a critical valuc of } f. \end{array}$$

The variants of Theorems 19.3.1, 19.3.2 and 19.3.3 can be easily proven.

Theorem 19.5.1 *For any real Banach space E we have*

$$(PS1)_c \Rightarrow (\mathbf{PS})_c \Leftrightarrow (PS0)_c \Rightarrow (PS2)_c.$$

Theorem 19.5.2 *Suppose E is a real separable Banach space and $f \in C^1(E, \mathbb{R})$. Then f satisfies $(\mathbf{PS})_c$ if and only if f satisfies $(PS1)_c$.*

Theorem 19.5.3 *If E has finite dimension, then*

$$(PS1)_c \Leftrightarrow (\mathbf{PS})_c \Leftrightarrow (PS0)_c \Leftrightarrow (PS2)_c.$$

Remark 19.5.1 Example 19.3.1 also shows that condition $(PS2)_c$ does generalize $(\mathbf{PS})_c$ and $(PS1)_c$ when $c = 0$.

19.6 Mountain Pass Theorems

Some important results in Critical Point Theory, such as the Mountain Pass Theorems and some variants of Ekeland's Variational Principle, can be obtained using a *deformation technique*. This technique was introduced in 1934 by Lusternik and Schnirelman [8] and consists in deforming a given C^1 functional outside the set of critical points. In 1983 Willem proved in [14] the Quantitative Deformation Lemma; a very technical lemma that involves the concept of pseudo-gradient vector field (see [14] or [9]).

For $S \subseteq E$, $\alpha \in \mathbb{R}^+$ and $c \in \mathbb{R}$, we use the following notations

$$f^c := \{x \in E : \ f(x) \leq c\} \quad \text{and} \quad S_\alpha := \{x \in E : \ dist(x, S) \leq \alpha\}$$

where

$$dist(x, S) = \inf\{\|x - y\| : \ y \in S\}.$$

Lemma 19.6.1 (Quantitative Deformation Lemma) *Let $f \in C^1(E, \mathbb{R})$, $S \subseteq E$, $c \in \mathbb{R}$, $\varepsilon, \delta \in \mathbb{R}^+$ be such that*

$$\forall y \in f^{-1}\big([c - 2\varepsilon, c + 2\varepsilon]\big) \cap S_{2\delta} \quad \Big[\ \|f'(y)\| \geq \frac{8\varepsilon}{\delta} \ \Big].$$

Then there exists $\eta \in C([0,1] \times E, E)$ such that

1. $\eta(0, y) = y$;
2. $\eta(t, y) = y$ *if* $y \notin f^{-1}([c - 2\varepsilon, c + 2\varepsilon]) \cap S_{2\delta}$;
3. $\eta(1, f^{c+\varepsilon} \cap S) \subseteq f^{c-\varepsilon}$;
4. *for all* $t \in [0,1]$, $\eta(t, \cdot) : E \to E$ *is a homeomorphism;*
5. $\forall t \in [0,1], \forall y \in E \ \ \|\eta(t, y) - y\| \leq \delta$;
6. *for each* $y \in E$, $f(\eta(\cdot, y))$ *is non increasing;*
7. $\forall t \in]0,1], \forall y \in f^c \cap S_\delta \ \ f(\eta(t, y)) < c$.

An easy consequence of the Quantitative Deformation Lemma is the following variant of Ekeland's Variational Principle (see [9, page 14]).

Corollary 19.6.1 *Let $f \in C^1(E, \mathbb{R})$ be bounded from below. Then, for any $\varepsilon \in \mathbb{R}^+$, there exists $u \in E$ such that*

$$f(u) \leq \inf_{x \in E} f(x) + \varepsilon \quad \text{and} \quad \|f'(u)\| < \sqrt{\varepsilon}.$$

Applying the Transfer Principle to Corollary 19.6.1 we obtain

Corollary 19.6.2 *Let $f \in C^1(E,\mathbb{R})$ be bounded from below. Then there exists a point $u \in {}^*E$ such that*

$$f(u) \approx \inf_{x\in E} f(x) \quad \textit{and} \quad f'(u) \approx 0.$$

Now we can deduce

Theorem 19.6.1 *Let $f \in C^1(E,\mathbb{R})$ be bounded below and $c = \inf_{x\in E} f(x)$. If f satisfies $(PS2)_c$ then c is a critical value of f.*

Proof. By Corollary 19.6.2 we conclude that there exists $u \in {}^*E$ such that $f(u) \approx c$ and $f'(u) \approx 0$. Since f satisfies $(PS2)_c$, $c = \mathbf{st}(f(u))$ is a critical value of f. □

Theorem 19.6.1 is a generalization of the classical result (see [7, page 16]):

Theorem 19.6.2 *Let $f \in C^1(E,\mathbb{R})$ be bounded below and $c = \inf_{x\in E} f(x)$. If f satisfies $(\mathbf{PS})_c$ then c is a critical value of f.*

We now state the Mountain Pass Theorem introduced by Ambrosetti and Rabinowitz in 1973 [1].

Theorem 19.6.3 (Mountain Pass Theorem, Ambrosetti-Rabinowitz) *Let E be a real Banach space and $f \in C^1(E,\mathbb{R})$. Suppose that*

1. *there exists $e \in E$ and $r \in \mathbb{R}^+$ such that $\|e\| > r$ and*

$$\max\{f(0), f(e)\} \ < \ b := \inf_{\|x\|=r} f(x);$$

2. $\Gamma = \{\gamma \in C([0,1],E) : \ \gamma(0) = 0 \wedge \gamma(1) = e\}$ *and* $c := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} f(\gamma(t))$;

3. *f satisfies* (**PS**).

Then $c \geq b$ and c is a critical value of f.

We usually say that if $f : E \to \mathbb{R}$ satisfies condition *1* of Theorem 19.6.3, then f satisfies the **mountain pass geometry** (with respect to 0 and e).

Remark 19.6.1 Conditions *1* and *2* of Theorem 19.6.3 are not enough to imply that c is a critical value of f as we can see with the following example (see [7, page 36]). The function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x,y) = x^2 + (x+1)^3 y^2$ satisfies the mountain pass geometry with $0 = (0,0)$, $e = (-2,3)$ and $r = \frac{1}{2}$. $(0,0)$ is a strict local minimizer and is the only critical point of f. Therefore, there is no $z \in \mathbb{R}^2$ such that $f(z) = c > 0$ and $f'(z) = 0$.

In 1980 Brézis, Coron and Nirenberg obtained in [3] a generalization of the Mountain Pass Theorem of Ambrosetti-Rabinowitz for functionals satisfying the $(\mathbf{PS})_c$ condition:

Theorem 19.6.4 (Mountain Pass Theorem, Brézis-Coron-Nirenberg) *Let E be a real Banach space and $f \in C^1(E, \mathbb{R})$. Suppose that*

1. *there exists $e \in E$ and $r \in \mathbb{R}^+$ such that $\|e\| > r$ and*

$$\max\{f(0), f(e)\} \ < \ b := \inf_{\|x\|=r} f(x);$$

2. $\Gamma = \{\gamma \in C([0,1], E) : \ \gamma(0) = 0 \wedge \gamma(1) = e\}$ *and* $c := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} f(\gamma(t))$;

3. *f satisfies $(\mathbf{PS})_c$.*

Then $c \geq b$ and c is a critical value of f.

This result and the Mountain Pass Theorem of Ambrosetti-Rabinowitz are easy consequences of the Quantitative Deformation Lemma, since it is possible to prove that for every C^1 functional that satisfies conditions *1* and *2* of both theorems, there exists a Palais-Smale sequence of level c (see [9, page 19]).

We now deduce the following generalization of the Mountain Pass Theorem of Brézis-Coron-Nirenberg.

Theorem 19.6.5 *Let E be a real Banach space and $f \in C^1(E, \mathbb{R})$. Suppose that*

1. *there exists $e \in E$ and $r \in \mathbb{R}^+$ such that $\|e\| > r$ and*

$$\max\{f(0), f(e)\} \ < \ b := \inf_{\|x\|=r} f(x);$$

2. $\Gamma = \{\gamma \in C([0,1], E) : \gamma(0) = 0 \wedge \gamma(1) = e\}$ *and* $c := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} f(\gamma(t))$;

3. *f satisfies $(PS4)_c$.*

Then $c \geq b$ and c is a critical value of f.

Proof. From the Quantitative Deformation Lemma there exists a Palais-Smale sequence of level c. Since f satisfies $(PS4)_c$, c is a critical value of f. □

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20

Averaging for ordinary differential equations and functional differential equations

Tewfik Sari*

Abstract

A nonstandard approach to averaging theory for ordinary differential equations and functional differential equations is developed. We define a notion of perturbation and we obtain averaging results under weaker conditions than the results in the literature. The classical averaging theorems approximate the solutions of the system by the solutions of the averaged system, for Lipschitz continuous vector fields, and when the solutions exist on the same interval as the solutions of the averaged system. We extend these results to perturbations of vector fields which are uniformly continuous in the spatial variable with respect to the time variable and without any restriction on the interval of existence of the solution.

20.1 Introduction

In the early seventies, Georges Reeb, who learned about Abraham Robinson's *Nonstandard Analysis* (NSA) [29], was convinced that NSA gives a language which is well adapted to the study of perturbation theory of differential equations (see [6, p. 374] or [25]). The axiomatic presentation *Internal Set Theory* (IST) [26] of NSA given by E. Nelson corresponded more to the Reeb's dream and was in agreement with his conviction *"Les entiers naïfs ne remplissent pas* $\mathbb{N}$*"*. Indeed, no formalism can recover exactly all the actual phenomena, and *nonstandard objects* which may be considered as a formalization of *non-naïve objects* are already elements of our usual (standard) sets. We do not need any use of *stars* and *enlargements*. Thus, the Reebian school adopted IST. For more informations about Reeb's dream and convictions see the Reeb's preface of Lutz and Goze's book [25], Stewart's book [40] p. 72, or Lobry's book [23].

The Reebian school of *nonstandard perturbation theory of differential equations* produced various and numerous studies and new results as attested by a

*Université de Haute Alsace, Mulhouse, France.
T.Sari@uha.fr

lot of books and proceedings (see [2, 3, 4, 7, 8, 9, 10, 12, 23, 25, 30, 37, 42] and their references). It has become today a well-established tool in asymptotic theory, see, for instance, [17, 18, 20, 24, 39] and the five-digits classification 34E18 of the 2000 Mathematical Subject Classification. Canards and rivers (or Ducks and Streams [7]) are the most famous discoveries of the Reebian school.

The classical perturbation theory of differential equations studies deformations, instead of perturbations, of differential equations (see Section 2.1). Classically the phenomena are described asymptotically, when the parameter of the deformation tends to some fixed value. The first benefit of NSA is a natural and useful notion of perturbation. A perturbed equation becomes a simple nonstandard object, whose properties can be investigated directly. This aspect of NSA was clearly described by E. Nelson in his paper *Mathematical Mythologies* [30], p. 159, when he said *"For me, the most exciting aspect of nonstandard analysis is that concrete phenomena, such as ducks and streams, that classically can only be described awkwardly as asymptotic phenomena, become mythologized as simple nonstandard objects."*

The aim of this paper is to present some of the basic nonstandard techniques for averaging in Ordinary Differential Equations (ODEs), that I obtained in [32, 36], and their extensions, obtained by M. Lakrib [19], to Functional Differential Equations (FDEs). This paper is organized as follows. In Section 20.2 we define the notion of *perturbation* of a vector field. The main problem of perturbation theory of differential equations is to describe the behavior of trajectories of perturbed vector fields. We define a standard topology on the set of vector fields, with the property that f is a perturbation of a standard vector field f_0 if and only if f is infinitely close to f_0 for this topology. In Section 20.3 we present the Stroboscopic Method for ODEs and we show how to use it in the proof of the averaging theorem for ODEs. In Section 20.4 we present the Stroboscopic Method for FDEs and we show how to use it in the proofs of the averaging theorem for FDEs. The nonstandard approaches of averaging are rather similar in structure both in ODEs and FDEs. It should be noticed that the usual approaches of averaging make use of different concepts for ODEs and for FDEs: compare with [5, 31] for averaging in ODEs and [13, 14, 15, 22] for averaging in FDEs.

20.2 Deformations and perturbations

20.2.1 Deformations

The classical *perturbation theory of differential equations* studies families of differential equations

$$\dot{x} = F(x, \varepsilon), \tag{20.2.1}$$

where x belongs to an open subset U of $\mathbb{R}^n$, called *phase space*, and ε belongs to a subset B of $\mathbb{R}^k$, called *space of parameters*.

The family (20.2.1) of differential equations is said to be a *k-parameters deformation* of the vector field $F_0(x) = F(x, \varepsilon_0)$, where ε_0 is some fixed value of ε. The main problem of the perturbation theory of differential equations is to investigate the behavior of the vector fields $F(x, \varepsilon)$ when ε tends to ε_0.

The intuitive notion of a *perturbation* of the vector field F_0 which would mean any vector field which is *close to* F_0 does not appear in the theory. The situation is similar in the theory of *almost periodic functions* which, classically, do not have *almost periods*. The nonstandard approach permits to give a very natural notion of almost period (see [16, 28, 33, 41]). The *classical perturbation theory of differential equations* considers *deformations* instead of *perturbations* and would be better called *deformation theory of differential equations*. Actually the vector field $F(x, \varepsilon)$ when ε is sufficiently close to ε_0 is called a perturbation of the vector field $F_0(x)$. In other words, the differential equation

$$\dot{x} = F_0(x) \tag{20.2.2}$$

is said to be the *unperturbed equation* and equation (20.2.1), for a fixed value of ε, is called the *perturbed equation*. This notion of *perturbation* is not very satisfactory since many of the results obtained for the family (20.2.1) of differential equations take place in all systems that are close to the unperturbed equation (20.2.2). Noticing this fact, V. I. Arnold (see [1], footnote page 157) suggested to study a neighborhood of the unperturbed vector field $F_0(x)$ in a suitable function space. For the sake of mathematical convenience, instead of neighborhoods, one considers deformations. According to V. I. Arnold, the situation is similar with the historical development of variational concepts, where the directional derivative (Gâteaux differential) preceded the derivative of a mapping (Fréchet differential). Nonstandard analysis permits to define a notion of perturbation. To say that a vector f is a *perturbation* of a standard vector field f_0 is equivalent to say that f is *infinitely close* to f_0 is a suitable function space, that is f *is in any standard neighborhood* of f_0. Thus, studying perturbations in our sense is nothing than studying neighborhoods, as suggested by V. I. Arnold.

20.2.2 Perturbations

Let X be a standard topological space. A point $x \in X$ is said to be infinitely close to a standard point $x_0 \in X$, which is denoted by $x \simeq x_0$, if x is in any standard neighborhood of x_0. Let A be a subset of X. A point $x \in X$ is said to be *nearstandard in* A if there is a standard $x_0 \in A$ such that $x \simeq x_0$.

Let us denote by

$$^{NS}A = \{x \in X : \exists^{st} x_0 \in A \ x \simeq x_0\},$$

the *external-set* of nearstandard points in A [34]. Let E be a standard uniform space. The points $x \in E$ and $y \in E$ are said to be infinitely close, which is denoted by $x \simeq y$, if (x, y) lies in every standard entourage. If E is a standard metric space, with metric d, then $x \simeq y$ is equivalent to $d(x, y)$ infinitesimal.

Definition 1 *Let X be a standard topological space X. Let E be a standard uniform space. Let D and D_0 be open subsets of X, D_0 standard. Let $f : D \to E$ and $f_0 : D_0 \to E$ be mappings, f_0 standard. The mapping f is said to be a perturbation of the mapping f_0, which is denoted by $f \simeq f_0$, if $^{NS}D_0 \subset D$ and $f(x) \simeq f_0(x)$ for all $x \in {}^{NS}D_0$.*

Let $\mathcal{F}_{X,E}$ be the set of mappings defined on open subsets of X to E :

$$\mathcal{F}_{X,E} = \{(f, D) : D \text{ open subset of } X \text{ and } f : D \to E\}.$$

Let us consider the topology on this set defined as follows. Let $(f_0, D_0) \in \mathcal{F}_{X,E}$. The family of sets of the form

$$\{(f, D) \in \mathcal{F}_{X,E} : K \subset D \ \forall x \in K \ (f(x), f_0(x)) \in U\},$$

where K is a compact subset of D_0, and U is an entourage of the uniform space E, is a basis of the system of neighborhoods of (f_0, D_0). Let us call this topology the *topology of uniform convergence on compacta.* If all the mappings are defined on the same open set D, this topology is the usual topology of uniform convergence on compacta on the set of functions on D to E.

Proposition 1 *Assume X is locally compact. The mapping f is a perturbation of the standard mapping f_0 if and only if f is infinitely close to f_0 for the topology of uniform convergence on compacta.*

Proof. Let $f : D \to E$ be a perturbation of $f_0 : D_0 \to E$. Let K be a standard compact subset of D_0. Let U be a standard entourage. Then $K \subset D$ and $f(x) \simeq f_0(x)$ for all $x \in K$. Hence $(f(x), f_0(x)) \in U$. Thus $f \simeq f_0$ for the topology of uniform convergence on compacta. Conversely, let f be infinitely close to f_0 for the topology of uniform convergence on compacta. Let $x \in {}^{NS}D_0$. There exists a standard $x_0 \in D_0$ such that $x \simeq x_0$. Let K be a standard compact neighborhood of x_0, such that $K \subset D_0$ (such a neighborhood exists since X is locally compact). Then $x \in K \subset D$ and $(f(x), f_0(x)) \in U$ for all standard entourages U, that is $^{NS}D_0 \subset D$ and $f(x) \simeq f_0(x)$ on $^{NS}D_0$. Hence f is a perturbation of f_0. □

The notion of perturbation can be used to formulate Tikhonov's theorem on slow and fast systems whose fast dynamics has asymptotically stable equilibrium points [24], and Pontryagin and Rodygin's theorem on slow and fast systems whose fast dynamics has asymptotically stable cycles [39]. In the following section we use it to formulate the theorem of Krilov, Bogolyubov and Mitropolski of averaging for ODEs. All these theorems belong to the singular perturbation theory. In this paper, by a solution of an Initial Value Problem (IVP) associated to an ODE we mean a maximal (i.e. noncontinuable) solution. The fundamental nonstandard result of the regular perturbation theory of ODEs is called the Short Shadow Lemma. It can be stated as follows [36, 37].

Let $g : D \to \mathbb{R}^d$ and $g_0 : D_0 \to \mathbb{R}^d$ be continuous vector fields, $D, D_0 \subset \mathbb{R}_+ \times \mathbb{R}^d$. Let a_0^0 and a^0 be initial conditions. Assume that g_0 and a_0^0 are standard. The IVP

$$dX/dT = g(T, X), \quad X(0) = a^0 \tag{20.2.3}$$

is said to be a perturbation of the standard IVP

$$dX/dT = g_0(T, X), \quad X(0) = a_0^0, \tag{20.2.4}$$

if $g \simeq g_0$ and $a^0 \simeq a_0^0$. To avoid inessential complications we assume that equation $dX/dT = g_0(T, X)$ has the uniqueness of the solutions. Let ϕ_0 be the solution of the IVP (20.2.4). Let I be its maximal interval of definition. Then, by the following theorem any solution of problem (20.2.3) also exist on I and is infinitely close to ϕ_0.

Theorem 1 (Short Shadow Lemma) *Let problem (20.2.3) be a perturbation of problem (20.2.4). Every solution ϕ of problem (20.2.3) is a perturbation of the solution ϕ_0 of problem (20.2.4), that is, for all nearstandard t in I, $\phi(t)$ is defined and satisfies $\phi(t) \simeq \phi_0(t)$.*

Let us consider the restriction ψ of ϕ to ${}^{NS}I$. By the Short Shadow Lemma, for standard $t \in I$, it takes nearstandard values $\psi(t) \simeq \phi_0(t)$. Thus its shadow, which is the unique standard mapping which associate to each standard t the standard part of $\psi(t)$, is equal to ϕ_0. In general the shadow of ϕ is not equal to ϕ_0. Thus, the Short Shadow Lemma describes only the "short time behaviour" of the solutions.

20.3 Averaging in ordinary differential equations

The method of averaging is well-known for ODEs. The fundamental result of this theory asserts that, for small $\varepsilon > 0$, the solutions of a nonautonomous system

$$\dot{x} = f\left(t/\varepsilon, x, \varepsilon\right), \quad \text{where} \quad \dot{x} = dx/dt, \tag{20.3.1}$$

are approximated by the solutions of the averaged autonomous system

$$\dot{y} = F(y), \quad \text{where} \quad F(x) = \lim_{T\to\infty} \frac{1}{T} \int_0^T f(t, x, 0)\, dt. \tag{20.3.2}$$

The approximation of the solutions of (20.3.1) by the solutions of (20.3.2) means that if $x(t, \varepsilon)$ is a solution of (20.3.1) and $y(t)$ is the solution of the averaged equation (20.3.2) with the same initial condition, which is assumed to be defined on some interval $[0, T]$, then for $\varepsilon \simeq 0$ and for all $t \in [0, T]$, we have $x(t, \varepsilon) \simeq y(t)$.

The change of variable $z(\tau) = x(\varepsilon\tau)$ transforms equation (20.3.1) into equation

$$z' = \varepsilon f(\tau, z, \varepsilon), \quad \text{where} \quad z' = dz/d\tau. \tag{20.3.3}$$

Thus, the method of averaging can be stated for equation (20.3.3), that is, if ε is infinitesimal and $0 \leq \tau \leq T/\varepsilon$ then $z(\tau, \varepsilon) \simeq y(\varepsilon\tau)$.

Classical results were obtained by Krilov, Bogolyubov, Mitropolski, Eckhaus, Sanders, Verhulst (see [5, 31] and the references therein). The theory is very delicate. The dependence of $f(t, x, \varepsilon)$ in ε introduces many complications in the formulations of the conditions under which averaging is justified. In the classical approach, averaging is justified for systems (20.3.1) for which the vector field f is Lipschitz continuous in x. Our aim in this section is first to formulate this problem with the concept of perturbations of vector fields and then to give a theorem of averaging under hypothesis less restrictive than the usual hypothesis. In our approach, averaging is justified for all perturbations of a continuous vector field which is continuous in x uniformly with respect to t. This assumption is of course less restrictive than Lipschitz continuity with respect to x.

20.3.1 KBM vector fields

Definition 2 *Let U_0 be an open subset of $\mathbb{R}^d$. The continuous vector field $f_0 : \mathbb{R}_+ \times U_0 \to \mathbb{R}^d$ is said to be a Krilov-Bogolyubov-Mitropolski (KBM) vector field if it satisfies the following conditions*

1. *The function $x \to f_0(t, x)$ is continuous in x uniformly with respect to the variable t.*

2. *For all $x \in U_0$ the limit $F(x) = \lim_{T\to\infty} \frac{1}{T} \int_0^T f_0(t, x)\, dt$ exists.*

3. *The averaged equation $\dot{y}(t) = F(y(t))$ has the uniqueness of the solution with prescribed initial condition.*

Notice that, in the previous definition, conditions (1) and (2) imply that the function F is continuous, so that the averaged equation considered in condition (3) is well defined. In the case of non autonomous ODEs, the definition of a perturbation given in Section 20.2 must be stated as follows.

Definition 3 *Let U_0 and U be open subsets of $\mathbb{R}^d$. A continuous vector field $f : \mathbb{R}_+ \times U \to \mathbb{R}^d$ is said to be a perturbation of the standard continuous vector field $f_0 : \mathbb{R}_+ \times U_0 \to \mathbb{R}^d$ if U contains all the nearstandard points in U_0, and $f(s,x) \simeq f_0(s,x)$ for all $s \in \mathbb{R}_+$ and all nearstandard x in U_0.*

Theorem 2 *Let $f_0 : \mathbb{R}_+ \times U_0 \to \mathbb{R}^d$ be a standard KBM vector field and let $a_0 \in U_0$ be standard. Let $y(t)$ be the solution of the IVP*

$$\dot{y}(t) = F(y(t)), \qquad y(0) = a_0, \tag{20.3.4}$$

defined on the interval $[0,\omega[$, $0 < \omega \leq \infty$. Let $f : \mathbb{R}_+ \times U \to \mathbb{R}^d$ be a perturbation of f_0. Let $\varepsilon > 0$ be infinitesimal and $a \simeq a_0$. Then every solution $x(t)$ of the IVP

$$\dot{x}(t) = f\left(t/\varepsilon, x(t)\right), \qquad x(0) = a, \tag{20.3.5}$$

is a perturbation of $y(t)$, that is, for all nearstandard t in $[0,\omega[$, $x(t)$ is defined and satisfies $x(t) \simeq y(t)$.

The proof, in the particular case of almost periodic vector fields, is given in Section 20.3.4. The proof in the general case is given in Section 20.3.5.

20.3.2 Almost solutions

The notion of almost solution of an ODE is related to the classical notion of *ε-almost solution.*

Definition 4 *A function $x(t)$ is said to be an almost solution of the standard differential equation $\dot{x} = G(t,x)$ on the standard interval $[0,L]$ if there exists a finite sequence $0 = t_0 < \cdots < t_{N+1} = L$ such that for $n = 0, \cdots, N$ we have*

$$t_{n+1} \simeq t_n, \quad x(t) \simeq x(t_n) \quad \textit{for} \quad t \in [t_n, t_{n+1}],$$
$$\textit{and} \quad \frac{x(t_{n+1}) - x(t_n)}{t_{n+1} - t_n} \simeq G(t_n, x(t_n)).$$

The aim of the following result is to show that an almost solution of a standard ODE is infinitely close to a solution of the equation. This result which was first established by J. L. Callot (see [11, 27]) is a direct consequence of the nonstandard proof of the existence of solutions of continuous ODEs [26].

Theorem 3 *If $x(t)$ is an almost solution of the standard differential equation $\dot{x} = G(t,x)$ on the standard interval $[0,L]$, $x(0) \simeq y_0$, with y_0 standard, and the IVP $\dot{y} = G(t,y)$, $y(0) = y_0$, has a unique solution $y(t)$, then $y(t)$ is defined at least on $[0,L]$ and we have $x(t) \simeq y(t)$, for all $t \in [0,L]$.*

Proof. See [11, 36] □

Let us apply this theorem to obtain an averaging result for an ODE which does not satisfy all the hypothesis of Theorem 2.

Consider the ODE (see [11, 27, 36])

$$\dot{x}(t) = \sin \frac{tx}{\varepsilon}. \tag{20.3.6}$$

The conditions (2) and (3) in Definition 2 are satisfied with $F(x) = 0$. Thus, the solutions of the averaged equation are constant. But condition (1) of the definition is not satisfied, since the function $f(t,x) = \sin(tx)$ is not continuous in x uniformly with respect to t. Hence Theorem 2 does not apply. In fact the solutions of (20.3.6) are not nearly constant and we have the following result:

Proposition 2 *If $\varepsilon > 0$ is infinitesimal then, in the region $t \geq x > 0$ the solutions of (20.3.6) are infinitely close to hyperbolas $tx =$ constant. In the region $x > t \geq 0$, they are infinitely close to the solutions of the ODE*

$$\dot{x} = G(t,x), \quad \text{where} \quad G(t,x) = \frac{\sqrt{x^2 - t^2} - x}{t}. \tag{20.3.7}$$

Proof. The isocline curves $I_k = \{(t,x) : tx = 2k\pi\varepsilon\}$ and $I'_k = \left\{(t,x) : tx = \left(2k + \frac{3}{2}\right)\pi\varepsilon\right\}$ define, in the region $t \geq x > 0$, tubes in which the trajectories are trapped. Thus for $t \geq x > 0$ the solutions are infinitely close to the hyperbolas $tx =$ constant. This argument does not work for $x > t \geq 0$. In this region, we consider the microscope

$$T = \frac{t - t_k}{\varepsilon}, \qquad X = \frac{x - x_k}{\varepsilon},$$

where (t_k, x_k) are the points where a solution $x(t)$ of (20.3.6) crosses the curve I_k. Then we have

$$\frac{dX}{dT} = \sin(x_k T + t_k X + \varepsilon T X), \qquad X(0) = 0.$$

By the Short Shadow Lemma (Theorem 1), $X(T)$ is infinitely close to a solution of $dX/dT = \sin(x_k T + t_k X)$. By straightforward computations we have

$$\frac{x_{k+1} - x_k}{t_{k+1} - t_k} \simeq G(t_k, x_k).$$

Hence, in the region $x > t \geq 0$, the function $x(t)$ is an almost solution of the ODE (20.3.7). By Theorem 3, the solutions of (20.3.6) are infinitely close to the solutions of (20.3.7). □

20.3.3 The stroboscopic method for ODEs

In this section we denote by $G : \mathbb{R}_+ \times D \to \mathbb{R}^d$ a standard continuous function, where D is a standard open subset of $\mathbb{R}^d$. Let $x : I \to \mathbb{R}^d$ be a function such that $0 \in I \subset \mathbb{R}_+$.

Definition 5 *We say that x satisfies the Strong Stroboscopic Property with respect to G if there exists $\mu > 0$ such that for every positive limited $t_0 \in I$ with $x(t_0)$ nearstandard in D, there exists $t_1 \in I$ such that $\mu < t_1 - t_0 \simeq 0$, $[t_0, t_1] \subset I$, $x(t) \simeq x(t_0)$ for all $t \in [t_0, t_1]$, and*

$$\frac{x(t_1) - x(t_0)}{t_1 - t_0} \simeq G(t_0, x(t_0)).$$

The real numbers t_0 and t_1 are called *successive instants of observation* of the stroboscopic method.

Theorem 4 (Stroboscopic Lemma for ODEs) *Let $a_0 \in D$ be standard. Assume that the IVP $\dot{y}(t) = G(t, y(t))$, $y(0) = a_0$, has a unique solution y defined on some standard interval $[0, L]$. Assume that $x(0) \simeq a_0$ and x satisfies the Strong Stroboscopic Property with respect to G. Then x is defined at least on $[0, L]$ and satisfies $x(t) \simeq y(t)$ for all $t \in [0, L]$.*

Proof. Since x satisfies the Strong Stroboscopic Property with respect to G, it is an almost solution of the ODE $\dot{x} = G(t, x)$. By Theorem 3 we have $x(t) \simeq y(t)$ for all $t \in [0, L]$. The details of the proof can be found in [36]. □

The Stroboscopic Lemma has many applications in the perturbation theory of differential equations (see [11, 32, 35, 36, 38, 39]). Let us use this lemma to obtain a proof of Theorem 2.

20.3.4 Proof of Theorem 2 for almost periodic vector fields

Suppose that f_0 is an almost periodic in t, then any of its translates $f_0(s + \cdot, x_0)$ is a nearstandard function, and f_0 has an average F which satisfies [16, 28, 33, 41]

$$F(x) = \lim_{T \to \infty} \frac{1}{T} \int_s^{s+T} f_0(t, x)\, dt,$$

uniformly with respect to $s \in \mathbb{R}_+$. Since F is standard and continuous, we have

$$F(x) \simeq \frac{1}{T} \int_s^{s+T} f_0(t, x)\, dt, \tag{20.3.8}$$

for all $s \in \mathbb{R}_+$, all $T \simeq \infty$ and all nearstandard x in U_0. Let $x : I \to U$ be a solution of problem (20.3.5). Let t_0 be an instant of observation: t_0 is limited in I, and $x_0 = x(t_0)$ is nearstandard in U_0. The change of variables

$$X = \frac{x(t_0 + \varepsilon T) - x_0}{\varepsilon},$$

transforms (20.3.5) into

$$dX/dT = f(s + T, x_0 + \varepsilon X), \quad \text{where} \quad s = t_0/\varepsilon.$$

By the Short Shadow Lemma (Theorem 1), applied to $g(T, X) = f(s + T, x_0 + \varepsilon X)$ and $g_0(T, X) = f_0(s + T, x_0)$, for all limited $T > 0$, we have $X(T) \simeq \int_0^T f_0(s + r, x_0)dr$. By Robinson's Lemma this property is true for some unlimited T which can be chosen such that $\varepsilon T \simeq 0$. Define $t_1 = t_0 + \varepsilon T$. Then we have

$$\frac{x(t_1) - x(t_0)}{t_1 - t_0} = \frac{X(T)}{T} \simeq \frac{1}{T}\int_0^T f_0(s + r, x_0)\, dr = \frac{1}{T}\int_s^{s+T} f_0(t, x_0)\, dt \simeq F(x_0).$$

Thus x satisfies the Strong Stroboscopic Property with respect to F. Using the Stroboscopic Lemma for ODEs (Theorem 4) we conclude that $x(t)$ is infinitely close to a solution of the averaged ODE (20.3.4).

20.3.5 Proof of Theorem 2 for KBM vector fields

Let f_0 be a KBM vector field. From condition (2) of Definition 2 we deduce that for all $s \in \mathbb{R}_+$, we have $F(x) = \lim_{T\to\infty} \frac{1}{T}\int_s^{s+T} f_0(t, x)\, dt$, but the limit is not uniform on s. Thus for unlimited positive s, the property (20.3.8) does not hold for all unlimited T, as it was the case for almost periodic vector fields. However, using also the uniform continuity of f_0 in x with respect to t we can show that (20.3.8) holds for some unlimited T which are not very large. This result is stated in the following technical lemma [36].

Lemma 1 *Let $g : \mathbb{R}_+ \times \mathcal{M} \to \mathbb{R}^d$ be a standard continuous function where $\mathcal{M}$ is a standard metric space. We assume that g is continuous in $m \in \mathcal{M}$ uniformly with respect to $t \in \mathbb{R}_+$ and that g has an average $G(m) = \lim_{T\to\infty} \frac{1}{T}\int_0^T g(t, m)\, dt$. Let $\varepsilon > 0$ be infinitesimal. Let $t \in \mathbb{R}_+$ be limited. Let m be nearstandard in $\mathcal{M}$. Then there exists $\alpha > \varepsilon$, $\alpha \simeq 0$ such that, for all limited $T \geq 0$ we have*

$$\frac{1}{S}\int_s^{s+TS} g(r, m)\, dr \simeq TG(m), \quad \text{where} \quad s = t/\varepsilon, \quad S = \alpha/\varepsilon.$$

The proof of Theorem 2 needs another technical lemma whose proof can be found also in [36].

Lemma 2 *Let $g : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ and $h : \mathbb{R}_+ \to \mathbb{R}^d$ be continuous functions. Suppose that $g(T, X) \simeq h(T)$ holds for all limited $T \in \mathbb{R}_+$ and all limited $X \in \mathbb{R}^d$, and $\int_0^T h(r)\,dr$ is limited for all limited $T \in \mathbb{R}_+$. Then, any solution $X(T)$ of the IVP $dX/dT = g(T, X)$, $X(0) = 0$, is defined for all limited $T \in \mathbb{R}_+$ and satisfies $X(T) \simeq \int_0^T h(r)\,dr$.*

Proof of Theorem 2. Let $x : I \to U$ be a solution of problem (20.3.5). Let $t_0 \in I$ be limited, such that $x_0 = x(t_0)$ is nearstandard in U_0. By Lemma 1, applied to $g = f_0$, $G = F$ and $m = x(t_0)$, there is $\alpha > 0$, $\alpha \simeq 0$ such that for all limited $T \geq 0$ we have

$$\frac{1}{S}\int_s^{s+TS} f_0(r, x_0)\,dr \simeq TF(x_0), \quad \text{where } s = t_0/\varepsilon,\ S = \alpha/\varepsilon. \tag{20.3.9}$$

The change of variables

$$X(T) = \frac{x\,(t_0 + \alpha T) - x_0}{\alpha}$$

transforms (20.3.5) into

$$dX/dT = f(s + ST, x_0 + \alpha X).$$

By Lemma 2, applied to $g(T, X) = f(s + ST, x_0 + \alpha X)$ and $h(T) = f_0(s + ST, x_0)$, and (20.3.9), for all limited $T > 0$, we have

$$X(T) \simeq \int_0^T f_0(s + Sr, x_0)\,dr = \frac{1}{S}\int_s^{s+TS} f_0(r, x_0)\,dr \simeq TF(x_0).$$

Define the successive instant of observation of the stroboscopic method t_1 by $t_1 = t_0 + \alpha$. Then we have

$$\frac{x(t_1) - x(t_0)}{t_1 - t_0} = X(1) \simeq F(x_0).$$

Since $t_1 - t_0 = \alpha > \varepsilon$ and $x(t) - x(t_0) = \alpha X(T) \simeq 0$ for all $t \in [t_0, t_1]$, we have proved that the function x satisfies the Strong Stroboscopic Property with respect to F. By the Stroboscopic Lemma, for any nearstandard $t \in [0, \omega[$, $x(t)$ is defined and satisfies $x(t) \simeq y(t)$. □

20.4 Functional differential equations

Let $\mathcal{C} = \mathcal{C}([-r, 0], \mathbb{R}^d)$, where $r > 0$, denote the Banach space of continuous functions with the norm $\|\phi\| = \sup\{\|\phi(\theta)\| : \theta \in [-r, 0]\}$, where $\|\cdot\|$ is a norm of $\mathbb{R}^d$. Let $L \geq t_0$. If $x : [-r, L] \to \mathbb{R}^d$ is continuous, we define $x_t \in \mathcal{C}$ by setting $x_t(\theta) = x(t + \theta)$, $\theta \in [-r, 0]$ for each $t \in [0, L]$. Let $g : \mathbb{R}_+ \times \mathcal{C} \to \mathbb{R}^d$, $(t, u) \mapsto g(t, u)$, be a continuous function. Let $\phi \in \mathcal{C}$ be an initial condition. A Functional Differential Equation (FDE) is an equation of the form

$$\dot{x}(t) = g\,(t, x_t)\,, \qquad x_0 = \phi.$$

This type of equation includes differential equations with delays of the form

$$\dot{x}(t) = G(t, x(t), x(t - r)),$$

where $G : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$. Here we have $g(t, u) = G(t, u(0), u(-r))$.

The method of averaging was extended [13, 22] to the case of FDEs of the form

$$z'(\tau) = \varepsilon f\,(\tau, z_\tau)\,, \tag{20.4.1}$$

where ε is a small parameter. In that case the averaged equation is the ODE

$$y'(\tau) = \varepsilon F(y(\tau)), \tag{20.4.2}$$

where F is the average of f. It was also extended [14] to the case of FDEs of the form

$$\dot{x}(t) = f\,(t/\varepsilon, x_t)\,. \tag{20.4.3}$$

In that case the averaged equation is the FDE

$$\dot{y}(t) = F\,(y_t)\,. \tag{20.4.4}$$

Notice that the change of variables $x(t) = z(t/\varepsilon)$ does not transform equation (20.4.1) into equation (20.4.3), as it was the case for ODEs (20.3.3) and (20.3.1), so that the results obtained for (20.4.1) cannot be applied to (20.4.3). In the case of FDEs of the form (20.4.1) or (20.4.3), the classical averaging theorems require that the vector field f is Lipschitz continuous in x uniformly with respect to t. In our approach, this condition is weakened and we only assume that the vector field f is continuous in x uniformly with respect to t. Also in the classical averaging theorems it is assumed that the solutions $z(\tau, \varepsilon)$ of (20.4.1) and $y(\tau)$ of (20.4.2) exist in the same interval $[0, T/\varepsilon]$ or that the solutions $x(t, \varepsilon)$ of (20.4.3) and $y(t)$ of (20.4.4) exist in the same interval $[0, T]$. In our approach, we assume only that the solution of the averaged equation is defined on some interval and we give conditions on the vector field f so that, for ε sufficiently small, the solution $x(t, \varepsilon)$ of the system exists at least on the same interval.

20.4.1 Averaging for FDEs in the form $z'(\tau) = \varepsilon f(\tau, z_\tau)$

We consider the IVP, where ε is a small parameter

$$z'(\tau) = \varepsilon f(\tau, z_\tau), \qquad z_0 = \phi.$$

The change of variable $x(t) = z(t/\varepsilon)$ transforms this equation in

$$\dot{x}(t) = f(t/\varepsilon, x_{t,\varepsilon}), \qquad x(t) = \phi(t/\varepsilon), \quad t \in [-\varepsilon r, 0], \tag{20.4.5}$$

where $x_{t,\varepsilon} \in \mathcal{C}$ is defined by $x_{t,\varepsilon}(\theta) = x(t + \varepsilon\theta)$ for $\theta \in [-r, 0]$.

Let $f : \mathbb{R}_+ \times \mathcal{C} \to \mathbb{R}^d$ be a standard continuous function. We assume that

(H1) The function $f : u \mapsto f(t, u)$ is continuous in u uniformly with respect to the variable t.

(H2) For all $u \in \mathcal{C}$ the limit $F(u) = \lim_{T\to\infty} \frac{1}{T} \int_0^T f(t, u)\, dt$ exists.

We identify $\mathbb{R}^d$ to the subset of constant functions in $\mathcal{C}$, and for any vector $c \in \mathbb{R}^d$, we denote by the same letter, the constant function $u \in \mathcal{C}$ defined by $u(\theta) = c$, $\theta \in [-r, 0]$. Averaging consists in approximating the solutions $x(t, \varepsilon)$ of (20.4.5) by the solution $y(t)$ of the averaged ODE

$$\dot{y}(t) = F(y(t)), \qquad y(0) = \phi(0). \tag{20.4.6}$$

According to our convention, $y(t)$, in the right-hand side of this equation, is the constant function $u^t \in \mathcal{C}$ defined by $u^t(\theta) = y(t)$, $\theta \in [-r, 0]$. Since F is continuous, this equation is well defined. We assume that

(H3) The averaged ODE (20.4.6) has the uniqueness of the solution with prescribed initial condition.

(H4) The function f is quasi-bounded in the variable u uniformly with respect to the variable t, that is, for every $t \in \mathbb{R}_+$ and every limited $u \in \mathcal{C}$, $f(t, u)$ is limited in $\mathbb{R}^d$.

Notice that conditions (H1), (H2) and (H3) are similar to conditions (1), (2) and (3) of Definition 2. In the case of FDEs we need also condition (H4). In classical words, the *uniform quasi boundedness* means that for every bounded subset B of $\mathcal{C}$, $f(\mathbb{R}_+ \times B)$ is a bounded subset of $\mathbb{R}^d$. This property is strongly related to the continuation properties of the solutions of FDEs (see Sections 2.3 and 3.1 of [15]).

Theorem 5 *Let $f : \mathbb{R}_+ \times \mathcal{C} \to \mathbb{R}^d$ be a standard continuous function satisfying the conditions (H1)-(H4). Let ϕ be standard in $\mathcal{C}$. Let $L > 0$ be standard and let $y : [0, L] \to \mathbb{R}^d$ be the solution of (20.4.6). Let $\varepsilon > 0$ be infinitesimal. Then every solution $x(t)$ of the problem (20.4.5) is defined at least on $[-\varepsilon r, L]$ and satisfies $x(t) \simeq y(t)$ for all $t \in [0, L]$.*

20.4.2 The stroboscopic method for ODEs revisited

In this section we give another formulation of the stroboscopic method for ODEs which is well adapted to the proof of Theorem 5. Moreover, this formulation of the Stroboscopic Method will be easily extended to FDEs (see Section 20.4.4). We denote by $G : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$, a standard continuous function. Let $x : I \to \mathbb{R}^d$ be a function such that $0 \in I \subset \mathbb{R}_+$.

Definition 6 *We say that x satisfies the Stroboscopic Property with respect to G if there exists $\mu > 0$ such that for every positive limited $t_0 \in I$, satisfying $[0, t_0] \subset I$ and $x(t)$ is limited for all $t \in [0, t_0]$, there exists $t_1 \in I$ such that $\mu < t_1 - t_0 \simeq 0$, $[t_0, t_1] \subset I$, $x(t) \simeq x(t_0)$ for all $t \in [t_0, t_1]$, and*

$$\frac{x(t_1) - x(t_0)}{t_1 - t_0} \simeq G\left(t_0, x(t_0)\right).$$

The difference with the Strong Stroboscopic Property with respect to G considered in Section 20.3.3 is that now we assume that the successive instant of observation t_1 exists only for those values t_0 for which $x(t)$ is limited for all $t \in [0, t_0]$. In Definition 5, in which we take $D = \mathbb{R}^d$, we assumed the stronger hypothesis that t_1 exists for all limited t_0 for which $x(t_0)$ is limited.

Theorem 6 (Second Stroboscopic Lemma for ODEs) *Let $a_0 \in D$ be standard. Assume that the IVP $\dot{y}(t) = G\left(t, y(t)\right)$, $y(0) = a_0$, has a unique solution y defined on some standard interval $[0, L]$. Assume that $x(0) \simeq a_0$ and x satisfies the Stroboscopic Property with respect to G. Then x is defined at least on $[0, L]$ and satisfies $x(t) \simeq y(t)$ for all $t \in [0, L]$.*

Proof. Since x satisfies the Stroboscopic Property with respect to G, it is an almost solution of the ODE $\dot{x} = G(t, x)$. By Theorem 3 we have $x(t) \simeq y(t)$ for all $t \in [0, L]$. The details of the proof can found in [19] or [21]. □

Proof of Theorem 5. Let $x : I \to \mathbb{R}^d$ be a solution of problem (20.4.5). Let $t_0 \in I$ be limited, such that $x(t)$ is limited for all $t \in [0, t_0]$. By Lemma 1, applied to $g = f$, $G = F$ and the constant function $m = x(t_0)$, there is $\alpha > 0$, $\alpha \simeq 0$ such that for all limited $T \geq 0$ we have

$$\frac{1}{S} \int_s^{s+TS} f(r, x(t_0))\, dr \simeq TF(x(t_0)), \quad \text{where } s = t_0/\varepsilon,\ S = \alpha/\varepsilon. \tag{20.4.7}$$

Using the uniform quasi boundedness of f we can show (for the details see [19] or [21]) that $x(t)$ is defined and limited for all $t \simeq t_0$. Hence the function

$$X(\theta, T) = \frac{x(t_0 + \alpha T + \varepsilon\theta) - x(t_0)}{\alpha}, \quad \theta \in [-r, 0], \quad T \in [0, 1],$$

is well defined. In the variable $X(\cdot, T)$ system (20.4.5) becomes

$$\frac{\partial X}{\partial T}(0,T) = f(s+ST, x(t_0) + \alpha X(\cdot, T)).$$

Using assumptions (H1) and (H4) together with (20.4.7), we obtain after some computations that for all $T \in [0,1]$, we have

$$X(0,T) \simeq \int_0^T f(s+Sr, x(t_0))\, dr = \frac{1}{S}\int_s^{s+TS} f(r, x(t_0))\, dr \simeq TF(x(t_0)).$$

Define the successive instant of observation of the stroboscopic method t_1 by $t_1 = t_0 + \alpha$. Then we have

$$\frac{x(t_1) - x(t_0)}{t_1 - t_0} = X(0,1) \simeq F(x(t_0)).$$

Since $t_1 - t_0 = \alpha > \varepsilon$ and $x(t) - x(t_0) = \alpha X(0,T) \simeq 0$ for all $t \in [t_0, t_1]$, we have proved that the function x satisfies the Stroboscopic Property with respect to F. By the Second Stroboscopic Lemma for ODEs, for any $t \in [0, L]$, $x(t)$ is defined and satisfies $x(t) \simeq y(t)$. □

20.4.3 Averaging for FDEs in the form $\dot{x}(t) = f\,(t/\varepsilon, x_t)$

We consider the IVP, where ε is a small parameter

$$\dot{x}(t) = f\,(t/\varepsilon, x_t)\,, \qquad x_0 = \phi, \tag{20.4.8}$$

We assume that f satisfies conditions (H1), (H2) and (H4) of Section 20.4.1. Now, the averaged equation is not the ODE (20.4.6), but the FDE

$$\dot{y}(t) = F\,(y_t)\,, \qquad y_0 = \phi. \tag{20.4.9}$$

Averaging consists in approximating the solutions $x(t, \varepsilon)$ of (20.4.8) by the solution $y(t)$ of the averaged FDE (20.4.9). Condition (H3) in Section 20.4.1 must be restated as follows

(H3) The averaged FDE (20.4.9) has the uniqueness of the solution with prescribed initial condition.

Theorem 7 *Let $f : \mathbb{R}_+ \times \mathcal{C} \to \mathbb{R}^d$ be a standard continuous function satisfying the conditions (H1)-(H4). Let ϕ be standard in $\mathcal{C}$. Let $L > 0$ be standard and let $y : [0, L] \to \mathbb{R}^d$ be the solution of problem (20.4.9). Let $\varepsilon > 0$ be infinitesimal. Then every solution $x(t)$ of the problem (20.4.8) is defined at least on $[-r, L]$ and satisfies $x(t) \simeq y(t)$ for all $t \in [-r, L]$.*

20.4.4 The stroboscopic method for FDEs

Since the averaged equation (20.4.9) is an FDE, we need an extension of the stroboscopic method for ODEs given in Section 20.4.2. In this section we denote by $G : \mathbb{R}_+ \times \mathcal{C} \to \mathbb{R}^d$, a standard continuous function. Let $x : I \to \mathbb{R}^d$ be a function such that $[-r, 0] \subset I \subset \mathbb{R}_+$.

Definition 7 *We say that x satisfies the Stroboscopic Property with respect to G if there exists $\mu > 0$ such that for every positive limited $t_0 \in I$, satisfying $[0, t_0] \subset I$ and $x(t)$ and $G(t, x_t)$ are limited for all $t \in [0, t_0]$, there exists $t_1 \in I$ such that $\mu < t_1 - t_0 \simeq 0$, $[t_0, t_1] \subset I$, $x(t) \simeq x(t_0)$ for all $t \in [t_0, t_1]$, and*

$$\frac{x(t_1) - x(t_0)}{t_1 - t_0} \simeq G\left(t_0, x_{t_0}\right).$$

Notice that now we assume that the successive instant of observation t_1 exists for those values t_0 for which both $x(t)$ and $G(t, x_t)$ are limited for all $t \in [0, t_0]$. In the limit case $r = 0$, the Banach space $\mathcal{C}$ is identified with $\mathbb{R}^d$ and the function x_t is identified with $x(t)$ so that, $G(t, x_t)$ is limited, for all limited $x(t)$. Hence the "Stroboscopic Property with respect to G" considered in the previous definition is a natural extension to FDEs of the "Stroboscopic Property with respect to G" considered in Definition 6.

Theorem 8 (Stroboscopic Lemma for FDEs) *Let $\phi \in \mathcal{C}$ be standard. Assume that the IVP $\dot{y}(t) = G\left(t, y_t\right)$, $y_0 = \phi$, has a unique solution y defined on some standard interval $[-r, L]$. Assume that the function x satisfies the Stroboscopic Property with respect to G and $x_0 \simeq \phi$. Then x is defined at least on $[-r, L]$ and satisfies $x(t) \simeq y(t)$ for all $t \in [-r, L]$.*

Proof. Since x satisfies the Stroboscopic Property with respect to G, it is an almost solution of the FDE $\dot{x} = G(t, x_t)$. For FDEs, we have to our disposal an analog of Theorem 3. Thus $x(t) \simeq y(t)$ for all $t \in [0, L]$. The details of the proof can found in [19] or [21]. □

Proof of Theorem 7. Let $x : I \to \mathbb{R}^d$ be a solution of problem (20.4.8). Let $t_0 \in I$ be limited, such that both $x(t)$ and $F(x_t)$ are limited for all $t \in [0, t_0]$. From the uniform quasi boundedness of f we deduce that $x(t)$ is S-continuous on $[0, t_0]$. Thus x_t is nearstandard for all $t \in [0, t_0]$. By Lemma 1, applied to $g = f$, $G = F$ and $m = x_{t_0}$, there is $\alpha > 0$, $\alpha \simeq 0$ such that for all limited $T \geq 0$ we have

$$\frac{1}{S} \int_s^{s+TS} f(r, x_{t_0})\, dr \simeq TF\left(x_{t_0}\right), \quad \text{where } s = t_0/\varepsilon,\ S = \alpha/\varepsilon. \tag{20.4.10}$$

Using the uniform quasi boundedness of f we can show (for the details see [19] or [21]) that $x(t)$ is defined and limited for all $t \simeq t_0$. Hence the function

$$X(\theta, T) = \frac{x(t_0 + \alpha T + \theta) - x(t_0 + \theta)}{\alpha}, \quad \theta \in [-r, 0], \quad T \in [0, 1],$$

is well defined. In the variable $X(\cdot, T)$ system (20.4.8) becomes

$$\frac{\partial X}{\partial T}(0, T) = f(s + ST, x_{t_0} + \alpha X(\cdot, T)).$$

Using assumptions (H1) and (H4) together with (20.4.10), we obtain that for all $T \in [0, 1]$, we have

$$X(0, T) \simeq \int_0^T f\left(s + Sr, x_{t_0}\right) dr = \frac{1}{S} \int_s^{s+TS} f\left(r, x_{t_0}\right) dr \simeq TF\left(x_{t_0}\right).$$

Define the successive instant of observation of the stroboscopic method t_1 by $t_1 = t_0 + \alpha$. Then we have

$$\frac{x(t_1) - x(t_0)}{t_1 - t_0} = X(0, 1) \simeq F\left(x_{t_0}\right)$$

Since $t_1 - t_0 = \alpha > \varepsilon$ and $x(t) - x(t_0) = \alpha X(0, T) \simeq 0$ for all $t \in [t_0, t_1]$, we have proved that the function x satisfies the Stroboscopic Property with respect to F. By the Stroboscopic Lemma for FDEs, for any $t \in [0, L]$, $x(t)$ is defined and satisfies $x(t) \simeq y(t)$. □

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21

Path-space measure for stochastic differential equation with a coefficient of polynomial growth

Toru Nakamura*

Abstract

A σ-additive measure over a space of paths is constructed to give the solution to the Fokker-Planck equation associated with a stochastic differential equation with coefficient function of polynomial growth by making use of nonstandard analysis.

21.1 Heuristic arguments and definitions

Consider a stochastic differential equation,

$$dx(t) = f\big(x(t)\big)\,dt + db(t), \tag{21.1.1}$$

where $f(x)$ is a real-valued function and $b(t)$ a Brownian motion with variance $2Dt$ for a time interval t. We wish to construct a measure over a space of paths for (21.1.1). To my knowledge, the coefficient function has been assumed to have at most linear growth, that is $|f(x)| \leq \text{const}\cdot|x|$ for sufficiently large x, otherwise some paths explode to infinity in finite times. As an example, let $f(x) = |x|^{1+\delta}$ $(\delta > 0)$ and define *explosion time* $\mathfrak{e}$ for each continuous path $x(t)$ by $\lim_{t\to\mathfrak{e}-0} x(t) = \pm\infty$. Then, it is proved that $P(\mathfrak{e} = \infty) < 1$ and more strongly $P(\mathfrak{e} = \infty) = 0$. For general case, see Feller's test for explosion in [1].

Despite the explosion, we shall consider $f(x)$ of polynomial growth of an arbitrary order and define a measure over a space of paths. We use nonstandard analysis because it has a very convenient theory, Loeb measure theory [2, 3], which enables us to construct a standard σ-additive measure in a simple way.

*Department of Mathematics, Sundai Preparatory School, Kanda-Surugadai, Chiyoda-ku, Tokyo 101-0062, Japan.

Let us interpret (21.1.1) as a law for a particle momentum $x = x(t)$ at time t with the force $f\big(x(t)\big)$ for drift and the random force $db(t)/dt$ acting on the particle. By a time $t > 0$ some particles may disappear to infinity along the exploding paths, but others still exist with finite momentum so that they should make up a "probability" density of a particle momentum. We wish to use the word "probability" though its total value may be less than 1 because of the disappearance of particles to infinity.

Especially when $f\big(x(t)\big)$ is a repulsive force, that is, its sign is the same as that of $x(t)$, a particle can get larger momentum compared to the case where $f\big(x(t)\big)$ is absent. However, once it gets very large momentum, it can hardly come back to the former one because the drift force $f\big(x(t)\big)$ acts on the particle so as to increase its momentum. Thus, the particles which have gone to infinity by a time $t > 0$ could not contribute to the probability density at t. This consideration suggests that we can introduce a cutoff at an infinite number in momentum space if it is necessary in order to define a measure over a space of paths so that the probability density should be constructed by a path integral with respect to the measure.

Eq. (21.1.1) gives the forward Fokker-Planck equation for the probability density $U(t, x)$ of a particle momentum x at time t,

$$\frac{\partial}{\partial t}U(t,x) = D\frac{\partial^2}{\partial x^2}U(t,x) - \frac{\partial}{\partial x}\big\{f(x)U(t,x)\big\}. \tag{21.1.2}$$

We assume that the drift coefficient $f(x)$ and the initial function $U(0, x)$ satisfy the following conditions:

(A1) For some natural number $n \in \mathbb{N}$, $|f(x)| \leq \text{const}\cdot|x|^n$ for sufficiently large x.

(A2) $f(x) \in \mathcal{C}^2(\mathbb{R})$.

(A3) $f(x)^2/(4D) + f'(x)/2$ is bounded from below. We denote the bound by

$$c = \min\{f(x)^2/(4D) + f'(x)/2 \mid x \in \mathbb{R}\}.$$

(A4) $U(0, x) \in \mathcal{C}^2(\mathbb{R})$ and its support is a bounded set.

Rewrite (21.1.2) into a difference equation with infinitesimal time-spacing ε and momentum-spacing $\delta = \sqrt{2D\varepsilon}$ using a forward difference quotient

$$\frac{\partial}{\partial t}U(t,x) \Rightarrow \frac{1}{\varepsilon}\big\{U(t+\varepsilon,x) - U(t,x)\big\}$$

for time-derivative, and central ones

$$\frac{\partial}{\partial x}U(t,x) \Rightarrow \frac{1}{2\delta}\big\{U(t,x+\delta) - U(t,x-\delta)\big\}$$

and

$$\frac{\partial^2}{\partial x^2}U(t,x) \Rightarrow \frac{1}{\delta^2}\big\{U(t,x+\delta)+U(t,x-\delta)-2U(t,x)\big\}$$

for space-derivative. The result is

$$\begin{aligned} U(t+\varepsilon,x) &= \frac{1}{2}\Big\{1+f(x-\delta)\sqrt{\varepsilon/(2D)}\Big\}U(t,x-\delta) \\ &\quad+\frac{1}{2}\Big\{1+f(x+\delta)\big(-\sqrt{\varepsilon/(2D)}\big)\Big\}U(t,x+\delta), \end{aligned} \tag{21.1.3}$$

which indicates that the coefficients $\frac{1}{2}\big\{1+f(x-\delta)\sqrt{\varepsilon/(2D)}\big\}$ should be assigned to the infinitesimal line-segment with end points $\big(t,\,x(t)\big)=\big(t,\,x(t+\varepsilon)-\delta\big)$ and $\big(t+\varepsilon,\,x(t+\varepsilon)\big)$ of a $*$-polygonal path $x(s)$, and $\frac{1}{2}\big\{1+f(x+\delta)\big(-\sqrt{\varepsilon/(2D)}\big)\big\}$ to the segment with end points $\big(t,\,x(t)\big)=\big(t,\,x(t+\varepsilon)+\delta\big)$ and $\big(t+\varepsilon,\,x(t+\varepsilon)\big)$.

Taking into account the approximation

$$\begin{aligned} \log\big[1+f\big(x(t)\big)\big(\pm\sqrt{\varepsilon/(2D)}\big)\big] &\simeq \frac{1}{2D}f\big(x(t)\big)\big(\pm\sqrt{2D\varepsilon}\big)-\frac{1}{4D}f\big(x(t)\big)^2\varepsilon \\ &= \frac{1}{2D}f\big(x(t)\big)\big\{x(t+\varepsilon)-x(t)\big\}-\frac{1}{4D}f\big(x(t)\big)^2\varepsilon, \end{aligned}$$

let us interpret the coefficients as

$$\begin{aligned} &\frac{1}{2}\Big\{1+f\big(x(t)\big)\big(\pm\sqrt{\varepsilon/(2D)}\big)\Big\} \\ &\qquad\simeq \frac{1}{2}\exp\left[\int_t^{t+\varepsilon}\frac{1}{2D}f\big(x(s)\big)\,db(s)-\int_t^{t+\varepsilon}\frac{1}{4D}f\big(x(s)\big)^2\,ds\right]. \end{aligned}$$

The first integral on the right-hand side is the Ito-integral. Thus, we define $*$-path ω, $*$-measure μ for each ω, and $\mathcal{U}(t,x)$ by

Definition 1

(1) Let ν be $[t/\varepsilon]$ with Gauss' parenthesis. For each internal function $\alpha : \{0,1,\cdots,\nu-1\} \to \{-1,1\}$ and $y \in \mathbb{R}$, define x_k by $x_k = y+\sum_{i=0}^{k-1}\alpha(i)\delta$, and ω by the $*$-polygonal path with vertices $(0,y)$, $(\varepsilon,x_1),\cdots,(\nu\varepsilon,x_\nu)$.

(2) Define $*$-measure μ by

$$\mu(\omega)=\frac{1}{2^\nu}\exp\left[\int_0^t\frac{1}{2D}f\big(\omega(s)\big)\,db(s)-\int_0^t\frac{1}{4D}f\big(\omega(s)\big)^2\,ds\right] \tag{21.1.4}$$

with the first integral in the exponent being the Ito integral, and

$$\mathcal{U}(t,x)=\sum_{\omega}U\big(0,\omega(0)\big)\mu(\omega) \tag{21.1.5}$$

where the sum is taken over all ω satisfying $\omega(\nu\varepsilon)=x$.

We note that the $*$-measure (21.1.4) corresponds to the Girsanov formula [4].

21.2 Bounds for the $*$-measure and the $*$-Green function

Making use of the Ito formula

$$\int_0^t \frac{1}{2D} f\big(\omega(s)\big)\, db(s) = \frac{1}{2D}\Big\{F\big(\omega(t)\big) - F\big(\omega(0)\big)\Big\} - \frac{1}{2}\int_0^t f'\big(\omega(s)\big)\, ds$$

where $F'(x) = f(x)$, we obtain a bound for μ as

$$\begin{aligned}\mu(\omega) &= \frac{1}{2^\nu}\exp\Big[\frac{1}{2D}\Big\{F\big(\omega(t)\big) - F\big(\omega(0)\big)\Big\} - \int_0^t \Big\{\frac{1}{4D} f\big(\omega(s)\big)^2 + \frac{1}{2} f'\big(\omega(s)\big)\Big\} ds\Big] \\ &\le \frac{1}{2^\nu}\exp\Big[\frac{1}{2D}\Big\{F\big(\omega(t)\big) - F\big(\omega(0)\big)\Big\} - ct\Big]. \end{aligned} \tag{21.2.1}$$

The constant c in the last line was given in the assumption (A3).

Let us fix the end points of ω at finite numbers y and x, and consider a space of $*$-paths,

$$P(t, x : 0, y) = \big\{\, \omega \;\big|\; \omega(0) = y \quad \text{and} \quad \omega\big(\varepsilon[t/\varepsilon]\big) = x \,\big\},$$

and define the $*$-Green function for the interval $[0,\, t]$ by

$$\mathcal{G}(t, x : 0, y) = \frac{1}{2\delta} \sum_{\omega \in P(t,x:0,y)} \mu(\omega). \tag{21.2.2}$$

Then, $\mathcal{U}(t, x)$ defined in (21.1.5) is written as

$$\mathcal{U}(t, x) = \sum_y U(0, y)\mathcal{G}(t, x : 0, y) 2\delta. \tag{21.2.3}$$

The infinitesimal spacing corresponding to dy is not δ but 2δ in (21.2.3) because only the paths that start every other point y could reach the end point x. Since

$$\sum_{\omega \in P(t,x:0,y)} \frac{1}{2^\nu} = \frac{2\delta}{(4\pi D t)^{1/2}} \exp\Big[-\frac{(x-y)^2}{4Dt}\Big]\Big(1 + \mathcal{O}(\varepsilon^{1/2})\Big), \tag{21.2.4}$$

the $*$-Green function is bounded as

$$\begin{aligned}\mathcal{G}(t, x : 0, y) \le \exp\Big[\frac{F(x)}{2D} - \frac{F(y)}{2D} - ct\Big] \\ \times \frac{1}{(4\pi D t)^{1/2}} \exp\Big[-\frac{(x-y)^2}{4Dt}\Big]\Big(1 + \mathcal{O}(\varepsilon^{1/2})\Big),\end{aligned} \tag{21.2.5}$$

and hence

$$|\mathcal{U}(t,x)| \le \sum_y |U(0,y)| \exp\Big[\frac{F(x)}{2D} - \frac{F(y)}{2D} - ct\Big] \times \frac{1}{(4\pi Dt)^{1/2}} \exp\Big[-\frac{(x-y)^2}{4Dt}\Big]\Big(1+\mathcal{O}(\varepsilon^{1/2})\Big)2\delta. \tag{21.2.6}$$

Since the support of $U(0,y)$ is assumed in (A4) to be a bounded set, the right-hand side of (21.2.6) is near-standard.

To define a standard function U as the standard part of $\mathcal{U}$, we introduce two time scales of different orders, one for the Brownian motion, ε, and the other, τ, for the changes in the drift term in (21.1.1); we choose $\varepsilon = \mathcal{O}(\tau^3)$, for example. The spacing ε is finer and stands for the time-spacing of $*$-random walks, and τ is long enough to cover many steps of the $*$-random walks, yet short enough for the change in the drift term to be small. Define a standard function $U(t,x)$ as the standard part of the value of $\mathcal{U}$ at the coarse-grained lattice point of time $\underline{t}$:

$$U(t,x) = \operatorname{st} \mathcal{U}(\underline{t},x) \quad \text{where} \quad \underline{t} = \tau\big[t/\tau\big], \tag{21.2.7}$$

which we expect to be the solution to the Fokker-Planck equation (21.1.2).

21.3 Solution to the Fokker-Planck equation

In order to prove that $U(t,x)$ in (21.2.7) is the solution to (21.1.2), or more concretely to estimate I_1 in (21.3.7) below, we should truncate the $*$-paths at an infinite number A as

$$P_A(\underline{t},x:0,y) = \big\{\, \omega \in P(\underline{t},x:0,y) \mid \ \forall s \in \big[0,\underline{t}\big] \ \ |\omega(s)| < A \,\big\}.$$

The magnitude of A will be determined later in (21.3.5). Then the corresponding $*$-Green function $\mathcal{G}_A$ and $\mathcal{U}_A$ are defined by

$$\mathcal{G}_A(\underline{t},x:0,y) = \frac{1}{2\delta} \sum_{\omega \in P_A(\underline{t},x:0,y)} \mu(\omega) \tag{21.3.1}$$

and

$$\mathcal{U}_A(\underline{t},x) = \sum_y U(0,y)\mathcal{G}_A(\underline{t},x:0,y)2\delta. \tag{21.3.2}$$

Let us first calculate the difference between (21.2.2) and (21.3.1). Consider a $*$-path $\omega \in P(\underline{t},x:0,y) \setminus P_A(\underline{t},x:0,y)$ and put $\mathfrak{t}(\omega) = \min\{s \mid |\omega(s)| = A\}$. Define a $*$-path ω' by turning upside down the section of the path $\omega(s)$ for

the interval $0 \le s \le \mathrm{t}(\omega)$ so that ω' should start $2A - y$ or $-2A - y$ at time $s = 0$. The section of $\omega'(s)$ for $\mathrm{t}(\omega) \le s \le t$ is that of $\omega(s)$ for the same interval. In this way, we can define a one-to-one correspondence from $\omega \in P(\underline{t}, x : 0, y) \setminus P_A(\underline{t}, x : 0, y)$ to $\omega' \in P(\underline{t}, x : 0, 2A-y) \cup P(\underline{t}, x : 0, -2A-y)$. Then by (21.2.4),

$$\sum_{\omega \in P(\underline{t},x:0,y)\setminus P_A(\underline{t},x:0,y)} \frac{1}{2^\nu} = \frac{2\delta}{(4\pi Dt)^{1/2}} \Big\{ \exp\Big[-\frac{(2A-y-x)^2}{4Dt}\Big] + \\ + \exp\Big[-\frac{(-2A-y-x)^2}{4Dt}\Big] \Big\} \times \Big(1 + \mathcal{O}(\varepsilon^{1/2})\Big),$$

and hence

$$\big|\mathcal{G}(\underline{t}, x : 0, y) - \mathcal{G}_A(\underline{t}, x : 0, y)\big| \le \exp\Big[\frac{F(x)}{2D} - \frac{F(y)}{2D} - ct\Big] \frac{1}{(4\pi Dt)^{1/2}} \times \\ \Big\{ \exp\Big[-\frac{(2A-y-x)^2}{4Dt}\Big] + \exp\Big[-\frac{(-2A-y-x)^2}{4Dt}\Big] \Big\} \times \Big(1 + (\varepsilon^{1/2})\Big), \tag{21.3.3}$$

which implies

$$\big|\mathcal{U}(\underline{t}, x) - \mathcal{U}_A(\underline{t}, x)\big| = \mathcal{O}\Big(\exp\Big[-\frac{A^2}{Dt}\Big]\Big) \tag{21.3.4}$$

for any finite x.

Now, we are ready to prove that $U(t, x)$ is the solution to (21.1.2). We wish to evaluate the standard part of

$$\frac{1}{\sigma}\Big\{\mathcal{U}(t+\sigma, x) - \mathcal{U}(t, x)\Big\}$$

for an infinitesimal $\sigma = k\tau$ ($k \in {}^*\mathbb{N}$) given arbitrarily. Choose the truncation parameter A as

$$A = (D/\beta)^{1/2} \big|\log \beta\sigma\big| \tag{21.3.5}$$

where $\beta > 0$, a standard constant, is just introduced to make the argument of logarithm dimensionless. Then by (21.3.4),

$$\mathcal{U}(\underline{t}, x) - \mathcal{U}_A(\underline{t}, x) = o(\sigma^n) \quad \text{and} \quad \mathcal{U}(\underline{t}+\sigma, x) - \mathcal{U}_A(\underline{t}+\sigma, x) = o(\sigma^n) \tag{21.3.6}$$

for any standard natural number n, meaning that $\mathcal{U}_A(\underline{t}, x)$ can be identified with $\mathcal{U}(\underline{t}, x)$ up to negligible error. Therefore we shall hereafter deal with the truncated $\mathcal{U}_A$ instead of $\mathcal{U}$.

Then the difference quotient we should calculate is

$$\frac{1}{\sigma}\Big\{\mathcal{U}_A(\underline{t}+\sigma, x) - \mathcal{U}_A(\underline{t}, x)\Big\} = \frac{1}{\sigma}(I_1 + I_2) + o(1)$$

where

$$I_1 = \sum_{\xi} \mathcal{U}_A(\underline{t}, x+\xi) \times \Big\{ \mathcal{G}_A(\underline{t}+\sigma, x : \underline{t}, x+\xi) - \frac{1}{(4\pi D\sigma)^{1/2}} \exp\Big[-\frac{\xi^2}{4D\sigma}\Big] \Big\} 2\delta, \tag{21.3.7}$$

$$I_2 = \sum_{\xi} \Big\{ \mathcal{U}_A(\underline{t}, x+\xi) - \mathcal{U}_A(\underline{t}, x) \Big\} \frac{1}{(4\pi D\sigma)^{1/2}} \exp\Big[-\frac{\xi^2}{4D\sigma}\Big] 2\delta. \tag{21.3.8}$$

In order to calculate the sum in I_1, we wish to expand the summand as power series in ξ except for the exponential function, which is possible if ξ is sufficiently small. Since

$$\mathcal{G}_A(\underline{t}+\sigma, x : \underline{t}, x+\xi) = \frac{1}{2\delta} \exp\Big[\frac{1}{2D}\big\{F(x) - F(x+\xi)\big\}\Big] \times \sum_{\omega \in P_A(\underline{t}+\sigma, x : \underline{t}, x+\xi)} \frac{1}{2^{\nu'}} \exp\Big[- \int_{\underline{t}}^{\underline{t}+\sigma} \Big\{ \frac{1}{4D} f\big(\omega(s)\big)^2 + \frac{1}{2} f'\big(\omega(s)\big) \Big\} ds \Big] \tag{21.3.9}$$

satisfies

$$\mathcal{G}_A(\underline{t}+\sigma, x : \underline{t}, x+\xi) \le \exp\Big[\frac{1}{2D}\big\{F(x) - F(x+\xi)\big\} - c\sigma\Big] \times \frac{1}{(4\pi D\sigma)^{1/2}} \exp\Big[-\frac{\xi^2}{4D\sigma}\Big], \tag{21.3.10}$$

the summand in (21.3.7) contains $\exp\big[-\frac{\xi^2}{4D\sigma}\big]$ as a factor which enables us to restrict ξ to be sufficiently small, $|\xi| = \mathcal{O}(\sigma^{1/2})$, in rough estimation. In reality, it is restricted to

$$|\xi| < (D/\beta)^{\frac{1}{2}} (\beta\sigma)^{\frac{1}{2}-a}$$

for some small a such as $1/10$ as shown in the following. Note that

$$\exp\Big[\frac{1}{2D}\big\{F(x) - F(x+\xi) - ct\big\}\Big] = \mathcal{O}\big(e^{|\log \beta\sigma|^m}\big) \tag{21.3.11}$$

for some $m \in \mathbb{N}$ by the truncation at $A = \mathcal{O}(|\log \beta\sigma|)$ and the assumption (A1). Then by (21.3.10),

$$\Big| \mathcal{U}_A(\underline{t}, x+\xi) \Big\{ \mathcal{G}_A(\underline{t}+\sigma, x : \underline{t}, x+\xi) - \frac{1}{(4\pi D\sigma)^{1/2}} \exp\Big[-\frac{\xi^2}{4D\sigma}\Big] \Big\} \Big| \le \text{const.}\, e^{|\log \beta\sigma|^m} \frac{1}{(4\pi D\sigma)^{1/2}} \exp\Big[-\frac{\xi^2}{4D\sigma}\Big].$$

If $|\xi| \geq (D/\beta)^{\frac{1}{2}}(\beta\sigma)^{\frac{1}{2}-a}$, the infinitely large number $e^{|\log\beta\sigma|^m}$ in the last line can be controlled by the half of the factor $e^{-\xi^2/(4D\sigma)}$. In fact,

$$\exp\Big[-\frac{\xi^2}{8D\sigma}\Big] \leq \exp\Big[-\frac{1}{8(\beta\sigma)^{2a}}\Big] = o\big(e^{-|\log\beta\sigma|^m}\big).$$

Hence, the sum $\sum_\xi$ in (21.3.7) for such ξ is estimated as

$$\text{const}\cdot \int_{|\xi|\geq (D/\beta)^{1/2}(\beta\sigma)^{1/2-a}} \frac{1}{(4\pi D\sigma)^{1/2}} \exp\Big[-\frac{\xi^2}{8D\sigma}\Big] d\xi = \mathcal{O}\Big(\sigma^a e^{-\frac{1}{8(\beta\sigma)^{2a}}}\Big) = o(\sigma^n)$$

for any $n \in \mathbb{N}$, so that it can be neglected. Now, we have only to consider ξ of infinitesimal order equal or less than $\sigma^{\frac{1}{2}-a}$, so that we can expand

$$F(x) - F(x+\xi) = -\xi f(x) - \frac{\xi^2}{2!} f'(x) + \mathcal{O}(\sigma^{\frac{3}{2}-3a}). \tag{21.3.12}$$

Next, we wish to replace the integral in (21.3.9) by

$$\frac{1}{4D}\int_t^{t+\sigma} f\big(\omega(s)\big)^2 ds + \frac{1}{2}\int_t^{t+\sigma} f'\big(\omega(s)\big)\, ds \;\simeq\; \frac{\sigma}{4D} f(x)^2 + \frac{\sigma}{2} f'(x). \tag{21.3.13}$$

It is justifiable if paths going very far at some time $s \in [\underline{t}, \underline{t}+\sigma]$ can be neglected. Let us take an infinitesimal

$$A' = (D/\beta)^{\frac{1}{2}}(\beta\sigma)^{\frac{1}{2}-a'}$$

where a' is small but $a' > a$, for example, $a' = 2/10$ if we take $a = 1/10$. Then, the sum over ω which satisfies $|\omega(s) - x| \geq A'$ for some $s \in [t, t+\sigma]$ is bounded by

$$\frac{\text{const}}{(4\pi D\sigma)^{1/2}}\Big\{\exp\Big[-\frac{(2A'-\xi)^2}{4D\sigma}\Big] + \exp\Big[-\frac{(2A'+\xi)^2}{4D\sigma}\Big]\Big\} = \mathcal{O}\big(\sigma^{-1/2} e^{-\frac{1}{(\beta\sigma)^{2a}}}\big) = o(\sigma^n) \tag{21.3.14}$$

for any $n \in \mathbb{N}$ in the same way as (21.3.3), and hence negligible. Here, we have used the fact $\xi = \mathcal{O}(\sigma^{\frac{1}{2}-a}) = o(A')$ because $a' > a$. Thus, we have only to consider ω satisfying

$$|\omega(s) - x| \leq (D/\beta)^{\frac{1}{2}}(\beta\sigma)^{\frac{1}{2}-a'}$$

for all $s \in [t,\, t+\sigma]$, and hence

$$\begin{aligned}
&\exp\Big[-\int_{\underline{t}}^{\underline{t}+\sigma}\Big\{\frac{1}{4D}f\big(\omega(s)\big)^2+\frac{1}{2}\int_{\underline{t}}^{\underline{t}+\sigma}f'\big(\omega(s)\big)\Big\}ds\Big]=\\
&=\exp\Big[-\frac{\sigma}{4D}f(x)^2-\frac{\sigma}{2}f'(x)\Big]\times\\
&\quad\times\exp\Big[-\frac{1}{4D}\int_{\underline{t}}^{\underline{t}+\sigma}\big\{f\big(\omega(s)\big)^2-f(x)^2\big\}\,ds\\
&\qquad\qquad-\frac{1}{2}\int_{\underline{t}}^{\underline{t}+\sigma}\big\{f'\big(\omega(s)\big)ds-f'(x)\big\}ds\Big]=\\
&=\exp\Big[-\frac{\sigma}{4D}f(x)^2-\frac{\sigma}{2}f'(x)\Big]\Big(1+\mathcal{O}\big(\sigma^{\frac{3}{2}-a'}\big)\Big).
\end{aligned}\tag{21.3.15}$$

Putting (21.3.12), (21.3.15) and

$$\sum_{\omega\in P_A(\underline{t}+\sigma,x:\underline{t},x+\xi)}\frac{1}{2^{\nu'}}=\frac{2\delta}{(4\pi D\sigma)^{1/2}}\exp\Big[-\frac{\xi^2}{4D\sigma}\Big]\Big(1+\mathcal{O}(\sigma^{3/2})\Big)$$

into (21.3.9), we obtain

$$\begin{aligned}
&\mathcal{G}_A(\underline{t}+\sigma,x:\underline{t},x+\xi)-\frac{1}{(4\pi D\sigma)^{1/2}}\exp\Big[-\frac{\xi^2}{4D\sigma}\Big]=\\
&=\Big\{-\frac{\xi}{2D}f(x)-\frac{\xi^2+2D\sigma}{4D}f'(x)+\frac{\xi^2-2D\sigma}{8D^2}f(x)^2+o(\sigma)\Big\}\times\\
&\qquad\qquad\times\frac{1}{(4\pi D\sigma)^{1/2}}\exp\Big[-\frac{\xi^2}{4D\sigma}\Big].
\end{aligned}\tag{21.3.16}$$

Lastly, we use the expansion

$$\mathcal{U}_A(\underline{t},x+\xi)=\mathcal{U}_A(\underline{t},x)+\xi D_\delta\mathcal{U}_A(\underline{t},x)+\mathcal{O}(\sigma^{1-2a})\tag{21.3.17}$$

with

$$D_\delta\mathcal{U}_A(\underline{t},x)=\frac{1}{\delta}\Big\{\mathcal{U}_A(\underline{t},x+\delta)-\mathcal{U}_A(\underline{t},x)\Big\}$$

for I_1. For I_2, we use the expansion taken one step further,

$$\mathcal{U}_A(\underline{t},x+\xi)=\mathcal{U}_A(\underline{t},x)+\xi D_\delta\mathcal{U}_A(\underline{t},x)+\frac{\xi^2}{2!}D_\delta^2\mathcal{U}_A(\underline{t},x)+o(\sigma)\tag{21.3.18}$$

with

$$D_\delta^2\mathcal{U}_A(\underline{t},x)=\frac{1}{\delta^2}\Big\{\mathcal{U}_A(\underline{t},x+\delta)+\mathcal{U}_A(\underline{t},x-\delta)-2\mathcal{U}_A(\underline{t},x)\Big\},$$

because (21.3.16) contains a factor of order $\mathcal{O}(\sigma^{\frac{1}{2}-a})$, whereas there is no such factor in I_2. We shall not give the proof of these expansions, for it requires elementary but a little long estimations. Estimations similar to this case can be found in [5, 6, 7].

Putting (21.3.16) and (21.3.17) into (21.3.7) and replacing the sum by integral which is permitted because $\delta = o(\tau^{3/2})$, we obtain

$$\begin{aligned} I_1 = & \int\limits_{|\xi|\le (D/\beta)^{1/2}(\beta\sigma)^{1/2-a}} \Big\{-\frac{\xi}{2D}f(x) - \frac{\xi^2+2D\sigma}{4D}f'(x) + \frac{\xi^2-2D\sigma}{8D^2}f(x)^2 + o(\sigma)\Big\} \\ & \times \Big\{\mathcal{U}_A(\underline{t},x) + \xi D_\delta \mathcal{U}_A(\underline{t},x)\Big\}\frac{1}{(4\pi D\sigma)^{1/2}}\exp\Big[-\frac{\xi^2}{4D\sigma}\Big]d\xi \\ = & \int\limits_{|\xi|\le (D/\beta)^{1/2}(\beta\sigma)^{1/2-a}} \Big\{\Big(-\frac{\xi}{2D}f(x) - \frac{\xi^2+2D\sigma}{4D}f'(x) + \frac{\xi^2-2D\sigma}{8D^2}f(x)^2\Big)\mathcal{U}_A(\underline{t},x) \\ & - \frac{\xi^2}{2D}f(x)D_\delta \mathcal{U}_A(\underline{t},x) + o(\sigma)\Big\}\frac{1}{(4\pi D\sigma)^{1/2}}\exp\Big[-\frac{\xi^2}{4D\sigma}\Big]d\xi. \end{aligned}$$

Since the domain of integration can be extended to ${}^*\mathbb{R}$ by the factor $\frac{1}{(4\pi D\sigma)^{1/2}}\exp\big[-\frac{\xi^2}{4D\sigma}\big]$,

$$\begin{aligned} I_1 = & \int\limits_{{}^*\mathbb{R}} \Big\{\Big(-\frac{\xi}{2D}f(x) - \frac{\xi^2+2D\sigma}{4D}f'(x) + \frac{\xi^2-2D\sigma}{8D^2}f(x)^2\Big)\mathcal{U}_A(\underline{t},x) \\ & - \frac{\xi^2}{2D}f(x)D_\delta \mathcal{U}_A(\underline{t},x) + o(\sigma)\Big\}\frac{1}{(4\pi D\sigma)^{1/2}}\exp\Big[-\frac{\xi^2}{4D\sigma}\Big]d\xi \qquad (21.3.19) \\ = & -\sigma\Big\{f'(x)\mathcal{U}_A(\underline{t},x) + f(x)D_\delta \mathcal{U}_A(\underline{t},x)\Big\} + o(\sigma). \end{aligned}$$

Similarly, putting (21.3.18) into (21.3.8), we obtain

$$I_2 = D\sigma D_\delta^2 \mathcal{U}_A(\underline{t},x) + o(\sigma). \qquad (21.3.20)$$

Therefore,

$$\begin{aligned} & \frac{1}{\sigma}\Big\{\mathcal{U}_A(\underline{t}+\sigma,x) - \mathcal{U}_A(\underline{t},x)\Big\} = \\ & DD_\delta^2 \mathcal{U}_A(\underline{t},x) - \Big\{f'(x)\mathcal{U}_A(\underline{t},x) + f(x)D_\delta \mathcal{U}_A(\underline{t},x)\Big\} + o(1) \end{aligned} \qquad (21.3.21)$$

holds for any infinitesimal $\sigma = k\tau$, which means that $U(t,x) = \operatorname{st}\mathcal{U}_A(\underline{t},x)$ is differentiable with respect to t and its t-derivative is the standard part of

the right-hand side of (21.3.21). Though we omit the proof, it is proved that $U(t,x)$ is twice differentiable with respect to x and their values are

$$\frac{\partial}{\partial x}U(t,x) = \mathrm{st}\, D_\delta \mathcal{U}_A(\underline{t},x) \quad \text{and} \quad \frac{\partial^2}{\partial x^2}U(t,x) = \mathrm{st}\, D_\delta^2 \mathcal{U}_A(\underline{t},x).$$

Taking the standard part of the both-hand sides of (21.3.21), we obtain

$$\frac{\partial}{\partial t}U(t,x) = D\frac{\partial^2}{\partial x^2}U(t,x) - \frac{\partial}{\partial x}\big\{f(x)U(t,x)\big\}, \tag{21.3.22}$$

namely $U(t,x)$ is the solution to the Fokker-Planck equation (21.1.2).

Let us finally note that, by Loeb measure theory, a standard σ-additive measure over a space of paths can be derived from the $*$-measure μ we have constructed in this paper so far.

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22

Optimal control for Navier-Stokes equations

Nigel J. Cutland* and Katarzyna Grzesiak**

Abstract

We survey recent results on existence of optimal controls for stochastic Navier-Stokes equations in 2 and 3 dimensions using Loeb space methods.

22.1 Introduction

In this paper we give a brief survey of recent results[1] concerning the existence of optimal controls for the stochastic Navier-Stokes equations (NSE) in a bounded domain D in 2 and 3 space dimensions; that is, $D \subset \mathbb{R}^d$ with $d = 2$ or 3. The controlled equations in their most general form are as follows (see the next section for details):

$$\begin{aligned} u(t) = u_0 &+ \int_0^t \{-\nu Au(s) - B(u(s)) + f(s, u(s), \theta(s, u))\}\, ds \\ &+ \int_0^t g(s, u(s))\, dw(s) \end{aligned} \tag{22.1.1}$$

Here the evolving velocity field $u = u(t, \omega)$ is a stochastic process with values in the Hilbert space $\mathbf{H} \subseteq \mathbf{L}^d(D)$ of divergence free functions with domain D; this gives the (random) velocity $u(t, x, \omega) \in \mathbb{R}^d$ of the fluid at any time t and point $x \in D$. The most general kind of control θ that we consider acts through the external forcing term f, and takes the form $\theta : [0, T] \times \mathcal{H} \to M$ where $\mathcal{H}$ is the space of paths in $\mathbf{H}$ and the control space M is a compact metric space.

*Mathematics Department, University of York, UK.
nc507@york.ac.uk
**Department of Finance, Wyzsza Szkola Biznesu, National-Louis University, Nowy Sacz, Poland.
jermakowicz@wsb-nlu.edu.pl

[1] The results reported here are developed from the second author's PhD thesis [16] written under the supervision of the first author. Full details may be found in the papers [9] and [10].

In certain settings however it is necessary to restrict to controls of the form $\theta : [0,T] \to M$ that involve no feedback, or those where the feedback only takes account of the instantaneous velocity $u(t)$.

The terms νA, B in the equations are the classical terms representing the effect of viscosity and the interaction of the particles of fluid respectively; the term B is quadratic in u and is the cause of the difficulties associated with solving the Navier-Stokes equations (even in the deterministic case $g = 0$). The final term in the equation represents noisy external forces, with w denoting an infinite dimensional Wiener process.

The methods involve the Loeb space techniques that were employed in [4, 6] to solve the stochastic Navier-Stokes equations with general force and multiplicative noise (that is, with noise $g(s, u(s))$ involving feedback of the solution u), combined with the nonstandard ideas used earlier in the study of optimal control of finite dimensional equations (see [7] for example). For the 3-d case it is necessary to utilize the idea of approximate solutions developed in [11] for the study of attractors.

We assume a fixed time horizon for the problem, and then the aim is to establish the existence of an optimal control that minimizes a general cost J that has a running component and a terminal cost, modelled by

$$\mathcal{J}(\theta, u) = \mathbb{E}\left(\int_0^T h\left(t, u(t), \theta(t,u)\right) dt + \bar{h}(u(T))\right). \tag{22.1.2}$$

The existence of an optimal control (in some cases, a generalized or *relaxed* control) can be established in a number of settings. The results for $d = 2$ are somewhat stronger than for $d = 3$, which is a reflection of the well-known distinction between these two cases even for the uncontrolled equations: in dimension $d = 2$ there is uniqueness of solutions and the solutions are strong (essentially this means that the field $u(t,x)$ is differentiable in x for each t) whereas for $d = 3$ the solutions that exist for all time are weak and uniqueness is a major open problem[2]. Consequently even the formulation of the optimal control problem is more difficult, and optimal controls are obtained using our methods for more restricted classes of controls compared to the results for $d = 2$.

The plan of the paper is as follows. First (Section 22.2) we provide some details of the Hilbert space setup for the equations and their solution, and then recall the basic nonstandard ideas concerning controls. The results for dimension $d = 2$ are then outlined in Section 22.3, and in the final section we do the same for $d = 3$. We omit proofs, referring the interested reader to [9] and [10], although in some cases where new ideas are introduced we provided sketches.

Alternative approaches to control theory for stochastic Navier-Stokes equations have been studied in [3, 13, 18], where the controls are assumed to be

[2]In fact this is one of the Millennium problems. Uniqueness *is* known for strong solutions but for $d = 3$ such solutions can only be obtained for short times.

stochastic processes

$$\theta : [0,T] \times \Omega \to \mathbf{H}$$

adapted to the information filter given by complete observations of the fluid velocity. In [3] for example, the author assumes that there is a constant $\rho > 0$ such that $|\theta(t,\omega)| < \rho$ for all $t \in [0,T]$ and $\omega \in \Omega$, while in [13, 18] it is assumed that the controls satisfy the condition

$$\mathbb{E}\left(\int_0^T |\theta(t)|^2 dt\right) < \infty.$$

In these papers the control is subject to the action of a linear operator. The stochastic minimum principle is derived providing a necessary condition for an optimal control, and a dynamic programming approach is presented to give a sufficient condition. It is shown that the minimum value function is a viscosity solution to the Hamilton-Jacobi-Bellman equation associated with the problem. In [13] the authors obtain smooth solutions to the HJB equation which justifies the dynamic programming approach.

22.2 Preliminaries

22.2.1 Nonstandard analysis

For optimal control theory, nonstandard analysis is a natural tool to use because one can always find a "nonstandard" optimal control — this is simply θ_N for any infinite N, where $(\theta_n)_{n\in\mathbb{N}}$ is a minimizing sequence of controls. The task then is to see whether θ_N can be transformed into a *standard* optimal control of some kind. For finite dimensional DEs and SDEs this idea was developed in [7] and related papers.

In the study of the Navier-Stokes equations (particularly the *stochastic* version) Loeb space techniques have proved very powerful, providing for example the first general existence proof for the stochastic Navier-Stokes equations in dimensions up to 4 (see [6]) and more recently new results concerning the existence of attractors (see [5], [11] and [12]).

In the work reported here the nonstandard techniques used in these two areas are combined. We work in a standard universe $\mathbb{V} = \mathbb{V}(S)$ where S is a base set that contains all the objects of interest, and take an $\aleph_1$-saturated extension ${}^*\mathbb{V}(S) \subset \mathbb{V}({}^*S)$. For ease of reference we gather together in an Appendix the most important facts about the nonstandard representation of the spaces used in the study of the NSE equations.

22.2.2 The stochastic Navier-Stokes equations

The classical form of the uncontrolled stochastic Navier-Stokes equations with zero boundary condition on a time interval $0 \le t \le T$ is as follows:

$$\begin{cases} du = \{\nu\Delta u - < u, \nabla > u - \nabla p + f(t,u)\}dt + g(t,u)dw(t) \\ \operatorname{div} u = 0 \\ u|_{\partial D} = 0 \\ u(0,x) = u_0(x) \end{cases}$$

These equations are considered in a bounded domain $D \subset \mathbb{R}^d$ $(d = 2,3)$ which is fixed throughout the paper, with the boundary ∂D of class $\mathcal{C}^2$. Here $u : [0,T] \times D \times \Omega \to \mathbb{R}^3$ is the random velocity field, ν is the viscosity, p is the pressure, and f represents external forces; u_0 is the initial condition. The diffusion term g together with the driving Wiener process w represents additional *random* external forces, or noise. Underlying this model is a filtered probability space $\mathbf{\Omega} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$.

For the conventional Hilbert space formulation, write $\mathbf{L}^2(D) = (L^2(D))^d$ and let

$$\mathcal{V} = \{u \in \mathcal{C}_0^\infty(D, \mathbb{R}^d) \colon \operatorname{div} u = 0\}.$$

Then $\mathbf{H}$ is the closure of $\mathcal{V}$ in $\mathbf{L}^2(D)$ with the norm given by $|u|^2 = (u,u)$, where

$$(u,v) = \sum_{i=1}^{d} \int_D u^i(x)v^i(x)dx.$$

and $\mathbf{V}$ is the closure of $\mathcal{V}$ in the norm $|u| + \|u\|$ where $\|u\|^2 = ((u,u))$ and

$$((u,v)) = \sum_{j=1}^{d} \left(\frac{\partial u}{\partial x_j}, \frac{\partial v}{\partial x_j}\right).$$

$\mathbf{H}$ and $\mathbf{V}$ are real Hilbert spaces, $\mathbf{V}$ dense in $\mathbf{H}$. The dual space to $\mathbf{V}$ is denoted by $\mathbf{V}'$ with the duality extending the scalar product in $\mathbf{H}$ and

$$\mathbf{V} \subset \mathbf{H} \equiv \mathbf{H}' \subset \mathbf{V}'.$$

Write A for the Stokes operator on $\mathbf{H}$ (the self-adjoint extension of the projection of $-\Delta$) which is densely defined in $\mathbf{H}$; it can be extended to $A : \mathbf{V} \to \mathbf{V}'$ by $Au[v] = ((u,v))$ for $u, v \in \mathbf{V}$. The operator A has an orthonormal basis of eigenfunctions $\{e_k\}_{k\in\mathbb{N}} = \mathcal{E} \subset \mathbf{H}$ with eigenvalues $0 < \lambda_k \nearrow \infty$. For $u \in \mathbf{H}$ write $u = \sum u_k e_k$. Write $\mathbf{H}_n$ for the finite dimensional subspace $\mathbf{H}_n = \operatorname{span}\{e_1, e_2, \ldots, e_n\}$ and $\Pr_n$ for the projection onto $\mathbf{H}_n$.

A family of spaces $\mathbf{H}^r$ for $r \in \mathbb{R}$ is defined as follows: for $r \geq 0$

$$\mathbf{H}^r = \left\{u \in \mathbf{H} : \sum_{k=1}^{\infty} \lambda_k^r u_k^2 < \infty\right\}$$

with the norm given by

$$|u|_r^2 = \sum_{k=1}^{\infty} \lambda_k^r u_k^2.$$

and $\mathbf{H}^{-r}$ is the dual of $\mathbf{H}^r$. We may represent $\mathbf{H}^{-r}$ by

$$\mathbf{H}^{-r} = \left\{ (u_k)_{k\in\mathbb{N}} : \sum_{k=1}^{\infty} \lambda_k^{-r} u_k^2 < \infty \right\}$$

In terms of this family we have $\mathbf{H}^0 = \mathbf{H}$, $\mathbf{H}^1 = \mathbf{V}$, and $\mathbf{H}^{-1} = \mathbf{V}'$ with the norms $|u| = |u|_0$ and $\|u\| = |u|_1$.

The quadratic function B is given by $\big(B(u), y\big) = b(u, u, y)$, where

$$b(u, v, y) = \sum_{i,j=1}^{d} \int_D u^i(x) \frac{\partial v^j}{\partial x_i}(x) y^j(x)\, dx$$

whenever the integral is defined. This describes the nonlinear inertia term in the equation. The trilinear form b has many properties [19] including the following that we need here.

$$b(u, v, v) = 0, \tag{22.2.1}$$

$$|b(u, v, y)| \le c|u|^{\frac{1}{4}} \|u\|^{\frac{3}{4}} |v|^{\frac{1}{4}} \|v\|^{\frac{3}{4}} \|y\| \tag{22.2.2}$$

There are additional properties that hold only in dimension $d = 2$.

The inequality (22.2.2) gives the second of the following properties.

Proposition 22.2.1 *For $u \in \mathbf{V}$*

$$|Au|_{V'} = \|u\|$$
$$|B(u)|_{V'} \le c|u|^{\frac{1}{2}} \|u\|^{\frac{3}{2}}$$

In the above setting the evolution form of the Navier-Stokes equations (without explicit control) in the space $\mathbf{V}'$ (the vector $\nabla p = 0$ in this space) is given by

$$\begin{aligned} u(t) = u_0 &+ \int_0^t \{-\nu Au(s) - B\left(u(s)\right) + f\left(s, u(s)\right)\} ds \\ &+ \int_0^t g\left(s, u(s)\right) dw(s) \end{aligned} \tag{22.2.3}$$

with the initial condition $u_0 \in \mathbf{H}$. The first integrals are Bochner integrals in $\mathbf{V}'$. The driving noise process w is an $\mathbf{H}$-valued Wiener process with covariance Q, a fixed non-negative trace class operator (see [6, 14] for details), and

the stochastic integral is the extension of the Itô integral to Hilbert spaces due to Ichikawa [17].

In the study of the Navier-Stokes equation there are several types of solution. The most general is a *weak solution*, but in the case $d = 2$ we also have *strong solutions*. The definitions are as follows (see [6]):

Definition 22.2.2

(a) An adapted process $u : [0,T] \times \Omega \to H$ is a ***weak solution*** to the stochastic Navier-Stokes equations (22.2.3) if

(i) for P-a.a. ω the path $u(\cdot, \omega)$ has

$$u(\cdot,\omega) \in L^\infty([0,T];\mathbf{H}) \cap L^2([0,T];\mathbf{V}) \cap C([0,T];\mathbf{H}_{\text{weak}}),$$

(ii) for all $t \in [0,T]$ the equation (22.2.3) holds as an identity in V',

(iii) u satisfies the energy inequality

$$\mathbb{E}\left(\sup_{t\in[0,T]} |u(t)|^2 + \int_0^T \|u(t)\|^2 dt\right) < \infty. \tag{22.2.4}$$

By *solution* henceforth in this paper we mean *weak solution*.

(b) A weak solution is ***strong*** if

(i) for P-a.a. ω the path $u(\cdot, \omega)$ has

$$u(\cdot,\omega) \in L^\infty([0,T];\mathbf{V}) \cap L^2([0,T];\mathbf{H}^2) \cap C([0,T];\mathbf{V}_{\text{weak}}), \tag{22.2.5}$$

(which implies that $u(\cdot, \omega) \in C([0,T]; H)$);

(ii) for P-a.a. ω the path $u(\cdot, \omega)$ has

$$\sup_{t\in[0,T]} \|u(t)\|^2 + \int_0^T Au(t)^2 dt < \infty. \tag{22.2.6}$$

When controls are introduced into the forcing term f the equation (22.2.3) takes the form

$$\begin{aligned} u(t) = u_0 &+ \int_0^t \{-\nu Au(s) - B(u(s)) + f(s, u(s), \theta(s,u))\}\, ds \\ &+ \int_0^t g(s, u(s))\, dw(s). \end{aligned}$$

In the next section we discuss the types of control that we consider.

The following general existence result for solutions to the sNSE was first proved in [4] (see also [6] for an exposition), using Loeb space methods.

First define $K_m = \{v : \|v\| \leq m\} \subseteq \mathbf{V}$, with the strong topology of $\mathbf{H}$. In the theorem below, continuity on each K_m turns out to be the appropriate condition for the coefficients f, g; this is weaker than continuity on $\mathbf{V}$ in either the $\mathbf{H}$-norm or the weak topology of $\mathbf{V}$.

Theorem 22.2.3 *Suppose that* $u_0 \in \mathbf{H}$ *and*

$$f : [0,\infty) \times \mathbf{V} \to \mathbf{V}', \qquad g : [0,\infty) \times \mathbf{V} \to L(\mathbf{H},\mathbf{H})$$

are jointly measurable functions with the following properties

(i) $f(t,\cdot) \in C(K_m, \mathbf{V}'_{\text{weak}})$ *for all* m,

(ii) $g(t,\cdot) \in C(K_m, L(\mathbf{H},\mathbf{H})_{\text{weak}})$ *for all* m,

(iii) $|f(t,u)|_{\mathbf{V}'} + |g(t,u)|_{\mathbf{H},\mathbf{H}} \leq a(t)(1+|u|)$ *where* $a \in L^2(0,T)$ *for all* T.

Then equation (22.2.3) has a solution u *on a filtered Loeb space.*

In dimension $d = 2$ a stronger existence theorem (and uniqueness, given Lipschitz coefficients) has been established — this will be noted when required in Section 22.3.

22.2.3 Controls

The simplest controls considered in this paper are those with no feedback, as follows. As is customary in optimal control theory, it is often necessary to extend this class to its natural completion, which is the class of generalized or *relaxed* controls. These, and the topology on them are as defined thus (as in [7]). The compact metric space M is fixed for the whole paper.

Definition 22.2.4

(i) The class C of ***ordinary controls*** is the set of measurable functions $\theta : [0,T] \to M$, where M is a fixed compact metric space.

(ii) *The class* D of ***relaxed controls*** is the set of measurable functions $\varphi : [0,T] \to M_1(M)$, where $M_1(M)$ is the set of probability measures on M. (We regard $C \subseteq D$ by identifying $a \in M$ with the Dirac measure δ_a).

The *weak* or *narrow* topology on $\mathfrak{C}$ is given by means of the set $\mathcal{K}$ of bounded measurable functions $z : [0,T] \times M \to \mathbb{R}$ with $z(t,\cdot)$ continuous for all $t \in [0,T]$. The action of $\theta \in \mathfrak{C}$ on $z \in \mathcal{K}$ is defined by

$$\theta(z) = \int_0^T z(t, \theta(t))dt.$$

Then the topology on $\mathfrak{C}$ is defined by specifying as subbase of open neighbourhoods the sets

$$\{\theta : \ |\theta(z)| < \varepsilon\}_{\varepsilon>0,\, z\in\mathcal{K}}.$$

The effect of a relaxed control on a function $z \in \mathcal{K}$ is obtain by first extending the domain of each $z \in \mathcal{K}$ to $[0,T] \times \mathcal{M}_1(M)$ as follows: for a probability measure $\mu \in \mathcal{M}_1(M)$ define

$$z(t,\mu) = \int_M z(t,m)d\mu(m).$$

The weak (or narrow) topology is then extended to $\mathfrak{D}$ by defining the action of $\varphi \in \mathfrak{D}$ on $z \in \mathcal{K}$ to be

$$\varphi(z) = \int_0^T z\big(t, \varphi(t)\big)dt = \int_0^T \left(\int_M z(t,m)d\varphi_t(m)\right) dt$$

where for brevity we use $\varphi_t := \varphi(t)$.

The fundamental results about controls that we shall use are summarized below; for details consult for example [7], [15], [2] or [20] (noting that a relaxed control is a particular case of a *Young measure*).

Theorem 22.2.5 *The set of relaxed controls $\mathfrak{D}$ is compact.*

This means that for an internal ("nonstandard") control Φ (*ordinary or *relaxed) there is a well defined *standard part* ${}^\circ\Phi \in \mathfrak{D}$. (In [7] it is shown how this can be defined explicitly using Loeb measures.) Note that for a control $\varphi \in \mathfrak{D}$ we have ${}^\circ({}^*\varphi) = \varphi$.

For the next result we define a *uniform step control* to be an ordinary control θ such that there is a partition of $[0,T]$ into intervals of constant length with θ constant on each partition interval.

Theorem 22.2.6 *The set of uniform step controls $\mathfrak{C}^S$ is dense in $\mathfrak{D}$.*

To handle controls in a nonstandard setting the next definition and result is basic.

Definition 22.2.7 $Z : {}^*[0,T] \times {}^*M \to {}^*R$ is a **bounded uniform lifting** of $z \in \mathcal{K}$ if Z is an internal *measurable function such that

(i) Z is finitely bounded

(ii) for a.a. $\tau \in {}^*[0,T]$, for all $\alpha \in {}^*M$: $Z(\tau,\alpha) \approx z({}^\circ\tau, {}^\circ\alpha)$.

Theorem 22.2.8 ([7]) *Let Φ be an internal control with $\varphi = {}^\circ\Phi$ and Z be a bounded uniform lifting of z. Then $\Phi(Z) \approx \varphi(z)$, i.e.*

$$\int_0^T Z(\tau, \hat{\varphi}_\tau) d\tau \approx \int_0^T z(t, \varphi_t) dt$$

In later sections we will define wider classes of controls of the form $\theta = \theta(t,u)$ for $u \in \mathbf{H}$ or even $\theta = \theta(t, u(\cdot))$ where $u(\cdot)$ denotes the trajectory $\{u(s) : 0 \le s \le t\}$ up to time t.

22.3 Optimal control for $d = 2$

Throughout this section we take $d = 2$ and utilize the stronger existence and uniqueness results that obtain in this case.

22.3.1 Controls with no feedback

First we consider the simplest kind of optimal control problem, taking the set of controls to be the no-feedback controls $\mathfrak{C}$ and its generalization $\mathfrak{D}$. The controlled equation then takes the form

$$\begin{aligned} u(t) = u_0 &+ \int_0^t \{-\nu A u(s) - B(u(s)) + f(s, u(s), \varphi(s))\} ds \\ &+ \int_0^t g(s, u(s)) dw(s) \end{aligned} \tag{22.3.1}$$

for a control in $\mathfrak{D}$. Provided f and g are suitably Lipschitz then solutions to this equation are unique. Theorem 22.2.3 is refined as follows.

Theorem 22.3.1 *Let $d = 2$. Consider the following conditions on the functions f, g: there exist constants $c > 0$ and $a(\cdot) \in L^2(0,T)$ such that:*

(a) *$f : [0,T] \times \mathbf{H} \times M \to \mathbf{V}'$ is jointly measurable and*

- **(i)** $|f(t,u,m) - f(t,v,m)|_{\mathbf{V}'} \le c|u - v|$
- **(ii)** *$f(t,u,\cdot)$ is continuous*
- **(iii)** *for a.a. $t \in [0,T]$*

$$|f(t,u,m)|_{\mathbf{V}'} \le a(t)(1 + |u|)$$

for all $u, v \in \mathbf{H}$ and $m \in M$.

(b) $g : [0,T] \times \mathbf{H} \to L(\mathbf{H},\mathbf{H})$ *is jointly measurable and*

(i) $|g(t,u) - g(t,v)|_{\mathbf{H},\mathbf{H}} \leq c|u-v|$

(ii) *for a.a.* $t \in [0,T]$

$$|g(t,u)|_{\mathbf{H},\mathbf{H}} \leq a(t)(1+|u|)$$

for all $u, v \in \mathbf{H}$.

There is a filtered probability space $\mathbf{\Omega} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ *carrying a Wiener process* w*, with the universal property that for any* f*,* g *satisfying the above conditions (for any* $a(t)$ *and* c*) and any ordinary or relaxed control the equation (22.3.1) has a unique solution on* $\mathbf{\Omega}$ *satisfying the energy inequality*

$$\mathbb{E}\left(\sup_{t\in[0,T]} |u(t)|^2 + \int_0^T \|u(t)\|^2 dt\right) < E \tag{22.3.2}$$

with the constant E *uniform over the set of controls; in fact* $E = E\,(u_0, a(\cdot))$.

The uniqueness of solutions requires the property

$$|b(u,v,y)| \leq c\,|u|^{\frac{1}{2}}\|u\|^{\frac{1}{2}}\,|v|^{\frac{1}{2}}\,\|v\|^{\frac{1}{2}}\,\|y\|$$

of the form b which is only valid in dimension 2.

We now fix a space $\mathbf{\Omega}$ as given by this theorem. This may be a Loeb space (as in [6, Theorem 6.4.1 and 6.6.2]), but for the current purpose this is not essential[3]: we simply assume that there is such a space $\mathbf{\Omega}$ in the basic standard universe of discourse and do not care about its provenance. (In Section 22.3.11 it will be necessary to specify the space $\mathbf{\Omega}$ more precisely.) For a given control φ and initial condition u_0 we define

$$u^\varphi = \text{the unique solution for the control } \varphi \text{ with } u(0) = u_0$$

(Strictly we should write $u^\varphi_{\mathbf{\Omega}}$ to denote the underlying space, but where not mentioned this will be $\mathbf{\Omega}$.)

22.3.2 Costs

To formulate an optimal control problem it is necessary to consider the cost of a control. Here we consider a general cost comprising a running cost and a terminal cost, defined as follows.

[3]This is in contrast to Section 22.3.11 and also when working in $d = 3$, where it will be necessary to specify the space $\mathbf{\Omega}$ more precisely.

Definition 22.3.2 We assume given (and fixed for the paper) two functions $h : [0,T] \times H \times M \to R$ and $\bar{h} : H \to R$ with the following properties.

(i) h is jointly measurable, non-negative, continuous in the second and the third variables and satisfies the following growth condition: for a.a. $t \in [0,T]$, all $u \in H$ and $m \in M$

$$|h(t,u,m)| \leq a(t)(1+|u|).$$

(ii) $\bar{h} : H \to R$ is non-negative, continuous and has linear growth: that is,

$$|\bar{h}(u)| \leq c(1+|u|)$$

for all $u \in H$.

The ***cost*** for a control φ (ordinary or relaxed) is $J(\varphi)$ defined by

$$\mathcal{J}(\varphi) = \mathbb{E}\left(\int_0^T h\left(t, u^{\varphi}(t), \varphi(t)\right) dt + \bar{h}\left(u^{\varphi}(T)\right)\right)$$

(Strictly we should write $J_{\mathbf{\Omega}}(\varphi)$ to denote the underlying space, but if the space is not mentioned then we mean Ω.)

The minimal cost for ordinary controls is

$$\mathcal{J}_0 = \inf\{\mathcal{J}(\theta) : \theta \in \mathfrak{C}\}$$

and the minimum cost for relaxed controls is

$$\hat{\mathcal{J}}_0 = \inf\{\mathcal{J}(\varphi) : \varphi \in \mathfrak{D}\}$$

22.3.3 Solutions for internal controls

In the present context we are not concerned with proving existence of solutions to the stochastic Navier-Stokes equations (sNSE) — either by Loeb space methods or any other technique: we are simply assuming the basic existence result above. However, for the purposes of obtaining an optimal control we wish to transfer this result to give the existence of an *internal* solution U^{Φ} to the internal equation on the internal space ${}^*\mathbf{\Omega}$ controlled by an internal control Φ. The idea then is to take the standard part to give a standard solution to the equation controlled by the standard part ${}^{\circ}\Phi$; of course the solution will live on the adapted Loeb space $L({}^*\mathbf{\Omega})$ rather than the original space $\mathbf{\Omega}$, and later we see how we can come back to $\mathbf{\Omega}$.

The proof of the following result, making the first part of the above procedure precise, is almost identical to that involved in the proof of existence — the main difference being that in the existence proof the internal process U lives in $\mathbf{H}_N$ whereas here the process U^{Φ} lives in ${}^*\mathbf{H}$. Here is the result needed.

Theorem 22.3.3 *Let $u_0 \in \mathbf{H}$ and f, g, h and $\bar{h}$ satisfy the conditions of Theorem 22.3.1 and Definition 22.3.2, and suppose that Φ is an internal control. Let U^Φ be the unique solution (which exists by transfer of Theorem 22.3.1) on ${}^*\mathbf{\Omega}$ to the internal equation *(22.3.1) with control Φ; that is*

$$U^\Phi(\tau) = {}^*u_0 + \int_0^\tau \left\{-\nu\, {}^*AU^\Phi(\sigma) - {}^*B\left(U^\Phi(\sigma)\right) + {}^*f\left(\sigma, U^\Phi(\sigma), \Phi(\sigma)\right)\right\} d\sigma + \\ + \int_0^\tau {}^*g\left(\sigma, U^\Phi(\sigma)\right) dW(\sigma) \tag{22.3.3}$$

and satisfying the internal energy inequality

$$\mathbb{E}_{{}^*P}\left(\sup_{\tau\in{}^*[0,T]} |U^\Phi(\tau)|^2 + \int_0^T \|U^\Phi(\tau)\|^2 d\tau\right) < E. \tag{22.3.4}$$

Then U^Φ has a standard part $u = {}^\circ U^\Phi$, which is a solution on $L({}^*\mathbf{\Omega})$ to the standard equation (22.3.1) with control ${}^\circ\Phi$, i.e. $u = U^{{}^\circ\Phi}$ and

$${}^*\mathcal{J}_{{}^*\Omega}(\Phi) \approx \mathcal{J}_{L({}^*\Omega)}({}^\circ\Phi). \tag{22.3.5}$$

22.3.4 Optimal controls

The main theorem for controls with no feedback is now easy to derive.

Theorem 22.3.4 *Let $u_0 \in \mathbf{H}$ and f, g, h and $\bar{h}$ satisfy the conditions of Theorem 22.3.1 and Definition 22.3.2. Then there is an optimal relaxed control φ_0, i.e. such that $\mathcal{J}(\varphi_0) = \mathcal{J}_0$*

Proof. Take a sequence of controls $(\theta_k)_{k\in\mathbb{N}} \subset \mathfrak{C}$ such that

$$\lim_{k\to\infty} \mathcal{J}(\theta_k) = \mathcal{J}_0 \quad \text{and} \quad \mathcal{J}(\theta_k) \searrow \mathcal{J}_0.$$

Fix an infinite $K \in {}^*\mathbb{N}$; then ${}^*\mathcal{J}_{{}^*\Omega}(\theta_K) \approx \mathcal{J}_0$ for the internal ordinary control $\theta_K : {}^*[0,T] \to {}^*M$. Corresponding to this control is the unique internal solution U^{θ_K} to the internal controlled Navier-Stokes equation (22.3.3) which obeys the energy inequality (22.3.4).

Let $\varphi_0 := {}^\circ\theta_K : [0,T] \to \mathcal{M}_1(M)$ and apply Theorem 22.3.3 to the control $\Phi = \theta_K$ and to the solution U^{θ_K}. Then ${}^\circ U^{\theta_K} = u^{\varphi_0}_{L({}^*\mathbf{\Omega})}$ and

$${}^*\mathcal{J}_{{}^*\mathbf{\Omega}}(\theta_K) \approx \mathcal{J}_{L({}^*\mathbf{\Omega})}(\varphi_0)$$

so

$$\mathcal{J}_{L({}^*\mathbf{\Omega})}(\varphi_0) = \mathcal{J}_0.$$

It only remains to show that $\mathcal{J}_{L(^*\mathbf{\Omega})}(\varphi_0) = \mathcal{J}(\varphi_0)$. First notice that $\mathcal{J}(\varphi_0)$ is a real number so $\mathcal{J}(\varphi_0) = {}^*\mathcal{J}_{^*\mathbf{\Omega}}({}^*\varphi_0)$. Apply Theorem 22.3.3 to the internal control ${}^*\varphi_0$ and the internal solution $u^{{}^*\varphi_0}$ on ${}^*\mathbf{\Omega}$ giving

$$\mathcal{J}(\varphi_0) = {}^*\mathcal{J}_{^*\mathbf{\Omega}}({}^*\varphi_0) \approx \mathcal{J}_{L(^*\mathbf{\Omega})}({}^\circ({}^*\varphi_0)) = \mathcal{J}_{L(^*\mathbf{\Omega})}(\varphi_0) = \mathcal{J}_0$$

so φ_0 is optimal. □

Actually the above theorem is easily derived from the following more general consequence of Theorem 22.3.3.

Theorem 22.3.5 *The function $\mathcal{J}$ is continuous with respect to the narrow topology on $\mathfrak{D}$.*

Proof. This is routine using the nonstandard criterion for continuity, together with Theorem 22.3.3 and the argument in the final part of the above proof. □

This result, together with the density of $\mathfrak{C}$ in $\mathfrak{D}$ (Theorem 22.2.6) means that $\hat{\mathcal{J}}_0 = \mathcal{J}_0$ and so we have

Theorem 22.3.6 *The optimal relaxed control φ_0 is also optimal in the class of relaxed controls.*

As usual in optimal control theory, if the force term f is suitably convex then from an optimal relaxed control it is easy to obtain an optimal ordinary control.

22.3.5 Hölder continuous feedback controls ($d = 2$)

The natural generalization of the control problem for the stochastic 2D Navier-Stokes equation is to allow θ to depend on the instantaneous fluid velocity as well as on time:

$$\begin{aligned} u(t) = u_0 + \int_0^t \left\{-\nu A u(s) - B\left(u(s)\right) + f\big(s, u(s), \theta(s, u(s))\big)\right\} ds + \\ + \int_0^t g\left(s, u(s)\right) dw(s). \end{aligned} \tag{22.3.6}$$

It is straightforward to extend the approach outlined above for no-feedback controls to the special situation when the controls considered are *Hölder continuous feedback controls* with uniform constants, as follows.

Definition 22.3.7 For a given set of constants $(\alpha, \beta, \gamma, c_1, c_2)$ with $\alpha, \beta \in (0,1)$ and $\gamma, c_1, c_2 > 0$ the set of ***Hölder continuous feedback controls*** $C(\alpha, \beta, \gamma, c_1, c_2)$ is the set of jointly measurable functions

$$\theta : [0,T] \times \mathbf{H} \to M$$

which are locally Hölder continuous in both variables with these constants; that is:

$$|\theta(t,u) - \theta(s,v)| \leq c_1 |t-s|^\alpha + c_2 |u-v|^\beta \tag{22.3.7}$$

for all $t, s \in [0,T]$ such that $|t-s| \leq \gamma$ and all $u, v \in H$ such that $|u-v| \leq \gamma$.

We will see that this definition guarantees the existence of an optimal ordinary control; the Hölder continuity allows us to construct the standard part of an internal control as an ordinary control without the need to consider measure-valued (i.e. relaxed) controls.

The cost of a control is as before:

$$\mathcal{J}(\theta) = \mathbb{E} \int_0^T h\big(t, u^\theta(t), \theta(t, u^\theta(t))\big)\, dt + \bar{h}(u^\theta(T)) \tag{22.3.8}$$

where h and $\bar{h}$ satisfy the conditions of Definition 22.3.2. The minimal cost is now defined as

$$\mathcal{J}_0 = \inf\{\mathcal{J}(\theta) : \ \theta \in \mathfrak{C}(\alpha, \beta, \gamma, c_1, c_2)\}$$

(of course, strictly this should be $\mathcal{J}_0(\alpha, \beta, \gamma, c_1, c_2)$).

The standard part of an internal control is defined in the obvious way:

Definition 22.3.8 Suppose that $\Theta \in {}^*\mathfrak{C}(\alpha, \beta, \gamma, c_1, c_2)$, so $\Theta : {}^*[0,T] \times {}^*\mathbf{H} \to {}^*M$. Define $\theta \in \mathfrak{C}(\alpha, \beta, \gamma, c_1, c_2)$ by

$$\theta(t,u) = {}^\circ\Theta(t,u) = {}^\circ\Theta(\tau, U)$$

for any $\tau \approx t$ and $U \approx u$ in $\mathbf{H}$ (because of the condition (22.3.7)). Write $\theta = {}^\circ\Theta$.

Now we have the counterpart of Theorem 22.3.3 for Hölder continuous controls.

Theorem 22.3.9 *Let $\Theta \in {}^*\mathfrak{C}(\alpha, \beta, \gamma, c_1, c_2)$ be an internal control and f, g satisfy the conditions of Theorem 22.3.1. Let U^Θ be the unique solution on ${}^*\mathbf{\Omega}$ to the internal equation with control Θ:*

$$\begin{aligned} U^\Theta(\tau) &= {}^*u_0 + \\ &+ \int_0^\tau \left\{ -\nu {}^*A U^\Theta(\tau) - {}^*B\left(U^\Theta(\tau)\right) + {}^*f\left(\tau, U^\Theta(\tau), \Theta(\tau, U^\Theta(\tau))\right) \right\} d\tau \\ &+ \int_0^\tau {}^*g\left(\tau, U^\Theta(\tau)\right) dW(\tau) \end{aligned} \tag{22.3.9}$$

and satisfying the internal energy inequality

$$\mathbb{E}_{*P}\left(\sup_{\tau\in{}^*[0,T]}|U^{\Theta}(\tau)|^2+\int_0^T\|U^{\Theta}(\tau)\|^2d\tau\right)<E.$$

Then its standard part $u = {}^\circ U^{\Theta}$ is a solution on $L({}^*\mathbf{\Omega})$ to the standard equation (22.3.6) with control ${}^\circ\Theta$ (i.e. $u = u^{{}^\circ\Theta}$), and

$${}^*\mathcal{J}_{*\Omega}(\Theta)\approx\mathcal{J}_{L({}^*\Omega)}({}^\circ\Theta).$$

This gives the following with proof almost identical to that of Theorem 22.3.4.

Theorem 22.3.10 *Let f, g satisfy the conditions of Theorem 22.3.1. There exists an optimal admissible control $\theta_0 \in \mathfrak{C}(c_1, c_2, \alpha, \beta, \gamma)$, i.e. such that $\mathcal{J}(\theta_0) = \mathcal{J}_0$.*

22.3.6 Controls based on digital observations ($d = 2$)

The most general results for optimal control of 2D stochastic Navier-Stokes equations involves feedback controls that depend on the entire past of the path of the process u through digital observations made at a fixed finite number of points of time. This model of control was discussed in [1] and [7] for finite dimensional stochastic equations; in the latter paper digital observations were made at random times, but we do not include that feature here.

The controlled system takes the form

$$\begin{aligned}du(t) =& \{-\nu Au(t) - B(u(t), u(t)) + f(t, u, \theta(y(u), t))\}\, dt\\ &+ g(t, u)dw(t)\end{aligned}\tag{22.3.10}$$

with initial condition $u(0) = u_0 \in \mathbf{H}$. In this equation $u = u(\cdot)$ denotes the path of the process $u(t)$ up to the present time (which will be clear from the context). The function $y(u)$ in the control θ denotes the *digital observations* or *read-out* given by the path u, described in the next sections.

22.3.7 The space $\mathcal{H}$

For this system the conditions on f, g will be strengthened slightly so that the solutions to (22.3.10) are strong, and thus the paths of solutions to the equation belong to the space $\mathcal{H}$ where

$$\mathcal{H} = C([0, T]; \mathbf{H}) \cap L^\infty([0, T]; \mathbf{V}).$$

Let $|u|$ denote the uniform norm on this space; that is

$$|u| = \sup_{0\le t\le T}|u(t)|$$

and note that with this norm $\mathcal{H}$ is separable.

22.3.8 The observations

Controls will be based on the information received from digital observations of the solution path $u \in \mathcal{H}$ made at **observation times**

$$0 < t_1 < \ldots < t_p < T$$

that are fixed for the problem. For completeness we write $t_0 = 0$ and $t_{p+1} = T$, but observations are not made at these times. The **digital observation** (taking its value in $\mathbb{N}$) at time t_i is denoted by y_i where

$$y_i : \mathcal{H} \to \mathbb{N} \quad (i = 1, \ldots, p)$$

The complete set of observations for a path u is recorded as

$$y(u) = (y_1(u), \ldots, y_p(u))$$

The i^{th} observation y_i is assumed to be *non-anticipating* in the sense that $y_i(u)$ depends only on the past $u \restriction [0, t_i]$ of the path u up to time t_i.

The digital observation functions y_i are fixed for the whole problem.

22.3.9 Ordinary and relaxed feedback controls for digital observations

The feedback controls considered here are as follows. M is the fixed compact metric space as before.

Definition 22.3.11 An ***ordinary feedback control based on digital observations*** is a measurable function

$$\theta : \mathbb{N}^p \times [0, T] \to M$$

where θ is non-anticipating in the sense that if $t_k \leq t < t_{k+1}$ for $k = 1, \ldots, p$, then $\theta(y, t)$ depends on t and only the first k components of y, namely $(y_1, \ldots, y_k)$. For $t \in [0, t_1)$ a control θ depends only on t.

A ***relaxed feedback control based on digital observations*** is a measurable function

$$\varphi : \mathbb{N}^p \times [0, T] \to \mathcal{M}_1(M)$$

where $M_1(M)$ is the space of probability measures on M and φ is non-anticipating in the same sense as for an ordinary control.

Write $\hat{C}$ and $\hat{D}$ for these sets of ordinary and relaxed feedback controls.

For brevity, in the current context, by a *control* (ordinary or relaxed) we mean a feedback control based on digital observations. The interpretation of a relaxed control is as before.

Nonstandard (i.e. internal) controls have standard parts that are given as follows, using the compactness of non-feedback controls (Theorem 22.2.5). Let Φ be an internal control:

$$\Phi : {}^*\mathbb{N}^p \times {}^*[0,T] \to {}^*\mathcal{M}(M)$$

It is sufficient to construct the standard part of Φ only for $\mathbf{n} \in \mathbb{N}^p$. For $\tau \in {}^*[0,t_1)$ the control Φ is a function of time only (so it is a no-feedback control as in Section 22.2.3) and hence on the first subinterval $[0,t_1)$ we define its standard part as $\varphi = {}^\circ\Phi$ in the sense of the weak topology on $\mathfrak{D}$ (see the remark following Theorem 22.2.5). For $k = 1,\dots,p$ and $\mathbf{n} \in \mathbb{N}^p$ write $\mathbf{n} \restriction k = (n_1,\dots,n_k)$ and notice that

$$\Phi(\mathbf{n},\tau) = \Phi(\mathbf{n} \restriction k, \tau)$$

for $t_k \le \tau < t_{k+1}$. So define $\Phi_{\mathbf{n}\restriction k} : {}^*[t_k, t_{k+1}[\to {}^*\mathcal{M}_1(M)$ by

$$\Phi_{\mathbf{n}\restriction k}(\tau) = \Phi(\mathbf{n} \restriction k, \tau).$$

Each $\Phi_{\mathbf{n}\restriction k}$ is an internal relaxed control from ${}^*\mathfrak{D}$ (restricted to ${}^*[t_k, t_{k+1}[\,)$, so, using Theorem 22.2.5 it has a standard part:

$$\varphi_{\mathbf{n}\restriction k} := {}^\circ\Phi_{\mathbf{n}\restriction k} : [t_k, t_{k+1}] \to \mathcal{M}_1(M)$$

From this we define the standard part $\varphi = {}^\circ\Phi$ to be the standard relaxed control given by

$$\varphi(\mathbf{n},t) = \varphi_{\mathbf{n}\restriction k}(t)$$

for $t \in [t_k, t_{k+1}[$. (Note that for fixed $\mathbf{n} \in \mathbb{N}^p$ if we write $\Psi(\tau) = \Phi(\mathbf{n},\tau)$ and $\psi(\tau) = \varphi(\mathbf{n},\tau)$ then ${}^\circ\Psi = \psi$.)

The weak or narrow topology on $\mathfrak{C}$ and $\mathfrak{D}$ is extended to $\hat{\mathfrak{C}}$ and $\hat{\mathfrak{D}}$ in the natural way (by regarding these as subsets of the products of $\mathfrak{C}^{\mathbb{N}^p}$ and $\mathfrak{D}^{\mathbb{N}^p}$ with the product topology) and then the standard part we have constructed is the standard part with respect to this topology — see [7] for details. The above together with the results of that paper give:

Theorem 22.3.12 *$\hat{\mathfrak{D}}$ is compact and the subset $\hat{\mathfrak{C}}^S$ of uniform step controls in $\mathfrak{C}$ is dense in $\hat{\mathfrak{D}}$.*

The following is the extension of Theorem 22.2.8 to show how the standard part of a digital feedback control acts.

Proposition 22.3.13 *Let Φ be an internal feedback control with $\varphi = {}^\circ\Phi$. If $z : [0,T] \times M \to \mathbb{R}$ is a bounded, measurable function continuous on M and $Z : {}^*[0,T] \times {}^*M \to {}^*\mathbb{R}$ is a bounded uniform lifting (according to Definition 22.2.7) then for any $\mathbf{n} \in \mathbb{N}^p$*

$$\int_0^T Z\big(\tau, \Phi(\mathbf{n},\tau)\big)d\tau \approx \int_0^T z\big(t, \varphi(\mathbf{n},t)\big)dt.$$

22.3.10 Costs for digitally observed controls

The cost of a control is defined in the expected way: for a control φ (ordinary or relaxed) define

$$\mathcal{J}(\varphi, u) = \mathbb{E}\left(\int_0^T h\left(t, u, \varphi(y(u),t)\right)dt + \bar{h}(u)\right) \tag{22.3.11}$$

where u is any solution to the equation with control φ. The cost functions h and $\bar{h}$ are given, with properties set out below. As usual the aim is to find a control which minimizes the cost over the set of controls. We will see that the solution to the equation is unique for a given control so we can define the cost $\mathcal{J}(\varphi) = \mathcal{J}(\varphi, u^\varphi)$ (where u^φ is the solution for the control φ). Then the optimal control problem is reduced to finding a control such that

$$\mathcal{J}_0 := \inf\{\mathcal{J}(\theta) : \theta \text{ an ordinary feedback control}\}$$

is obtained. As is to be expected, in general the set of ordinary feedback controls does not have sufficient closure properties and we prove existence of an optimal *relaxed* control.

22.3.11 Solution of the equations

As noted above the conditions on the functions f, g, h, $\bar{h}$ need to be strengthened (and modified to take account of the form of feedback). We fix the following set of assumptions for the current discussion.

Conditions 22.3.14 *There exist constants $a > 0$ and $L > 0$ such that:*

(a) *$f : [0,T] \times \mathcal{H} \times M \to \mathbf{H}$ is jointly measurable, bounded, non-anticipating, Lipschitz continuous in the second variable and continuous in the third variable. Specifically*

- **(i)** $|f(t,u,m)| \le a$
- **(ii)** $u \restriction t = v \restriction t \Longrightarrow f(t,u,m) = f(t,v,m)$
- **(iii)** $|f(t,u,m) - f(t,v,m)|_{\mathbf{H}} \le L|u-v|$

(iv) *$f(t,u,\cdot)$ continuous for a.a. $t \in [0,T]$, for all $u, v \in \mathcal{H}$ and $m \in M$.*

(b) *$g : [0,T] \times \mathcal{H} \to L(\mathbf{H},\mathbf{V})$ is jointly measurable, bounded, non-anticipating and Lipschitz continuous in the second variable. Specifically*

(i) $|g(t,u)|_{\mathbf{H},\mathbf{V}} \leq a$

(ii) $u \restriction t = v \restriction t \Longrightarrow g(t,u) = g(t,v)$

(iii) $|g(t,u) - g(t,v)|_{\mathbf{H},\mathbf{H}} \leq L|u - v|$ *for a.a. $t \in [0,T]$ and all $u, v \in \mathcal{H}$. For convenience we also assume that g is diagonal, i.e. $\forall n \in \mathbb{N}$ $g_n = \mathrm{Pr}_n(g \restriction \mathbf{H}_n) : [0,T] \times \mathcal{H} \longrightarrow S_d(n)$ where $S_d(n)$ denotes the space of $n \times n$ diagonal matrices (that is, $a_{ij} \neq 0$ iff $i \neq j$).*

(c) *$g(t,u)$ is invertible and*

$$|g^{-1}(t,u)f(t,u,m)| \leq a$$

for a.a. $t \in [0,T]$, all $u \in \mathcal{H}$ and $m \in M$.

(d) *$h : [0,T] \times \mathcal{H} \times M \to \mathbb{R}$ is jointly measurable, non-negative, bounded, non-anticipating in the second variable and continuous in the second and third variables. Specifically*

(i) $|h(t,u,m)| \leq a$

(ii) $u \restriction t = v \restriction t \Longrightarrow h(t,u,m) = h(t,v,m)$

(iii) *$h(t,\cdot,\cdot)$ is continuous for a.a. $t \in [0,T]$, all $u, v \in \mathcal{H}$ and $m \in M$.*

(e) *$\overline{h} : \mathcal{H} \to \mathbb{R}$ is non-negative, continuous and bounded. Specifically*

$$|\overline{h}(u)| \leq a$$

for all $u \in \mathcal{H}$.

Existence and uniqueness of solutions to the equation With the information, control and cost structures now defined, we return to the equations (22.3.10) under consideration. Since these equations involve feedback of the entire past of the process both directly in the coefficients f, g and through the control φ there are new considerations concerning existence and uniqueness. The first task therefore is to show how the basic existence theorem of [6] can be extended to prove the following.

Theorem 22.3.15 *There is an adapted Loeb space $\mathbf{\Omega} = \big(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P\big)$ carrying a Wiener process w of covariance Q, with the following property. For any f, g satisfying Conditions 22.3.14(a,b,c), any initial condition $u_0 \in \mathbf{H}$ and any feedback control φ (ordinary or relaxed) the equation (22.3.10) has a unique strong solution $u = u^{\varphi}$ on $\mathbf{\Omega}$.*

The proof of this is rather long and somewhat complicated technically. The essence is to solve the internal hyperfinite dimensional Galerkin approximation in $\mathbf{H}_N$ with the internal control ${}^*\varphi$ giving an internal solution $U^{{}^*\varphi}$, and then take the standard parts. The latter involves transferring the Girsanov theorem to the hyperfinite setting, and the use of special topologies defined on the spaces $C(\mathcal{H}, \mathbf{H})$ and $C(\mathcal{H} \times M, \mathbf{H})$ to make them separable so that Anderson's Luzin Theorem can be applied.

22.3.12 Optimal control

For the rest of this section the space $\mathbf{\Omega} = \big(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P\big)$ and Wiener process w are fixed as those given by Theorem 22.3.15. The following is proved by taking the internal control in the proof of the previous theorem to be an arbitrary internal control Φ instead of ${}^*\varphi$.

Theorem 22.3.16 *Suppose that Φ is an internal control and U^Φ is the solution to the following equation in $\mathbf{H}_N$ on the internal space $\bar{\mathbf{\Omega}} = \big(\Omega, \mathcal{A}, (\mathcal{A}_\tau)_{\tau\in{}^*[0,T]}, \Pi\big)$ of the previous theorem:*

$$\begin{aligned} &dU(\tau) \\ &= \Big\{-\nu {}^*\!AU(\tau) - B_N\big(U(\tau), U(\tau)\big) + F_N(\tau, U, \Phi(Y(U), \tau), U)\Big\}d\tau \\ &+ G_N(\tau, U)dW(\tau) \end{aligned} \tag{22.3.12}$$

with $V(0) = U(0) = U_0 = \mathrm{Pr}_N{}^*u_0$ and B_N, F_N, G_N as before, and $Y = (Y_i)$ a lifting of y with respect to μ. Then $u = {}^\circ U^\Phi$ is the unique strong solution to the controlled equation (22.3.10) with $\varphi = {}^\circ\Phi$ and

$$J(\varphi) = {}^\circ\hat{J}(\Phi)$$

where $\hat{J}(\Phi)$ is defined by

$$\hat{J}(\Phi) = {}^*\mathbb{E}\left(\int_0^T {}^*h\left(\tau, U^\Phi, \Phi(Y(U^\Phi), \tau)\right) dt + {}^*\bar{h}(U^\Phi)\right)$$

The existence of an optimal relaxed control now follows easily:

Theorem 22.3.17 *The cost function J is continuous on $\mathfrak{D}$*

Proof. Suppose that $\varphi_n \to \varphi$ in $\mathfrak{D}$. Theorem 22.3.16 shows that $\hat{J}({}^*\varphi_n) \approx J(\varphi_n)$ and so $\hat{J}(\varphi_M) \approx {}^*J(\varphi_M)$ for small infinite M.

Again using Theorem 22.3.16 we have $\hat{J}(\varphi_M) \approx J({}^\circ\varphi_M) = J(\varphi)$. Thus ${}^*J(\varphi_M) \approx J(\varphi)$ for small infinite M and so $J(\varphi_n) \to J(\varphi)$ as required. □

The compactness of $\mathfrak{D}$ and the density of step-controls now gives:

Theorem 22.3.18 *For feedback controls based on digital observations $J_0 = \hat{J}_0$ (the infimum for relaxed controls) and there is an optimal relaxed control φ_0 with $J(\varphi_0) = J_0 = \hat{J}_0$.*

22.4 Optimal control for $d = 3$

The extra complications of the 3D stochastic Navier-Stokes equations mean that we are only able to consider the simplest kind of controls. The controlled equation considered takes the form

$$u(t) = u_0 + \int_0^t \{-\nu Au(s) - B(u(s)) + f(s, u(s), \theta(s))\}\, ds + \\ + \int_0^t g(s, u(s))\, dw(s) \tag{22.4.1}$$

which has the same appearance as equation (22.3.1) for no-feedback controls in dimension $d = 2$.

The possible non-uniqueness of solutions means that the very definition of cost and optimality has to be refined — see below — and this requires a space that is rich enough to support essentially all possible solutions for any given control. For this it is necessary to employ a Loeb space.

First we fix the conditions on the coefficients f, g in the equation. These are the natural extension of the basic conditions for the fundamental existence of solutions in Theorem 22.2.3.

Conditions 22.4.1 *There exists an L^2 function $a(t) \geq 0$ such that*

(a) *$f : [0,T] \times \mathbf{H} \times M \longrightarrow \mathbf{V}'$ is jointly measurable with linear growth, continuous in the second variable on each K_n and continuous in the third variable, i.e.*

- **(i)** $|f(t,u,m)|_{\mathbf{V}'} \leq a(t)(1 + |u|)$,
- **(ii)** $f(t,\cdot,m) \in C(K_n; \mathbf{V}'_{\text{weak}})$ *for all finite n, where $K_n = \{u \in \mathbf{V} : \|u\| \leq n\}$ as before (with the strong $\mathbf{H}$-topology).*
- **(iii)** $f(t,u,\cdot)$ *continuous*

for a.a. $t \in [0,T]$, all $u \in \mathbf{H}$ and all $m \in M$.

(b) *$g : [0,T] \times \mathbf{H} \longrightarrow L(\mathbf{H},\mathbf{H})$ is jointly measurable with of linear growth, and continuous in the second variable on each K_n, i.e.*

- **(i)** $|g(t,u)|_{\mathbf{H},\mathbf{H}} \leq a(t)(1 + |u|)$,
- **(ii)** $g(t,\cdot) \in C(K_n; L(\mathbf{H},\mathbf{H})_{\text{weak}})$ *for all finite n,*

for a.a. $t \in [0,T]$ and all $u \in \mathbf{H}$.

22.4.1 Existence of solutions for any control

Theorem 22.2.3 then gives the following, where as before the key feature to note is the uniform energy bound.

Theorem 22.4.2 *There is a filtered probability space* $\mathbf{\Omega} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ *with the universal property that for any* f, g *satisfying Conditions 22.4.1(a,b) and any ordinary or relaxed control, the equation (22.4.1) has a solution satisfying the energy inequality*

$$\mathbb{E}\left(\sup_{t\in[0,T]}|u(t)|^2 + \int_0^T \|u(t)\|^2 dt\right) < E \tag{22.4.2}$$

with a constant $E = E(u_0, a)$ *uniform over the set of controls.*

The space fulfilling this result in [6, Theorem 6.4.1] is a particular Loeb space, which we need to specify later. Note that in the current 3D setting solutions may not be unique (this is an open problem).

22.4.2 The control problem for 3D stochastic Navier-Stokes equations

In order to formulate the optimal control problem for the 3D stochastic Navier-Stokes equations it is necessary to have a space $\mathbf{\Omega}$ as in Theorem 22.4.2 that has solutions for all controls. Fix such a space; then bearing in mind the possible non-uniqueness of solutions, the optimal control problem is formulated as follows. For a given control φ define

$$\begin{aligned}\mathcal{U}^{\varphi} := \{u :\ & u \text{ is weak solution on } \mathbf{\Omega} \\ & \text{to (22.4.1) for } \varphi \text{ and satisfying (22.4.2)}\}\end{aligned} \tag{22.4.3}$$

Then, taking the functions $h, \bar{h}$ satisfying the conditions of Definition 22.3.2 let

$$\mathcal{J}(\varphi, u) = \mathbb{E}\left(\int_0^T h\big(t, u(t), \varphi(t)\big)dt + \bar{h}(u(T))\right). \tag{22.4.4}$$

The cost for φ is then defined by

$$\mathcal{J}(\varphi) := \inf\{\mathcal{J}(\varphi, u) :\ u \in \mathcal{U}^{\varphi}\}. \tag{22.4.5}$$

Now set

$$\mathcal{J}_0 = \inf\big\{\mathcal{J}(\theta, u) :\ \theta \in \mathfrak{C}, u \in \mathcal{U}^{\theta}\big\} = \inf\{\mathcal{J}(\theta) :\ \theta \in \mathfrak{C}\}$$

which is the optimal cost for ordinary controls. Likewise

$$\hat{\mathcal{J}}_0 = \inf\{\mathcal{J}(\varphi, u) : \varphi \in \mathfrak{D}, u \in \mathcal{U}^\varphi\} = \inf\{\mathcal{J}(\varphi) : \varphi \in \mathfrak{D}\}$$

is the optimal cost for relaxed controls. The optimal control problem is to find if possible an optimal control θ and optimal solution $u \in \mathcal{U}^\theta$ such that

$$\mathcal{J}(\theta, u) = \mathcal{J}(\theta) = \mathcal{J}_0$$

and similarly for relaxed controls

$$\mathcal{J}(\varphi, u) = \mathcal{J}(\varphi) = \hat{\mathcal{J}}_0.$$

Remark The above definitions are of course relative to the space $\mathbf{\Omega}$ — so strictly this should be acknowledged by writing $\mathcal{J}_0^{\mathbf{\Omega}}$, etc. However, the space $\mathbf{\Omega}$ defined in the next section will be fixed for the rest of the paper.

22.4.3 The space Ω

In order to obtain an optimal control and optimal solution to the 3D stochastic Navier-Stokes equations the space must be rich enough to carry a good supply of solutions for each given control; we will see that the Loeb space used in [4] (see also [6]) to solve the general existence problem for the stochastic Navier-Stokes equations in dimensions ≤ 4 is just what is needed. For the rest of this paper we take $\mathbf{\Omega}$ to be this space, defined as follows.

Fix an infinite natural number N and take the internal space

$$\bar{\mathbf{\Omega}} = \big(\Omega, \mathcal{A}, (\mathcal{A}_\tau)_{\tau \in {}^*[0,T]}, \Pi\big)$$

where Ω is the canonical space of continuous functions ${}^*C\big({}^*[0,T]; \mathbf{H}_N\big)$, and Π is the Wiener measure induced by the canonical Wiener process $W(\tau) \in \mathbf{H}_N$ having covariance operator Q_N $(= \mathrm{Pr}_N\, {}^*Q\, \mathrm{Pr}_N)$. The filtration $(\mathcal{A}_\tau)_{\tau \in {}^*[0,T]}$ is that generated by the internal process $W(\tau)$.

Take the Loeb measure and Loeb algebra $P = L(\Pi)$, $\mathcal{F} = L(\mathcal{A})$ and put

$$\mathcal{F}_t = \bigcap_{t < {}^\circ\tau} \sigma(\mathcal{A}_\tau) \vee \mathcal{N}$$

where $\mathcal{N}$ denotes the family of P-null sets. This gives the adapted Loeb space

$$\mathbf{\Omega} = \big(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, P\big)$$

which carries the process $w = {}^\circ W$, which is a Wiener process in $\mathbf{H}$ with covariance Q.

The fundamental existence result for the 3D stochastic Navier-Stokes equations on the space $\mathbf{\Omega}$ in [4] or [6] is proved by taking the internal solution $U(\tau, \omega)$ to the Galerkin approximation of dimension N, so $U(\tau, \omega)$ is an internal process with values in $\mathbf{H}_N$. The existence of a uniform *finite* energy bound allows the definition of the standard part process $u = {}^\circ U$ in $\mathbf{H}$, which is the required solution. We do not need to repeat this here in order to establish existence, but the proof of the key theorem below concerning approximate solutions uses similar techniques.

22.4.4 Approximate solutions

Recall the main idea for using nonstandard methods in deterministic optimal control theory used in 2-dimensions. Take a minimizing sequence of controls θ_n and consider a nonstandard control $\Theta = {}^*\theta_K$ for infinite K and a solution $U = U^\Theta$ to the equation for this control. Then $u = {}^\circ U$ is a standard solution for the control $\varphi = {}^\circ\Theta$ which is thus optimal. The problem with this approach in the 3D stochastic setting is that the solution U associated with Θ would live in ${}^*\mathbf{H}$ and be carried by the nonstandard space ${}^*\mathbf{\Omega}$ (provided we can make sense of that). We showed in Section 22.3 how the uniqueness of solutions for $d = 2$ allows us to move back to the space $\mathbf{\Omega}$ to complete the story, but here an alternative approach is needed. This is provided by the notion of *approximate solutions*, first introduced in [11] in order to establish results on attractors for 3D stochastic Navier-Stokes equations. The rather technical notion required in [11] has been adjusted below to suit the current needs — and is somewhat simpler than the corresponding notion in [11].

Definition 22.4.3 (Approximate solutions) Fix an initial condition $u_0 \in H$ and let $E = E(u_0, \bar{a})$. Let $\Phi \in {}^*D$ be an internal control (ordinary or relaxed) and define sets X_j^Φ and X^Φ as follows.

(a) For each $j \in {}^*N$ denote by X_j^Φ the internal class of internal processes $U : {}^*[0,T] \times \Omega \to H_N$ that are *adapted to the filtration (A_τ), with the following properties:

(i) With Π-probability $\geq 1 - \frac{1}{j}$ on Ω, for all $\tau \in {}^*[0,T]$ and all $k \leq j$:

$$\begin{aligned} &\Big| U_k(\tau) - U_k(0) - \int_0^\tau \Big\{ -\nu A U_k(\sigma) - B_k\big(U(\sigma), U(\sigma)\big) \Big\} d\sigma \\ &- \int_0^\tau F_k^\Phi\big(\sigma, U(\sigma)\big) d\sigma - \int_0^\tau G_k\big(\sigma, U(\sigma)\big) dW(\sigma) \Big| \leq 2^{-j} \end{aligned} \tag{22.4.6}$$

where $B_k = ({}^*B, {}^*e_k)$, $F_k^\Phi\big(\sigma, U(\sigma)\big) = \Big({}^*f\big(\sigma, U(\sigma), \Phi(\sigma)\big), {}^*e_k\Big)$ and $G_k = ({}^*g, {}^*e_k)$.

(ii)

$$\mathrm{E}_\Pi\left(\sup_{\sigma\in{}^*[0,T]}|U(\sigma)|^2+\nu\int_0^T\|U(\sigma)\|^2d\sigma\right)\le E+\frac{1}{j} \tag{22.4.7}$$

(iii)

$$|U(0)-u_0|\le\frac{1}{j} \tag{22.4.8}$$

(b) Define $X^\Phi=\bigcap_{j\in\mathbb{N}}X_j^\Phi$. This is the set of ***approximate solutions*** to the controlled stochastic Navier-Stokes equations for control Φ.

Remark The sequence of internal sets $\mathcal{X}_j^\Phi$ is decreasing with j and since the set $\mathcal{X}^\Phi$ involves $\mathcal{X}_j^\Phi$ only for finite j then we have $\mathcal{X}_K^\Phi\subseteq\mathcal{X}^\Phi$ for all $K\in{}^*\mathbb{N}\setminus\mathbb{N}$.

The theory of [11], adapted to the modified definition of approximate solution here gives the following key result. First we must define the internal cost $\bar{\mathcal{J}}(\Phi,U)$ for an internal process $U\in\mathbf{H}_N$ on $\mathbf{\Omega}$ and control Φ by

$$\bar{\mathcal{J}}(\Phi,U)=\mathbb{E}_\Pi\left(\int_0^T{}^*h\big(\tau,U(\tau),\Phi(\tau)\big)d\tau+{}^*\bar{h}\big(U(\tau)\big)\right)$$

which is different from ${}^*\mathcal{J}_{*\Omega}(\Phi,U)$ (which doesn't actually make sense).

Theorem 22.4.4

(a) *Let Φ be an internal control (that is, $\Phi\in{}^*\mathfrak{D}$) and $U\in\mathcal{X}^\Phi$. Then for P-a.a. $\omega\in\Omega$, $|U(\tau,\omega)|$ is finite for all $\tau\in{}^*[0,T]$ and $U(\cdot,\omega)$ is weakly S-continuous. The process defined by*

$$u(t,\omega)={}^\circ U(\tau,\omega)$$

for $\tau\approx t$ belongs to $\mathcal{U}^{{}^\circ\Phi}$ (where ${}^\circ\Phi$ is the standard part of Φ in the weak topology on $\mathfrak{D}$ as described in Section 22.2.3) and

$$\bar{\mathcal{J}}(\Phi,U)\approx\mathcal{J}({}^\circ\Phi,u).$$

(b) *Let $\varphi\in\mathfrak{D}$ be a standard control and $u\in\mathcal{U}^\varphi$. Then there exists $U\in\mathcal{X}^{{}^*\varphi}$ with $u={}^\circ U$ as defined in part (a), and hence*

$$\bar{\mathcal{J}}({}^*\varphi,U)\approx\mathcal{J}(\varphi,u).$$

Proof (Sketch). The proof of (a) follows quite closely the proof of Theorem 6.4.1 in [6]. The main difference here compared with that proof is (i) the presence of the control in the drift term f, which is dealt with routinely using

Anderson's Luzin Theorem, and (ii) the fact that internally we only have an approximate equation. That is, we have $\approx$ instead of $=$ in the internal equation for each finite co-ordinate of the solution. That is, from the approximate equality (22.4.6) with $j = K$ for some infinite K, using overspill, we have

$$U_k(\tau) \approx U_k(0) + \int_0^\tau \{-\nu A U_k(\sigma) - B_k(U(\sigma), U(\sigma))\} d\sigma + \int_0^\tau F_k^\Phi(\sigma, U(\sigma)) d\sigma + \int_0^\tau G_k(\sigma, U(\sigma)) dW(\sigma)$$

for finite k. This gives equality after taking standard parts.

For (b) the theory in [11] can be adapted easily to obtain suitable liftings of the solution u, giving an internal process $U \in \mathcal{X}^{*\varphi}$ such that $u = {}^\circ U$. The remaining fact about the cost follows from part **(a)** with $\Phi = {}^*\varphi$ and using the fact that ${}^\circ({}^*\theta) = \theta$. □

22.4.5 Optimal control

The results concerning optimal control of the stochastic 3D equation follow from the next more general theorem.

Theorem 22.4.5 *Let f, g, h and $\bar{h}$ satisfy Conditions 22.4.1.*

(a) *For every control φ there is an optimal solution $u^\varphi \in \mathcal{U}^\varphi$ with $\mathcal{J}(\varphi, u^\varphi) = \mathcal{J}(\varphi)$*

(b) *Let φ_n be a sequence of controls and $u_n \in \mathcal{U}^{\varphi_n}$, with $\mathcal{J}(\varphi_n, u_n)$ converging. Then there is a control φ and $u \in \mathcal{U}^\varphi$ with $\mathcal{J}(\varphi, u) = \lim_{n\to\infty} \mathcal{J}(\varphi_n, u_n)$.*

Proof. (a) For a control $\varphi \in \mathfrak{D}$ take a minimizing sequence $(u_k)_{k\in\mathbb{N}} \subset \mathcal{U}^\varphi$ such that $\mathcal{J}(\varphi, u_k) \searrow \mathcal{J}(\varphi)$. Using Theorem 22.4.4 (b), for each k take an approximate solution $U_k \in \mathcal{X}^{*\varphi}$ such that $u_k = {}^\circ U_k$ and $\bar{\mathcal{J}}({}^*\varphi, U_k) \approx \mathcal{J}(\varphi, u_k)$. Weakening this gives

$$U_k \in \mathcal{X}_k^{*\varphi}$$

and

$$|\bar{\mathcal{J}}({}^*\varphi, U_k) - \mathcal{J}(\varphi, u_k)| < \frac{1}{k}$$

for each finite k so by $\aleph_1$-saturation there is an infinite K with $U_K \in \mathcal{X}_K^{*\varphi} \subseteq \mathcal{X}^{*\varphi}$ and $\bar{\mathcal{J}}({}^*\varphi, U_K) \approx \mathcal{J}(\varphi, u_K)$. Then Theorem 22.4.4 (a) gives ${}^\circ U_K \in \mathcal{U}^\varphi$ with $\mathcal{J}(\varphi, {}^\circ U_K) = \mathcal{J}(\varphi)$, so we may take $u^\varphi = {}^\circ U_K$.

(b) The proof is similar to the proof of (a). For each k take an approximate solution $U_k \in \mathcal{X}^{*\varphi_k}$ such that $u_k = {}^\circ U_k$ and $\bar{\mathcal{J}}({}^*\varphi_k, U_k) \approx \mathcal{J}(\varphi_k, u_k)$. Using $\aleph_1$-saturation there is an infinite K with $U_K \in \mathcal{X}_K^{*\varphi_K} \subseteq \mathcal{X}^{*\varphi_K}$ and $\bar{\mathcal{J}}({}^*\varphi_K, U_K) \approx \mathcal{J}_1 = \lim_{n\to\infty} \mathcal{J}(\varphi_n, u_n)$.

Now let $\varphi = {}^\circ({}^*\varphi_K)$ and $u = {}^\circ U_K$; Theorem 22.4.4 (a) gives $u \in \mathcal{U}^\varphi$ and $\mathcal{J}(\varphi, u) = {}^\circ\bar{\mathcal{J}}({}^*\varphi_K, U_K) = \mathcal{J}_1$ □

Corollary 22.4.6

(a) *There exists a relaxed control φ_0 (and solution $u^{\varphi_0} \in \mathcal{U}^{\varphi_0}$) that achieves the minimum cost for ordinary controls (i.e. such that $\mathcal{J}(\varphi_0, u^{\varphi_0}) = \mathcal{J}(\varphi_0) = \mathcal{J}_0$);*

(b) *There is an optimal relaxed control $\hat{\varphi}_0$ and optimal solution $u^{\hat{\varphi}_0} \in \mathcal{U}^{\hat{\varphi}_0}$ with $\mathcal{J}(\hat{\varphi}_0, u^{\hat{\varphi}_0}) = \mathcal{J}(\hat{\varphi}_0) = \hat{\mathcal{J}}_0$.*

Proof. For (a) take a minimizing sequence of ordinary controls (and solutions) and for (b) take a minimizing sequence of relaxed controls and solutions. Then apply (b) of the previous theorem. □

Remark 22.4.7 One consequence of the possible non-uniqueness of solutions is that unlike in dimension 2 we cannot prove that $\hat{J}_0 = J_0$, so this theory is rather less satisfactory than that for 2D.

22.4.6 Hölder continuous feedback controls ($d = 3$)

Recall that in dimension $d = 2$, the existence of optimal ordinary feedback controls in a general setting was ensured by taking *Hölder continuous feedback controls* (Section 22.3.5). The technique of approximate solutions allows the extension of this idea to the 3D stochastic setting. Controls take the form $\theta(t, u(t))$, and the controlled equation (22.4.1) becomes

$$\begin{aligned} u(t) = u_0 + \int_0^t \big\{ -\nu Au(s) - B\left(u(s)\right) + f\left(s, u(s), \theta(s, u(s))\right) \big\} ds + \\ + \int_0^t g\left(s, u(s)\right) dw(s) \end{aligned} \tag{22.4.9}$$

with cost function

$$\mathcal{J}(\theta, u) = \mathbb{E}\left(\int_0^T h\big(t, u(t), \theta(t, u(t))\big) dt + \bar{h}(u(T)) \right) \tag{22.4.10}$$

for $u \in \mathcal{U}^\varphi$

We maintain the Conditions 22.4.1 on the coefficients $f, g, h, \bar{h}$.

The controls considered are the Hölder continuous feedback controls $\mathfrak{C}(\alpha, \beta, \gamma, c_1, c_2)$ given by Definition 22.3.7.

The controls we consider are *Hölder continuous feedback controls* defined as follows. So we fix constants $(\alpha, \beta, \gamma, c_1, c_2)$ and as before write $\mathfrak{C}^H = \mathfrak{C}(\alpha, \beta, \gamma, c_1, c_2)$. We continue to work with the space $\mathbf{\Omega}$ as defined above (Section 22.4.3); it is routine to see that for any control $\theta \in \mathfrak{C}^H$ there is a solution to (22.4.9) having the energy bound (22.4.2). As before write $\mathcal{U}^\theta$ for the set of all such solutions, and then the minimal cost is defined as expected: for $\theta \in \mathfrak{C}^H$ extend the earlier definition to give:

$$\mathcal{J}(\theta) = \inf\{\mathcal{J}(\theta, u) : \ u \in \mathcal{U}^\theta\}$$

and then set

$$\mathcal{J}_0^H = \inf\{\mathcal{J}(\theta) : \ \theta \in \mathfrak{C}^H\}.$$

The standard part of an internal control is defined in the natural way:

Definition 22.4.8 Suppose that $\Theta \in {}^*\mathfrak{C}^H$, so $\Theta : {}^*[0,T] \times {}^*\mathbf{H} \to {}^*M$. Define ${}^\circ\Theta = \theta \in \mathfrak{C}^H$ by

$$\theta(t,u) = {}^\circ\Theta(t,u) = {}^\circ\Theta(\tau, U)$$

for any $\tau \approx t$ and $U \approx u$ in $\mathbf{H}$ (this makes sense because of the condition (22.3.7)).

22.4.7 Approximate solutions for Hölder continuous controls

For an initial condition $u_0 \in \mathbf{H}$ and an internal control $\Theta \in {}^*\mathfrak{C}^H$ the sets $\mathcal{X}_j^\Theta$ and $\mathcal{X}^\Theta$ are defined as in Definition 22.4.3 with F_k^Θ modified in the obvious way:

$$F_k^\Theta\big(\sigma, U(\sigma)\big) = \Big({}^*f\big(\sigma, U(\sigma), \Theta(\sigma, U(\sigma))\big), {}^*e_k\Big)$$

Then we have the counterpart of Theorem 22.4.4 for Hölder continuous controls.

Theorem 22.4.9

(a) *Let Θ be an internal Hölder continuous control (that is, $\Theta \in {}^*\mathfrak{C}^H$) and $U \in \mathcal{X}^\Theta$. Then for a.a. $\omega \in \Omega$, $|U(\tau, \omega)|$ is finite for all $\tau \in {}^*[0,T]$ and $U(\cdot, \omega)$ is weakly S-continuous. The process defined by*

$$u(t, \omega) = {}^\circ U(\tau, \omega)$$

for $\tau \approx t$ belongs to $\mathcal{U}^{{}^\circ\Theta}$ and

$$\bar{\mathcal{J}}(\Theta, U) \approx \mathcal{J}({}^\circ\Theta, u).$$

(b) *Let $\theta \in \mathfrak{C}^H$ be a standard Hölder continuous control and $u \in \mathcal{U}^\theta$. Then there exists $U \in \mathcal{X}^{*\theta}$ with $u = {}^\circ U$ as defined in part (a), and hence*

$$\bar{\mathcal{J}}({}^*\theta, U) \approx \mathcal{J}(\theta, u).$$

The proof is similar to the proof of Theorem 22.4.4 except for dealing with the terms that contain the control, and this is routine using the Hölder continuity property.

Finally we have the counterpart of Theorem 22.4.5.

Theorem 22.4.10 *Let f, g, h and $\bar{h}$ satisfy Conditions 22.4.1.*

(a) *For every Hölder continuous control θ there is an optimal solution $u^\theta \in \mathcal{U}^\theta$ with $\mathcal{J}(\theta, u^\theta) = \mathcal{J}(\theta)$.*

(b) *Let θ_n be a sequence of controls in $\mathfrak{C}^H$ and $u_n \in \mathcal{U}^{\theta_n}$, with $\mathcal{J}(\theta_n, u_n)$ converging. Then there is a control $\theta \in \mathfrak{C}^H$ and $u \in \mathcal{U}^\theta$ with $\mathcal{J}(\theta, u) = \lim_{n\to\infty} \mathcal{J}(\theta_n, u_n)$.*

Proof. Proved from Theorem 22.4.9 just as Theorem 22.4.5 followed from Theorem 22.4.4. □

The existence of an optimal control now follows.

Corollary 22.4.11 *There is an optimal Hölder continuous control θ_0 and optimal solution $u^{\theta_0} \in \mathcal{U}^{\theta_0}$ with $\mathcal{J}(\theta_0, u^{\theta_0}) = \mathcal{J}(\theta_0) = \mathcal{J}_0^H$.*

Proof. Take a minimizing sequence θ_n of controls in $\mathfrak{C}^H$ and solutions $u_n \in \mathcal{U}^\theta$ with $\mathcal{J}(\theta_n, u^{\theta_n}) \to \mathcal{J}_0^H$ and apply (b) of the previous theorem. □

Appendix: Nonstandard representations of the spaces $\mathbf{H}^r$

For ease of reference the most important facts about the nonstandard representation of members of the spaces $\mathbf{H}^r$ are summarized here, together with other crucial nonstandard matters. For full details consult [6].

The space ${}^*\mathbf{H}$ has a basis $({}^*e_n)_{n\in{}^*\mathbb{N}} = (E_n)_{n\in{}^*\mathbb{N}}$ given by the nonstandard extension ${}^*e$ of the function $e : \mathbb{N} \to \mathbf{H}$. For each $U \in {}^*\mathbf{H}$ there is a unique internal sequence of hyperreals $(U_n)_{n\in{}^*\mathbb{N}}$ such that

$$U = \sum_{n=1}^{{}^*\infty} U_n E_n.$$

and the subspace $\mathbf{H}_N$ of ${}^*\mathbf{H}$ defined by

$$\mathbf{H}_N = \Big\{ \sum_{n=1}^{N} U_n E_n : \; U \in {}^*\mathbb{R}^N, \; U \text{ internal} \Big\}$$

On $\mathbf{H}$ (and also $\mathbf{V}$ or any of the spaces $\mathbf{H}^r$) there are both weak and strong topologies. Here are the most important nonstandard facts.

Proposition 22.4.12 *Let $U \in {}^*\mathbf{H}$ (or $\mathbf{H}_N$) Then*

(a) *U is (strongly) nearstandard to u in $\mathbf{H}$ (denoted $U \approx u$) if $|U - {}^*u| \approx 0$ (and then $|U| \approx |u|$).*

(a) *U is weakly nearstandard to u in $\mathbf{H}$ (denoted $U \approx_w u$) if $(U, {}^*v) \approx (u, v)$ for all $v \in \mathbf{H}$ and $|u| \leq {}^\circ|U|$ (allowing this to be ∞).*

(b) *If U is strongly nearstandard then it is weakly nearstandard and the standard parts agree, so we write ${}^\circ U$ for the standard part in whichever topology it may be nearstandard.*

(c) *If $|U|$ is finite then U is weakly nearstandard and $({}^\circ U)_n = {}^\circ(U_n)$ for finite n.*

(d) *If $\|U\|$ is finite then U is strongly nearstandard in $\mathbf{H}$.*

(e) *If $|U|, |V| < \infty$ then*

$$U \approx_w V \iff U_i \approx V_i \quad \forall\, i \in \mathbb{N}.$$

Similar facts obtain for other spaces in the spectrum $\mathbf{H}^r$.

From (22.2.2) we have the following lemma (a slight extension of the Crucial Lemma (Lemma 2.7.7) of [6]), which is crucial for taking standard parts of the quadratic term $B(U)$, which is the source of many of the difficulties when discussing the Navier-Stokes equations.

Lemma 22.4.13 (Crucial Lemma) *If $U, V \in {}^*\mathbf{V}$ with $\|U\|$ and $\|V\|$ both finite, and $z \in \mathbf{V}$ then*

$${}^*b(U, V, {}^*z) \approx b(u, v, z)$$

where $u = {}^\circ U$ and $v = {}^\circ V$ (with $u, v \in \mathbf{V}$.) Hence, if $\|U\| < \infty$

$${}^*B(U) \approx b(u) \text{ in } \mathbf{V}' \text{ (weakly)}$$

For the proof consult any of [6, 9, 10].

Taking standard parts of the term AU needs the following observation.

Lemma 22.4.14 *If $U \in {}^*\mathbf{V}$ with $\|U\|$ finite with $u = {}^\circ U$ then*

$${}^*AU \approx Au \text{ in } \mathbf{V}' \text{ (weakly)}$$

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23

Local-in-time existence of strong solutions of the n-dimensional Burgers equation via discretizations

João Paulo Teixeira*

Abstract

Consider the equation:

$$u_t = \nu\Delta u - (u\cdot\nabla)u + f \quad \text{for} \quad x \in [0,1]^n \ \text{ and } \ t \in (0,\infty),$$

together with periodic boundary conditions and initial condition $u(t,0) = g(x)$. This corresponds a Navier-Stokes problem where the incompressibility condition has been dropped. The major difficulty in existence proofs for this simplified problem is the unbounded advection term, $(u\cdot\nabla)u$.

We present a proof of local-in-time existence of a smooth solution based on a discretization by a suitable Euler scheme. It will be shown that this solution exists in an interval $[0,T)$, where $T \leq \frac{1}{C}$, with C depending only on n and the values of the Lipschitz constants of f and u at time 0. The argument given is based directly on local estimates of the solutions of the discretized problem.

23.1 Introduction

The Burgers equation

$$u_t - \nu\Delta u + (u\cdot\nabla)u = f \qquad x \in D \subset \mathbb{R}^n,\ t > 0$$

provides an example of a model for flows that takes into account the interaction between diffusion and (nonlinear) advection. This is probably the simplest nonlinear physical model for turbulence.

*Instituto Superior Técnico, Lisbon, Portugal.
jteix@math.ist.utl.pt

This equation is a simplification of the Navier-Stokes equations where the incompressibility constraint on the flow has been dropped. The Navier-Stokes equations are usually studied by taking projected solutions of the Burgers equations onto a subspace of divergence free functions. The main difficulty in the analysis of nonlinear flows, the advection term, $(u \cdot \nabla)u$, is present in both equations.

It is a consequence of the incompressibility condition that there are only trivial Navier-Stokes flows for $n = 1$. This is not the case in Burgers flows, where the case $n = 1$ is already nontrivial. Cole [6] and Hopf [10] have studied the problem in the real line:

$$\begin{cases} u_t - \nu u_{xx} + u u_x = 0 & x \in \mathbb{R},\ t > 0 \\ u(x,t) = u_0(x) & x \in \mathbb{R}. \end{cases} \tag{23.1.1}$$

Using the transformation of variables

$$u = -2\nu \frac{v}{v_x}, \tag{23.1.2}$$

(23.1.1) was reduced to and initial value problem for the heat equation:

$$\begin{cases} v_t - \nu v_{xx} = 0 & x \in \mathbb{R},\ t > 0 \\ v(x,t) = v_0(x) & x \in \mathbb{R}. \end{cases} \tag{23.1.3}$$

(Where $u_0' = -2\nu \frac{v}{v_0'}$). This enabled them to get a formula for a solution of problem (23.1.1), in terms of a Gaussian integral. Thus, and under very mild conditions on u_0, a solution, u, exists for all $x \in \mathbb{R}$ and $t > 0$, and is smooth.

It would be reasonable to expect that the $n > 1$ case should behave in a similar way. The usual standard theory for this type of equations uses a weak formulation of the problem in Sobolev spaces, a Galerkin approximation to show existence of weak solutions and, finally, regularity estimates. However, these methods only lead to partial results. For well-posed problems with generic initial and boundary data, the best that can be obtained is local in-time existence of regular solutions. It can also be shown that regular (strong) solutions are unique (if they exist). See [4, 13, 9, 12, 16] for accounts on these methods.

To gain some new insight into this problem, we develop a hyperfinite approach to the following model problem on a compact domain.

Model Problem: *Let $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ be an n-dimensional torus. Assume that f is locally Lipschitz continuous on $\mathbb{T}^n \times [0,\infty)$ and $u_0 \in C^{2,1}(\mathbb{T}^n)$ (that is, u_0 is twice differentiable and all its second partial derivatives are Lipschitz continuous on $\mathbb{T}^n$). Let $\mathcal{D} = \mathbb{T}^n \times (0,\infty)$. Let $\nu \in \mathbb{R}^+$. Let $f : \mathbb{R}^n/\mathbb{Z}^n \to \mathbb{R}$,*

$u_0 : \mathbb{R}^n/\mathbb{Z}^n \to \mathbb{R}^n$. *Our task is to study the initial value problem for the Burgers equations:*

$$\begin{cases} u_t - \nu\Delta u + (u \cdot \nabla)u = f & \text{in } \mathcal{D} \\ u = u_0 & \text{on } \mathbb{T}^n \times \{0\}. \end{cases} \tag{23.1.4}$$

A (strong) solution of this problem is a sufficiently smooth u satisfying (23.1.4). By "sufficiently smooth u" we mean that $u : \mathbb{T}^n \times [0, \infty) \to \mathbb{R}$ is such that $u(\cdot, t) \in C^2(\mathbb{T}^n)$ for all $t \in [0, \infty)$ and $u(x, \cdot) \in C^1([0, \infty))$ for all $x \in \mathbb{T}^n$.

23.2 A discretization for the diffusion-advection equations in the torus

We now look for a discretized version of problem (23.1.4). Since we are mainly interested in the existence result, we will do this in the simplest way possible. First we work in the standard universe. To discretize $\mathbb{T}^n$, we introduce an h-spaced grid on $\mathbb{T}^n$. Choose $M \in \mathbb{N}_1$, and let $h = \frac{1}{M}$. Then, let:

$$\mathbb{T}^n_M = \Big\{0, h, 2h, \ldots, (M-1)h, 1\Big\}^n = h\,(\mathbb{Z} \bmod M)^n.$$

Consistently with our interpretation of $\mathbb{T}^n_M$ as a discrete version of the torus, $\mathbb{T}^n$, we define addition in $\mathbb{T}^n_M$ as follows: given any $x = (m_1, m_2, \ldots, m_n)h$ and $y = (l_1, l_2, \ldots, l_n)h$ in $\mathbb{T}^n_M$, let:

$$x + y = \Big((m_1 + l_1) \bmod M, (m_2 + l_2) \bmod M, \ldots, (m_n + l_n) \bmod M\Big)h$$

This makes addition well-defined in $\mathbb{T}^n_M$; furthermore, it will behave in a similar way to addition in $\mathbb{T}^n$. In particular, the set of grid-neighbors of any $x \in \mathbb{T}^n_M$,

$$\{x \pm he_i : \ i = 1, 2, \ldots, n\}$$

is well-defined.

As for the discretization of time, consider $T \in \mathbb{R}^+$ and $K \in \mathbb{N}_1$. Let $k = \frac{T}{K}$, and define:

$$I^T_K = \Big\{0, k, 2k, \ldots, (K-1)k, T\Big\} = k\,(\mathbb{N} \cap [0, K));$$

To each triple $d = (M, K, T)$, with $M, N \in \mathbb{N}_1$ and $T \in \mathbb{R}^+$, we associate a discretization as defined above. We will, later on, introduce some restrictions on the set of admissible d. For now, given any $d = (M, K, T)$, with $M, N \in \mathbb{N}_1$ and $T \in \mathbb{R}^+$, we let:

$$\mathcal{D}_d = \mathbb{T}^n_M \times I^T_K$$

$$\overline{\mathcal{D}}_d = \mathbb{T}_M^n \times \left(I_K^T \cup \{T\}\right)$$

The elements of $\overline{\mathcal{D}}_d$ are called gridpoints. Any function U whose domain is a subset of $\overline{\mathcal{D}}_d$ is called a gridfunction.

The discrete Laplacian can be defined as a map $\Delta_d : \mathbb{R}^{\overline{\mathcal{D}}_d} \to \mathbb{R}^{\overline{\mathcal{D}}_d}$ given by:

$$\Delta_d U(x,t) = \frac{1}{h^2} \sum_{i=1}^{n} \Big(U(x + he_i, t) - 2U(x,t) + U(x - he_i, t) \Big). \tag{23.2.1}$$

The definition of addition in $\mathbb{T}_M^n$ makes this well-defined.

The discrete version of the nonlinear parabolic operator, P, that occurs in the diffusion-advection equations is defined as the map $P_d : \mathbb{R}^{\overline{\mathcal{D}}_d} \to \mathbb{R}^{\mathcal{D}_d}$ given by:

$$\begin{aligned} P_d U(x,t) &= \frac{U(x,t+k) - U(x,t)}{k} - \nu \Delta_d U(x,t) \\ &\quad + \sum_{i=1}^{n} U_i(x,t) \frac{U(x + he_i, t) - U(x - he_i, t)}{2h}. \end{aligned}$$

With

$$\lambda = \frac{k}{h^2} = \frac{TM^2}{K} \in \mathbb{R}^+,$$

we get:

$$\begin{aligned} &P_d U(x,t) = \\ &= \frac{1}{\lambda h^2} \Bigg(U(x,t+k) - (1 - 2n\nu\lambda) U(x,t) - \\ &- \lambda \sum_{i=1}^{n} \Big(\Big(\nu - \frac{h}{2} U_i(x,t) \Big) U(x + he_i, t) + \Big(\nu + \frac{h}{2} U_i(x,t) \Big) U(x - he_i, t) \Big) \Bigg) \end{aligned} \tag{23.2.2}$$

The discretized version of problem (23.1.4) is, then:

$$\begin{cases} P_d U(x,t) = f(x,t) & \text{if } (x,t) \in \mathcal{D}_d \\ U(x,0) = u_0(x,0) & \text{if } x \in \mathbb{T}_M^n. \end{cases} \tag{23.2.3}$$

Let us look more closely at the finite difference equation in (23.2.3). If we solve for $U(x, t+k)$, we get:

$$\begin{aligned} &U(x,t+k) = \big(1 - 2n\nu\lambda\big) U(x,t) \\ &+ \lambda \sum_{i=1}^{n} \Big(\Big(\nu - \frac{h}{2} U_i(x,t) \Big) U(x + he_i, t) + \Big(\nu + \frac{h}{2} U_i(x,t) \Big) U(x - he_i, t) \Big) \\ &+ \lambda h^2 f(x,t) \end{aligned} \tag{23.2.4}$$

Equation (23.2.4), together with the initial condition in (23.2.3), gives a recursive formula for the unique solution of problem (23.2.3).

Define a map

$$(\mathbb{R}^n)^{\overline{\mathcal{D}}_d} \times (\mathbb{R}^n)^{\overline{\mathcal{D}}_d} \ni (U, V) \stackrel{\Phi}{\longmapsto} \Phi(U, V) \in (\mathbb{R}^n)^{\overline{\mathcal{D}}_d},$$

by:

$$\begin{aligned} \Phi(U,V)(x,t) \;&=\; \big(1 - 2n\nu\lambda\big)\, U(x,t) + \\ &+ \lambda \sum_{i=1}^{n} \left(\left(\nu - \frac{h}{2} V_i(x,t) \right) U(x + he_i, t) + \left(\nu + \frac{h}{2} V_i(x,t) \right) U(x - he_i, t) \right), \end{aligned} \tag{23.2.5}$$

whenever $(x,t) \in \overline{\mathcal{D}}_d$. Equation (23.2.4) can now be written as:

$$U(x, t+k) = \Phi(U,U)(x,t) + \lambda h^2 f(x,t)$$

In what follows, we will require the right-hand side of (23.2.5) to be in the form of a weighted average, that is, all coefficients multiplying $U(x,t)$ and $U(x \pm he_i, t)$ must be positive. For this, we will always assume that:

$$\lambda < \frac{1}{2n\nu} \tag{23.2.6}$$

We also require that:

$$|U_d(x,t)| < \frac{2\nu}{h} \qquad \text{for all } \; (x,t) \in \overline{\mathcal{D}}_d, \tag{23.2.7}$$

where U_d is the solution of (23.2.3), relative to d.

This means that the set of admissible discretizations is:

$$\left\{ (M,K,T) \in \mathbb{N}_1 \times \mathbb{N}_1 \times \mathbb{R}^+ : \; \lambda < \frac{1}{2n\nu} \;\wedge\; \forall (x,t) \in \overline{\mathcal{D}}_d \; |U_d(x,t)| < \frac{2\nu}{h} \right\}$$

Note that condition (23.2.7) is easily satisfied when we work in the nonstandard universe. Whenever h is infinitesimal, $\frac{2\nu}{h}$ will be infinitely large; so, if U is kept finite, then condition (23.2.7) holds.

23.3 Some standard estimates for the solution of the discrete problem

In this section, we derive estimates that will not require a nonstandard discretization.

Consider any (standard) discretization, d. Given $U : \overline{\mathcal{D}}_d \to \mathbb{R}^n$ and $A \subset \overline{\mathcal{D}}_d$, let:

$$\begin{aligned} \|U\|_{L_d^\infty(A)} &= \max_{(x,t)\in A} |U(x,t)| \\ [U]_{L_d^\infty(A)} &= \max_{(x,t),(y,t)\in A \; x\neq y} \frac{|U(x,t)-U(y,t)|}{|x-y|} \\ [[U]]_{L_d^\infty(A)} &= \max_{(x,t),(x,t+k)\in A} \frac{|U(x,t+k)-U(x,t)|}{k} \end{aligned}$$

Here, $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^j$.

We begin by showing some properties of Φ.

Lemma 1 *Let $U, V \in (\mathbb{R}^n)^{\overline{\mathcal{D}}_d}$. Let $h > 0$ be such that:*

$$h < \frac{2\nu}{\|V\|_{L_d^\infty(\overline{\mathcal{D}}_d)}}. \tag{23.3.1}$$

Then, for all $(x,t) \in \overline{\mathcal{D}}_d$:

$$|\Phi(U,V)(x,t)| \leq \|U\|_{L_d^\infty(\overline{\mathcal{D}}_d)}.$$

Proof. For any $(x,t) \in \overline{\mathcal{D}}_d$:

$$\begin{aligned} &\left|\Phi(U,V)(x,t)\right| \leq \left|\left(1-2n\nu\lambda\right)U(x,t)\right| + \\ &+\lambda\sum_{i=1}^{n}\left(\left|\left(\nu-\frac{h}{2}V_i(x,t)\right)U(x+he_i,t)\right| + \left|\left(\nu+\frac{h}{2}V_i(x,t)\right)U(x-he_i,t)\right|\right). \end{aligned}$$

Let $M = \|U\|_{L_d^\infty(\overline{\mathcal{D}}_d)}$. By condition (23.3.1):

$$\nu \pm \frac{h}{2}V(x,t) > 0.$$

Thus:

$$\begin{aligned} &\left|\Phi(U,V)(x,t)\right| \leq \left(1-2n\nu\lambda\right)|U(x,t)| + \\ &+\lambda\sum_{i=1}^{n}\left(\left(\nu-\frac{h}{2}V_i(x,t)\right)|U(x+he_i,t)| + \left(\nu+\frac{h}{2}V_i(x,t)\right)|U(x-he_i,t)|\right). \end{aligned}$$

Since $|U(x,t)| \leq M$, we get:

$$\begin{aligned} |\Phi(U,V)(x,t)| &\leq \left(1-2n\nu\lambda\right)M + \lambda M\sum_{i=1}^{n}\left(\nu-\frac{h}{2}V_i(x,t)+\nu+\frac{h}{2}V_i(x,t)\right) \\ &\leq \left(1-2n\nu\lambda\right)M + \lambda M 2\nu n = M. \end{aligned}$$

□

Lemma 2 *Let $U, V, W, Z \in (\mathbb{R}^n)^{\overline{\mathcal{D}}_d}$. Then, for all $(x,t) \in \overline{\mathcal{D}}_d$:*

$$\Phi(U,W)(x,t) - \Phi(V,Z)(x,t) = \Phi(U-V,W)(x,t) + \\ + \frac{\lambda h}{2} \sum_{i=1}^{n} \Big(Z_i(x,t) - W_i(x,t)\Big)\Big(V(x+he_i,t) - V(x-he_i,t)\Big).$$

Proof. For all $(x,t) \in \overline{\mathcal{D}}_d$:

$$\Phi(U,W)(x,t) - \Phi(V,Z)(x,t) = (1-2n\lambda\nu)\Big(U(x,t) - V(x,t)\Big) + \\ + \lambda \sum_{i=1}^{n} \left(\left(\nu - \frac{h}{2} W_i(x,t)\right) U(x+he_i,t) - \left(\nu - \frac{h}{2} Z_i(x,t)\right) V(x+he_i,t) + \right. \\ \left. + \left(\nu + \frac{h}{2} W_i(x,t)\right) U(x-he_i,t) - \left(\nu + \frac{h}{2} Z_i(x,t)\right) V(x-he_i,t) \right) \\ = (1-2n\lambda\nu)\Big(U(x,t) - V(x,t)\Big) + \\ + \lambda \sum_{i=1}^{n} \Big(\Big(\nu - \frac{h}{2} W_i(x,t)\Big)\Big(U(x+he_i,t) - V(x+he_i,t)\Big) + \\ + \Big(\nu + \frac{h}{2} W_i(x,t)\Big)\Big(U(x-he_i,t) - V(x-he_i,t)\Big)\Big) + \\ + \frac{\lambda h}{2} \sum_{i=1}^{n} \Big(\big(Z_i(x,t) - W_i(x,t)\big) V(x+he_i,t) - \\ - \big(Z_i(x,t) - W_i(x,t)\big) V(x-he_i,t) \Big) \\ = \Phi(U-V,W)(x,t) + \\ \frac{\lambda h}{2} \sum_{i=1}^{n} \Big(Z_i(x,t) - W_i(x,t)\Big)\Big(V(x+he_i,t) - V(x-he_i,t)\Big). \qquad \square$$

Lemma 3 *Let d be a discretization such that:*

$$h < \frac{2\nu}{\|u_0\|_{L^\infty} + T\|f\|_{L^\infty}}$$

If U is the solution of the discrete problem, (23.2.3), then:

$$\|U\|_{L^\infty_d(\overline{\mathcal{D}}_d)} \leq \|u_0\|_{L^\infty} + T\|f\|_{L^\infty}$$

In particular, d is an admissible discretization.

Proof. Let $M = \|u_0\|_{L^\infty}$ and $L = \|f\|_{L^\infty}$. We show, by induction on $t = 0, k, \ldots, T$, that for all $\tau = 0, k, \ldots, t$ and for all $x \in \mathbb{T}_M^n$:

$$|U(x,t)| \leq M + tL.$$

By the initial condition in problem (23.2.3), the result holds for $t = 0$. Assuming it is valid for some $t \in I_K^T$, we have, by the hypothesis on h and the induction hypothesis, that:

$$h < \frac{2\nu}{M+TL} < \frac{2\nu}{M+tL} < \frac{2\nu}{|U(x,t)|},$$

for all $(x,t) \in \mathbb{T}_M^n \times \{0, k, 2k, \ldots, t\}$. Hence, by Lemma 2:

$$|U(x,t+k)| \leq |\Phi(U,U)(x,t)| + k|f(x,t)| \leq M + tL + kL = M + (t+k)L. \quad \square$$

23.4 Main estimates on the hyperfinite discrete problem

The nonstandard analytical setup we now use is as follows. We work in a superstructure $\langle V(\mathbb{R}), {}^*V(\mathbb{R}), * \rangle$. We will omit the stars on all standard functions of one or several variables and usual binary relations. Any $x \in {}^*\mathbb{R}^n$ is called finite iff there exists $m \in \mathbb{N}$ such that $|x| < m$. Each finite $x \in {}^*\mathbb{R}$ can be uniquely decomposed as $x = r + \varepsilon$, where $r \in \mathbb{R}$ and ε is an infinitesimal; r is called the standard part of x, and denoted by $\operatorname{st} x$. If $x, y \in {}^*\mathbb{R}$ are such that $x - y$ is infinitesimal, then we say that x is infinitely close to y, and write $x \approx y$. Similarly, if $x, y \in {}^*\mathbb{R}^n$:

$$x \approx y \quad \text{iff} \quad |x - y| \approx 0 \quad \text{iff} \quad x_i \approx y_i, \quad \text{for each } i = 1, \ldots, n.$$

Let $j, l \in \mathbb{N}$. If $x \in {}^*\mathbb{R}^j$ is finite then let:

$$\operatorname{st} x = \operatorname{st}(x_1, \ldots, x_j) = (\operatorname{st} x_1, \ldots, \operatorname{st} x_j).$$

If $F : A \subset {}^*\mathbb{R}^j \to {}^*\mathbb{R}^l$ is an S-continuous function, define ${}^\circ F$ by:

$${}^\circ F(\operatorname{st} x) = \operatorname{st}(F(x)) \quad \forall x \in A. \tag{23.4.1}$$

For sets $A \in \mathbb{R}^j$, let:

$${}^\circ A = \Big\{ \operatorname{st} x : \text{ “}x \text{ is finite” and } x \in A \Big\}.$$

Each “circle” map as introduced above is sometimes called a standard part map.

In this, and other similar definitions of this work, we consider ${}^*\mathcal{V}$, where $\mathcal{V} = \{\Delta_d\}$. ${}^*\mathcal{V}$ is an internal *family of linear maps indexed in the internal set of all admissible d; each $\Delta_d \in {}^*\mathcal{U}$ is a *linear map acting on the vector space of internal gridfunctions $U : \overline{\mathcal{D}}_d \to {}^*\mathbb{R}$.

Similarly, we can consider ${}^*\mathcal{U}$, where

$$\mathcal{U} = \left\{ P_d : \ M \in {}^*\mathbb{N}, T \in {}^*\mathbb{R}^+, K \in {}^*\mathbb{N} \ \text{ and } \ \frac{TM^2}{K} < \frac{1}{2n\nu} \right\},$$

${}^*\mathcal{U}$ is now an internal family of linear maps, indexed on an internal set. Each $H^T_{MK} \in {}^*\mathcal{U}$ is then a *linear map acting on the vector space of internal gridfunctions.

For the norms and seminorms introduced in the previous Section, we can proceed similarly. We get families of internal norms and seminorms which, by transfer, satisfy the following. Given $U : \overline{\mathcal{D}}_d \to {}^*\mathbb{R}^n$ and $A \subset \overline{\mathcal{D}}_d$:

$$\|U\|_{L^\infty_d(A)} = {}^*\max \left\{ |U(x,t)| : \ (x,t) \in A \right\},$$

$$[U]_{L^\infty_d(A)} = {}^*\max \left\{ \frac{|U(x,t) - U(y,t)|}{|x-y|} : \ (x,t),(y,t) \in A \text{ and } x \neq y \right\},$$

$$[[U]]_{L^\infty_d(A)} = {}^*\max \left\{ \frac{|U(x,t+k) - U(x,t)|}{k} : \ (x,t),(x,t+k) \in A \right\}.$$

Lemma 4 *Let d be an admissible discretization such that h is infinitesimal. Let*

$$L_0 = \max \left([u_0]_{L^\infty_d(\mathbb{T}^n_M)}, \frac{1}{n} ([f]_{L^\infty_d(\overline{\mathcal{D}}_d)})^{1/2} \right).$$

If U is the solution of the discrete problem (23.2.3) then, for $T < \frac{1}{2nL_0}$:

$$[U]_{L^\infty_d(\overline{\mathcal{D}}_d)} \lesssim \frac{L_0}{1 - 2nL_0T}$$

Proof. To work internally, we will begin by requiring a weaker condition on h, namely:

$$h < \frac{2\nu}{\|u_0\|_{L^\infty} + T\|f\|_{L^\infty}}. \tag{23.4.2}$$

Lemma 3 shows that this is a sufficient condition for admissibility of d, which is all that is needed for now.

Fix $z \in \mathbb{T}_M^n$ and write $V(x,t) = U(x+z,t)$. Then, by Lemma 2:

$$\begin{aligned} U(x+z,t+k) - U(x,t+k) &= \Phi(V,V)(x,t) - \Phi(U,U)(x,t) + \\ &\quad + \lambda h^2(f(x+z,t) - f(x,t)) \\ &= \Phi(V-U,V)(x,t) + \\ &\quad + \frac{\lambda h}{2}\sum_{i=1}^{n}\Big(U_i(x,t) - V_i(x,t)\Big)\Big(U(x+he_i,t) - U(x-he_i,t)\Big) + \\ &\quad + \lambda h^2(f(x+z,t) - f(x,t)) \end{aligned}$$

By *recursion, we will construct a function $L : \{0,k,2k,\dots,T\} \to {}^*\mathbb{R}$ such that, for all $t = 0,k,\dots T$ and $x,z \in \mathbb{T}_M^n$:

$$|U(x+z,t) - U(x,t)| \le L(t)|z| \tag{23.4.3}$$

Let $L(t)$ be given by:

$$\begin{cases} L(0) &= L_0 \\ L(t+k) &= L(t) + nk(L(t))^2 + nkL_0^2 \end{cases}$$

For $t = 0$, we get:

$$|U(x+z,0) - U(x,0)| = |u_0(x+z) - u_0(x)| \le [u_0]_{L_d^\infty(\mathbb{T}_M^n)}\,|z| \le L_0|z|.$$

Now, we assume that for all $\tau = 0,k,\dots,t$ and $x \in \mathbb{T}_M^n$

$$|U(x+z,\tau) - U(x,\tau)| \le L(t)|z|.$$

Since d is admissible (and so $|V(x,\tau)| = |U(x+z,\tau)| < \frac{2\nu}{h}$), Lemma 1 implies that $|\Phi(V-U,V)| \le L(t)|z|$. Hence:

$$\begin{aligned} |U(x+z,t+k) - U(x,t+k)| &\le L(t)|z| \\ &\quad + \frac{\lambda h}{2}\sum_{i=1}^{n} L(t)|z|L(t)2h + \lambda h^2[f]_{L^\infty}|z| \\ &\le \Big(L(t) + nk(L(t))^2 + nkL_0^2\Big)|z|. \end{aligned}$$

This shows inequality (23.4.3) by *induction.

Note that $L(t)$ defines the Euler iterates for the standard ODE initial value problem:

$$\begin{cases} y' &= ny^2 + nL_0^2 \\ y(0) &= L_0 \end{cases} \tag{23.4.4}$$

Since $y' \geq 0$, $y(t) \geq L_0$. Hence, $y' \leq 2ny^2$, and thus:

$$y(t) \leq \frac{L_0}{1 - 2nL_0 t}$$

Now, use the fact that h is infinitesimal. By the convergence of the Euler iterates to the solutions of problem (23.4.4), we conclude that:

$$L(t) \approx y(t) \leq \frac{L_0}{1 - 2nL_0 t}$$

This estimate is valid for t in any compact interval where problem (23.4.4) has solution; in particular, the estimate will be valid for $t < \frac{1}{2nL_0}$. □

Lemma 5 *Let d be an admissible discretization such that h is infinitesimal. Let L_0 be as in Lemma 4 and*

$$M_0 = \max\left([[u_0]]_{L_d^\infty(\mathbb{T}_M^n)}, \frac{1}{n}\Big([[f]]_{L_d^\infty(\overline{\mathcal{D}}_d)} \Big)^{1/2}, L_0 \right), \quad \text{with}$$

$$[[u_0]]_{L_d^\infty(\mathbb{T}_M^n)} = {}^*\max\left\{ \nu \Delta u_0(x) - (u \cdot \nabla) u_0(x) + f(x,0) : \ x \in \mathbb{T}_M^n \right\}$$

If U is the solution of the discrete problem (23.2.3) then, for $T < \frac{1}{2nL_0}$:

$$[[U]]_{L_d^\infty(\overline{\mathcal{D}}_d)} \lesssim \frac{M_0}{1 - 2nL_0 T}$$

Proof. Again we begin by requiring that h satisfies the weaker statement:

$$h < \frac{2\nu}{\|u_0\|_{L^\infty} + T\|f\|_{L^\infty}} \tag{23.4.5}$$

Write $V(x,t) = U(x, t+k)$. Then, by Lemma 2:

$$\begin{aligned} U(x,t+2k) - U(x,t+k) &= \Phi(V,V)(x,t) - \Phi(U,U)(x,t) + \\ &\qquad + \lambda h^2 (f(x,t+k) - f(x,t)) \\ &= \Phi(V-U, V)(x,t) + \\ &\quad + \frac{\lambda h}{2} \sum_{i=1}^{n} \Big(U_i(x,t) - V_i(x,t) \Big) \Big(U(x + he_i, t) - U(x - he_i, t) \Big) \\ &\quad + \lambda h^2 (f(x,t+k) - f(x,t)) \end{aligned}$$

By *recursion, we again construct a function $M : \{0, k, 2k, \ldots, T\} \to {}^*\mathbb{R}$ such that, for all $t = 0, k, \ldots T$ and $x \in \mathbb{T}_M^n$:

$$|U(x,t+k) - U(x,t)| \leq M(t)k. \tag{23.4.6}$$

Let $M(t)$ be given by:

$$\begin{cases} M(0) &= M_0 \\ M(t+k) &= M(t) + nkM(t)L(t) + nkM_0^2 \end{cases}$$

Here, L is the function introduced in the proof of Lemma 4.

For $t = 0$, we get:

$$\begin{aligned} \frac{U(x,k) - U(x,0)}{k} &= \nu \sum_{i=1}^{n} \delta^0_{h,i,i} U - \sum_{i=1}^{n} U_i \delta^0_{h,i} U + f(x,0) \\ &\approx \nu \Delta u_0(x) - (u_0 \cdot \nabla) u_0(x) + f(x,0) \\ &\leq [[u_0]]_{L^\infty_d(\mathbb{T}^n_M)} \\ &\leq M_0. \end{aligned}$$

Now, we assume that for all $\tau = 0, k, \ldots, t$ and $x \in \mathbb{T}^n_M$

$$|U(x, \tau + k) - U(x, \tau)| \leq M(t)k.$$

Since d is admissible (and so $|V(x,\tau)| = |U(x,\tau+k)| < \frac{2\nu}{h}$), Lemma 1 implies that $|\Phi(V - U, V)| \leq M(t)k$. Hence:

$$\begin{aligned} |U(x,t+2k) - U(x,t+k)| &\leq M(t)k + \frac{\lambda h}{2} \sum_{i=1}^{n} M(t)kL(t)2h + \lambda h^2 [[f]]_{L^\infty} h \\ &\leq \Big(M(t) + nkM(t)L(t) + nkM_0^2 \Big) k \\ &= M(t+k)k. \end{aligned}$$

This shows inequality (23.4.6) by *induction.

Now, we note that $M(t)$ defines the Euler iterates for the standard ODE initial value problem:

$$\begin{cases} z' &= n\, z\, y + nM_0^2 \\ z(0) &= M_0. \end{cases} \tag{23.4.7}$$

Here, $y(t)$ is the solution of problem (23.4.7), as given in the proof of Lemma 4.

Since $z' \geq 0$, $z(t) \geq M_0$, so $nz(t)y(t) \geq nM_0L_0 \geq nM_0^2$. Hence, $z'(t) \leq 2nz(t)y(t)$, and thus:

$$z(t) \leq \frac{M_0}{1 - 2nL_0 t}$$

Now, use the fact that h is infinitesimal. By the convergence of the Euler iterates to the solutions of problem (23.4.4), we conclude that:

$$M(t) \approx z(t) \leq \frac{M_0}{1 - 2nL_0 t}$$

This estimate is valid for t in any compact interval where problem (23.4.4) has solution; in particular, the estimate will be valid for $t < \frac{1}{2nL_0}$. □

23.5 Existence and uniqueness of solution

By the results of section 23.4, the function $U(x,t)$ is S-continuous for $t \in \{0, k, \ldots, T\}$, with $T < \frac{1}{2nL_0}$. Note that the set of admissible values for T depends only on L_0, that is, on the size of the Lipschitz constants of u_0 and f.

Therefore,

$$u(\operatorname{st} x, \operatorname{st} t) = \operatorname{st} U(x,t) \tag{23.5.1}$$

is well-defined and continuous on $\mathbb{T}^n \times [0,T]$. Also, Lemmas 4 and 5 imply that u is globally Lipschitz on $\mathbb{T}^n \times [0,T]$.

To show that u solves (locally in time) the problem and also to prove a uniqueness result, we need the following:

Lemma 6 *Let $a : \mathbb{T}^n \times [0,T] \to \mathbb{R}$ be Lipschitz continuous. Then, the problem,*

$$\begin{cases} v_t - \nu\Delta v + (a \cdot \nabla)v = f & \text{in } \mathcal{D} \\ v = u_0 & \text{on } \mathbb{T}^n \times \{0\}. \end{cases} \tag{23.5.2}$$

has a unique solution $v \in C^{2,1}(\mathbb{T}^n \times [0,T])$. Furthermore, if v is a solution of problem (23.5.2) then (in the nonstandard universe) v satisfies: for all $(x,t) \in \overline{\mathcal{D}}_d$, there exists an $\varepsilon \approx 0$ such that

$$v(x, t+k) = \Phi(v,a)(x,t) + \lambda h^2 f(x,t) + \varepsilon\lambda h^2.$$

Proof. Consider problem (23.5.2). Since a is given, the vectorial differential equation $v_t - \nu\Delta v + (a \cdot \nabla)v = f$ is just an uncoupled system of n scalar second order parabolic equations. Since a is a Lipschitz continuous function on $\mathbb{T}^n \times [0,T]$ and $u_0 \in C^{2,1}(\mathbb{T}^n)$, by the theory of parabolic equations (e.g. Friedman, [8]), the problem has a unique (strong) solution, $v \in C^{2,1}(\mathbb{T}^n \times [0,T])$. By the regularity of v:

$$\begin{aligned} f(x,t) &= v_t(x,t) - \nu\Delta v(x,t) + (a(x,t) \cdot \nabla)v(x,t) \approx P_d v(x,t) \\ &= \frac{v(x,t+k) - v(x,t)}{k} - \nu\Delta_d v(x,t) + \sum_{i=1}^{n} a_i(x,t) \frac{v(x+he_i,t) - v(x-he_i,t)}{2h} \\ &= \frac{1}{\lambda h^2} \Bigg(v(x,t+k) - (1 - 2n\nu\lambda)v(x,t) - \\ &\quad - \lambda \sum_{i=1}^{n} \left(\left(\nu - \frac{h}{2} a_i(x,t) \right) v(x+he_i,t) + \left(\nu + \frac{h}{2} a_i(x,t) \right) v(x-he_i,t) \right) \Bigg) \end{aligned}$$

Consequently, for each $(x,t) \in \overline{\mathcal{D}}_d$, there exists an $\varepsilon \approx 0$ such that:

$$\begin{aligned} v(x,t+k) &= \big(1-2n\nu\lambda\big)\, v(x,t) + \\ &+ \lambda \sum_{i=1}^{n} \left(\left(\nu - \frac{h}{2} a_i(x,t) \right) v(x+he_i,t) + \left(\nu + \frac{h}{2} a_i(x,t) \right) v(x-he_i,t) \right) + \\ &+ \lambda h^2 f(x,t) + \varepsilon \lambda h^2 \\ &= \Phi(v,a)(x,t) + \lambda h^2 f(x,t) + \varepsilon \lambda h^2. \end{aligned}$$

□

Here is our existence lemma:

Lemma 7 *Let u and T be as given by equation (23.5.1). Then $u \in C^{(2;1)}(\mathbb{T}^n \times [0,T])$ and u is a (strong) solution of the Burgers equation problem (23.1.4).*

Proof. Consider the problem:

$$\begin{cases} v_t - \nu \Delta v + (u \cdot \nabla) v = f & \text{in } \mathcal{D} \\ v = u_0 & \text{on } \mathbb{T}^n \times \{0\}. \end{cases}$$

By Lemma 6 the problem has a unique (strong) solution, $u \in C^{2,1}(\mathbb{T}^n \times [0,T])$ and, in the nonstandard universe, for all $(x,t) \in \overline{\mathcal{D}}_d$, there exists an $\varepsilon \approx 0$ such that:

$$v(x,t+k) = \Phi(v,a)(x,t) + \lambda h^2 f(x,t) + \varepsilon \lambda h^2.$$

Now, we show that $u = v$.

Let $r \in \mathbb{R}^+$ be arbitrary. We will show by induction on $t = 0, k, 2k, \ldots, T$ that for all $\tau = 0, k, 2k, \ldots, t$ and $x \in \mathbb{T}_M^n$:

$$|U(x,t) - {}^*v(x,t)| \le rt.$$

This implies that, for all $(x,t) \in \overline{\mathcal{D}}_d$:

$$u(\operatorname{st} x, \operatorname{st} t) = \operatorname{st} U(x,t) = \operatorname{st} {}^*v(x,t) = v(\operatorname{st} x, \operatorname{st} t),$$

as wanted.

(As usual, from this point on we omit stars on ${}^*v$). For $t = 0$ the statement follows from the initial conditions. Now, assuming the statement holds true for some $t = jk$, $j \in {}^*\mathbb{N}$:

$$\begin{aligned} U(x,t+k) - v(x,t+k) &= \Phi(U,U)(x,t) - \Phi(v,u)(x,t) - \varepsilon \lambda h^2 \\ &= \Phi(U-v,u)(x,t) - \varepsilon k + \\ &+ \frac{\lambda h}{2} \sum_{i=1}^{n} \Big(u_i(x,t) - U_i(x,t) \Big) \Big(v(x+he_i,t) - v(x-he_i,t) \Big). \end{aligned}$$

Since $v(\cdot,t)$ is a C^2 function on the compact set $\mathbb{T}^n$, there exists a finite C so that $|v(x+he_i,t)-v(x-he_i,t)| \leq 2hC$. Also, by the definition and continuity of u, $u_i(x,t) - U_i(x,t) = \delta \approx 0$. By the induction hypothesis and Lemma 3, $|\Phi(U-v,u)(x,t)| \leq rt$. Therefore:

$$\begin{aligned}\Big|U(x,t+k) - v(x,t+k)\Big| &\leq rt + \varepsilon k + \frac{\lambda h}{2}\, n\delta\, 2hC \\ &= rt + k\big(\varepsilon + nC\delta\big) \\ &\leq r(t+k). \qquad \square\end{aligned}$$

As consequence of the above Lemma, we get:

Theorem 1 *Let f be locally Lipschitz continuous on $\mathbb{T}^n \times [0,\infty)$ and $u_0 \in C^{2,1}(\mathbb{T}^n)$. Let*

$$L = \max\left([u_0]_{L_d^\infty(\mathbb{T}^n_M)}, \frac{1}{n}([f]_{L_d^\infty(\overline{\mathcal{D}}_d)})^{1/2}, [[u_0]]_{L_d^\infty(\mathbb{T}^n_M)}, \frac{1}{n}([[f]]_{L_d^\infty(\overline{\mathcal{D}}_d)})^{1/2}\right).$$

Here, $[[u_0]]_{L_d^\infty(\mathbb{T}^n_M)}$ is an upper-bound for the Lipschitz constant of the function $\Delta u_0(x) + (u_0 \cdot \nabla)u_0(x) + f(x,0)$. Then, for any $T < \frac{1}{2nL}$, the problem

$$\begin{cases} u_t = \nu\Delta u - (u\cdot\nabla)u + f & \text{in } \mathcal{D} \\ u = u_0 & \text{on } \mathbb{T}^n \times \{0\}. \end{cases}$$

has a strong solution, $u \in C^{2,1}(\mathbb{T}^n \times [0,T])$.

The uniqueness theorem follows from:

Lemma 8 *Let u and v be solutions of the Burgers equation problem (23.1.4) on $\mathbb{T}^n \times [0,T]$, for some $T > 0$. Let $L_v = [v]_{L_d^\infty(\mathbb{T}^n \times [0,T])}$. Then:*

$$u(x,t) = v(x,t) \qquad \text{for all } \ x \in \mathbb{T}^n, \ \ 0 \leq t < \min\left\{\frac{1}{2nL_v}, T\right\}$$

Proof. Using the nonstandard statement of Lemma 6, (note that u and v are given functions), we conclude that for all $(x,t) \in \overline{\mathcal{D}}_d$, there exist infinitesimal δ_1 and δ_2 such that:

$$u(x,t+k) = \Phi(u,u)(x,t) + \lambda h^2 f(x,t) + \delta_1 \lambda h^2$$

and

$$v(x,t+k) = \Phi(v,v)(x,t) + \lambda h^2 f(x,t) + \delta_2 \lambda h^2.$$

Let Θ be the largest ik, $i \in {}^*\mathbb{N}$, such that $ik \leq \min\left\{\frac{1}{2nL_v}, T\right\}$. Let $r \in \mathbb{R}^+$ be arbitrary. We will show by induction on $t = 0, k, 2k, \ldots, \Theta$ that for all $\tau = 0, k, 2k, \ldots, t$ and $x \in \mathbb{T}_M^n$:

$$|{}^*u(x,t) - {}^*v(x,t)| \leq rt.$$

This implies that, for all $(x,t) \in \overline{\mathcal{D}}_d$:

$$u(\mathrm{st}\, x, \mathrm{st}\, t) = v(\mathrm{st}\, x, \mathrm{st}\, t),$$

as wanted.

(From hereon, we omit stars on ${}^*u$ and ${}^*v$). For $t = 0$ the statement follows from the initial conditions. Now, assuming the statement holds true for some $t = jk$, $j \in {}^*\mathbb{N}$:

$$u(x,t+k) - v(x,t+k) = \Phi(u,u)(x,t) - \Phi(v,v)(x,t) + (\delta_1 - \delta_2)\lambda h^2$$

Let $\varepsilon = \delta_1 - \delta_2 \approx 0$. Using Lemma 2:

$$\begin{aligned} u(x,t+k) - v(x,t+k) \;&=\; \Phi(u-v,v)(x,t) + \varepsilon k \\ &+ \frac{\lambda h}{2} \sum_{i=1}^{n} \Big(v_i(x,t) - u_i(x,t)\Big)\Big(v(x+he_i,t) - v(x-he_i,t)\Big) \end{aligned}$$

Note that $|v(x+he_i,t) - v(x-he_i,t)| \leq 2hL_v$. Using the induction hypothesis $|u_i(x,t) - v_i(x,t)| \leq rt$. Also, by Lemma 3 and the induction hypothesis, $|\Phi(u-v,v)(x,t)| \leq rt$. Therefore:

$$\begin{aligned} \Big|u(x,t+k) - v(x,t+k)\Big| \;&\leq\; rt + \varepsilon k + \frac{\lambda h}{2}\, nrt\, 2hL_v \\ &=\; rt + k\Big(\varepsilon + rntL_v\Big). \end{aligned}$$

Since $t \leq \Theta \leq \frac{1}{2nL_v}$, it follows that:

$$|u(x,t+k) - v(x,t+k)| \leq r(t+k). \qquad \square$$

Theorem 2 *Let u and v be solutions of our Model Problem (23.1.4) on $\mathbb{T}^n \times [0,T]$, for some $T > 0$. Then $u = v$.*

Proof. Assume there exists $t \in [0,T]$ such that, for some $x \in \mathbb{T}^n$, $u(x,t) \neq v(x,t)$; let θ be the infimum of such t.

Note that:

$$u(x,\theta) = v(x,\theta), \quad \text{for all } x \in \mathbb{T}^n. \tag{23.5.3}$$

If $\theta = 0$ then equality (23.5.3) follows from the initial conditions; otherwise, it follows from continuity of u and v. Consider the problem:

$$\begin{cases} w_t = \nu \Delta w - (w \cdot \nabla) w + g & \text{in } \mathcal{D} \\ w = u(x, \theta) & \text{on } \mathbb{T}^n \times \{0\}. \end{cases} \tag{23.5.4}$$

where $g(x,t) = f(x, \theta + t)$, for all $(x,t) \in \mathbb{T}^n \times [0, T - \theta]$. Consider $w_1, w_2 : \mathbb{T}^n \times [0, T - \theta] \to \mathbb{R}$ given by $w_1(x,t) = u(x, \theta + t)$ and $w_1(x,t) = v(x, \theta + t)$. But w_1 and w_2 are solutions of problem (23.5.4) such that, for arbitrarily small $t > 0$:

$$w_1(x,t) \neq w_2(x,t) \quad \text{for some } x \in \mathbb{T}^n.$$

This contradicts Lemma 8. □

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Part V

Infinitesimals and education

24

Calculus with infinitesimals

Keith D. Stroyan*

24.1 Intuitive proofs with "small" quantities

Abraham Robinson discovered a rigorous approach to calculus with infinitesimals in 1960 and published it in [9]. This solved a 300 year old problem dating to Leibniz and Newton. Extending the ordered field of (Dedekind) "real" numbers to include infinitesimals is not difficult algebraically, but calculus depends on approximations with transcendental functions. Robinson used mathematical logic to show how to extend all real functions in a way that preserves their properties in a precise sense. These properties can be used to develop calculus with infinitesimals. Infinitesimal numbers have always fit basic intuitive approximation when certain quantities are "small enough," but Leibniz, Euler, and many others could not make the approach free of contradiction. Section 1 of this article uses some intuitive approximations to derive a few fundamental results of analysis. We use approximate equality, $x \approx y$, only in an intuitive sense that "x is sufficiently close to y".

H. Jerome Keisler developed simpler approaches to Robinson's logic and began using infinitesimals in beginning U.S. calculus courses in 1969. The experimental and first edition of his book were used widely in the 1970's. Section 2 of this article completes the intuitive proofs of Section 1 using Keisler's approach to infinitesimals from [6].

24.1.1 Continuity and extreme values

Theorem 1 ***The Extreme Value Theorem***

Suppose a function $f[x]$ is continuous on a compact interval $[a, b]$. Then $f[x]$ attains both a maximum and minimum, that is, there are points x_{MAX} and x_{min} in $[a, b]$, so that for every other x in $[a, b]$, $f[x_{\text{min}}] \leq f[x] \leq f[x_{\text{MAX}}]$.

*University of Iowa, USA.
keith-stroyan@uiowa.edu

Formulating the meaning of "continuous" is a large part of making this result precise. We will take the intuitive definition that $f[x]$ is continuous means that if an input value x_1 is close to another, x_2, then the output values are close. We summarize this as: $f[x]$ is continuous if and only if $a \leq x_1 \approx x_2 \leq b \Longrightarrow f[x_1] \approx f[x_2]$.

Given this property of $f[x]$, if we partition $[a, b]$ into tiny increments,

$$a < a + \frac{1(b-a)}{H} < a + \frac{2(b-a)}{H} < \cdots < a + \frac{k(b-a)}{H} < \cdots < b$$

the maximum of the function on the finite partition occurs at one (or more) of the points $x_M = a + \frac{k(b-a)}{H}$. This means that for any other partition point $x_1 = a + \frac{j(b-a)}{H}$, $f[x_M] \geq f[x_1]$.

Any point $a \leq x \leq b$ is within $\frac{1}{H}$ of a partition point $x_1 = a + \frac{j(b-a)}{H}$, so if H is very large, $x \approx x_1$ and

$$f[x_M] \geq f[x_1] \approx f[x]$$

and we have found the approximate maximum.

It is not hard to make this idea into a sequential argument where $x_{M[H]}$ depends on H, but there is quite some trouble to make the sequence $x_{M[H]}$ converge (using some form of compactness of $[a, b]$.) Robinson's theory simply shows that the hyperreal x_M chosen when $1/H$ is infinitesimal, is infinitely near an ordinary real number where the maximum occurs. We complete this proof as a simple example of Keisler's Axioms in Section 2.

24.1.2 Microscopic tangency in one variable

In beginning calculus you learned that the derivative measures the slope of the line tangent to a curve $y = f[x]$ at a particular point, $(x, f[x])$. We begin by setting up convenient "local variables" to use to discuss this problem. If we fix a particular $(x, f[x])$ in the x-y-coordinates, we can define new parallel coordinates (dx, dy) through this point. The (dx, dy)-origin is the point of tangency to the curve.

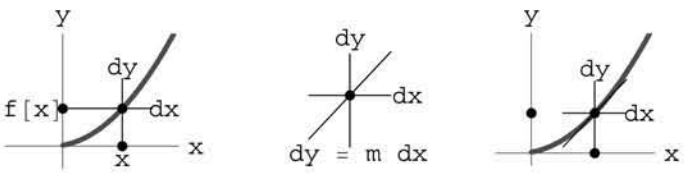

Figure 24.1.1: Microscopic Tangency

A line in the local coordinates through the local origin has equation $dy = mdx$ for some slope m. Of course we seek the proper value of m to make $dy = mdx$ tangent to $y = f[x]$.

You probably learned the derivative from the approximation

$$\lim_{\Delta x \longrightarrow 0} \frac{f[x + \Delta x] - f[x]}{\Delta x} = f'[x].$$

If we write the error in this limit explicitly, the approximation can be expressed as $\frac{f[x+\Delta x]-f[x]}{\Delta x} = f'[x] + \varepsilon$ or $f[x + \Delta x] - f[x] = f'[x] \cdot \Delta x + \varepsilon \cdot \Delta x$. Intuitively we say $f[x]$ is smooth if there is a function $f'[x]$ that makes the error small, $\varepsilon \approx 0$, in the formula

$$f[x + \delta x] - f[x] = f'[x] \cdot \delta + \varepsilon \cdot \delta x \tag{24.1.1}$$

when the change in input is small, $\delta x \approx 0$. The main point of this article is to show that requiring ε to be infinitesimal whenever δx is infinitesimal and x is near standard gives an intuitive and simple direct meaning to $\mathcal{C}^1$ smooth. Requiring only that ε tend to zero only for fixed real x, or pointwise, is not sufficient to capture the intuitive approximations of our proofs.

The nonlinear change on the left side of (24.1.1) equals a linear change, $m \cdot \delta x$, with $m = f'[x]$, plus a term that is small compared with the input change.

The error ε has a direct graphical interpretation as the error measured above $x + \delta x$ after magnification by $1/\delta x$. This magnification makes the small change δx appear unit size and the term $\varepsilon \cdot \delta x$ measures ε after magnification.

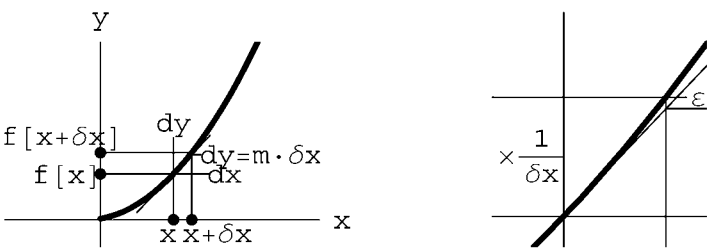

Figure 24.1.2: Magnified Error

When we focus a powerful microscope at the point $(x, f[x])$ we only see the linear curve $dy = m \cdot dx$, because $\varepsilon \approx 0$ is smaller than the thickness of the line. The figure below shows a small box magnified on the right.

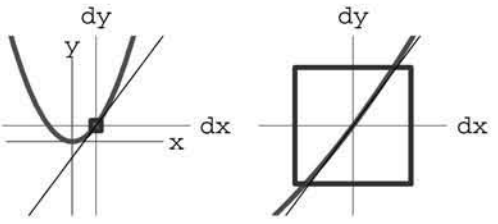

Figure 24.1.3: A Magnified Tangent

24.1.3 The Fundamental Theorem of Integral Calculus

Now we use the intuitive microscope approximation (24.1.1) to prove:

Theorem 2 ***The Fundamental Theorem of Integral Calculus: Part I***
Suppose we want to find $\int_a^b f[x]\,dx$. *If we can find another function* $F[x]$ *so that the derivative satisfies* $F'[x] = f[x]$ *for every* x, $a \le x \le b$, *then*

$$\int_a^b f[x]\,dx = F[b] - F[a]$$

The definition of the integral we use is the real number approximated by a sum of small slices,

$$\int_a^b f[x]\,dx \approx \sum_{\substack{x=a \\ \text{step } \delta x}}^{b-\delta x} f[x] \cdot \delta x, \text{ when } \delta x \approx 0$$

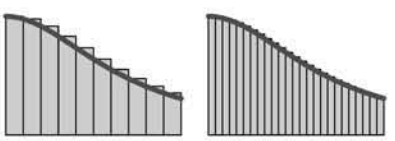

Figure 24.1.4: Sum Approximations

24.1.4 Telescoping sums and derivatives

We know that if $F[x]$ has derivative $F'[x] = f[x]$, the differential approximation above says,

$$F[x + \delta x] - F[x] = f[x] \cdot \delta x + \varepsilon \cdot \delta x$$

so we can sum both sides

$$\sum_{\substack{x=a\\ \text{step } \delta x}}^{b-\delta x} F[x+\delta x]-F[x]=\sum_{\substack{x=a\\ \text{step } \delta x}}^{b-\delta} f[x]\cdot\delta x+\sum_{\substack{x=a\\ \text{step } \delta x}}^{b-\delta x}\varepsilon\cdot\delta x$$

The telescoping sum satisfies,

$$\sum_{\substack{x=a\\ \text{step } \delta x}}^{b-\delta x} F[x+\delta x]-F[x]=F[b']-F[a]$$

so we obtain the approximation,

$$\int_a^b f[x]\,dx\approx\sum_{\substack{x=a\\ \text{step } \delta x}}^{b-\delta x} f[x]\cdot\delta x=F[b']-F[a]-\sum_{\substack{x=a\\ \text{step } \delta x}}^{b-\delta x}\varepsilon\cdot\delta x$$

This gives,

$$\begin{aligned}\left|\sum_{\substack{x=a\\ \text{step } \delta x}}^{b-\delta x} f[x]\cdot\delta x-(F[b']-F[a])\right| &\leq \left|\sum_{\substack{x=a\\ \text{step } \delta x}}^{b-\delta x}\varepsilon\cdot\delta x\right|\leq\sum_{\substack{x=a\\ \text{step } \delta x}}^{b-\delta x}|\varepsilon|\cdot\delta x\\ &\leq \max[|\varepsilon|]\cdot\sum_{\substack{x=a\\ \text{step } \delta x}}^{b-\delta x}\delta x=\max[|\varepsilon|]\cdot(b'-a)\\ &\approx 0\end{aligned}$$

or $\int_a^b f[x]\,dx\approx\sum_{\substack{x=a\\ \text{step } \delta x}}^{b-\delta x} f[x]\cdot\delta x\approx F[b']-F[a]$. Since $F[x]$ is continuous, $F[b']\approx F[b]$, so $\int_a^b f[x]\,dx=F[b]-F[a]$.

We need to know that all the epsilons above are small when the step size is small, $\varepsilon\approx 0$, when $\delta x\approx 0$ for all $x=a,a+\delta x,a+2\delta,\cdots$. This is a uniform condition that has a simple appearance in Robinson's theory. There is something to explain here because the theorem stated above is false if we take the usual pointwise notion of derivative and the Riemann integral. (There are pointwise differentiable functions whose derivative is not Riemann integrable. See [5] Example 35, Chapter 8, p.107 ff.)

The condition needed to make this proof complete is natural geometrically and plays a role in the intuitive proof of the inverse function theorem in the next example.

24.1.5 Continuity of the derivative

We show now that the differential approximation

$$f[x+\delta x]-f[x]=f'[x]\cdot\delta+\varepsilon\cdot\delta x$$

forces the derivative function $f'[x]$ to be continuous,

$$x_1 \approx x_2 \Longrightarrow f'[x_1] \approx f'[x_2]$$

Let $x_1 \approx x_2$, but $x_1 \neq x_2$. Use the differential approximation with $x = x_1$ and $\delta x = x_2 - x_1$ and also with $x = x_2$ and $\delta x = x_1 - x_2$, geometrically looking at the tangent approximation from both endpoints.

$$f[x_2] - f[x_1] = f'[x_1] \cdot (x_2 - x_1) + \varepsilon_1 \cdot (x_2 - x_1)$$
$$f[x_1] - f[x_2] = f'[x_2] \cdot (x_1 - x_2) + \varepsilon_2 \cdot (x_1 - x_2)$$

Adding these equations, we obtain

$$0 = \big((f'[x_1] - f'[x_2]) + (\varepsilon_1 - \varepsilon_2)\big) \cdot (x_2 - x_1)$$

Dividing by the nonzero term $(x_2 - x_1)$ and adding $f'[x_2]$ to both sides, we obtain, $f'[x_2] = f'[x_1] + (\varepsilon_1 - \varepsilon_2)$ or $f'[x_2] \approx f'[x_1]$, since the difference between two small errors is small.

This fact can be used to prove:

Theorem 3 ***The Inverse Function Theorem***

If $f'[x_0] \neq 0$ then $f[x]$ has an inverse function in a small neighborhood of x_0, that is, if $y \approx y_0 = f[x_0]$, then there is a unique $x \approx x_0$ so that $y = f[x]$.

We saw above that the differential approximation makes a microscopic view of the graph look linear. If $y \approx y_1$ the linear equation $dy = m \cdot dx$ with $m = f'[x_1]$ can be inverted to find a first approximation to the inverse,

$$y - y_0 = m \cdot (x_1 - x_0)$$
$$x_1 = x_0 + \tfrac{1}{m}(y - y_0)$$

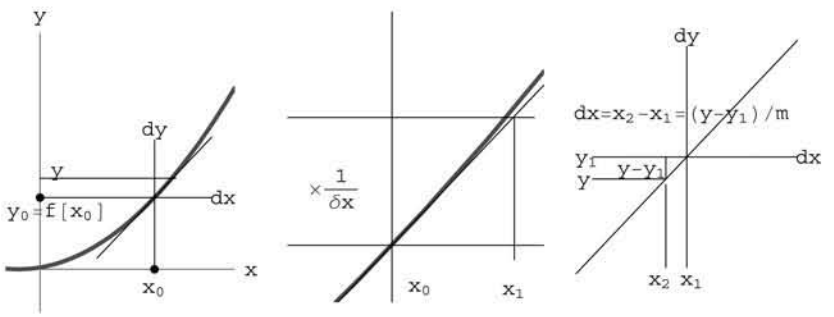

Figure 24.1.5: Approximate Inverse

We test to see if $f[x_1] = y$. If not, examine the graph microscopically at $(x_1, y_1) = (x_1, f[x_1])$. Since the graph appears the same as its tangent to within ε and since $m = f'[x_1] \approx f'[x_2]$, the local coordinates at (x_1, y_1) look like a line of slope m. Solving for the linear x-value which gives output y, we get

$$\begin{aligned} y - y_1 &= m \cdot (x_2 - x_1) \\ x_2 &= x_1 + \tfrac{1}{m}(y - f[x_1]) \end{aligned}$$

Continue in this way generating a sequence of approximations, $x_1 = x_0 + \frac{1}{m}(y - y_0)$, $x_{n+1} = G[x_n]$, where the recursion function is $G[\xi] = x + \frac{1}{m}(y - f[\xi])$. The distance between successive approximations is

$$\begin{aligned} |x_2 - x_1| &= \big|G[x_1] - G[x_0]\big| \le \frac{1}{2} \cdot |x_1 - x_0| \\ |x_3 - x_2| &= \big|G[x_2] - G[x_1]\big| \le \frac{1}{2} \cdot |x_2 - x_1| \le \frac{1}{2} \cdot \frac{1}{2} \cdot |x_1 - x_0| \end{aligned}$$

by the Differential Approximation for $G\,[x]$. Notice that $G'[\xi] = 1 - f'[\xi]/m \approx 0$, for $\xi \approx x_0$, so $|G'[\xi]| < 1/2$ in particular, and

$$\begin{aligned} |x_2 - x_1| &= \big|G[x_1] - G[x_0]\big| \le \frac{1}{2} \cdot |x_1 - x_0| \\ |x_3 - x_2| &= \big|G[x_2] - G[x_1]\big| \le \frac{1}{2} \cdot |x_2 - x_1| \le \frac{1}{2^2} \cdot |x_1 - x_0| \\ &\vdots \\ |x_{n+1} - x_n| &\le \frac{1}{2^n} \cdot |x_1 - x_0| \\ |x_{n+1} - x_0| &\le |x_{n+1} - x_n| + |x_n - x_{n-1}| + \cdots + |x_1 - x_0| \\ &\le \left(1 + \frac{1}{2} + \cdots + \frac{1}{2^n}\right) \cdot |x_1 - x_0| \end{aligned}$$

A geometric series estimate shows that the series converges, $x_n \to x \approx x_0$ and $f[x] = y$.

To complete this proof we need to show that $G[\xi]$ is a contraction on some noninfinitesimal interval. The precise definition of the derivative matters because the result is false if $f'[x]$ is defined by a pointwise limit. The function $f[x] = x + x^2 \sin[\pi/x]$ with $f[0] = 0$ has pointwise derivative 1 at zero, but is not increasing in any neighborhood of zero.

24.1.6 Trig, polar coordinates, and Holditch's formula

Calculus depends on small approximations with transcendental functions like sine, cosine, and the natural logarithm. Following are some intuitive approximations with non-algebraic functions.

Sine and cosine in radian measure give the x and y location of a point on the unit circle measured a distance θ along the circle. Now make a small change in the angle and magnify by $1/\delta\theta$ to make the change appear unit size.

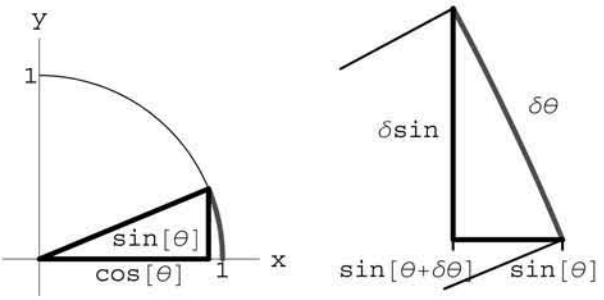

Figure 24.1.6: Sine increments

Since magnification does not change lines, the radial segments from the origin to the tiny increment of the circle meet the circle at right angles and appear under magnification to be parallel lines. Smoothness of the circle means that under powerful magnification, the circular increment appears to be a line. The difference in values of the sine is the long vertical leg of the increment "triangle" above on the right. The apparent hypotenuse with length $\delta\theta$ is the circular increment.

Since the radial lines meet the circle at right angles the large triangle on the unit circle at the left with long leg $\cos[\theta]$ and hypotenuse 1 is similar to the increment triangle, giving

$$\frac{\cos[\theta]}{1} \approx \frac{\delta \sin}{\delta\theta}$$

We write approximate similarity because the increment "triangle" actually has one circular side that is $\approx$-straight. In any case, this is a convincing argument that $\frac{d\sin}{d\theta}[\theta] = \cos[\theta]$. A similar geometric argument on the increment triangle shows that $\frac{d\cos}{d\theta}[\theta] = -\sin[\theta]$.

24.1.7 The polar area differential

The derivation of sine and cosine is related to the area differential in polar coordinates. If we take an angle increment of $\delta\theta$ and magnify a view of the circular arc on a circle of radius r, the length of the circular increment is $r \cdot \delta\theta$, by similarity.

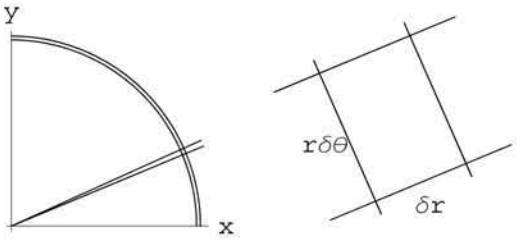

Figure 24.1.7: Polar increment

A magnified view of circles of radii r and $r+\delta r$ between the rays at angles θ and $\theta+\delta\theta$ appears to be a rectangle with sides of lengths δr and $r\cdot\delta\theta$. If this were a true rectangle, its area would be $r\cdot\delta\theta\cdot\delta r$, but it is only an approximate rectangle. Technically, we can show that the area of this region is $r\cdot\delta\theta\cdot\delta r$ plus a term that is small compared with this infinitesimal,

$$\delta A = r\delta\theta\delta r + \varepsilon\cdot\delta\theta\delta r,\ \varepsilon\approx 0.$$

Keisler's Infinite Sum Theorem [6] assures us that we can neglect this size error and integrate with respect to $rd\theta dr$.

Theorem 4 ***Holditch's formula***

The area swept out by a tangent of length R as it traverses an arbitrary convex curve in the plane is $A=\pi R^2$.

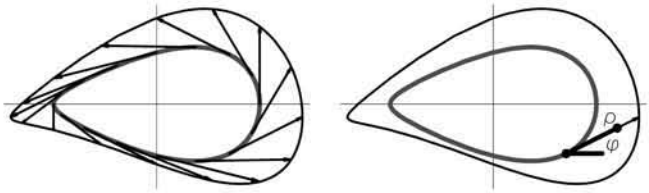

Figure 24.1.8: ρ-φ-coordinates

We can see this interesting result by using a variation of polar coordinates and the infinitesimal polar area increment above. Since the curve is convex, each tangent meets the curve at a unique angle, φ, and each point in the region swept out by the tangents is a distance ρ along that tangent.

We look at an infinitesimal increment of the region in ρ-φ-coordinates, first holding the φ-base point on the curve and changing φ. Microscopically this produces an increment like the polar increment:

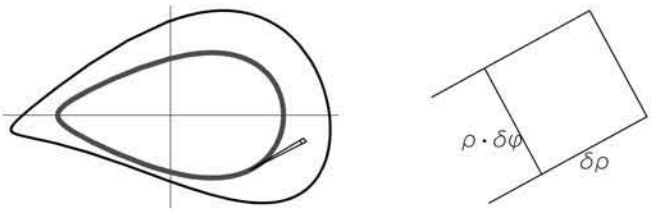

Figure 24.1.9: ρ-only increment

Next, looking at the base point of the tangent on the curve, moving to the correct $\varphi + \delta\varphi$-base point, moves along the curve. Microscopically this looks like translation along the tangent line (by smoothness) as shown on the left. Including this near-translation in the infinitesimal area increment produces a parallelogram:

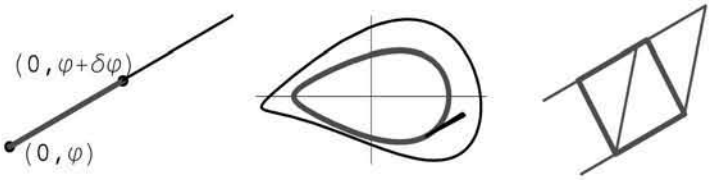

Figure 24.1.10: φ and ρ increment

of height $\rho \cdot \delta\varphi$ and base $\delta\rho$, or area $\delta A = \rho \cdot \delta\varphi \cdot \delta\rho$:

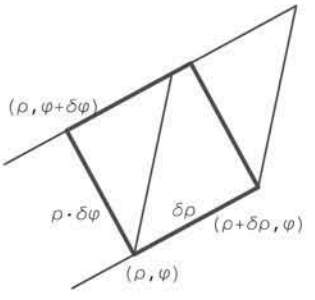

Figure 24.1.11: $da = \rho \cdot \delta\varphi \cdot \delta\rho$

Integrating gives the total area of the region

$$\int_0^R \int_0^{2\pi} \rho d\varphi d\rho = \pi R^2.$$

24.1.8 Leibniz's formula for radius of curvature

The radius r of a circle drawn through three infinitely nearby points on a curve in the (x, y)-plane satisfies

$$\frac{1}{r} = -\frac{d}{dx}\left(\frac{dy}{ds}\right)$$

where s denotes the arclength.

For example, if $y = f[x]$, so $ds = \sqrt{1 + (f'[x])^2}dx$, then

$$\frac{1}{r} = -\frac{d}{dx}\left(\frac{f'[x]}{\sqrt{1 + (f'[x])^2}}\right) = -\frac{f''[x]}{(1 + f'[x]^2)^{3/2}}$$

If the curve is given parametrically, $y = y[t]$ and $x = x[t]$, so $ds = \sqrt{x'[t]^2 + y'[t]^2}dt$, then

$$\frac{1}{r} = -\frac{d\left(\frac{dy}{ds}\right)}{dx} = \frac{y'[t]x''[t] - x'[t]y''[t]}{(x'[t]^2 + y'[t]^2)^{3/2}}$$

24.1.9 Changes

Consider three points on a curve $\mathbb{C}$ with equal distances Δs between the points. Let α_I and α_{II} denote the angles between the horizontal and the segments connecting the points as shown. We have the relation between the changes in y and α:

$$\sin[\alpha] = \frac{\Delta y}{\Delta s} \tag{24.1.2}$$

The difference between these angles, $\Delta\alpha$, is shown near p_{III} (figure 24.1.12).

The angle between the perpendicular bisectors of the connecting segments is also $\Delta\alpha$, because they meet the connecting segments at right angles.

These bisectors meet at the center of a circle through the three points on the curve whose radius we denote r. The small triangle with hypotenuse r gives

$$\sin\left[\frac{\Delta\alpha}{2}\right] = \frac{\Delta s/2}{r} \tag{24.1.3}$$

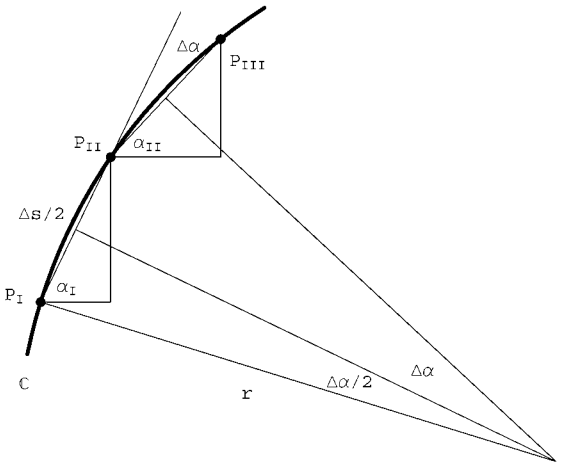

Figure 24.1.12: Changes in s and a

24.1.10 Small changes

Now we apply these relations when the distance between the successive points is an infinitesimal δs. The change

$$-\delta \sin[\alpha] = -\delta\left(\frac{\delta y}{\delta s}\right) = \sin[\alpha] - \sin[\alpha - \delta\alpha] = \cos[\alpha]\cdot\delta\alpha + \vartheta\cdot\delta\alpha, \quad (24.1.4)$$

with $\vartheta \approx 0$, by smoothness of sine (see above). Smoothness of sine also gives,

$$\sin\left[\frac{\delta\alpha}{2}\right] = \frac{\delta\alpha}{2} + \eta\cdot\delta\alpha, \text{ with } \eta \approx 0.$$

Combining this with formula (24.1.3) for the infinitesimal case (assuming $r \neq 0$), we get

$$\delta\alpha = \frac{\delta s}{r} + \iota\cdot\delta\alpha, \text{ with } \iota \approx 0.$$

Now substitute this in (24.1.4) to obtain

$$-\delta\left(\frac{\delta y}{\delta s}\right) = \cos[\alpha]\frac{\delta s}{r} + \zeta\cdot\delta s, \text{ with } \zeta \approx 0.$$

By trigonometry, $\cos[\alpha] = \delta x/\delta s$, so

$$-\frac{\delta\left(\frac{\delta y}{\delta s}\right)}{\delta x} = \frac{1}{r} + \zeta\cdot\frac{\delta s}{\delta x} \approx \frac{1}{r},$$

as long as $\frac{\delta s}{\delta x}$ is not infinitely large.

Keisler's Function Extension Axiom allows us to apply formulas (24.1.3) and (24.1.4) when the change is infinitesimal, as we shall see. We still have a gap to fill in order to know that we may replace infinitesimal differences with differentials (or derivatives), especially because we have a difference of a quotient of differences.

First differences and derivatives have a fairly simple rigorous version in Robinson's theory, just using the differential approximation (24.1.1). This can be used to derive many classical differential equations like the tractrix, catenary, and isochrone, see: Chapter 5, Differential Equations from Increment Geometry in [11].

Second differences and second derivatives have a complicated history. See [2]. This is a very interesting paper that begins with a course in calculus as Leibniz might have presented it.

24.1.11 The natural exponential

The natural exponential function satisfies

$$\begin{aligned} y[0] &= 1 \\ \frac{dy}{dx} &= y \end{aligned}$$

We can use (24.1.1) to find an approximate solution,

$$y[\delta x] = y[0] + y'[0] \cdot \delta = 1 + \delta x$$

Recursively,

$$\begin{aligned} y[2\delta x] &= y[\delta x] + y'[\delta x] \cdot \delta x = y[\delta] \cdot (1 + \delta x) = (1 + \delta x)^2 \\ y[3\delta x] &= y[2\delta x] + y'[2\delta x] \cdot \delta x = y[2\delta x] \cdot (1 + \delta x) = (1 + \delta x)^3 \\ &\vdots \\ y[x] &= (1 + \delta x)^{x/\delta x}, \text{ for } x = 0, \delta x, 2\delta x, 3\delta x, \cdots \end{aligned}$$

This is the product expansion $e \approx (1 + \delta)^{1/\delta x}$, for $\delta x \approx 0$.

No introduction to calculus is complete without mention of this sort of "infinite algebra" as championed by Euler as in [3]. A wonderful modern interpretation of these sorts of computations is in [7]. W. A. J. Luxemburg's reformulation of the proof of one of Euler's central formulas $\sin[z] = z \prod_{k=1}^{\infty} \left(1 - \left(\frac{z}{k\pi}\right)^2\right)$ appears in our monograph, [13].

24.1.12 Concerning the history of the calculus

Chapter X of Robinson's monograph [10] begins:

The history of a subject is usually written in the light of later developments. For over half a century now, accounts of the history of the Differential and Integral Calculus have been based on the belief that even though the idea of a number system containing infinitely small and infinitely large elements might be consistent, it is useless for the development of Mathematical Analysis. In consequence, there is in the writings of this period a noticeable contrast between the severity with which the ideas of Leibniz and his successors are treated and the leniency accorded to the lapses of the early proponents of the doctrine of limits. We do not propose here to subject any of these works to a detailed criticism. However, it will serve as a starting point for our discussion to try to give a fair summary of the contemporary impression of the history of the Calculus...

I recommend that you read Robinson's Chapter X. I have often wondered if mathematicians in the time of Weierstrass said things like, 'Karl's epsilon-delta stuff isn't very interesting. All he does is re-prove old formulas of Euler.'

I have a non-standard interest in the history of infinitesimal calculus. It really is not historical. Some of the old derivations like Bernoulli's derivation of Leibniz' formula for the radius of curvature seem to me to have a compelling clarity. Robinson's theory of infinitesimals offers me an opportunity to see what is needed to complete these arguments with a contemporary standard of rigor.

Working on such problems has led me to believe that the best theory to use to underly calculus when we present it to beginners is one based on the kind of derivatives described in Section 2 and not the pointwise approach that is the current custom in the U.S. I believe we want a theory that supports intuitive reasoning like the examples above and pointwise defined derivatives do not.

24.2 Keisler's axioms

The following presentation of Keisler's foundations for Robinson's Theory of Infinitesimals is explained in more detail in either of the (free .pdf) files [12] and the Epilog to Keisler's text [6].

24.2.1 Small, medium, and large hyperreal numbers

A field of numbers is a system that satisfies the associative, commutative and distributive laws and has additive inverses and multiplicative inverses for nonzero elements. (See the above references for details.) Essentially this means the laws of high school algebra apply. The binomial expansion that follows is

a consequence of the field axioms. Hence this formula holds for any pair of numbers x and Δx in a field.

$$(x + \Delta x)^3 = x^3 + 3x^2\Delta x + ((3x + \Delta x) \cdot \Delta x) \cdot \Delta x$$

To compare sizes of numbers we need an ordering. An ordered field has a transitive order that is compatible with the field operations in the sense that if $a < b$, then $a + c < b + c$ and if $0 < a$ and $0 < b$, then $0 < a \cdot b$. (The complex numbers can't be ordered compatibly because $i^2 = -1$.)

An ***infinitesimal*** is a number satisfying $|\delta| < 1/m$ for any ordinary natural counting number, $m = 1, 2, 3, \cdots$. Archimedes' Axiom is precisely the statement that the (Dedekind) "real" numbers have no positive infinitesimals. (We take 0 as infinitesimal.) Keisler's Algebra Axiom is the following:

24.2.2 Keisler's algebra axiom

Axiom 5 *The hyperreal numbers are an ordered field extension of the real numbers. In particular, there is a positive hyperreal infinitesimal, δ.*

Any ordered field extending the reals has an infinitesimal, but we just include this fact in the axiom. There are many different infinitesimals. For example, the law $a < b \Rightarrow a + c < b + c$ applied to $a = 0$ and $b = c = \delta$ says $\delta < 2\delta$. If k is a natural number, $k\delta < \frac{1}{m}$, for any natural m, because $\delta < \frac{1}{k \cdot m}$ when δ is infinitesimal. All the ordinary integer multiples of δ are distinct infinitesimals,

$$\cdots < -3\delta < -2\delta < -\delta < 0 < \delta < 2\delta < 3\delta < \cdots$$

Magnification at center c with power $1/\delta$ is simply the transformation $x \to (x - c)/\delta$, so by laws of algebra, integer multiples of δ end up the same integers apart after magnification by $1/\delta$ centered at zero.

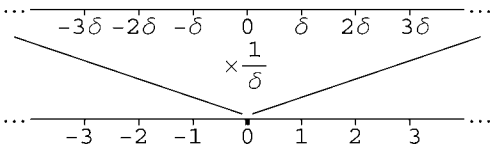

Figure 24.2.1: Magnification by $\frac{1}{\delta}$

Similar reasoning lets us place $\frac{\delta}{2}, \frac{\delta}{3}, \cdots$ on a magnified line at one half the distance to δ, one third the distance, etc.

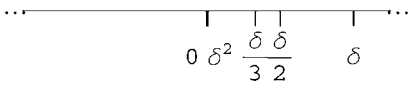

Figure 24.2.2: Fractions of δ

Where should we place the numbers δ^2, $\delta^3 \cdots$? On a scale of δ, they are infinitely near zero, $\frac{1}{\delta}(\delta^2 - 0) = \delta \approx 0$. Magnification by $1/\delta^2$ reveals δ^2, but moves δ infinitely far to the right, $\frac{1}{\delta^2}(\delta - 0) = \frac{1}{\delta} > m$ for all natural $m = 1, 2, 3, \cdots$

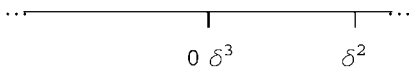

Figure 24.2.3: δ^3 on a δ^2 scale

Laws of algebra dictate many "orders of infinitesimal" such as $0 < \cdots < \delta^3 < \delta^2 < \delta$. The laws of algebra also show that near every real number there are many hyperreals, say near $\pi = 3.14159\cdots$

$$\cdots < \pi - 3\delta < \pi - 2\delta < \pi - \delta < \pi < \pi + \delta < \pi + 2\delta < \pi + 3\delta < \cdots$$

A hyperreal number x is called ***limited*** (or "finite in magnitude") if there is a natural number m so that $|x| < m$. If there is no natural bound for a hyperreal number it is called ***unlimited*** (or "infinite"). Infinitesimal numbers are limited, being bounded by 1.

Theorem 6 ***Standard Parts of Limited Hyperreal Numbers***

Every limited hyperreal number x differs from some real number by an infinitesimal, that is, there is a real r so that $x \approx r$. This number is called the "standard part" of x, $r = st[x]$.

Proof. Define a Dedekind cut in the real numbers by $A = \{s : \ s \leq x\}$ and $B = \{s : \ x < s\}$. st$[x]$ is the real number defined by this cut. □

The fancy way to state the next theorem is to say the limited numbers are an ordered ring with the infinitesimals as a maximal ideal. This amounts to simple rules like "infsml × limited = infsml."

Theorem 7 ***Computation rules for small, medium, and large***

(*a*) *If p and q are limited, so are $p + q$ and $p \cdot q$*

(*b*) *If* ε *and* δ *are infinitesimal, so is* $\varepsilon + \delta$.

(*c*) *If* $\delta \approx 0$ *and* q *is limited, then* $q \cdot \delta \approx 0$.

(*d*) $1/0$ *is still undefined and* $1/x$ *is unlimited only when* $x \approx 0$.

Proof. These rules are easy to prove as we illustrate with (c). If q is limited, there is a natural number with $|q| < k$. the condition $\delta \approx 0$ means $|\delta| < \frac{1}{k \cdot m}$, so $|q \cdot \delta| < \frac{1}{m}$ proving that $q \cdot \delta \approx 0$. □

24.2.3 The uniform derivative of x^3

Let's apply these rules to show that $f[x] = x^3$ satisfies the differential approximation with $f'[x] = 3x^2$ when x is limited. We know by laws of algebra that

$$(x + \delta x)^3 - x^3 = 3x^2 \delta x + \varepsilon \cdot \delta x, \text{ with } \varepsilon = ((3x + \delta x) \cdot \delta x)$$

If x is limited and $\delta x \approx 0$, (a) shows that $3x$ is limited and that $3x + \delta x$ is also limited. Condition (c) then shows that $\varepsilon = ((3x + \delta x) \cdot \delta x) \approx 0$ proving that for all limited x

$$f[x + \delta x] - f[x] = f'[x] \cdot \delta x + \varepsilon \cdot \delta x$$

with $\varepsilon \approx 0$ whenever $\delta x \approx 0$.

Below we will see that this computation is logically equivalent to the statement that $\lim_{\Delta x \longrightarrow 0} \frac{(x+\Delta x)^3 - x^3}{\Delta x} = 3x^2$, uniformly on compact sets of the real line. It is really no surprise that we can differentiate algebraic functions using algebraic properties of numbers. You should try this yourself with $f[x] = x^n$, $f[x] = 1/x$, $f[x] = \sqrt{x}$, etc. This does not solve the problem of finding sound foundations for calculus using infinitesimals because we need to treat transcendental functions like sine, cosine, log.

24.2.4 Keisler's function extension axiom

Keisler's Function Extension Axiom says that all real functions have extensions to the hyperreal numbers and these "natural" extensions obey the same identities and inequalities as the original function. Some familiar identities are

$$\sin[\alpha + \beta] = \sin[\alpha] \cos[\beta] + \cos[\alpha] \sin[\beta]$$

$$\log[x \cdot y] = \log[x] + \log[y]$$

The log identity only holds when x and y are positive. Keisler's Function Extension Axiom is formulated so that we can apply it to the Log identity in the form of the implication

$$\begin{aligned}&(x > 0 \text{ and } y > 0) \Rightarrow \log[x] \text{ and } \log[y] \text{ are defined}\\ &\qquad \text{and}\\ &\log[x \cdot y] = \log[x] + \log[y]\end{aligned}$$

The Function Extension Axiom guarantees that the natural extension of $\log[\cdot]$ is defined for all positive hyperreals and its identities hold for hyperreal numbers satisfying $x > 0$ and $y > 0$. We can state the addition formula for sine as the implication

$$\begin{aligned}&(\alpha = \alpha \text{ and } \beta = \beta) \Rightarrow \sin[\alpha], \sin[\beta], \sin[\alpha + \beta], \cos[\alpha], \cos[\beta] \text{ are defined}\\ &\qquad \text{and}\\ &\sin[\alpha + \beta] = \sin[\alpha]\cos[\beta] + \cos[\alpha]\sin[\beta]\end{aligned}$$

The addition formula is always true so we make the logical real statement (see 24.2.7 below) begin with $(\alpha = \alpha \text{ and } \beta = \beta)$.

24.2.5 Logical real expressions

Logical real expressions are built up from numbers and variables using functions.

(a) A real number is a real expression.

(b) A variable standing alone is a real expression.

(c) If $e_1, e_2, \cdots, e_n$ are a real expressions and $f[x_1, x_2, \cdots, x_n]$ is a real function of n variables, then $f[e_1, e_2, \cdots, e_n]$ is a real expression.

24.2.6 Logical real formulas

A logical real formula is one of the following:

(i) An equation between real expressions, $E_1 = E_2$.

(ii) An inequality between real expressions, $E_1 < E_2$, $E_1 \leq E_2$, $E_1 > E_2$, $E_1 \geq E_2$, or $E_1 \neq E_2$.

(iii) A statement of the form "E *is defined*" or of the form "E *is undefined.*"

24.2.7 Logical real statements

Let S and T be finite sets of real formulas. A logical real statement is an implication of the form,

$$S \Rightarrow T.$$

The functional identities for sine and log given above are logical real statements.

Axiom 8 ***Keisler's Function Extension Axiom***
Every real function $f[x_1, x_2, \cdots, x_n]$ has a "natural" extension to the hyperreals such that every logical real statement that holds for all real numbers also holds for all hyperreal numbers when the real functions in the statement are replaced by their natural extensions.

There are two general uses of the Function Extension Axiom that underlie most of the theoretical problems in calculus. These involve extension of the discrete maximum and extension of finite summation. The proof of the Extreme Value Theorem below uses a hyperfinite maximum, while the proof of the Fundamental Theorem of Integral Calculus uses hyperfinite summation and a maximum.

Theorem 9 ***Simple Equivalency of Limits and Infinitesimals***
Let $f[x]$ be a real valued function defined for $0 < |x-a| < \Delta$ with Δ a fixed positive real number. Let b be a real number. Then the following are equivalent:

(a) *Whenever the hyperreal number x satisfies $a \neq x \approx a$, the natural extension function satisfies*

$$f[x] \approx b.$$

(b) *For every real accuracy tolerance θ there is a sufficiently small positive real number γ such that if the real number x satisfies $0 < |x-a| < \gamma$, then*

$$|f[x] - b| < \theta.$$

Condition (b) is the familiar Weierstrass "epsilon-delta" condition (written with θ and γ.) Notice that the condition $f[x] \approx b$ is NOT a logical *real* statement because the infinitesimal relation is NOT included in the formation rules for forming logical *real* statements.

Proof. We show that (a) $\Rightarrow$ (b) by proving that not (b) implies not (a), the contrapositive. Assume (b) fails. Then there is a real $\theta > 0$ such that for every real $\gamma > 0$ there is a real x satisfying $0 < |x - a| < \gamma$ and $|f[x] - b| \geq \theta$. Let $X[\gamma] = x$ be a real function that chooses such an x for a particular γ. Then we have the equivalence

$$\gamma > 0 \Leftrightarrow (X[\gamma] \text{ is defined}, 0 < |X[\gamma] - a| < \gamma, |f[X[\gamma]] - b| \geq \theta)$$

By the Function Extension Axiom this equivalence holds for hyperreal numbers and the natural extensions of the real functions $X[\cdot]$ and $f[\cdot]$. In particular, choose a positive infinitesimal γ and apply the equivalence. We have $0 < |X[\gamma] - a| < \gamma$ and $|f[X[\gamma]] - b| > \theta$ and θ is a positive real number. Hence, $f[X[\gamma]]$ is not infinitely close to b, proving not (a) and completing the proof that (a) implies (b).

Conversely, suppose that (b) holds. Then for every positive real θ, there is a positive real γ such that $0 < |x - a| < \gamma$ implies $|f[x] - b| < \theta$. By the Function Extension Axiom, this implication holds for hyperreal numbers. If $\xi \approx a$, then $0 < |\xi - a| < \gamma$ for every real γ, so $|f[\xi] - b| < \theta$ for every real positive θ. In other words, $f[\xi] \approx b$, showing that (b) implies (a) and completing the proof of the theorem. □

Other examples of uses of the Function Extension Axiom to complete the proofs of the basic results of Section 1 follow.

24.2.8 Continuity and extreme values

We follow the idea of the proof in Section 1 for a real function $f[x]$ on a real interval $[a, b]$. Coding our proof in terms of real functions.

There is a real function $x_M[h]$ so that for each natural number h the maximum of the values $f[x]$ for $x = a + k\Delta x$, $k = 1, 2, \cdots, h$ and $\Delta x = (b - a)/h$ occurs at $x_M[h]$. We can express this in terms of real functions using a real function indicating whether a real number is a natural number,

$$I[x] = \begin{cases} 0, & \text{if } x \neq 1, 2, 3, \cdots \\ 1, & \text{if } x = 1, 2, 3, \cdots \end{cases}$$

When the natural extension of the indicator function satisfies $I[k] = 1$, we say that k is a hyperinteger. (Every limited hyperinteger is an ordinary positive integer. As you can show with these functions.) The maximum of the partition can be described by

$$\left(a \leq x \leq b \;\&\; I\left[h\frac{x-a}{b-a}\right] = 1\right) \Rightarrow f[x] \leq f[x_M[h]]$$

We want to extend this function to unlimited "hypernatural" numbers. The greatest integer function $\text{Floor}[x]$ satisfies, $I[\text{Floor}[x]] = 1$, $0 \leq x - \text{Floor}[x] \leq 1$. The unlimited number $1/\delta$, for $\delta \approx 0$ gives an unlimited $H = \text{Floor}[x]$ with $I[H] = 1$ and

$$\left(a \leq x \leq b \;\&\; I\left[H\frac{x-a}{b-a}\right] = 1\right) \Rightarrow f[x] \leq f[x_M[H]]$$

There is a greatest partition point of any number in $[a,b]$, $P[h,x] = a + \text{Floor}\big[h\frac{x-a}{b-a}\big]\frac{b-a}{h}$ with $a \leq P[h,x] \leq b$ & $I\big[h\frac{P[h,x]-a}{b-a}\big] = 1$ and $0 \leq x - P[h,x] \leq 1/h$. When we take the unlimited hypernatural number H we have $x - P[H,x] \leq 1/H \approx 0$ and $P[H,x]$ a partition point in the sense that $(a \leq x \leq b$ & $I\big[H\frac{P[H,x]-a}{b-a}\big] = 1)$, so we have

$$f[P[H,x]] \leq f[x_M[H]].$$

Let $r_M = \text{st}[x_M[H]]$, the standard part. Since $a \leq x_M[H] \leq b$, $a \leq r_M \leq b$. Continuity of the function in the sense $x_1 \approx x_2 \Rightarrow f[x_1] \approx f[x_2]$ gives

$$f[x] \approx f[P[H,x]] \leq f[x_M[H]] \approx f[r_M], \text{ so } f[x] \leq f[r_M]$$

for any real x in$[a,b]$.

One important comment about the proof of the Extreme Value Theorem is this. The simple fact that the standard part of every hyperreal x satisfying $a \leq x \leq b$ is in the original real interval $[a,b]$ is the form that topological compactness takes in Robinson's theory: A standard topological space is compact if and only if every point in its extension is near a standard point, that is, has a standard part and that standard part is in the original space.

24.2.9 Microscopic tangency in one variable

Suppose $f[x]$ and $f'[x]$ are real functions defined on the interval (a,b), if we know that for all hyperreal numbers x with $a < x < b$ and $a \not\approx x \not\approx b$, $f[x+\delta x] - f[x] = f'[x]\cdot\delta x + \varepsilon\cdot\delta x$ with $\varepsilon \approx 0$ whenever $\delta x \approx 0$, then arguments like the proof of the simple equivalency of limits and infinitesimals above show that $f'[x]$ is a uniform limit of the difference quotient functions on compact subintervals $[\alpha,\beta] \subset (a,b)$. More generally, in [12] we show:

Theorem 10 ***Uniform Differentiability***

Suppose $f[x]$ and $f'[x]$ are real functions defined on the open real interval (a,b). The following are equivalent definitions of, "The function $f[x]$ is smooth with continuous derivative $f'[x]$ on (a,b)."

(*a*) *Whenever a hyperreal x satisfies $a < x < b$ and x is not infinitely near a or b, then an infinitesimal increment of the extended dependent variable is approximately linear on a scale of the change, that is, whenever $\delta x \approx 0$*

$$f[x+\delta x] - f[x] = f'[x]\cdot\delta x + \varepsilon\cdot\delta x \text{ with } \varepsilon \approx 0.$$

(*b*) *For every compact subinterval $[\alpha,\beta] \subset (a,b)$, the real limit*

$$\lim_{\Delta x \longrightarrow 0} \frac{f[x+\Delta x] - f[x]}{\Delta} = f'[x] \text{ uniformly for } \alpha \leq x \leq \beta.$$

(c) *For every pair of hyperreal* $x_1 \approx x_2$ *with* $a < \mathrm{st}[x_i] = c < b$, $\frac{f[x_2]-f[x_1]}{x_2-x_1} \approx f'[c]$.

(d) *For every* c *in* (a, b), *the real double limit,* $\lim_{x_1 \longrightarrow c, x_2 \longrightarrow c} \frac{f[x_2]-f[x_1]}{x_2-x_1} = f'[c]$.

(e) *The traditional pointwise defined derivative* $D_x f = \lim_{\Delta x \longrightarrow 0} \frac{f[x+\Delta x]-f[x]}{\Delta x}$ *is continuous on* (a, b).

Continuity of the derivative follows rigorously from the argument of Section 1, approximating the increment $f[x_1]-f[x_2]$ from both ends of the interval $[x_1, x_2]$. It certainly is geometrically natural to treat both endpoints equally, but this is a "locally uniform" approximation in real-only terms because the x values are hyperreal.

In his *General Investigations of Curved Surfaces* (original in draft of 1825, published in Latin 1827, English translation by Morehead and Hiltebeitel, Princeton, NJ, 1902 and reprinted by the University of Michigan Library), Gauss begins as follows:

A curved surface is said to possess continuous curvature at one of its points A, if the directions of all straight lines drawn from A to points of the surface at an infinitely small distance from A are deflected infinitely little from one and the same plane passing through A. This plane is said to touch the surface at the point A.

In [4], Chapter A6, we show that this can be interpreted as C^1-*embedded* if we apply the condition to all points in the natural extension of the surface.

24.2.10 The Fundamental Theorem of Integral Calculus

The definite integral $\int_a^b f[x]\ dx$ is approximated in real terms by taking sums of slices of the form

$$f[a] \cdot \Delta x + f[a + \Delta x] \cdot \Delta x + f[a + 2\Delta x] \cdot \Delta x + \cdots + f[b'] \cdot \Delta x,$$

where $b' = a + h \cdot \Delta x$ and $a + (h + 1) \cdot \Delta x > b$

Given a real function $f[x]$ defined on $[a, b]$ we can define a new real function $S[a, b, \Delta x]$ by

$$S[a, b, \Delta x] = f[a] \cdot \Delta x + f[a + \Delta x] \cdot \Delta x + f[a + 2\Delta x] \cdot \Delta x + \cdots + f[b'] \cdot \Delta x,$$

where $b' = a + h \cdot \Delta x$ and $a + (h + 1) \cdot \Delta x > b$. This function has the properties of summation such as

$$\big|S[a, b, \Delta x]\big| \leq |f[a]| \cdot \Delta x + |f[a+\Delta x]| \cdot \Delta x + |f[a+2\Delta x]| \cdot \Delta x + \cdots + |f[b']| \cdot \Delta x$$

$$\big|S[a, b, \Delta x]\big| \leq \mathrm{Max}\Big[\big|f[x]\big| : x = a, a + \Delta x, a + 2\Delta x, \cdots, b'\Big] \cdot (b - a)$$

We can say we have a sum of infinitesimal slices when we apply this function to an infinitesimal δx,

$$\int_a^b f[x]\ dx \approx \sum_{\substack{x=a \\ \text{step } \delta x}}^{b-\delta} f[x] \cdot \delta x$$

or

$$\int_a^b f[x]\ dx = \text{st}\left[\sum_{\substack{x=a \\ \text{step } \delta x}}^{b-\delta x} f[x] \cdot \delta x \right], \text{ when } \delta x \approx 0$$

Officially, we code the various summations with the functions like $S[a, b, \delta x]$ (in order to remove the function $f[x]$ as a variable.) We need to show that this is well-defined, that is, gives the same real standard part for every infinitesimal, $S[a, b, \delta x] \approx S[a, b, \iota]$ and both are limited (so they have a common standard part.)

When $f[x]$ is continuous, we can show this "existence," but in the case of the Fundamental Theorem, if we know a real function $F[x]$ with $F'[x] = f[x]$ for all $a \leq x \leq b$, the proof in Section 1 interpreted with the extended summation functions and extended maximum functions proves this "existence" at the same time it shows that the value is $F[b] - F[a]$. The only ingredient needed to make this work is that

$$\max\left[|\varepsilon[x, \delta x]| :\ x = a, a + \delta x, a + 2\delta x, \cdots, b'\right] = \varepsilon[a + k\,\delta x, \delta x] \approx 0$$

This follows from the Uniform Differentiability Theorem above when we take one of the equivalent conditions as the definition of "$F'[x] = f[x]$ for all $a \leq x \leq b$."

Notice that $\varepsilon[x, \Delta x]$ is the real function $\frac{f[x+\Delta x]-f[x]}{\Delta x} - f'[x]$, so we can define an infinite sum by extending the real function

$$S_\varepsilon[a, b, \Delta x] = \sum_{\substack{x=a \\ \text{step } \delta x}}^{b-\delta x} |\varepsilon| \cdot \delta x$$

24.2.11 The Local Inverse Function Theorem

In [1] Michael Behrens noticed that the inverse function theorem is true for a function with a uniform derivative even just at one point. (It is NOT true for a pointwise derivative.) Specifically, condition (d) of the Uniform Differentiability Theorem makes the intuitive proof of Section 1 work.

Theorem 11 ***The Inverse Function Theorem***

If m is a nonzero real number and the real function $f[x]$ is defined for all $x \approx x_0$, a real x_0 with $y_0 = f[x_0]$ and $f[x]$ satisfies

$$\frac{f[x_2] - f[x_1]}{x_2 - x_1} \approx m \text{ whenever } x_1 \approx x_2 \approx x_0$$

then $f[x]$ has an inverse function in a small neighborhood of x_0, that is, there is a real number $\Delta > 0$ and a smooth real function $g[y]$ defined when $|y - y_0| < \Delta$ with $f[g[y]] = y$ and there is a real $\varepsilon > 0$ such that if $|x - x_0| < \varepsilon$, then $|f[x] - y_0| < \Delta$ and $g[f[x]] = x$.

Proof. This proof introduces a "permanence principle." When a logical real formula is true for all infinitesimals, it must remain true out to some positive real number. We know that the statement "$|x - x_0| < \delta \Rightarrow f[x]$ is defined" is true whenever $\delta \approx 0$. Suppose that for every positive real number Δ there was a real point r with $|r - x_0| < \Delta$ where $f[r]$ was not defined. We could define a real function $U[\Delta] = r$. Then the logical real statement

$$\Delta > 0 \Rightarrow \big(r = U[\Delta],\ |r - x_0| < \Delta,\ f[r] \text{ is undefined}\big)$$

is true. The Function Extension Axiom means it must also be true with $\Delta = \delta \approx 0$, a contradiction, hence, there is a positive real Δ so that $f[x]$ is defined whenever $|x - x_0| < \Delta$.

We complete the proof of the Inverse Function Theorem by a permanence principle on the domain of y-values where we can invert $f[x]$. The intuitive proof of Section 1 shows that whenever $|y - y_0| < \delta \approx 0$, we have $|x_1 - x_0| \approx 0$, and for every natural n and k,

$$|x_n - x_0| < 2|x_1 - x_0|,\ |x_{n+k} - x_k| < \tfrac{1}{2^{k-1}}\,|x_1 - x_0|\,,\ f[x_n] \text{ is defined,}$$
$$\big|y - f[x_{n+1}]\big| < \tfrac{1}{2}\big|y - f[x_n]\big|$$

Recall that we re-focus our infinitesimal microscope after each step in the recursion. This is where the uniform condition is used.

Now by the permanence principle, there is a real $\Delta > 0$ so that whenever $|y - y_0| < \Delta$, the properties above hold, making the sequence x_n convergent. Define $g[y] = \operatorname{Lim}_{n \longrightarrow \infty} x_n$. □

24.2.12 Second differences and higher order smoothness

In Section 1 we derived Leibniz' second derivative formula for the radius of curvature of a curve. We actually used infinitesimal second differences, rather than second derivatives and a complete justification requires some more work.

One way to re-state the Uniform First Derivative Theorem above is: The curve $y = f[x]$ is smooth if and only if the line through any two pairs of infinitely close points on the curve is near the same real line,

$$x_1 \approx x_2 \Rightarrow \frac{f[x_1] - f[x_2]}{x_1 - x_2} \approx m$$

A natural way to extend this is to ask: What is the parabola through three infinitely close points? Is the (standard part) of it independent of the choice of the triple? In [8], Vítor Neves and I show:

Theorem 12 ***Theorem on Higher Order Smoothness***

Let $f[x]$ be a real function defined on a real open interval (α, ω). Then $f[x]$ is n-times continuously differentiable on (α, ω) if and only if the n^{th}-order differences $\delta^n f$ are S-continuous on (α, ω). In this case, the coefficients of the interpolating polynomial are near the coefficients of the Taylor polynomial,

$$\delta^n f[x_0, ..., x_n] \approx \frac{1}{n!} f^{(n)}[b]$$

whenever the interpolating points satisfy $x_1 \approx \cdots \approx x_n \approx b$.

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25

Pre-University Analysis

Richard O'Donovan*

Abstract

This paper is a follow-up of K. Hrbacek's article showing how his approach can be pedagogically helpful when introducing analysis at pre-university level.

Conceptual difficulties arise in elementary pedagogical approaches. In most cases it remains difficult to explain at pre-university level how the derivative is calculated at nonstandard values or how an internal function is defined. Hrbacek provides a modified version of IST [8] (rather Péraire's RIST) which seems to reduce all these difficulties. This system is briefly presented here in its pedagogical form with an application to the derivative. It must be understood as a state-of-the-art report[1].

25.1 Introduction

Infinitesimals are interesting when teaching analysis because they give meaning to symbols such as dx, dy or $df(x)$ and all related formulae. Also, symbolic manipulations are usually simpler than with limits.

In secondary school, real numbers are introduced with no formal justification. It is sometimes shown that $\sqrt{2} \notin \mathbb{Q}$. It is traditional to quote that π is not a rational either (with no proof)... and therefore the set of real numbers exists. The algebraic rules of the rationals are applied to the new numbers — students would never imagine that it could be otherwise.

On the one hand, it is possible to use the students' intuition of infinity and infinitesimals to make them find out most of the computational rules of a system containing infinitesimals, as shown in [10]. On the other hand,

*Collège André-Chavanne, Genève, Switzerland.
richard.o-donovan@edu.ge.ch

[1]The author has been teaching analysis with infinitesimals at pre-university level for several years. This paper has been possible thanks to many exchanges with Karel Hrbacek and also many helpful remarks by Keith Stroyan.

Hoskins writes: "the logical basis of NSA needs to be made clear before it can be used safely" [6]. Most published books on the subject seem to confirm this view. Di Nasso, Benci and Forti state: "Roughly, nonstandard analysis consists of two fundamental tools: the star-map and the transfer principle" [2]. Foundational aspects are fundamental but they should not be a prerequisite to study infinitesimals. Frege and Wittgenstein are not studied in kindergarten prior to learning how to add natural numbers.

Our motivations may have a more pedagogical origin than those exposed by Hrbacek in [8], but globally the reasons why we feel unsatisfied with the available textbooks are the same. Because of these difficulties, some colleagues have been tempted to work with infinitesimals by ignoring the question of transfer altogether. But without transfer, analysis with infinitesimals is analysis in a non-archimedean field, and these fields are known to have very special and unwanted properties as shown in [4] and [13]. Nonstandard analysis ensures that the properties of completeness are transferred to a certain class of objects.

25.2 Standard part

A major difficulty in nonstandard analysis is the existence of external functions. Consider the "updown" function:

$$f : x \mapsto 2 \cdot \mathbf{st}(x) - x$$

At standard scale it is indistinguishable from the identity function but at the infinitesimal scale the function is everywhere decreasing with a rate of change (slope?) equal to -1 *and* it is S-continuous and satisfies the intermediate value property! Why does it not satisfy that if a continuous function has a negative slope everywhere on an interval, then the function is decreasing on that interval?

It appears difficult to explain to students that statements which use the standardness predicate are "external" but that we can nonetheless use it safely for the definition of the derivative. How can a student understand when it is acceptable to use $\mathbf{st}[\ ]$ and when is it not? Some students have pointed out that if the derivative is given by $\mathbf{st}\left(\frac{\Delta f(x)}{h}\right)$ for all x, then $f'(x+\delta) = f'(x)$. The classical textbooks such as [9], [14] or [5] use an indirect definition for the derivative at nonstandard points which makes a drastic distinction in nature between standard and nonstandard numbers: at standard points, the derivative is calculated using that point and other nonstandard neighbours but at nonstandard points some form of transfer is used. How can we be convincing when explaining why there are two different definitions?

Even Stroyan's uniform derivative [14] is not totally satisfying here because it requires that we recognise a "real function", which in turn means that we need a definition of what such a function is, and what an extension is.

25.3 Stratified analysis

Stratified analysis may be an answer. An interesting aspect is that the theory is adapted for elementary teaching at the same time as it is being developed. The interactions are mutual.

In the adaptation for high-school, we have deliberately avoided references to the words "standard" and "nonstandard" for reasons already discussed in [10].

Instead of the binary relation $\sqsubseteq$ introduced by Hrbacek, we have suggested to use a concept of level, symbolised by $\mathbf{v}(x)$. If $a \sqsubseteq b$ then $a \in \mathbf{v}(b)$: a is relatively standard to b means that a is at the level of b. For high-school, we feel that this is a simpler concept. All of the adaptations have been discussed with Hrbacek so that they satisfy both the educational requirements and the theoretical consistency.

The following is a possible description given to the students:

Levels

We get to know the familiar natural numbers $1, 2, 3, \ldots$ when we learn to count. But it would be a rare child that actually counts (in steps of 1) to more than a thousand or so. It is not necessary to keep going further, because somewhere at this point one gets the idea that the counting process never ends (this is called: potential infinity), and one is capable of forming the set of natural numbers $\mathbb{N}$, i.e. it is possible to consider the collection of all natural numbers as one "thing" (this is called: actual infinity). It is neither necessary nor possible to count all natural numbers to do it!

Also, sums, products, differences, quotients, powers, roots, etc. of these familiar numbers are or can be known at this stage.

All these familiar numbers are at the same **level**.

If x is at the **level** of y, it means that x can be known when y can be known; written

$$x \in \mathbf{v}(y)$$

Familiar numbers are all at the **coarsest** level.

$$\mathbf{v}(0) = \mathbf{v}(1) = \mathbf{v}\left(\frac{3}{2}\right) = \mathbf{v}(\sqrt{3})$$

But then there are natural numbers unknown at this level, numbers that are so large (we will say "unlimited") that they do not belong to $\mathbf{v}(0)$.

Let K be such an unlimited natural number, i.e. $K \notin \mathbf{v}(0)$. It takes a higher level of knowledge to know K, but once we do, we also know $2K, K+1\ldots$ and many other objects which all belong to $\mathbf{v}(K)$. In increasing our knowledge about numbers, we do not lose former knowledge, so $1, 2, 3, \ldots$ are also at $\mathbf{v}(K)$: they remain known. Also, the reciprocal: $\frac{1}{K}$ is an "infinitesimal" that becomes known at the level of K. Hence a number such as, say, $3+\frac{5}{31\cdot K}$ is at $\mathbf{v}(K)$ (a **finer** level) and is infinitely close to 3.

But again, this level of knowledge does not exhaust the set $\mathbb{N}$, so there are numbers not at $\mathbf{v}(K)$ — hence not at $\mathbf{v}(0)$, etc. The levels potentially expand forever and never fully exhaust $\mathbb{N}$.

- There are many levels, each making new numbers known.
- At $\mathbf{v}(0)$ there are no infinitesimals.

Infinitesimals are defined relatively to a level. α-infinitesimals and α-unlimited are numbers which are not at $\mathbf{v}(\alpha)$ and which are, respectively, less in modulus, greater in modulus, than any nonzero number at $\mathbf{v}(\alpha)$. For an α-limited number ξ, the α-shadow (noted $\mathbf{sh}_\alpha(\xi)$) is the unique number at $\mathbf{v}(\alpha)$ which is α-infinitesimally close to ξ. The x-shadow replaces the concept of standard part in this relativistic framework[2].

We give a more restricted definition of levels than Hrbacek's. We only include numbers in the levels, not sets. This slight difference can be considered as a restriction of the definition given by Hrbacek not a contradiction: the only sets we use explicitly in our course are intervals, $\mathbb{N}$ or $\mathbb{R}$.

One of the immediate advantages of stratified analysis is the possibility to easily identify "acceptable" statements — those that will eventually transfer. A statement about x is acceptable if it makes either no reference to levels or only to the level of x. The same holds if instead of x there is a list $\overline{x} = x_1, x_2, \ldots$

A function f is defined by a rule that tells us what $f(x)$ is when x is in the domain of f. The rule may depend on some parameters, $p_1, p_2, \ldots$ A function is acceptable if the rule either does not refer to levels or refers only to $\mathbf{v}(x, p_1, p_2, \ldots)$. (Although probably not necessary at this school level, transfer ensures that if there are two different rules using parameters from different levels which give the same value for all values of the variable, then they define

[2]Again, for reasons related to politics in the educational realm, shadow has been preferred to "standard part". It also yields a fairly intuitive image that numbers cast a shadow on coarser levels (provided they are finite with respect to that level); shadows contain less information.

the same function and its level is the coarsest one where the function can be defined).

$f : x \mapsto 2 \cdot \mathbf{sh}_0(x) - x$ is not an acceptable function because there is an absolute reference to $\mathbf{v}(0)$ in the shadow predicate. If adapted as $2 \cdot \mathbf{sh}_x(x) - x$ then, as $\mathbf{sh}_x(x) = x$, it simplifies to the identity function. Theorems about continuity are for (acceptable) continuous functions and this function does not satisfy the conditions. The student can see this with no "external" knowledge.

It seems reasonable to state that from then on, acceptable functions will be simply called functions (so the statements of theorems will look very much like the classical ones).

25.4 Derivative

The derivative of a function is defined by:

Let $f :]a; b[\to \mathbb{R}$ be a function and $x \in]a; b[$

f is differentiable at x iff there is an $L \in \mathbf{v}(x, f)$ such that, for any $< x, f >$-infinitesimal h, with $x + h \in]a; b[$,

$$\mathbf{sh}_{<x,f>}\left(\frac{f(x+h) - f(x)}{h}\right) = L$$

then the derivative is

$$f'(x) = L$$

It is a direct observation that the derivative is "acceptable".

For $f : x \mapsto x^2$ at $x = 2$ the derivative is simple and works exactly as in any other nonstandard method. For x in general, the quotient simplifies to $2x + h$. The only parameter of f is $2 \in \mathbf{v}(0)$ hence $\mathbf{v}(x, f) = \mathbf{v}(x)$. As h is x-infinitesimal, $\mathbf{sh}_x(2x + h) = 2x$. For a direct calculation of $f'(2 + \delta)$, we have $2 + \delta \in \mathbf{v}(\delta)$ hence a $[2 + \delta]$-infinitesimal is also a δ-infinitesimal. Let h be a δ-infinitesimal.

$$\mathbf{sh}_\delta\left(\tfrac{(2+\delta+h)^2-(2+\delta)^2}{h}\right) = \mathbf{sh}_\delta\left(\tfrac{4h+2\delta h+h^2}{h}\right) = \mathbf{sh}_\delta(4 + 2\delta + h) = 4 + 2\delta.$$

Thus stratified analysis satisfies one of our major requirements: a single definition of the derivative which applies to all numbers.

In [10] it is shown that using the definition of infinitesimals, the students could work out the rules of computation as exercises and find that for standard a and infinitesimal δ, $a \cdot \delta$ is infinitesimal. The same exercises adapted to acceptable statements yield that if δ is a-infinitesimal, then $a \cdot \delta$ is a-infinitesimal.

25.5 Transfer and closure

Some definitions have been re-written several times and the definitions we use today may not be final. All the proofs of theorems in the syllabus must be checked and cross-checked.

The only principles that are needed are Hrbacek's closure principle: If f is an acceptable function, then $f(x) \in \mathbf{v}(x, f)$ for all x in the domain of f. This extends the observation that operations with "familiar" numbers do not yield infinitesimals. The other is a simple form of transfer which states that an acceptable statement is true for $\mathbf{v}(\alpha)$ iff it is true for all $\mathbf{v}(\beta)$ with $\alpha \in \mathbf{v}(\beta)$.

This adaptation to pre-university level is not completed yet. We hope to finish a first version of a handout, with proofs, within a year or so. Our goal is still the same: for most people, infinitesimals "are there", but can they be used to make maths easier to teach and learn and still remain rigorous? We hope to be able to contribute an answer to this question in a not too distant future.

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