

# Invertibility in Groupoid $C^*$ -algebras

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**Abstract.** Given a second-countable, Hausdorff, étale, amenable groupoid  $\mathcal{G}$  with compact unit space, we show that an element  $a$  in  $C^*(\mathcal{G})$  is invertible if and only if  $\lambda_x(a)$  is invertible for every  $x$  in the unit space of  $\mathcal{G}$ , where  $\lambda_x$  refers to the *regular representation* of  $C^*(\mathcal{G})$  on  $\ell_2(\mathcal{G}_x)$ . We also prove that, for every  $a$  in  $C^*(\mathcal{G})$ , there exists some  $x \in \mathcal{G}^{(0)}$  such that  $\|a\| = \|\lambda_x(a)\|$ .

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## 1. Introduction

The structure of certain  $C^*$ -algebras is often best studied via large families of  $*$ -representations. According to this point of view, one tries to deduce the properties of any given element of the algebra by means of the properties of its images under the representations provided. Here we shall mostly be interested in *invertibility* questions, and thus on families of representations of a given  $C^*$ -algebra which are large enough to determine when an element is invertible.

One of the first, and arguably also the most influential such result is the Allan–Douglas local principle [1, Corollary 2.10], [4, Theorem 7.47], which asserts that an element in a unital Banach algebra is invertible if and only if it is invertible modulo certain ideals associated to the points of the spectrum of a given central subalgebra. This principle has been generalized to *nonlocal* algebras (see [7] and the references given there) and has successfully been applied to study Fredholm singular integral operators with semi-almost periodic coefficients [3].

The present paper is an attempt to transpose the *local-trajectory method* of [7] to the context of groupoid  $C^*$ -algebras. Since invertibility only makes sense on unital algebras, and since the  $C^*$ -algebra of a groupoid is unital only when the groupoid is étale and has a compact unit space, we restrict ourselves to this

case (however our work suggests questions that might be relevant for more general groupoids). To be precise, our main result, Theorem 2.10 applies to second-countable, Hausdorff, étale, amenable groupoids with compact unit space. Given such a groupoid  $\mathcal{G}$ , we show that an element  $a$  in the groupoid  $C^*$ -algebra  $C^*(\mathcal{G})$  is invertible if and only, for every  $x$  in the unit space of  $\mathcal{G}$ , one has that  $\lambda_x(a)$  is invertible, where  $\lambda_x$  is the *regular representation* of  $C^*(\mathcal{G})$  on  $\ell_2(\mathcal{G}_x)$ .

A crucial tool used to prove our main result is the theory of induced representations started by Renault in [9, Chap. II, §2] and improved by Ionescu and Williams in [5] and [6].

Since  $\mathcal{G}$  is amenable, we have by [2, Theorem 6.1.4(iii)] that

$$\|a\| = \sup_{x \in \mathcal{G}^{(0)}} \|\lambda_x(a)\|, \quad \forall a \in C^*(\mathcal{G}). \tag{1.1}$$

As a byproduct of our work we have found a small improvement of this result, namely Corollary 3.3, below, which asserts that

$$\|a\| = \max_{x \in \mathcal{G}^{(0)}} \|\lambda_x(a)\|, \quad \forall a \in C^*(\mathcal{G}), \tag{1.2}$$

which is to say that the supremum in (1.1) is in fact *attained* for every  $a$ . The proof of this fact is a straightforward combination of Theorem 2.10 with a result of S. Roch [10], which we carefully describe below.

Even though the invertibility question treated in Theorem 2.10 only makes sense for groupoids with compact unit space, (1.2) applies to a wider context. A sensible question to be asked at this point is therefore whether or not (1.2) holds in the absence of the compactness hypothesis.

Dropping the assumption that  $\mathcal{G}$  is amenable, it is well known that (1.1) holds as long as we replace the full by the reduced groupoid  $C^*$ -algebra. So it makes sense to ask whether or not

$$\|a\| = \max_{x \in \mathcal{G}^{(0)}} \|\lambda_x(a)\|, \quad \forall a \in C_r^*(\mathcal{G}) ? \tag{1.3}$$

Unfortunately we have not been able to answer any of these questions, which we are then forced to leave as open problems.

Attaining the supremum is a well-known property of continuous functions on compact spaces, so a proof of (1.3) could be obtained, at least in the case of a compact unit space, should we be able to prove that the function

$$x \mapsto \|\lambda_x(a)\|$$

is continuous for every  $a \in C_r^*(\mathcal{G})$ . However sensible this appears to be, we have not been able to determine its validity.

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## 2. Sufficient family of representations

Let  $A$  be a unital  $C^*$ -algebra. The following concept appears in [10, Section 5].

**Definition 2.1.** A family  $\mathcal{F}$  of non-degenerated representations (always assumed to preserve the involution) of  $A$  is called *sufficient* if, for every  $a$  in  $A$ , one has that

$$a \text{ is invertible} \iff \pi(a) \text{ is invertible for all } \pi \in \mathcal{F}.$$

Observe that the implication “ $\Rightarrow$ ” is always true, so the relevant property conveyed by this definition is the implication “ $\Leftarrow$ ”.

**Proposition 2.2.** *The set of all irreducible representations of  $A$  is a sufficient family of representations.*

*Proof.* If  $a$  is a non-invertible element of  $A$ , then either  $a^*a$  or  $aa^*$  are non-invertible. So we may assume, without loss of generality that  $a^*a$  is non-invertible. Let  $B$  be the closed  $*$ -subalgebra of  $A$  generated by  $a^*a$  and  $1$ , and let  $X$  be the compact spectrum of  $B$ . Since  $a^*a$  is non-invertible, there exists some point  $x_0$  in  $X$  such that  $\widehat{a^*a}(x_0) = 0$ , where the hat indicates the Gelfand transform.

The map

$$\phi : b \in B \mapsto \hat{b}(x_0) \in \mathbb{C}$$

is therefore a pure state of  $B$ , which may be extended to a pure state  $\psi$  on  $A$ . Let  $\pi$  be the GNS representation associated to  $\psi$ , so that  $\pi$  is an irreducible representation. If  $\xi$  is the associated cyclic vector we have

$$\|\pi(a)\xi\|^2 = \langle \pi(a)\xi, \pi(a)\xi \rangle = \langle \pi(a^*a)\xi, \xi \rangle = \psi(a^*a) = \phi(a^*a) = \widehat{a^*a}(x_0) = 0.$$

It follows that the operator  $\pi(a)$  is not injective and hence non-invertible. □

▷ From now on we will be interested in the question of sufficiency for groupoid  $C^*$ -algebras. We therefore fix a second-countable, Hausdorff, étale groupoid  $\mathcal{G}$ , with source and range maps denoted by “ $s$ ” and “ $r$ ”, respectively.

Given  $x$  in the unit space  $\mathcal{G}^{(0)}$  of  $\mathcal{G}$ , we shall use the following standard notations:

$$\begin{aligned} \mathcal{G}_x &= \{\gamma \in \mathcal{G} : s(\gamma) = x\}, \\ \mathcal{G}^x &= \{\gamma \in \mathcal{G} : r(\gamma) = x\}, \quad \text{and} \\ \mathcal{G}(x) &= \mathcal{G}_x \cap \mathcal{G}^x. \end{aligned}$$

Consider the Hilbert space  $H_x = \ell_2(\mathcal{G}_x)$  and the *regular representation*  $\lambda_x$  of  $C_c(\mathcal{G})$  on  $H_x$ , given by

$$\lambda_x(f)\xi|_\gamma = \sum_{\gamma'\gamma''=\gamma} f(\gamma')\xi(\gamma''), \quad \forall f \in C_c(\mathcal{G}), \quad \forall \xi \in H_x, \quad \forall \gamma \in \mathcal{G}_x,$$

which is well known to extend to  $C^*(\mathcal{G})$ . For each  $\gamma$  in  $\mathcal{G}_x$ , let  $e_\gamma$  be the basis vector of  $H_x$  corresponding to  $\gamma$ .

**Proposition 2.3.** *For every  $\gamma_1$  and  $\gamma_2$  in  $\mathcal{G}_x$ , and all  $f$  in  $C_c(\mathcal{G})$ , one has that*

$$\langle \lambda_x(f)e_{\gamma_1}, e_{\gamma_2} \rangle = f(\gamma_2\gamma_1^{-1}).$$

*Proof.* We have

$$\begin{aligned} \langle \lambda_x(f)e_{\gamma_1}, e_{\gamma_2} \rangle &= \lambda_x(f)e_{\gamma_1^{-1}\gamma_2} = \sum_{\gamma'\gamma''=\gamma_2} f(\gamma')e_{\gamma_1}(\gamma'') \\ &= \sum_{\gamma'\gamma_1=\gamma_2} f(\gamma') = f(\gamma_2\gamma_1^{-1}). \end{aligned} \quad \square$$

**Proposition 2.4.** *Let  $\mathcal{H}$  be a closed sub-groupoid of  $\mathcal{G}$ , viewed as a topological groupoid with the relative topology. Then the following are equivalent:*

- (i) *the restriction of the range map  $r$  to  $\mathcal{H}$ , viewed as a mapping*

$$r|_{\mathcal{H}} : \mathcal{H} \rightarrow \mathcal{H}^{(0)},$$

*is an open mapping,*

- (ii)  *$\mathcal{H}$  is étale.*

*Proof.* Assuming (i), let  $\gamma \in \mathcal{H}$  and choose an open set  $U \subseteq \mathcal{G}$  such that  $r$  is a homeomorphism from  $U$  onto the open set  $r(U) \subseteq \mathcal{G}^{(0)}$ . Then  $U \cap \mathcal{H}$  is open in the relative topology of  $\mathcal{H}$  and, by (i), we have that  $r(U \cap \mathcal{H})$  is open in  $\mathcal{H}^{(0)}$ . It is then clear that  $r$  is a homeomorphism from  $U \cap \mathcal{H}$  to  $r(U \cap \mathcal{H})$ , showing that  $r|_{\mathcal{H}}$  is a local homeomorphism and hence that  $\mathcal{H}$  is étale. The converse is evident.  $\square$

▷ From now on we fix a closed sub-groupoid  $\mathcal{H} \subseteq \mathcal{G}$ , satisfying the equivalent conditions above. We will denote the unit spaces of  $\mathcal{G}$  and  $\mathcal{H}$  as follows

$$X := \mathcal{G}^{(0)}, \quad \text{and} \quad Y := \mathcal{H}^{(0)}.$$

Since  $H$  is closed in  $\mathcal{G}$  and since  $Y = \mathcal{H} \cap X$ , we see that  $Y$  is a closed subspace of  $X$ .

Let us briefly describe the process of inducing representations from  $C^*(\mathcal{H})$  to  $C^*(\mathcal{G})$ , cf. [9, Chap. II, §2] and [6, Section 2]. Given a representation  $L$  of  $C^*(\mathcal{H})$  on a Hilbert space  $H_L$ , we want to produce a representation  $\text{Ind}_H^{\mathcal{G}} L$  of  $C^*(\mathcal{G})$  on a Hilbert space  $H_{\text{Ind} L}$ . In order to do so, consider the closed subset of  $\mathcal{G}$  given by

$$\mathcal{G}_Y = s^{-1}(Y) = \{\gamma \in \mathcal{G} : s(\gamma) \in Y\}.$$

For  $\varphi$  and  $\psi$  in  $C_c(\mathcal{G}_Y)$ , define  $\langle \varphi, \psi \rangle_*$  in  $C_c(\mathcal{H})$ , by

$$\langle \varphi, \psi \rangle_*(\zeta) = \sum_{\gamma_1\gamma_2=\zeta} \overline{\varphi(\gamma_1^{-1})}\psi(\gamma_2), \quad \forall \zeta \in \mathcal{H}.$$

It should be noticed that the above sum ranges over all pairs of elements  $\gamma_1$  and  $\gamma_2$  in  $\mathcal{G}$  (as opposed to  $\mathcal{H}$ ), whose product equals  $\zeta$ . In this case notice that both  $r(\gamma_1)$  and  $s(\gamma_2)$  lie in  $Y$ , so that  $\gamma_1^{-1}$  and  $\gamma_2$  indeed belong to the domain of  $\varphi$  and  $\psi$ , respectively.

By [8, Theorem 2.8], one has that in fact  $C_c(\mathcal{G}_Y)$  may be completed to a right  $C^*(\mathcal{H})$ -Hilbert module, which we will denote by  $M$ , the appropriate right

multiplication being that which is described in [8, page 11]. It is therefore profitable to view  $\langle \cdot, \cdot \rangle_*$  as a  $C^*(\mathcal{H})$ -valued map.

The space  $H_{\text{Ind } L}$ , on which the induced representation will act, is then defined to be the completion of

$$C_c(\mathcal{G}_Y) \otimes H_L,$$

relative to the inner-product

$$\langle \varphi \otimes \xi, \psi \otimes \eta \rangle := \langle L(\langle \psi, \varphi \rangle_*) \xi, \eta \rangle, \quad \forall \varphi, \psi \in C_c(\mathcal{G}_Y), \quad \forall \xi, \eta \in H_L.$$

One next gives  $C_c(\mathcal{G}_Y)$  the structure of a left  $C_c(\mathcal{G})$ -module by setting

$$(f * \varphi)(\gamma) := \sum_{\gamma_1 \gamma_2 = \gamma} f(\gamma_1) \varphi(\gamma_2), \quad \forall f \in C_c(\mathcal{G}), \quad \forall \varphi \in C_c(\mathcal{G}_Y), \quad \forall \gamma \in \mathcal{G}_Y.$$

Again by [8, Theorem 2.8], the above left-module structure may be extended to a bounded multiplication operation

$$(a, x) \in C^*(\mathcal{G}) \times M \mapsto ax \in M.$$

In order to define the induced representation one may either work with the completion  $M$  described above or take the more pedestrian point of view of sticking to compactly supported functions. Taking the latter approach, for  $f \in C_c(\mathcal{G})$  one initially defines  $\text{Ind}_H^{\mathcal{G}} L(f)$  on the dense subspace  $C_c(\mathcal{G}_Y) \otimes H_L \subseteq H_{\text{Ind } L}$ , by the formula

$$\text{Ind}_H^{\mathcal{G}} L(f)(\varphi \otimes \xi) := (f * \varphi) \otimes \xi, \quad \forall \varphi \in C_c(\mathcal{G}_Y), \quad \forall \xi \in H_L,$$

and then extend it by continuity to  $H_{\text{Ind } L}$ . This provides a  $*$ -representation of  $C_c(\mathcal{G})$  on  $H_{\text{Ind } L}$  which, in turn, may be extended to the whole of  $C^*(\mathcal{G})$ .

The resulting representation of  $C^*(\mathcal{G})$  on  $H_{\text{Ind } L}$  is denoted by  $\text{Ind}_H^{\mathcal{G}} L$ , and is called the *representation induced by  $L$  from  $\mathcal{H}$  up to  $\mathcal{G}$* . For more details, see [9, Chap. II, §2] and [6, Section 2].

▷ Fix, for the time being, an element  $x \in X$ .

We would now like to consider the question of inducing representations from  $\mathcal{H} := \mathcal{G}(x)$  up to  $\mathcal{G}$ . Observing that

$$Y = \mathcal{G}(x)^{(0)} = \{x\},$$

we have that  $\mathcal{G}_Y = \mathcal{G}_x$ , which is a discrete topological space. Consequently  $C_c(\mathcal{G}_Y)$  is linearly generated by the set

$$\{e_\gamma : \gamma \in \mathcal{G}_x\},$$

where  $e_\gamma$  denotes the characteristic function of the singleton  $\{\gamma\}$ .

**Proposition 2.5.** *Given  $\gamma, \gamma' \in \mathcal{G}_x$ , we have that*

$$\langle e_\gamma, e_{\gamma'} \rangle_* = \begin{cases} \delta_{\gamma^{-1}\gamma'}, & \text{if } r(\gamma) = r(\gamma'), \\ 0, & \text{otherwise,} \end{cases}$$

where, for each  $h \in \mathcal{G}(x)$ , we denote by  $\delta_h$  the characteristic function of the singleton  $\{h\}$ , viewed as an element of  $C_c(\mathcal{G}(x)) \subseteq C^*(\mathcal{G}(x))$ .

*Proof.* We have, for every  $\zeta \in \mathcal{G}(x)$ , that

$$\langle e_\gamma, e_{\gamma'} \rangle_* (\zeta) = \sum_{\gamma_1 \gamma_2 = \zeta} \overline{e_\gamma(\gamma_1^{-1})} e_{\gamma'}(\gamma_2) = [\gamma^{-1}\gamma' = \zeta]$$

where the brackets denote the Boolean value of the expression inside, with the convention that a syntactically incorrect expression, e.g., when the multiplication  $\gamma^{-1}\gamma'$  is illegal, the value is zero.

Thus, when  $r(\gamma) = r(\gamma')$ , we have that the product  $\gamma^{-1}\gamma'$  is defined, evidently giving an element of  $\mathcal{G}(x)$  and, in this case,

$$\langle e_\gamma, e_{\gamma'} \rangle_* = \delta_{\gamma^{-1}\gamma'},$$

On the other hand, when  $r(\gamma) \neq r(\gamma')$ , we clearly have that  $\langle e_\gamma, e_{\gamma'} \rangle_* = 0$ . □

The following elementary result is included in order to illustrate a simple example.

**Proposition 2.6.** *Let  $\Lambda$  be the left-regular representation of  $C^*(\mathcal{G}(x))$  on  $\ell_2(\mathcal{G}(x))$ . Then  $\text{Ind}_{\mathcal{G}(x)}^\mathcal{G} \Lambda$  is unitarily equivalent to  $\lambda_x$ .*

*Proof.* For each element  $\gamma \in \mathcal{G}_x$ , and each  $g \in \mathcal{G}(x)$ , consider the element

$$\varphi_{\gamma,g} = e_\gamma \otimes e_g \in C_c(\mathcal{G}_x) \otimes \ell_2(\mathcal{G}(x)) \subseteq H_{\text{Ind} \Lambda}.$$

We first claim that

$$\langle \varphi_{\gamma,g}, \varphi_{\gamma',g'} \rangle = [\gamma g = \gamma' g'], \quad \forall \gamma, \gamma' \in \mathcal{G}_x, \quad \forall g, g' \in \mathcal{G}(x). \tag{2.1}$$

In fact, we have

$$\langle \varphi_{\gamma,g}, \varphi_{\gamma',g'} \rangle = \langle e_\gamma \otimes e_r(\gamma), e_{\gamma'} \otimes e_{r(\gamma')} \rangle = \langle \Lambda(\langle e_{\gamma'}, e_\gamma \rangle_*) e_{r(\gamma)}, e_{r(\gamma')} \rangle = (\dagger)$$

Consequently, if  $r(\gamma) \neq r(\gamma')$  we have by Proposition 2.5 that  $\langle \varphi_{\gamma,g}, \varphi_{\gamma',g'} \rangle = 0$ , which proves (2.1) in this case. If  $r(\gamma) = r(\gamma')$  then, again by Proposition 2.5, it follows that

$$(\dagger) = \langle \Lambda(\delta_{\gamma'^{-1}\gamma}) e_g, e_{g'} \rangle = \langle e_{\gamma'^{-1}\gamma g}, e_{g'} \rangle = [\gamma'^{-1}\gamma g = g'] = [\gamma g = \gamma' g'],$$

proving (2.1). In particular, this implies that

$$\langle \varphi_{\gamma,g}, \varphi_{\gamma',g'} \rangle = \langle \varphi_{\gamma g, x}, \varphi_{\gamma' g', x} \rangle,$$

and since the collection of all  $\varphi_{\gamma',g'}$  evidently spans  $H_{\text{Ind} \Lambda}$ , we have that  $\varphi_{\gamma,g} = \varphi_{\gamma g, x}$ , and it is then clear that the mapping

$$e_\gamma \mapsto \varphi_{\gamma, x}$$

extends to a unitary operator  $U : H_x \rightarrow H_{\text{Ind} \Lambda}$ . Given  $f \in C_c(\mathcal{G})$ , we claim that

$$\langle U^*(\text{Ind}_H^\mathcal{G} \Lambda(f)) U e_\gamma, e_{\gamma'} \rangle = \langle \lambda_x(f) e_\gamma, e_{\gamma'} \rangle, \quad \forall \gamma, \gamma' \in \mathcal{G}_x. \tag{2.2}$$

In order to verify it observe that the left-hand side equals

$$\begin{aligned} \langle \text{Ind}_H^\mathcal{G} \Lambda(f)(\varphi_{\gamma, x}), \varphi_{\gamma', x} \rangle &= \langle (f * e_\gamma) \otimes e_x, e_{\gamma'} \otimes e_{x'} \rangle \\ &= \langle \Lambda(\langle e_{\gamma'}, f * e_\gamma \rangle_*) e_x, e_x \rangle = (\diamond) \end{aligned}$$

After checking that

$$f * e_\gamma = \sum_{\eta \in \mathcal{G}_x} f(\eta\gamma^{-1})e_\eta,$$

we conclude that

$$\begin{aligned} (\diamond) &= \sum_{\eta \in \mathcal{G}_x} f(\eta\gamma^{-1})\langle \Lambda(\langle e_{\gamma'}, e_\eta \rangle_*) e_x, e_x \rangle \\ &= \sum_{\substack{\eta \in \mathcal{G}_x \\ r(\gamma')=r(\eta)}} f(\eta\gamma^{-1})\langle \Lambda(\gamma\gamma'^{-1}\eta) e_x, e_x \rangle \\ &= \sum_{\substack{\eta \in \mathcal{G}_x \\ r(\gamma')=r(\eta)}} f(\eta\gamma^{-1})\langle e_{\gamma'^{-1}\eta}, e_x \rangle = f(\gamma'\gamma^{-1}) \stackrel{\text{P.2.3}}{=} \langle \lambda_x(f)e_\gamma, e_{\gamma'} \rangle. \end{aligned}$$

This proves (2.2), and taking into account that  $\gamma$  and  $\gamma'$  are arbitrary, we conclude that  $U^*(\text{Ind}_H^G \Lambda(f))U = \lambda_x(f)$ , finishing the proof.  $\square$

Notice that there are two completions of  $C_c(\mathcal{G}_x)$  which are relevant to us. On the one hand  $M$  is the completion under the  $C^*(\mathcal{G}(x))$ -valued inner-product  $\langle \cdot, \cdot \rangle_*$ , and, on the other,  $H_x$  is the completion for the 2-norm. These two spaces are related to each other by the following.

**Proposition 2.7.** *There is a bounded linear map*

$$j : M \rightarrow H_x,$$

such that  $j(\varphi) = \varphi$ , for every  $\varphi \in C_c(\mathcal{G}_x)$ .

*Proof.* Given  $\varphi \in C_c(\mathcal{G}_x)$ , notice that

$$\|\varphi\|_2^2 = \sum_{\gamma \in \mathcal{G}_x} \overline{\varphi(\gamma)}\varphi(\gamma) = \langle \varphi, \varphi \rangle_*(1) \leq \|\langle \varphi, \varphi \rangle_*\|_{C^*(\mathcal{G}(x))} = \|\varphi\|_M^2.$$

This implies that the identity map on  $C_c(\mathcal{G}_x)$  is continuous for  $\|\cdot\|_M$  on its domain and the 2-norm on its codomain. The required map is then obtained by a continuous extension.  $\square$

If  $\zeta \in \mathcal{G}(x)$ , we have a well-defined bijective map

$$\gamma \in \mathcal{G}_x \mapsto \gamma\zeta \in \mathcal{G}_x,$$

and hence the map

$$R_\zeta : H_x \rightarrow H_x,$$

defined by

$$R_\zeta(\xi)_\gamma = \xi(\gamma\zeta), \quad \forall \xi \in H_x, \quad \forall \gamma \in \mathcal{G}_x,$$

is a unitary operator. It is also easy to see that  $R_{\zeta_1} \circ R_{\zeta_2} = R_{\zeta_1\zeta_2}$ , which is to say that  $R$  is a unitary representation of  $\mathcal{G}(x)$  on  $H_x$ .

This representation will play an important role in our next result, but before stating it, we need to introduce a notation.

Given any discrete group  $G$ , and any  $\zeta \in G$ , the map

$$f \in C_c(G) \mapsto f(\zeta) \in \mathbb{C}$$

is well known to extend to a bounded linear functional on  $C^*(G)$ , which we will denote by

$$a \in C^*(G) \mapsto \hat{a}(\zeta) \in \mathbb{C}.$$

**Proposition 2.8.** *For every  $a \in C^*(\mathcal{G})$ , every  $x, y \in M$ , and every  $\zeta \in \mathcal{G}(x)$ , we have that*

$$\widehat{\langle x, ay \rangle}_*(\zeta) = \left\langle \lambda_x(a) R_\zeta(j(y)), j(x) \right\rangle.$$

*Proof.* Given  $f \in C_c(\mathcal{G})$ , and  $\psi, \varphi \in C_c(\mathcal{G}_x)$ , we have

$$\langle \varphi, f * \psi \rangle_*(\zeta) = \sum_{\gamma_1 \gamma_2 = \zeta} \overline{\varphi(\gamma_1^{-1})} (f * \psi)(\gamma_2) = \sum_{\gamma_1 \gamma_2 \gamma_3 = \zeta} \overline{\varphi(\gamma_1^{-1})} f(\gamma_2) \psi(\gamma_3) = \dots$$

With the change of variables “ $\gamma'_3 = \gamma_3 \zeta^{-1}$ ” the above equals

$$\begin{aligned} \dots &= \sum_{\gamma_1 \gamma_2 \gamma'_3 = x} \overline{\varphi(\gamma_1^{-1})} f(\gamma_2) \psi(\gamma'_3 \zeta) \\ &= \sum_{\gamma_1 \gamma_2 \gamma'_3 = x} \overline{\varphi(\gamma_1^{-1})} f(\gamma_2) R_\zeta(\psi)(\gamma'_3) = \langle f * R_\zeta(\psi), \varphi \rangle. \end{aligned}$$

This gives that

$$\langle \varphi, f * \psi \rangle_*(\zeta) = \langle f * R_\zeta(\psi), \varphi \rangle,$$

and the proof is concluded upon replacing

- $f$  by the terms of a sequence  $\{f_n\}_n$  converging to  $a$  in  $C^*(\mathcal{G}(x))$ ,
- $\varphi$  by the terms of a sequence  $\{\varphi_n\}_n$  converging to  $x$  in  $M$ , and finally
- $\psi$  by the terms of a sequence  $\{\psi_n\}_n$  converging to  $y$  in  $M$ . □

**Corollary 2.9.** *Given  $x \in X$ , suppose that  $a$  is an element of  $C^*(\mathcal{G})$  such that  $\lambda_x(a) = 0$ . Then*

$$\text{Ind}_{\mathcal{G}(x)}^{\mathcal{G}} L(a) = 0,$$

for any representation  $L$  of  $C^*(\mathcal{G}(x))$  which is weakly contained in  $\Lambda$ .

*Proof.* By Proposition 2.8, we deduce that

$$\widehat{\langle x, ay \rangle}_*(\zeta) = 0, \quad \forall \zeta \in \mathcal{G}(x), \quad \forall x, y \in M.$$

Temporarily fixing  $x$  and  $y$ , we then deduce that  $\Lambda(\langle x, ay \rangle_*) = 0$ , and hence that

$$L(\langle x, ay \rangle_*) = 0, \tag{2.3}$$

for any  $L$  as in the statement. Given  $f \in C_c(\mathcal{G})$ ,  $\varphi, \psi \in C_c(\mathcal{G}_x)$  and  $\xi, \eta \in H_L$ , we have that

$$\langle \text{Ind}_{\mathcal{G}(x)}^{\mathcal{G}} L(f)(\varphi \otimes \xi), \psi \otimes \eta \rangle = \langle (f * \varphi) \otimes \xi, \psi \otimes \eta \rangle = \langle L(\langle \psi, f * \varphi \rangle_*) \xi, \eta \rangle.$$



Applying this for  $f$  ranging in a sequence  $\{f_n\}_n$  converging to  $a$  in  $C^*(\mathcal{G}(x))$ , we conclude that

$$\langle \text{Ind}_{\mathcal{G}(x)}^{\mathcal{G}} L(a)(\varphi \otimes \xi), \psi \otimes \eta \rangle = \langle L(\langle \psi, a\varphi \rangle_* \xi), \eta \rangle \stackrel{(2.3)}{=} 0,$$

from where the conclusion follows easily. □

We may now prove our main result:

**Theorem 2.10.** *Let  $\mathcal{G}$  be a second-countable, Hausdorff, étale groupoid, such that  $\mathcal{G}^{(0)}$  is compact. Suppose moreover that  $\mathcal{G}$  is amenable. Then  $\{\lambda_x\}_{x \in \mathcal{G}^{(0)}}$  is a sufficient family of representations for  $C^*(\mathcal{G})$ . In other words, if  $a \in C^*(\mathcal{G})$  is such that  $\lambda_x(a)$  is invertible for every  $x$  in the unit space of  $\mathcal{G}$ , then  $a$  is necessarily invertible.*

*Proof.* Suppose, by way of contradiction, that  $a$  is non-invertible. By Proposition 2.2 there exists an irreducible representation  $\pi$  of  $C^*(\mathcal{G})$  such that  $\pi(a)$  is non-invertible. Employing [5, Theorem 2.1] we have that, for some  $x \in \mathcal{G}^{(0)}$ , there exists an irreducible representation  $L$  of  $C^*(\mathcal{G}(x))$  such that  $\pi$  and  $\text{Ind}_{\mathcal{G}(x)}^{\mathcal{G}} L$  share null spaces.

Since  $\mathcal{G}$  is amenable we have that  $\mathcal{G}(x)$  is also amenable by [2, Proposition 5.1.1], and hence that  $L$  is weakly contained in the left-regular representation. We may therefore employ Corollary 2.9 to conclude that

$$\text{Ker}(\lambda_x) \subseteq \text{Ker}(\text{Ind}_{\mathcal{G}(x)}^{\mathcal{G}} L) = \text{Ker}(\pi).$$

By hypothesis  $a$  is invertible modulo  $\text{Ker}(\lambda_x)$ , and hence it must also be invertible modulo  $\text{Ker}(\pi)$ , a contradiction. □

### 3. Strictly norming family of representations

A family  $\mathcal{F}$  of representations of a  $C^*$ -algebra  $A$  is often called *norming*, when

$$\|a\| = \sup_{\pi \in \mathcal{F}} \|\pi(a)\|, \quad \forall a \in A. \tag{3.1}$$

As an example, the family  $\{\lambda_x\}_{x \in \mathcal{G}^{(0)}}$  is norming for the reduced groupoid  $C^*$ -algebra  $C_r^*(\mathcal{G})$ , for every (non-necessarily amenable) groupoid  $\mathcal{G}$ . Based on this concept, let us give the following:

**Definition 3.1.** A family  $\mathcal{F}$  of representations of a  $C^*$ -algebra  $A$  will be called *strictly norming* when it is norming and, in addition, the supremum in (3.1) is attained for every  $a$  in  $A$ .

The next result, due to Roch, relates strictly norming and sufficient families in an interesting way. Its proof is included for the convenience of the reader and also because it is slightly simpler than the proof given by Roch in [10].

**Theorem 3.2 ([10, Theorem 5.7]).** *Let  $\mathcal{F}$  be a family of non-degenerated representations of a unital  $C^*$ -algebra  $A$ . Then  $\mathcal{F}$  is strictly norming if and only if it is sufficient.*

*Proof.* Arguing by contradiction, suppose that  $\mathcal{F}$  is sufficient, but there exists  $a \in A$  such that  $\|\pi(a)\| < \|a\|$ , for all  $\pi$  in  $\mathcal{F}$ . Replacing  $a$  by  $a^*a$ , we may assume that  $a$  is positive. For every  $\pi$  in  $\mathcal{F}$ , we then have that

$$\sigma(\pi(a)) \subseteq [0, \|\pi(a)\|] \subseteq [0, \|a\|].$$

Setting  $b = a - \|a\|$ , we then have by the spectral mapping theorem that

$$\sigma(\pi(b)) = \sigma(\pi(a) - \|a\|) = \sigma(\pi(a)) - \|a\| \subseteq [-\|a\|, 0].$$

It follows that  $0 \notin \sigma(\pi(b))$ , and hence that  $\pi(b)$  is invertible for every  $\pi$  in  $\mathcal{F}$ , but, since  $\|a\|$  belongs to the spectrum of  $a$ , we see that  $b$  is not invertible, a contradiction.

To verify the “only if” part of the statement, let  $a$  be non-invertible. We thus need to find some  $\pi \in \mathcal{F}$ , such that  $\pi(a)$  is non-invertible.

Since  $a$  is non-invertible, then either  $a^*a$  or  $aa^*$  is non-invertible. We suppose without loss of generality that the former is true, that is, that the element  $c := a^*a$  is non-invertible. We then have that

$$0 \in \sigma(c) \subseteq [0, \|c\|].$$

With  $b = \|c\| - c$ , we conclude from the spectral mapping theorem that

$$\|c\| \in \sigma(b) \subseteq \|c\| - [0, \|c\|] = [0, \|c\|],$$

so  $\|b\| = \|c\|$ , and by hypothesis there exists  $\pi \in \mathcal{F}$ , such that  $\|\pi(b)\| = \|c\|$ . Since  $\pi(b)$  is positive, this implies that  $\|c\|$  lies in its spectrum, which is to say that  $\|c\| - \pi(b)$  is non-invertible, but

$$\|c\| - \pi(b) = \pi(c),$$

so  $\pi(c)$  is non-invertible which implies that  $\pi(a)$  is non-invertible. □

Putting Theorem 2.10 and Theorem 3.2 together, we therefore deduce the following important consequence:

**Corollary 3.3.** *Suppose we are given a second-countable, Hausdorff, étale, amenable groupoid  $\mathcal{G}$  with  $\mathcal{G}^{(0)}$  compact. Then, for every  $a \in C^*(\mathcal{G})$ , there exists  $x \in \mathcal{G}^{(0)}$ , such that*

$$\|a\| = \|\lambda_x(a)\|.$$

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