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Boundedness of the Maximal and Singular Operators on Generalized Orlicz–Morrey Spaces

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Dedicated to Professor António Ferreira dos Santos

Abstract. We consider generalized Orlicz–Morrey spaces $M_{\Phi,\varphi}(\mathbb{R}^n)$ including their weak versions. In these generalized spaces we prove the boundedness of the Hardy–Littlewood maximal operator and Calderón–Zygmund singular operators with standard kernel. In all the cases the conditions for the boundedness are given either in terms of Zygmund-type integral inequalities on $\varphi(r)$ without assuming any monotonicity property of $\varphi(r)$, or in terms of supremal operators, related to $\varphi(r)$.

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1. Introduction

Inequalities involving classical operators of harmonic analysis, such as maximal functions, fractional integrals and singular integrals of convolution type have been extensively investigated in various function spaces. Results on weak and strong type inequalities for operators of this kind in Lebesgue spaces are classical and can be found for example in [3, 41, 42, 44]. Generalizations of these results to Zygmund spaces are presented in [3]. An exhaustive treatment of the problem of boundedness of such operators in Lorentz and Lorentz–Zygmund spaces is given in [2]. See also [10, 11] for further extensions in the framework of generalized

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Lorentz–Zygmund spaces. As far as Orlicz spaces are concerned, we refer to the books [21, 23, 37] and note that a characterization of Young functions A for which the Hardy–Littlewood maximal operator or the Hilbert and Riesz transforms are of weak or strong type in Orlicz space L_A is known (see for example [5, 21]). In [33, 44] conditions on Young functions A and B are given for the fractional integral operator to be bounded from L_A into L_B under some restrictions involving the growth and certain monotonicity properties of A and B (see also [5]).

Orlicz spaces, introduced in [34, 35], are generalizations of Lebesgue spaces L_p. They are useful tools in harmonic analysis and its applications. For example, the Hardy–Littlewood maximal operator is bounded on L_p for $1 < p < \infty$, but not on L_1. Using Orlicz spaces, we can investigate the boundedness of the maximal operator near $p = 1$ more precisely (see [17, 18] and [5]).

In the study of local properties of solutions to of partial differential equations, together with weighted Lebesgue spaces, Morrey spaces $M_{p,\lambda}(\mathbb{R}^n)$ play an important role, see [12]. Introduced by C. Morrey [29] in 1938, they are defined by the norm

$$\|f\|_{M_{p,\lambda}} := \sup_{x,\, r>0} r^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,r))}, \tag{1.1}$$

where $0 \le \lambda \le n$, $1 \le p < \infty$. Here and everywhere in the sequel $B(x, r)$ stands for the ball in $\mathbb{R}^n$ of radius r centered at x. Let $|B(x, r)|$ be the Lebesgue measure of the ball $B(x, r)$ and $|B(x, r)| = v_n r^n$, where $v_n = |B(0, 1)|$.

Note that $M_{p,0} = L_p(\mathbb{R}^n)$ and $M_{p,n} = L_\infty(\mathbb{R}^n)$. If $\lambda < 0$ or $\lambda > n$, then $M_{p,\lambda} = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.

We also denote by $WM_{p,\lambda} \equiv WM_{p,\lambda}(\mathbb{R}^n)$ the weak Morrey space of all functions $f \in WL_p^{\text{loc}}(\mathbb{R}^n)$ for which

$$\|f\|_{WM_{p,\lambda}} = \sup_{x\in\mathbb{R}^n,\, r>0} r^{-\frac{\lambda}{p}} \|f\|_{WL_p(B(x,r))} < \infty.$$

We refer in particular to [24] for the classical Morrey spaces. Observe that Morrey spaces with r^λ replaced by a function $\varphi(r)$ first appeared in [9] and [46]; we also refer to the survey paper [36] for more various definitions of generalized Morrey spaces and note that study of classical operators of harmonic analysis in generalized Morrey spaces started in [13], [14], [30], up to authors' knowledge.

Last two decades there is an increasing interest to the study of variable exponent spaces and operators with variable parameters, in such spaces, we refer to the recent books [6], [8] and surveying papers [7], [20], [22], [38].

Orlicz–Morrey spaces and maximal and singular operators in such spaces were studied in [31], [32]. The most general spaces of such a type, Musielak–Orlicz–Morrey spaces, unifying the classical and variable exponent approaches, were studied in the recent paper [28], where potential operators were studied together with the corresponding Sobolev embeddings.

In this paper we study the maximal and singular operators in Orlicz–Morrey spaces, introduced in a less generality, but advance in the following two directions:

1) we make minimal assumptions on the functions defining the space avoiding any kind of monotonicity or growth condition, required, for instance, in [30], [31], [32],
2) we prove weak-type inequalities.

Our conditions for the boundedness are sufficient. We do not discuss their necessity in this paper but hope to do that in another paper.

We define the generalized Orlicz–Morrey space $M_{\Phi,\varphi}(\mathbb{R}^n)$ in question by the norm

$$\|f\|_{M_{\Phi,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} \, \|f\|_{L_\Phi(B(x,r))}.$$

where $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and Φ a Young function, but refer to Section 2 for all the precise definitions and comparison with other norms.

The main purpose of this paper is to find sufficient conditions on general Young function Φ and functions φ_1, φ_2 ensuring that the operators under consideration are of weak or strong type from generalized Orlicz–Morrey spaces $M_{\Phi,\varphi_1}(\mathbb{R}^n)$ into $M_{\Phi,\varphi_2}(\mathbb{R}^n)$. Our results for the maximal operator are presented in Section 4, while Section 5 deals with singular integrals.

1.1. Operators under consideration

We study the following operators: *the maximal operator*

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|dy,$$

Calderón–Zygmund type singular operators; by this we mean operators bounded in $L^2(\mathbb{R}^n)$ of the form

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y)f(y)dy,$$

where $K(x,y)$ is a “standard singular kernel”, that is, a continuous function defined on $\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : x \neq y\}$ and satisfying the estimates

$$|K(x,y)| \le C|x-y|^{-n} \qquad \text{for all } x \neq y,$$

$$|K(x,y) - K(x,z)| \le C\frac{|y-z|^\sigma}{|x-y|^{n+\sigma}}, \;\; \sigma > 0, \;\; \text{if } \; |x-y| > 2|y-z|,$$

$$|K(x,y) - K(\xi,y)| \le C\frac{|x-\xi|^\sigma}{|x-y|^{n+\sigma}}, \;\; \sigma > 0, \;\; \text{if } \; |x-y| > 2|x-\xi|.$$

Our main results are obtained in Theorems 4.6 and 5.5, where we use recent results presented in Theorems 2.11 and 2.12 to obtain a generalization of known conditions for the boundedness of maximal and singular operators in Orlicz–Morrey spaces, it is given in terms of conditions (4.8) and (5.7), respectively, without any assumption of monotonicity type on the functions φ_1 and φ_2 as, for instance, used in [28], [31] and other sources.

2. Some preliminaries on Orlicz and Orlicz–Morrey spaces

Definition 2.1. A function $\Phi : [0,+\infty] \to [0,\infty]$ is called a Young function if Φ is convex, left-continuous, $\lim\limits_{r\to+0} \Phi(r) = \Phi(0) = 0$ and $\lim\limits_{r\to+\infty} \Phi(r) = \Phi(\infty) = \infty$.

From the convexity and $\Phi(0) = 0$ it follows that any Young function is increasing. If there exists $s \in (0,+\infty)$ such that $\Phi(s) = +\infty$, then $\Phi(r) = +\infty$ for $r \geq s$.

We say that $\Phi \in \Delta_2$, if for any $a > 1$, there exists a constant $C_a > 0$ such that $\Phi(at) \leq C_a\Phi(t)$ for all $t > 0$. A Young function Φ is said to satisfy the ∇_2-condition, denoted also by $\Phi \in \nabla_2$, if

$$\Phi(r) \leq \frac{1}{2k}\Phi(kr), \qquad r \geq 0,$$

for some $k > 1$. The function $\Phi(r) = r$ satisfies the Δ_2-condition but does not satisfy the ∇_2-condition. If $1 < p < \infty$, then $\Phi(r) = r^p$ satisfies both the conditions. The function $\Phi(r) = e^r - r - 1$ satisfies the ∇_2-condition but does not satisfy the Δ_2-condition. The following two indices

$$q_\Phi = \inf_{t>0} \frac{t\varphi(t)}{\Phi(t)}, \qquad p_\Phi = \sup_{t>0} \frac{t\varphi(t)}{\Phi(t)},$$

of Φ, where $\varphi(t)$ is the right-continuous derivative of Φ, are well known in the theory of Orlicz spaces. As is well known,

$$p_\Phi < \infty \qquad \Longleftrightarrow \qquad \Phi \in \Delta_2,$$

and the function Φ is strictly convex if and only if $q_\Phi > 1$. If $0 < q_\Phi \leq p_\Phi < \infty$, then $\frac{\Phi(t)}{t^{q_\Phi}}$ is increasing and $\frac{\Phi(t)}{t^{p_\Phi}}$ is decreasing on $(0,\infty)$.

Lemma 2.2. ([21], *Lemma* 1.3.2) *Let $\Phi \in \Delta_2$. Then there exist $p > 1$ and $b > 1$ such that*

$$\frac{\Phi(t_2)}{t_2^p} \leq \frac{b\Phi(t_1)}{t_1^p}$$

for $0 < t_1 < t_2$.

Recall that a function Φ is said to be quasiconvex if there exist a convex function ω and a constant $c > 0$ such that

$$\omega(t) \leq \Phi(t) \leq c\omega(ct), \; t \in [0,\infty).$$

Let $\mathcal{Y}$ be the set of all Young functions Φ such that

$$0 < \Phi(r) < +\infty \qquad \text{for} \qquad 0 < r < +\infty \tag{2.1}$$

If $\Phi \in \mathcal{Y}$, then Φ is absolutely continuous on every closed interval in $[0,+\infty)$ and bijective from $[0,+\infty)$ to itself.

Definition 2.3 (Orlicz Space). For a Young function Φ, the set

$$L_\Phi(\mathbb{R}^n) = \left\{ f \in L_1^{loc}(\mathbb{R}^n) : \int_{\mathbb{R}^n} \Phi(k|f(x)|)dx < +\infty \text{ for some } k > 0 \right\}$$

is called Orlicz space. The space $L^{\mathrm{loc}}_{\Phi}(\mathbb{R}^n)$ endowed with the natural topology is defined as the set of all functions f such that $f\chi_B \in L_\Phi(\mathbb{R}^n)$ for all balls $B \subset \mathbb{R}^n$.

Note that, $L_\Phi(\mathbb{R}^n)$ is a Banach space with respect to the norm

$$\|f\|_{L_\Phi} = \inf\Big\{\lambda > 0 : \int_{\mathbb{R}^n} \Phi\Big(\frac{|f(x)|}{\lambda}\Big)dx \le 1\Big\},$$

see, for example, [37], Section 3, Theorem 10, so that

$$\int_{\mathbb{R}^n} \Phi\Big(\frac{|f(x)|}{\|f\|_{L_\Phi}}\Big)dx \le 1.$$

For a measurable set $\Omega \subset \mathbb{R}^n$, a measurable function f and $t > 0$, let

$$m(\Omega,\ f,\ t) = |\{x \in \Omega : |f(x)| > t\}|.$$

In the case $\Omega = \mathbb{R}^n$, we shortly denote it by $m(f,\ t)$.

Definition 2.4. The weak Orlicz space

$$WL_\Phi(\mathbb{R}^n) = \{f \in L^{\mathrm{loc}}_1(\mathbb{R}^n) : \|f\|_{WL_\Phi} < +\infty\}$$

is defined by the norm

$$\|f\|_{WL_\Phi} = \inf\Big\{\lambda > 0\ :\ \sup_{t>0} \Phi(t) m\Big(\frac{f}{\lambda},\ t\Big)\ \le 1\Big\}.$$

For Young functions Φ and Ψ, we write $\Phi \approx \Psi$ if there exists a constant $C \ge 1$ such that

$$\Phi(C^{-1}r) \le \Psi(r) \le \Phi(Cr) \qquad \text{for all } r \ge 0$$

If $\Phi \approx \Psi$, then $L_\Phi(\mathbb{R}^n) = L_\Psi(\mathbb{R}^n)$ with equivalent norms. We note that, for Young functions Φ and Ψ, if there exist $C, R \ge 1$ such that

$$\Phi(C^{-1}r) \le \Psi(r) \le \Phi(Cr) \qquad \text{for } r \in (0, R^{-1}) \cup (R, \infty),$$

then $\Phi \approx \Psi$.

For a Young function Φ and $0 \le s \le +\infty$, let

$$\Phi^{-1}(s) = \inf\{r \ge 0 : \Phi(r) > s\} \qquad (\inf \emptyset = +\infty).$$

If $\Phi \in \mathcal{Y}$, then Φ^{-1} is the usual inverse function of Φ. We note that

$$\Phi(\Phi^{-1}(r)) \le r \le \Phi^{-1}(\Phi(r)) \quad \text{for } 0 \le r < +\infty.$$

For a Young function Φ, the complementary function $\widetilde{\Phi}(r)$ is defined by

$$\widetilde{\Phi}(r) = \begin{cases} \sup\{rs - \Phi(s) : s \in [0, \infty)\} & , \quad r \in [0, \infty) \\ +\infty & , \quad r = +\infty. \end{cases} \tag{2.2}$$

The complementary function $\widetilde{\Phi}$ is also a Young function and $\widetilde{\widetilde{\Phi}} = \Phi$. If $\Phi(r) = r$, then $\widetilde{\Phi}(r) = 0$ for $0 \le r \le 1$ and $\widetilde{\Phi}(r) = +\infty$ for $r > 1$. If $1 < p < \infty$, $1/p + 1/p' = 1$ and $\Phi(r) = r^p/p$, then $\widetilde{\Phi}(r) = r^{p'}/p'$. If $\Phi(r) = e^r - r - 1$, then $\widetilde{\Phi}(r) = (1 + r)\log(1 + r) - r$.

Remark 2.5. Note that $\Phi \in \nabla_2$ if and only if $\widetilde{\Phi} \in \Delta_2$. Also, if Φ is a Young function, then $\Phi \in \nabla_2$ if and only if Φ^γ be quasiconvex for some $\gamma \in (0,1)$ (see, for example, [21], p. 15).

It is known that

$$r \le \Phi^{-1}(r)\widetilde{\Phi}^{-1}(r) \le 2r \qquad \text{for } r \ge 0. \tag{2.3}$$

The following analogue of the Hölder inequality is known, see [45].

Theorem 2.6 ([45]). *For a Young function Φ and its complementary function $\widetilde{\Phi}$, the following inequality is valid*

$$\|fg\|_{L_1(\mathbb{R}^n)} \le 2\|f\|_{L_\Phi}\|g\|_{L_{\widetilde{\Phi}}}.$$

Note that Young functions satisfy the property

$$\Phi(\alpha t) \le \alpha\Phi(t) \tag{2.4}$$

for all $0 < \alpha < 1$ and $0 \le t < \infty$, which is a consequence of the convexity: $\Phi(\alpha t) = \Phi(\alpha t + (1-\alpha)0) \le \alpha\Phi(t) + (1-\alpha)\Phi(0) = \alpha\Phi(t)$.

The following lemma is valid.

Lemma 2.7 ([3, 25]). *Let Φ be a Young function and B a set in $\mathbb{R}^n$ with finite Lebesgue measure. Then*

$$\|\chi_B\|_{WL_\Phi(\mathbb{R}^n)} = \|\chi_B\|_{L_\Phi(\mathbb{R}^n)} = \frac{1}{\Phi^{-1}\left(|B|^{-1}\right)}.$$

In the next sections where we prove our main estimates, we use the following lemma, which follows from Theorem 2.6 and Lemma 2.7.

Lemma 2.8. *For a Young function Φ and $B = B(x,r)$, the following inequality is valid*

$$\|f\|_{L_1(B)} \le 2|B|\Phi^{-1}\left(|B|^{-1}\right)\|f\|_{L_\Phi(B)}.$$

Definition 2.9. (Orlicz–Morrey space). For a Young function Φ and $0 \le \lambda \le n$, we denote by $M_{\Phi,\lambda}(\mathbb{R}^n)$ the Orlicz–Morrey space, defined as the space of all functions $f \in L_\Phi^{\mathrm{loc}}(\mathbb{R}^n)$ with finite quasinorm

$$\|f\|_{M_{\Phi,\lambda}} = \sup_{x\in\mathbb{R}^n,\ r>0} \Phi^{-1}\big(r^{-\lambda}\big)\|f\|_{L_\Phi(B(x,r))}.$$

Note that $M_{\Phi,\lambda}\big|_{\lambda=0} = L_\Phi(\mathbb{R}^n)$.

We also denote by $WM_{\Phi,\lambda}(\mathbb{R}^n)$ the weak Morrey space which consists of all functions $f \in WL_\Phi^{\mathrm{loc}}(\mathbb{R}^n)$ for which

$$\|f\|_{WM_{\Phi,\lambda}} = \sup_{x\in\mathbb{R}^n,\ r>0} \Phi^{-1}\big(r^{-\lambda}\big)\|f\|_{WL_\Phi(B(x,r))} < \infty,$$

where $WL_\Phi(B(x,r))$ denotes the weak L_Φ-space of measurable functions f for which

$$\|f\|_{WL_\Phi(B(x,r))} \equiv \|f\chi_{B(x,r)}\|_{WL_\Phi(\mathbb{R}^n)}.$$

Definition 2.10 (Generalized Orlicz–Morrey Space). Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and Φ any Young function. We denote by $M_{\Phi,\varphi}(\mathbb{R}^n)$ the generalized Morrey space, the space of all functions $f \in L_\Phi^{\mathrm{loc}}(\mathbb{R}^n)$ with finite quasinorm

$$\|f\|_{M_{\Phi,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1}\, \|f\|_{L_\Phi(B(x,r))}.$$

It may be easily shown that $\|f\|_{M_{\Phi,\varphi}}$ is a norm and $M_{\Phi,\varphi}$ is a Banach space, for any Young function Φ.

By $WM_{\Phi,\varphi}(\mathbb{R}^n)$ we denote the weak generalized Morrey space of all functions $f \in WL_\Phi^{\mathrm{loc}}(\mathbb{R}^n)$ for which

$$\|f\|_{WM_{\Phi,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1}\, \|f\|_{WL_\Phi(B(x,r))} < \infty.$$

If Φ satisfies the Δ_2-condition, then the norm $\|f\|_{M_{\Phi,\varphi}}$ is equivalent (see [28], p. 416) to the norm

$$\|f\|_{\overline{M}_{\Phi,\varphi}} = \inf\Big\{\lambda > 0 : \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} \int_{B(x,r)} \Phi\Big(\frac{|f(x)|}{\lambda}\Big) dx \le 1\Big\}.$$

The latter was used in [28, 31, 32, 39], see also references there.

Definition 2.10 recovers the spaces $M_{\Phi,\lambda}$ and $WM_{\Phi,\lambda}$ under the choice $\varphi(x,r) = 1/\Phi^{-1}\big(r^{-\lambda}\big)$ and the spaces $M_{p,\varphi}$ and $WM_{p,\varphi}$ under the choice $\Phi(r) = r^p$.

The following statement was proved in [1] (see also [4]).

Theorem 2.11. *Let $1 \le p < \infty$ and (φ_1, φ_2) satisfies the condition*

$$\sup_{r<t<\infty} \operatorname{ess\,sup} \inf_{t<s<\infty} \varphi_1(x,s)\, t^{-n/p} \le C\, \varphi_2(x,r)\, r^{-n/p}, \tag{2.5}$$

where C does not depend on x and r. Then the maximal operator M is bounded from M_{p,φ_1} to M_{p,φ_2} for $p > 1$ and from M_{1,φ_1} to WM_{1,φ_2}. Moreover, for $p > 1$

$$\|Mf\|_{M_{p,\varphi_2}} \lesssim \|f\|_{M_{p,\varphi_1}}, \quad \text{and for } p = 1 \quad \|Mf\|_{WM_{1,\varphi_2}} \lesssim \|f\|_{M_{1,\varphi_1}}.$$

The following statement, containing results obtained in [13, 14, 15, 27, 30] was proved in [1] (see also [16]).

Theorem 2.12. *Let $1 \le p < \infty$ and (φ_1, φ_2) satisfies the condition*

$$\int_r^\infty \operatorname{ess\,sup} \inf_{t<s<\infty} \varphi_1(x,s)\, t^{-n/p} \frac{dt}{t} \le C\, \varphi_2(x,r)\, r^{-n/p},$$

where C does not depend on x and r. Then the singular operator T is bounded from M_{p,φ_1} to M_{p,φ_2} for $p > 1$ and from M_{1,φ_1} to WM_{1,φ_2}. Moreover, for $p > 1$

$$\|Tf\|_{M_{p,\varphi_2}} \lesssim \|f\|_{M_{p,\varphi_1}}, \quad \text{and for } p = 1 \quad \|Tf\|_{WM_{1,\varphi_2}} \lesssim \|f\|_{M_{1,\varphi_1}}.$$

3. Some supremal and Hardy type inequalities

Let v be a weight. We denote by $L_{\infty,v}(0,\infty)$ the space of all functions $g(t)$, $t>0$ with finite norm

$$\|g\|_{L_{\infty,v}(0,\infty)} = \sup_{t>0} v(t)|g(t)|$$

and $L_\infty(0,\infty) \equiv L_{\infty,1}(0,\infty)$. Let $\mathfrak{M}(0,\infty)$ be the set of all Lebesgue-measurable functions on $(0,\infty)$ and $\mathfrak{M}^+(0,\infty)$ its subset of all nonnegative functions on $(0,\infty)$. We denote by $\mathfrak{M}^+(0,\infty;\uparrow)$ the cone of all functions in $\mathfrak{M}^+(0,\infty)$ which are non-decreasing on $(0,\infty)$ and

$$\mathcal{A} = \left\{ \varphi \in \mathfrak{M}^+(0,\infty;\uparrow) : \lim_{t\to 0+} \varphi(t) = 0 \right\}.$$

Let u be a continuous and non-negative function on $(0,\infty)$. We define the supremal operator $\overline{S}_u$ on $g \in \mathfrak{M}(0,\infty)$ by

$$(\overline{S}_u g)(t) := \|u\, g\|_{L_\infty(t,\infty)},\ t \in (0,\infty).$$

The following theorem was proved in [4].

Theorem 3.1. *Let v_1, v_2 be non-negative measurable functions satisfying $0 < \|v_1\|_{L_\infty(t,\infty)} < \infty$ for any $t>0$ and let u be a continuous non-negative function on $(0,\infty)$. Then the operator $\overline{S}_u$ is bounded from $L_{\infty,v_1}(0,\infty)$ to $L_{\infty,v_2}(0,\infty)$ on the cone $\mathcal{A}$ if and only if*

$$\left\| v_2 \overline{S}_u \left(\|v_1\|^{-1}_{L_\infty(\cdot,\infty)} \right) \right\|_{L_\infty(0,\infty)} < \infty. \tag{3.1}$$

We will use the following statement on the boundedness of the weighted Hardy operator

$$H^*_w g(t) := \int_t^\infty g(s)w(s)ds, \quad 0<t<\infty,$$

where w is a weight.

The following theorem in the case $w=1$ was proved in [4].

Theorem 3.2. *Let v_1, v_2 and w be weights on $(0,\infty)$ and $v_1(t)$ be bounded outside a neighborhood of the origin. The inequality*

$$\sup_{t>0} v_2(t) H^*_w g(t) \le C \sup_{t>0} v_1(t) g(t) \tag{3.2}$$

holds for some $C>0$ for all non-negative and non-decreasing g on $(0,\infty)$ if and only if

$$B := \sup_{t>0} v_2(t) \int_t^\infty \frac{w(s)ds}{\sup_{s<\tau<\infty} v_1(\tau)} < \infty. \tag{3.3}$$

Moreover, the value $C=B$ is the best constant for (3.2).

Remark 3.3. In (3.2) and (3.3) it is assumed that $\frac{1}{\infty}=0$ and $0\cdot\infty=0$.

Proof. Sufficiency. Suppose that (3.3) holds. Whenever F, G are non-negative functions on $(0,\infty)$ and F is non-decreasing, then

$$\sup_{t>0} F(t)G(t) = \sup_{t>0} F(t) \sup_{s>t} G(s),\ t>0. \tag{3.4}$$

By (3.4) we have

$$\begin{aligned} \sup_{t>0} v_2(t) H_w^* g(t) &= \sup_{t>0} v_2(t) \int_t^\infty g(s)w(s)\, \frac{\sup_{s<\tau<\infty} v_1(\tau)}{\sup_{s<\tau<\infty} v_1(\tau)}\, ds \\ &\le \sup_{t>0} v_2(t) \int_t^\infty \frac{w(s)ds}{\sup_{s<\tau<\infty} v_1(\tau)}\, \sup_{t>0} g(t) \sup_{t<\tau<\infty} v_1(\tau) \\ &= \sup_{t>0} v_2(t) \int_t^\infty \frac{w(s)ds}{\sup_{s<\tau<\infty} v_1(\tau)}\, \sup_{t>0} g(t) v_1(t) \\ &\le B \sup_{t>0} g(t) v_1(t), \end{aligned}$$

so that (3.2) holds with $C = B$.

Necessity. Suppose that the inequality (3.2) holds with some $C>0$. The function

$$g(t) = \frac{1}{\sup_{t<\tau<\infty} v_1(\tau)},\ t>0$$

is nonnegative and non-decreasing on $(0,\infty)$. Thus

$$B = \sup_{t>0} v_2(t) \int_t^\infty \frac{w(s)ds}{\sup_{s<\tau<\infty} v_1(\tau)} \le C \sup_{t>0} \frac{v_1(t)}{\sup_{t<\tau<\infty} v_1(\tau)} \le C,$$

which competes the proof. □

4. Boundedness of the maximal operator in the spaces $M_{\Phi,\varphi}(\mathbb{R}^n)$

In this section sufficient conditions on φ for the boundedness of M in generalized Orlicz–Morrey spaces $M_{\Phi,\varphi}(\mathbb{R}^n)$ have been obtained.

Definition 4.1. The operator T is said to be of strong type (Φ,Ψ) if there exists a positive constant k such that

$$\|Tf\|_{L_\Psi} \le k\|f\|_{L_\Phi}$$

for all $f \in L_\Phi(\mathbb{R}^n)$.

The operator T is said to be of weak type (Φ,Ψ) if there exists a positive constant k such that

$$|\{y\in\mathbb{R}^n : |Tf(y)|>t\}| \le 1/\Psi\left(\frac{t}{k\|f\|_{L_\Phi}}\right)$$

for all $t>0$ and all $f \in L_\Phi(\mathbb{R}^n)$.

Necessary and sufficient conditions on Φ for the boundedness of M in Orlicz spaces $L_\Phi(\mathbb{R}^n)$ have been obtained in [19], Theorem 2.1 and [21], Theorem 1.2.1. With Remark 2.5 taken into account, the known boundedness statement runs as follows.

The strong estimate in the following theorem is well known, proved in fact in [21], [5], although not stated directly in the form we need (they may be also derived from the Lorentz–Shimogaki theorem (see [3], p. 154) on the boundedness of the maximal operator on rearrangement invariant spaces and Boyd's interpolation theorem. So we present the proof only of the weak estimate.

Theorem 4.2. *Let Φ be a Young function. Then the maximal operator M is bounded from $L_\Phi(\mathbb{R}^n)$ to $WL_\Phi(\mathbb{R}^n)$ and for $\Phi \in \nabla_2$ bounded in $L_\Phi(\mathbb{R}^n)$.*

Proof. To prove the weak estimate, we take $f \in L_\Phi$ satisfying $\|f\|_{L_\Phi} = 1$ so that $\rho_\Phi(f) := \int_{\mathbb{R}^n} \Phi(|f(x)|)dx \le 1$. By Jensen's inequality,

$$\Phi\left(\frac{1}{|B|}\int_B |f(y)|dy\right) \le \frac{1}{|B|}\int_B \Phi(|f(y)|)dy \tag{4.1}$$

for all balls B. Using (4.1) and definition of the maximal operator, we have

$$\Phi(Mf(x)) \le M[(\Phi \circ f)(x)]. \tag{4.2}$$

Then by (4.2) and the weak $(1,1)$-boundedness of the maximal operator we get

$$|\{x : Mf(x) > t\}| = |\{x : \Phi(Mf(x)) > \Phi(t)\}| \le |\{x : M(\Phi \circ f)(x) > \Phi(t)\}|$$
$$\le \frac{C}{\Phi(t)}\int_{\mathbb{R}^n} \Phi(|f(x)|)dx \le \frac{C}{\Phi(t)} \le \frac{1}{\Phi(\frac{t}{C\|f\|_{L_\Phi}})},$$

since $\|f\|_{L_\Phi} = 1$ and $\frac{1}{C}\Phi(t) \ge \Phi\left(\frac{t}{C}\right)$, $C \ge 1$. By the homogeneity of the norm $\|\cdot\|_{L_\Phi}$, we then have

$$|\{x : Mf(x) > t\}| \le \frac{1}{\Phi(\frac{t}{C\|f\|_{L_\Phi}})}$$

for every $f \in L_\Phi$, which completes the proof. □

The following lemma is valid.

Lemma 4.3. *Let $f \in L_\Phi^{\rm loc}(\mathbb{R}^n)$ and $B = B(x,r)$. Then*

$$\|Mf\|_{L_\Phi(B)} \lesssim \|f\|_{L_\Phi(B(x,2r))} + \frac{1}{\Phi^{-1}\big(r^{-n}\big)} \sup_{t>2r} t^{-n}\|f\|_{L_1(B(x,t))}, \tag{4.3}$$

for any Young function $\Phi \in \nabla_2$ and

$$\|Mf\|_{WL_\Phi(B)} \lesssim \|f\|_{L_\Phi(B(x,2r))} + \frac{1}{\Phi^{-1}\big(r^{-n}\big)} \sup_{t>2r} t^{-n}\|f\|_{L_1(B(x,t))} \tag{4.4}$$

for any Young function Φ.

Proof. Let $\Phi \in \nabla_2$. We put $f = f_1 + f_2$, where $f_1 = f\chi_{B(x,2r)}$ and $f_2 = f\chi_{\complement_{B(x,2r)}}$ and have

$$\|Mf\|_{L_\Phi(B)} \leq \|Mf_1\|_{L_\Phi(B)} + \|Mf_2\|_{L_\Phi(B)}.$$

By the boundedness of the operator M on $L_\Phi(\mathbb{R}^n)$ provided by Theorem 4.2 we have

$$\|Mf_1\|_{L_\Phi(B)} \lesssim \|f\|_{L_\Phi(B(x,2r))}.$$

Let y be an arbitrary point from B. If $B(y,t) \cap {}^\complement(B(x,2r)) \neq \emptyset$, then $t > r$. Indeed, if $z \in B(y,t) \cap {}^\complement(B(x,2r))$, then $t > |y-z| \geq |x-z| - |x-y| > 2r - r = r$.

On the other hand, $B(y,t) \cap {}^\complement(B(x,2r)) \subset B(x,2t)$. Indeed, if $z \in B(y,t) \cap {}^\complement(B(x,2r))$, then we get $|x-z| \leq |y-z| + |x-y| < t + r < 2t$.

Hence

$$\begin{aligned} Mf_2(y) &= \sup_{t>0} \frac{1}{|B(y,t)|} \int_{B(y,t)\cap {}^\complement(B(x,2r))} |f(z)|dz \\ &\leq 2^n \sup_{t>r} \frac{1}{|B(x,2t)|} \int_{B(x,2t)} |f(z)|dz = 2^n \sup_{t>2r} \frac{1}{|B(x,t)|} \int_{B(x,t)} |f(z)|dz. \end{aligned}$$

Therefore, for all $y \in B$ we have

$$Mf_2(y) \leq 2^n \sup_{t>2r} \frac{1}{|B(x,t)|} \int_{B(x,t)} |f(z)|dz. \tag{4.5}$$

Thus

$$\|Mf\|_{L_\Phi(B)} \lesssim \|f\|_{L_\Phi(B(x,2r))} + \frac{1}{\Phi^{-1}\big(r^{-n}\big)} \left(\sup_{t>2r} \frac{1}{|B(x,t)|} \int_{B(x,t)} |f(z)|dz \right).$$

Let now Φ be an arbitrary Young function. It is obvious that

$$\|Mf\|_{WL_\Phi(B)} \leq \|Mf_1\|_{WL_\Phi(B)} + \|Mf_2\|_{WL_\Phi(B)}$$

for every ball $B = B(x,r)$.

By the boundedness of the operator M from $L_\Phi(\mathbb{R}^n)$ to $WL_\Phi(\mathbb{R}^n)$, provided by Theorem 4.2, we have

$$\|Mf_1\|_{WL_\Phi(B)} \lesssim \|f\|_{L_\Phi(B(x,2r))}.$$

Then by (4.5) we get the inequality (4.4). □

Lemma 4.4. *Let $f \in L^{\rm loc}_\Phi(\mathbb{R}^n)$ and $B = B(x,r)$. Then*

$$\|Mf\|_{L_\Phi(B)} \lesssim \frac{1}{\Phi^{-1}\big(r^{-n}\big)} \sup_{t>2r} \Phi^{-1}\big(t^{-n}\big) \|f\|_{L_\Phi(B(x,t))} \tag{4.6}$$

for any Young function $\Phi \in \nabla_2$ and

$$\|Mf\|_{WL_\Phi(B)} \lesssim \frac{1}{\Phi^{-1}\big(r^{-n}\big)} \sup_{t>2r} \Phi^{-1}\big(t^{-n}\big) \|f\|_{L_\Phi(B(x,t))} \tag{4.7}$$

for any Young function Φ.

Proof. Let $\Phi \in \nabla_2$. Denote

$$\mathcal{M}_1 := \frac{1}{\Phi^{-1}\big(r^{-n}\big)} \left(\sup_{t>2r} \frac{1}{|B(x,t)|} \int_{B(x,t)} |f(z)| dz \right),$$
$$\mathcal{M}_2 := \|f\|_{L_\Phi(B(x,2r))}.$$

Applying Hölder's inequality provided by Lemma 2.8, we get

$$\mathcal{M}_1 \lesssim \frac{1}{\Phi^{-1}\big(r^{-n}\big)} \sup_{t>2r} \frac{1}{|B(x,t)|} \|f\|_{L_\Phi(B(x,t))} \|1\|_{L_{\widetilde{\Phi}}(B(x,t))}$$
$$\lesssim \frac{1}{\Phi^{-1}\big(r^{-n}\big)} \sup_{t>2r} \Phi^{-1}\big(t^{-n}\big) \|f\|_{L_\Phi(B(x,t))}.$$

On the other hand,

$$\frac{1}{\Phi^{-1}\big(r^{-n}\big)} \sup_{t>2r} \Phi^{-1}\big(t^{-n}\big) \|f\|_{L_\Phi(B(x,t))}$$
$$\gtrsim \frac{1}{\Phi^{-1}\big(r^{-n}\big)} \sup_{t>2r} \Phi^{-1}\big(t^{-n}\big) \|f\|_{L_\Phi(B(x,2r))} \approx \mathcal{M}_2.$$

Since $\|Mf\|_{L_\Phi(B)} \le \mathcal{M}_1 + \mathcal{M}_2$ by Lemma 4.3, we arrive at (4.6). Finally, when Φ is an arbitrary Young function. the inequality (4.7) directly follows from (4.4). □

Corollary 4.5. [1] *Let $1 \le p < \infty$ and $f \in L_p^{\text{loc}}(\mathbb{R}^n)$, $B = B(x_0, r)$, $x_0 \in \mathbb{R}^n, r > 0$. Then, for $1 < p < \infty$*

$$\|Mf\|_{L_p(B(x_0,r))} \lesssim r^{\frac{n}{p}} \sup_{t>2r} t^{-\frac{n}{p}} \|f\|_{L_p(B(x_0,t))}$$

and for $p = 1$

$$\|Mf\|_{WL_1(B(x_0,r))} \lesssim r^n \sup_{t>2r} t^{-n} \|f\|_{L_1(B(x_0,t))}.$$

Theorem 4.6. *Let Φ be a Young function, the functions φ_1, φ_2 and Φ satisfy the condition*

$$\sup_{r<t<\infty} \operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s)\, \Phi^{-1}\big(t^{-n}\big) \le C\, \varphi_2(x,r)\, \Phi^{-1}\big(r^{-n}\big), \tag{4.8}$$

where C does not depend on x and r. Then the maximal operator M is bounded from $M_{\Phi,\varphi_1}(\mathbb{R}^n)$ to $WM_{\Phi,\varphi_2}(\mathbb{R}^n)$ and for $\Phi \in \nabla_2$, the operator M is bounded from $M_{\Phi,\varphi_1}(\mathbb{R}^n)$ to $M_{\Phi,\varphi_2}(\mathbb{R}^n)$.

Proof. By Lemma 4.4 and Theorem 3.1 with $u(r) = \Phi^{-1}(r^{-n})$, $v_1(r) = \varphi_1(x,r)^{-1}$, $v_2(r) = \frac{1}{\varphi_2(x,r)\Phi^{-1}(r^{-n})}$ and $g(r) = \|f\|_{L_\Phi(B(x,r))}$ we get

$$\|Mf\|_{M_{\Phi,\varphi_2}} \lesssim \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{\varphi_2(x,r)\Phi^{-1}\big(r^{-n}\big)} \sup_{t>r} \Phi^{-1}\big(t^{-n}\big) \|f\|_{L_\Phi(B(x,t))}$$
$$\lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_1(x,r)^{-1} \|f\|_{L_\Phi(B(x,r))}$$
$$= \|f\|_{M_{\Phi,\varphi_1}},$$

if $\Phi \in \nabla_2$ and

$$\begin{aligned}\|Mf\|_{WM_{\Phi,\varphi_2}} &\lesssim \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{\varphi_2(x,r)\Phi^{-1}\big(r^{-n}\big)} \sup_{t>r} \Phi^{-1}\big(t^{-n}\big)\|f\|_{L_\Phi(B(x,t))} \\ &\lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_1(x,r)^{-1}\,\|f\|_{L_\Phi(B(x,r))} = \|f\|_{M_{\Phi,\varphi_1}},\end{aligned}$$

if Φ is an arbitrary Young function. □

Remark 4.7. Note that, in the case $\Phi(t) = t^p$ from Theorem 4.6 we get Theorem 2.11.

In the case $\varphi_1(x,r) = \frac{1}{\Phi^{-1}\left(r^{-\lambda_1}\right)}$, $\varphi_2(x,r) = \frac{1}{\Phi^{-1}\left(r^{-\lambda_2}\right)}$ of Orlicz–Morrey spaces from Theorem 4.6 we get

Corollary 4.8. *Let Φ be any Young function, $0 \le \lambda_1, \lambda_2 < n$ and*

$$\sup_{r<t<\infty} \frac{\Phi^{-1}\big(t^{-n}\big)}{\Phi^{-1}\big(t^{-\lambda_1}\big)} \le C\,\frac{\Phi^{-1}\big(r^{-n}\big)}{\Phi^{-1}\big(r^{-\lambda_2}\big)}. \tag{4.9}$$

Then the maximal operator M is bounded from $M_{\Phi,\lambda_1}(\mathbb{R}^n)$ to $WM_{\Phi,\lambda_2}(\mathbb{R}^n)$ and for $\Phi \in \nabla_2$ the operator M is bounded from $M_{\Phi,\lambda_1}(\mathbb{R}^n)$ to $M_{\Phi,\lambda_2}(\mathbb{R}^n)$.

5. Calderón–Zygmund operators in the spaces $M_{\Phi,\varphi}$

In this section, sufficient conditions on φ for the boundedness of the operator T in generalized Orlicz–Morrey spaces $M_{\Phi,\varphi}(\mathbb{R}^n)$ are obtained.

Sufficient conditions on Φ for the boundedness of the operator T in Orlicz spaces $L_\Phi(\mathbb{R}^n)$, as stated in the following theorem are known, see [21], Theorem 1.4.3 and [43], Theorem 3.3, and also [40]; in the next Theorem 5.2 we present the proof of the corresponding weak estimate.

Theorem 5.1. *Let Φ be a Young function and T a singular integral operator. If $\Phi \in \Delta_2 \bigcap \nabla_2$, then the operator T is bounded on $L_\Phi(\mathbb{R}^n)$.*

Theorem 5.2. *Let Φ be a Young function and T a singular integral operator. If $\Phi \in \Delta_2$, then the operator T is bounded from $L_\Phi(\mathbb{R}^n)$ to $WL_\Phi(\mathbb{R}^n)$.*

Proof. Let $\|f\|_{L_\Phi} = 1$. Fix $\lambda > 0$ and put $f = f_1 + f_2$, where $f_1 = \chi_{\{|f|>\lambda\}} \cdot f$ and $f_2 = \chi_{\{|f|\le\lambda\}} \cdot f$. We have

$$|\{|Tf| > \lambda\}| \le |\{|Tf_1| > \lambda/2\}| + |\{|Tf_2| > \lambda/2\}|$$

and

$$\Phi(\lambda)|\{|Tf| > \lambda\}| \le |\Phi(\lambda)\{|Tf_1| > \lambda/2\}| + \Phi(\lambda)|\{|Tf_2| > \lambda/2\}|.$$

By the weak (p,p)-boundedness of T, $p \geq 1$ we get

$$\{|T(\chi_{\{|f|>\lambda\}} \cdot f)| > \lambda\}| \lesssim \frac{1}{\lambda} \int_{\{|f|>\lambda\}} |f|,$$
$$\{|T(\chi_{\{|f|\leq\lambda\}} \cdot f)| > \lambda\}| \lesssim \frac{1}{\lambda^p} \int_{\{|f|\leq\lambda\}} |f|^p.$$

Since $f_1 \in WL_1(\mathbb{R}^n)$ and $\frac{\Phi(\lambda)}{\lambda}$ is increasing, we have

$$\begin{aligned}\Phi(\lambda)\big|\big\{x \in \mathbb{R}^n : |Tf_1(x)| > \frac{\lambda}{2}\big\}\big| &\lesssim \frac{\Phi(\lambda)}{\lambda} \int_{\mathbb{R}^n} |f_1(x)|dx \\ &= \frac{\Phi(\lambda)}{\lambda} \int_{\{x\in\mathbb{R}^n:|f(x)|>\lambda\}} |f(x)|dx \\ &\lesssim \int_{\mathbb{R}^n} |f(x)|\frac{\Phi(|f(x)|)}{|f(x)|}dx = \int_{\mathbb{R}^n} \Phi(|f(x)|)dx.\end{aligned}$$

By Lemma 2.2 we have

$$\begin{aligned}\Phi(\lambda)\,\big|\big\{x \in \mathbb{R}^n : |Tf_2(x)| > \frac{\lambda}{2}\big\}\big| &\lesssim \frac{\Phi(\lambda)}{\lambda^p} \int_{\mathbb{R}^n} |f_2(x)|^p dx \\ &= \frac{\Phi(\lambda)}{\lambda^p} \int_{\{x\in\mathbb{R}^n:|f(x)|\leq\lambda\}} |f(x)|^p dx \\ &\lesssim \int_{\mathbb{R}^n} |f(x)|^p \frac{\Phi(|f(x)|)}{|f(x)|^p}dx = \int_{\mathbb{R}^n} \Phi(|f(x)|)dx.\end{aligned}$$

Thus we get

$$|\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\}| \leq \frac{C}{\Phi(\lambda)} \int_{\mathbb{R}^n} \Phi(|f(x)|)dx \leq \frac{1}{\Phi\left(\frac{\lambda}{C\|f\|_{L_\Phi}}\right)}. \qquad \square$$

Lemma 5.3. *Let Φ be any Young function and $f \in L^{\rm loc}_\Phi(\mathbb{R}^n)$, $B = B(x_0, r)$, $x_0 \in \mathbb{R}^n, r > 0$ and T a singular integral operator. Then*

$$\|Tf\|_{L_\Phi(B)} \lesssim \frac{1}{\Phi^{-1}\big(r^{-n}\big)} \int_{2r}^{\infty} \|f\|_{L_\Phi(B(x_0,t))}\Phi^{-1}\big(t^{-n}\big)\frac{dt}{t},$$

when $\Phi \in \Delta_2 \bigcap \nabla_2$ and

$$\|Tf\|_{WL_\Phi(B)} \lesssim \frac{1}{\Phi^{-1}\big(r^{-n}\big)} \int_{2r}^{\infty} \|f\|_{L_\Phi(B(x_0,t))}\Phi^{-1}\big(t^{-n}\big)\frac{dt}{t}, \tag{5.1}$$

when $\Phi \in \Delta_2$.

Proof. Let $\Phi \in \Delta_2 \bigcap \nabla_2$ first. With the notation $2B = B(x_0, 2r)$, we represent f as

$$f = f_1 + f_2, \quad f_1(y) = f(y)\chi_{2B}(y), \quad f_2(y) = f(y)\chi_{\complement(2B)}(y), \tag{5.2}$$

and then

$$\|Tf\|_{L_\Phi(B)} \leq \|Tf_1\|_{L_\Phi(B)} + \|Tf_2\|_{L_\Phi(B)}.$$

Since $f_1 \in L_\Phi(\mathbb{R}^n)$, by the boundedness of T in $L_\Phi(\mathbb{R}^n)$ provided by Theorem 5.2, it follows that

$$\|Tf_1\|_{L_\Phi(B)} \le \|Tf_1\|_{L_\Phi(\mathbb{R}^n)} \le C\|f_1\|_{L_\Phi(\mathbb{R}^n)} = C\|f\|_{L_\Phi(2B)}.$$

Next, observe that the inclusions $x \in B$, $y \in {}^{\complement}(2B)$ imply $\frac{1}{2}|x_0-y| \le |x-y| \le \frac{3}{2}|x_0 - y|$. Then we get

$$|Tf_2(x)| \le C\int_{{}^{\complement}(2B)} \frac{|f(y)|}{|x_0 - y|^n} dy.$$

By Fubini's theorem we have

$$\begin{aligned}\int_{{}^{\complement}(2B)} \frac{|f(y)|}{|x_0 - y|^n} dy &\approx \int_{{}^{\complement}(2B)} |f(y)| \int_{|x_0-y|}^{\infty} \frac{dt}{t^{n+1}} dy \\ &\approx \int_{2r}^{\infty} \int_{2r\le|x_0-y|<t} |f(y)|dy\frac{dt}{t^{n+1}} \lesssim \int_{2r}^{\infty} \int_{B(x_0,t)} |f(y)|dy\frac{dt}{t^{n+1}}.\end{aligned}$$

Applying the Hölder's inequality (see, Lemma 2.8), we get

$$\begin{aligned}\int_{{}^{\complement}(2B)} \frac{|f(y)|}{|x_0 - y|^n} dy &\lesssim \int_{2r}^{\infty} \|f\|_{L_\Phi(B(x_0,t))} \|1\|_{L_{\widetilde{\Phi}}(B(x_0,t))} \frac{dt}{t^{n+1}} \\ &= \int_{2r}^{\infty} \|f\|_{L_\Phi(B(x_0,t))} \frac{1}{\widetilde{\Phi}^{-1}(|B(x_0,t)|^{-1})} \frac{dt}{t^{n+1}} \\ &\approx \int_{2r}^{\infty} \|f\|_{L_\Phi(B(x_0,t))} \Phi^{-1}\big(t^{-n}\big) \frac{dt}{t}.\end{aligned} \tag{5.3}$$

Moreover,

$$\|Tf_2\|_{L_\Phi(B)} \lesssim \frac{1}{\Phi^{-1}\big(r^{-n}\big)} \int_{2r}^{\infty} \|f\|_{L_\Phi(B(x_0,t))} \Phi^{-1}\big(t^{-n}\big) \frac{dt}{t}. \tag{5.4}$$

Thus

$$\|Tf\|_{L_\Phi(B)} \lesssim \|f\|_{L_\Phi(2B)} + \frac{1}{\Phi^{-1}\big(r^{-n}\big)} \int_{2r}^{\infty} \|f\|_{L_\Phi(B(x_0,t))} \Phi^{-1}\big(t^{-n}\big) \frac{dt}{t}.$$

On the other hand, by (2.3) we get

$$\Phi^{-1}\big(r^{-n}\big) \approx \Phi^{-1}\big(r^{-n}\big) r^n \int_{2r}^{\infty} \frac{dt}{t^{n+1}} \lesssim \int_{2r}^{\infty} \Phi^{-1}\big(t^{-n}\big) \frac{dt}{t}$$

and then

$$\|f\|_{L_\Phi(2B)} \lesssim \frac{1}{\Phi^{-1}\big(r^{-n}\big)} \int_{2r}^{\infty} \|f\|_{L_\Phi(B(x_0,t))} \Phi^{-1}\big(t^{-n}\big) \frac{dt}{t}. \tag{5.5}$$

Thus

$$\|Tf\|_{L_\Phi(B)} \lesssim \frac{1}{\Phi^{-1}\big(r^{-n}\big)} \int_{2r}^{\infty} \|f\|_{L_\Phi(B(x_0,t))} \Phi^{-1}\big(t^{-n}\big) \frac{dt}{t}.$$

Let $\Phi \in \Delta_2$. By the weak boundedness of T on Orlicz space and (5.5) it follows that:

$$\|Tf_1\|_{WL_\Phi(B)} \le \|Tf_1\|_{WL_\Phi(\mathbb{R}^n)} \lesssim \|f_1\|_{L_\Phi(\mathbb{R}^n)} = \|f\|_{L_\Phi(2B)} \lesssim \frac{1}{\Phi^{-1}\big(r^{-n}\big)} \int_{2r}^{\infty} \|f\|_{L_\Phi(B(x_0,t))} \Phi^{-1}\big(t^{-n}\big) \frac{dt}{t}. \tag{5.6}$$

Then by (5.4) and (5.6) we get the inequality (5.1). □

Corollary 5.4. [13, 14, 15] *Let $1 \le p < \infty$ and $f \in L_p^{\rm loc}(\mathbb{R}^n)$, $B = B(x_0, r)$, $x_0 \in \mathbb{R}^n, r > 0$ and T a singular integral operator. Then, for $1 < p < \infty$*

$$\|Tf\|_{L_p(B(x_0,r))} \lesssim r^{\frac{n}{p}} \int_{2r}^{\infty} t^{-\frac{n}{p}-1} \|f\|_{L_p(B(x_0,t))} dt$$

and for $p = 1$

$$\|Tf\|_{WL_1(B(x_0,r))} \lesssim r^{n} \int_{2r}^{\infty} t^{-n-1} \|f\|_{L_1(B(x_0,t))} dt.$$

The following theorem contains Theorem 2.12 under the choice in the case $\Phi(t) = t^p$.

Theorem 5.5. *Let Φ any Young function, φ_1, φ_2 and Φ satisfy the condition*

$$\sup_{x\in\mathbb{R}^n,\, r>0} \frac{1}{\varphi_2(x,r)\Phi^{-1}\big(r^{-n}\big)} \int_{r}^{\infty} \operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s)\, \Phi^{-1}\big(t^{-n}\big) \frac{dt}{t} < \infty. \tag{5.7}$$

Then the operator T is bounded from $M_{\Phi,\varphi_1}(\mathbb{R}^n)$ to $M_{\Phi,\varphi_2}(\mathbb{R}^n)$ for $\Phi \in \Delta_2 \cap \nabla_2$ and from $M_{\Phi,\varphi_1}(\mathbb{R}^n)$ to $WM_{\Phi,\varphi_2}(\mathbb{R}^n)$ for $\Phi \in \Delta_2$.

Proof. By Lemma 5.3 and Theorem 3.2 with $w(r) = \frac{\Phi^{-1}(r^{-n})}{r}$, $v_1(r) = \varphi_1(x,r)^{-1}$, $v_2(r) = \frac{1}{\varphi_2(x,r)\Phi^{-1}(r^{-n})}$ and $g(r) = \|f\|_{L_\Phi(B(x,r))}$, we have

$$\begin{aligned}
\|Tf\|_{M_{\Phi,\varphi_2}} &= \sup_{x\in\mathbb{R}^n,\, r>0} \varphi_2(x,r)^{-1} \|Tf\|_{L_\Phi(B(x,r))} \\
&\lesssim \sup_{x\in\mathbb{R}^n,\, r>0} \frac{1}{\varphi_2(x,r)\Phi^{-1}\big(r^{-n}\big)} \int_{r}^{\infty} \|f\|_{L_\Phi(B(x,t))} \Phi^{-1}\big(t^{-n}\big) \frac{dt}{t} \\
&\lesssim \sup_{x\in\mathbb{R}^n,\, r>0} \varphi_1(x,r)^{-1} \|f\|_{L_\Phi(B(x,r))} \lesssim \|f\|_{M_{\Phi,\varphi_1}}.
\end{aligned}$$

if $\Phi \in \Delta_2 \cap \nabla_2$ and

$$\begin{aligned}
\|Tf\|_{WM_{\Phi,\varphi_2}} &= \sup_{x\in\mathbb{R}^n,\, r>0} \varphi_2(x,r)^{-1} \|Tf\|_{WL_\Phi(B(x,r))} \\
&\lesssim \sup_{x\in\mathbb{R}^n,\, r>0} \frac{1}{\varphi_2(x,r)\Phi^{-1}\big(r^{-n}\big)} \int_{r}^{\infty} \|f\|_{L_\Phi(B(x,t))} \Phi^{-1}\big(t^{-n}\big) \frac{dt}{t} \\
&\lesssim \sup_{x\in\mathbb{R}^n,\, r>0} \varphi_1(x,r)^{-1} \|f\|_{L_\Phi(B(x,r))} \lesssim \|f\|_{M_{\Phi,\varphi_1}}.
\end{aligned}$$

if $\Phi \in \nabla_2$. □

Remark 5.6. The condition (4.8) is weaker than (5.7). Indeed, (5.7) implies (4.8):

$$\begin{aligned}\varphi_2(x,r)\Phi^{-1}\big(r^{-n}\big) &\gtrsim \int_r^\infty \operatorname*{ess\,inf}_{t<\tau<\infty} \varphi_1(x,\tau)\,\Phi^{-1}\big(t^{-n}\big)\frac{dt}{t}\\ &\gtrsim \int_s^\infty \operatorname*{ess\,inf}_{t<\tau<\infty} \varphi_1(x,\tau)\,\Phi^{-1}\big(t^{-n}\big)\frac{dt}{t}\\ &\gtrsim \operatorname*{ess\,inf}_{s<\tau<\infty} \varphi_1(x,\tau)\int_s^\infty \Phi^{-1}\big(t^{-n}\big)\frac{dt}{t}\\ &\approx \operatorname*{ess\,inf}_{s<\tau<\infty} \varphi_1(x,\tau)\Phi^{-1}\big(s^{-n}\big),\end{aligned}$$

where we took $s \in (r,\infty)$, so that

$$\sup_{s>r} \operatorname*{ess\,inf}_{s<\tau<\infty} \varphi_1(x,\tau)\Phi^{-1}\big(s^{-n}\big) \lesssim \varphi_2(x,r)\Phi^{-1}\big(r^{-n}\big).$$

On the other hand the functions $\varphi_1(x,t)=\varphi_2(x,t)=\frac{1}{\Phi^{-1}\left(t^{-n}\right)}$ satisfy the condition (4.8), but do not satisfy the condition (5.7).

Corollary 5.7. *Let Φ be any Young function, $0 \le \lambda_1,\lambda_2 < n$ and*

$$\int_r^\infty \frac{\Phi^{-1}\big(t^{-n}\big)}{\Phi^{-1}\big(t^{-\lambda_1}\big)}\frac{dt}{t} \le C\,\frac{\Phi^{-1}\big(r^{-n}\big)}{\Phi^{-1}\big(r^{-\lambda_2}\big)}. \tag{5.8}$$

Then for $\Phi \in \Delta_2 \cap \nabla_2$, the singular operator T is bounded from $M_{\Phi,\lambda_1}(\mathbb{R}^n)$ to $M_{\Phi,\lambda_2}(\mathbb{R}^n)$ and for $\Phi \in \Delta_2$ is bounded from $M_{\Phi,\lambda_1}(\mathbb{R}^n)$ to $WM_{\Phi,\lambda_2}(\mathbb{R}^n)$.

Proof. Choose $\varphi_1(x,r)=\frac{1}{\Phi^{-1}\left(r^{-\lambda_1}\right)}$, $\varphi_2(x,r)=\frac{1}{\Phi^{-1}\left(r^{-\lambda_2}\right)}$ in Theorem 5.5. □

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