# Chapter 1

# New Structures on Valuations and Applications

*Semyon Alesker*

## Introduction

The theory of valuations on convex sets is a classical part of the topic of onvexity, with traditionally strong relations to integral geometry. During the roughly last 15 years a considerable progress was made in valuation theory and its applications to integral geometry. The progress is both conceptual and technical: several new structures on valuations have been discovered, new classification results of various special classes of valuations have been obtained, the tools used in valuation theory and its relations with other parts of mathematics have become much more diverse —besides convexity and integral geometry, one can mention representation theory, geometric measure theory, elements of contact geometry, and complex and quaternionic analysis. This progress in valuation theory has led to new developments in integral geometry, particularly in Hermitian spaces. Some of the new structures turned out to encode in an elegant and useful way important integral geometric information: for example, the product operation on valuations encodes somehow the principal kinematic formulas in various spaces.

Quite recently, generalizations of the classical theory of valuations on convex sets to the context of manifolds were initiated; this development extends the applicability of valuation theory beyond affine spaces, and also covers a broader scope of integral geometric problems. In particular, the theory of valuations on manifolds provides a common point of view on three classical and previously unrelated directions of integral geometry: Crofton-style integral geometry, dealing with integral geometric and differential geometric invariants of sets and their intersections, and with projections to lower-dimensional subspaces; Gelfand-style integral geometry,



S. Alesker, J.H.G. Fu, *Integral Geometry and Valuations*, Advanced Courses in Mathematics - CRM Barcelona, DOI 10.1007/978-3-0348-0874-3_1

dealing with the Radon transform on smooth functions on various spaces; and, less classical but still well known, the Radon transform with respect to the Euler characteristic on constructible functions.

Although the relations between valuation theory and Crofton-style integral geometry have been known since the works of Blaschke and especially Hadwiger, the new developments have enriched both subjects and, in fact, more progress is expected. The relations of valuation theory to the two other types of integral geometry are new.

Besides new notions, theorems, and applications, these recent developments contain a fair amount of new intuition on the subject. However, when one tries to make this intuition formally precise, the clarity of basic ideas is often lost among numerous technical details; moreover, in a few cases this formalization has not been done yet. Here, in several places, I take the opportunity to use the somewhat informal format of lecture notes to explain the new intuition in a heuristic way, leaving the technicalities aside. Nevertheless, I clearly separate formal rigorous statements from such heuristic discussions.

The goal of my and Joe Fu's lectures is to provide an introduction to these modern developments. These two sets of lectures complement each other. My lectures concentrate mostly on valuation theory itself and provide a general background for Fu's lectures. In my lectures the discussion of the relations between valuation theory and integral geometry is usually relatively brief, and its goal is to give simple illustrations of general notions. The important exceptions are Sections 1.2.11 and 1.2.12, where new integral geometric results are discussed, namely a Radon-type transform on valuations. A much more thorough discussion of applications to Crofton-style integral geometry, especially in Hermitian spaces, will be offered in Fu's lectures.

My lectures consist of two main parts. The first part discusses the theory of valuations on convex sets and the second part discusses its recent generalizations to manifolds. The theory of valuations on convex sets is a very classical and much studied area. In these lectures, I mention only several facts from these classical developments which are necessary for our purposes; I refer to the surveys [54, 55] for further details and history.

The exposition contains almost no proofs. I tried to give the necessary definitions and list the main properties and sometimes present constructions of the principal objects and some intuition behind. Among important new operations on valuations are product, convolution, Fourier-type transform, pull-back, push-forward, and the Radon-type transform on valuations; all of them are relevant to integral geometry and are discussed in these notes.

Several interesting recent developments in valuation theory are not discussed here. The main omissions are a series of investigations by M. Ludwig with collaborators of valuations with weaker assumptions on continuity and various symmetries (see, e.g., [48, 50, 51]) and convex-bodies-valued valuations (see, e.g., [47, 49, 60]). Particularly, let me mention the surprising Ludwig–Reitzner characterization [50] of the affine surface area as the only example (up to the Euler characteristic, volume, and a non-negative multiplicative factor) of upper semi-continuous convex valuation invariant under all affine volume-preserving transformations.

**Acknowledgements**

These are notes of my lectures given at Centre de Recerca Matemàtica during the Advanced Course on Integral Geometry and Valuation Theory; I thank this institution and the organizers of the course E. Gallego, X. Gual, G. Solanes, and E. Teufel, for the invitation to give these lectures. I thank A. Bernig for his remarks on the first version of the notes, and E. Gallego, F. Schuster, and G. Solanes for a very careful reading of them and numerous remarks.

# 1.1 Translation-invariant valuations on convex sets

## 1.1.1 Definitions

Let $V$ be a finite-dimensional vector space of dimension $n$. Throughout these notes we will denote by $\mathcal{K}(V)$ the family of all convex compact non-empty subsets of $V$.

**Definition 1.1.1.** A complex-valued functional

$$\phi\colon \mathcal{K}(V) \longrightarrow \mathbb{C}$$

is called a *valuation* if

$$\phi(A \cup B) = \phi(A) + \phi(B) - \phi(A \cap B)$$

whenever $A$, $B$, $A \cup B \in \mathcal{K}(V)$.

**Remark 1.1.2.** In Section 1.2 we will introduce a different but closely related notion of valuation on a smooth manifold. To avoid confusion, we will sometimes call valuations on convex sets from Definition 1.1.1 *convex valuations*, though this is not a traditional terminology. But when there is no danger of confusion, we will call them just valuations. In fact, all valuations from Section 1.1 will be convex, while those from Section 1.2 will not, unless otherwise stated.

**Examples 1.1.3.** (1) Any $\mathbb{C}$-valued measure on $V$ is a convex valuation. In particular, the Lebesgue measure is such.

(2) The Euler characteristic $\chi$, defined by $\chi(K) = 1$ for any $K \in \mathcal{K}(V)$, is a convex valuation.

(3) Let $\phi$ be a convex valuation. Let $C \in \mathcal{K}(V)$ be fixed. Define

$$\psi(K) := \phi(K + C).$$

Then $\psi$ is a convex valuation. (Here $K + C := \{k + c \mid k \in K,\, c \in C\}$ is the Minkowski sum.) Indeed, $(A \cup B) + C = (A + C) \cup (B + C)$, and if $A$, $B$, $A \cup B \in \mathcal{K}(V)$, then

$$(A \cap B) + C = (A + C) \cap (B + C).$$

Let us define a very important class of continuous convex valuations. Fix a Euclidean metric on $V$. The Hausdorff distance on $\mathcal{K}(V)$ is defined by

$$\mathrm{dist}_{\mathrm{H}}(A,B) := \inf\big\{\varepsilon > 0 \mid A \subset (B)_\varepsilon,\, B \subset (A)_\varepsilon\big\},$$

where $(A)_\varepsilon$ denotes the $\varepsilon$-neighborhood of $A$ in the Euclidean metric. It is well known (see, e.g., [58]) that $\mathcal{K}(V)$ equipped with $\mathrm{dist}_{\mathrm{H}}$ is a metric locally compact space in which any closed bounded set is compact.

**Definition 1.1.4.** A convex valuation $\phi\colon \mathcal{K}(V) \to \mathbb{C}$ is called *continuous* if $\phi$ is continuous in the Hausdorff metric.

This notion of continuity of a valuation is readily seen to be independent of the choice of a Euclidean metric on $V$.

**Definition 1.1.5.** A convex valuation $\phi\colon \mathcal{K}(V) \to \mathbb{C}$ is called *translation-invariant* if

$$\phi(K + x) = \phi(K) \text{ for any } K \in \mathcal{K}(V),\, x \in V.$$

The linear space of translation-invariant continuous convex valuations will be denoted by $\mathrm{Val}(V)$. If equipped with the topology of uniform convergence on compact subsets of $\mathcal{K}(V)$, $\mathrm{Val}(V)$ is a Fréchet space. In fact it follows from McMullen's decomposition (Corollary 1.1.7 below) that $\mathrm{Val}(V)$ with this topology is a Banach space, with the norm given by

$$||\phi|| := \sup_{K \subset D} |\phi(K)|,$$

where $D \subset V$ is the unit Euclidean ball for some auxiliary Euclidean metric.

### 1.1.2 McMullen's theorem and mixed volumes

The following result due to McMullen [52] is very important.

**Theorem 1.1.6.** *Let $\phi\colon \mathcal{K}(V) \to \mathbb{C}$ be a translation-invariant continuous convex valuation. Then for any convex compact sets $A_1, \dots, A_s \in \mathcal{K}(V)$ the function*

$$f(\lambda_1, \dots, \lambda_s) = \phi(\lambda_1 A_1 + \cdots + \lambda_s A_s),$$

*defined for $\lambda_1, \dots, \lambda_s \geq 0$, is a polynomial of degree at most $n = \dim V$.*

The special case $s = 1$ is already non-trivial and important. It means that, for $\lambda \geq 0$,

$$\phi(\lambda K) = \phi_0(K) + \lambda\phi_1(K) + \cdots + \lambda^n \phi_n(K).$$

It is easy to see that the coefficients $\phi_0, \phi_1, \dots, \phi_n$ are also continuous translation-invariant convex valuations. Moreover, $\phi_i$ is homogeneous of degree $i$ (or $i$-homogeneous for brevity). By definition, a valuation $\psi$ is called $i$-homogeneous if for any $K \in \mathcal{K}(V)$ and any $\lambda \geq 0$ one has

$$\phi(\lambda K) = \lambda^i \phi(K).$$

Let us denote by $\mathrm{Val}_i(V)$ the subspace in $\mathrm{Val}(V)$ of $i$-homogeneous valuations. We immediately get the following corollary:

**Corollary 1.1.7** (McMullen's decomposition)**.**

$$\mathrm{Val}(V) = \bigoplus_{i=0}^{n} \mathrm{Val}_i(V).$$

**Remark 1.1.8.** Clearly $\mathrm{Val}_0(V)$ is one-dimensional and is spanned by the Euler characteristic. Actually, $\mathrm{Val}_n(V)$ is also one-dimensional and is spanned by a Lebesgue measure; this fact is not obvious and was proved by Hadwiger [39].

Let us now recall the definition of (Minkowski's) mixed volumes, which provide interesting examples of translation-invariant continuous convex valuations. Fix a Lebesgue measure on $V$, denoted vol. For any $n$-tuple of convex compact sets $A_1, \ldots, A_n$ consider the function

$$f(\lambda_1, \ldots, \lambda_n) = \mathrm{vol}(\lambda_1 A_1 + \cdots + \lambda_n A_n).$$

This is a homogeneous polynomial in $\lambda_i \geq 0$ of degree $n$. Of course, this fact follows from McMullen's theorem (Theorem 1.1.6) and the $n$-homogeneity of the volume, though originally it was discovered much earlier by Minkowski, and in this particular case there is a simpler proof (see, e.g., [58]).

**Definition 1.1.9.** The coefficient of the monomial $\lambda_1 \cdots \lambda_n$ in $f(\lambda_1, \ldots, \lambda_n)$ divided by $n!$ is called the *mixed volume* of $A_1, \ldots, A_n$ and is denoted by $V(A_1, \ldots, A_n)$.

The normalization of the coefficient is chosen in such a way that $\mathrm{vol}(A) = V(A, \ldots, A)$. Mixed volumes have a number of interesting properties; in particular they satisfy the Aleksandrov–Fenchel inequality [58]. The property relevant for us, however, is the valuation property. Fix $1 \leq s \leq n-1$ and an $s$-tuple of convex compact sets $A_1, \ldots, A_s$. Define

$$\phi(K) = V\big(K[n-s], A_1, \ldots, A_s\big), \tag{1.1.1}$$

where $K[n-s]$ means that $K$ is taken $n-s$ times. Then $\phi$ is a translation-invariant continuous valuation. This easily follows from Example 1.1.3(3).

### 1.1.3 Hadwiger's theorem

One of the most famous and classical results of valuation theory is Hadwiger's classification of isometry-invariant continuous convex valuations on the Euclidean space $\mathbb{R}^n$. To formulate it, let us denote by $V_i$ the $i$th intrinsic volume, which by definition is

$$V_i(K) = c_{n,i} V\big(K[i], D[n-i]\big),$$

where $c_{n,i}$ is an explicitly written constant which is just a standard normalization (see [58]). In particular, $V_0 = \chi$ is the Euler characteristic and $V_n = \mathrm{vol}$ is the Lebesgue measure normalized so that the volume of the unit cube is equal to 1. Clearly $V_i \in \mathrm{Val}_i$ is an $O(n)$-invariant valuation.

**Theorem 1.1.10** (Hadwiger's classification [39]). *Any $SO(n)$-invariant translation-invariant continuous convex valuation is a linear combination of $V_0, V_1, \dots, V_n$. (In particular, it is $O(n)$-invariant.)*

In 1995, Klain [43] obtained a simplified proof of this deep result as an easy consequence of his classification of simple even valuations discussed below in Section 1.1.5. Hadwiger's theorem turned out to be very useful in integral geometry of the Euclidean space. This will be discussed in more detail in J. Fu's lectures. We also refer to the book [45].

## 1.1.4 Irreducibility theorem

One of the basic questions in valuation theory is to describe valuations with given properties. Hadwiger's theorem is one example of such a result of great importance. In recent years, a number of classification results of various classes of valuations have been obtained. The case of continuous translation-invariant valuations will be discussed in detail in these lectures and in the lectures by Fu.

The question is whether it is possible or not to give a reasonable description of all translation-invariant continuous convex valuations. In 1980, P. McMullen [53] formulated a more precise conjecture which says that linear combinations of mixed volumes (as in (1.1.1)) are dense in Val. This conjecture was proved in the positive by the author [2] in a stronger form which later on turned out to be important in further developments and applications.

To describe the result, let us make a few more remarks. We say that a valuation $\phi$ is *even* (respectively *odd*) if $\phi(-K) = \phi(K)$ (respectively $\phi(-K) = -\phi(K)$) for any $K \in \mathcal{K}(V)$. The subspace of even (respectively odd) $i$-homogeneous valuations will be denoted by $\mathrm{Val}_i^+$ (respectively $\mathrm{Val}_i^-$). Clearly,

$$\mathrm{Val}_i = \mathrm{Val}_i^+ \oplus \mathrm{Val}_i^- \,. \tag{1.1.2}$$

Next observe that the group $GL(V)$ of all invertible linear transformations acts linearly on Val by

$$(g\phi)(K) = \phi(g^{-1}K).$$

**Theorem 1.1.11** (Irreducibility theorem [2]). *For each $i$, the spaces $\mathrm{Val}_i^+$ and $\mathrm{Val}_i^-$ are irreducible representations of $GL(V)$, i.e., they do not have proper invariant closed subspaces.*

**Remark 1.1.12.** By Remark 1.1.8, $\mathrm{Val}_0^+ = \mathrm{Val}_0$ and $\mathrm{Val}_n^+ = \mathrm{Val}_n$ are one-dimensional. But for $1 \le i \le n-1$ the spaces $\mathrm{Val}_i^\pm$ are infinite-dimensional. $\mathrm{Val}_{n-1}$ was explicitly described by McMullen [53]; we state his result in Section 1.1.5 below.

Theorem 1.1.11 immediately implies McMullen's conjecture. Indeed, it is easy to see that the closure of the linear span of mixed volumes is a $GL(V)$-invariant subspace, and its intersection with any $\mathrm{Val}_i^\pm$ is non-zero. Hence, by the irreducibility theorem, any such intersection should be equal to the whole space $\mathrm{Val}_i^\pm$.

The irreducibility theorem will be used in these lectures several times. The proof of this result uses a number of deep results from valuation theory in combination with representation-theoretical techniques. A particularly important such result of high independent interest is the Klain–Schneider classification of *simple* translation-invariant continuous convex valuations; it is discussed in the next section.

### 1.1.5 Klain–Schneider characterization of simple valuations

**Definition 1.1.13.** A convex valuation $\phi \in \mathrm{Val}$ is called *simple* if $\phi(K) = 0$ for any $K \in \mathcal{K}(V)$ with $\dim K < n := \dim V$.

**Theorem 1.1.14.** (i) (Klain [43]). *Any simple even valuation from* Val *is proportional to the Lebesgue measure.*

(ii) (Schneider [59]). *Any simple odd valuation from* Val *is* $(n-1)$*-homogeneous.*

Clearly, any simple valuation is the sum of a simple even one and a simple odd one. Hence, in order to complete the description of simple valuations, it remains to classify simple $(n-1)$-homogeneous valuations. Fortunately, McMullen [53] has previously described $\mathrm{Val}_{n-1}$ very explicitly. His result was used in Schneider's proof, and it is worthwhile to state it explicitly as it is of independent interest.

First let us recall the definition of the area measure $S_{n-1}(K, \cdot\,)$ of a convex compact set $K$. Though it is not strictly necessary, it is convenient and common to fix a Euclidean metric on $V$. After this choice, $S_{n-1}(K, \cdot\,)$ is a measure on the unit sphere $S^{n-1}$ defined as follows. First let us assume that $K$ is a polytope. For any $(n-1)$-face $F$, let us denote by $n_F$ the unit outer normal to $F$. Then, by definition,

$$S_{n-1}(K, \cdot\,) = \sum_F \mathrm{vol}_{n-1}(F)\delta_{n_F},$$

where the sum runs over all $(n-1)$-faces of $K$, and $\delta_{n_F}$ denotes the delta measure supported at $n_F$. Then the claim is that the area measure extends uniquely by weak continuity to the class of all convex compact sets: if $K_N \to K$ in the Hausdorff metric, then $S_{n-1}(K_N, \cdot\,) \to S_{n-1}(K, \cdot\,)$ weakly in the sense of measures (see [58, §4.2]).

**Theorem 1.1.15** (McMullen [53])**.** *Let* $\phi \in \mathrm{Val}_{n-1}$ *and* $n = \dim V$*. Then there exists a continuous function* $f\colon S^{n-1} \to \mathbb{C}$ *such that*

$$\phi(K) = \int_{S^{n-1}} f(x)\, dS_{n-1}(K, x).$$

*Conversely, any expression of this form with* $f$ *continuous is a valuation from* $\mathrm{Val}_{n-1}$*. Moreover, two continuous functions* $f$ *and* $g$ *define the same valuation if and only if the difference* $f - g$ *is the restriction of a linear functional on* $V$ *to the unit sphere.*

Now we can state the classification of simple valuations.

**Theorem 1.1.16** (Klain–Schneider). *Simple translation-invariant continuous valuations are precisely of the form*

$$K \longmapsto c \cdot \mathrm{vol}_n(K) + \int_{S^{n-1}} f(x)\, dS_{n-1}(K,x),$$

*where $f\colon S^{n-1} \to \mathbb{C}$ is an odd continuous function and $c$ is a constant. Moreover, the constant $c$ is defined uniquely, while $f$ is defined up to a linear functional.*

### 1.1.6 Smooth translation-invariant valuations

We are going to describe an important subclass of Val, that of *smooth* valuations. They form a dense subspace in Val and carry a number of extra structures (e.g., product, convolution, Fourier transform) which do not extend by continuity to the whole space Val of continuous valuations. Moreover, main examples relevant to integral geometry are in fact smooth valuations.

**Definition 1.1.17.** A valuation $\phi \in \mathrm{Val}(V)$ is called *smooth* if the Banach space valued map $GL(V) \to \mathrm{Val}(V)$ given by $g \mapsto g(\phi)$ is infinitely differentiable.

By a very general and elementary representation-theoretical reasoning, the subset of smooth valuations, denoted by $\mathrm{Val}^{\mathrm{sm}}(V)$, is a linear dense subspace of $\mathrm{Val}(V)$ invariant under the natural action of $GL(V)$. Also, $\mathrm{Val}^{\mathrm{sm}}(V)$ carries a linear topology which is stronger than that induced from $\mathrm{Val}(V)$, and with respect to which it is a Fréchet space. This is often called the *Gårding topology*, and we will tacitly assume that $\mathrm{Val}^{\mathrm{sm}}(V)$ is equipped with it. Of course, $\mathrm{Val}^{\mathrm{sm}}$ also satisfies McMullen's decomposition and the irreducibility theorem.

Let us give some examples of smooth valuations and non-smooth valuations. Let $A, A_1, \ldots, A_s \in \mathcal{K}(V)$ be full-dimensional convex bodies with infinitely smooth strictly convex boundary. Then $K \mapsto \mathrm{vol}(K + A)$ is a smooth valuation, and consequently the mixed volumes $K \mapsto V(K[n-s], A_1, \ldots, A_s)$ are also smooth valuations. A simple geometric example of a non-smooth continuous valuation is the volume of a projection to a subspace of $V$.

For future applications to integral geometry, the following result will be important.

**Proposition 1.1.18** ([4]). *Let $G$ be a compact subgroup of the orthogonal group of a Euclidean space $V$. Assume that $G$ acts transitively on the unit sphere of $V$. Then any $G$-invariant valuation from* $\mathrm{Val}(V)$ *is smooth.*

### 1.1.7 Product on smooth translation-invariant valuations and Poincaré duality

In this section we discuss the product operation on translation-invariant smooth valuations introduced in [4]. This structure turned out to be intimately related to integral geometric formulas discussed in detail in J. Fu's lectures.

First we introduce the exterior product on valuations.

**Theorem 1.1.19** ([4]). *Let $V$ and $W$ be finite-dimensional real vector spaces. There exists a continuous bilinear map, called exterior product,*

$$\mathrm{Val}^{sm}(V) \times \mathrm{Val}^{sm}(W) \longrightarrow \mathrm{Val}(V \times W)$$

*which is uniquely characterized by the following property: fix $A \in \mathcal{K}(V)$ and $B \in \mathcal{K}(W)$ such that both of them have smooth boundaries with positive curvature. Let $\mathrm{vol}_V$ and $\mathrm{vol}_W$ be Lebesgue measures on $V$ and $W$, respectively. Let $\phi(K) = \mathrm{vol}_V(K + A)$ and $\psi(K) = \mathrm{vol}_W(K + B)$. Then their exterior product, denoted by $\phi \boxtimes \psi$, is*

$$(\phi \boxtimes \psi)(K) = \mathrm{vol}_{V\times W}(K + (A \times B)) \text{ for any } K \in \mathcal{K}(V \times W),$$

*where $\mathrm{vol}_{V\times W}$ is the product measure of $\mathrm{vol}_V$ and $\mathrm{vol}_W$.*

Notice that the uniqueness in this theorem follows immediately from the (proved) McMullen's conjecture, since linear combinations of valuations of the form $\mathrm{vol}(\,\cdot + A)$ are dense in valuations.

Let us emphasize that the exterior product is defined on smooth valuations, but it takes values not in smooth but just continuous valuations. Usually the exterior product is not smooth. Let us give some examples.

**Examples 1.1.20.** (1) Obviously, the exterior product of Lebesgue measures in the sense of valuations coincides obviously with their measure-theoretical product.

(2) The exterior product of Euler characteristics is the Euler characteristic on $V \times W$. This can be seen as follows. First it is easy to verify that the exterior product commutes with the natural action of the group $GL(V) \times GL(W)$. Hence $\chi_V \boxtimes \chi_W$ is invariant under this group, and in particular 0-homogeneous. Thus it must be proportional to the Euler characteristic. Explicit evaluation of this product on a point shows that the coefficient of proportionality must be equal to 1. This evaluation can be done by observing that

$$\chi_V(K) = \frac{1}{n!\,\mathrm{vol}_V(D)} \frac{d^n}{d\varepsilon^n}\Big|_{\varepsilon=0} \mathrm{vol}_V(K + \varepsilon D),$$

where $D$ is a Euclidean ball (or any convex compact set of non-zero volume) and $n = \dim V$.

(3) Let $\mathrm{vol}_V$ be a Lebesgue measure on $V$ and $\chi_W$ be the Euler characteristic on $W$. Then the exterior product $\mathrm{vol}_V \boxtimes \chi_W$ is the volume of the projection to $V$:

$$(\mathrm{vol}_V \boxtimes \chi_W)(K) = \mathrm{vol}_V(\mathrm{pr}_V(K)) \text{ for any } K \in \mathcal{K}(V \times W),$$

where $\mathrm{pr}_V : V \times W \to V$ is the natural projection. Observe that this valuation is not smooth (in contrast to the first two examples.)

The first non-trivial point in Theorem 1.1.19 is that the exterior product is well defined; the second one is continuity. We do not give here any proof. However, let us give an incomplete, but instructive, explanation of why the exterior product is well defined. There are of course many different ways to write a valuation as a linear combination of $\mathrm{vol}(\,\cdot\,+A)$. Let us check that the exterior product of finite linear combinations of such expressions is independent of the particular choice of a linear combination. Since the situation is symmetric with respect to both valuations, we may assume that $\phi(\,\cdot\,)=\sum_i c_i\cdot \mathrm{vol}_V(\,\cdot\,+A_i)$ and $\psi(\,\cdot\,)=\mathrm{vol}_W(\,\cdot\,+B)$. Then using Fubini's theorem and the equality $A_i\times B=(A_i\times 0)+(0\times B)$ we get

$$\begin{aligned}(\phi\boxtimes\psi)(K)&=\sum_i c_i\cdot \mathrm{vol}_{V\times W}(K+(A_i\times B))\\ &=\sum_i c_i\cdot\int_{w\in W}\mathrm{vol}_V\left[\{(K+(0\times B))\cap(V\times\{w\})\}+A_i\right]d\,\mathrm{vol}_W(w)\\ &=\int_{w\in W}\phi\big[(K+(0\times B))\cap(V\times\{w\})\big]\,d\,\mathrm{vol}_W(w).\end{aligned}$$

Clearly the last expression is independent of the form of presentation of $\phi$.

Now let us define the product on $\mathrm{Val}^{\mathrm{sm}}$. Let us denote by

$$\Delta\colon V\longrightarrow V\times V$$

the diagonal imbedding. The product of $\phi,\,\psi\in \mathrm{Val}^{\mathrm{sm}}(V)$ is defined by

$$(\phi\cdot\psi)(K):=(\phi\boxtimes\psi)(\Delta(K)).$$

It turns out that the product of smooth valuations is again smooth.

**Theorem 1.1.21** ([4])**.** *The product of smooth valuations* $\mathrm{Val}^{sm}(V)\times\mathrm{Val}^{sm}(V)\to \mathrm{Val}^{sm}(V)$ *is continuous (in the Gårding topology), associative, commutative, and distributive. Accordingly,* $\mathrm{Val}^{sm}(V)$ *becomes an algebra over* $\mathbb{C}$ *with unit, which is the Euler characteristic. Moreover, the product respects the degree of homogeneity:*

$$\mathrm{Val}_i^{sm}\cdot\mathrm{Val}_j^{sm}\subset\mathrm{Val}_{i+j}^{sm}\,.$$

**Example 1.1.22.** The product of intrinsic volumes $V_i\cdot V_j$ with $i+j\le n$ is a non-zero multiple of $V_{i+j}$: by the Hadwiger theorem it is clear that the product should be proportional to $V_{i+j}$; the constant of proportionality can be computed explicitly.

Let explain why the Euler characteristic is a unit in $\mathrm{Val}^{\mathrm{sm}}(V)$. Let $\phi(K)=\mathrm{vol}_n(K+A)$. Fix a convex body $B$ of non-zero volume. Then

$$\chi(K)=\frac{1}{n!\,\mathrm{vol}_n(B)}\,\frac{d^n}{d\varepsilon^n}\Big|_{\varepsilon=0}\mathrm{vol}_n(K+\varepsilon B).$$

Consequently,

$$(\chi \cdot \phi)(K) = \frac{1}{n! \operatorname{vol}_n(B)} \frac{d^n}{d\varepsilon^n}\Big|_{\varepsilon=0} \operatorname{vol}_{2n}\left((\Delta(K) + (0 \times A)) + \varepsilon B \times 0\right). \quad (1.1.3)$$

It is well known (prove it, or see an equivalent formula (5.3.23) in [58]) that for a convex compact subset $U$ in a $k$-dimensional linear subspace $E_k$ of the Euclidean space $\mathbb{R}^N$ and any convex compact set $M \subset \mathbb{R}^N$, one has

$$\frac{1}{k! \operatorname{vol}_k(U)} \frac{d^k}{d\varepsilon^k}\Big|_{\varepsilon=0} \operatorname{vol}_N(M + \varepsilon U) = \operatorname{vol}_{N-k}(p_{E_k^\perp} M).$$

This and (1.1.3) imply that

$$(\chi \cdot \phi)(K) = \operatorname{vol}_n(p_1(\Delta(K) + (0 \times A))) = \operatorname{vol}_n(K + A) = \phi(K),$$

where $p_1 \colon V \times V \to V$ is the natural projection to the first component.

An interesting property of the product is a version of Poincaré duality.

**Theorem 1.1.23** ([4]). *For any $i = 0, 1, \ldots, n = \dim V$ the bilinear map*

$$\operatorname{Val}_i^{sm}(V) \times \operatorname{Val}_{n-i}^{sm}(V) \longrightarrow \operatorname{Val}_n^{sm}(V)$$

*is a perfect pairing, namely for any non-zero valuation $\phi \in \operatorname{Val}_i^{sm}(V)$ there exists $\psi \in \operatorname{Val}_{n-i}^{sm}(V)$ such that $\phi \cdot \psi \neq 0$.*

This result follows easily from the irreducibility theorem (Theorem 1.1.11). Indeed, it suffices to prove the statement for valuations of fixed parity $\varepsilon = \pm 1$. Then the kernel of the above pairing in $\operatorname{Val}_i^{\varepsilon,\mathrm{sm}}(V)$ is a $GL(V)$-invariant closed subspace. Hence it must be either zero or everything. But it cannot be everything, since then for any valuation $\psi \in \operatorname{Val}_{n-i}^{\mathrm{sm}}(V)$ one would have $\psi \cdot \operatorname{Val}_i^{\varepsilon,\mathrm{sm}}(V) = 0$. But this is not the case, as can be easily proved by constructing an explicit example. (Say in the even case, the product of the intrinsic volumes $V_i \cdot V_{n-i}$ is a non-zero multiple of the Lebesgue measure.)

Thus $\operatorname{Val}^{\mathrm{sm}}(V)$ is a graded algebra that satisfies Poincaré duality. In Section 1.1.10 we will also see that it satisfies two versions of the hard Lefschetz theorem.

### 1.1.8 Pull-back and push-forward of translation-invariant valuations

In this section we describe the operations of pull-back and push-forward on translation-invariant valuations under linear maps.

Let $f \colon V \to W$ be a linear map. We define [15] a continuous linear map,

$$f^* \colon \operatorname{Val}(W) \longrightarrow \operatorname{Val}(V)$$

called *pull-back*, as usual by $(f^*\phi)(K) = \phi(f(K))$. It is easy to see that $f^*\phi$ is indeed a continuous translation-invariant convex valuation. The following result is evident.

**Proposition 1.1.24.** (i) *$f^*$ preserves the degree of homogeneity and the parity.*
(ii) *$(f \circ g)^* = g^* \circ f^*$.*

Notice that the product on valuations can be expressed via the exterior product and the pull-back as

$$\phi \cdot \psi = \Delta^*(\phi \boxtimes \psi),$$

where $\Delta$ is the diagonal imbedding.

A somewhat less obvious operation is the *push-forward* $f_*$, introduced in [15]. In some non-precise sense $f_*$ is dual to $f^*$. In these notes it will be used only to give an alternative description of the convolution on valuations in Section 1.1.9 and to clarify some properties of the Fourier-type transform on valuations in Section 1.1.11; the reader not interested in these subjects may skip the rest of this section.

Canonically, the push-forward map acts not between spaces of valuations, but between their tensor product (twist) with an appropriate one-dimensional space of Lebesgue measures. To be more precise, let us denote by $D(V^*)$ the one-dimensional space of ($\mathbb{C}$-valued) Lebesgue measures on $V^*$. Then $f_*$ is a linear continuous map

$$f_*\colon \operatorname{Val}(V) \otimes D(V^*) \longrightarrow \operatorname{Val}(W) \otimes D(W^*).$$

In order to define this map, we will split its construction into the cases of $f$ being injective, surjective, and a general linear map.

*Case* 1: Let $f$ be injective. Thus we may assume that $V \subset W$. In order to simplify the notation, we choose a splitting $W = V \oplus L$ and we fix Lebesgue measures on $V$ and $L$. Then on $W$ we have the product measure. These choices induce isomorphisms $D(V^*) \simeq \mathbb{C}$ and $D(W^*) \simeq \mathbb{C}$. We leave for the reader to check that the construction of $f_*$ is independent of these choices.

Let $\phi \in \operatorname{Val}(V)$. Define

$$(f_*\phi)(K) = \int_{l \in L} \phi(K \cap (l + V))\, dl.$$

It is easy to see that $f_*\colon \operatorname{Val}(V) \to \operatorname{Val}(W)$ is a continuous linear map.

*Case* 2: Let $f$ be surjective. Again it will be convenient to assume that $f$ is just a projection to a subspace, and fix a splitting $V = W \oplus M$. Again fix Lebesgue measures on $W, M$, and hence on $V$. Let us also fix a set $S \in \mathcal{K}(M)$ of unit measure. Set $m := \dim M$. Then define

$$(f_*\phi)(K) = \frac{1}{m!}\, \frac{d^m}{d\varepsilon^m} \phi(K + \varepsilon \cdot S)\Big|_{\varepsilon=0}.$$

Recall that by McMullen's theorem $\phi(K+\varepsilon\cdot S)$ is a polynomial in $\varepsilon \geq 0$. Moreover, its degree is at most $m$. Indeed, when $K$ is fixed, this expression is a translation-invariant continuous valuation with respect to $S \in \mathcal{K}(M)$. The coefficient of $\varepsilon^m$ is an $m$-homogeneous valuation with respect to $S \subset M$, and hence, by Hadwiger's theorem (see Remark 1.1.8), it must be proportional to the Lebesgue measure on $M$ with a constant depending on $K$. By our definition, this coefficient is exactly $(f_*\phi)(K)$ —in particular it does not depend on $S$. In fact, it also does not depend on the choice of Lebesgue measures and the splitting.

*Case* 3: Let $f$ be a general linear map. Let us choose a factorization $f = g \circ h$, where $h\colon V \to Z$ is surjective and $g\colon Z \to W$ is injective. Then define $f_* := g_* \circ h_*$. One can show that $f_*$ is independent of the choice of such a factorization.

**Proposition 1.1.25** ([15, Section 3.2])**.**

(i) *The map* $f_*\colon \operatorname{Val}(V) \otimes D(V^*) \to \operatorname{Val}(W) \otimes D(W^*)$ *is a continuous linear operator.*
(ii) $(f \circ g)_* = f_* \circ g_*$.
(iii) $f_* \left(\operatorname{Val}_i(V) \otimes D(V^*)\right) \subset \operatorname{Val}_{i-\dim V+\dim W}(W) \otimes D(W^*)$.

### 1.1.9 Convolution

In this section we describe another interesting operation on valuations: a convolution introduced by Bernig and Fu [24]. This is a continuous product on $\operatorname{Val}^{\mathrm{sm}} \otimes D(V^*)$. To simplify the notation, let us fix a Lebesgue measure vol on $V$; it induces a Lebesgue measure on $V^*$. With these identifications, convolution is going to be defined on $\operatorname{Val}^{\mathrm{sm}}(V)$ (without the twist by $D(V^*)$).

**Theorem 1.1.26** ([24])**.** *There exists a unique continuous bilinear map, called* convolution*,*

$$*\colon \operatorname{Val}^{sm}(V) \times \operatorname{Val}^{sm}(V) \longrightarrow \operatorname{Val}^{sm}(V),$$

*such that*

$$\operatorname{vol}(\,\cdot\, + A) * \operatorname{vol}(\,\cdot\, + B) = \operatorname{vol}(\,\cdot\, + A + B).$$

*This product makes* $\operatorname{Val}^{sm}(V)$ *a commutative associative algebra with unit element* vol*. Moreover,* $\operatorname{Val}_i^{sm} * \operatorname{Val}_j^{sm} \subset \operatorname{Val}_{i+j-n}^{sm}$.

The above result characterizes the convolution uniquely, and allows to compute it in some examples. We can give, however, one more description of it using the previously introduced operations. Namely, let $a\colon V \times V \to V$ be the addition map, $a(x, y) = x + y$. Then, by [15, Proposition 3.3.2], one has

$$\phi * \psi = a_*(\phi \boxtimes \psi).$$

The product and convolution will be transformed into one another in Section 1.1.11 by another useful operation, the Fourier-type transform.

### 1.1.10 Hard Lefschetz type theorems

The product and the convolution on valuations enjoy another non-trivial property, analogous to the hard Lefschetz theorem from algebraic geometry [36]. Let us fix a Euclidean metric on $V$. Consider the operator

$$L\colon \operatorname{Val}_*^{\mathrm{sm}} \longrightarrow \operatorname{Val}_{*+1}^{\mathrm{sm}}$$

given by $L\phi := \phi \cdot V_1$, where $V_1$ is the first intrinsic volume introduced in Section 1.1.3. Consider also another operator

$$\Lambda\colon \operatorname{Val}_*^{\mathrm{sm}} \longrightarrow \operatorname{Val}_{*-1}^{\mathrm{sm}},$$

defined by $(\Lambda\phi)(K) = \frac{d}{d\varepsilon}\phi(K+\varepsilon\cdot D)\big|_{\varepsilon=0}$, where $D$ is the unit ball (here we use again McMullen's theorem that $\phi(K+\varepsilon\cdot D)$ is a polynomial). Notice that up to a normalizing constant, the operator $\Lambda$ is equal to the convolution with $V_{n-1}$, as was observed by Bernig and Fu [24] (here is a hint: use Theorem 1.1.26 and the fact that $\Lambda(\mathrm{vol}_n)$ is equal to a constant times $V_{n-1}$).

**Theorem 1.1.27.** (i) *Let $0 \le i < \frac{1}{2}n$. Then $L^{n-2i}\colon \operatorname{Val}_i^{sm} \to \operatorname{Val}_{n-i}^{sm}$ is an isomorphism.*

(ii) *Let $\frac{1}{2}n < i \le n$. Then $\Lambda^{2i-n}\colon \operatorname{Val}_i^{sm} \to \operatorname{Val}_{n-i}^{sm}$ is an isomorphism.*

Several people have contributed to the proof of this theorem. First the author proved (i) and (ii) in the even case [3, 6], using previous joint work with Bernstein [17] and integral geometry on Grassmannians (Radon and cosine transforms). Then Bernig and Bröcker [23] proved part (ii) in the odd case, using a very different method: the Laplacian acting on differential forms on the sphere bundle and some results from complex geometry (Kähler identities). Next Bernig and Fu have shown [24] that, in the even case, the two versions of the hard Lefschetz theorem are in fact equivalent via the Fourier transform (which was at that time defined only for even valuations). Finally, the author extended in [15] the Fourier transform to odd valuations and derived version (i) of the hard Lefschetz theorem in the odd case from version (ii).

### 1.1.11 A Fourier-type transform on translation-invariant convex valuations

A Fourier-type transform on translation-invariant smooth valuations is another useful operation. First it was introduced in [3] (under the different name of duality) for even valuations and was applied there to Hermitian integral geometry in order to construct a new basis in the space of $U(n)$-invariant valuations on $\mathbb{C}^n$. In the odd case it was constructed in [15]. Some recent applications and non-trivial computations of the Fourier transform in Hermitian integral geometry are due to Bernig and Fu [25].

In this section we will describe the general properties of the Fourier transform and its relation to the product and convolution described above. We will present a construction of the Fourier transform in the even case only. The construction in the odd case is more technical. Notice that the even case will suffice for a reader interested mostly in applications to integral geometry of affine spaces, since by a result of Bernig [20] any $G$-invariant valuation from Val must be even, provided $G$ is a compact subgroup of the orthogonal group acting transitively on the unit sphere.

The main general properties of the Fourier transform are summarized in the following theorem. Part (2) for even valuations was proved in [24], while the general case and parts (1) and (3) were proved in [15].

**Theorem 1.1.28.** *There exists an isomorphism of linear topological spaces*

$$\mathbb{F}\colon \mathrm{Val}^{sm}(V) \longrightarrow \mathrm{Val}^{sm}(V^*) \otimes D(V)$$

*which satisfies the following properties:*

(1) $\mathbb{F}$ *commutes with the natural actions of the group $GL(V)$ on the two spaces.*
(2) $\mathbb{F}$ *is an isomorphism of algebras when the source is equipped with the product and the target with the convolution.*
(3) *A Plancherel-type inversion formula holds for $\mathbb{F}$, as explained below.*

In order to describe the Plancherel-type formula and present a more explicit description of the Fourier transform, it is convenient (but not necessary) to fix a Euclidean metric on $V$. This induces identifications $V^* \simeq V$ and $D(V^*) \simeq \mathbb{C}$. With these identifications, $\mathbb{F}\colon \mathrm{Val}^{\mathrm{sm}}(V) \to \mathrm{Val}^{\mathrm{sm}}(V)$; actually it changes the degree of homogeneity:

$$\mathbb{F}\colon \mathrm{Val}_i^{\mathrm{sm}} \xrightarrow{\sim} \mathrm{Val}_{n-i}^{\mathrm{sm}}.$$

The Plancherel-type formula says, under these identifications, that $(\mathbb{F}^2\phi)(K) = \phi(-K)$.

Here are a few simple examples: $\mathbb{F}(\mathrm{vol}) = \chi$; $\mathbb{F}(\chi) = \mathrm{vol}$; $\mathbb{F}(V_i) = c_{n,i}V_{n-i}$, where $c_{n,i} > 0$ is a normalizing constant that can be computed explicitly. (Notice that the last fact, except for the positivity of $c_{n,i}$, is an immediate consequence of the fact that $\mathbb{F}$ commutes with the action of $O(n)$ and Hadwiger's theorem.)

The Fourier transform on a 2-dimensional space has an explicit description which we are going to give now. We will work for simplicity in $\mathbb{R}^2$ with the standard Euclidean metric and standard orientation. With the identifications induced by the metric as above, $\mathbb{F}\colon \mathrm{Val}^{\mathrm{sm}}(\mathbb{R}^2) \to \mathrm{Val}^{\mathrm{sm}}(\mathbb{R}^2)$. It remains to describe $\mathbb{F}$ on 1-homogeneous valuations. Every such smooth valuation $\phi$ can be written uniquely in the form

$$\phi(K) = \int_{S^1} h(\omega)\, dS_1(K,\omega),$$

where $h\colon S^1 \to \mathbb{C}$ is a smooth function which is orthogonal on $S^1$ to the 2-dimensional space of linear functionals. Let us decompose $h$ into the even and odd parts:

$$h = h_+ + h_-.$$

Let us decompose further the odd part $h_-$ into "holomorphic" and "anti-holomorphic" parts,

$$h_- = h_-^{\mathrm{hol}} + h_-^{\mathrm{anti}},$$

as follows. Expand $h_-$ in the usual Fourier series on the circle $S^1$:

$$h_-(\omega) = \sum_k \hat{h}_-(k) e^{ik\omega}.$$

By definition,

$$h_-^{\mathrm{hol}}(\omega) := \sum_{k>0} \hat{h}_-(k) e^{ik\omega},$$
$$h_-^{\mathrm{anti}}(\omega) := \sum_{k<0} \hat{h}_-(k) e^{ik\omega}.$$

Then the Fourier transform of the valuation $\phi$ is equal to

$$(\mathbb{F}\phi)(K) = \int_{S^1} (h_+(J\omega) + h_-^{\mathrm{hol}}(J\omega))\, dS_1(K,\omega) - \int_{S^1} h_-^{\mathrm{anti}}(J\omega)\, dS_1(K,\omega),$$

where $J$ is the counterclockwise rotation of $\mathbb{R}^2$ by $\pi/2$. (Notice the minus sign in front of the second integral.) Observe that $\mathbb{F}$ preserves the class of real-valued even valuations, but not that of odd real-valued valuations. This phenomenon also holds in higher dimensions.

Let us consider even valuations in arbitrary dimension. We again fix a Euclidean metric on $V$. A useful tool in studying even valuations is an imbedding of $\mathrm{Val}_i^+(V)$ into the space of continuous functions on the Grassmannian $\mathrm{Gr}_i(V)$ of $i$-dimensional subspaces of $V$. It was constructed by Klain [44] as an easy consequence of his classification of simple even valuations (Theorem 1.1.14(i)). Define the map

$$\mathrm{Kl}_i \colon \mathrm{Val}_i^+(V) \longrightarrow C(\mathrm{Gr}_i(V))$$

as follows. Let $\phi \in \mathrm{Val}_i^+(V)$. For any $E \in \mathrm{Gr}_i(V)$, the restriction of $\phi$ to $E$ is a valuation of maximal degree of homogeneity. Hence, by a result of Hadwiger, it must be proportional to the Lebesgue measure $\mathrm{vol}_E$ induced by the Euclidean metric. Thus, by definition,

$$\phi|_E = (\mathrm{Kl}_i(\phi))\,(E) \cdot \mathrm{vol}_E\,.$$

Clearly, $\mathrm{Kl}_i(\phi)$ is a continuous function and $\mathrm{Kl}_i$ is a continuous linear $O(n)$-equivariant linear map. The non-trivial fact is that $\mathrm{Kl}_i$ is *injective*. For that, we observe that if $\phi \in \ker(\mathrm{Kl}_i)$, then the restriction of $\phi$ to any $(i+1)$-dimensional subspace is a simple, even, $i$-homogeneous valuation. Hence it vanishes by Klain's theorem. Proceeding by induction, one sees that $\phi = 0$.

Next it is not hard to see that smooth valuations are mapped under $\mathrm{Kl}_i$ to infinitely smooth functions on $\mathrm{Gr}_i(V)$. Let us define the Fourier transform $\mathbb{F}\colon \mathrm{Val}_i^{+,sm} \to \mathrm{Val}_{n-i}^{+,sm}$ by the following property: for any subspace $E \in \mathrm{Gr}_{n-i}(V)$,

$$(\mathrm{Kl}_{n-i}(\mathbb{F}\phi))(E) = (\mathrm{Kl}_i(\phi))(E^\perp),$$

where as usual $E^\perp$ denotes the orthogonal complement. This condition characterizes $\mathbb{F}$ uniquely in the even case. The non-trivial point is the existence of $\mathbb{F}$ with this property. The problem is that the Klain imbedding $\mathrm{Kl}_i\colon \mathrm{Val}_i^{+,sm}(V) \hookrightarrow C^\infty(\mathrm{Gr}_i(V))$ is not onto (for $i \neq 1, n-1$). The main point is to show that this image is invariant under taking the orthogonal complement. It was shown by Bernstein and the author [17] that the image of $\mathrm{Kl}_i$ coincides with the image of the so-called cosine transform on Grassmannians; this step used also the irreducibility theorem. From the definition of the cosine transform (which we do not reproduce here) it is easy to see that its image is invariant under taking the orthogonal complement.

Let us add a couple of words on the odd case. There is a version of Klain's imbedding for odd valuations, though it is more complicated: $\mathrm{Val}_i^{-,sm}(V)$ is realized as a quotient of a subspace of functions on a manifold of partial flags (here instead of Klain's characterization of simple even valuations one has to use Schneider's version for odd valuations —Theorem 1.1.14(ii)). We call it Schneider's imbedding. However, there is no direct analogue of the "cosine transform" description of the image of it. A more delicate analysis is required; it is based (besides the irreducibility theorem) on a more detailed study of the action of $GL(n,\mathbb{R})$ on valuations and on functions (or, rather, sections of an equivariant line bundle) on partial flags. This requires more tools from infinite-dimensional representation theory of the group $GL(n,\mathbb{R})$. We refer for the details to [15].

Another important property of the Fourier transform is that it intertwines the pull-back and push-forward on valuations. We will formulate this property in a non-rigorous way to avoid various technicalities making the formal statement heavier (see [14]). Let $f\colon V \to W$ be a linear map. Let $f^\vee\colon W^* \to V^*$ be the dual map between the dual spaces. Then the claim is that one should have

$$\mathbb{F}_V \circ f^* = (f^\vee)_* \circ \mathbb{F}_W, \tag{1.1.4}$$

where $f^*$ is the pull-back under $f$, $(f^\vee)_*$ is the push-forward under $f^\vee$, and $\mathbb{F}_V$ and $\mathbb{F}_W$ are the Fourier transforms on $V$ and $W$, respectively. Notice that the equality (1.1.4) is formally ill-defined if $f$ is not an isomorphism. This is due to the fact that $\mathbb{F}$ is formally defined only on the class of smooth valuations, while $f^*$ and $(f^\vee)_*$ do not preserve this class. Nevertheless, in principle this equality should be true, yet technically one should be more accurate.

Moreover, one expects that in some sense the Fourier transform should commute with the exterior product:

$$\mathbb{F}(\phi \boxtimes \psi) = \mathbb{F}\phi \boxtimes \mathbb{F}\psi.$$

The difficulty here is that the exterior product of smooth valuations is usually not smooth.

As a last remark, let us mention that it would be desirable to have a more direct construction of the Fourier transform. For example, we still do not know how to describe it in terms of the construction of valuations using integration with respect to the normal cycle discussed below in Section 1.1.12.

### 1.1.12 General constructions of translation-invariant convex valuations

So far the only construction of valuations we have seen is the mixed volume. In this section we review some other general constructions of translation-invariant continuous convex valuations. In Section 1.1.12 we describe briefly an array of examples coming from integral geometry —a more complete treatment will be given in Fu's lectures. In Section 1.1.12 we describe another very general and useful construction via integration over the normal cycle of a set; this construction will be generalized appropriately to the context of valuations on manifolds in Section 1.2. There is yet another construction of valuations based on complex and quaternionic pluripotential theory. It is somewhat more specialized and will not be discussed here; we refer to [7, 13] and the survey [12].

#### Integral geometry

Let us give a few basic examples which arise naturally in (Crofton-style) integral geometry. The classical reference to this type of integral geometry is Santaló's book [56]. For further discussions of this type of integral geometry and its relations to valuation theory we refer to Fu's lectures, the book [45], and the articles [3, 19, 20, 21, 25, 33] (these recent results are surveyed by Bernig [22]).

Let $V = \mathbb{R}^n$ be the standard Euclidean space. Let $\mathrm{Gr}_{k,n}$ denote the Grassmannian of all linear $k$-dimensional subspaces of $V$, and let $\overline{\mathrm{Gr}}_{k,n}$ denote the Grassmannian of affine $k$-dimensional subspaces. It is not hard to check that the following expressions are continuous valuations invariant with respect to all isometries of $\mathbb{R}^n$:

$$\phi(K) = \int_{E\in \mathrm{Gr}_{k,n}} V_i(\mathrm{pr}_E(K))\, dE, \tag{1.1.5}$$

$$\psi(K) = \int_{E\in \overline{\mathrm{Gr}}_{k,n}} V_i(K\cap E)\, dE, \tag{1.1.6}$$

where $dE$ denotes in both formulas a Haar measure on the corresponding Grassmannian, and $\mathrm{pr}_E\colon \mathbb{R}^n \to E$ denotes the orthogonal projection. These expressions have been studied quite extensively in the classical integral geometry; they can be computed as integrals of certain expressions of the principal curvatures of the boundary $\partial K$, at least under appropriate smoothness assumptions on $\partial K$; see, e.g., [45, 56]. Notice that Hadwiger's theorem implies that these valuations can be written as linear combinations of intrinsic volumes $V_0, V_1, \ldots, V_n$, with coefficients that can be computed explicitly.

Let us present analogous expressions from the Hermitian integral geometry of $\mathbb{C}^n$. Despite the obvious similarity to the Euclidean case, these expressions have been studied in depth only quite recently [3, 25, 33]. Let ${}^{\mathbf{C}}\mathrm{Gr}_{k,n}$ denote the Grassmannian of complex linear $k$-dimensional subspaces of $\mathbb{C}^n$, and ${}^{\mathbf{C}}\overline{\mathrm{Gr}}_{k,n}$ the Grassmannian of complex affine $k$-dimensional subspaces. Let us define, in analogy with (1.1.5)–(1.1.6),

$$\phi(K) = \int_{E \in {}^{\mathbf{C}}\mathrm{Gr}_{k,n}} V_i(\mathrm{pr}_E(K))\, dE, \tag{1.1.7}$$

$$\psi(K) = \int_{E \in {}^{\mathbf{C}}\overline{\mathrm{Gr}}_{k,n}} V_i(K \cap E)\, dE, \tag{1.1.8}$$

where $dE$ again denotes a Haar measure on the appropriate complex Grassmannian. It was shown in [3] that from valuations of the form (1.1.7) (or alternatively, (1.1.8)) one can choose a basis of unitarily invariant valuations in $\mathrm{Val}(\mathbb{C}^n)$. Moreover, in the same paper it was shown that the Fourier transform of a valuation of the form (1.1.7) has the form (1.1.8) with appropriately chosen $i$ and $k$, and vice versa. Some different bases in unitarily invariant valuations have been constructed by Bernig and Fu in [25], where they also computed several integral geometric formulas in $\mathbb{C}^n$, in particular the principal kinematic formula.

### Normal cycle

In this section we recall the notion of the normal cycle of a convex set and use it to construct translation-invariant smooth valuations on convex sets. In fact, we will see that all such valuations can be obtained using this construction. One of the important aspects of this construction is that it generalizes to a broader context of valuations on manifolds to be discussed in Section 1.2.

In this section we will fix again a Euclidean metric and an orientation on a vector space $V$ with $\dim V = n$, for the convenience of a geometrically oriented reader. However, this metric is not necessary, and in Section 1.2.1 we describe an extension of the construction of a normal cycle to any smooth manifold without any additional structure (not for convex sets of course, but for compact submanifolds with corners).

Let $K \in \mathcal{K}(V)$ be a convex compact subset of $V$. For any point $x \in K$ let us define the normal cone of $K$ at $x$ as a subset of the unit sphere $S^{n-1}$ (see, e.g, [58, p. 70]):

$$N(K,x) := \big\{ u \in S^{n-1} \mid (u, y-x) \le 0 \text{ for any } y \in K \big\}.$$

It is clear that $N(K,x)$ is non-empty if and only if $x$ belongs to the boundary of $K$. Now define the normal cycle of $K$ by

$$N(K) := \bigcup_{x\in K} \{(x,u) \mid u \in N(K,x)\}.$$

It is not hard to see that $N(K)$ is a closed subset of $V \times S^{n-1}$. Moreover, it is locally bi-Lipschitz equivalent[1] to $\mathbb{R}^{n-1}$, and hence integration of smooth differential $(n-1)$-forms on $V \times S^{n-1}$ over $N(K)$ defines a continuous linear functional on such forms (more precisely, $N(K)$ can be considered as an integral $(n-1)$-current). A proof of the following result can be found in [18]; it is based on some geometric measure theory and previous work of Fu [28, 29, 30, 32] and other people [62, 63] on normal cycles (the references can be found in [18]).

**Proposition 1.1.29.** *Let $\omega$ be an infinitely smooth $(n-1)$-form on $V \times S^{n-1}$. Then the functional*

$$K \longmapsto \int_{N(K)} \omega$$

*is a continuous valuation on $\mathcal{K}(V)$. If moreover $\omega$ is invariant with respect to translations in $V$, then the above expression is a smooth translation-invariant valuation in the sense of Definition* 1.1.17.

Let us denote by $\Omega^{n-1}_{\mathrm{tr}}(V \times S^{n-1})$ the space of infinitely smooth $(n-1)$-forms on $V \times S^{n-1}$ which are invariant with respect to translations on $V$.

**Proposition 1.1.30** ([8, Theorem 5.2.1])**.** *The linear map $\mathbb{C} \oplus \Omega^{n-1}_{\mathrm{tr}}(V \times S^{n-1}) \to \mathrm{Val}^{sm}(V)$ given by*

$$(a,\omega) \longmapsto a \cdot \mathrm{vol}(K) + \int_{N(K)} \omega$$

*is continuous and onto.*

The proof of this theorem is based on the observation that the map in the proposition can be rewritten in metric-free terms so that it will commute with the action of the full linear group $GL(V)$. The irreducibility theorem implies that the image of this map is dense in $\mathrm{Val}^{\mathrm{sm}}(V)$. The fact that the image is closed follows from a rather general representation-theoretical result due to Casselman and Wallach, which says that any morphism between two $GL(V)$-representations in Fréchet spaces satisfying appropriate technical conditions has a closed image (see [8, Theorem 1.1.5] for a precise statement and references).

The kernel of this map was described by Bernig and Bröcker [23] by a system of differential and integral equations. Bernig has applied very successfully this description in classification problems of translation-invariant valuations invariant under various groups acting transitively on spheres [19, 20, 21].

[1]This fact was communicated to me by Joe Fu. Unfortunately, I have no reference to it.

### 1.1.13 Valuations invariant under a group

Let $G$ be a compact subgroup of the group of orthogonal transformations of a Euclidean $n$-dimensional space $V$. We denote by $\mathrm{Val}^G$ the subspace of $\mathrm{Val}(V)$ of $G$-invariant valuations. When $G = SO(n)$, the space $\mathrm{Val}^G$ was described by Hadwiger (see Section 1.1.3). There are examples of other groups, such as the unitary group $U(n/2)$, of particular interest to integral geometry. In fact, whenever the space $\mathrm{Val}^G$ turns out to be finite-dimensional, we may hope to classify it explicitly in geometric terms and then apply this classification to integral geometry, for example to obtain generalizations of Crofton and principal kinematic formulas. The first general result in this direction is as follows.

**Proposition 1.1.31** ([1])**.** *Let $G$ be a compact subgroup of the orthogonal group. The space $\mathrm{Val}^G$ is finite-dimensional if and only if $G$ acts transitively on the unit sphere.*

Recall also that, by Proposition 1.1.18, if $G$ acts transitively on the sphere, then $\mathrm{Val}^G \subset \mathrm{Val}^{\mathrm{sm}}$. This equips $\mathrm{Val}^G$ with the product. Evidently, we have also McMullen's decomposition

$$\mathrm{Val}^G = \bigoplus_{i=0}^{n} \mathrm{Val}_i^G .$$

Thus $\mathrm{Val}^G$ becomes a finite-dimensional commutative associative graded algebra with unit. It satisfies Poincaré duality and two versions of the hard Lefschetz theorem as in Section 1.1.10. Moreover, it was shown by Bernig [20] that for such $G$ all $G$-invariant valuations are even. Next, $\mathrm{Val}_1^G = \mathbb{C} \cdot V_1$ and $\mathrm{Val}_{n-1}^G = \mathbb{C} \cdot V_{n-1}$ by [4].

In topology there is an explicit classification of compact connected Lie groups acting transitively and effectively on spheres due to A. Borel and Montgomery and Samelson. There are 6 infinite lists

$$SO(n),\ U(n),\ SU(n),\ Sp(n),\ Sp(n) \cdot Sp(1),\ Sp(n) \cdot U(1),$$

and three exceptional groups

$$\mathrm{Spin}(7),\ \mathrm{Spin}(9),\ \text{and } G_2.$$

Valuations in the case of $SO(n)$ were completely studied by Hadwiger [39]. The next interesting case is the unitary group $U(n)$. This case turned out to be more complicated than $SO(n)$ and in recent years there was considerable progress in it. There is a complete geometric classification [3], the description of the algebra structure [33], and the principal kinematic formula [25]. This is discussed in detail in Fu's lectures. For most of the other groups, new strong results with applications to integral geometry were obtained recently by Bernig in a series of articles. We refer to his survey [22] reporting on the progress.

## 1.2 Valuations on manifolds

The notion of valuation on smooth manifolds was introduced by the author in [9]. The goal of this section is to describe this notion, its properties, and some applications to integral geometry established in [8, 9, 10, 14, 16, 18]. In particular, we extend the product construction to the setting of valuations on manifolds and explain its intuitive meaning. This intuitive interpretation is based on another useful notion of generalized valuation, which establishes an explicit link between valuation theory and a better studied notion of constructible functions. The usefulness of this comparison will be illustrated with several other examples. Next we introduce operations of pull-back and push-forward under smooth maps of manifolds in a number of important special cases, generalizing familiar operations on smooth functions, measures, and constructible functions. All these structures are eventually used to define a general Radon-type transform on valuations which generalizes the classical Radon transforms on smooth and constructible functions.

### 1.2.1 Definition of smooth valuations on manifolds and basic examples

The original approach [9] to define smooth valuations on smooth manifolds was rather technical. In these notes we will follow a different, more direct and actually equivalent approach, which however might look less natural and less motivated.

Let $X$ be a smooth manifold[2] of dimension $n$. We describe a certain class of finitely additive measures on nice subsets of $X$ (to be more precise, on compact submanifolds with corners). In our current approach this class is defined by the explicit construction of integration of a differential form with respect to the normal cycle. While in the original approach [8] this description was a theorem rather than a definition, it seems to be faster not to repeat all the intermediate steps leading to it. A reader may take Proposition 1.1.30 above as a possible justification for the current approach.

A submanifold with corners of $X$ is a closed subset $P \subset X$ which is locally diffeomorphic to $\mathbb{R}^i_{\geq 0} \times \mathbb{R}^j$ (then necessarily $0 \leq i+j \leq n$). We denote by $\mathcal{P}(X)$ the family of all compact submanifolds with corners. Basic examples from $\mathcal{P}(X)$ are compact smooth submanifolds, possibly with boundary. When $X = \mathbb{R}^n$, simplices, or more generally, simple polytopes of any dimension belong to $\mathcal{P}(X)$; however, non-simplicial polytopes (such as the octahedron in $\mathbb{R}^3$) do not.

We are going to define the normal cycle of $P \in \mathcal{P}(X)$. Let $T^*X$ denote the cotangent bundle of $X$. Let $\mathbb{P}_X$ denote the oriented projectivization of $T^*X$, namely $\mathbb{P}_X := (T^*X \backslash \underline{0})/\mathbb{R}_{>0}$, where $\underline{0}$ is the zero-section of $T^*X$, and $\mathbb{R}_{>0}$ is the multiplicative group of positive real numbers acting on $T^*X$ by multiplication on the cotangent vectors. We call $\mathbb{P}_X$ the *cosphere bundle* since if one fixes a Riemann-

[2] All manifolds are assumed to be countable at infinity, i.e., presentable as a union of countably many compact subsets.

ian metric on $X$, then it induces an identification of $\mathbb{P}_X$ with the unit (co)tangent bundle.

Let $P \in \mathcal{P}(X)$, and let $x \in P$ be a point. The *tangent cone* to $P$ at $x$ is the subset of the tangent space $T_xX$ consisting of all $\xi$ such that there exists a $C^1$-smooth curve $\gamma\colon [0,1] \to P$ such that $\gamma(0) = x$ and $\gamma'(0) = \xi$. It it not hard to see that $T_xP \subset T_xX$ is a closed convex polyhedral cone. Let $(T_xP)^o$ denote the dual cone, namely

$$(T_xP)^o := \big\{\eta \in T_x^*X \mid \langle \eta, \xi\rangle \leq 0 \text{ for any } \xi \in T_xP\big\}.$$

This is a closed convex cone in $T^*X$. Now define the *normal cycle* of $P$ by

$$N(P) := \bigcup_{x\in P} \big(((T_xP)^o\backslash\{0\})/\mathbb{R}_{>0}\big).$$

It is well known (and easy to see) that $N(P)$ is a compact $(n-1)$-dimensional submanifold of $\mathbb{P}_X$ with singularities (warning: it is not a manifold with corners in general; it is smooth outside of a subset of codimension one). Also it is Legendrian with respect to the canonical contact structure on $\mathbb{P}_X$, though this fact will not be used explicitly in these lectures.

**Remark 1.2.1.** If $X = \mathbb{R}^n$ and $P \in \mathcal{P}(\mathbb{R}^n)$ is convex, then this definition of the normal cycle coincides with the definition of the normal cycle from Section 1.1.12. Actually, the normal cycle can be defined for other classes of sets: sets of positive reach (which includes convex compact sets in the case $X = \mathbb{R}^n$), and subanalytic sets when $X$ is a real analytic manifold (see Fu [32], which is based on [28, 29, 30, 31] and develops further [27, 62, 63]). An essentially equivalent notion of characteristic cycle was developed in [42] for subanalytic sets using a different approach.

Below in this exposition we will assume for simplicity of exposition that $X$ is oriented; this assumption can be easily removed. The orientation of $X$ induces an orientation of the normal cycle of every subset.

**Definition 1.2.2.** A map $\phi\colon \mathcal{P}(X) \to \mathbb{C}$ is called a *smooth valuation* if there exist a measure $\mu$ on $X$ and an $(n-1)$-form $\omega$ on $\mathbb{P}_X$, both infinitely smooth, such that

$$\phi(P) = \mu(P) + \int_{N(P)} \omega$$

for any subset $P \in \mathcal{P}(X)$.

**Remark 1.2.3.** This definition should be compared with Proposition 1.1.30. It can be shown that any translation-invariant *convex* valuation on $\mathbb{R}^n$ which is smooth in the sense of Definition 1.1.17 can be naturally extended to a broader class of sets: to compact sets of positive reach and also to relatively compact subanalytic subsets of $\mathbb{R}^n$. This is done as follows: given a convex valuation $\phi \in \mathrm{Val}^{\mathrm{sm}}(\mathbb{R}^n)$, let us represent it (non-uniquely) in the form

$$\phi(K) = a \cdot \mathrm{vol}(K) + \int_{N(K)} \omega,$$

where $\omega$ is a smooth translation-invariant form. Then $\phi$ can be extended by the same formula to any subset from the above broader class; this extension is independent of the choice of the form $\omega$ and the constant $a$ (see [9, Lemma 2.4.7]).

It can be shown that every smooth valuation is a finitely additive functional in some precise sense [9].

Let us denote by $V^\infty(X)$ the space of all smooth valuations. The space $V^\infty(X)$ is the main object of study in what follows.

**Examples 1.2.4.** (1) Any smooth measure on $X$ is a smooth valuation. Indeed, take $\omega = 0$ in Definition 1.2.2.

(2) The Euler characteristic $\chi$ is also a smooth valuation. This fact is less obvious. In the current approach, it is a reformulation of a version of the Gauss–Bonnet formula due to Chern [26], who has constructed $\mu$ and $\omega$ to represent the Euler characteristic; his construction depends on the choice of a Riemannian metric on $X$.

(3) The next example is very typical for integral geometry. Let $X = \mathbb{CP}^n$ be the complex projective space. Let ${}^{\mathbf{C}}\mathrm{Gr}$ denote the Grassmannian of all complex projective subspaces of $\mathbb{CP}^n$ of a fixed complex dimension $k$. It is well known that ${}^{\mathbf{C}}\mathrm{Gr}$ has a unique probability measure $dE$ invariant under the group $U(n+1)$. Consider the functional

$$\phi(P) = \int_{E \in {}^{\mathbf{C}}\mathrm{Gr}} \chi(P \cap E)\, dE.$$

Then $\phi \in V^\infty(\mathbb{CP}^n)$ —this follows, e.g., from Fu [31].

$V^\infty(X)$ is naturally a Fréchet space. Indeed, it is a quotient space of the direct sum of Fréchet spaces $\mathcal{M}^\infty(X) \oplus \Omega^{n-1}(\mathbb{P}_X)$ by a closed subspace, where $\mathcal{M}^\infty(X)$ denotes the space of infinitely smooth measures. The subspace of pairs $(\mu, \omega)$ representing the zero valuation was described by Bernig and Bröcker [23] in terms of a system of differential and integral equations.

One can show [9] that smooth valuations form a sheaf. This means that

(1) we have the natural restriction map $V^\infty(U) \to V^\infty(V)$ for any open subsets $V \subset U \subset X$;

(2) given an open covering $\{U_\alpha\}$ of an open subset $U$, and $\phi \in V^\infty(U)$ such that the restriction $\phi|_{U_\alpha}$ of $\phi$ to all $U_\alpha$ vanishes, one has $\phi = 0$;

(3) given an open covering $\{U_\alpha\}$ of an open subset $U$ and $\phi_\alpha \in V^\infty(U_\alpha)$ for any $\alpha$ such that $\phi_\alpha|_{U_\alpha \cap U_\beta} = \phi_\beta|_{U_\alpha \cap U_\beta}$ for all $\alpha, \beta$, there exists (uniquely by (2)) $\phi \in V^\infty(U)$ such that $\phi|_{U_\alpha} = \phi_\alpha$.

### 1.2.2 Canonical filtration on smooth valuations

The space of smooth valuations carries a canonical filtration by closed subspaces. In this section we summarize its main properties without giving a precise definition, for which we refer to [9]. The important property of this filtration is that it partly allows to reduce the study of valuations on manifolds to the more familiar case of translation-invariant convex valuations. A more explicit geometric property of this filtration is given at the end of Section 1.2.5.

Let us denote by $\mathrm{Val}(TX)$ the (infinite-dimensional) vector bundle over $X$ such that its fiber over a point $x \in X$ is equal to the space $\mathrm{Val}^\infty(T_xX)$ of smooth translation-invariant convex valuations on $T_xX$. By McMullen's theorem, it has a grading by the degree of homogeneity: $\mathrm{Val}^\infty(TX) = \bigoplus_{i=0}^n \mathrm{Val}_i^\infty(TX)$.

**Theorem 1.2.5.** *There exists a canonical filtration of $V^\infty(X)$ by closed subspaces*

$$V^\infty(X) = W_0 \supset W_1 \supset \cdots \supset W_n, \quad n = \dim X,$$

*such that the associated graded space $\bigoplus_{i=0}^n W_i/W_{i+1}$ is canonically isomorphic to the space of smooth sections $C^\infty(X, \mathrm{Val}_i^\infty(TX))$.*

**Remarks 1.2.6.** (1) For $i = n$, the above isomorphism means that $W_n$ coincides with the space of smooth measures on $X$.

(2) For $i = 0$, the above isomorphism means that $W_0/W_1$ is canonically isomorphic to the space of smooth functions $C^\infty(X)$. The epimorphism $V^\infty(X) \to C^\infty(X)$ with kernel $W_1$ is just the point evaluation map

$$\phi \longmapsto [x \longmapsto \phi(\{x\})].$$

Thus $W_1$ consists precisely of valuations vanishing on all points.

(3) Actually $U \mapsto W_i(U)$ defines a subsheaf $\mathcal{W}_i$ of the sheaf of valuations.

### 1.2.3 Integration functional

Let $V_c^\infty(X)$ denote the subspace of $V^\infty(X)$ of compactly supported valuations. (The definition is obvious: a valuation $\phi$ is said to have compact support if there exists a compact subset $A \subset X$ such that the restriction $\phi|_{X\setminus A}$ is zero.) Clearly if $X$ is compact then $V_c^\infty(X) = V^\infty(X)$. The space $V_c^\infty(X)$ carries a natural locally convex topology such that the natural imbedding to $V^\infty(X)$ is continuous (however in general this is not a Fréchet space, but rather a strict inductive limit of Fréchet spaces; see [10, Section 5.1]).

The integration functional

$$\int_X : V_c^\infty(X) \longrightarrow \mathbb{C}$$

is defined by $\int_X \phi := \phi(X)$.

Formally speaking, $\phi(X)$ is not defined when $X$ is not compact. The formal way to define it is to choose first a large compact domain $A$ containing the support of $\phi$ and set $\int_X \phi := \phi(A)$. Then one can show that this definition is independent of the large subset $A$. Moreover, $\int_X$ is a continuous linear functional.

### 1.2.4 Product operation on smooth valuations on manifolds and Poincaré duality

The product on smooth translation-invariant convex valuations, which was discussed in Section 1.1.7, can be extended to the case of smooth valuations on manifolds. We will describe below its main properties, and in Section 1.2.7 we will explain its intuitive meaning. However we present no construction of it in these notes.

For the moment there are two different constructions of the product, both rather technical. The first one was done in several steps. Initially, the product was constructed by the author [8] on $\mathbb{R}^n$ (earlier the same construction was carried out in an even more specific situation [4] of convex valuations that are polynomial with respect to translations). Then this construction was extended by Fu and the author [18] to any smooth manifold: it was shown that the product can be defined locally choosing of a diffeomorphism of $X$ with $\mathbb{R}^n$ and applying the above construction, and the main technical point was to show that the product is independent of the choice of this local diffeomorphism.

The second and rather different construction of the product was given recently by Bernig and the author [16]. This construction describes the product of valuations directly in terms of the forms $\mu$ and $\omega$ defining the valuations; it uses the Rumin operator and some other standard operations on differential forms. Compared to the first construction, the second one has the advantage of being independent of extra structures on $X$ (such as a coordinate system) and also some other technical advantages. However, it is less intuitive than the first one. We summarize basic properties of the product as follows.

**Theorem 1.2.7.** *There exists a canonical product $V^\infty(X) \times V^\infty(X) \to V^\infty(X)$ such that*

(1) *it is continuous;*
(2) *it is commutative and associative;*
(3) *the filtration $W_\bullet$ is compatible with it:*

$$W_i \cdot W_j \subset W_{i+j},$$

*where we set $W_k = 0$ for $k > n = \dim X$;*

(4) *the Euler characteristic $\chi$ is the unit in the algebra $V^\infty(X)$;*
(5) *it commutes with restrictions to open and closed submanifolds.*

*Thus $V^\infty(X)$ is a commutative associative filtered unital algebra over $\mathbb{C}$.*

Let us also add that the point evaluation map $V^\infty(X) \to C^\infty(X)$ defined in Remark 1.2.6(2) is an epimorphism of algebras when $C^\infty(X)$ is equipped with the usual pointwise product.

An important property of the product is a version of Poincaré duality. Consider the bilinear map

$$V^\infty(X) \times V_{\mathrm{c}}^\infty(X) \longrightarrow \mathbb{C}$$

defined by $(\phi, \psi) \mapsto \int_X \phi \cdot \psi$.

**Theorem 1.2.8.** *This bilinear form is a perfect pairing. In other words, the induced map*

$$V^\infty(X) \longrightarrow (V_{\mathrm{c}}^\infty(X))^*$$

*is injective and has a dense image with respect to the weak topology.*

### 1.2.5 Generalized valuations and constructible functions

**Definition 1.2.9.** The space of generalized valuations is defined as

$$V^{-\infty}(X) := (V_{\mathrm{c}}^\infty(X))^*,$$

equipped with the weak topology. Elements of this space are called *generalized valuations*.

By Theorem 1.2.8 we have the canonical imbedding with dense image

$$V^\infty(X) \hookrightarrow V^{-\infty}(X).$$

Informally speaking, at least when $X$ is compact, the space of valuations is essentially self-dual (up to completion). This imbedding also means that $V^{-\infty}(X)$ is a completion of $V^\infty(X)$ in the weak topology. Every smooth valuation can be considered as a generalized one.

The advantage of working with generalized valuations is that they contain the constructible functions (described below) as a dense subspace. This gives a completely different point of view on valuations which is often useful, especially on a heuristic level. Constructible functions have been studied quite extensively by methods of algebraic topology (sheaf theory; see the book [42, Ch. 9]). We will illustrate this below while discussing again the product on valuations, a Radon-type transform, and the Euler–Verdier involution.

Let us define the space of constructible functions on $X$. In the literature there are various, slightly different definitions of this notion, but the differences are technical rather than conceptual. For simplicity of exposition, we will assume in these notes, while talking about constructible functions, that $X$ is a real analytic manifold.

**Definition 1.2.10.** A function $f\colon X \to \mathbb{C}$ on a real analytic manifold $X$ is called *constructible* if it takes finitely many values and for any $a \in \mathbb{C}$ the level set $f^{-1}(a)$ is subanalytic.

For the definition of a subanalytic set, see [10, Section 1.2], or for more details [42, §8.2]. Constructible functions with compact support form a linear space which will be denoted by $\mathcal{F}(X)$. Moreover, $\mathcal{F}(X)$ is an algebra with pointwise product.

An important property of constructible functions is that they also admit a normal cycle such that if $P \in \mathcal{P}(X)$ is subanalytic, then the normal cycle of the indicator function $\mathbb{1}_P$ is equal to the normal cycle of $P$ (see [32] and [42, Ch. 9]). Using this notion we define the map

$$\Xi\colon \mathcal{F}(X) \longrightarrow V^{-\infty}(X)$$

as follows. Let $\phi \in V_c^\infty(X)$ be given by $\phi(P) = \mu(P) + \int_{N(P)} \omega$, with smooth $\mu$ and $\omega$. Then, for any $f \in \mathcal{F}(X)$,

$$\langle \Xi(f), \phi \rangle = \int_X f \cdot d\mu + \int_{N(f)} \omega.$$

The map $\Xi$ is well defined, i.e., it is independent of the particular choice of $\mu$ and $\omega$ representing $\phi$. Moreover, $\Xi$ is linear injective with dense image [10, Section 8.1].

To summarize, we have a large space of generalized valuations with two completely different dense subspaces,

$$V^\infty(X) \subset V^{-\infty}(X) \supset \mathcal{F}(X). \tag{1.2.1}$$

Notice that, when $X$ is compact, the image of the constant function $1 \in \mathcal{F}(X)$ in $V^{-\infty}(X)$ coincides with the image of the Euler characteristic $\chi \in V^\infty(X)$. The two subspaces $V^\infty(X)$ and $\mathcal{F}(X)$ are very different: thus, for connected $X$, their intersection is spanned by $\chi$.

While working with valuations it is useful to keep in mind the imbeddings (1.2.1). The role of constructible functions in the theory of valuations is somewhat analogous to the role of delta-functions in the classical theory of generalized functions (distributions). It is often instructive to compare various structures on valuations with their analogues on constructible functions. We will see several examples of this below. Here we will show how this works for the integration functional and the filtration $W_\bullet$.

It was shown in [10] that the integration functional $\int_X : V_c^\infty \to \mathbb{C}$ extends uniquely by continuity in the weak topology to generalized valuations with compact support,

$$\int_X : V_c^{-\infty}(X) \longrightarrow \mathbb{C}.$$

Let us restrict this functional to the subspace $\mathcal{F}_c(X)$ of constructible functions with compact support. It turns out that this restriction coincides with the integration with respect to the Euler characteristic; this operation is uniquely characterized by the property that, for any compact subanalytic subset $P \subset X$,

$$\int_X \mathbb{1}_P = \chi(P).$$

Let us consider now the filtration $W_\bullet$ on $V^\infty(X)$. Let $W_i'$ denote the closure of $W_i$ in $V^{-\infty}(X)$ with respect to the weak topology. By [10] the restriction of $W_i'$ back to $V^\infty(X)$ coincides with $W_i$, i.e., $W_i' \cap V^\infty(X) = W_i$. Consider the induced filtration on constructible functions, namely

$$\mathcal{F}(X) = \mathcal{F}(X) \cap W_0' \supset \mathcal{F}(X) \cap W_1' \supset \cdots \supset \mathcal{F}(X) \cap W_n'.$$

It was shown in [10] that $\mathcal{F}(X) \cap W_i'$ consists of constructible functions whose support has codimension at most $i$. In particular, $\mathcal{F}(X) \cap W_n'$ consists of functions with discrete support.

### 1.2.6 Euler–Verdier involution

Let us give another example of an application of the comparison with constructible functions. The space of constructible functions has a canonical linear involution, called the Verdier involution (see, e.g., [42]). In the special case of functions on $\mathbb{R}^n$ which are constructible in a narrower (polyhedral) sense, this involution has been known to convexity experts under the name of Euler involution. We will see that it extends naturally to valuations, and this extension will be called the Euler–Verdier involution.

Here we will choose a sign normalization different from the standard one. Let us describe the Verdier involution $\sigma$ (with a different sign convention) in the special case when a constructible function has the form $1\!\!1_P$, where $P$ is a compact subanalytic submanifold with corners (the general case is not very far from this one using the linearity property of it). Then

$$\sigma(1\!\!1_P) = (-1)^{n-\dim P} 1\!\!1_{\operatorname{int} P},$$

where $\operatorname{int} P$ denotes the relative interior of $P$. One has $\sigma^2 = \mathrm{id}$.

**Theorem 1.2.11** ([10]). (1) *The involution $\sigma$ extends (uniquely) by continuity to $V^{-\infty}(X)$ in the weak topology. This extension is also denoted by $\sigma$.*

(2) *$\sigma$ preserves the class of smooth valuations and $\sigma\colon V^\infty(X) \to V^\infty(X)$ is a continuous linear operator (in the Fréchet topology).*

(3) $\sigma^2 = \mathrm{id}$.

(4) *$\sigma\colon V^\infty(X) \to V^\infty(X)$ is an algebra automorphism.*

(5) *$\sigma$ preserves the filtration $W_\bullet$, namely $\sigma(W_i) = W_i$.*

(6) *For any smooth homogeneous translation-invariant valuation $\phi$ on $\mathbb{R}^n$ one has*

$$(\sigma\phi)(K) = (-1)^{\deg \phi} \phi(-K),$$

*where $\deg \phi$ denotes the degree of homogeneity of $\phi$.*

(7) *$\sigma$ commutes with restrictions to open subsets (both for smooth and generalized valuations).*

**Remark 1.2.12.** Although the involution $\sigma$ was defined above only on a real analytic manifold $X$, it can be defined on any smooth manifold as a continuous linear operator $\sigma\colon V^{-\infty}(X) \to V^{-\infty}(X)$. Then it satisfies the properties (2)–(7) of Theorem 1.2.11.

### 1.2.7 Partial product operation on generalized valuations

In this section we discuss the promised intuitive meaning of the product on valuations. This interpretation was conjectured by the author [11] and proved rigorously by Bernig and the author [16]. It provides yet another example of the relevance of constructible functions to valuations.

Recall again that we have the imbedding of smooth valuations into the generalized ones

$$V^{\infty}(X) \subset V^{-\infty}(X).$$

One could try to extend the product on smooth valuations to $V^{-\infty}(X)$, say by continuity. Unfortunately, this is not possible. The situation here is much analogous to what is known in the classical theory of generalized functions (see, e.g., [41]). There the space of smooth functions $C^{\infty}(X)$ is naturally imbedded into the larger space of generalized functions $C^{-\infty}(X)$, which is the completion of the former in the weak topology. The space $C^{\infty}(X)$ has its usual pointwise product. However, this product does not extend to $C^{-\infty}(X)$ by continuity: for example, no rigorous way is known to take the square of the delta-function on $X = \mathbb{R}$. Nevertheless it is still possible to define a *partial* product on $C^{-\infty}(X)$. This roughly means that one can define a product of two generalized functions whose "singularities" are disjoint. The precise technical condition is formulated in the language of the wave front sets of generalized functions in the sense of Hörmander and Sato; we will not reproduce it here, but rather refer to [41]. This partial product is natural and enjoys some continuity properties [37, Ch. VI §3].

In the case of valuations we have the following result.

**Theorem 1.2.13** ([16])**.** *There exists a partial product on $V^{-\infty}(X)$ extending the product on $V^{\infty}(X)$. It is commutative and associative.*

We refer to [16] for the precise technical formulation when the partial product of two generalized valuations is defined. We notice only that the condition is also formulated in the language of wave front sets.

Now we can try to restrict the partial product on generalized valuations to constructible functions and see what we get. The answer is very natural: we just get their pointwise product (under certain technical assumptions on the functions guaranteeing that their product in $V^{-\infty}(X)$ is well defined). More precisely, we have the following result.

**Theorem 1.2.14** ([16])**.** *Let $P$, $Q \subset X$ be compact submanifolds with corners which intersect transversally. Then the product of $\mathbb{1}_P$ and $\mathbb{1}_Q$ in the sense of generalized* *valuations is well defined and is equal to* $\mathbb{1}_{P\cap Q}$ *(notice that, under the transversality assumption,* $P \cap Q$ *is also a compact submanifold with corners).*

We did not give a formal definition of transversality of two submanifolds with corners. In the special case of submanifolds without corners, the definition is the usual one. In the general case, the precise definition is given in [16]. Notice only that any two compact submanifolds with corners can be made transversal to each other by applying to one of them a generic diffeomorphism which is arbitrarily close (in the $C^\infty$-topology) to the identity map.

### 1.2.8 A heuristic remark

In this section we will make a general heuristic remark on valuations. In the next Section 1.2.9 we will show how it can be informally used in applications to integral geometry.

Let $\psi$ be a generalized valuation on a (say, real analytic) manifold $X$. Can we consider it as a finitely additive measure on $X$? The answer is "essentially" yes. This measure is partially defined: its value on a compact submanifold with corners or compact real analytic subset $P$, which is "in generic position" to $\psi$, is equal to $\int_X \psi \cdot \mathbb{1}_P$. Notice that, once the product is defined, the integral is defined too. When the valuation $\psi$ is smooth, this integral has a very clear meaning, namely one has

$$\int_X \psi \cdot \mathbb{1}_P = \psi(P). \tag{1.2.2}$$

Let us prove the last identity. First, by the definition of $\mathbb{1}_P$, $\psi(P) = \langle \mathbb{1}_P, \psi\rangle$. But for any generalized valuation $\Phi$ with compact support one has

$$\langle \Phi, \psi\rangle = \int_X \Phi \cdot \psi. \tag{1.2.3}$$

To show this, let us observe that for fixed $\psi$ both sides are continuous in $\Phi$ in the weak topology; hence, it suffices to show (1.2.3) for smooth $\Phi$. But in this case this equality is just the definition of the imbedding $V^\infty(X) \hookrightarrow V^{-\infty}(X)$.

Let us specialize the above discussion to the case $\psi = \mathbb{1}_A$. Then we get a finitely additive partially defined measure $P \mapsto \chi(A \cap P)$, where $P$ should be in a generic position with respect to $A$. Indeed,

$$P \longmapsto \int_X \mathbb{1}_A \cdot \mathbb{1}_P = \int_X \mathbb{1}_{A\cap P} = \chi(A \cap P).$$

### 1.2.9 A few examples of computation of the product in integral geometry

In this section we give examples of computation of the product of valuations in the complex projective space $\mathbb{CP}^n$. These examples are very typical in integral geometry. We will use the heuristic discussion of the previous Section 1.2.8 since hopefully it will clarify the intuition behind the product in applications.

Let now $X = \mathbb{CP}^n$ with the Fubini–Study metric. Let us denote by ${}^{\mathbb{C}}\mathrm{G}_l$ the Grassmannian of $l$-dimensional complex projective subspaces of $\mathbb{CP}^n$. Clearly it is equal to the Grassmannian of $(l+1)$-dimensional complex linear subspaces in $\mathbb{C}^{n+1}$. Let us consider the smooth $U(n+1)$-invariant valuations

$$\phi_l(K) := \int_{{}^{\mathbb{C}}\mathrm{G}_l} \chi(K \cap E)\, dE, \tag{1.2.4}$$

where $dE$ is the Haar measure on ${}^{\mathbb{C}}\mathrm{G}$ normalized in some way (we do not care about normalization constants). We claim that

$$\phi_l \cdot \phi_m = \begin{cases} c \cdot \phi_{l+m-n}, & l+m \geq n, \\ 0, & l+m < n, \end{cases} \tag{1.2.5}$$

where $c \neq 0$ is a normalizing constant depending on normalizations of Haar measures and $l, m, n$.

Let us give a heuristic proof of this equality. Using the discussion from the previous Section 1.2.8, we observe that

$$\phi_l(K) = \left( \int_{{}^{\mathbb{C}}\mathrm{G}_l} \mathbb{1}_E\, dE \right)(K),$$

where $\mathbb{1}_E$ is considered as a generalized valuation. Hence

$$\phi_l \cdot \phi_m = \int_{(E,F) \in {}^{\mathbb{C}}\mathrm{G}_l \times {}^{\mathbb{C}}\mathrm{G}_m} \mathbb{1}_E \cdot \mathbb{1}_F\, dE\, dF = \int_{{}^{\mathbb{C}}\mathrm{G}_l \times {}^{\mathbb{C}}\mathrm{G}_m} \mathbb{1}_{E \cap F}\, dE\, dF,$$

where the last equality is due to Theorem 1.2.14. Since for generic projective subspaces $E$ and $F$ their intersection $E \cap F$ is a projective subspace of dimension $l+m-n$ for $l+m \geq n$ and empty otherwise, it follows that

$$\int_{{}^{\mathbb{C}}\mathrm{G}_l \times {}^{\mathbb{C}}\mathrm{G}_m} \mathbb{1}_{E \cap F}\, dE\, dF = c \int_{{}^{\mathbb{C}}\mathrm{G}_{l+m-n}} \mathbb{1}_M\, dM = c \cdot \phi_{l+m-n}.$$

Thus the equality (1.2.5) is proved.

Let us consider another important example of the product on $\mathbb{CP}^n$. We claim that the $U(n+1)$-invariant valuation

$$K \longmapsto \int_{{}^{\mathbb{C}}\mathrm{G}_l} V_i(K \cap E)\, dE \tag{1.2.6}$$

is equal to $\phi_l \cdot V_i$, where $\phi_l$ is defined by (1.2.4). First observe that, by (1.2.2), one has

$$V_i(K \cap E) = \int \mathbb{1}_{K \cap E} \cdot V_i,$$

where $1\!\!1_{K\cap E}$ is considered as a generalized valuation and $\int$ in the last expression is the integration functional (i.e., evaluation on the whole space $\mathbb{CP}^n$). Now we use again Theorem 1.2.14 to write (under transversality assumptions) $1\!\!1_{K\cap E} = 1\!\!1_K \cdot 1\!\!1_E$. Thus

$$\int 1\!\!1_{K\cap E} \cdot V_i = \int 1\!\!1_K \cdot 1\!\!1_E \cdot V_i = (1\!\!1_E \cdot V_i)(K),$$

where the last equality follows from the heuristic discussion of Section 1.2.8. Thus the valuation (1.2.6) is equal to

$$\int_{{}^{\mathbb{C}}\mathrm{G}_l} 1\!\!1_E \cdot V_i \, dE = \left( \int_{{}^{\mathbb{C}}\mathrm{G}_l} 1\!\!1_E \, dE \right) \cdot V_i = \phi_l \cdot V_i,$$

as claimed.

Finally let us compute a generalization of the two previous examples. We claim that

$$\left( \int_{{}^{\mathbb{C}}\mathrm{G}_l} V_i(\,\cdot\, \cap E)\, dE \right) \cdot \left( \int_{{}^{\mathbb{C}}\mathrm{G}_m} V_j(\,\cdot\, \cap F)\, dF \right) = c' \cdot \int_{{}^{\mathbb{C}}\mathrm{G}_{l+m-n}} V_{i+j}(\,\cdot\, \cap M)\, dM, \tag{1.2.7}$$

where $c'$ is a constant which can be computed explicitly. By the previous two examples of this section, Example 1.1.22 from Section 1.1.7, and using the associativity and the commutativity of the product, we see that the left-hand side of (1.2.7) is equal to

$$(\phi_l \cdot V_i) \cdot (\phi_m \cdot V_j) = (\phi_l \cdot \phi_m) \cdot (V_i \cdot V_j) = c' \cdot \phi_{l+m-n} \cdot V_{i+j} = \text{r.h.s. of (1.2.7)}.$$

Thus (1.2.7) is proved.

### 1.2.10 Functorial properties of valuations

We describe the operations of pull-back and push-forward on valuations under smooth maps of manifolds. These operations generalize the well-known operation of pull-back on smooth and constructible functions, the operation of push-forward on measures, and integration with respect to the Euler characteristic along the fibers (also called push-forward) on constructible functions. However, for the moment this is done rigorously only in several special cases of maps (say submersions and immersions). We believe, however, that these constructions can be extended to "generic" smooth maps as partially defined maps on valuations. The precise conditions under which the maps could be defined might be rather technical. For this reason we describe first the general picture heuristically. This picture should be considered as conjectural. Then we formulate several rigorous results with precise conditions under which one can define pull-back and push-forward on valuations. These special cases turn out to be sufficient to define rigorously the Radon-type transform on valuations (again under some conditions) in the next section. The results of this section have been obtained by the author in [14].

Let us start with the heuristic picture. Denote by $V(X)$ a space of valuations on a manifold $X$ without specifying exactly the class of smoothness (smooth, generalized, or something else). $V_{\mathrm{c}}(X)$ denotes the subspace of $V(X)$ of compactly supported valuations. Let $f\colon X \to Y$ be a smooth map of manifolds. There should exist a partially defined linear map, called push-forward,

$$f_*\colon V_{\mathrm{c}}(X) \dashrightarrow V_{\mathrm{c}}(Y),$$

such that, for any nice subset $P \subset Y$,

$$(f_*\phi)(P) = \phi(f^{-1}(P)). \tag{1.2.8}$$

Since (smooth) measures are contained in $V(X)$, we can effect their push-forward in the sense of valuations. Clearly this operation should coincide with the classical push-forward of measures.

It immediately follows from (1.2.8) that for the composition of maps we should have

$$(f \circ g)_* = f_* \circ g_*. \tag{1.2.9}$$

We expect that the following interesting property of push-forward $f_*$ holds. It should extend somehow to a partially defined map on generalized valuations. Hence $f_*$ can be restricted to a partially defined map on constructible functions; it should be defined on constructible functions which are "in generic position" with respect to the map $f\colon X \to Y$. We expect that when $f$ is a proper map (i.e., preimages of compact sets are compact), then on constructible functions $f_*$ coincides with integration with respect to the Euler characteristic along the fibers.

Let us recall how this operation is defined assuming that $X$ and $Y$ are real analytic manifolds and $f$ is a proper real analytic map. It is uniquely characterized by the following property: Let $P \subset X$ be a subanalytic compact subset. Then $(f_*\mathbb{1}_P)(y) = \chi(P \cap f^{-1}(y))$ for any point $y \in Y$. One can show that $f_*$ maps constructible functions to constructible ones. We refer to [42, Ch. 9] for further details.

The push-forward should be related to the filtration on valuations in the following way:

$$f_*(W_i) \subset W_{i-\dim X+\dim Y}.$$

Also, $f_*$ should commute (up to a sign) with the Euler–Verdier involution.

Let us now discuss the pull-back operation

$$f^*\colon V(Y) \dashrightarrow V(X),$$

which should be a partially defined linear map in the opposite direction. Heuristically, $f^*$ should be the dual map to $f_*$ (recall from Section 1.2.4 that $V_{\mathrm{c}}(X)$ and

$V(X)$ are essentially dual to each other). The pull-back $f^*$ should be a homomorphism of algebras of valuations (again, the product might be partially defined). We expect that $f^*\chi = \chi$. Also $f^*$ should preserve the filtration

$$f^*(W_i) \subset W_i,$$

and $f^*$ should commute with the Euler–Verdier involution. Notice that, since in particular $f^*(W_1) \subset W_1$, $f^*$ induces a map between the quotients

$$f^*\colon V(Y)/W_1 \longrightarrow V(X)/W_1.$$

But by Remark 1.2.6(2), $V(Y)/W_1$ coincides with functions on $Y$ of an appropriate class of smoothness. In particular, we should get a map

$$f^*\colon C^\infty(Y) \longrightarrow C^{-\infty}(X).$$

We expect that this is the usual pull-back on smooth functions, i.e.,

$$f^*(F) = F \circ f. \tag{1.2.10}$$

Now let us restrict $f^*$ to constructible functions. We expect that it coincides again with the usual pull-back on constructible functions, which is defined by the same formula (1.2.10).

Finally, consider the restriction of $f^*$ to (say smooth) measures on $Y$. In classical measure theory, the operation of pull-back of a measure does not exist. Nevertheless, it is possible to define such a pull-back as a valuation, at least under appropriate technical conditions on the map $f$. Let $\mu$ be a smooth measure on $Y$. Then, leaving all the technicalities aside, one should have

$$(f^*\mu)(P) = \int_{y\in Y} \chi(P \cap f^{-1}(y))\, d\mu(y).$$

In particular, if $f\colon X \to Y$ is a linear projection of vector spaces and $P \subset X$ is a convex compact subset, then $(f^*\mu)(P) = \mu(f(P))$ is the measure of the projection of $P$.

Now let us describe several rigorous results which will be used later. Let $f\colon X \to Y$ be a smooth map.

*Case* 1: Assume that $f$ is a closed imbedding. Then the obvious restriction map $V^\infty(Y) \to V^\infty(X)$ defines the pull-back map $f^*$, which is a linear continuous operator. Dualizing it, we get the push-forward map

$$f_*\colon V^{-\infty}(X) \longrightarrow V^{-\infty}(Y),$$

which is a linear continuous operator (in the weak topology). Notice that in this situation $f_*$ does not preserve the class of smooth valuations.

It was shown in [14] that in this case $f_*(\mathbb{1}_P) = \mathbb{1}_{f(P)}$ for any compact submanifold with corners $P \subset X$. It was also shown that $f^*$ can be extended to a partially defined map $V^{-\infty}(Y) \dashrightarrow V^{-\infty}(X)$ such that if $Q \subset Y$ is a compact submanifold with corners which is transversal to $X$, then $f^*1\!\!1_Q$ is well defined in the sense of valuations and is equal to $1\!\!1_{X\cap Q}$, i.e., the pull-back on valuations is compatible with the pull-back on constructible functions.

*Case* 2: Assume that $f$ is a proper submersion. Let us define the push-forward $f_*\colon V_c^\infty(X) \to V_c^\infty(Y)$ by $(f_*\phi)(P) = \phi(f^{-1}(P))$ for any compact submanifold with corners $P \subset Y$. Notice that in this case $f^{-1}(P)$ is a compact submanifold with corners, and $f_*\phi$ is indeed a smooth valuation. The map constructed is linear and continuous. Taking the dual map, we define the pull-back map

$$f^*\colon V^{-\infty}(Y) \longrightarrow V^{-\infty}(X).$$

It was shown in [14] that in this case for any compact submanifold with corners $P \subset Y$ one has $f^*(1\!\!1_P) = 1\!\!1_P \circ f = 1\!\!1_{f^{-1}(P)}$. It was also shown that the push-forward $f_*$ extends to a partially defined map on generalized valuations. However, its compatibility with integration with respect to the Euler characteristic along the fibers was proved only under rather unpleasant restrictions on the class of constructible functions.

### 1.2.11 Radon transform on valuations on manifolds

In this section we combine the product, pull-back, and push-forward on valuations to define a Radon-type transform on them. Before we introduce this notion, it is instructive to recall the general Radon transform on smooth functions following Gelfand, and less classical but still known Radon transform on constructible functions. These two completely different transforms can be considered as special cases of the general Radon transform on valuations. In our opinion, this is the most interesting property of the new Radon transform on valuations.

**Definition 1.2.15.** A *double fibration* is a triple of smooth manifolds $X$, $Y$, and $Z$ with two submersive maps

$$X \xleftarrow{p} Z \xrightarrow{q} Y$$

such that the map $p \times q\colon Z \longrightarrow X \times Y$ is a closed imbedding.

To define a general Radon transform on smooth functions, let us fix a double fibration as above and an infinitely smooth measure $\gamma$ on $Z$. Let us also assume that $q\colon Z \to Y$ is proper. The Radon transform is the operator $R_\gamma\colon C_c^\infty(X) \to \mathcal{M}^\infty(Y)$ (where $\mathcal{M}^\infty(Y)$ denotes the space of smooth measures) defined by

$$R_\gamma f := q_*(\gamma \cdot p^* f), \tag{1.2.11}$$

where $p^*f = f \circ p$ is the usual pull-back on smooth functions, the product is just the usual product of a measure by a function, and $q_*$ is the usual push-forward on measures. Notice that all classical Radon transforms on smooth functions have such a form. For example, let us take $X = \mathbb{R}^n$, $Y$ the Grassmannian of affine $k$-dimensional subspaces, and $Z$ the incidence variety, i.e.,

$$Z = \{(x, E) \in X \times Y \mid x \in E\}.$$

Let $\gamma$ be a Haar measure on $Z$ invariant under the group of all isometries of $\mathbb{R}^n$. Then $R_\gamma$ is the classical Radon transform, given by integration of a function on $\mathbb{R}^n$ over all affine $k$-dimensional subspaces. There is a very extensive literature on this subject; see, e.g., [34, 35, 40].

Let us recall the Radon transform with respect to the Euler characteristic on constructible functions. It was studied for the real projective space and a somewhat more restrictive class of constructible functions by Khovanskii and Pukhlikov [46]; their work has been motivated by the earlier work of Viro [61] on the Radon transform on complex constructible functions on complex projective spaces. We will discuss and generalize the Khovanskii–Pukhlikov result in the next section. For subanalytic constructible functions and other spaces, the Radon transform with respect to the Euler characteristic was studied by Schapira [57]. Thus let

$$X \xleftarrow{p} Z \xrightarrow{q} Y$$

be a double fibration of real analytic spaces with real analytic maps $p$ and $q$. We assume again that $q$ is proper. Let us denote by $\mathcal{F}(X)$ the space of constructible functions, as defined in Section 1.2.5. Then one defines the Radon transform $R\colon \mathcal{F}(X) \to \mathcal{F}(Y)$ by

$$Rf := q_* p^*(f), \tag{1.2.12}$$

where $p^*$ denotes the usual pull-back on (constructible) functions, and $q_*$ is integration with respect to the Euler characteristic along the fibers of $q$.

With these preliminaries, let us introduce the Radon transform on valuations. We fix a double fibration as above with the map $q$ being proper. Let us fix a smooth valuation $\gamma \in V^\infty(Z)$. We define the Radon transform on valuations $\mathcal{R}_\gamma\colon V^\infty(X) \to V^{-\infty}(Y)$ by

$$\mathcal{R}_\gamma(\phi) = q_*(\gamma \cdot p^*\phi),$$

where $p^*$ and $q_*$ are the pull-back and push-forward on valuations, respectively, and the product with $\gamma$ is taken in the sense of valuations. It was shown in [14] that $\mathcal{R}_\gamma$ is a well-defined continuous linear operator.

Let us comment on some of the technical difficulties in this construction. Usually $p^*\phi$ is not a smooth valuation, though $\phi$ is. Thus we have to multiply the smooth valuation $\gamma$ by the non-smooth $p^*\phi$. This is always possible in the class of generalized valuations, but the product is not a smooth valuation. Next we have to take the push-forward of this generalized valuation. The push-forward of a generalized valuation under a general proper submersion is not always defined; it is so only under a rather technical condition of "generic position" of "singularities" of the valuation with respect to the map $q$. Fortunately, this technical condition is satisfied for valuations of the form $\gamma\cdot p^*\phi$ with smooth $\phi$. It was also shown in [14] that under extra assumptions on the double fibration, the image $\mathcal{R}_\gamma(V^\infty(X))$ is contained in smooth valuations. Also under a similar extra assumption $\mathcal{R}_\gamma$ can be extended uniquely by continuity in the weak topology to generalized valuations $V^{-\infty}(X)$. An example satisfying both assumptions will be considered in the next section.

Let us discuss now the relation of the new Radon transform on valuations to the classical Radon transforms discussed above in this section. First let us assume that the valuation $\gamma\in V^\infty(Z)$ is in fact a smooth measure considered as a smooth valuation. Then the Radon transform

$$\mathcal{R}_\gamma\colon V^\infty(X)\longrightarrow V^{-\infty}(Y)$$

vanishes on $W_1\subset V^\infty(X)$. Indeed $p^*(W_1)\subset W_1$ and $\gamma\cdot W_1=0$, since $\gamma$ is a measure. Hence $\mathcal{R}_\gamma$ factorizes (uniquely) via the quotient $V^\infty(X)/W_1=C^\infty(X)$. Notice also that in this case $\mathcal{R}_\gamma$ takes values in measures, in fact in infinitely smooth ones. Hence we get a map $C^\infty(X)\to\mathcal{M}^\infty(Y)$. It was shown in [14] that this map coincides with the classical Radon transform $R_\gamma$ defined by (1.2.11).

Let us consider another extremal case of $\mathcal{R}_\gamma$ with $\gamma=\chi$ being the Euler characteristic. In this case our discussion will be less rigorous. First assume that $\mathcal{R}_\gamma$ extends naturally to a partially defined map on generalized valuations $V^{-\infty}(X)\dashrightarrow V^{-\infty}(Y)$. We expect that its restriction to the class of constructible functions coincides with the Radon transform with respect to the Euler characteristic defined previously by (1.2.12). This result was proved rigorously in [14] in very special circumstances. It is desirable to make the result rigorous under more general assumptions.

### 1.2.12 Khovanskii–Pukhlikov-type inversion formula for the Radon transform on valuations on $\mathbb{RP}^n$

Let us consider the Radon-type transform on valuations in the following special case. Let $X=\mathbb{RP}^n$ be the real projective space, i.e., the manifold of lines in $\mathbb{R}^{n+1}$ passing through the origin. Let $Y=\mathbb{RP}^{n\vee}$ be the dual projective space, i.e., the manifold of linear hyperplanes in $\mathbb{R}^{n+1}$. Let $Z\subset X\times Y$ be the incidence variety

$$Z:=\big\{(l,E)\in\mathbb{RP}^n\times\mathbb{RP}^{n\vee}\mid l\subset E\big\}.$$

We have the double fibration

$$\mathbb{RP}^n\xleftarrow{\;p\;}Z\xrightarrow{\;q\;}\mathbb{RP}^{n\vee},$$

where $p$ and $q$ are the obvious projections. All the manifolds and maps are real analytic.

We consider the Radon transform

$$\mathcal{R}_\chi\colon V^\infty(\mathbb{RP}^n) \longrightarrow V^{-\infty}(\mathbb{RP}^{n\vee})$$

with the kernel $\gamma = \chi$ being the Euler characteristic on $Z$. In this case

$$\mathcal{R}_\chi = q_* p^*.$$

It was shown in [14] that the image of this transformation is contained in smooth valuations, and that $\mathcal{R}_\chi\colon V^\infty(\mathbb{RP}^n) \to V^\infty(\mathbb{RP}^{n\vee})$ is continuous. Moreover, this operator extends (uniquely) to a continuous linear operator, also denoted by $\mathcal{R}_\chi$, on generalized valuations equipped, as usual, with the weak topology:

$$\mathcal{R}_\chi\colon V^{-\infty}(\mathbb{RP}^n) \longrightarrow V^{-\infty}(\mathbb{RP}^{n\vee}).$$

It was shown in [14] that $\mathcal{R}_\chi$ is invertible for odd $n$, and for even $n$ its kernel consists precisely of multiples of the Euler characteristic. In both cases there is an explicit inversion formula (in the latter case, up to a multiple of the Euler characteristic); it generalizes and was motivated by the Khovanskii–Pukhlikov inversion formula for constructible functions [46]. In order to state the result, let us consider the analogous operator in the opposite direction,

$$\mathcal{R}^t_\chi\colon V^{-\infty}(\mathbb{RP}^{n\vee}) \longrightarrow V^{-\infty}(\mathbb{RP}^n),$$

namely

$$\mathcal{R}^t_\chi := p_* q^*.$$

**Theorem 1.2.16** ([14])**.** *For any generalized valuation $\phi \in V^{-\infty}(\mathbb{RP}^n)$ one has*

$$(-1)^{n-1}\mathcal{R}^t_\chi \mathcal{R}_\chi(\phi) = \phi + \frac{1}{2}\big((-1)^{n-1} - 1\big)\left(\int_{\mathbb{RP}^n} \phi\right)\cdot \chi. \tag{1.2.13}$$

Let us say a few words about the proof of this theorem. After all the operators involved were defined, the next technically non-trivial step was to show that the restriction of $\mathcal{R}_\chi$ to a rather special class of constructible functions, which is still dense in $V^{-\infty}(\mathbb{RP}^n)$, coincides with the Radon transform with respect to the Euler characteristic on constructible functions; also, an analogous result holds for $\mathcal{R}^t_\chi$. Then Theorem 1.2.16 follows immediately by continuity from the Khovanskii–Pukhlikov inversion formula for constructible functions, which claims precisely the identity (1.2.13) for such functions in place of $\phi$.

# Bibliography


[1] Alesker, S.: On P. McMullen's conjecture on translation invariant valuations. *Adv. Math.* **155** (2000), no. 2, 239–263.

[2] Alesker, S.: Description of translation invariant valuations on convex sets with solution of P. McMullen's conjecture. *Geom. Funct. Anal.* **11** (2001), no. 2, 244–272.

[3] Alesker, S.: Hard Lefschetz theorem for valuations, complex integral geometry, and unitarily invariant valuations. *J. Differential Geom.* **63** (2003), no. 1, 63–95.

[4] Alesker, S.: The multiplicative structure on polynomial continuous valuations. *Geom. Funct. Anal.* **14** (2004), no. 1, 1–26.

[5] Alesker, S.: $SU(2)$-invariant valuations. In: *Geometric Aspects of Functional Analysis*, 21–29, Lecture Notes in Math. 1850, Springer, Berlin, 2004.

[6] Alesker, S.: Hard Lefschetz theorem for valuations and related questions of integral geometry. In: *Geometric Aspects of Functional Analysis*, 9–20, Lecture Notes in Math. 1850, Springer, Berlin, 2004.

[7] Alesker, S.: Valuations on convex sets, non-commutative determinants, and pluripotential theory. *Adv. Math.* **195** (2005), no. 2, 561–595.

[8] Alesker, S.: Theory of valuations on manifolds, I. Linear spaces. *Israel J. Math.* **156** (2006), 311–339.

[9] Alesker, S.: Theory of valuations on manifolds, II. *Adv. Math.* **207** (2006), no. 1, 420–454.

[10] Alesker, S.: Theory of valuations on manifolds, IV. New properties of the multiplicative structure. In: *Geometric Aspects of Functional Analysis*, 1–44, Lecture Notes in Math. 1910, Springer, Berlin, 2007.

[11] Alesker, S.: Theory of valuations on manifolds: a survey. *Geom. Funct. Anal.* **17** (2007), no. 4, 1321–1341.

[12] Alesker, S.: Quaternionic plurisubharmonic functions and their applications to convexity. *St. Petersburg Math. J.* **19** (2008), no. 1, 1–13.

[13] Alesker, S.: Plurisubharmonic functions on the octonionic plane and Spin(9)-invariant valuations on convex sets. *J. Geom. Anal.* **18** (2008), no. 3, 651–686.

[14] Alesker, S.: Valuations on manifolds and integral geometry. *Geom. Funct. Anal.* **20** (2010), no. 5, 1073–1143.

[15] Alesker, S.: A Fourier type transform on translation invariant valuations on convex sets. *Israel J. Math.* **181** (2011), 189–294.

[16] Alesker, S. and Bernig, A.: The product on smooth and generalized valuations. *Amer. J. Math.* **134** (2012), no. 2, 507–560.

[17] Alesker, S. and Bernstein, J.: Range characterization of the cosine transform on higher Grassmannians. *Adv. Math.* **184** (2004), no. 2, 367–379.

[18] Alesker, S. and Fu, J. H. G.: Theory of valuations on manifolds, III. Multiplicative structure in the general case. *Trans. Amer. Math. Soc.* **360** (2008), no. 4, 1951–1981.

[19] Bernig, A.: A Hadwiger-type theorem for the special unitary group. *Geom. Func. Anal.* **19** (2009), 356–372.

[20] Bernig, A.: Integral geometry under $G_2$ and Spin(7). *Israel J. Math.* **184** (2011), 301–316.

[21] Bernig, A.: Invariant valuations on quaternionic vector spaces. *J. Inst. Math. Jussieu* **11** (2012), 467–499.

[22] Bernig, A.: Algebraic integral geometry. In: *Global Differential Geometry* (C. Bär, J. Lohkamp and M. Schwarz, eds.), 107–145, Springer Proceedings in Math. Vol. 17, Springer, Berlin, 2012.

[23] Bernig, A. and Bröcker, L.: Valuations on manifolds and Rumin cohomology. *J. Differential Geom.* **75** (2007), 433–457.

[24] Bernig, A. and Fu, J. H. G.: Convolution of convex valuations. *Geom. Dedicata* **123** (2006), 153–169.

[25] Bernig, A. and Fu, J. H. G.: Hermitian integral geometry. *Ann. of Math.* **173** (2011), 907–945.

[26] Chern, S.-S.: On the curvatura integra in a Riemannian manifold. *Ann. of Math.* **46** (1945), 674–684.

[27] Federer, H.: Curvature measures. *Trans. Amer. Math. Soc.* **93** (1959), 418–491.

[28] Fu, J. H. G.: Curvature measures and generalized Morse theory, *J. Differential Geom.* **30** (1989), 619–642.

[29] Fu, J. H. G.: Monge–Ampère functions, I. *Indiana Univ. Math. J.* **38** (1989), no. 3, 745–771.

[30] Fu, J. H. G.: Monge–Ampère functions, II. *Indiana Univ. Math. J.* **38** (1989), no. 3, 773–789.

[31] Fu, J. H. G.: Kinematic formulas in integral geometry. *Indiana Univ. Math. J.* **39** (1990), no. 4, 1115–1154.

[32] Fu, J. H. G.: Curvature measures of subanalytic sets. *Amer. J. Math.* **116** (1994), no. 4, 819–880.

[33] Fu, J. H. G.: Structure of the unitary valuation algebra. *J. Differential Geom.* **72** (2006), no. 3, 509–533.

[34] Gelfand, I. M., Graev, M. I. and Roşu, R.: The problem of integral geometry and intertwining operators for a pair of real Grassmannian manifolds. *J. Operator Theory* **12** (1984), no. 2, 359–383.

[35] Gelfand, I. M., Graev, M. I. and Šapiro, Z. Ja.: Integral geometry on $k$-dimensional planes. (Russian) *Funkcional. Anal. i Priložen* **1** (1967), 15–31.

[36] Griffiths, P. and Harris, J.: *Principles of Algebraic Geometry.* Reprint of the 1978 original. Wiley Classics Library. John Wiley & Sons, Inc., New York, 1994.

[37] Guillemin, V. and Sternberg, S.: *Geometric Asymptotics.* Mathematical Surveys, no. 14. American Mathematical Society, Providence, R.I., 1977.

[38] Hadwiger, H.: Translationsinvariante, additive und stetige Eibereichfunktionale. (German) *Publ. Math. Debrecen* **2** (1951), 81–94.

[39] Hadwiger, H.: *Vorlesungen über Inhalt, Oberfläche und Isoperimetrie.* (German) Springer-Verlag, Berlin-Göttingen-Heidelberg, 1957.

[40] Helgason, S.: *The Radon Transform.* Second edition. Progress in Math. 5. Birkhäuser Boston, Inc., Boston, MA, 1999.

[41] Hörmander, L.: *The Analysis of Linear Partial Differential Operators. I. Distribution Theory and Fourier Analysis.* Reprint of the second (1990) edition. Classics Math. Springer-Verlag, Berlin, 2003.

[42] Kashiwara, M. and Schapira, P.: *Sheaves on Manifolds.* Grundlehren Math. Wiss. 292. Springer-Verlag, Berlin, 1990.

[43] Klain, D. A.: A short proof of Hadwiger's characterization theorem. Mathematika **42** (1995), no. 2, 329–339.

[44] Klain, D. A.: Even valuations on convex bodies. *Trans. Amer. Math. Soc.* **352** (2000), no. 1, 71–93.

[45] Klain, D. A. and Rota, G.-C.: *Introduction to Geometric Probability.* Lezioni Lincee. Cambridge University Press, Cambridge, 1997.

[46] Khovanskii, A. G. and Pukhlikov, A. V.: Finitely additive measures of virtual polyhedra. (Russian) Algebra i Analiz 4 (1992), no. 2, 161–185; translation in *St. Petersburg Math. J.* **4** (1993), no. 2, 337–356.

[47] Ludwig, M.: Projection bodies and valuations. *Adv. Math.* **172** (2002), no. 2, 158–168.

[48] Ludwig, M.: Ellipsoids and matrix-valued valuations. *Duke Math. J.* **119** (2003), no. 1, 159–188.

[49] Ludwig, M.: Intersection bodies and valuations. *Amer. J. Math.* **128** (2006), no. 6, 1409–1428.

[50] Ludwig, M. and Reitzner, M.: A characterization of affine surface area. *Adv. Math.* **147** (1999), no. 1, 138–172.

[51] Ludwig, M. and Reitzner, M.: A classification of $SL(n)$ invariant valuations. *Ann. of Math.* **172** (2010), no. 2, 1219–1267.

[52] McMullen, P.: Valuations and Euler-type relations on certain classes of convex polytopes. *Proc. London Math. Soc.* **35** (1977), no. 1, 113–135.

[53] McMullen, P.: Continuous translation-invariant valuations on the space of compact convex sets. *Arch. Math. (Basel)* **34** (1980), no. 4, 377–384.

[54] McMullen, P.: Valuations and dissections. In: *Handbook of Convex Geometry*, Vol. B, 933–988, North-Holland, Amsterdam, 1993.

[55] McMullen, P. and Schneider, R.: Valuations on convex bodies. *Convexity and its applications*, 170–247, Birkhäuser, Basel, 1983.

[56] Santaló, L. A.: *Integral Geometry and Geometric Probability.* Encyclopedia Math. Appl., 1. Addison-Wesley Publishing Co., Reading, Mass.-London-Amsterdam, 1976.

[57] Schapira, P.: Tomography of constructible functions. In: *Applied Algebra, Algebraic Algorithms and Error-Correcting Codes (Paris, 1995)*, 427–435, Lecture Notes in Comput. Sci. 948, Springer, Berlin, 1995.

[58] Schneider, R.: *Convex Bodies: the Brunn–Minkowski Theory.* Encyclopedia Math. Appl., 44. Cambridge University Press, Cambridge, 1993.

[59] Schneider, R.: Simple valuations on convex bodies. *Mathematika* **43** (1996), no. 1, 32–39.

[60] Schneider, R. and Schuster, F. E.: Rotation equivariant Minkowski valuations. *Int. Math. Res. Not.* 2006, Art. ID 72894, 20 pp.

[61] Viro, O.: Some integral calculus based on Euler characteristic. In: *Topology and Geometry – Rohlin Seminar*, 127–138, Lecture Notes in Math. 1346, Springer, Berlin, 1988.

[62] Wintgen, P.: Normal cycle and integral curvature for polyhedra in Riemannian Manifolds. In: *Differential Geometry* (G. Soos and J. Szenthe, eds.), North-Holland, Amsterdam, 1982.

[63] Zähle, M.: Approximation and characterization of generalised Lipschitz–Killing curvatures. *Ann. Global Anal. Geom.* **8** (1990), no. 3, 249–260.