Chapter 2

Central Configurations

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The topic of the present chapter is one of my favorites: central configurations of the n-body problem. I gave a course on the same subject in Trieste in 1994 and wrote up some notes (by hand) which can be found on my website [23]. For the new course, I tried to focus on some new ideas and techniques which have been developed in the intervening twenty years. In particular, I consider space dimensions bigger than three. There are still a lot of open problems and it remains an attractive area for mathematical research.

2.1 The n-body problem

The Newtonian n-body problem is the study of the dynamics of n point particles with masses $m_i > 0$ and positions $x_i \in \mathbb{R}^d$, moving according to Newton's laws of motion:

$$m_j \ddot{x}_j = \sum_{i \neq j} \frac{m_i m_j (x_i - x_j)}{r_{ij}^3}, \quad 1 \leq j \leq n, \tag{2.1}$$

where $r_{ij} = |x_i - x_j|$ is the Euclidean distance between x_i and x_j. Although we are mainly interested in dimensions $d \leq 3$, it is illuminating and entertaining to consider higher dimensions as well.

Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^{dn}$ be the *configuration vector* and let

$$U(x) = \sum_{i<j} \frac{m_i m_j}{r_{ij}} \tag{2.2}$$

be the *Newtonian potential*. Then we have

$$m_j \ddot{x}_j = \nabla_j U(x), \quad 1 \leq j \leq n, \tag{2.3}$$

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where ∇_j denotes the d-dimensional partial gradient with respect to x_j or

$$M\ddot{x} = \nabla U(x), \tag{2.4}$$

where ∇ is the dn-dimensional gradient and $M = \mathrm{diag}(m_1, \ldots, m_n)$ is the matrix with d copies of each mass along the diagonal. (Later there will be an $n \times n$ mass matrix, also called M.)

Let $v_j = \dot{x}_j \in \mathbb{R}^d$ be the velocity vectors and $v = (v_1, \ldots, v_n) \in \mathbb{R}^{dn}$. Then there is an equivalent first-order system

$$\begin{aligned}\dot{x} &= v,\\ \dot{v} &= M^{-1}\nabla U(x).\end{aligned}$$

Since Newtonian potential is singular at collisions, we have to restrict x to the *configuration space* $\mathbb{R}^{nd} \setminus \Delta$, where

$$\Delta = \{x : x_i = x_j \text{ for some } i \neq j\} \tag{2.5}$$

is the *singular set.*

The phase space for the first-order system is $(\mathbb{R}^{nd} \setminus \Delta) \times \mathbb{R}^{nd}$. Newton's equations are conservative. The *total energy*

$$H = K(v) - U(x), \quad K = \sum_{j=1}^{n} m_j |v_j|^2$$

is constant along solutions in phase space.

Even though we are considering the n-body problem in $\mathbb{R}^d$, it may happen that the motion that takes place in a subspace $\mathcal{W}$. In fact, let $\mathcal{W} \subset \mathbb{R}^d$ be any subspace. If all of the positions and velocities satisfy $x_j, v_j \in \mathcal{W}$, equation (2.1) shows that the acceleration vectors are also in $\mathcal{W}$. It follows that $\mathcal{W}^n \setminus \Delta \times \mathcal{W}^n$ is an invariant set for the flow in phase space. In particular we can consider the smallest subspace containing all of the positions and velocities,

$$\mathcal{S}(x, v) = \mathrm{span}\{x_j, v_j : j = 1, \ldots, n\} \subset \mathbb{R}^d.$$

If $(x(t), v(t))$ is any solution, then $\mathcal{S}(x(t), v(t))$ is independent of t. It will be called the *motion space* of the solution.

2.2 Symmetries and integrals

Newton's equations are invariant under simultaneous translations and rotations of all of the positions and velocities $x_j, v_j \in \mathbb{R}^d$. Symmetry under translations gives rise, via Nöther's Theorem [5], to conservation of the *total momentum* vector

$$p = m_1 v_1 + \cdots + m_n v_n.$$

Let

$$c = \frac{1}{m_0}\left(m_1 x_1 + \cdots + m_n x_n\right), \quad m_0 = m_1 + \cdots + m_n \tag{2.6}$$

be the *center of mass*, where m_0 is the *total mass*. Then

$$\dot{c} = p/m_0,$$
$$\dot{p} = 0$$

so $c(t)$ moves in a straight line with constant velocity. It follows that the positions relative to the center of mass, $y_j(t) = x_j(t) - c(t)$, are also solutions of Newton's equations. These have center of mass at the origin and total momentum zero. A solution with this property will be called *centered*. We will use the notation

$$x - c = (x_1 - c, \ldots, x_n - c) \in \mathbb{R}^{dn}$$

for the configuration relative to the center of mass.

For any configuration x, the vectors $x_j - c$, $j = 1, \ldots, n$, span a subspace of $\mathbb{R}^d$ which we will call the *centered position space* and denote by $\mathcal{C}(x)$. It is natural to define the *dimension of a configuration* to be $\dim(x) = \dim \mathcal{C}(x)$. The maximum possible dimension of a configuration of the n-body problem is $n-1$. For example, every configuration of the three-body problem has dimension 1 (collinear) or 2 (planar).

The rotation group $\mathrm{SO}(d)$ in $\mathbb{R}^d$ has dimension $\binom{d}{2} = \frac{d(d-1)}{2}$. The Lie algebra $\mathrm{so}(d)$ consists of all anti-symmetric $d \times d$ matrices. If $Q(t)$ is a one parameter subgroup, it can be written as a matrix exponential

$$Q(t) = e^{t\alpha}, \quad \alpha \in \mathrm{so}(d).$$

From linear algebra we know that there is a rotation $S \in \mathrm{SO}(d)$ putting α into the normal form:

$$S^{-1}\alpha S = \mathrm{diag}(a_1 j, \ldots, a_k j, 0 \ldots, 0), \quad j = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

where $a_i \in \mathbb{R}$. Then α has even rank, say $2k$. The one-parameter group satisfies

$$S^{-1}Q(t)S = \mathrm{diag}(\rho(a_1 t), \ldots, \rho(a_k t), 1, \ldots, 1), \quad \rho(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

Thus $Q(t)$ acts by rotation at different rates in k orthogonal planes while fixing the part of $\mathbb{R}^d$ orthogonal to these planes.

For example, in $\mathbb{R}^3$, an angular velocity matrix can be written

$$\alpha = \begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix}$$

and the block-diagonal normal form is

$$S^{-1}\alpha S = \begin{bmatrix} 0 & -a_1 & 0 \\ a_1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad a_1 = \pm\sqrt{a^2+b^2+c^2}.$$

The corresponding one-parameter group is a rotation around the angular velocity vector (a, b, c) with constant angular speed $|a_1|$.

Symmetry under rotations implies that the *angular momentum* is preserved. The angular momentum (with respect to the origin) can be represented by an anti-symmetric $d \times d$ matrix $\omega(x, v)$ with entries

$$\omega_{kl} = \sum_{j=1}^{n} m_j(x_{jk}v_{jl} - x_{jl}v_{jk}), \tag{2.7}$$

where x_{jk}, v_{jk} denote the k-th components of the vectors $x_j, y_j \in \mathbb{R}^d$. In case $d = 2$, the angular momentum reduces to a scalar ω_{12}, while if $d = 3$ it can be viewed as a vector

$$\omega = (\omega_{23}, \omega_{31}, \omega_{12}) = \sum_{j=1}^{n} m_j x_j \times v_j,$$

where $\times$ denotes the cross product in $\mathbb{R}^3$.

The Newtonian potential is homogeneous of degree -1 and its gradient is homogeneous of degree -2. It follows that if $x(t)$ is any solutions of (2.1) and if $\lambda > 0$ is constant, then $\tilde{x}(t) = \lambda^2 x(\lambda^{-3}t)$ is also a solution. This will be called the *scaling symmetry* of the n-body problem.

For any configuration x, the *moment of inertia with respect to the center of mass* is

$$I(x) = (x-c)^T M(x-c) = \sum_j m_j |x_j - c|^2, \tag{2.8}$$

where y is the corresponding centered configuration. $I(x)$ is homogeneous of degree 2 with respect to the scaling symmetry. The following alternative formula in terms of mutual distances is also useful:

$$I(x) = \frac{1}{m_0} \sum_{i<j} m_i m_j r_{ij}^2. \tag{2.9}$$

2.3 Central configurations and self-similar solutions

At this point we can define the concept which will be the main focus of the present notes.

Definition 2.3.1. *A* central configuration (CC) *for masses $m_1, \ldots, m_n$ is an arrangement of the n point masses whose configuration vector satisfies*

$$\nabla U(x) + \lambda M(x - c) = 0 \tag{2.10}$$

for some real constant λ.

Multiplying (2.10) on the left by $(x-c)^T$ and using the translation invariance and homogeneity of $U(x)$ shows that

$$\lambda = \frac{U(x)}{I(x)} > 0,$$

where $I(x)$ is the moment of inertia with respect to c from (2.8). If x is a central configuration then the gravitational acceleration on the j-th body due to the other bodies is

$$\ddot{x}_j = \tfrac{1}{m_j}\nabla_j U(x) = -\lambda(x_j - c).$$

In other words, all of the accelerations are pointing towards the center of mass, c, and are proportional to the distance from c. We will see that this delicate balancing of the gravitational forces gives rise to some remarkably simple solutions of the n-body problem. Before describing some of these, we will briefly consider the question of existence of central configurations.

For given masses $m_1, \ldots, m_n$, it is far from clear that (2.10) has any solutions at all. We will consider this question in due course. For now we just note the existence of symmetrical examples for equal masses. If all n masses are equal we can arrange the bodies at the vertices of a regular polygon, polyhedron or polytope. Then it follows from symmetry that the acceleration vectors of each mass must point toward the barycenter of the configuration. This is the condition for a central configuration, i.e., there will be some λ for which the CC equations hold.

In $\mathbb{R}^2$ we can put three equal masses at the vertices of an equilateral triangle or n equal masses at the vertices of a regular n-gon to get simple examples. One can also put an arbitrary mass at the center of a regular n-gon of equal masses as in Figure 2.1 (left). In $\mathbb{R}^3$ we have the five regular Platonic solids, the tetrahedron, cube, octahedron, dodecahedron and icosahedron. It is not clear what to do if $n \neq 4, 6, 8, 12, 20$, however. It turns out that there are six kinds of regular, convex four-dimensional polytopes but in higher dimensions there are only three, namely the obvious generalization of the tetrahedron, cube and octahedron [9, 16].

The regular d-simplex provides an example of a central configuration of $d+1$ equal masses in $\mathbb{R}^d$ generalizing the equilateral triangle and tetrahedron. Remarkably, these turn out to be central configurations even when the masses are not equal (see Proposition 2.8.6) so we do indeed have at least one CC for any choice of masses, provided we are willing to work in high-dimensional spaces. As a special case, note that for the two-body problem, every configuration is a regular simplex, i.e., a line segment. So every configuration of $n = 2$ bodies is a central configuration.

Less obvious examples can be found by numerically solving (2.10), for example the asymmetrical CC of 8 equal masses shown in Figure 2.1 (right).

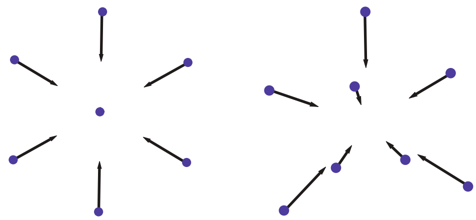

Figure 2.1: Central configurations.

Central configurations can be used to construct simple, special solutions of the n-body problem where the shape of the figure formed by the bodies remains constant. The configuration changes only by simultaneous translation, rotation and scaling. In other words, the configurations $x(t)$ at different times are all *similar*. In this case the configuration relative to the center of mass will change only by scaling and rotation.

Definition 2.3.2. *A solution of the n-body problem is* self-similar *or* homographic *if it satisfies*

$$x(t) - c(t) = r(t)Q(t)(x_0 - c_0), \tag{2.11}$$

where x_0 is a constant configuration, $r(t) > 0$ is a real scaling factor, and $Q(t) \in \mathrm{SO}(d)$ is a rotation. Here $c(t), c_0$ are the centers of mass of $x(t), x_0$.

Two special cases are the homothetic *solutions, where*

$$x(t) - c(t) = r(t)(x_0 - c_0), \tag{2.12}$$

and the rigid motions *or* relative equilibrium *solutions, where*

$$x(t) - c(t) = Q(t)(x_0 - c_0). \tag{2.13}$$

The simplest of these are the *homothetic* solutions. For example, if put three equal masses at the vertices of an equilateral triangle and release them with initial velocities all zero, it seems clear that the triangle will just collapse to the center of mass with each particle just moving on a line towards the center. It turns out that such a solution is possible only when x_0 is a central configuration.

Proposition 2.3.3. *If x_0 is a central configuration with constant λ and if $r(t)$ is any solution of the one-dimensional Kepler problem*

$$\ddot{r}(t) = -\frac{\lambda}{r(t)^2}, \tag{2.14}$$

then $x(t)$ as in (2.12) is a homothetic solution of the n-body problem, and every homothetic solution is of this form.

Proof. Substituting $x(t)$ from (2.12) into Newton's equation (2.4) gives

$$\ddot{r}(t)M(x_0 - c_0) = \nabla U(x(t)) = r(t)^{-2}\nabla U(x_0).$$

Now $\nabla U(x_0) \neq 0$ for all x_0, so this equation is satisfied if and only if there is some constant, call it $-\lambda$, such that

$$\ddot{r}(t)r(t)^2 = -\lambda, \quad -\lambda M(x_0 - c_0) = \nabla U(x_0). \qquad \square$$

The one-dimensional Kepler problem (2.14) describes the motion of a point on a line gravitationally attracted to a mass λ at the origin. It is easy to see qualitatively what will happen even without solving it. For example, the solution $r(t)$ with initial velocity $\dot{r}(0) = 0$ collapses to the origin in both forward and backward time. The corresponding homothetic solutions maintain the shape of the underlying central configuration x_0 while collapsing to a total collision at the center of mass in both forward and backward time (see Figure 2.2 for the forward-time half). Each body moves along a straight line towards the collision. From the examples of central configurations mentioned above we see that we can have homothetically collapsing solutions in the shape of an equilateral triangle, regular n-gon or regular polytope.

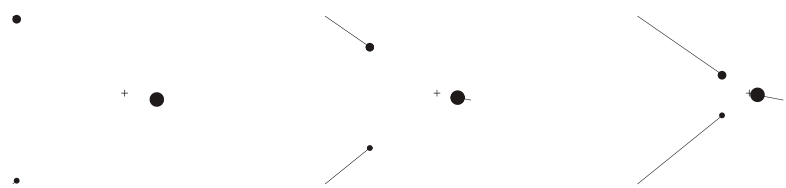

Figure 2.2: The forward-time half of a homothetic solution based on Lagrange's equilateral triangle with masses 10, 2, and 1. Released with zero velocity, the masses collapse to the center of mass (indicated by the + symbol) along straight lines, maintaining the equilateral shape.

It turns out that central configurations also lead to rigid motions and more general homographic solutions. We will postpone a general discussion of homographic solutions in $\mathbb{R}^d$ to later sections. For now we will consider the case of planar motions. Let $d = 2$ and suppose $x_0 \in \mathbb{R}^{2n}$ is a central configuration. Let

$$Q(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \in \mathrm{SO}(2).$$

The most general planar homographic motion would be of the form

$$x(t) - c(t) = r(t)Q(\theta(t))(x_0 - c_0) \qquad (2.15)$$

for some functions $r(t) > 0, \theta(t)$. Substituting this into Newton's equation leads, after some simplifications, to

$$(\ddot{r} - r\,\dot{\theta}^2)M(x_0 - c_0) + (r\,\ddot{\theta} + 2\,\dot{r}\,\dot{\theta})JM(x_0 - c_0) = r^{-2}\nabla U(x_0),$$

where J is the $2n \times 2n$ matrix

$$J = \operatorname{diag}(j, \ldots, j), \quad j = Q(\theta)^{-1}Q'(\theta) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Now $(x_0 - c_0)$ and $J(x_0 - c_0)$ are nonzero, orthogonal vectors in $\mathbb{R}^{2n}$ and the latter is also orthogonal to $\nabla U(x_0)$. Therefore, there must be some constant $-\lambda$ such that

$$\begin{aligned} \ddot{r}(t) - r(t)\dot{\theta}(t)^2 &= -\frac{\lambda}{r(t)^2}, \\ r(t)\ddot{\theta}(t) + 2\dot{r}(t)\dot{\theta}(t) &= 0, \end{aligned} \tag{2.16}$$

and

$$-\lambda M(x_0 - c_0) = \nabla U(x_0).$$

The differential equation is just the two-dimensional Kepler problem in polar coordinates whose solutions are of the familiar elliptical, parabolic or hyperbolic types and the last equation is the CC equation.

Proposition 2.3.4. *If x_0 is a planar central configuration with constant λ and if $r(t), \theta(t)$ is any solution of the two-dimensional Kepler problem* (2.16)*, then* (2.15) *is a planar homographic solution and every such solution is of this form.*

As a special case, we could take a circular solution of the Kepler problem with $r(t) = 1$. Then we get a rigid motion or relative equilibrium solution where the planar central configuration just rotates at constant angular speed around the center of mass. This is the most general relative equilibrium solution in the plane. In particular, nonuniform rotations are not possible.

In higher dimensions, the situation regarding rigid solutions and nonhomothetic homographic solutions is more complicated, mainly due to the increased complexity of the rotation group $\mathrm{SO}(d)$. The next few sections describe an approach to the general case developed by Albouy and Chenciner.

2.4 Matrix equations of motion

We will now describe an interesting reformulation of the n-body problem due to Albouy and Chenciner [2, 3, 7] which is very convenient for studying symmetric solutions. Let

$$X = \left[x_1|\cdots|x_n\right], \quad V = \left[v_1|\cdots|v_n\right],$$

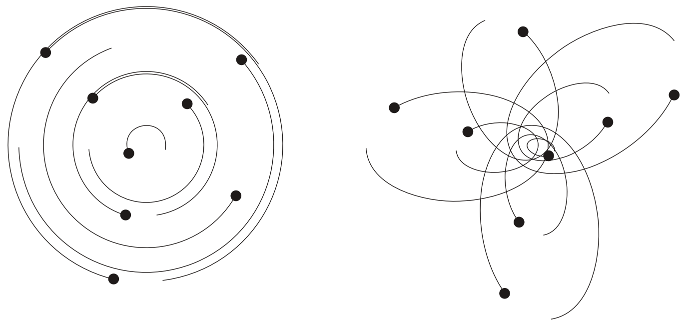

Figure 2.3: Planar homographic motions based on a central configuration of eight equal masses from Figure 2.1. On the left is a relative equilibrium solution while the solution on the right features elliptical orbits of eccentricity 0.8.

be the $d \times n$ matrix whose columns are the positions and velocities of the n bodies. For example, the matrix

$$X = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.17}$$

represents a configuration of $n = 3$ bodies in $d = 4$ dimensions arranged at the vertices of an equilateral triangle.

We will view X, V as linear maps $X, V : \mathbb{R}^n \to \mathbb{R}^d$. The domain of these maps has no particular physical meaning; it is just a space of $n \times 1$ column vectors ξ with one coordinate for each of the n-bodies. We can think of the standard basis vectors $e_1, \ldots, e_n$ as representing the different bodies.

While the columns of X, V have an immediate dynamical meaning, it is not clear what to think about the rows. These are $1 \times n$ vectors which we will view as elements of the dual space $\mathbb{R}^{n*}$, another nonphysical space. For example, the first row $\begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$ of the matrix above gives the coefficients of a linear function whose values on the basis vectors e_1, e_2, e_3 of $\mathbb{R}^3$ are the first coordinates of the three bodies in $\mathbb{R}^4$.

To get the matrix version of the laws of motion, write the j-th acceleration vector from Newton's equations (2.1) as a linear combination of the position vectors:

$$\ddot{x}_j = \frac{1}{m_j} \nabla_j U(x) = \sum_{i \neq j} \frac{m_i (x_i - x_j)}{r_{ij}^3} = \sum_{i \neq j} x_i \frac{m_i}{r_{ij}^3} - x_j \left(\sum_{i \neq j} \frac{m_i}{r_{ij}^3} \right).$$

So we get the matrix equation:

$$\ddot{X} = XA(X), \tag{2.18}$$

where $A(X)$ is the $n \times n$ matrix

$$A(X) = \begin{bmatrix} A_{11} & \frac{m_1}{r_{12}^3} & \cdots & \frac{m_1}{r_{1n}^3} \\ \frac{m_2}{r_{12}^3} & A_{22} & \cdots & \frac{m_2}{r_{2n}^3} \\ \vdots & & & \vdots \\ \frac{m_n}{r_{1n}^3} & \frac{m_n}{r_{2n}^3} & \cdots & A_{nn} \end{bmatrix}, \quad A_{jj} = -\sum_{i \neq j} A_{ij} = -\sum_{i \neq j} \frac{m_i}{r_{ij}^3}. \tag{2.19}$$

Note that $A(X)$ is invariant under translations and rotations, since it involves only the mutual distances. It is independent of the space dimension d. For example, consider the three-body problem in $\mathbb{R}^d$ where we have the 3×3 matrix

$$A = \begin{bmatrix} -\frac{m_2}{r_{12}^3} - \frac{m_3}{r_{13}^3} & \frac{m_1}{r_{12}^3} & \frac{m_1}{r_{13}^3} \\ \frac{m_2}{r_{12}^3} & -\frac{m_1}{r_{12}^3} - \frac{m_3}{r_{23}^3} & \frac{m_2}{r_{23}^3} \\ \frac{m_3}{r_{13}^3} & \frac{m_3}{r_{23}^3} & -\frac{m_1}{r_{13}^3} - \frac{m_2}{r_{23}^3} \end{bmatrix}.$$

$A(X)$ has some other useful properties. Let $M = \operatorname{diag}(m_1, \ldots, m_n)$ be an $n \times n$ version of the mass matrix. Then we have

$$XA(X)M = \begin{bmatrix} \nabla_1 U(X) & \cdots & \nabla_n U(X) \end{bmatrix}.$$

In addition, $A(X)M$ is symmetric:

$$AM = (AM)^T = MA^T.$$

Finally, $A(X)M$ is negative semi-definite. Indeed, for any $\xi \in \mathbb{R}^n$ one can check that

$$\xi^T AM\xi = -\sum_{i<j} \frac{m_i m_j}{r_{ij}^3} (\xi_i - \xi_j)^2.$$

We will also need a matrix version of the first-order differential equations of the n-body problem:

$$\begin{aligned} \dot{X} &= V, \\ \dot{V} &= XA(X). \end{aligned} \tag{2.20}$$

The $d \times 2n$ matrix $Z = \begin{bmatrix} X & V \end{bmatrix}$ will be called the *state matrix*.

It is interesting to look at the symmetries and integrals of the n-body problem from the matrix point of view. Let $k \in \mathbb{R}^d$ be a $d \times 1$ column vector. The translation $x_j \mapsto x_j + k$ has the effect of adding $k_i L$ to the i-th row of X, where

$$L = \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix} \in \mathbb{R}^{n*}$$

is the $1 \times n$ row vector of 1's. In other words the configuration matrix transforms by addition of the $d \times n$ matrix kL,

$$X \mapsto X + kL. \tag{2.21}$$

We call two $d \times n$ matrices X, Y *translation equivalent* if $Y = X + kL$ for some $k \in \mathbb{R}^d$. If X, Y are translation equivalent then the corresponding linear maps $X, Y\colon \mathbb{R}^n \to \mathbb{R}^d$ take the same values when restricted to the hyperplane

$$\mathcal{D}^* = L^\perp = \{\xi \in \mathbb{R}^n : L\xi = \xi_1 + \cdots + \xi_n = 0\}.$$

The converse also holds so translation equivalence amounts to saying that

$$X|_{\mathcal{D}^*} = Y|_{\mathcal{D}^*}.$$

The notation $\mathcal{D}^*$, due to Albouy–Chenciner [3], is explained as follows. The quotient vector space $\mathbb{R}^{n^*}/L$ is called the *disposition space* and denoted by $\mathcal{D}$. Then $L^\perp$ can be identified with its dual vector space.

With this notation, the total mass and center of mass can be written

$$m_0 = Lm, \quad c = \frac{1}{m_0} Xm, \tag{2.22}$$

where m is the $n \times 1$ column vector

$$m = \begin{bmatrix} m_1 & \cdots & m_n \end{bmatrix}^T.$$

A state will have center of mass at the origin and total momentum zero if

$$Xm = Vm = 0.$$

We will call a $d \times n$ matrix X *centered* if $Xm = 0$.

Proposition 2.4.1. *Given a $d \times n$ matrix X, there is a unique centered matrix Y translation equivalent to X, namely*

$$Y = X - C, \quad C = cL,$$

where c is the center of mass (2.22). *Moreover,*

$$Y = XP, \quad P = I - \frac{1}{m_0} mL.$$

The $n \times n$ matrix P represents the orthogonal projection of $\mathbb{R}^n$ onto the hyperplane $\mathcal{D}^*$ with respect to the inverse mass inner product *on $\mathbb{R}^n$.*

Proof. Let $Y = X - cL$. Then Y is translation equivalent to X and is centered if and only if c is given by (2.22). In this case it is easy to check that $Y = XP$ where P is as claimed. We have

$$P^2 = P, \quad LP = 0.$$

Hence, the linear map $P\colon \mathbb{R}^n \to \mathbb{R}^n$ is a projection map of $\mathbb{R}^n$ onto $\mathcal{D}^*$. One can also check that P is an M^{-1}-symmetric matrix:

$$P^T M^{-1} = M^{-1} P,$$

where M is the mass matrix. It follows that P represents the orthogonal projection onto $\mathcal{D}^*$ with respect to the inner product $\langle \xi, \eta \rangle = \xi^T M^{-1} \eta$. □

If the matrices $X(t), V(t)$ solve Newton's equations (2.20) so do the centered matrices

$$Y(t) = X(t) - C(t) = X(t)P, \quad W(t) = V(t)P$$

which describe the dynamics relative to the center of mass. This was shown already in Section 2.2 but it can also be verified directly from (2.20) with the help of the following easily verified formula:

$$A(X) = A(XP) = A(X - C) = A(X)P = PA(X). \tag{2.23}$$

The following facts about the right-hand side of Newton's equation are also useful

$$CA(X) = 0, \quad XA(X) = (X - C)A(X - C). \tag{2.24}$$

We will use the matrix formulation to study central configurations and homographic solutions in $\mathbb{R}^d$. The factorization (2.18) of the equations of motion is very useful for understanding symmetrical solutions. The CC equation (2.10) for configuration vectors gives the following equation for configuration matrices:

$$XA(X) + \lambda(X - C) = 0. \tag{2.25}$$

2.5 Homographic motions of central configurations in $\mathbb{R}^d$

We have already defined homographic, homothetic and rigid solutions. The configuration matrix of a homographic solution will satisfy

$$X(t) - C(t) = r(t)Q(t)(X_0 - C_0). \tag{2.26}$$

Homothetic and rigid solutions are of the same form but with $Q(t) = I$ and $r(t) = 1$, respectively.

We have seen in Proposition 2.3.3 that every homothetic motion comes from a CC, x_0, with $r(t)$ a solution of the one-dimensional Kepler problem. Also, Proposition 2.3.4 shows that planar CC's can execute Keplerian homographic motions. The next result treats *Keplerian homographic motions* of central configurations in $\mathbb{R}^d$.

Proposition 2.5.1. *Let X_0 be the configuration matrix of a central configuration with constant λ and let $\mathcal{C}(x_0) = \mathrm{im}(X_0 - C_0)$ be its centered position subspace. Suppose there is an antisymmetric $d \times d$ matrix J such that $J^2|_{\mathcal{C}} = -I|_{\mathcal{C}}$. Then for any solution $r(t), \theta(t)$ of the planar Kepler problem* (2.16) *there is a homographic solution of the form* (2.26) *with*

$$Q(t) = \exp(\theta(t)J).$$

Proof. Since X_0 is a CC, the right-hand side of (2.18) is

$$rQ(X_0 - C_0)A(rQ(X_0 - C_0)) = r^{-2}QX_0A(X_0) = -\frac{\lambda}{r^2}Q(X_0 - C_0),$$

where we have used the homogeneity and the translation and rotation invariance of A. The left-hand side is

$$\ddot{X} = \ddot{r}Q(X_0 - C_0) + 2\dot{r}\dot{Q}(X_0 - C_0) + r\ddot{Q}(X_0 - C_0).$$

We have

$$\dot{Q} = \dot{\theta}(t)JQ, \quad \ddot{Q}(t) = \ddot{\theta}(t)JQ + (\dot{\theta}(t))^2J^2Q.$$

Since J and Q commute and $J^2(X_0 - C_0) = -(X_0 - C_0)$, we get

$$\ddot{X} = (\ddot{r} - r(\dot{\theta})^2)Q(X_0 - C_0) + (r\ddot{\theta} + 2\dot{r}\dot{\theta})QJ(X_0 - C_0).$$

Since $r(t), \theta(t)$ are solutions of the Kepler problem, this reduces to

$$\ddot{X} = -\frac{\lambda}{r^2}Q(X_0 - C_0)$$

as required. □

Recall that a *complex structure* on a vector space $\mathcal{S}$ is given by a linear map $J\colon \mathcal{S} \to \mathcal{S}$ with $J^2 = -I$. If there is an inner product with respect to which J is antisymmetric then we have a *Hermitian structure*. An antisymmetric matrix J as above with $J^2|_{\mathcal{C}} = -I_{\mathcal{C}}$ determines a Hermitian structure on the larger space

$$\mathcal{S} = \mathcal{C} + J\mathcal{C}.$$

To see this, note that $\mathcal{S}$ is J-invariant. If $\eta \in J\mathcal{C}$ then $\eta = J\xi$ for some $\xi \in \mathcal{C}$ and we get

$$J^2\eta = J^3\xi = J(-\xi) = -\eta.$$

Thus, we actually have

$$J^2|_{\mathcal{S}} = -I\,|_{\mathcal{S}}.$$

Since J is antisymmetric, it has even rank and so $\dim \mathcal{S}$ must be even. In the proposition, $\mathcal{S}$ is the motion space of the Keplerian homographic motion.

Thus a necessary condition that a CC x_0 admits a matrix J as above is that $\mathcal{C}(x_0)$ be contained in an even dimensional subspace of $\mathbb{R}^d$. Since any even-dimensional subspace of the Euclidean space $\mathbb{R}^d$ has a natural Hermitian structure where J is rotation by $\pi/2$ in k mutually orthogonal planes, this condition is also sufficient. This will always be possible if either d is even or $\dim \mathcal{C} < d$. The only bad case is when $d = \dim \mathcal{C}$ is odd. For example, if we have a collinear central configuration in $\mathbb{R}^1$ or a nonplanar configuration in $\mathbb{R}^3$, we will not be able to find such an even-dimensional subspace.

Example 2.5.2. Consider the equilateral triangle in $\mathbb{R}^4$ whose configuration matrix X is given by (2.17). Then $\dim \mathcal{C} = \operatorname{rank} X = 2$. We could choose J to be a rotation by $\pi/2$ in the plane $\mathcal{C}$ which fixes the orthogonal complement. Then the motion space is also $\mathcal{S} = \mathcal{C}$ and the triangle rotates rigidly in its own plane.

On the other hand we could choose

$$J = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Now the motion space will be $\mathcal{S} = \mathbb{R}^4$. Each body moves in a planar Keplerian orbit, but the orbits are in different planes. Indeed, we have

$$X(t) = r(t)\cos\theta(t) \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + r(t)\sin\theta(t) \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}.$$

The i-th body moves in the plane spanned by the i-th columns in the two matrices.

On the other hand a regular tetrahedron in $\mathbb{R}^3$ is not contained in any even-dimensional subspace. But if we put it in $\mathbb{R}^4$ we can choose any 4×4 matrix J with $J^2 = -I$, such as the one in the last paragraph, and proceed to construct Keplerian homographic motions. Figure 2.4 shows a projection of such a motion onto the first three coordinate axes. Each body moves on a circular orbit at constant speed, but the circles are in different planes. In this projection the circles look like ellipses on a vertical cylinder. Initially, the projected shape is a regular tetrahedron as in the figure but later the projected bodies will form a square in the horizontal plane. Of course it is still a regular tetrahedron in $\mathbb{R}^4$.

Note that, on the centered position space $\mathcal{C}(X_0)$, the matrix exponential in Proposition 2.5.1 can be written

$$Q(t) = \exp(\theta(t)J) = \cos\theta(t)I + \sin\theta(t)J.$$

It follows that, for a Keplerian homographic solution as in the proposition, the j-th body moves in the two-dimensional plane spanned by the vectors x_{j_0}, Jx_{j_0}. All of

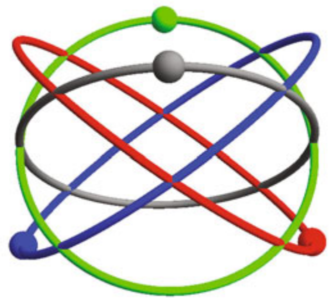

Figure 2.4: Three-dimensional projection of a rigid motion of a central configuration in $\mathbb{R}^4$. Four equal masses are at the vertices of a regular tetrahedron. Each body moves on a circle in $\mathbb{R}^4$ but the circles are in different planes. In the projection, the circles become ellipses.

the bodies describe similar Keplerian orbits and the overall configuration remains similar to the CC X_0 throughout the motion. In particular, for each admissible choice of J we get a family of periodic solutions with elliptical orbits of different eccentricities. Eccentricity zero gives the uniform rigid motions and eccentricity one gives the homothetic solutions.

2.6 Albouy–Chenciner reduction and relative equilibria in $\mathbb{R}^d$

The matrix formulation of Newton's equations leads to an elegant way to reduce by the rotational symmetry. The reduced equations lead to a deeper understanding of the most general rigid and homographic motions. This section is based on the works Albouy–Chenciner [3] and Chenciner [7]. The Albouy–Chenciner method of reducing the equations of motion is a far-reaching generalization of Lagrange's reduction method for the three-body problem [15].

Starting from the matrix equations of motion (2.20), we can eliminate the rotational symmetry of the n-body problem by passing to Gram matrices.

$$B(X) = X^T X, \quad C(X,V) = X^T V, \quad D(V) = V^T V.$$

The entries of these matrices are the dot products of the position and velocity vectors:

$$B_{ij} = x_i \cdot x_j, \quad C_{ij} = x_i \cdot v_j, \quad D_{ij} = v_i \cdot v_j.$$

It follows that the matrices are invariant under simultaneous rotation of all positions and velocities in $\mathbb{R}^d$. In other words, if $Q \in \mathrm{SO}(d)$ is any rotation matrix then

$$B(QX) = B(X), \quad C(QX, QV) = C(X, V), \quad D(QV) = D(V).$$

Note also that $B(X), D(V)$ are symmetric and positive semi-definite.

To eliminate the translational symmetry we can work with the centered matrices $Y = X - C = XP$ and $W = VP$.

Definition 2.6.1. *Given configuration and velocity matrices X and V, $B(XP) = B(X - C)$ is the* relative configuration *matrix and $B(XP)$, $C(XP, VP)$, $D(VP)$ are the* relative state *matrices. If $X(t), V(t)$ is a solution, we will write $B(t)$, $C(t)$ and $D(t)$ for the corresponding relative state matrices.*

An alternative approach to eliminating the center of mass is just to view all of these matrices as representations of bilinear forms on the hyperplane $\mathcal{D}^*$. In other words, only the values $\xi^T B\eta$ for $\xi, \eta \in \mathcal{D}^*$ are significant. Let's call two $n \times n$ matrices *translation equivalent* if they define the same bilinear form on $\mathcal{D}^*$. Then, for example, $B(X) = X^T X$ and $B(X - C) = (X - C)^T (X - C)$ are translation equivalent. In fact any matrix obtained from B by adding multiples of L to the rows and multiples of L^T to the columns will be translation equivalent. Starting from $B(X)$ we get a particularly simple representative by adding subtracting $\frac{1}{2}|x_i|^2 L$ from the i-th row and $\frac{1}{2}|x_j|^2 L$ column. The diagonal entries of the new matrix are 0 and the off diagonals are

$$x_i^T x_j - \frac{1}{2}|x_i|^2 - \frac{1}{2}|x_j|^2 = -\frac{1}{2}|x_i - x_j|^2 = -\frac{1}{2}r_{ij}^2.$$

Thus the following matrix is translation equivalent to $B(X)$ and $B(X - C)$:

$$\hat{B}(X) = -\frac{1}{2}\begin{bmatrix} 0 & r_{12}^2 & \cdots & r_{1n}^2 \\ r_{21}^2 & 0 & \cdots & r_{2n}^2 \\ \vdots & & & \vdots \\ r_{n1}^2 & \cdots & r_{n(n-1)}^2 & 0 \end{bmatrix}. \tag{2.27}$$

Using (2.20) it is easy to derive differential equations for the matrices B, C, D. One finds

$$\begin{aligned} \dot{B} &= C + C^T, \\ \dot{C} &= D + BA, \\ \dot{D} &= C^T A + A^T C. \end{aligned} \tag{2.28}$$

These apply equally to the original Gram matrices $B(X), C(X, V), D(V)$ and to the translation reduced versions.

Recall that $A(X) = A(X - C)$ depends only on the mutual distances r_{ij}. The mutual distances can be expressed in terms of the Gram matrix B, since

$$r_{ij}^2 = |x_i - x_j|^2 = |x_i|^2 + |x_j|^2 - 2x_i \cdot x_j = B_{ii} + B_{jj} - 2B_{ij}.$$

Hence we can view A as a function $A(B)$. Then the system (2.28) could be used to find the time evolution of the relative state matrices B, C, D without reference to the actual state variables X, V.

The angular momentum is equivariant with respect to rotations

$$\omega(QX, QV) = Q\omega(X,V)Q^T = Q\omega(X,V)Q^{-1},$$

for $Q \in \mathrm{SO}(d)$. The eigenvalues of $\omega(X,V)$ are rotation invariant and provide constants of motion for the relative equations.

At this point we can write down the reduced version of the CC equation.

Proposition 2.6.2. *Let X be a $d \times n$ configuration matrix. Then X is a CC with constant λ if and only if the relative configuration matrix $B(X - C)$ satisfies*

$$BA(B) + \lambda B = 0. \tag{2.29}$$

Proof. By hypothesis, we have $XA(X) + \lambda(X - C) = 0$. Multiplying by $(X - C)^T$ and using the translation and rotation invariance of A, we get (2.29). Conversely, if (2.29) holds we get

$$(X - C)^T \left(XA(X) + \lambda(X - C)\right) = 0.$$

To eliminate $(X - C)^T$ note that the matrix in parentheses has range contained in $\mathrm{im}(X - C)$. Since $\mathrm{im}(X - C) \cap \ker(X - C)^T = \{0\}$, it must vanish. □

Next we will use the reduced equations to study general rigid motions of the n-body problem. For a rigid motion we have

$$X(t) - C(t) = Q(t)(X_0 - C_0) \tag{2.30}$$

for some $Q(t) \in \mathrm{SO}(d)$, and the relative configuration matrix

$$B(t) = B(X - C)$$

is constant. Conversely, if $B(t)$ is constant then all of the mutual distances are constant and (2.30) holds for some $Q(t) \in \mathrm{SO}(d)$. Thus rigid motions are characterized by the constancy of $B(t)$. It turns out that the other relative state matrices are also constant, so we have an equilibrium point of (2.28).

Proposition 2.6.3. *$X(t), V(t)$ are the state matrices of a rigid motion solution of the n-body problem in $\mathbb{R}^d$ if and only if the relative state matrices $B(t), C(t), D(t)$ are constant.*

Proof. We have seen that $X(t), V(t)$ is a rigid motion if and only if $B(t)$ is constant. It remains to show that the constancy of B implies that of C and D. Assuming $\dot{B} = 0$ we also get $\dot{A} = \dot{A(B)} = 0$. Now use (2.28) to calculate the derivatives of $B(t)$:

$$\begin{aligned}
\dot{B} &= C + C^T = 0,\\
\ddot{B} &= \dot{C} + \dot{C}^T = 2D + BA + A^T B = 0,\\
\dddot{B} &= 2\dot{D} = 2(C^T A + A^T C) = 0.
\end{aligned}$$

So we have $\dot{D} = 0$ and also find that $2D = -(BA + A^T B)$ which implies

$$\dot{C} = \frac{1}{2}(A^T B - BA).$$

We need to show that this vanishes. Computing one more derivative gives

$$\ddddot{B} = 2(\dot{C}^T A + A^T \dot{C}) = (A^T B - BA)A - A^T(A^T B - BA) = 0.$$

It turns out that this equation can hold only when the quantity in parentheses is already zero.

To see this we use the fact that AM is a symmetric matrix so $AM = MA^T$. We have

$$M(A^T B - BA) = MA^T B - MBA = A^T MB - MBA = -[MB, A],$$

the commutator of MB and A. Similarly,

$$M\left((A^T B - BA)A - A^T(A^T B - BA)\right) = -[[MB, A], A].$$

Now the symmetry of AM also gives $A^T M^{-1} = M^{-1}A$, i.e., A is M^{-1}-symmetric. This implies that A is diagonalizable with respect to some M^{-1} orthogonal basis. Choose such a basis and let the matrix representing A be $\mathrm{diag}(a_1, \dots, a_n)$ and that representing MB have entries b'_{ij}. Then the entries of $[MB, A]$ and $[[MB, A], A]$ are

$$b'_{ij}(a_i - a_j), \quad b'_{ij}(a_i - a_j)^2,$$

respectively. Thus $[[MB, A], A] = 0$ if and only if $[MB, A] = 0$ as claimed. Hence, $\ddddot{B} = 0$ implies $\dot{C} = 0$ completing the proof. □

This result justifies the terminology *relative equilibrium solution (RE)* applied to rigid motion solutions. We really do have an equilibrium of the relative equations of motion (2.28). We have seen how to construct a uniformly rotating relative equilibrium solution based on a central configuration. But it is not at all clear that this is the only kind and, indeed, we will see that rotations of certain noncentral configurations are possible. However, it is true that every rigid motion is a uniform rotation.

Proposition 2.6.4. *Let $X(t), V(t)$ be any rigid motion (RE) solution. Then there is a configuration matrix X_0 (not necessarily central) and a constant antisymmetric $d \times d$ matrix α such that*

$$X(t) - C(t) = Q(t)(X_0 - C_0),$$

where $Q(t) = \exp(t\alpha)$.

We will call α the *angular velocity matrix*. The proof uses the following fact from linear algebra.

Lemma 2.6.5. *Let L_1, L_2 be $d \times k$ matrices such that* $\ker L_1 \subset \ker L_2$*. Then there is a $d \times d$ matrix J such that*

$$L_2 = JL_1.$$

Moreover, if $\operatorname{im} L_2 \subset \operatorname{im} L_1$ *and if the $k \times k$ matrix $L_1^T L_2$ is symmetric (antisymmetric), then J can be chosen to be symmetric (antisymmetric).*

Proof. The hypothesis about the kernels implies that we get a well-defined linear map $\operatorname{im} L_1 \to \mathbb{R}^d$ by setting $J\xi = L_2 u$ when $L_1 u = \xi$. We can extend it to $J: \mathbb{R}^d \to \mathbb{R}^d$ by making it vanish on the Euclidean orthogonal complement $(\operatorname{im} L_1)^\perp$ and this choice makes the extension unique.

If $\operatorname{im} L_2 \subset \operatorname{im} L_1$ then $J(\operatorname{im} L_1) \subset \operatorname{im} L_1$. Let ξ, η be two vectors in $\operatorname{im} L_1$ and write $\xi = L_1 u, \eta = L_1 v$. Then, by definition of J,

$$\xi^T J\eta = u^T L_1^T L_2 v.$$

If $L_1^T L_2$ is symmetric (antisymmetric), this shows that the restriction of J to $\operatorname{im} L_1$ is also symmetric (antisymmetric). Since we extended trivially on the orthogonal complement, it is easy to see that the extension has the same symmetry. □

Proof of Proposition 2.6.4. Let $Z(t) = \begin{bmatrix} X(t)P & V(t)P \end{bmatrix} = \begin{bmatrix} Y(t) & W(t) \end{bmatrix}$ be the $d \times 2n$ centered state matrix and note that the $2n \times 2n$ Gram matrix

$$Z^T Z = \begin{bmatrix} B & C^T \\ C & D \end{bmatrix}$$

encodes the relative state matrices B, C, D. For a RE solution this matrix is constant so

$$Z^T \dot{Z} + \dot{Z}^T Z = 0.$$

In other words, the $2n \times 2n$ matrix

$$Z(t)^T \dot{Z}(t)$$

is antisymmetric. Now apply Lemma 2.6.5 with $L_1 = Z(t)$ and $L_2 = \dot{Z}(t)$ to get an antisymmetric $d \times d$ matrix $\alpha(t)$ such that $\dot{Z}(t) = \alpha(t)Z(t)$, i.e.,

$$\dot{Y}(t) = \alpha(t)Y(t), \quad \dot{W}(t) = \alpha(t)W(t).$$

In particular, at $t = 0$, we have

$$\dot{Y}(0) = W_0 = \alpha_0 Y_0, \quad \dot{W}(0) = Y_0 A(Y_0) = \alpha_0 W_0 = \alpha_0^2 Y_0. \tag{2.31}$$

To complete the proof, we will show

$$Y(t) = Q(t)Y_0, \quad Q(t) = \exp(t\alpha_0).$$

Since this function has the right initial conditions, we need only to show that it is a solution of Newton's equations. We have

$$\ddot{Y}(t) = \alpha_0^2 Y(t),$$

so we need to show that

$$\alpha_0^2 Y(t) = Y(t)A(Y(t)). \tag{2.32}$$

From (2.31) we have

$$\alpha_0^2 Y_0 = Y_0 A(Y_0) \tag{2.33}$$

so (2.36) holds when $t = 0$. It follows for other times by multiplying by $Q(t)$ and using the rotation invariance of A. □

It follows from this result that if X_0 is a CC, then the most general possible rigid motions with shape X_0 are the circular Keplerian ones from Proposition 2.5.1. Comparing the antisymmetric matrices which appear in the two propositions, we should have $t\alpha = \theta(t)J$. Now for the circular Kepler orbit of radius $r = 1$ we have $\dot{\theta}^2 = \lambda$. With

$$\alpha = \pm\sqrt{\lambda}\, J \tag{2.34}$$

then one can check that, for the solution of Proposition 2.5.1, $\dot{Z} = \alpha Z$ holds.

The formulas in the last proof suggest a way to construct rigid motions whose configurations are not central. The condition (2.33) is enough to guarantee that a corresponding rigid solution exists.

Definition 2.6.6. *A configuration x is* balanced in $\mathbb{R}^d$ *or d-*balanced *if there is a $d \times d$ antisymmetric matrix α such that*

$$XA(X) - \alpha^2(X - C) = 0 \tag{2.35}$$

or, equivalently, if

$$\nabla_j U(x) - \alpha^2 M(x_j - c) = 0. \tag{2.36}$$

It is called balanced *if it is d-balanced for d sufficiently large.*

The definition of balanced configurations in [3] is equivalent to the one given here. The proof of Proposition 2.6.4 shows that every balanced configuration gives rise to a uniformly rotating relative equilibrium solution (2.30), with $Q(t) = \exp(t\alpha)$ in the appropriate ambient space $\mathbb{R}^d$. From (2.34) we see that every central configuration is balanced provided it is contained in an even-dimensional subspace hence, certainly, in $\mathbb{R}^d$ or in $\mathbb{R}^{d+1}$. However, there exist balanced configurations which are not central.

Before presenting an example we will derive a couple of equivalent versions of the concept of balance. Note that if X is balanced then the matrix $S = -\alpha^2$ is symmetric and positive semi-definite.

Proposition 2.6.7. *A configuration is balanced if and only if its configuration matrix satisfies*

$$XA(X) + S(X - C) = 0 \tag{2.37}$$

for some positive semi-definite matrix S. Equivalently, the relative configuration matrix $B(X - C)$ should satisfy

$$BA = (BA)^T. \tag{2.38}$$

Proof. If X is balanced in $\mathbb{R}^d$ then (2.37) holds with $S = -\alpha^2$. Conversely suppose (2.37) holds for a configuration in $\mathbb{R}^d$. If we double the dimension of the space, padding X with rows of zeros and replace S by the $2d \times 2d$ matrix $\hat{S} = \text{diag}(S, S)$, then we can solve the equation $\hat{S} = -\alpha^2$ for the antisymmetric matrix α. To see this, assume without loss of generality that $\hat{S} = \text{diag}(\sigma_1^2, \ldots, \sigma_d^2, \sigma_1^2, \ldots, \sigma_d^2)$. Then we can use the block matrix

$$\alpha = \begin{bmatrix} 0 & -\sigma \\ \sigma & 0 \end{bmatrix} \qquad \sigma = \text{diag}(\sigma_1, \ldots, \sigma_d).$$

Thus, if X satisfies (2.37), it will give rise to a rigid motion in $\mathbb{R}^{2d}$, i.e., it will be $2d$-balanced.

Multiplying (2.37) by $(X - C)^T$ and using (2.24) shows that BA is symmetric. Conversely, suppose $BA = (X - C)^T(X - C)A(X)$ is symmetric. Using Lemma 2.6.5 with $L_1 = X - C$ and $L_2 = (X - C)A(X)$ gives a symmetric $d \times d$ matrix $-S$ with

$$(X - C)A(X) = -S(X - C),$$

as required. □

In the following example we will use (2.38) to check for balance. Moreover, we can avoid explicitly shifting the center of mass by just requiring $BA = (BA)^T$ on $\mathcal{D}^*$.

Figure 2.5: Three-dimensional projection of a rigid motion of a balanced configuration in $\mathbb{R}^4$. An isosceles triangle with edges $1, \sqrt{3}/2, \sqrt{3}/2$ and masses $1, 1, (\sqrt{5} - 1)/2$ is rotating with different frequencies in two orthogonal planes in $\mathbb{R}^4$. The mass on the symmetry axis is in one of the planes and moves on a circle while the other two masses move on a torus.

Example 2.6.8. Consider a triangle with sides $r_{12} = r$, $r_{13} = s$, $r_{23} = t$. We will investigate the *inverse problem*: given a configuration, find which masses make it balanced or central. We have

$$A(X) = \begin{bmatrix} -\frac{m_2}{r^3} - \frac{m_3}{s^3} & \frac{m_1}{r^3} & \frac{m_1}{s^3} \\ \frac{m_2}{r^3} & -\frac{m_1}{r^3} - \frac{m_3}{t^3} & \frac{m_2}{t^3} \\ \frac{m_3}{s^3} & \frac{m_3}{t^3} & -\frac{m_1}{s^3} - \frac{m_2}{t^3} \end{bmatrix}$$

and

$$\hat{B}(X) = -\frac{1}{2}\begin{bmatrix} 0 & r^2 & s^2 \\ r^2 & 0 & t^2 \\ s^2 & t^2 & 0 \end{bmatrix}.$$

The condition for a balanced triangle is that the restriction of BA to $\mathcal{D}^*$ be symmetric. To avoid explicitly shifting the center of mass, we calculate the commutator $\hat{B}A - A^T\hat{B}$ and require that $e_i^T(BA - AB)e_j = 0$ for some basis e_1, e_2 of the plane $\mathcal{D}^*$; for example, we could use $e_1 = (1, -1, 0), e_2 = (1, 0, -1)$. The result is a 2×2 antisymmetric matrix so there is only one equation which turns out to be

$$\begin{aligned} m_1(s^{-3} - r^{-3})(t^2 - r^2 - s^2) + m_2(r^{-3} - t^{-3})(s^2 - r^2 - t^2) \\ + m_3(t^{-3} - s^{-3})(r^2 - s^2 - t^2) = 0. \end{aligned} \tag{2.39}$$

For the equilateral triangle $r = s = t$ the equation is trivial, so the triangle is balanced for all choices of the masses. Of course we already knew this since it is a CC for all masses (and is even-dimensional). For any nonequilateral triangle (2.39) gives a two-dimensional plane of masses. This plane always intersects the positive octant, so every triangle is balanced for some two-dimensional cone of masses. For example, the isosceles triangle with $(r, s, t) = (r, s, s)$ is balanced for all mass vectors with $m_1 = m_2$ and arbitrary m_3. On the other hand, the right triangle with $(r, s, t) = (3, 4, 5)$ is balanced for $183m_2 = 392m_3$ with m_1 (the mass at the right angle) arbitrary. Since the only non-collinear CC is the equilateral triangle, there are plenty of triangles which are balanced but not central.

To investigate the possible rigid motions of such triangles we need to work with configuration matrices X and find the corresponding antisymmetric angular velocity matrices, α. For the isosceles case in $\mathbb{R}^2$ we can take

$$X = \begin{bmatrix} 0 & 0 & x \\ y & -y & 0 \end{bmatrix}$$

and in $\mathbb{R}^d$ we can just add rows of zeros. With masses $m_1 = m_2 = 1$ we find that

$$XA(X) + S(X - C) = 0, \quad S = \mathrm{diag}(\frac{2 + m_3}{s^3}, \frac{1}{4y^3} + \frac{m_3}{s^3}), \quad s = \sqrt{x^2 + y^2}.$$

We need a $d \times d$ antisymmetric matrix with $\alpha^2 = -S$. This is only possible in $\mathbb{R}^2$ when $S = \lambda I$, that is, only for the equilateral CC case. In the nonequilateral case with $d = 4$ the only valid angular velocity matrices are the block-diagonal matrices

$$J = \begin{bmatrix} 0 & -\sigma \\ \sigma & 0 \end{bmatrix}, \quad \sigma = \mathrm{diag}(\sigma_1, \sigma_2), \quad \sigma_1^2 = \frac{2+m_3}{s^3}, \quad \sigma_2^2 = \frac{1}{4y^3} + \frac{m_3}{s^3}.$$

The isosceles triangle rotates around its symmetry axis and simultaneously around an orthogonal axis with two different frequencies, the two planes of rotation being orthogonal. The motion of the mass on the symmetry axis is planar and periodic but the other two masses move on a torus which spans $\mathbb{R}^4$ (see Figure 2.5). For fixed $m_3 > 0$ one can check that the eigenvalue ratio σ_2^2/σ_1^2 of S varies over $((1+4m_3)/(8+4m_3), \infty)$ as the angle at m_3 of the isosceles shape decreases from π to 0.

2.7 Homographic motions in $\mathbb{R}^d$

Next we will show that the orbits described in Proposition 2.5.1 are actually the most general, *nonrigid* homographic motions. In particular, only central configurations give rise to such motions.

Proposition 2.7.1. *Every nonrigid homographic solution of the n-body problem in $\mathbb{R}^d$ is of the form*

$$X(t) - C(t) = r(t)Q(t)(X_0 - C_0), \quad Q(t) = \exp(\theta(t)J),$$

where X_0 is a central configuration with constant λ, $(r(t), \theta(t))$ is a solution of the Kepler problem (2.16)*, and J is an antisymmetric $d \times d$ matrix with $J^2|_{\mathcal{C}(X_0)} = -I|_{\mathcal{C}(X_0)}$.*

Proof following [7]. Since the motion is homographic, the right-hand side of equation (2.18) is

$$X(t)A(X(t)) = r(t)^{-3}X(t)A(X_0).$$

The fact that the $n \times n$ matrix $A(X_0)$ is M^{-1}-symmetric implies that it is diagonalizable. One of the eigenvalues is zero since the mass vector m is in the kernel, and the others are nonpositive because of the negative semi-definiteness of AM. Let R be an invertible $n \times n$ matrix with

$$R^{-1}A(X_0)R = \mathrm{diag}(-\lambda_1, -\lambda_2, \ldots, -\lambda_n).$$

If $W(t) = (X(t) - C(t))R$ then Newton's equations give

$$\ddot{W} = r(t)^{-3}X(t)A(X_0)R = r(t)^{-3}W(t)R^{-1}A(X_0)R$$

and so the columns $w_j(t)$ of $W(t)$ satisfy

$$\ddot{w}_j(t) = -\frac{\lambda_j w_j(t)}{r(t)^3}.$$

Since the solution is homographic, we have $W(t) = r(t)Q(t)W_0$ where $W_0 = (X_0 - C_0)R$. It follows that the columns of W, W_0 satisfy

$$|w_j(t)| = r(t)|w_{0j}|, \quad j = 1, \ldots, n.$$

For each column such that $|w_{0j}| \neq 0$, define $u_j(t) = w_j(t)/|w_{0j}|$. Then $|u_j(t)| = r(t)$ for $j = 1, \ldots, n$ and

$$\ddot{u}_j(t) = -\frac{\lambda_j u_j(t)}{|u_j(t)|^3},$$

i.e., the normalized nonzero columns solve Kepler's equations with constant λ_j. Moreover, they all have the same norm $r(t)$. It follows that each of these $u_j(t)$ moves in a plane and can be represented with respect to polar coordinates in that plane by functions $r(t), \theta(t)$ satisfying (2.16) with $\lambda = \lambda_j$.

Lemma 2.7.2. *If $r(t), \theta(t)$ solves* (2.16) *and $r(t)$ is not constant, then λ and $\dot{\theta}(t)$ are uniquely determined by $r(t)$.*

Proof. Exercise. □

Continuing with the proof of the proposition, we now see that all of the λ_j corresponding to nonzero columns of $W(t)$ are equal. Then we have

$$X_0 A(X_0) = W_0 \operatorname{diag}(-\lambda_1, \ldots, -\lambda_n) S^{-1} = -\lambda W_0 S^{-1} = -\lambda(X_0 - C_0),$$

where the second equation holds because changing λ_j to λ for a column $w_j = 0$ does no harm. This shows that X_0 is a central configuration.

To get the rest we will use the reduced equations of motion (2.28). Since we are assuming that $X(t)$ is homographic, the relative state matrices have a particularly simple form. Let $Y(t) = X(t) - C(t) = X(t)P$ and $W(t) = V(t)P$ be the centered position and velocity matrices. Then $Y(t) = r(t)Q(t)Y_0$ and $W(t) = \dot{r}(t)Q(t)Y_0 + r(t)\dot{Q}(t)Y_0$. The relative state matrices are

$$B(t) = r(t)^2 B_0, \quad C(t) = r(t)\dot{r}(t)B_0, \quad D(t) = \dot{r}(t)^2 B_0 - r(t)^2 Y_0^T \Omega(t)^2 Y_0,$$

where $\Omega(t) = Q(t)^T \dot{Q}(t) \in \mathrm{so}(d)$. The antisymmetry of this matrix implies that terms involving $Y_0^T \Omega(t) Y_0$ in the calculation of these matrices vanish. Now calculating $\dot{C}(t)$ and comparing with (2.28) gives

$$(r\ddot{r} + \dot{r}^2)B_0 = D + BA = D - \lambda r^2 B_0, \tag{2.40}$$

where we used (2.29).

Now we already found that $r(t), \theta(t)$ are solutions of Kepler's equation. By rescaling X_0 and choosing the origin of time, we may assume that $r(0) = 1$ and $\dot{r}(0) = 0$. The second assumption certainly holds at the perihelion of the Kepler orbit. At this point the velocities and positions are orthogonal. Evaluating (2.40) at $t = 0$ and using Kepler's equation (2.16) we get

$$D_0 = \dot{\theta}_0^2 B_0. \tag{2.41}$$

We also have $C_0 = 0$.

Let $Z_0 = \begin{bmatrix} Y_0 & W_0 \end{bmatrix}$ be the initial state matrix and consider the matrices

$$L_1 = Z_0, \quad L_2 = \begin{bmatrix} \dot\theta_0^{-1} W_0 & -\dot\theta_0 Y_0 \end{bmatrix}.$$

We have

$$L_1^T L_2 = \begin{bmatrix} \dot\theta_0^{-1} C_0 & -\dot\theta_0 B_0 \\ \dot\theta_0^{-1} D_0 & -\dot\theta_0 C_0 \end{bmatrix} = \begin{bmatrix} 0 & -\dot\theta_0 B_0 \\ \dot\theta_0^{-1} D_0 & 0 \end{bmatrix}.$$

This $2n \times 2n$ matrix is antisymmetric by (2.41) so, by Lemma 2.6.5, there is an antisymmetric $d \times d$ matrix J such that

$$W_0 = \dot\theta_0 J y_0, \quad Y_0 = -\dot\theta_0^{-1} W_0 = -J^2 Y_0.$$

By Proposition 2.5.1, $\tilde Y(t) = \exp(\theta(t)J)Y_0$ is a homographic solution and its initial conditions

$$\tilde Y(0) = Y_0, \quad \tilde W(0) = \dot\theta_0 J Y_0 = W_0$$

are the same as those of the given homographic solution, $Y(t)$. Therefore, $Y(t) = \exp(\theta(t)J)Y_0$ as claimed. □

Although we have made a point of studying the special solutions of the n-body problem in $\mathbb{R}^d$, we will summarize the results for the physical case $d = 3$. The homographic solutions in $\mathbb{R}^3$ are of the following types. For any central configuration and any solution of the one-dimensional Kepler problem there is a homothetic solution. For any central configuration which is contained in some two-dimensional subspace and any solution of the two-dimensional Kepler problem, there is a homographic solution for which the bodies remain in the same plane. This is a uniform planar rigid motion if we take the circular solution of the Kepler problem. There are no other homographic motions. In particular, a nonplanar CC does not lead to any rigid or homographic, nonhomothetic solutions. A configuration which is balanced but not central is not balanced in $\mathbb{R}^3$ so does not give rise to a RE solution in $\mathbb{R}^3$.

2.8 Central configurations as critical points

Now that we have some motivation for studying central configurations, lots of interesting questions arise. Fixing the masses m_i we can ask whether central configurations exist and if so, how many there are up to symmetry. Working with configuration vectors $x \in \mathbb{R}^{dn}$ we need to study solutions of the CC equation

$$\nabla U(x) + \lambda M(x - c) = 0. \tag{2.42}$$

If x is a CC then so is any configuration y obtained from x by translations and rotations. In particular, the centered configuration $x - c$ is also a CC. If $k > 0$

then kx is also a central configuration but with a different λ. Recall that $\lambda(x) = U(x)/I(x)$, where $I(x)$ is the moment of inertia around the center of mass. So

$$\lambda(kx) = \lambda(x)/k^3.$$

We will view such CC's as equivalent and refer to similarity classes of CC's.

The key idea in this section is to interpret CC's as constrained critical points of the Newtonian potential. The constraint is just to fix the moment of inertia. Since $\nabla I(x) = 2M(x - c)$, the CC equation can be written

$$\nabla U(x) + \tfrac{1}{2}\lambda \nabla I(x) = 0.$$

Interpreting $\lambda/2$ as Lagrange multiplier, we get:

Proposition 2.8.1. *A configuration vector x_0 is a central configuration if and only if it is a critical point of $U(x)$ subject to the constraint $I(x) = k$, where $k = I(x_0)$.*

It is useful for existence proofs to have a compact constraint set. We can use the scaling symmetry to normalize the moment of inertia to be $I(x) = 1$ but, because of the translation invariance, $\{x : I(x) = 1\}$ is not compact. We can eliminate the translation symmetry by fixing the center of mass.

Just as in the matrix formulation of the problem, we can view the passage from x to $x - c$ as an orthogonal projection. In fact

$$x - c = \hat{P}x,$$

where $\hat{P}\colon \mathbb{R}^{dn} \to \mathbb{R}^{dn}$ is the orthogonal projection onto the subspace where $m_1x_1 + \cdots + m_nx_n = 0 \in \mathbb{R}^d$ with respect to the mass inner product v^TMw. The matrix of $\hat{P}$ is

$$\hat{P} = I - \frac{1}{m_0}\hat{L}^T\hat{L}M, \quad \hat{L} = \begin{bmatrix} I & I & \cdots & I \end{bmatrix}, \tag{2.43}$$

where $\hat{L}$ is $d \times dn$ with blocks of $d \times d$ identity matrices. One can check that $\hat{P}$ is an M-symmetric projection matrix.

Define the *normalized configuration space* as

$$\mathcal{N} = \{x : c = \hat{L}Mx = 0, I(x) = 1\}.$$

Any configuration x determines a unique *normalized* configuration with $c = 0$ and $I = 1$. Note that the center of mass condition defines a subspace of $\mathbb{R}^{dn}$ of dimension $d(n-1)$ and then $I = 1$ gives an ellipsoid in this subspace. Hence $\mathcal{N}$ is a smooth compact manifold diffeomorphic to a sphere, $\mathcal{N} \simeq \mathbf{S}^{d(n-1)-1}$.

Proposition 2.8.2. *A configuration vector x is a central configuration if and only if the corresponding normalized configuration is a critical point of the Newtonian potential $U(x)$ restricted to $\mathcal{N}$.*

Proof. If x is a CC, so is the corresponding normalized configuration. Proposition 2.8.1 shows that this normalized configuration is a critical point of $U(x)$ with the constraint $I(x) = 1$, so it is still a critical point if we add the center of mass constraint defining $\mathcal{N}$.

Conversely, suppose x is a critical point of $U(x)$ restricted to $\mathcal{N}$. We need to show that it is still a critical point if we remove the center of mass constraint. This can be checked using the orthogonal projection $\hat{P}$. Note that $\mathcal{N}$ is a smooth codimension one submanifold of the subspace $\ker \hat{L}M \subset \mathbb{R}^{dn}$. Therefore $x \in \mathcal{N}$ is a critical point of $U|_{\mathcal{N}}$ if and only if

$$(DU(x) + kDI_S(x))\, v = 0$$

for all $v \in \ker \hat{L}M$, where $k \in \mathbb{R}$ is a Lagrange multiplier. Equivalently we need

$$(DU(x) + kDI_S(x))\, \hat{P} = 0,$$

where $\hat{P}$ is the orthogonal projection onto $\ker \hat{L}M$ from (2.43). By translation invariance $U(\hat{P}x) = U(x)$, and differentiation gives $DU(x)\hat{P} = DU(x)$ for $x \in \mathcal{N}$. Similarly, $DI(x)\hat{P} = DI(x)$. So we can drop $\hat{P}$ from the last equation and take transposes to get

$$\nabla U(x) + k\,\nabla I(x) = 0,$$

which is the CC equation. □

An alternative approach is based on the moment of inertia with respect to the origin,

$$I_0(x) = x^T M x = \sum_{j=1}^{n} m_j |x_j|^2.$$

For configurations with $c = 0$, $I(x) = I_0(x)$ and the CC equation becomes

$$\nabla U(x) + \lambda M x = 0. \tag{2.44}$$

This is the critical point equation with fixed $I_0(x)$. It turns out that (2.44) forces $c = 0$ and we have:

Proposition 2.8.3. *The point x is a critical point of $U(x)$ on $\{x : I_0(x) = 1\}$ if and only if x is a normalized central configuration.*

Proof. If $x \in \mathcal{N}$ then $c = 0$ and $I(x) = I_0(x) = 1$. If it is also a central configuration then (2.44) holds, so it is a critical point of $U(x)$ on $\{I_0 = 1\}$. Conversely, suppose x is a critical point of $U(x)$ on $\{I_0 = 1\}$. Then (2.44) holds. We will show that this implies $c = 0$ and it follows that $x \in \mathcal{N}$ and that the CC equation (2.42) holds.

Equation (2.44) gives

$$\lambda m_j x_j = -\nabla_j U(x) = \sum_{i \neq j} F_{ji},$$

where $F_{ji} = (m_i m_j (x_i - x_j))/r_{ij}^3$ is the force on body j due to body i. Summing over j and dividing by the total mass gives

$$\lambda c = \frac{1}{m_0} \sum_{i<j} F_{ij}.$$

The terms in this sum cancel out in pairs because $F_{ij} = -F_{ji}$. Since $\lambda > 0$ we get $c = 0$ as required. □

The manifold $\{x : I_0(x) = 1\}$ is diffeomorphic to the sphere $\mathbf{S}^{dn-1}$ so this approach gives compactness without explicitly imposing the center of mass constraint. The critical points will automatically lie in our previous constraint manifold $\mathcal{N}$.

It is also possible to treat balanced configurations as critical points. Modify the vector version of the balance equation (2.37) by introducing a constant λ to get

$$\nabla U(x) + \lambda \hat{S} M(x - c) = 0. \tag{2.45}$$

Here $\lambda \in \mathbb{R}$ and $\hat{S} = \operatorname{diag}(S, \ldots, S)$ is a $dn \times dn$ block-diagonal matrix with identical $d \times d$ blocks S, the positive semi-definite, symmetric matrix from Proposition 2.6.7. We will call x an *S-balanced configuration (SBC)* if (2.45) holds for some λ. CC's are a special case with $S = I$. By putting a λ into (2.45) we can say that x and kx are both S-balanced. The equation is also invariant under translations but generally not invariant under rotations. In fact, the matrix S transforms under rotations and scalings via

$$S(kQx) = k^{-3} Q S Q^T.$$

In the CC case we have $S = I$ and we get rotation invariance. The other extreme would be that S has d distinct eigenvalues and then it is not stabilized by any rotation. By choosing an appropriate rotation Q we can get

$$QSQ^T = \operatorname{diag}(\sigma_1^2, \sigma_2^2, \ldots, \sigma_d^2).$$

It is no loss of generality to assume S positive definite since it is definite on $\mathcal{C}(x)$ and we could extend it arbitrarily on $\mathcal{C}(x)^\perp$.

To handle SBC's in a similar way to CC's, we will define an S-weighted moment of inertia. Assuming that S is positive definite, we can use it to define a new inner product and norm on $\mathbb{R}^d$,

$$\langle \xi, \eta \rangle_S = \xi^T S \eta, \quad |\xi|_S^2 = \xi^T S \xi.$$

Then set

$$I_S(x) = (x - c)^T \hat{S} M (x - c) = \sum_{j=1}^{n} m_j |x_j - c|_S^2.$$

As in the CC case, the constant λ in (2.45) is $\lambda = U(x)/I_S(x)$. Define the *S-normalized configuration space*

$$\mathcal{N}(S) = \{x : c = \hat{L}Mx = 0, I_S(x) = 1\}.$$

Then, as for CC's, we have:

Proposition 2.8.4. *A configuration vector x is a S-balanced configuration if and only if the corresponding normalized configuration is a critical point of $U(x)$ restricted to $\mathcal{N}(S)$.*

One of the main applications of the characterization of CC's and SBC's as critical points are the existence proofs. For example:

Corollary 2.8.5. *For every choice of masses $m_i > 0$ in the n-body problem in $\mathbb{R}^d$, there is at least one central configuration. For every choice of masses and every $d \times d$ positive definite symmetric matrix S, there exists at least one S-balanced configuration.*

Proof. It suffices to consider SBC's, since CC's are a special case. Note that $\mathcal{N}(S)$ is a compact submanifold of $\mathbb{R}^{dn}$. The Newtonian potential defines a smooth function $U\colon \mathcal{N}(S) \setminus \Delta \to \mathbb{R}$. The singular set $\mathcal{N}(S) \cap \Delta$ is compact and $U(x) \to \infty$ as $x \to \Delta$. It follows that U attains a minimum at some point $x \in \mathcal{N}(S) \setminus \Delta$ and this point will be an S-balanced configuration. □

Although restricting to the compact space $\mathcal{N}$ or $\mathcal{N}(S)$ is useful, there are a couple of alternative variational characterizations of CC's and SBC's as unconstrained critical points. The first version is obtained by normalizing the constant λ instead of the moment of inertia. For every solution of (2.42) or (2.45), there is a rescaled solution with $\lambda = k$, where $k > 0$ is any positive constant. If we choose $k = 2$ then this rescaled configuration will be a critical point of the function

$$F(x) = U(x) + I_S(x)$$

on $\mathbb{R}^{dn}$, i.e., with no constraint on x. Or, we can impose the linear constraint $c = 0$. Another variational approach is to avoid normalization altogether and look for critical points of the homogeneous function

$$G(x) = \sqrt{I_S(x)}\, U(x) \quad \text{or} \quad I_S(x)U(x)^2.$$

One can check that if x is a solution of (2.45) we get a ray of critical points kx, $k > 0$, for these functions.

In the CC case, the Newtonian potential determines a function on the quotient space

$$\mathcal{M} = (\mathcal{N} \setminus \Delta)/\operatorname{SO}(d).$$

However, for $d > 2$ the action of the rotation group is not free and the quotient space is not a manifold. We can get a manifold by restricting to the configurations of a given dimension.

An amusing application of the variational approach on a reduced space is the study of central configurations of maximal dimension. For any configuration of n-bodies, the centered position space has $\dim C(x) \leq n-1$. We will look for CC's with $\dim C(x) = n-1$.

Proposition 2.8.6. *The only central configuration of n-bodies with $\dim C(x) = n-1$ is the regular n-simplex and it is a central configuration for all choices of the masses.*

Proof. Without loss of generality we can consider the n-body problem in $\mathbb{R}^{n-1}$. The configuration space is $\mathbb{R}^{n(n-1)} \setminus \Delta$ and the centered configurations form a subspace of dimension $n(n-1)-(n-1) = (n-1)^2$. The subset of configurations with $\dim C(x) = n-1$ is an open subset. The rotation group $\mathrm{SO}(n-1)$ acts freely on this open set and we can look for critical points on the quotient space which will be a smooth manifold of dimension

$$(n-1)^2 - \frac{(n-1)(n-2)}{2} = \frac{n(n-1)}{2}.$$

The dimension suggests using the mutual distances $r_{ij}, 1 \leq i < j \leq n$, as local coordinates. We will look for unconstrained critical points of $U(x)+I(x)$, where we express both terms as functions of the r_{ij} using (2.2) and (2.9). We get

$$\frac{\partial U}{\partial r_{ij}} + \frac{\partial I}{\partial r_{ij}} = -\frac{m_i m_j}{r_{ij}^2} + \frac{2m_i m_j r_{ij}}{m_0} = 0.$$

The masses cancel out and the mutual distances are equal to $r_{ij}^3 = m_0/2$. □

The variational characterization suggests using the gradient flow of the Newtonian potential to understand central or balanced configurations. Generically, a smooth function on a smooth manifold is a Morse function, i.e., it has isolated critical points which are nondegenerate. Due to the rotational symmetry, critical points of $U|_{\mathcal{N}}$ will never be isolated for $d \geq 2$. One can try to eliminate the rotational symmetry or just work with the similarity classes of critical points. We can still hope for these classes to be isolated from one another or nondegenerate in some sense.

First we deal with another problematic aspect of the gradient flow, the lack of compactness. The manifold $\mathcal{N}(S)$ is compact, but the flow is only defined on the open subset $\mathcal{N}(S) \setminus \Delta$. The next result, known as Shub's lemma [32], shows that CC's and SBC's are bounded away from Δ.

Proposition 2.8.7. *For fixed masses $m_1, \ldots, m_n$ and a fixed positive definite symmetric matrix S, there is a neighborhood of Δ in $\mathcal{N}(S)$ which contains no S-balanced configurations.*

Proof. Otherwise, there would be some $\bar{x} \in \mathcal{N}(S) \cap \Delta$ and a sequence of SBC's $x^k \in \mathcal{N}(S)$ with $x^k \to \bar{x}$ as $k \to \infty$. The collision configuration $\bar{x}$ defines a partition of the bodies into *clusters*, where m_i, m_j are in the same cluster if $\bar{x}_i = \bar{x}_j$. For k

large, the bodies in each cluster will be close to each other but the clusters will be bounded away from one another.

Let $F_i(x^k) = \nabla_i U(x^k)$ be the force on the i-th body. Since x^k is a normalized SBC, we have

$$F_i = -\lambda_k m_i S x_i^k, \quad \lambda_k = U(x^k).$$

Let $\gamma \subset \{1, \ldots, n\}$ be the set of subscripts of one of the clusters. Then,

$$\sum_{i \in \gamma} F_i = -\lambda_k S \sum_{i \in \gamma} m_i x_i^k. \tag{2.46}$$

As $k \to \infty$, we have $\lambda_k = U(x^k) \to \infty$ since $\bar{x} \in \Delta$. On the other hand

$$S \sum_{i \in \gamma} m_i x_i^k \to S m_\gamma \bar{x}_\gamma,$$

where m_γ is the total mass of the cluster and $\bar{x}_\gamma$ is the common value of the limiting positions $\bar{x}_i, i \in \gamma$. We will show below that the left-hand side of (2.46) is bounded. It follows that we must have $\bar{x}_\gamma = 0$ for all of the clusters. In other words, there could be only one cluster and it would have to be at the origin. But this is impossible since $I_S(\bar{x}) = 1$.

To see that the left-hand side of (2.46) is bounded, we can split the sum as

$$\sum_{i \in \gamma} F_i = \sum_{\substack{i,j \in \gamma \\ i \neq j}} F_{ij} + \sum_{\substack{i \in \gamma \\ l \notin \gamma}} F_{il},$$

where $F_{ij} = (m_i m_j (x_j - x_i))/r_{ij}^3$ is the force on body i due to body j. The first sum is identically zero since $F_{ij} = -F_{ji}$, and the second is bounded by definition of cluster. □

It follows from Shub's lemma that if the similarity classes of CC's or SBC's are isolated then there are only finitely many of them. To see this, let U denote a neighborhood of Δ in $\mathcal{N}(S)$ which contains no SBC's. Since the complement $\mathcal{N}(S) \setminus U$ is compact, a hypothetical infinite sequence of distinct, similarity classes would have normalized representatives with a convergent subsequence. The limiting configuration would be a nonisolated SBC.

If we allow the masses to vary, it is possible to find a sequence of CC's, say $\bar{x}_k$, converging to Δ. This idea was used by Xia in [36], and further explored in [21]. The masses in each nontrivial cluster all tend to zero. The limiting shapes of the clusters are governed by equations similar to the CC equation.

It is interesting to classify CC's and SBC's by their Morse index. Recall that if x is a critical point of a smooth function V on a manifold $\mathcal{N}$, there is a *Hessian* quadratic form on the tangent space $T_x\mathcal{N}$ which is given in local coordinates by the symmetric matrix of second partial derivatives,

$$H(x)(v) = v^T D^2 V(x) v.$$

Alternatively, if $\gamma(t)$ is any smooth curve in $\mathcal{N}$ with $\gamma(0) = x$ and $\gamma'(0) = v$ then

$$H(x)(v) = \frac{1}{2}\frac{d^2}{dt^2}V(\gamma(t))|_{t=0}.$$

The *Morse index* ind(x) is the maximum dimension of a subspace of $T_x\mathcal{N}$ on which $H(x)$ is negative definite. The *nullity* is the dimension of

$$\ker H(x) = \{v : H(q)(v,w) = 0 \text{ for all } w \in T_x\mathcal{N}\},$$

where $H(x)(v,w) = v^T D^2V(x)w$ is the symmetric bilinear form associated to $H(x)$. We are interested in the function $V = U|_{\mathcal{N}(S)}$ given by restricting the Newtonian potential to the normalized configuration space.

Instead of working in local coordinates, we want to represent the Hessian by a $dn \times dn$ matrix, also called $H(x)$, whose restriction to $T_x\mathcal{N}(S)$ gives the correct values.

Proposition 2.8.8. *The Hessian of* $V : \mathcal{N}(S) \to \mathbb{R}$ *at a critical point* x *is given by* $H(x)(v) = v^T H(x)v$, *where* $H(x)$ *is the* $dn \times dn$ *matrix*

$$H(x) = D^2U(x) + U(x)\hat{S}M. \tag{2.47}$$

Proof. A critical point of V is also an unconstrained critical point of $G(x) = \sqrt{I_S(x)}U(x)$ in $\mathbb{R}^{dn}$. Since $G|_{\mathcal{N}(S)} = U|_{\mathcal{N}(S)} = V$, their Hessians on $T_x\mathcal{N}(S)$ agree.

To calculate D^2G first recall that $I_S(x) = x^T\hat{P}^T\hat{S}M\hat{P}x$. For any vector $w \in \mathbb{R}^{dn}$, we have

$$DI_S(x)w = 2x^T\hat{P}^T\hat{S}M\hat{P}w.$$

Hence,

$$DG(x)w = I_S(x)^{\frac{1}{2}}\, DU(x)w + I_S(x)^{-\frac{1}{2}}U(x)x^T\hat{P}^T\hat{S}M\hat{P}w.$$

We are only interested in computing $D^2G(x)(v,w)$ where $v, w \in T_x\mathcal{N}(S)$. In that case we have

$$I_S(x) = 1, \quad \hat{P}v = v, \quad x^T\hat{P}^T\hat{S}M\hat{P}v = 0,$$

and analogous equations for w. Differentiating G again and using these equations we get

$$D^2G(x)(v,w) = D^2U(x)(v,w) + U(x)v^T\hat{S}Mw,$$

as claimed. □

It is straightforward to calculate the $dn \times dn$ matrix $D^2U(x)$ with the result

$$D^2U(x) = \begin{bmatrix} D_{11} & D_{12} & \cdots & D_{1n} \\ D_{21} & D_{12} & \cdots & D_{2n} \\ \vdots & & & \vdots \end{bmatrix}, \tag{2.48}$$

where the $d \times d$ blocks are

$$D_{ij} = \frac{m_i m_j}{r_{ij}^3}\left(I - 3u_{ij}u_{ij}^T\right), \quad u_{ij} = \frac{x_i - x_j}{r_{ij}}, \quad \text{for } i \neq j,$$

and

$$D_{ii} = -\sum_{j \neq i} D_{ij}.$$

The following formula for the value of the Hessian quadratic form on a vector $v \in \mathbb{R}^{dn}$ is sometimes useful:

$$H(x)(v,v) = \sum_{i<j} \frac{m_i m_j}{r_{ij}^3}\left(-|v_{ij}|^2 + 3(u_{ij}\cdot v_{ij})^2\right) + U(q)v^T M v, \tag{2.49}$$

where $v_{ij} = v_i - v_j \in \mathbb{R}^d$.

As noted above, the rotational symmetry implies that CC's are always degenerate as critical points for $d \geq 2$. The following result describes the minimal degeneracy.

Proposition 2.8.9. *Let $x \in \mathcal{N}$ be a CC in $\mathbb{R}^d$. Then the nullity of x as a critical point $U|_{\mathcal{N}}$ satisfies*

$$\text{null}(x) \geq \frac{d(d-1)}{2} - \frac{k(k-1)}{2}, \quad k = d - \dim(x) = d - \dim \mathcal{C}(x). \tag{2.50}$$

Proof. The formula just gives the dimension of the subspace of $T_x\mathcal{N}$ consisting of tangent vectors to the action of the rotation group, i.e., the subspace

$$\{v = \alpha x : \alpha \in \text{so}(d)\}.$$

To see this, first note that the manifold $\mathcal{N}$ is rotation invariant. For any curve of rotations $Q(t) \in \text{SO}(d)$ with $Q(0) = I$, we have

$$\dot{Q}(t)x|_{t=0} = \alpha x \in T_x\mathcal{N}.$$

But x is stabilized by rotations which fix the subspace $\mathcal{C}(x)$. This stabilizer is isomorphic to the rotation group of the orthogonal complement $\mathcal{C}(x)^\perp$ which has dimension k. □

For SBC's the corresponding minimal nullity will depend on how the rotation group acts on the symmetric matrix S. If S has distinct eigenvalues, it is possible for SBC's to be nondegenerate. For example, recall that for masses $m_1 = m_2 = 1$ and $m_3 > 0$ any isosceles triangle is balanced with the eigenvalues of S varying with the shape. One can check using a computer that generic choices of isosceles shape lead to nondegenerate SBC's.

In all cases, it is natural to call a critical point nondegenerate if its nullity is as small as possible given the rotational symmetry.

Definition 2.8.10. *A CC or SBC in* $\mathbb{R}^d$ *is* nondegenerate *if the nullity of the corresponding critical point is as small as possible consistent with the rotational symmetry. For CC's this means that equality should hold in* (2.50).

For example in $\mathbb{R}^3$ a nondegenerate collinear CC has nullity 2, while nondegenerate planar and spatial CC's have nullity 3.

2.9 Collinear central configurations

The first central configurations were discover by Euler in 1767, see [11]. He studied the collinear three-body problem where he found collinear central configurations and the corresponding homothetic motions. Moulton investigated the central configurations of the collinear n-body problem in 1910, see [25]. The results are definitive in contrast to the state of the theory for $d \geq 2$. This section is devoted to proving Moulton's theorem:

Proposition 2.9.1 (Moulton's Theorem)**.** *Given masses* $m_i > 0$*, there is a unique normalized collinear central configuration for each ordering of the masses along the line.*

Note that when $d = 1$ there is no difference between CC's and SBC's due to the lack of variety in 1×1 symmetric matrices.

It is instructive to start with Euler's case $n = 3$. The normalized configuration space

$$\mathcal{N} = \{x \in \mathbb{R}^3 : m_1x_1 + m_2x_2 + m_3x_3 = 0,\ \ m_1x_1^2 + m_2x_2^2 + m_3x_3^2 = m_0\}$$

is the curve of intersection of a plane and an ellipsoid. The collision set consists of three planes:

$$\Delta = \{x_1 = x_2\} \cup \{x_1 = x_3\} \cup \{x_2 = x_3\}$$

which divide the curve into six arcs corresponding to the different orderings of the three masses along the line (see Figure 2.6). Since $U \to \infty$ at these points, there must be at least one critical point in each of the arcs. To see that there is only one requires more work.

The three mutual distances provide convenient coordinates, but we need to subject them to a collinearity constraint. If we fix the ordering of the bodies to be $x_1 < x_2 < x_3$ then the constraint is $r_{12} + r_{23} - r_{13} = 0$. Looking for critical points of the homogeneous function $F = U(r_{ij})^2 I(r_{ij})$ with this constraint, and then normalizing by setting $r_{12} = r, r_{13} = 1, r_{23} = 1 - r$ gives a degree five polynomial equation for r:

$$\begin{aligned}(m_2 + m_3)r^5 + (2m_2 + 3m_3)r^4 + (m_2 + 3m_3)r^3 \\ - (3m_1 + m_2)r^2 - (3m_1 + 2m_2)r - (m_1 + m_2) = 0.\end{aligned} \tag{2.51}$$

Fortunately there is a single sign change so Descartes' rule of signs implies there is a unique positive real root. Of course, there is no simple formula for how this

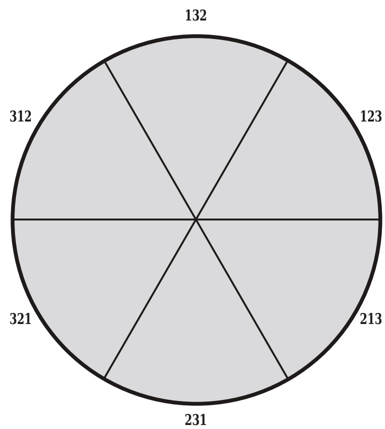

Figure 2.6: $\mathcal{N}$ for the collinear three-body problem is the boundary circle of the shaded disk which represents the set $I \leq 1$ in the plane of centered configurations. Δ intersects this plane in three lines which divide the circle into six arcs, one for each ordering of the bodies along the line.

root changes as a function of the masses. Euler's example is a shot over the bow about the CC equation. Even in the simplest nontrivial case, finding CC's for given masses involves solving complicated polynomial equations. Figure 2.7 shows a surface defined by Euler's quintic when one of the masses is normalized to 1. The surface lies over the mass plane in a complicated way making the uniqueness result for fixed positive masses all the more remarkable.

Before moving on to the proof of Moulton's theorem we will have a look at the geometry of the next case, $n = 4$. This time $\mathcal{N}$ is the intersection of a hyperplane and an ellipsoid in $\mathbb{R}^4$. So it is a two-dimensional surface diffeomorphic to $\mathbf{S}^2$. There are six collision planes which divide the sphere into $4! = 24$ triangles. Figure 2.8 shows how the collision planes divide the sphere.

Proof of Moulton's Theorem. The collision set Δ divides the ellipsoid $\mathcal{N}$ of normalized centered configurations into $n!$ components, one for each ordering of the bodies along the line. Let $\mathcal{V}$ denote any one of these components, an open set whose boundary is contained in Δ. The Newtonian potential gives a smooth function $U|_{\mathcal{V}} \colon \mathcal{V} \to \mathbb{R}$, and $U(x) \to \infty$ as $x \to \partial\mathcal{V}$. Hence $U|_{\mathcal{V}}$ attains its minimum at some $x_0 \in \mathcal{V}$ and x_0 is a CC with the given ordering of the bodies along the line.

Instead of working on the normalized space where $I(x) = 1$ we can study the function $F(x) = U(x) + I(x)$ on the cone $\tilde{\mathcal{V}}$ of all rays through the origin passing through $\mathcal{V}$ (in Figure 2.6 this would be an infinite triangular wedge based on one of the six arcs). Let $x, y \in \tilde{\mathcal{V}}$ and consider a line segment $p(t) = (1-t)x + ty$, $0 \leq t \leq 1$. Note that since the ordering is fixed, the sign of $p_i(t) - p_j(t) = (1-t)(x_i - x_j) + t(y_i - y_j)$ is equal to the common sign of $x_i - x_j$ and $y_i - y_j$. It

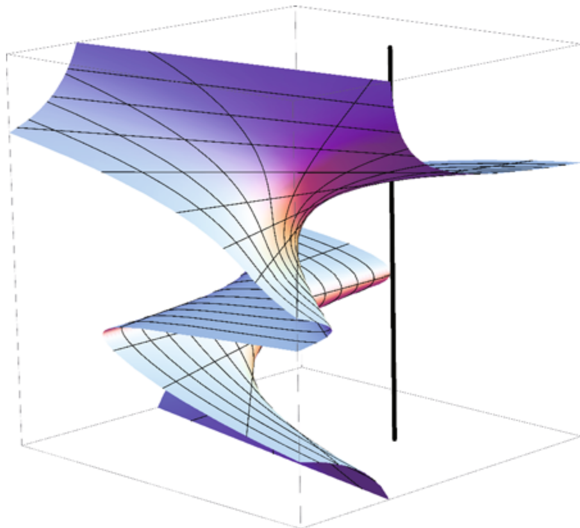

Figure 2.7: Surface defined by Euler's quintic equations in the product space of masses and configurations. Two mass parameters (horizontal) and one configuration variable r (vertical). Fixing the masses means looking for intersections of the surface with a vertical fiber, here a line segment. For positive masses, the segment cuts the surface just once.

follows that $p(t) \in \tilde{\mathcal{V}}$ for all t and so $\tilde{\mathcal{V}}$ is a convex set. We will show that if $x \neq y$ then $F(p(t))$ has a strictly positive second derivative. It follows that x, y cannot both be critical points of $F(x)$.

First consider $F(r_{ij})$ as a function of the mutual distances r_{ij} on $(\mathbb{R}^+)^{\frac{n(n-1)}{2}}$. We have

$$\frac{\partial^2 F}{\partial r_{ij}^2} = \frac{2m_i m_j}{r_{ij}^3} + \frac{2m_i m_j}{m_0} > 0.$$

Now since the configurations x, y are collinear, the mutual distances reduce to $r_{ij}(t) = |p_i(t) - p_j(t)|$ and, as the ordering is constant along the segment, this is a linear function of t. It follows that $F(p(t))''$ is a sum of terms

$$\frac{\partial^2 F}{\partial r_{ij}^2}(p(t)) \left(r'_{ij}(t)\right)^2.$$

These terms are all nonnegative and at least one is positive if $x \neq y$. □

Next we will take a look at the Hessian $H(x)$ of a collinear CC. Using the rotation invariance of U we get

$$H(Qx) = Q^T H(x) Q,$$

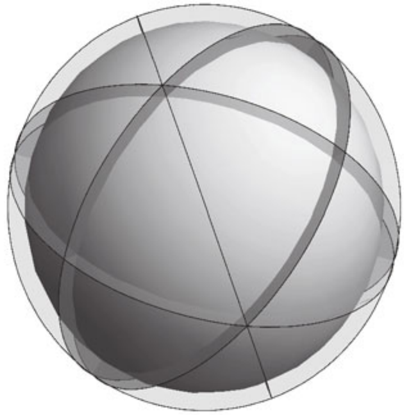

Figure 2.8: $\mathcal{N}$ for the collinear four-body problem. The collision planes divide the sphere into triangles representing the possible orderings of the bodies.

where $H(x)$ is given by (2.47) and $Q \in \mathrm{SO}(d)$ is any rotation. It follows that the index and nullity are unchanged by such rotations. If x is collinear, we can therefore assume that all of the bodies have positions $x_j \in \mathbb{R}^1 \times \{0\}^{d-1} \subset \mathbb{R}^d$. Then the unit vectors u_{ij} appearing in the formula (2.48) are all multiples of $e_1 = (1, 0, \ldots, 0)$. It follows that if we permute the components of configuration vectors into groups of n with all of the e_1 components first, the e_2 components next, etc., then $D^2U(x)$ will have a block-diagonal form

$$D^2U(x) = \mathrm{diag}(-2\tilde{A}, \tilde{A}, \ldots, \tilde{A}),$$

where

$$\tilde{A} = \begin{bmatrix} \tilde{A}_{11} & \frac{m_1 m_2}{r_{12}^3} & \cdots & \frac{m_1 m_n}{r_{1n}^3} \\ \frac{m_1 m_2}{r_{12}^3} & \tilde{A}_{22} & \cdots & \frac{m_2 m_n}{r_{2n}^3} \\ \vdots & & & \vdots \\ \frac{m_1 m_n}{r_{1n}^3} & \frac{m_2 m_n}{r_{2n}^3} & \cdots & \tilde{A}_{nn} \end{bmatrix}, \quad \tilde{A}_{jj} = -\sum_{i \neq j} \tilde{A}_{ij} = -\sum_{i \neq j} \frac{m_i}{r_{ij}^3}.$$

Note that $\tilde{A}$ is just the symmetric matrix $A(X)M$ from Section 2.4.

Let $v = (\xi_1, \xi_2, \ldots, \xi_d)^T$ denote a vector in $\mathbb{R}^{dn}$ with its coordinates permuted into groups of n as described above. Vectors of the form $v = (\xi_1, 0, \ldots, 0)^T$ will be called collinear vectors and those of the form $v = (0, \xi_2, \ldots, \xi_n)^T$ normal vectors. We are interested in the tangent space $T_x\mathcal{N}$ to the normalized configuration space. With these coordinates the center of mass subspace, $\ker \hat{L}M$, is given by

$$m \cdot \xi_i = 0, \quad i = 1, \ldots, d,$$

where $m \in \mathbb{R}^n$ is the mass vector. Since x is collinear, the equation $DI(x)v = 0$ affects only the first vector ξ_1:

$$m_1 x_{11}\xi_{11} + \cdots + m_n x_{n1}\xi_{1n} = 0.$$

Finally, the action of the rotation group leads to a $(d-1)$-dimensional subspace of vectors in the kernel of the Hessian. A basis is $\omega_2(x), \ldots, \omega_d(x)$, where $\omega_i(x)$ is the vector whose i-th group of n coordinates is the vector of first coordinates of the configuration, $(x_{11}, x_{21}, \ldots, x_{n1})$. For example, $\omega_2(x)$ is the tangent vector at x in the direction of a rotation in the $(1,2)$-coordinate plane.

Proposition 2.9.2. *Every collinear central configuration in* $\mathbb{R}^d$ *is nondegenerate with* $\operatorname{null}(x) = d-1$ *and* $\operatorname{ind}(x) = (d-1)(n-2)$*. In the collinear tangent directions,* $H(x)$ *is positive definite while in the normal directions it is negative semi-definite.*

Proof. We will analyze the Hessian block-by-block. The first block of the Hessian corresponds to the collinear directions and we have

$$\xi^T H(x)\xi = -2\xi^T \tilde{A}\xi + U(x)\xi^T M\xi,$$

where M is the $n \times n$ version of the mass matrix. We showed in Section 2.4 that the matrix $\tilde{A} = AM$ is negative semi-definite, so both terms here are nonnegative and the second is strictly positive for nonzero vectors. Therefore the collinear part of the Hessian is positive definite.

For each of the other blocks we have

$$\xi^T H(x)\xi = \xi^T \tilde{A}\xi + U(x)\xi^T M\xi.$$

The terms are of different signs and it is a subtle problem to see which is dominant. The following proof, due to Conley, appears in [27].

Instead of finding the index and nullity of $H(x)$ we will find the number of negative and zero eigenvalues of the linear map with matrix

$$M^{-1}H(x) = M^{-1}\tilde{A} + U(X)I.$$

It is possible to guess two eigenvalues and eigenvectors. Let $u_1 = \begin{bmatrix}1 & \cdots & 1\end{bmatrix}^T$. Since the row sums of $\tilde{A}$ are zero we have

$$M^{-1}Hu_1 = \lambda_1 u_1, \quad \lambda_1 = U(x) > 0.$$

However, this vector is orthogonal to the zero center of mass subspace so is not relevant for our index and nullity computation. Next we have $u_2 = x = \begin{bmatrix}x_1 & \cdots & x_n\end{bmatrix}^T$, where we have simplified the notation so $x_i \in \mathbb{R}$ denotes the position of the i-th body along the line. Then a short computation gives

$$M^{-1}\tilde{A}u_2 = M^{-1}\nabla U(x),$$

where ∇ is the gradient in $\mathbb{R}^n$. Since x is a normalized CC we have $M^{-1}\nabla U(x) = -U(x)x = -U(x)u_2$ and so

$$M^{-1}Hu_2 = (M^{-1}\tilde{A} + U(x)I)u_2 = -U(x)u_2 + U(x)u_2 = 0.$$

In other words u_2 is an eigenvector with eigenvalue $\lambda_2 = 0$. We have one such null vector for each of the last $d-1$ blocks. Note that u_2 is the vector $\omega_i(x)$ tangent to the rotation group action. If we can show that the other $n-2$ eigenvalues of $M^{-1}H$ are strictly negative, the proposition will be proved.

Conley's proof uses the dynamics of the linear flow of the differential equation

$$\dot{\xi} = M^{-1}\tilde{A}\xi.$$

Every linear flow determines a flow on the space of lines through the origin, and the eigenvector lines are exactly the equilibrium points. Moreover the equilibrium corresponding to the largest eigenvector is an attractor for this projectivized flow. If we can show that the line of the eigenvector $u_2 = x$ is an attractor, then it follows that all of the other eigenvalues of $M^{-1}\tilde{A}$ are strictly less than $-U(x)$ and so all of the other eigenvalues of $M^{-1}H(x)$ are negative.

Suppose that the ordering of the bodies along the line is $x_1 < x_2 < \cdots < x_n$. Define a cone in the zero center of mass subspace by

$$K = \{\xi : m \cdot \xi = 0, \xi_1 \leq \xi_2 \leq \cdots \leq \xi_n\}.$$

This cone contains the line spanned by the eigenvector u_2 in its interior and does not contain any two-dimensional subspaces. We will show that the flow carries K strictly inside itself. It follows that for the projectivized flow, u_2 is an attractor.

Now the boundary of K is the set where one or more of the inequalities in the definition is an equality. Consider a boundary point where, for some $i < j$, we have

$$u_{i-1} \leq u_i = \cdots = u_j \leq u_{j+1}.$$

The differential equation gives

$$\dot{u}_i = \sum_{k \neq i} \frac{m_k}{r_{ik}^3}(u_k - u_i), \quad \dot{u}_j = \sum_{k \neq j} \frac{m_k}{r_{jk}^3}(u_k - u_j).$$

Since $u_i = u_j$ the difference of these can be written:

$$\dot{u}_j - \dot{u}_i = \sum_{k \neq i,j} m_k(u_k - u_i)\left[\frac{1}{r_{jk}^3} - \frac{1}{r_{ik}^3}\right].$$

Every term in this sum is nonnegative:

$$\text{if } k < i, \quad u_k - u_i \leq 0 \ \text{ and } \ \tfrac{1}{r_{jk}^3} - \tfrac{1}{r_{ik}^3} < 0;$$

$$\text{if } i < k < j, \quad u_k - u_i = 0;$$

$$\text{if } j < k, \quad u_k - u_i \geq 0 \ \text{ and } \ \tfrac{1}{r_{jk}^3} - \tfrac{1}{r_{ik}^3} > 0.$$

Moreover, not all of the terms can vanish since otherwise u would be a multiple of $\begin{bmatrix}1 & \cdots & 1\end{bmatrix}^T$, which is not in the zero center of mass space. It follows that at this boundary point $\dot{u}_j - \dot{u}_i > 0$ so the point moves strictly inside the cone under the linear flow. It follows that the line determined by u_2 is an attractor, as required. □

2.10 Morse indices of non-collinear central configurations

Unfortunately, much less is known about the Morse indices of non-collinear CC's. The following result gives a weak lower bound on the index which, at least, shows that a minimum must have the maximum possible dimension.

Proposition 2.10.1. *Suppose x is a central configuration of the n-body problem in $\mathbb{R}^d$ with $\dim(x) < \min(d, n-1)$. Then the Morse index of the corresponding critical point satisfies $\operatorname{ind}(x) \geq d - \dim(x)$. In particular, the critical point is not a local minimum of $U|_{\mathcal{N}}$.*

As a corollary we get the existence of CC's of the n-body problem of all possible dimensions.

Corollary 2.10.2. *For the n-body problem in $\mathbb{R}^d$ and for any k with $1 \leq k \leq \min(d, n-1)$ there exists at least one central configuration with $\dim(x) = k$.*

Proof. We have seen that $U|_{\mathcal{N}}$ achieves a minimum at some CC x, and it follows from the proposition that $\dim(x) = \min(d, n-1)$. If $1 \leq k < \min(d, n-1)$ then we can further restrict U to a subspace of $\mathbb{R}^d$ of dimension k and get a CC of dimension $\min(k, n-1) = k$. □

Proof of Proposition 2.10.1. If $\dim(x) = k < \min(d, n-1)$ we can assume that all of the bodies have position vectors $x_j \in \mathcal{W} = \mathbb{R}^k \times \{0\}^{d-k}$. As in the last section we get a block decomposition of the Hessian $D^2U(x) = \operatorname{diag}(D^2(U|_{\mathcal{W}}), \tilde{A}, \ldots, \tilde{A})$, where $D^2(U|_{\mathcal{W}})$ is the $nk \times nk$ tangential part, and where there are $d-k$ copies of the familiar $n \times n$ block $\tilde{A}$. We will show that the matrix $M^{-1}\tilde{A} + U(x)I$ has at least one negative eigenvalue whose eigenvector has zero center of mass. Since the eigenvalue in the u_1-direction normal to the center of mass subspace is $\lambda_1 = U(x)$, it suffices to show that $\operatorname{tr}(M^{-1}\tilde{A} + U(x)I) < -U(x)$ or, equivalently,

$$\tau = -\operatorname{tr} M^{-1}\tilde{A} > (n-1)U(x).$$

Now

$$\tau = \sum_i \sum_{j \neq i} \frac{m_j}{r_{ij}^3} = \sum_{\substack{(i,j)\\ i<j}} \frac{m_i + m_j}{r_{ij}^3}.$$

The problem, of course, is that we do not have much control over the mutual distances. All we know is that we are at some CC. The following approach is due to Albouy [1].

We will use the reduced version of the CC equation (2.29). Viewing B as a bilinear form on the hyperplane $\mathcal{D}^*$, we can use the matrix representative $\hat{B}$ from (2.27). For each pair of standard basis vectors in $\mathbb{R}^n$, e_i, e_j, $i < j$, we have $e_i - e_j \in \mathcal{D}^*$. From (2.29),

$$(e_i - e_j)^T(\hat{B}A + \lambda\hat{B})(e_i - e_j) = 0, \quad i < j.$$

We have $(e_i - e_j)^T\hat{B}(e_i - e_j) = r_{ij}^2$. The other term is more complicated but with some effort we arrive at

$$2\lambda = \frac{2(m_i + m_j)}{r_{ij}^3} + \sum_{k\neq i,j} m_k\left(\frac{1}{r_{ik}^3} + \frac{1}{r_{jk}^3}\right) + \sum_{k\neq i,j} m_k(r_{ik}^2 - r_{jk}^2)\left(\frac{1}{r_{ik}^3} - \frac{1}{r_{jk}^3}\right).$$

Note that the two parentheses in the last sum always have opposite signs unless they are both zero. So the sum is strictly negative unless all of the mutual distances are equal. However, this would mean that the configuration was the regular simplex with $\dim(x) = n - 1$. By hypothesis, this is not the case, so we can drop the last sum to get a strict inequality. Summing this inequality over all pairs $i < j$ gives

$$n(n-1)\lambda < n\tau.$$

Since x is a normalized CC, we have $\lambda = U(x)$ and this is exactly the inequality we need. □

Upper bounds on the index are also of interest. For planar central configurations we have the following result of Palmore which shows that the collinear CC's have the maximum possible index.

Proposition 2.10.3. *If x is a central configuration of the n-body problem in $\mathbb{R}^2$, then* $\operatorname{ind}(x) \leq n - 2$.

Proof. For the planar problem, the dimension of the normalized configuration space is $\dim\mathcal{N} = 2n - 3$. The tangent space $T_x\mathcal{N}$ is given by

$$x^T Mv = 0, \quad \hat{L}Mv = 0,$$

where $\hat{L}$ is the $2 \times 2n$ matrix consisting of n copies of the 2×2 identity matrix.

Let j be the rotation of the plane by $\pi/2$ and let it act on vectors $v = (v_1, \ldots, v_n) \in \mathbb{R}^{2n}$ by $jv = (jv_1, \ldots, jv_n)$, as usual. The vector $v_0 = jx$ is in the tangent space and is tangent to the action of the rotation group SO(2) so $v_0 \in \ker H(x)$. The orthogonal complement $v_0^\perp$ is a $(2n-4)$-dimensional subspace of $T_x\mathcal{N}$ and is invariant under the action of j.

For $v \in T_x\mathcal{N}$, it turns out that $H(x)(v, v) + H(x)(jv, jv) > 0$. To see this we will use formula (2.49). The inner product terms are

$$3(u_{ij} \cdot v_{ij})^2 + 3(u_{ij} \cdot jv_{ij})^2 = 3|v_{ij}|^2$$

since the vectors u_{ij} and ju_{ij} form an orthonormal basis for $\mathbb{R}^2$. Then (2.49) gives

$$H(x)(v,v) + H(x)(jv,jv) = \sum_{i<j} \frac{m_i m_j}{r_{ij}^3} \left(|v_{ij}|^2\right) + 2U(q)v^T M v > 0.$$

Suppose $S \subset T_x\mathcal{N}$ is a maximal subspace on which $H(x)$ is negative semidefinite. We may as well assume that $S \subset v_0^\perp$. From the positivity of $H(x)(v,v) + H(x)(jv,jv)$ it follows that we must have $S \cap jS = \{0\}$ and hence $\operatorname{ind}(x) = \dim S \leq n-2$. □

For $d = 3$ it is known, at least, that $U|_\mathcal{N}$ does not have any local maxima. See [20, 23] for these results. I don't know if this is still true for $d > 3$.

2.11 Morse theory for CC's and SBC's

In this section we will describe how to use Morse theory to prove existence of CC's. This approach was initiated by Smale [34] and developed by Palmore [28] for the planar n-body problem, and then extended to three dimensions using equivariant Morse theory by Pacella [27]. An alternative approach to the three-dimensional case is due to Merkel [19].

Recall that central configurations in $\mathbb{R}^d$, $d \geq 2$, correspond to degenerate critical points of $U|_\mathcal{N}$ due to the action of the symmetry group $\mathrm{SO}(d)$. In the planar case, $\mathrm{SO}(2) \simeq \mathbf{S}^1$ acts freely on $\mathcal{N} \setminus \Delta$ and we can think of U as a smooth function on the quotient manifold

$$\mathcal{M} = (\mathcal{N} \setminus \Delta)/\,\mathrm{SO}(2).$$

We can still define such a quotient space when $d > 2$ but, due to the non-free action of $\mathrm{SO}(d)$, it will not be a manifold. In Section 2.8, we defined the concept of nondegeneracy for CC's with the symmetry group in mind so, using this terminology, a nondegenerate CC of the planar n-body problem determines a nondegenerate critical point in the manifold $\mathcal{M}$.

A generic smooth function on a manifold is a *Morse function*, that is, all of its critical points are nondegenerate. But it is difficult to actually verify this for particular functions like the Newtonian potential. From Proposition 2.9.2 we know that the collinear CC's are nondegenerate.

When $n = 3$ the only non-collinear CC's are the equilateral triangles and these are nondegenerate. The same holds for the regular simplex in the n-body problem.

Proposition 2.11.1. *For every choice of n positive masses, the regular simplex is a nondegenerate central configuration. It is a nondegenerate minimum of the potential in the quotient space $\mathcal{M}$.*

Proof. Suppose $d = n-1$. As noted above, $\mathrm{SO}(d)$ acts freely on the open subset of $\mathbb{R}^{n(n-1)} \setminus \Delta$ consisting of configurations with $\dim(x) = n-1$, and we can use the mutual distances r_{ij} as local coordinates in the corresponding open subset of the quotient space under rotations and translations. In these coordinates, the matrix of second derivatives of $F = I + U$ is diagonal and the partial derivatives $\partial^2 F/\partial r_{ij}^2$ are all positive.

Now suppose we have a curve $\gamma(t)$ of normalized configurations passing through the regular simplex when $t = 0$, and whose tangent vector $\gamma'(0)$ is not in the direction of the rotational symmetry. We would like to show that $U(\gamma(0))'' > 0$. The corresponding curve of mutual distances $r_{ij}(t)$ passes through the equal-distance point corresponding to the normalized regular simplex and we have $F(r_{ij}(t)) = 1 + U(\gamma(t))$. From the discussion in the previous paragraph we have $U(\gamma(0))'' = F(r_{ij}(0))'' > 0$, as required. □

It follows that for the planar three-body problem and for all choices of the three masses, the Newtonian potential determines a Morse function on $\mathcal{M}$. The space of normalized triangles is a three-dimensional ellipsoid. The quotient space under the rotation group is diffeomorphic to $\mathbf{S}^2$ and is called the *shape sphere* since it represents all possible shapes of triangles in the plane up to translation, rotation and scaling. $\mathcal{M}$ is the shape sphere with three collision shapes deleted. Figure 2.9 shows the level curves of the potential for two choices of the masses. The poles represent the equilateral triangles which are minima. On the equator, which represents the collinear shapes, there are the three collinear central configurations found by Euler, which are saddle points.

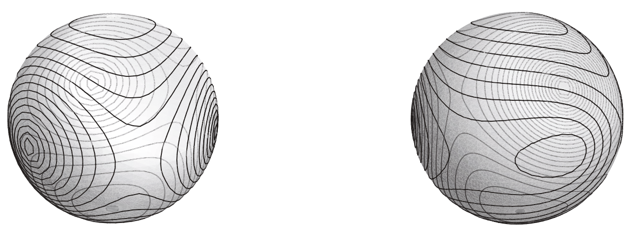

Figure 2.9: $\mathcal{M}$ for the planar three-body problem is the shape sphere. The Newtonian potential determines a Morse function with five critical points, shown here for the case of equal masses (left) and masses 1, 2, and 10 (right).

For $n > 3$, $d \geq 2$, it is much harder to check whether the critical points are nondegenerate. For the planar four-body problem Palmore showed that degenerate central configurations can occur for some choices of the masses and this is related to bifurcations in the number of central configurations as the masses are varied. Simó investigated the bifurcations numerically [33]. In Section 2.14 we will show that for generic choices of the masses in the planar four-body problem the potential determines a Morse function.

Now we will see what Morse theory tells us about the number of central configurations in the plane, taking the nondegeneracy of the critical points as an assumption. Morse theory is based on the gradient flow induced by a function on a Riemannian manifold. In our case the manifold is the quotient manifold $\mathcal{M}$, where we can use the restriction of the mass inner product as the Riemannian metric. First consider the gradient flow on $\mathcal{N} \setminus \Delta$. If the masses are fixed, Shub's lemma allows us to restrict to a compact set of the form $K = \{x \in \mathcal{N} : U(x) \leq U_0\}$ for some sufficiently large U_0. By definition, the gradient vector field of $U|_{\mathcal{N}}$ with respect to an inner product is the unique tangent vector field $\tilde{\nabla} U(x)$ with the property

$$\langle \tilde{\nabla} U(x), W \rangle = DU(x)W, \quad W \in T_x\mathcal{N}.$$

Using the mass inner product $\langle \xi, \eta \rangle = \xi^T M \eta$, one can check that the gradient vector field is the restriction of

$$\tilde{\nabla} U(x) = M^{-1} \nabla U(x) + U(x)x$$

to $\mathcal{N}$. By rotation invariance, this vector field determines a gradient flow on the quotient space $\mathcal{M}$. Orbits of the gradient flow cross the level sets of U orthogonally in the direction of increasing U. Orbits starting in the compact set K will continue to exist at least until they reach the exit level $U = U_0$.

The Morse inequalities relate the indices of the critical points of a Morse function on a manifold $\mathcal{M}$ to the topology of the manifold. They are most easily expressed in terms of polynomial generating functions. Define the *Morse polynomial* as

$$M(t) = \sum_k \gamma_k t^k,$$

where γ_k is the number of critical points of index k, and the *Poincaré polynomial* as

$$P(t) = \sum_k \beta_k t^k,$$

where β_k is the k-th Betti number of the manifold, i.e., the rank of the homology group $H_k(\mathcal{M}, \mathbb{R})$ with real (or rational) coefficients. Then the Morse inequalities can be written

$$M(t) = P(t) + (1+t)R(t), \tag{2.52}$$

where $R(t)$ is some polynomial with nonnegative integer coefficients. In particular, the Betti number β_k is a lower bound on the number of critical points of index k.

It turns out that the manifold $\mathcal{M}$ has a complicated topology so the Morse inequalities give interesting results. Recall that, for the n-body problem in $\mathbb{R}^d$, the space $\mathcal{N}$ of normalized configurations is an ellipsoid of dimension $d(n-1)-1$. It is the deletion of collision set Δ which produces the topological complexity.

Proposition 2.11.2. *For the n-body problem in $\mathbb{R}^d$, the Poincaré polynomial of $\mathcal{N} \setminus \Delta$ is*

$$\tilde{P}(t) = (1 + t^{d-1})(1 + 2t^{d-1}) \cdots (1 + (n-1)t^{d-1}).$$

In particular, for the planar three-body problem we have

$$\tilde{P}(t) = (1 + t)(1 + 2t) = 1 + 3t + 2t^2.$$

Proof. It suffices to find the Betti number of the unnormalized space $\mathbb{R}^{dn} \setminus \Delta$. To do this, note that the normalization of the center of mass and moment of inertia gives a diffeomorphism

$$\mathbb{R}^{dn} \setminus \Delta \simeq \mathbb{R}^d \times \mathbb{R}^+ \times (\mathcal{N} \setminus \Delta).$$

Now Künneth's theorem from algebraic topology shows that the Poincaré polynomial of a product space is the product of the Poincaré polynomials of the factors. Here, the first two factors are homologically trivial with Poincaré polynomials equal to 1.

The computation for $\mathbb{R}^{dn} \setminus \Delta$ is by induction on n. For $n = 1$ we have

$$\mathbb{R}^d \setminus \Delta = \mathbb{R}^d \setminus \{0\} \simeq \mathbb{R}^+ \times \mathbf{S}^{d-1}$$

and we get the Poincaré polynomial of a sphere, $\tilde{P}(t) = 1 + t^{d-1}$. For $n > 1$ we have a fiber bundle $\pi\colon \mathbb{R}^{dn} \setminus \Delta \to \mathbb{R}^{d(n-1)} \setminus \Delta$ where the projection just forgets the n-th body, $\pi(x_1, \ldots, x_n) = x_1, \ldots, x_{n-1})$. The fiber over a point $(x_1, \ldots, x_{n-1})$ is $\mathbb{R}^d \setminus \{n - 1 \text{ points}\}$ because the n-th body must avoid the other $n - 1$. Now this fiber bundle is not a product but it does satisfy certain topological conditions which guarantee that the Poincaré polynomials multiply. First, there is a cross-section map $\sigma\colon \mathbb{R}^{d(n-1)} \setminus \Delta \to \mathbb{R}^{dn} \setminus \Delta$ with $\pi \circ \sigma = id$. For example, we could let the n-th body of $\sigma(x_1, \ldots, x_{n-1})$ be at the point obtained by translating the barycenter of the other $n-1$ bodies a distance greater than the maximum distance between these bodies in the direction of the first coordinate axis. In addition, the fundamental group of the base acts trivially on the fiber (for $d \neq 2$ the base is simply connected). In any case, we go from the Poincaré polynomial for $(n - 1)$ bodies to the polynomial for n bodies by multiplying by the Poincaré polynomial of the fiber, namely $1 + (n - 1)t^{d-1}$. □

Next we restrict attention to the planar problem and pass to the quotient space $\mathcal{M}$ under the $\mathbf{S}^1$ action. The image of the normalized space $\mathcal{N} \simeq \mathbf{S}^{2n-3}$ is diffeomorphic to the complex projective space $\mathbb{CP}(n - 2)$ and the projection is a nontrivial circle bundle. But when we delete the collision set, the bundle becomes trivial. For example, there is a global cross-section to the circle action consisting of all noncollision configurations where the vector from x_1 to x_2 is the direction of the positive first coordinate axis. It follows that, in the planar case,

$$\mathcal{N} \setminus \Delta \simeq \mathbf{S}^1 \times \mathcal{M}.$$

Proposition 2.11.3. *For the n-body problem in $\mathbb{R}^2$, the Poincaré polynomial of the rotation reduced, normalized configuration space is*

$$P(t) = (1+2t)\cdots(1+(n-1)t).$$

Proof. Since $\mathcal{N} \setminus \Delta$ is the product of a circle and $\mathcal{M}$, we have $\tilde{P}(t) = (1+t)P(t)$. Then Proposition 2.11.2 with $d = 2$ gives the result. □

For example when $n = 3, 4$ we have, respectively,

$$P(t) = 1 + 2t, \quad P(t) = (1+2t)(1+3t) = 1 + 5t + 6t^2.$$

For $n = 3$, the Betti numbers $\beta_0 = 1$ and $\beta_1 = 2$ describe the homology of the shape sphere with the three collision points deleted which is diffeomorphic to the twice punctured plane.

To apply the Morse inequalities to the planar n-body problem first note that we have, after quotienting by rotations, $n!/2$ collinear central configurations. By Proposition 2.9.2, these have Morse index $n-2$. The next result, due to Palmore, uses this information to good effect.

Proposition 2.11.4. *Suppose that all of the central configurations are nondegenerate for a certain choice of masses in the planar n-body problem. Then there are at least*

$$\frac{(3n-4)(n-1)!}{2}$$

central configurations, of which at least

$$\frac{(2n-4)(n-1)!}{2}$$

are non-collinear.

Proof. The simplest lower bound on the number of critical points is obtained by setting $t = 1$ in (2.52):

$$\sum_k \gamma_k \geq \sum_k \beta_k = P(1) = \frac{n!}{2}.$$

But the information about the collinear configurations mentioned above shows that in the Morse polynomial, we have $\gamma_{n-2} \geq n!/2$. On the other hand, the coefficient of t^{n-2} in the Poincaré polynomial $P(t)$ is $\beta_{n-2} = 2 \cdot 3 \cdots (n-1) = (n-1)!$.

Let $R(t) = \sum_k r_k t^k$ be the residual polynomial in the Morse inequality (2.52). Then we have

$$r_{n-2} + r_{n-3} \geq \frac{n!}{2} - (n-1)!.$$

Setting $t = 1$ in (2.52) now gives

$$\sum_k \gamma_k \geq \frac{n!}{2} + 2(r_{n-2} + r_{n-3}) \geq \frac{3n!}{2} - 2(n-1)! = \frac{(3n-4)(n-1)!}{2}.$$

Subtracting $n!/2$ gives the non-collinear estimate. □

For example, when $n = 3$ the Morse estimate gives five critical points, which is exactly right. For $n = 4$ we have at least 24 CC's of including the 12 collinear ones, assuming nondegeneracy. The estimates increase rapidly with n —we expect there to be many CC's.

In the nonplanar case, the reduction of symmetry is more complicated and the quotient space is not a manifold. See [19, 27] for two approaches to the spatial case. We also mention the paper of McCord [18] which gives estimates based on Lyusternik–Schnirelmann theory instead of Morse theory.

Instead of pursuing this, we will just make a few remarks on what Morse theory can tell us about balanced configurations. Recall that these also admit a variational characterization as critical points of $U|_{\mathcal{N}(S)}$, where $\mathcal{N}(S)$ is the space of normalized configurations with respect to the metric based on the symmetric matrix S, $\langle \xi, \eta \rangle = \xi^T \hat{S} M \eta$. Now if we fix a symmetric matrix S with distinct eigenvalues, there is no longer any rotational symmetry and we can have nondegenerate critical points in $\mathcal{N}(S) \setminus \Delta$. The topology of this space is independent of S, so we can use the Poincaré polynomial $\tilde{P}(t)$ from Proposition 2.11.2.

This time there are more collinear configurations. If we fix any one of the d eigenlines of S we will find $n!$ collinear SBC's which are nondegenerate with Morse index $(d-1)(n-1)$. There are d eigenlines for a total of $d\,n!$ collinear SBC's. If we knew their indices, it might be possible to use the information to get strong Morse estimates for the number of non-collinear SBC's. It seems that the proof of Proposition 2.9.2 can be generalized to show that the collinear SBC's corresponding to the largest eigenvalue of S have index $(d-1)(n-1)$ which would give $\gamma_{(d-1)(n-1)} \geq n!$. Using this to estimate the residual polynomial as in the proof of Proposition 2.11.4 gives a lower bound

$$\sum_k \gamma_k \geq (3n-1)(n-1)!,$$

but this exceeds the known count of $d\,n!$ collinear configurations only for $d = 2$.

2.12 Dziobek configurations

In Section 2.9 we studied collinear central configurations. These are at the lower end of the dimension range for an n-body configuration, $1 \leq \dim(x) \leq n-1$. We also saw that the only CC with $\dim(x) = n-1$ is the regular simplex. In this section we consider the highest nontrivial dimension.

Definition 2.12.1. *A* Dziobek *configuration is a configuration of n bodies with* $\dim(x) = n-2$.

The physically interesting examples are collinear configurations of 3 bodies, planar but non-collinear configurations of 4 bodies, and spatial but nonplanar configurations of 5 bodies. They are named after Otto Dziobek who studied the planar four-body case [10]. We will be interested in finding Dziobek central configurations (DCC's).

We begin by studying the geometry of Dziobek configurations. We will assume that the dimension of the ambient space is $d = n - 2$ so any n-body configuration is given by $x = (x_1, \ldots, x_n)$ with $x_j \in \mathbb{R}^{n-2}$. It is useful to associate with x the so-called $(n-1) \times n$ *augmented configuration matrix*

$$\hat{X} = \left[\begin{array}{ccc} 1 & \cdots & 1 \\ \hline x_1 & \cdots & x_n \\ & & \end{array}\right]. \tag{2.53}$$

This is just the configuration matrix of Section 2.4 with a row of ones added to the top. Then it is easy to see that $\dim(x) = \operatorname{rank} \hat{X} - 1$. Note that, because of the row of ones, two configurations are translation equivalent if and only if their augmented configuration matrices have the same row space or, equivalently, the same kernel.

For a Dziobek configuration we have $\operatorname{rank} \hat{X} = n - 1$ and $\dim \ker \hat{X} = 1$. Hence there is a nonzero vector $\Delta = (\Delta_1, \ldots, \Delta_n)$, unique up to a constant multiple, such that

$$\begin{aligned} \Delta_1 + \cdots + \Delta_n &= 0, \\ x_1\Delta_1 + \cdots + x_n\Delta_n &= 0. \end{aligned} \tag{2.54}$$

There is a nice formula for a vector Δ satisfying (2.54). Let $\hat{X}_k$ be the $(n-1) \times (n-1)$ matrix obtained from X by deleting the k-th column and let $|\hat{X}_k|$ denote its determinant. Then,

$$\Delta = (|\hat{X}_1|, -|\hat{X}_2|, \ldots, (-1)^{k+1}|\hat{X}_k|, \ldots)^T \tag{2.55}$$

is a solution to (2.54). Moreover, since the determinants are proportional to the volumes of the $(n-2)$-simplices of the deleted configurations, at least one of them is nonzero in the Dziobek case.

Next we will reformulate the dimension criteria above in terms of the mutual distances r_{ij} or rather, their squares $s_{ij} = r_{ij}^2$. Using equations (2.54) we have

$$\sum_j s_{ij}\Delta_j = |x_i|^2 \sum_j \Delta_j - 2x_i \cdot \sum_j x_j\Delta_j + \sum_j |x_j|^2\Delta_j = \sum_j |x_j|^2\Delta_j, \tag{2.56}$$

where i is any fixed index and the sum over j runs from 1 to n (here, $s_{ii} = 0$). The result is independent of i and we denote it by $-\Delta_0$. Define the Cayley–Menger matrix and determinant by

$$CM(x) = \begin{bmatrix} 0 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & s_{12} & s_{13} & \cdots & s_{1n} \\ 1 & s_{12} & 0 & s_{23} & \cdots & s_{2n} \\ 1 & s_{13} & s_{23} & 0 & \cdots & s_{3n} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 1 & s_{1n} & s_{2n} & s_{3n} & \cdots & 0 \end{bmatrix}, \quad F(x) = |CM(x)|. \tag{2.57}$$

Then we have $CM(x)\Delta = 0$, where now $\Delta = (\Delta_0, \Delta_1, \ldots, \Delta_n)$. Consequently, we have

$$F(x) = |CM(x)| = 0$$

for any Dziobek configuration or, indeed, for any configuration with $\dim(x) \leq n-2$.

In order to find equations for Dziobek central configurations (DCC's), begin by setting $\lambda = m_0\lambda'$ in the standard equations (2.10). After some algebra we find that, for each $j = 1, \ldots, n$,

$$\sum_{i=1}^{n} m_i S_{ij} x_i = 0, \tag{2.58}$$

where

$$\begin{aligned} S_{ij} &= \frac{1}{r_{ij}^3} - \lambda', \quad i \neq j, \\ m_j S_{jj} &= -\sum_{i \neq j} m_i S_{ij}. \end{aligned} \tag{2.59}$$

Proposition 2.12.2. *Let x be a Dziobek central configuration of the n-body problem, let S_{ij} be given by* (2.59) *and let Δ be any nonzero solution of* (2.54)*. Then there is a real number $\kappa \neq 0$ such that*

$$m_i m_j S_{ij} = \kappa \Delta_i \Delta_j. \tag{2.60}$$

Moreover, at least two of the Δ_i are nonzero.

Proof. Equation (2.58) and the second equation of (2.59) show that for each $j = 1, \ldots, n$ the vector

$$(m_1 S_{1j}, m_2 S_{2j}, \ldots, m_n S_{nj})$$

is a solution to equations (2.54). Since the solution is unique up to a constant multiple, there must be constants k_j such that $m_i S_{ij} = k_j \Delta_i$. Since $S_{ij} = S_{ji}$, the vector $(k_1, \ldots, k_n)$ is a multiple $\kappa(\Delta_1/m_1, \ldots, \Delta_n/m_n)$ so we get (2.60) for some real number κ. If $\kappa = 0$ or if only one of the Δ_i were nonzero then all of the S_{ij}, $i \neq j$, would vanish and so all of the r_{ij} would be equal. But this only happens for the regular simplex, which is not a Dziobek configuration. □

Multiplying two of the equations (2.60) gives:

Corollary 2.12.3. *Let x be a Dziobek configuration and let S_{ij} be given by* (2.59). *Then for any four indices $i, j, k, l \in \{1, \dots, n\}$ we have*

$$S_{ij}S_{kl} = S_{il}S_{kj}.$$

These equations can be used to derive some mass independent constraints on the shapes of CC's. For example, when $n = 4$ we have two independent equations of the form

$$(r_{12}^3 - \lambda')(r_{34}^3 - \lambda') = (r_{13}^3 - \lambda')(r_{24}^3 - \lambda') = (r_{14}^3 - \lambda')(r_{23}^3 - \lambda').$$

Eliminating λ' gives a necessary condition on the distances, in addition to the vanishing of the Cayley–Menger determinant, for a configuration to be central for some choice of the masses.

2.13 Convex Dziobek central configurations

In this section we present an existence proof for convex Dziobek configurations based on ideas of Xia [37]. First we discuss the geometry of the space of convex configurations. Consider the n-body problem in $\mathbb{R}^{n-2}$ as in Section 2.12. The normalized configuration space $\mathcal{N}$ is diffeomorphic to a sphere of dimension $(n-1)(n-2)-1$. The Dziobek configurations form an open subset, but $\mathcal{N}$ also contains configurations with $\dim(x) < n-2$.

For each $x \in \mathcal{N}$, let $\Delta(x)$ be the vector of determinants (2.55) representing, up to a factor, the $(n-2)$-dimensional volumes of its $(n-1)$-body subconfigurations. Then $\Delta\colon \mathcal{N} \to \mathcal{V} \subset \mathbb{R}^n$, where $\mathcal{V}$ is the hyperplane $\Delta_1 + \cdots + \Delta_n = 0$. If x is a Dziobek configuration then at least two of the determinants Δ_i are nonzero and Δ determines a point $[\Delta]$ of the unit sphere $\mathbf{S}(\mathcal{V}) \simeq \mathbf{S}^{n-2}$ in $\mathcal{V}$. The planes $\Delta_i = 0$ divide the sphere into components where the signs of the Δ_i are constant.

The signs of the variables Δ_i provide a geometric classification of Dziobek configurations. Suppose, for example, that $\Delta_n \neq 0$ so that the first $n-1$ bodies span a nondegenerate simplex in $\mathbb{R}^{n-2}$ and the ratios $b_i = -\Delta_i/\Delta_n$, $i = 1, \dots, n-1$, are the barycentric coordinates of x_n with respect to this simplex [6]. In particular, x_n is in the interior of the simplex if and only if $b_i > 0$ for $i = 1, \dots, n-1$. This provides a simple characterization of when a Dziobek configuration is *nonconvex*, namely, we must have either exactly one $\Delta_i > 0$ and $\Delta_j < 0$ for $j \neq i$, or else exactly one $\Delta_i < 0$ and $\Delta_j > 0$ for $j \neq i$. Let $\mathcal{NCD} \subset \mathcal{N}$ denote the open set of nonconvex Dziobek configurations.

The complement $K = \mathcal{N} \setminus \mathcal{NCD}$ is a compact set containing all of the convex Dziobek configurations. There will be some point $x \in K$ where $U|_K$ achieves its minimum and we would like to conclude that x is a convex Dziobek central configuration. This entails showing that the minimum does not occur on the boundary ∂K. We will prove this for $n = 4$ and get existence of planar, noncollinear convex central configurations for the four-body problem, a result due to

MacMillan–Bartky [17]. Unfortunately, there seem to be problems extending the proof to higher dimensions. To highlight the difficulties, we will split the proof into two parts. First we consider the part of ∂K consisting of Dziobek configurations. This part of the proof works for all n.

Proposition 2.13.1. *Let $x \in \partial K$ be a Dziobek configuration. Then x is not the minimizer of $U|_K$.*

Proof. We will show that arbitrarily close to x, there are points of K with strictly smaller values of $U|_K$. Instead of working with normalized configurations and $U|_K$, we can forget the normalization and use the homogeneous function $G = I(x)U(x)^2$.

By hypothesis, there is a sequence of nonconvex Dziobek configurations $x^k \to x$. After re-indexing and taking a subsequence we may assume that for all k, the n-th body x_n^k is contained in the interior of the simplex formed by $x_1^k, \ldots, x_{n-1}^k$. Taking the limit we conclude that x_n is contained in the boundary of the closed simplex formed by $x_1, \ldots, x_{n-1}$. Since we are assuming that x is still a Dziobek configuration, $x_1, \ldots, x_{n-1}$ span a nondegenerate $(n-2)$-simplex. After re-indexing again, we may assume that x_n is contained in the facet of this simplex spanned by $x_2, \ldots, x_{n-1}$. Let x_{ik}, $k = 1, \ldots, n-2$, denote the coordinates of the bodies in the ambient space $\mathbb{R}^{n-2}$. After a rotation and translation we may assume $x_{11} > 0$ and $x_{i1} = 0$, $i = 2, \ldots, n-1$. In other words all of the bodies except x_1 lie in a coordinate plane with x_1 strictly to the right.

Consider the distances r_{1k} from x_1 to the other bodies. Since x_n is contained in the closed simplex spanned by $x_2, \ldots, x_{n-1}$, we will have $r_{1n} < r_{1k}$ for some $k \in \{2, \ldots, n-1\}$ and we may assume without loss of generality that $r_{1n} < r_{12}$. Then we will see that moving x_n a little to the left while moving x_2 a little to the right decreases G. Moreover these perturbed configurations are in K.

We will use mutual distance version of the moment of inertia (2.9) and the usual formula for $U(r_{ij})$. Note that if we move x_2, x_n in the direction of the first coordinate axis, the derivatives of the distances r_{ij}, $2 \le i < j \le n$, are all zero. Only r_{12} and r_{1n} change to first order. If we change the first coordinates of x_2, x_n by $\delta x_{21} = m_2^{-1}\xi$ and $\delta x_{n1} = -m_n^{-1}\xi$ for some small $\xi > 0$, a short computation shows that the first-order change in G is

$$\delta G = 2IUm_1x_{11}\xi(r_{12}^{-3} - r_{1n}^{-3}),$$

where $x_{11} > 0$ is the first coordinate of x_1. Since $r_{1n} < r_{12}$ and $\xi > 0$, we have $\delta G < 0$ as required. □

Next we need to consider boundary points $x \in \partial K$ with $\dim(x) < n-2$. It is easy to see that every configuration with $\dim(x) < n-2$ can be perturbed into both a convex and nonconvex Dziobek configuration, hence all such lower-dimensional configurations are in ∂K. Fix a dimension $k < n-2$ and let $\mathcal{N}_k \subset \mathcal{N}$ be the set of configurations with $\dim(x) \le k$. Since $\mathcal{N}_k \subset \partial K \subset K$ it follows that if $x \in \mathcal{N}_k$ is a minimizer of $U|_K$ then it is also a minimizer of $U|_{\mathcal{N}_k}$ and is therefore a lower-dimensional CC. Therefore, in order to rule out such boundary points we need to understand how the potential changes when we perturb x to a convex Dziobek configuration. We know from Proposition 2.10.1 that there will be some perturbation to a Dziobek configuration which lowers the potential, but we do not know that this perturbation moves us into K. When $n = 4$, however, the only lower-dimensional configurations are collinear and we have the stronger Proposition 2.9.2.

Proposition 2.13.2. *There exists at least one convex, planar, non-collinear central configuration of the four-body problem for each cyclic ordering of the bodies; hence, at least six in all, up to similarity in the plane.*

Proof. If $x \in \partial K$ is a collinear configuration, then Proposition 2.9.2 shows that every perturbation of x to a non-collinear configuration in $\mathcal{N}$ will lower the potential. In particular, perturbing x into K will lower the potential. On the other hand, Proposition 2.13.1 shows that the non-collinear boundary points also admit potential-lowering perturbations into K. So the minimizer of $U|_K$ is in the interior as required.

Note that there are six components of Dziobek configurations with Δ's having the convex sign patterns

$$(+,+,-,-), \quad (+,-,+,-), \quad (+,-,-,+),$$

and the three more with the signs reversed. These correspond to the distinct cyclic orderings. If K_0 is the closure of any one of these, we can apply the same argument to find a CC in its interior. We only need to note that the required potential-lowering perturbations can be made into K_0. □

In [37] it is claimed that the analogous result holds for $n = 5$, but as noted above, more information about the behavior of planar five-body CC's under perturbations into Dziobek configurations seems to be needed.

Given that convex Dziobek configurations exist, one can ask about their possible shapes. It is possible to use equations (2.60) together with the positivity of the masses and the signs of the Δ_i to derive some simple geometrical constraints, see [17, 30].

Finally, we can use the existence of at least six local minima to improve the Morse estimates for the planar four-body problem. Recall that Proposition 2.11.4 gives the existence of at least twenty four CC's, including the twelve collinear ones (assuming that all critical points are nondegenerate). The twelve collinear CC's have index 2 which is the maximum possible, and the six convex Dziobek configurations are minima so $\gamma_0 \geq 6$ if they are nondegenerate. The Morse inequalities become

$$\gamma_0 + \gamma_1 t + \gamma_2 t^2 = 1 + 5t + 6t^2 + (1+t)(r_0 + r_1 t),$$

where $\gamma_0 \geq 6$ and $\gamma_2 \geq 12$. It follows that $r_0 \geq 5$ and $r_1 \geq 6$. Setting $t = 1$ gives a lower bound for the total number of CC's of

$$\gamma_0 + \gamma_1 + \gamma_2 \geq 12 + 2(5+6) = 34.$$

This lower bound seems to be sharp although there can be as few as 32 in degenerate cases, see [12, 33].

2.14 Generic finiteness for Dziobek central configurations

In this section we will present a proof that there are at most finitely many similarity classes of Dziobek central configurations for generic choices of the masses; the proof is based on [22]. We will also sketch a proof that these central configurations are generically nondegenerate.

Proposition 2.14.1. *For generic choices of the masses, there are only finitely many Dziobek central configurations up to similarity. In fact there is a mass-independent bound on the number of such configurations valid whenever the number is finite.*

In particular, this applies to planar CC's of the four-body problem and spatial but nonplanar CC's of the five-body problem. For the four-body problem, the only non-Dziobek central configurations are the regular tetrahedron and the collinear CC's. So in this case it follows that the total number of CC's is generically finite. However, there is a stronger result [14]: the number of CC's is finite for *all* choices of positive masses and is at most 8472. This is proved by completely different methods which required extensive algebraic computations. Similar methods were applied to the spatial five-body problem in [13] with the result that the generic conditions on the masses mentioned in Proposition 2.14.1 are made explicit. For the planar five-body problem, Albouy and Kaloshin have recently proved generic finiteness with explicit genericity conditions, see [4]. It is still open whether or not there exist exceptional choices of five positive masses which admit infinitely many CC's, but Roberts has an example involving masses of different signs [29]. The problem of finiteness for planar CC's was singled out by Steve Smale as the sixth of eighteen problems for twenty-first century mathematics [35]. But for $n > 5$ even generic finiteness is open.

The rest of this section is devoted to the proof of Proposition 2.14.1. The key point is to find the dimension of the algebraic variety defined by the equations for Dziobek central configurations. If the dimension of the space of central configurations is the same as the dimension of the space of normalized mass parameters, then the generic finiteness will follow from general theorems of algebraic geometry. For example, in Figure 2.7, Euler's quintic equation defines a two-dimensional surface. The projection of the surface to the two-dimensional normalized mass space necessarily has zero-dimensional fibers, at least for generic masses. In this case, all of the fibers are finite.

We begin with equations (2.60) relating the quantities S_{ij} from (2.59) and the Δ_i variables. However, we will make a few modifications. First of all, it is theoretically advantageous to work with complex, projective algebraic varieties which are defined by homogeneous polynomial equations. Define a new variable r_0 such that $\lambda' = r_0^{-3}$ so that

$$S_{ij} = r_{ij}^{-3} - r_0^{-3}.$$

Let $p = n(n-1)/2$ be the number of mutual distance variables r_{ij}. We will think of the vector $r = (r_0, r_{12}, \ldots, r_{34}) \in \mathbb{C}^{p+1}$ as homogeneous coordinates for a point $[r] \in \mathbb{CP}(p)$, the complex projective space. Passing from r to $[r]$ can be viewed as an alternative way of normalizing the size of the configuration.

Next we suppress the mass variables from equations (2.60) by defining new variables $z_i = \Delta_i/m_i$. After clearing denominators we get polynomial equations

$$r_0^3 - r_{ij}^3 = \kappa z_i z_j r_0^3 r_{ij}^3. \tag{2.61}$$

The following proposition shows that by introducing another variable z_0 we can get a set of equations which are separately homogeneous in the variables r and $z = (z_0, z_1, \ldots, z_n) \in \mathbb{C}^{n+1}$. We will view z as a set of homogeneous coordinates for a point $[z] \in \mathbb{CP}(n)$.

Proposition 2.14.2. *Suppose r_{ij} are the mutual distances of a Dziobek central configuration for some choice of masses $m_i > 0$. Let $r_0^{-3} = \lambda'$, and let $[r] \in \mathbb{CP}(p)$ be the corresponding point in the projective space. Then there is a point $[z] \in \mathbb{CP}(n)$ such that*

$$z_0^2(r_0^3 - r_{ij}^3) = z_i z_j r_{ij}^3. \tag{2.62}$$

Moreover, the Cayley–Menger determinant vanishes, $F(r) = 0$.

Proof. It follows from Proposition 2.12.2 and the definition of r_0 that there exist $z_i, \kappa \in \mathbb{R}$ such that (2.61) holds. Since $\kappa \neq 0$ we can define $z_0 \in \mathbb{C}$ so that $\kappa z_0^3 = r_0^{-3}$ and then we get equations (2.62). □

Equations (2.62) and the Cayley–Menger determinant are separately homogeneous with respect to the variables r and z so they define a projective variety in the product space $\mathbb{CP}(p) \times \mathbb{CP}(n)$. As usual, we need to exclude the collision configurations. Let

$$\Sigma = \{([r],[z]) \in \mathbb{CP}(p) \times \mathbb{CP}(n) : z_0 r_0 \prod_{i<j} r_{ij} = 0\}.$$

Then we can define the variety

$$V = \{([r],[z]) \in \mathbb{CP}(p) \times \mathbb{CP}(n) \setminus \Sigma :\ F(r) = 0 \text{ and (2.62) hold }\},$$

which contains all of the Dziobek central configurations. We will also work with the subvarieties obtained by setting some of the $z_i = 0$. Let

$$V_k = \{([r],[z]) \in V : z_{k+1} = \cdots = z_n = 0\}.$$

These are *quasi-projective* varieties, that is, they are difference sets $V = X \setminus Y$ where X, Y are projective varieties. Much of the theory of complex, algebraic geometry applies to such difference sets. We will use [8, 26, 31] as references for this theory. One important point is that every quasi-projective variety has a projective closure, defined as the smallest projective variety containing V. In general, this is smaller than the variety X.

The following result is crucial for proving the generic finiteness theorem we are after. It shows that the variety V containing the Dziobek configurations has the same dimension as the normalized mass space.

Proposition 2.14.3. *The variety V satisfies* $\dim V = n-1$. *More generally,* $\dim V_k = k-1$, *for all* $k \geq 2$.

Proof. Let $\pi_2 \colon \mathbb{CP}(p) \times \mathbb{CP}(n) \to \mathbb{CP}(n)$ be the projection. The proof for V consists of analyzing the fibers and image of the mapping $\pi_2 \colon V \to \mathbb{CP}(n)$. Suppose $[z] \in \pi_2(V)$ and let $([r],[z]) \in V$. By definition of Σ we have $z_0 r_0 \neq 0$ so there will be a representative r of $[r]$ with $r_0^3 z_0^2 = 1$. Then r_{ij} satisfies

$$g_{ij} = (z_i z_j + z_0^2) r_{ij}^3 - 1 = 0. \tag{2.63}$$

It follows that $z_i z_j + z_0^2 \neq 0$ on $\pi_2(V)$, and that the mapping $\pi_2 \colon V \to \mathbb{CP}(n)$ has finite fibers. If we can show that the projective closure $W = \overline{\pi_2(V)}$ has dimension $\dim W = n-1$, general results from algebraic geometry will give $\dim V = n-1$ as well.

The main point is to show that there exists a nonzero homogeneous polynomial $H(z)$ which vanishes on $\pi_2(V)$. This implies $\dim W \leq n-1$. We have $p+1$ equations for the p variables r_{ij}, namely, equations (2.63) and the Cayley–Menger determinant. To construct $H(z)$, begin by taking the resultant with respect to r_{12} of the Cayley–Menger determinant $F(r)$ and the polynomial g_{12}. The result is a polynomial involving z and the variables r_{ij} but with r_{12} eliminated. Now take the resultant with respect to r_{13} of this new polynomial and g_{13}. Continuing in this way, we can eliminate all of the variables r_{ij} obtaining a homogeneous polynomial $H(z)$ in the z variables alone. It is conceivable that $H(z)$ is identically zero, and the next step is to show this is not the case.

Recall that the vanishing of the resultant is a necessary condition for two polynomials in a single variable to have a common complex root. The polynomials may involve other variables which can be viewed as parameters. If the parameters are such that the leading coefficient of at least one of the two polynomials is nonzero, then the vanishing of the resultant is also sufficient for the existence of a common root. It follows that if $H(z) = 0$ for some $z \in \mathbb{C}^{n+1}$ such that

$$z_i z_j + z_0^2 \neq 0, \quad 1 \leq i < j \leq n, \tag{2.64}$$

then there do exist $r_{ij} \in \mathbb{C}$ such that equations (2.63) and the Cayley–Menger condition hold. Therefore, to show that $H(z)$ is not identically zero, it suffices to find a single point z such that (2.64) hold, but for which the required r_{ij} do not exist.

To this end, choose z such that $z_0 = 1$, $z_i = 0, 3 \le i \le n$. Then for $3 \le i, j \le n$ we have $z_i z_j + z_0^2 = 1$ and the equations $g_{ij} = 0$ reduce to $r_{ij}^3 = 1$. So these r_{ij} and their squares s_{ij} are all third roots of unity. On the other hand, if we choose z_1, z_2 so that

$$z_1 z_2 + z_0^2 = 1/\sqrt{8}$$

then $r_{12}^3 = \sqrt{8}$ and s_{12} is twice a third root of unity. We will show that with this z, the Cayley–Menger determinant does not vanish.

Lemma 2.14.4. *Let $\omega_{ij} \in \mathbb{C}$, $0 \le i < j \le n$, be third roots of unity. Then*

$$\begin{vmatrix} 0 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 2\omega_{12} & \omega_{13} & \cdots & \omega_{1n} \\ 1 & 2\omega_{12} & 0 & \omega_{23} & \cdots & \omega_{2n} \\ 1 & \omega_{13} & \omega_{23} & 0 & \cdots & \omega_{3n} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 1 & \omega_{1n} & \omega_{2n} & \omega_{3n} & \cdots & 0 \end{vmatrix} \neq 0.$$

Proof. The determinant can be expanded as a sum of monomials in the ω_{ij} with integer coefficients. Each monomial is equal to an integer multiple of $1, \omega$ or ω^2 where $\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$. Therefore the determinant is of the form $\alpha + \beta\omega + \gamma\omega^2$ where α, β, γ are integers. An expression of this form vanishes if and only if it is a multiple of the minimal polynomial of ω, $1 + \omega + \omega^2$; that is, if and only if $\alpha = \beta = \gamma$. A necessary condition for this is that $\alpha + \beta + \gamma$ be divisible by 3. Now the sum $\alpha + \beta + \gamma$ is the value of the determinant with all $\omega_{ij} = 1$ which turns out to be $(-1)^n 4$. So the determinant cannot vanish. □

It follows that our homogeneous polynomial $H(z)$ is not identically zero. Therefore the subvariety $Z = \{[z] : H(z) = 0\} \subset \mathbb{CP}(n)$ has dimension $n - 1$. The the projection $\pi_2(V)$ is contained in Z. In fact,

$$\pi_2(V) = \{[z] \in Z : (2.64) \text{ hold}\}.$$

Since $\pi_2(V) \neq \emptyset$, at least some of the irreducible components of Z intersect the set where (2.64) holds. Let W denote the union of these irreducible components (W will be the zero set of those factors of $H(z)$ which are not divisible by any of the polynomials in (2.64)). Then $\dim W = n - 1$ and the complement $W \setminus \pi_2(V)$ is a lower-dimensional subvariety. It follows that W is the projective closure of $\pi_2(V)$ and that $\dim W = \dim V = n - 1$, as claimed.

The proof for V_k is similar, but we use the projection $\pi_2 \colon V_k \to \mathbb{CP}(k)$ where we view $\mathbb{CP}(k)$ as the subset of $\mathbb{CP}(n)$ with $z_{k+1} = \cdots = z_n = 0$. Again we need to see that the resultant $H(z)$ does not vanish identically on $\mathbb{CP}(k)$. This follows because the point z with $H(z) \neq 0$ which we constructed above is actually in $\mathbb{CP}(k)$, $k \ge 2$. □

So far, we have discussed the variety V of Dziobek central configurations without fixing the masses. Next we discuss the mapping from V to the normalized mass space. A nonzero mass vector $m = (m_1, m_2, \ldots, m_n)$ determines a point in the projective space $[m] \in \mathbb{RP}(n-1) \subset \mathbb{CP}(n-1)$. We will think of $\mathbb{CP}(n-1)$ as the normalized mass space. A generic mass vector will mean $[m] \in \mathbb{CP}(n-1) \setminus B$, where B is a proper subvarienty of $\mathbb{CP}(n-1)$. Note that if B is such a proper subvariety then $B \cap \mathbb{RP}(n-1)$ is also a proper subvariety. This follows since any complex polynomial which vanishes identically on $\mathbb{RP}(n-1)$ also vanishes identically on $\mathbb{CP}(n-1)$.

Relations between the variables $([r], [z]) \in V$ and the masses are derived from the fact that the vector

$$\Delta = (\Delta_0, \Delta_1, \ldots, \Delta_n) = (\Delta_0, m_1 z_1, \ldots, m_n z_n)$$

is in the kernel of the Cayley–Menger matrix $CM(r)$ from (2.57). Let $K \subset \mathbb{CP}(p)$ be the subvariety of projective vectors $[r]$ such that $\operatorname{rank} CM(r) < n$. If $[r] \in K$ then r_{ij} cannot be the mutual distance of a Dziobek configuration. Consider the decomposition of V into irreducible components. Call an irreducible component W a *Dziobek component* if $W \not\subset K$. To study generic finiteness for Dziobek configurations it suffices to consider each Dziobek component separately.

If $W \subset V$ is a Dziobek irreducible component, then outside the proper subvariety $W \cap K$, the vector Δ is uniquely determined up to a constant multiple. There are two cases depending on whether or not some of the variables z_i vanish identically on W, a possibility we will denote by $z_i \equiv 0$. If $z_i \not\equiv 0$ for all i then the subset $W_0 = \{([r], [z]) \in W : z_i = 0 \text{ for some } i\}$ is a proper subvariety of W. The uniqueness of Δ implies that $[m]$ is uniquely determined for $([r], [z]) \in W \setminus (W_0 \cup K)$. This means that we have a rational *mass mapping* $W \to \mathbb{CP}(n-1)$ assigning to each point of $W \setminus (W_0 \cup K)$ a unique, projective mass vector (in algebraic geometry, a rational map can be multivalued on a proper subvariety). Since $\dim W = n - 1 = \dim \mathbb{CP}(n-1)$ it follows that a generic $[m]$ has a finite number of preimages in W. More precisely, either the mass mapping takes W into a proper subvariety of the mass space or not. In the first case the generic mass point $[m]$ has no preimages in W. In the latter case, we say that the mapping is *dominant* and the generic point $[m]$ has a nonzero but finite number of preimages, the number being bounded by some bound which is independent of $[m]$.

On the other hand, if some $z_i \equiv 0$ on W we may assume without loss of generality that W is a component of V_k from Proposition 2.14.3. Since $z_{k+1} = \cdots = z_n = 0$, the $(n-k)$ masses $m_{k+1} = \cdots = m_n$ are arbitrary. But other masses are unique up to a constant factor. Then Proposition 2.14.3 shows that

$$\tilde{W} = \{([r], [z], [m] : ([r], [z]) \in W, CM(r)\Delta = 0\}$$

is a subvariety of the product $\mathbb{CP}(p) \times \mathbb{CP}(n) \times \mathbb{CP}(n-1)$ of dimension $(k - 1) + (n-k) = n - 1$. Projection onto the mass space defines a rational map

$\tilde{W} \to \mathbb{CP}(n-1)$, and the same reasoning as before shows that a generic mass point has a finite number of preimages in W. This completes the proof of generic finiteness.

The generic nondegeneracy of DCC's follows from another nice fact about rational maps of varieties. Consider a dominant rational map between varieties of the same dimension. Then for a generic $[m]$ in the range space, all of its preimages are smooth points (meaning that the variety is locally a complex manifold) and the mapping is a local diffeomorphism. If this holds for a map of complex manifolds then it also holds for the real parts. Applying this theory to the real part of the varieties $\tilde{W}$ in $\mathbb{RP}(p) \times \mathbb{RP}(n) \times \mathbb{RP}(n-1)$ shows that the variety of DCC's looks like a finite covering map near a generic real $[m]$.

On the other hand, consider Dziobek CC's as critical points of U in $\mathcal{M}$, the quotient space of $\mathcal{N}$ under the action of the rotation group. Since we are working in $\mathbb{R}^{n-2}$, the Dziobek configurations have top dimension and the quotient space is locally a manifold. The implicit function theorem shows that DCC has a unique smooth continuation to nearby masses with the map to mass space a local diffeomorphism if and only if it is a nondegenerate critical point in $\mathcal{M}$. So generic masses admit only nondegenerate DCC's.

2.15 Some open problems

We will close these notes by mentioning some open questions about central configurations. Perhaps the simplest one to state, if not to solve, is Smale's sixth problem about finiteness of the number of central configurations in the plane for fixed positive masses [35]. As noted in the last section, even the weaker question of generic finiteness is open for $n > 5$. One could also consider the same problem in higher dimensions or for S-balanced configurations with both the masses and the symmetric matrix S fixed. The generic finiteness problem seems more tractable in light of Roberts' example of a continuum of solutions for fixed nonpositive masses, and the difficulties preventing Albouy and Kaloshin from handling all positive masses in the five-body case. Perhaps opening up the problem to allow SBC's might make a positive mass counterexample possible.

Another type of open problem is about the Morse indices of CC's and SBC's. As noted in Section 2.10, not much is known about the Morse indices of noncollinear CC's and even about collinear SBC's. Good results about this would improve the Morse theoretical estimates of the total number of critical points. It was a lack of information about the Hessian in directions normal to the subspace occupied by the configurations which prevented us from extending the existence proof for convex Dziobek configurations to $n > 4$ bodies. The most natural conjecture, that the normal blocks of the Hessian are negative semi-definite, is not true in general. There are planar CC's for which the potential increases in certain normal directions [20, 24].

As far as we know, the convex Dziobek configurations of the four-body problem are unique given the ordering of the bodies, but no proof has been given. The problem of counting convex Dziobek configurations could be posed for $n > 4$ once the existence problem is solved.

Another group of open questions concerns a topic not treated in these notes, namely the dynamical stability of relative equilibrium and homographic motions. Given a planar CC we saw that we have a simple relative equilibrium solution where the bodies rigidly rotate around their center of mass. In rotating coordinates this becomes an equilibrium and one can ask about its linear stability. In particular, one can ask if there is any relation between the eigenvalues at the equilibrium point and the Morse index of the critical point. All of the known examples of linearly stable relative equilibria correspond to critical points which are local minima. Is this always the case? In light of Albouy–Chenciner's theory of higher-dimensional relative equilibria, one can generalize the problem to ask for the relationship between the properties of an SBC as a critical point and as an equilibrium point of the reduced equations of motion. In fact, the problem of linear stability of higher-dimensional relative equilibria seems to be completely open.

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