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On the Factorization of Some Block Triangular Almost Periodic Matrix Functions

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To Professor António Ferreira dos Santos, in celebration of his 70th birthday.

Abstract. Canonical factorization criterion is established for a class of block triangular almost periodic matrix functions. Explicit factorization formulas are also obtained, and the geometric mean of matrix functions in question is computed.

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1. Introduction

Factorization of matrix functions is a standard tool in solving systems of convolution type equations

$$k * \phi = f \tag{1.1}$$

on a half-line, going back to the classical paper [27] and known as the Wiener–Hopf technique; see, e.g., the monographs [6, 17, 16] for detailed presentation and further references. This technique was modified by Ganin [15] to allow for consideration of equations on intervals of finite length. Ganin's approach, however, gives rise to matrix functions

$$\begin{bmatrix} e_\lambda I_N & 0 \\ \widehat{k} & e_{-\lambda} I_N \end{bmatrix}, \tag{1.2}$$

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where λ is the length of the interval,

$$e_\lambda(x) = e^{i\lambda x}, \quad x \in \mathbb{R}, \tag{1.3}$$

and $\widehat{k}$ is the Fourier transform of the $N \times N$ kernel k. These matrix functions have a second kind discontinuity at ∞, even when $\widehat{k}$ behaves nicely. Besides, the size of the matrix doubles, so even scalar equations (1.1) yield a factorization problem for a 2×2 matrix function, more complicated than that for scalar functions.

It was shown by two of the authors [19] that for a rather wide class of kernels k the factorization of (1.2) reduces to that for matrix functions of the same block triangular structure in which the off-diagonal block is substituted by its so-called almost periodic representatives at $\pm\infty$. Thus emerged the factorization problem for almost periodic matrix functions G, with special interest in the case of

$$G_F = \begin{bmatrix} e_1 I_N & 0_N \\ F & e_{-1} I_N \end{bmatrix}. \tag{1.4}$$

(Note that the change of λ to 1 in (1.4) can be achieved by a simple change of variable and can therefore be adopted without any loss of generality.)

A systematic exposition of the factorization theory for such matrix functions can be found in [5], while some more recent results are in [2, 9, 10, 11, 12, 13, 18, 25]. Still, the theory is far from being complete, even for matrix functions (1.4).

The factorability criterion for matrix functions (1.4) in the case $N = 1$,

$$F = C_1 e_\alpha + C_{-1} e_{\alpha-1} + C_2 e_\beta + C_{-2} e_{\beta-1} \tag{1.5}$$

with $0 < \alpha < \beta < 1$ was established (in somewhat different terms) in [1], with an alternative approach and some generalization presented in [25]. In our previous paper [2], we provided explicit factorization formulas for this setting. For $N > 1$ the canonical factorability criterion and the factorization formulas are available if $C_1 = 0$ or $C_{-2} = 0$, see [24, Theorem 6.5]. If $C_2 = 0$ or $C_{-1} = 0$ then the respective results can be derived from [21, Theorem 6.1], but only under an additional assumption that the remaining matrix coefficients can be simultaneously put in a triangular form via the same equivalence transformation.

The goal of this paper is to establish respective results (that is, the canonical factorization criterion and explicit factorization formulas) under the similar "triangularizability" requirement on the coefficients C_j in (1.5) without supposing that either of them vanishes, thus extending the statements of [2]. This is done in Section 3. Section 2 contains necessary notation and background information, including a slight variation of a known result on factorability in decomposing algebras. Section 4 provides formulas for the so-called geometric mean of the matrices G_F when $N = 2$, with technical details delegated to the Appendix.

2. Preliminaries

2.1. Background information on AP factorization

For any algebra $\mathfrak{A}$, we denote by $\mathcal{G}\mathfrak{A}$ the group of its invertible elements, and by $\mathfrak{A}_{N\times N}$ the algebra of all $N \times N$ matrices with the entries in $\mathfrak{A}$.

Let APP be the algebra of almost periodic polynomials, that is, the set of all finite linear combinations of elements e_λ ($\lambda \in \mathbb{R}$), with e_λ defined by (1.3). The closure of APP with respect to the uniform norm is the C^*-algebra AP of almost periodic functions, and the closure of APP with respect to the stronger norm,

$$\|\textstyle\sum_\lambda c_\lambda e_\lambda\|_W = \sum_\lambda |c_\lambda|, \quad c_\lambda \in \mathbb{C},$$

is the Banach algebra APW.

The basic information about AP functions can be found in several monographs, including [4, 14] and [22]. For our purposes, the following will suffice.

For any $f \in AP$ there exists the Bohr mean value

$$\mathbf{M}(f) = \lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} f(x)\,dx.$$

The functions $f \in AP$ are defined uniquely by the Bohr–Fourier series

$$\sum_{\lambda\in\Omega(f)} \widehat{f}(\lambda) e_\lambda$$

where $\Omega(f) := \Big\{\lambda \in \mathbb{R} :\ \widehat{f}(\lambda) \neq 0\Big\}$ is the Bohr–Fourier spectrum of f and the numbers $\widehat{f}(\lambda) = \mathbf{M}(f e_{-\lambda})$ are referred to as the Bohr–Fourier coefficients of f.

Let

$$AP^\pm := \big\{f \in AP \colon \Omega(f) \subset \mathbb{R}_\pm\big\}, \quad APW^\pm := AP^\pm \cap APW,$$

$$APW_0^\pm := \big\{f \in APW^\pm \colon \widehat{f}(0) = 0\big\},$$

where, as usual, $\mathbb{R}_\pm = \{x \in \mathbb{R} \colon \pm\, x \geq 0\}$.

A function $f \in AP$ is invertible in AP if and only if it is invertible in $L_\infty(\mathbb{R})$, that is, if and only if $\inf_{x\in\mathbb{R}} |f(x)| > 0$. For every $f \in \mathcal{G}AP$, the following limits exist, are finite, equal and independent of the choice of a continuous branch of the argument of f:

$$\kappa(f) := \lim_{T\to+\infty} \frac{1}{2T} \Big\{\arg f(x)\Big\}_{-T}^{T} = \lim_{T\to\pm\infty} \frac{1}{T} \Big\{\arg f(x)\Big\}_{0}^{T}.$$

Their common value is called the *mean motion* (or the *AP index*) of f.

We say that $G \in AP_{N\times N}$ admits a canonical left AP factorization if

$$G = G_+ G_-^{-1} \tag{2.1}$$

with $G_\pm \in \mathcal{G}AP^\pm_{N\times N}$. If in fact $G_\pm \in \mathcal{G}APW^\pm_{N\times N}$, (2.1) is said to be a canonical left APW factorization of G. More generally, a left AP or APW factorization (not necessarily canonical) of G is a representation $G = G_+ D G_-^{-1}$ with $G_\pm$ as above and an extra middle factor $D = \operatorname{diag}[e_{\kappa_1}, \ldots, e_{\kappa_N}]$. The parameters $\kappa_j \in \mathbb{R}$ are defined by G uniquely up to a permutation whenever the factorization exists, and are called the (left) *partial AP indices* of G. Of course, condition $G \in \mathcal{G}AP_{N\times N}$ (resp., $G \in \mathcal{G}APW_{N\times N}$) is necessary in order for G to admit a left AP (resp., APW) factorization, and

$$\kappa_1 + \cdots + \kappa_N = \kappa(\det G).$$

A canonical AP factorization of $G \in APW_{N\times N}$ is automatically its (naturally, also canonical) APW factorization. For $N = 1$, any $G \in \mathcal{G}APW$ admits an APW factorization, and thus AP (and even APW) factorable functions form a dense subset of AP. As was discovered recently [8], this is not the case any more if $N > 1$.

However, for matrix functions of the form (1.4) with $N = 1$, that is,

$$G_f = \begin{bmatrix} e_\lambda & 0 \\ f & e_{-\lambda} \end{bmatrix}, \tag{2.2}$$

it is presently not known whether (and therefore still a priori possible that) the set of f for which G_f admits an AP factorization is dense in AP; see open problems in [7]. Let us denote by $\mathcal{E}$ the closure of this set, and say that $E \subset \mathbb{R}$ is *admissible* if

$$\Omega(f) \subset E \Longrightarrow f \in \mathcal{E}.$$

From previous work on the factorization theory it follows in particular that grids $-\nu + h\mathbb{Z}$ and sets E with a gap of length at least 1 inside $(-1, 1)$ are admissible.

The next result implies that the set of $f \in APW$ for which (2.2) admits a *canonical AP* factorization is dense in $\mathcal{E} \cap APW$.

Lemma 2.1. *Let G_f be APW factorable. Then in every neighborhood of f in APW metric there exist g for which G_g admit a canonical AP factorization.*

Proof. Step 1. It is a standard trick in AP factorization theory (see, e.g., [5, Proposition 13.4]) to consider along with G_f the matrix function

$$\begin{bmatrix} 1 & 0 \\ \phi_+ & 1 \end{bmatrix} G_f \begin{bmatrix} 1 & 0 \\ \phi_- & 1 \end{bmatrix} = G_{\widetilde{f}},$$

where $\widetilde{f} = f + e_\lambda\phi_+ + e_{-\lambda}\phi_-$. Obviously, G_f and $G_{\widetilde{f}}$ are APW factorable only simultaneously and have the same sets of partial AP indices, provided that $\phi_\pm \in APW^\pm$. Moreover, small perturbations of $\widetilde{f}$ are equivalent to small perturbations of f, for $\phi_\pm$ being fixed. So, choosing for $f \in APW$

$$\phi_\pm = -\sum_{\mu\in\Omega(f),\pm\mu\geq\lambda} \widehat{f}(\mu) e_{(\mu\mp\lambda)},$$

we reduce the general case to the situation when

$$\Omega(f) \subset (-\lambda, \lambda). \tag{2.3}$$

Step 2. Suppose that (2.3) holds and G_f admits an APW factorization $G_+DG_-^{-1}$. Then its partial AP indices are $\pm\nu$ for some $\nu \in [0, \lambda]$, $\Omega(G_+^{\pm 1}) \subset [0, \lambda]$ and

$\Omega(G_-^{\pm 1}) \subset [-\lambda, 0]$. Of course, only the case $\nu \neq 0$ is of interest. Thus, we may suppose that $D = \mathrm{diag}[e_\nu, e_{-\nu}]$ with $\nu \in (0, \lambda]$.

Since for $c \neq 0$

$$\begin{bmatrix} e_\nu & 0 \\ c & e_{-\nu} \end{bmatrix} = \begin{bmatrix} 1 & e_\nu \\ 0 & c \end{bmatrix} \begin{bmatrix} e_{-\nu} & 1 \\ -c & 0 \end{bmatrix}^{-1},$$

the matrix function

$$G_f + c\,G_+ \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} G_-^{-1} = G_+ \begin{bmatrix} e_\nu & 0 \\ c & e_{-\nu} \end{bmatrix} G_-^{-1}$$

admits a canonical *APW* factorization. So, there are arbitrarily small (in *APW* metric) perturbations of G_f by matrix functions with the Bohr–Fourier spectrum in $[-\lambda, \lambda]$ admitting a canonical *AP* factorization.

Step 3. Let $H \in APW_{2\times 2}$ be a small perturbation the existence of which was proved at Step 2, that is, $\Omega(H) \subset [-\lambda, \lambda]$ and $G_f + H$ admits a canonical *AP* factorization. Observe that

$$\begin{aligned} G_f + H &= \begin{bmatrix} e_\lambda(1 + e_{-\lambda}h_{11}) & h_{12} \\ f + h_{21} & e_{-\lambda}(1 + e_\lambda h_{22}) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 + e_\lambda h_{22} \end{bmatrix} \begin{bmatrix} e_\lambda & h_{12} \\ \widetilde{f} & e_{-\lambda} \end{bmatrix} \begin{bmatrix} 1 + e_{-\lambda}h_{11} & 0 \\ 0 & 1 \end{bmatrix}, \end{aligned} \tag{2.4}$$

where

$$\widetilde{f} = \frac{f + h_{21}}{(1 + e_{-\lambda}h_{11})(1 + e_\lambda h_{22})}. \tag{2.5}$$

Since $\Omega(e_\lambda h_{22}) \subset [0, 2\lambda]$, $1 + e_\lambda h_{22} \in \mathcal{G}APW^+$ provided that $\|H\|_W$ is sufficiently small. Similarly, $1 + e_{-\lambda}h_{11} \in \mathcal{G}APW^-$. From (2.4) we then conclude that the matrix function

$$\begin{bmatrix} e_\lambda & h_{12} \\ \widetilde{f} & e_{-\lambda} \end{bmatrix} \tag{2.6}$$

admits a canonical *AP* factorization, while (2.5) implies that $\widetilde{f}$ can be made arbitrarily close to f. In other words, the perturbation H can be made off-diagonal, with $\Omega(h_{12}) \subset [-\lambda, \lambda]$.

Step 4. Consider now a small perturbation of G_f the existence of which was established at Step 3, and represent it as

$$\begin{bmatrix} e_\lambda & h_{12} \\ f + h_{21} & e_{-\lambda} \end{bmatrix} = \begin{bmatrix} e_\lambda & 0 \\ f + h_{21} & e_{-\lambda}(1 - h_{12}(f + h_{21})) \end{bmatrix} \begin{bmatrix} 1 & e_{-\lambda}h_{12} \\ 0 & 1 \end{bmatrix}. \tag{2.7}$$

Since $\Omega(e_{-\lambda}h_{12}) \subset [-2\lambda, 0]$, the right factor in (2.7) belongs to $\mathcal{G}APW_{2\times 2}^-$. Thus, the left factor in the right-hand side of (2.7) admits a canonical *AP* factorization along with its left-hand side. In its turn, $1 - h_{12}(f + h_{21})$ is a function close to 1 in *APW* and therefore admitting a canonical factorization $g_+ g_-^{-1}$ with the multiples also close to 1. From here we conclude that the matrix function

$$\begin{bmatrix} e_\lambda & 0 \\ (f + h_{21})g_+^{-1} & e_{-\lambda} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & g_+^{-1} \end{bmatrix} \begin{bmatrix} e_\lambda & 0 \\ f + h_{21} & e_{-\lambda}(1 - h_{12}(f + h_{21})) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & g_- \end{bmatrix}$$

also admits a canonical AP factorization. It remains to observe that $(f + h_{21})g_+^{-1}$ can be made arbitrarily close to f by choosing $\|H\|_W$ small enough. $\square$

If $G \in AP_{N\times N}$ has a canonical left AP factorization then the matrix

$$\mathbf{d}(G) := \mathbf{M}(G_+)\mathbf{M}(G_-)^{-1}, \tag{2.8}$$

with $\mathbf{M}(G_\pm)$ understood entry-wise, does not depend on the particular choice of such a factorization and is called the *geometric mean* of G.

The value of $\mathbf{d}(G)$ plays an important role in the Fredholmness criteria for the related convolution type equations. It is known to depend continuously on G ([26], see also [5]).

2.2. Factorization in decomposing algebras

Let $\mathcal{B}$ be a decomposing unital Banach algebra with unit e, that is, $\mathcal{B}$ admits a direct sum decomposition

$$\mathcal{B} = \mathcal{B}_+ \oplus \mathcal{B}_- \tag{2.9}$$

with $\mathcal{B}_\pm$ being subalgebras of $\mathcal{B}$, and let $P_\pm$ be the complementary projections associated with this decomposition, $P_\pm\colon \mathcal{B} \to \mathcal{B}_\pm$.

We say that $b = e - a \in \mathcal{B}$ admits a canonical left factorization if

$$e - a = (e + b_+)\,(e + b_-), \tag{2.10}$$

where $e + b_\pm \in \mathcal{GB}$, $b_\pm \in \mathcal{B}_\pm$ and $(e + b_\pm)^{-1} - e \in \mathcal{B}_\pm$.

The existence of such a factorization when $\|a\|$ is sufficiently small is well known, see, e.g., [16, Chapter I, Lemma 5.1] or [17, Chapter XXIX, Theorem 9.1]. For our purposes we need a variation of this result presented below.

Consider the linear mappings

$$\begin{aligned} \mathcal{P}_a^+ &\colon \mathcal{B} \to \mathcal{B}_+, \quad x \mapsto P_+(xa),\\ \mathcal{P}_a^- &\colon \mathcal{B} \to \mathcal{B}_-, \quad x \mapsto P_-(ax). \end{aligned} \tag{2.11}$$

Theorem 2.2. *Let $\mathcal{B}$ be a decomposing unital Banach algebra with associated direct sum decomposition* (2.9) *and let $b = e - a \in \mathcal{B}$ be invertible in $\mathcal{B}$. If*

$$(\mathcal{P}_a^+)^{n_+}e = 0 \quad \textit{and} \quad (\mathcal{P}_a^-)^{n_-}e = 0 \quad \textit{for some} \quad n_\pm \in \mathbb{N}, \tag{2.12}$$

then $e - a$ admits the canonical left factorization (2.10)*, where*

$$e + b_+ = \left(\sum_{n=0}^{n_+-1} (\mathcal{P}_a^+)^n e\right)^{-1}, \quad e + b_- = \left(\sum_{n=0}^{n_--1} (\mathcal{P}_a^-)^n e\right)^{-1}. \tag{2.13}$$

Proof. Setting

$$\begin{aligned} e + c_- &:= e - P_-\left(\left(\sum_{n=0}^{n_+-1} (\mathcal{P}_a^+)^n e\right) a\right),\\ e + c_+ &:= e - P_+\left(a\left(\sum_{n=0}^{n_--1} (\mathcal{P}_a^-)^n e\right)\right), \end{aligned}$$

it is easily seen from (2.12) that

$$\left(\sum_{n=0}^{n_+-1}(\mathcal{P}_a^+)^n e\right)(e-a)=e+c_-, \tag{2.14}$$

$$(e-a)\left(\sum_{n=0}^{n_--1}(\mathcal{P}_a^-)^n e\right)=e+c_+. \tag{2.15}$$

In view of the invertibility of $e-a$, equation (2.15) is equivalent to

$$(e-a)^{-1}(e+c_+)=\sum_{n=0}^{n_--1}(\mathcal{P}_a^-)^n e. \tag{2.16}$$

Multiplying (2.14) and (2.16), we obtain

$$\left(\sum_{n=0}^{n_+-1}(\mathcal{P}_a^+)^n e\right)(e+c_+)=(e+c_-)\left(\sum_{n=0}^{n_--1}(\mathcal{P}_a^-)^n e\right) \tag{2.17}$$

or, equivalently,

$$c_+ + \left(\sum_{n=1}^{n_+-1}(\mathcal{P}_a^+)^n e\right)(e+c_+)=c_- + (e+c_-)\left(\sum_{n=1}^{n_--1}(\mathcal{P}_a^-)^n e\right), \tag{2.18}$$

where the expression on the left of (2.18) belongs to $\mathcal{B}_+$ and on the right of (2.18) belongs to $\mathcal{B}_-$. Since $\mathcal{B}_+\cap\mathcal{B}_-=\{0\}$, both sides of (2.18) equal zero. Hence, (2.17) can be rewritten in the form

$$\left(\sum_{n=0}^{n_+-1}(\mathcal{P}_a^+)^n e\right)(e+c_+)=(e+c_-)\left(\sum_{n=0}^{n_--1}(\mathcal{P}_a^-)^n e\right)=e, \tag{2.19}$$

which means that the elements $\sum_{n=0}^{n_\pm-1}(\mathcal{P}_a^\pm)^n e$ are one-sided inverses for the elements $e+c_\pm$, respectively.

Replacing a by λa, where $\lambda\in[0,1]$, and following the proof of [16, Chapter I, Lemma 5.1], we infer that all the multiples in (2.19) are two-sided invertible. Then (2.14) and (2.19) imply the canonical left factorization (2.10) with $e+b_\pm$ given by (2.13). □

2.3. *APW* factorization in the scalar quadrinomial case

In what follows we use the notation $\lfloor x\rfloor$ and $\lceil x\rceil$ for the best integer approximation to $x\in\mathbb{R}$ from below and above, respectively; $\{x\}$ denotes the fractional part of $x\in\mathbb{R}$: $\{x\}=x-\lfloor x\rfloor$. Also, as usual,

$$\mathbb{N}:=\{1,2,\ldots\},\quad \mathbb{N}_-:=\{-1,-2,\ldots\},\quad \mathbb{Z}_+:=\mathbb{N}\cup\{0\},\quad \mathbb{Z}_-:=\mathbb{N}_-\cup\{0\}.$$

The results of this subsection are not new, and are listed here for convenience of reference.

Theorem 2.3 ([1, 25]). *Let in* (2.2)

$$\lambda = 1 \quad \textit{and} \quad f = C_1 e_\alpha + C_{-1} e_{\alpha-1} + C_2 e_\beta + C_{-2} e_{\beta-1},\ C_{\pm 1}, C_{\pm 2} \in \mathbb{C}, \qquad (2.20)$$

where $0 < \alpha < \beta < 1$ *and the number* $\beta - \alpha$ *is irrational. Then* G_f *admits a canonical left AP factorization if and only if*

$$|C_2|^{1-\beta} |C_{-2}|^\beta \neq |C_1|^{1-\alpha} |C_{-1}|^\alpha .$$

Corollary 2.4. *Functions* f *of the form* (2.20) *belong to* $\mathcal{E}$, *with say, the sets* $\{\alpha, \beta, \alpha - 1, \beta - 1\}$ *are admissible.*

Note that the matrix function (2.2) with

$$|C_2|^{1-\beta} |C_{-2}|^\beta = |C_1|^{1-\alpha} |C_{-1}|^\alpha \neq 0$$

in (2.20) is not *AP* factorable [25] while for $C_2 C_{-2} = C_1 C_{-1} = 0$ its *APW* factorization exists but it is not canonical. Also, only the case of irrational $\beta - \alpha$ is of interest, since otherwise the distances between all the points in $\Omega(f)$ are commensurable. The latter situation, with an arbitrary number of terms in f, was covered earlier in [20] (see also [5, Section 14.4]).

The remaining portion of this subsection is a restatement of the results from [2] in a form convenient for our current purposes.

Theorem 2.5. *Suppose that* G *is given by* (2.20) *where* $0 < \alpha < \beta < 1$, *the number* $\beta - \alpha$ *is irrational, and*

$$|C_2|^{1-\beta} |C_{-2}|^\beta < |C_1|^{1-\alpha} |C_{-1}|^\alpha .$$

Then G *admits a canonical left APW factorization* (2.1) *where the matrix functions* $G_\pm, G_\pm^{-1} \in APW^\pm_{2\times 2}$ *are given by*

$$G_\pm = \begin{bmatrix} \varphi_1^\pm & \widetilde{\varphi}_1^\pm \\ \varphi_2^\pm & \widetilde{\varphi}_2^\pm \end{bmatrix}, \quad G_\pm^{-1} = \frac{1}{\det G_\pm} \begin{bmatrix} \widetilde{\varphi}_2^\pm & -\widetilde{\varphi}_1^\pm \\ -\varphi_2^\pm & \varphi_1^\pm \end{bmatrix},$$

$$\begin{aligned}
\varphi_1^+ &= e_1 + \sum\nolimits_{n=0}^{\infty} X_n e_{\{n(\beta-\alpha)\}}, \\
\varphi_1^- &= 1 + \sum\nolimits_{n=0}^{\infty} X_n e_{\{n(\beta-\alpha)\}-1}, \\
\varphi_2^+ &= C_1 e_\alpha + C_2 e_\beta + \sum_{\{n\in\mathbb{Z}_+ :\, 0 \le \{\alpha+n(\beta-\alpha)\} < \alpha\}} C_1 X_n e_{\{\alpha+n(\beta-\alpha)\}} \\
&\quad + \sum_{\{n\in\mathbb{Z}_+ :\, 0 \le \{\beta+n(\beta-\alpha)\} < \beta\}} C_2 X_n e_{\{\beta+n(\beta-\alpha)\}}, \qquad (2.21) \\
\varphi_2^- &= - \sum_{\{n\in\mathbb{Z}_+ :\, \{\alpha+n(\beta-\alpha)\} = 0\}} C_{-1} X_n - \sum_{\{n\in\mathbb{Z}_+ :\, \alpha \le \{\alpha+n(\beta-\alpha)\} < 1\}} C_{-1} X_n e_{\{\alpha+n(\beta-\alpha)\}-1} \\
&\quad - \sum_{\{n\in\mathbb{Z}_+ :\, \{\beta+n(\beta-\alpha)\} = 0\}} C_{-2} X_n - \sum_{\{n\in\mathbb{Z}_+ :\, \beta \le \{\beta+n(\beta-\alpha)\} < 1\}} C_{-2} X_n e_{\{\beta+n(\beta-\alpha)\}-1},
\end{aligned}$$

$$\begin{aligned}
\widetilde{\varphi}_1^+ &= \sum\nolimits_{n=0}^{\infty} \widetilde{X}_n e_{\{n(\beta-\alpha)-\alpha\}}, \\
\widetilde{\varphi}_1^- &= \sum\nolimits_{n=0}^{\infty} \widetilde{X}_n e_{\{n(\beta-\alpha)-\alpha\}-1}, \\
\widetilde{\varphi}_2^+ &= \sum_{\{n\in\mathbb{Z}_+:\ 0\le\{n(\beta-\alpha)\}<\alpha\}} C_1 \widetilde{X}_n e_{\{n(\beta-\alpha)\}} + \sum_{\{n\in\mathbb{Z}_+:\ 0\le\{(n+1)(\beta-\alpha)\}<\beta\}} C_2 \widetilde{X}_n e_{\{(n+1)(\beta-\alpha)\}}, \\
\widetilde{\varphi}_2^- &= -\sum_{\{n\in\mathbb{Z}_+:\ \alpha\le\{n(\beta-\alpha)\}<1\}} C_{-1} \widetilde{X}_n e_{\{n(\beta-\alpha)\}-1} \\
&\quad - \sum_{\{n\in\mathbb{Z}_+:\ \beta\le\{(n+1)(\beta-\alpha)\}<1\}} C_{-2} \widetilde{X}_n e_{\{(n+1)(\beta-\alpha)\}-1} - C_{-1}\widetilde{X}_0.
\end{aligned} \tag{2.22}$$

The coefficients X_n *and* $\widetilde{X}_n$ $(n \in \mathbb{Z}_+)$ *here are given by*

$$X_n = \begin{cases}
(-1)^{n-1}\left(\frac{C_{-2}}{C_1}\right)^k \left(\frac{C_2}{C_1}\right)^{-1+\left(n-\left\lceil\frac{k}{\beta-\alpha}\right\rceil\right)+\sum_{s=1}^{k}\left(\left\lceil\frac{s-\alpha}{\beta-\alpha}\right\rceil-\left\lceil\frac{s-1}{\beta-\alpha}\right\rceil-1\right)} \\
\times\left(\frac{C_{-2}}{C_{-1}}\right)^{\sum_{s=1}^{k}\left(\left\lceil\frac{s}{\beta-\alpha}\right\rceil-\left\lceil\frac{s-\alpha}{\beta-\alpha}\right\rceil\right)} X_1 \quad \textit{if} \quad n = \left\lceil\frac{k}{\beta-\alpha}\right\rceil+1,\dots,\left\lceil\frac{k+1-\alpha}{\beta-\alpha}\right\rceil-1, \\
(-1)^{n-1}\left(\frac{C_{-2}}{C_1}\right)^k \left(\frac{C_{-2}}{C_{-1}}\right)^{\left(n-\left\lceil\frac{k+1-\alpha}{\beta-\alpha}\right\rceil+1\right)+\sum_{s=1}^{k}\left(\left\lceil\frac{s}{\beta-\alpha}\right\rceil-\left\lceil\frac{s-\alpha}{\beta-\alpha}\right\rceil\right)} \\
\times\left(\frac{C_2}{C_1}\right)^{-1+\sum_{s=1}^{k+1}\left(\left\lceil\frac{s-\alpha}{\beta-\alpha}\right\rceil-\left\lceil\frac{s-1}{\beta-\alpha}\right\rceil-1\right)} X_1 \quad \textit{if} \quad n = \left\lceil\frac{k+1-\alpha}{\beta-\alpha}\right\rceil,\dots,\left\lceil\frac{k+1}{\beta-\alpha}\right\rceil-1, \\
(-1)^{n-1}\left(\frac{C_{-2}}{C_1}\right)^{k+1} \left(\frac{C_{-2}}{C_{-1}}\right)^{\sum_{s=1}^{k+1}\left(\left\lceil\frac{s}{\beta-\alpha}\right\rceil-\left\lceil\frac{s-\alpha}{\beta-\alpha}\right\rceil\right)} \\
\times\left(\frac{C_2}{C_1}\right)^{-1+\sum_{s=1}^{k+1}\left(\left\lceil\frac{s-\alpha}{\beta-\alpha}\right\rceil-\left\lceil\frac{s-1}{\beta-\alpha}\right\rceil-1\right)} X_1 \quad \textit{if} \quad n = \left\lceil\frac{k+1}{\beta-\alpha}\right\rceil
\end{cases} \tag{2.23}$$

and

$$\widetilde{X}_n = \begin{cases}
(-1)^{n}\left(\frac{C_{-2}}{C_{-1}}\right)^{\left(n-\left\lceil\frac{k}{\beta-\alpha}\right\rceil\right)+\sum_{s=0}^{k-1}\left(\left\lceil\frac{s+\alpha}{\beta-\alpha}\right\rceil-\left\lceil\frac{s}{\beta-\alpha}\right\rceil\right)} \left(\frac{C_{-2}}{C_1}\right)^k \\
\times\left(\frac{C_2}{C_1}\right)^{\sum_{s=1}^{k}\left(\left\lceil\frac{s}{\beta-\alpha}\right\rceil-\left\lceil\frac{s-1+\beta}{\beta-\alpha}\right\rceil\right)} \widetilde{X}_0 \quad \textit{if} \quad n = \left\lceil\frac{k}{\beta-\alpha}\right\rceil,\dots,\left\lceil\frac{k+\alpha}{\beta-\alpha}\right\rceil-1, \\
(-1)^{n}\left(\frac{C_{-2}}{C_{-1}}\right)^{-1+\sum_{s=0}^{k}\left(\left\lceil\frac{s+\alpha}{\beta-\alpha}\right\rceil-\left\lceil\frac{s}{\beta-\alpha}\right\rceil\right)} \left(\frac{C_{-2}}{C_1}\right)^{k+1} \\
\times\left(\frac{C_2}{C_1}\right)^{\sum_{s=1}^{k}\left(\left\lceil\frac{s}{\beta-\alpha}\right\rceil-\left\lceil\frac{s-1+\beta}{\beta-\alpha}\right\rceil\right)} \widetilde{X}_0 \quad \textit{if} \quad n = \left\lceil\frac{k+\alpha}{\beta-\alpha}\right\rceil = \left\lceil\frac{k+\beta}{\beta-\alpha}\right\rceil-1, \\
(-1)^{n}\left(\frac{C_{-2}}{C_1}\right)^{k+1} \left(\frac{C_2}{C_1}\right)^{\left(n-\left\lceil\frac{k+\beta}{\beta-\alpha}\right\rceil+1\right)+\sum_{s=1}^{k}\left(\left\lceil\frac{s}{\beta-\alpha}\right\rceil-\left\lceil\frac{s-1+\beta}{\beta-\alpha}\right\rceil\right)} \\
\times\left(\frac{C_{-2}}{C_{-1}}\right)^{-1+\sum_{s=0}^{k}\left(\left\lceil\frac{s+\alpha}{\beta-\alpha}\right\rceil-\left\lceil\frac{s}{\beta-\alpha}\right\rceil\right)} \widetilde{X}_0 \quad \textit{if} \quad n = \left\lceil\frac{k+\beta}{\beta-\alpha}\right\rceil,\dots,\left\lceil\frac{k+1}{\beta-\alpha}\right\rceil-1
\end{cases} \tag{2.24}$$

for $k = 0, 1, 2, \dots$, *with the initial conditions* $\widetilde{X}_0 = 1$,

$$X_0 = -\frac{C_{-1}}{C_1}, \quad X_1 = -\frac{C_{-2}}{C_1} + \frac{C_2 C_{-1}}{C_1^2}.$$

To simplify the forthcoming formulas, we let

$$\mathbb{N}_\gamma^\pm := \{n\colon \pm n \in \mathbb{N} \text{ and } \gamma + n(\beta-\alpha) \in \mathbb{Z}\} \quad \text{for} \quad \gamma = \pm\alpha, \beta. \tag{2.25}$$

Corollary 2.6. *In the setting of Theorem* 2.5 *we have*

$$\mathbf{M}(G_-) = \begin{bmatrix} 1 & 0 \\ -\sum_{n\in\mathbb{N}_\alpha^+} C_{-1}X_n - \sum_{n\in\mathbb{N}_\beta^+} C_{-2}X_n & -C_{-1} \end{bmatrix},$$

$$\mathbf{M}(G_+) = \begin{bmatrix} -C_{-1}C_1^{-1} & \sum_{n\in\mathbb{N}_{-\alpha}^+} \widetilde{X}_n \\ \sum_{n\in\mathbb{N}_\alpha^+} C_1X_n + \sum_{n\in\mathbb{N}_\beta^+} C_2X_n & C_1 \end{bmatrix},$$

and hence the geometric mean of G is given by

$$\mathbf{d}(G) = \begin{bmatrix} -C_{-1}C_1^{-1} & -\sum_{n\in\mathbb{N}_{-\alpha}^+} C_{-1}^{-1}\widetilde{X}_n \\ \sum_{n\in\mathbb{N}_\beta^+}(C_2 - C_1C_{-1}^{-1}C_{-2})X_n & -C_1C_{-1}^{-1} \end{bmatrix}.$$

Theorem 2.7. *Suppose that G is given by* (2.20) *where $0<\alpha<\beta<1$, the number $\beta-\alpha$ is irrational, and*

$$|C_2|^{1-\beta}|C_{-2}|^\beta > |C_1|^{1-\alpha}|C_{-1}|^\alpha.$$

Then G admits a canonical left APW factorization (2.1) *where the matrix functions $G_\pm, G_\pm^{-1} \in APW_{2\times 2}^\pm$ are given by*

$$G_\pm = \begin{bmatrix} \psi_1^\pm & \widetilde{\psi}_1^\pm \\ \psi_2^\pm & \widetilde{\psi}_2^\pm \end{bmatrix}, \quad G_\pm^{-1} = \frac{1}{\det G_\pm}\begin{bmatrix} \widetilde{\psi}_2^\pm & -\widetilde{\psi}_1^\pm \\ -\psi_2^\pm & \psi_1^\pm \end{bmatrix}, \tag{2.26}$$

with

$$\begin{aligned} \psi_1^+ &= e_1 + \sum\nolimits_{n=-\infty}^0 Y_n e_{\{n(\beta-\alpha)\}}, \\ \psi_1^- &= 1 + \sum\nolimits_{n=-\infty}^0 Y_n e_{\{n(\beta-\alpha)\}-1}, \\ \psi_2^+ &= C_1e_\alpha + C_2e_\beta + \sum_{\{n\in\mathbb{Z}_-\colon 0\le\{\alpha+n(\beta-\alpha)\}<\alpha\}} C_1Y_ne_{\{\alpha+n(\beta-\alpha)\}} \\ &\quad + \sum_{\{n\in\mathbb{Z}_-\colon 0\le\{\beta+n(\beta-\alpha)\}<\beta\}} C_2Y_ne_{\{\beta+n(\beta-\alpha)\}}, \\ \psi_2^- &= -\sum_{\{n\in\mathbb{Z}_-\colon \{\alpha+n(\beta-\alpha)\}=0\}} C_{-1}Y_n - \sum_{\{n\in\mathbb{Z}_-\colon \alpha\le\{\alpha+n(\beta-\alpha)\}<1\}} C_{-1}Y_ne_{\{\alpha+n(\beta-\alpha)\}-1} \\ &\quad - \sum_{\{n\in\mathbb{Z}_-\colon \{\beta+n(\beta-\alpha)\}=0\}} C_{-2}Y_n - \sum_{\{n\in\mathbb{Z}_-\colon \beta\le\{\beta+n(\beta-\alpha)\}<1\}} C_{-2}Y_ne_{\{\beta+n(\beta-\alpha)\}-1}, \end{aligned} \tag{2.27}$$

$$\begin{aligned}
\widetilde{\psi}_1^+ &= \sum\nolimits_{n=-\infty}^{-1} \widetilde{Y}_n e_{\{n(\beta-\alpha)-\alpha\}},\\
\widetilde{\psi}_1^- &= \sum\nolimits_{n=-\infty}^{-1} \widetilde{Y}_n e_{\{n(\beta-\alpha)-\alpha\}-1},\\
\widetilde{\psi}_2^+ &= \sum_{\{n\in\mathbb{N}_-:\, 0\le\{n(\beta-\alpha)\}<\alpha\}} C_1\widetilde{Y}_n e_{\{n(\beta-\alpha)\}} + \sum_{\{n\in\mathbb{N}_-:\, 0\le\{(n+1)(\beta-\alpha)\}<\beta\}} C_2\widetilde{Y}_n e_{\{(n+1)(\beta-\alpha)\}},\\
\widetilde{\psi}_2^- &= -\sum_{\{n\in\mathbb{N}_-:\, \alpha\le\{n(\beta-\alpha)\}<1\}} C_{-1}\widetilde{Y}_n e_{\{n(\beta-\alpha)\}-1}\\
&\quad -\sum_{\{n\in\mathbb{N}_-:\, \beta\le\{(n+1)(\beta-\alpha)\}<1\}} C_{-2}\widetilde{Y}_n e_{\{(n+1)(\beta-\alpha)\}-1} \quad - C_{-2}\widetilde{Y}_{-1}.
\end{aligned} \tag{2.28}$$

The coefficients Y_n $(n\in\mathbb{Z}_-)$ *and* $\widetilde{Y}_n$ $(n\in\mathbb{N}_-)$ *here are defined by*

$$Y_{n-1} = \begin{cases} (-1)^{|n|}\left(\frac{C_{-1}}{C_{-2}}\right)^{\left(|n|-\left\lceil\frac{k}{\beta-\alpha}\right\rceil\right)+\sum_{s=0}^{k-1}\left(\left\lceil\frac{s+\alpha}{\beta-\alpha}\right\rceil-\left\lceil\frac{s}{\beta-\alpha}\right\rceil\right)}\left(\frac{C_1}{C_{-2}}\right)^k\\ \times\left(\frac{C_1}{C_2}\right)^{\sum_{s=1}^{k}\left(\left\lceil\frac{s}{\beta-\alpha}\right\rceil-\left\lceil\frac{s-1+\alpha}{\beta-\alpha}\right\rceil-1\right)}Y_{-1} \quad \text{if } -n=\left\lceil\frac{k}{\beta-\alpha}\right\rceil,\dots,\left\lceil\frac{k+\alpha}{\beta-\alpha}\right\rceil-1,\\ (-1)^{|n|}\left(\frac{C_1}{C_2}\right)^{\left(|n|-\left\lceil\frac{k+\alpha}{\beta-\alpha}\right\rceil+1\right)+\sum_{s=1}^{k}\left(\left\lceil\frac{s}{\beta-\alpha}\right\rceil-\left\lceil\frac{s-1+\alpha}{\beta-\alpha}\right\rceil-1\right)}\left(\frac{C_1}{C_{-2}}\right)^k\\ \times\left(\frac{C_{-1}}{C_{-2}}\right)^{-1+\sum_{s=0}^{k}\left(\left\lceil\frac{s+\alpha}{\beta-\alpha}\right\rceil-\left\lceil\frac{s}{\beta-\alpha}\right\rceil\right)}Y_{-1} \quad \text{if } -n=\left\lceil\frac{k+\alpha}{\beta-\alpha}\right\rceil,\dots,\left\lceil\frac{k+1}{\beta-\alpha}\right\rceil-2,\\ (-1)^{|n|}\left(\frac{C_{-1}}{C_{-2}}\right)^{-1+\sum_{s=0}^{k}\left(\left\lceil\frac{s+\alpha}{\beta-\alpha}\right\rceil-\left\lceil\frac{s}{\beta-\alpha}\right\rceil\right)}\left(\frac{C_1}{C_{-2}}\right)^{k+1}\\ \times\left(\frac{C_1}{C_2}\right)^{\sum_{s=1}^{k+1}\left(\left\lceil\frac{s}{\beta-\alpha}\right\rceil-\left\lceil\frac{s-1+\alpha}{\beta-\alpha}\right\rceil-1\right)} Y_{-1} \quad \text{if } \; -n=\left\lceil\frac{k+1}{\beta-\alpha}\right\rceil-1 \end{cases} \tag{2.29}$$

and

$$\widetilde{Y}_{n-1} = \begin{cases} (-1)^{|n|}\left(\frac{C_1}{C_2}\right)^{\left(|n|-\left\lfloor\frac{k}{\beta-\alpha}\right\rfloor\right)+\sum_{s=1}^{k}\left(\left\lfloor\frac{s-\beta}{\beta-\alpha}\right\rfloor-\left\lfloor\frac{s-1}{\beta-\alpha}\right\rfloor\right)}\left(\frac{C_1}{C_{-2}}\right)^k\\ \times\left(\frac{C_{-1}}{C_{-2}}\right)^{\sum_{s=1}^{k}\left(\left\lfloor\frac{s}{\beta-\alpha}\right\rfloor-\left\lfloor\frac{s-\beta}{\beta-\alpha}\right\rfloor-1\right)}\widetilde{Y}_{-1} \quad \text{if } \; -n=\left\lceil\frac{k}{\beta-\alpha}\right\rceil,\dots,\left\lfloor\frac{k+1-\beta}{\beta-\alpha}\right\rfloor,\\ (-1)^{|n|}\left(\frac{C_1}{C_2}\right)^{\sum_{s=1}^{k+1}\left(\left\lfloor\frac{s-\beta}{\beta-\alpha}\right\rfloor-\left\lfloor\frac{s-1}{\beta-\alpha}\right\rfloor\right)}\left(\frac{C_1}{C_{-2}}\right)^{k+1}\\ \times\left(\frac{C_{-1}}{C_{-2}}\right)^{\sum_{s=1}^{k}\left(\left\lfloor\frac{s}{\beta-\alpha}\right\rfloor-\left\lfloor\frac{s-\beta}{\beta-\alpha}\right\rfloor-1\right)}\widetilde{Y}_{-1} \quad \text{if } \; -n=\left\lfloor\frac{k+1-\beta}{\beta-\alpha}\right\rfloor+1,\\ (-1)^{|n|}\left(\frac{C_1}{C_{-2}}\right)^{k+1}\left(\frac{C_{-1}}{C_{-2}}\right)^{\left(|n|-\left\lfloor\frac{k+1-\beta}{\beta-\alpha}\right\rfloor-1\right)+\sum_{s=1}^{k}\left(\left\lfloor\frac{s}{\beta-\alpha}\right\rfloor-\left\lfloor\frac{s-\beta}{\beta-\alpha}\right\rfloor-1\right)}\\ \times\left(\frac{C_1}{C_2}\right)^{\sum_{s=1}^{k+1}\left(\left\lfloor\frac{s-\beta}{\beta-\alpha}\right\rfloor-\left\lfloor\frac{s-1}{\beta-\alpha}\right\rfloor\right)}\widetilde{Y}_{-1} \quad \text{if } \; -n=\left\lfloor\frac{k+1-\beta}{\beta-\alpha}\right\rfloor+2,\dots,\left\lfloor\frac{k+1}{\beta-\alpha}\right\rfloor \end{cases} \tag{2.30}$$

for $k=0,1,2,\dots,$ *with the initial conditions* $\widetilde{Y}_{-1}=1$,

$$Y_0 = -\frac{C_{-2}}{C_2}, \quad Y_{-1} = -\frac{C_{-1}}{C_{-2}} + \frac{C_1}{C_2}.$$

In the notation (2.25), we have the following.

Corollary 2.8. *In the setting of Theorem* 2.7 *we have*

$$\mathbf{M}(G_-) = \begin{bmatrix} 1 & 0 \\ -\sum_{n\in\mathbb{N}_\alpha^-} C_{-1}Y_n - \sum_{n\in\mathbb{N}_\beta^-} C_{-2}Y_n & -C_{-2} \end{bmatrix},$$

$$\mathbf{M}(G_+) = \begin{bmatrix} -C_{-2}C_2^{-1} & \sum_{n\in\mathbb{N}_{-\alpha}^-} \widetilde{Y}_n \\ \sum_{n\in\mathbb{N}_\alpha^-} C_1Y_n + \sum_{n\in\mathbb{N}_\beta^-} C_2Y_n & C_2 \end{bmatrix},$$

and hence the geometric mean of G *is given by*

$$\mathbf{d}(G) = \begin{bmatrix} -C_{-2}C_2^{-1} & -\sum_{n\in\mathbb{N}_{-\alpha}^-} C_{-2}^{-1}\widetilde{Y}_n \\ \sum_{n\in\mathbb{N}_\alpha^-} (C_1 - C_2C_{-1}C_{-2}^{-1})Y_n & -C_2C_{-2}^{-1} \end{bmatrix}.$$

3. Factorization of some block triangular matrix functions

3.1. A conditional criterion of AP factorability

Factorability properties of $G \in AP_{N\times N}$ obviously do not change under multiplication on the left and on the right by matrices from $\mathcal{G}\mathbb{C}_{N\times N}$. In particular, G of the form (1.4) admits a left AP or APW factorization only simultaneously with

$$\begin{bmatrix} Q^{-1} & 0 \\ 0 & P \end{bmatrix} G \begin{bmatrix} Q & 0 \\ 0 & P^{-1} \end{bmatrix} = \begin{bmatrix} e_1 I_N & 0 \\ PFQ & e_{-1}I_N \end{bmatrix} \tag{3.1}$$

for any $P, Q \in \mathcal{G}\mathbb{C}_{N\times N}$, and the partial AP indices of G_F and G_{PFQ} coincide.

Proposition 3.1. *Let* $F \in APW_{N\times N}$ *be a triangular matrix function with the diagonal entries* f_j*. Then in order for* G_F *to admit a canonical AP factorization it is sufficient, and if* $f_j \in \mathcal{E}$ *for* $j = 1, \dots, N$ *also necessary, that all* 2×2 *matrix functions* G_{f_j} *admit such a factorization.*

Proof. Choosing $P = Q = [\delta_{j,N-j+1}]$ in (3.1), we can switch between lower and upper triangular F. So, without loss of generality we may suppose that F is lower triangular.

Sufficiency. Observe that

$$F = F_0 + \widetilde{F}, \tag{3.2}$$

where $F_0 = \mathrm{diag}[f_1, \dots, f_N]$ and $\widetilde{F}$ is lower triangular with zero diagonal. Letting now

$$P = Q^{-1} = \mathrm{diag}[1, \epsilon, \dots, \epsilon^{N-1}],$$

we can make the difference $PFQ - F_0$ arbitrarily small by an appropriate choice of ϵ. Since canonical AP factorable matrices form an open set, it suffices to show that G_{F_0} lies there. But the latter matrix is permutationally similar to

$$\mathrm{diag}[G_{f_1}, \dots, G_{f_N}],$$

and thus admits a left canonical AP factorization along with its diagonal 2×2 blocks.

Necessity. Suppose G_F admits a left canonical AP factorization. Consider h_j so close to f_j $(j = 2, \dots, N)$ that the matrix G_H with

$$H = \widetilde{F} + \operatorname{diag}[f_1, h_2, \dots, h_N]$$

still admits a left canonical AP factorization, while the matrices G_{h_j} also are AP factorable with zero partial AP indices, $j = 2, \dots, N$. (This is possible due to Lemma 2.1 since $f_j \in \mathcal{E} \cap APW$.) Via a permutational similarity corresponding to the permutation $\{1, N+1, 2, \dots, 2N\}$, the matrix G_H can be put in a block triangular form

$$\begin{bmatrix} G_{f_1} & 0 \\ * & G_{H_1} \end{bmatrix}, \tag{3.3}$$

where $H_1 \in APW_{(N-1)\times(N-1)}$ is lower triangular with the diagonal entries $h_2, \dots,$ h_N. By the already proven sufficiency, G_{H_1} admits a canonical AP factorization.

So, a block triangular matrix (3.3) and one of its diagonal blocks both admit a left canonical AP factorization. Since canonical AP factorability of APW matrices is equivalent to the invertibility of the respective Toeplitz operators, from here it follows that the other diagonal block of (3.3), that is, G_{f_1}, must admit a canonical AP factorization.

In its turn, the same permutational similarity can be used to rewrite the unperturbed matrix G_F in a block triangular form, with the diagonal blocks being G_{f_1} and G_{F_1}, where $F_1 \in APW_{(N-1)\times(N-1)}$ is simply F with the first row and column deleted. From the canonical AP factorability of G_F and G_{f_1} we now conclude that G_{F_1} also admits a canonical AP factorization. Since the statement is trivially correct for $N = 1$, the induction argument thus completes the proof. ☐

3.2. Quadrinomial case: Existence

We now pass to the case of matrix functions (1.4) with $N > 1$ and the off-diagonal block (1.5) such that its coefficients $C_j \in \mathbb{C}_{N\times N}$ can be put in a triangular form by the same transformation $C_j \mapsto PC_jQ$ with some $P, Q \in \mathcal{G}\mathbb{C}_{N\times N}$. This condition is satisfied, in particular, if C_j pairwise commute, in which case it is possible to choose $Q = P^{-1}$ (see, e.g., [23, Lemma 4.3]).

Since the matrix functions G_F given by (1.4) and G_{PFQ} admit a canonical factorization only simultaneously, we may without loss of generality suppose that C_i are themselves (lower) triangular:

$$C_i = \begin{bmatrix} (c_i)_{1,1} & 0 & 0 & \cdots & 0 & 0 \\ (c_i)_{2,1} & (c_i)_{2,2} & 0 & \cdots & 0 & 0 \\ (c_i)_{3,1} & (c_i)_{3,2} & (c_i)_{3,3} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ (c_i)_{N-1,1} & (c_i)_{N-1,2} & (c_i)_{N-1,3} & \cdots & (c_i)_{N-1,N-1} & 0 \\ (c_i)_{N,1} & (c_i)_{N,2} & (c_i)_{N,3} & \cdots & (c_i)_{N,N-1} & (c_i)_{N,N} \end{bmatrix} \tag{3.4}$$

$(i = \pm 1, \pm 2)$. In what follows, we will relabel the diagonal entries $(c_i)_{s,s}$ of the matrices (3.4) by $c_{i,s}$. Note that in the case of pairwise commuting (but a priori not necessarily triangular) matrices C_i, $c_{i,s}$ are their so-called *bonded eigenvalues*, in the terminology of [3].

Theorem 3.2. *Let $G = G_F$ be given by* (1.4), (1.5) *and* (3.4), *where* $0 < \alpha < \beta < 1$ *and the number $\beta-\alpha$ is irrational. Then G admits a canonical left AP factorization $G = G_+G_-^{-1}$ if and only if*

$$|c_{2,s}|^{1-\beta}|c_{-2,s}|^{\beta} \neq |c_{1,s}|^{1-\alpha}|c_{-1,s}|^{\alpha} \quad \textit{for all} \;\; s = 1, 2, \dots, N, \tag{3.5}$$

where $c_{i,s} := (c_i)_{s,s}$ for all $i = \pm1, \pm2$ and all $s = 1, 2, \dots, N$ are the diagonal entries of matrix coefficients (3.4) *in* (1.5).

Proof. Follows directly by combining Corollary 2.4 with Proposition 3.1. □

3.3. Quadrinomial case: Explicit factorization

We now turn to the explicit factorization construction of matrix functions G_F with F given by (1.5), (3.4) when its canonical factorization exists, that is, conditions (3.5) hold. Decomposition (3.2) in our setting yields matrices F_0 and $\widetilde{F}$ of the same structure (1.5) as F but with C_i replaced by $\operatorname{diag}[c_{i,1}, \dots, c_{i,N}]$ for F_0 and by

$$\widetilde{C}_i = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ (c_i)_{2,1} & 0 & 0 & \dots & 0 & 0 \\ (c_i)_{3,1} & (c_i)_{3,2} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ (c_i)_{N-1,1} & (c_i)_{N-1,2} & (c_i)_{N-1,3} & \dots & 0 & 0 \\ (c_i)_{N,1} & (c_i)_{N,2} & (c_i)_{N,3} & \dots & (c_i)_{N,N-1} & 0 \end{bmatrix} \tag{3.6}$$

for $\widetilde{F}$. Then $G_F = \mathcal{G} + K$, where $\mathcal{G}$ is an abbreviated notation for G_{F_0} and

$$K = \begin{bmatrix} 0_N & 0_N \\ \widetilde{F} & 0_N \end{bmatrix}.$$

Consider the matrix functions

$$\mathcal{G}_\pm = \begin{bmatrix} \mathcal{G}_1^\pm & \widetilde{\mathcal{G}}_1^\pm \\ \mathcal{G}_2^\pm & \widetilde{\mathcal{G}}_2^\pm \end{bmatrix}, \tag{3.7}$$

$$\mathcal{G}_1^\pm := \operatorname{diag}\{g_{s,1}^\pm\}_{s=1}^N, \qquad \widetilde{\mathcal{G}}_1^\pm := \operatorname{diag}\{\widetilde{g}_{s,1}^\pm\}_{s=1}^N,$$
$$\mathcal{G}_2^\pm := \operatorname{diag}\{g_{s,2}^\pm\}_{s=1}^N, \qquad \widetilde{\mathcal{G}}_2^\pm := \operatorname{diag}\{\widetilde{g}_{s,2}^\pm\}_{s=1}^N,$$

where for $s = 1, 2, \dots, N$ and $j = 1, 2$,

$$g_{s,j}^\pm := \left\{ \begin{array}{l} \varphi_{s,j}^\pm \\ \psi_{s,j}^\pm \end{array} \right., \quad \widetilde{g}_{s,j}^\pm := \left\{ \begin{array}{l} \widetilde{\varphi}_{s,j}^\pm \\ \widetilde{\psi}_{s,j}^\pm \end{array} \right. \quad \text{if} \quad \left\{ \begin{array}{l} |c_{2,s}|^{1-\beta}|c_{-2,s}|^{\beta} < |c_{1,s}|^{1-\alpha}|c_{-1,s}|^{\alpha}, \\ |c_{2,s}|^{1-\beta}|c_{-2,s}|^{\beta} > |c_{1,s}|^{1-\alpha}|c_{-1,s}|^{\alpha}, \end{array} \right. \tag{3.8}$$

$\varphi^{\pm}_{s,j}$ and $\widetilde{\varphi}^{\pm}_{s,j}$ are given by (2.21) and (2.22), respectively, with C_i replaced by $c_{i,s}$ for $i = \pm 1, \pm 2$, and with X_n and $\widetilde{X}_n$ $(n \in \mathbb{Z}_+)$ calculated by formulas (2.23), (2.24) (again with C_i replaced by $c_{i,s}$ for $i = \pm 1, \pm 2$), where

$$X_0 = -\frac{c_{-1,s}}{c_{1,s}}, \quad X_1 = -\frac{c_{-2,s}}{c_{1,s}} + \frac{c_{2,s}c_{-1,s}}{c_{1,s}^2}, \quad \widetilde{X}_0 = 1;$$

$\psi^{\pm}_{s,j}$ and $\widetilde{\psi}^{\pm}_{s,j}$ are given by (2.27) and (2.28), respectively, with C_i replaced by $c_{i,s}$ for $i = \pm 1, \pm 2$, and with Y_n $(n \in \mathbb{Z}_-)$ and $\widetilde{Y}_n$ $(n \in \mathbb{N}_-)$ calculated by formulas (2.29), (2.30) (once again, with C_i replaced by $c_{i,s}$ for $i = \pm 1, \pm 2$), where

$$Y_0 = -\frac{c_{-2,s}}{c_{2,s}}, \quad Y_{-1} = -\frac{c_{-1,s}}{c_{-2,s}} + \frac{c_{1,s}}{c_{2,s}}, \quad \widetilde{Y}_{-1} = 1.$$

Below we will denote these X_n, $\widetilde{X}_n$, Y_n and $\widetilde{Y}_n$ as $X_{n,s}$, $\widetilde{X}_{n,s}$, $Y_{n,s}$ and $\widetilde{Y}_{n,s}$.

For $s = 1, 2, \ldots, N$, we put

$$G^{\pm}_s = \begin{bmatrix} g^{\pm}_{s,1} & \widetilde{g}^{\pm}_{s,1} \\ g^{\pm}_{s,2} & \widetilde{g}^{\pm}_{s,2} \end{bmatrix}, \tag{3.9}$$

where $g^{\pm}_{s,j}$ and $\widetilde{g}^{\pm}_{s,j}$ for $j = 1, 2$ are given by (3.8). Note that

$$\det G^+_s = \det G^-_s = k_s \quad \text{where} \quad k_s = \mathbf{M}(\det G^{\pm}_s). \tag{3.10}$$

We also define the matrix

$$K_N = \operatorname{diag}\left[k_1^{-1}, \ldots, k_N^{-1}\right]. \tag{3.11}$$

By [2], the matrix function $\mathcal{G} = G_{F_0}$ admits the canonical left APW factorization

$$\mathcal{G} = \mathcal{G}_+ \mathcal{G}_-^{-1}. \tag{3.12}$$

It follows from (3.12) that

$$G = \mathcal{G}_+ \widetilde{G} \mathcal{G}_-^{-1},$$

where

$$\begin{aligned} \widetilde{G} = \mathcal{G}_+^{-1} G \mathcal{G}_- &= \begin{bmatrix} K_N \widetilde{\mathcal{G}}_2^+ & -K_N \widetilde{\mathcal{G}}_1^+ \\ -K_N \mathcal{G}_2^+ & K_N \mathcal{G}_1^+ \end{bmatrix} \begin{bmatrix} e_1 I_N & 0_N \\ F & e_{-1} I_N \end{bmatrix} \begin{bmatrix} \mathcal{G}_1^- & \widetilde{\mathcal{G}}_1^- \\ \mathcal{G}_2^- & \widetilde{\mathcal{G}}_2^- \end{bmatrix} \\ &= \begin{bmatrix} I_N & 0_N \\ 0_N & I_N \end{bmatrix} + \begin{bmatrix} -K_N \widetilde{\mathcal{G}}_1^+ \widetilde{F} \mathcal{G}_1^- & -K_N \widetilde{\mathcal{G}}_1^+ \widetilde{F} \widetilde{\mathcal{G}}_1^- \\ K_N \mathcal{G}_1^+ \widetilde{F} \mathcal{G}_1^- & K_N \mathcal{G}_1^+ \widetilde{F} \widetilde{\mathcal{G}}_1^- \end{bmatrix} \end{aligned} \tag{3.13}$$

and K_N is given by (3.11).

Consider now $APW_{N\times N}$ as the decomposing Banach algebra $\mathcal{B}$, with $\mathcal{B}_+ = APW^+_{N\times N}$ and $\mathcal{B}_- = (APW_0^-)_{N\times N}$. Letting

$$a := \begin{bmatrix} K_N \widetilde{\mathcal{G}}_1^+ \widetilde{F} \mathcal{G}_1^- & K_N \widetilde{\mathcal{G}}_1^+ \widetilde{F} \widetilde{\mathcal{G}}_1^- \\ -K_N \mathcal{G}_1^+ \widetilde{F} \mathcal{G}_1^- & -K_N \mathcal{G}_1^+ \widetilde{F} \widetilde{\mathcal{G}}_1^- \end{bmatrix} \tag{3.14}$$

and observing that each of its blocks is lower triangular with zero diagonal entries, we conclude that

$$(\mathcal{P}_a^+)^N I_{2N} = 0_{2N} \quad \text{and} \quad (\mathcal{P}_a^-)^N I_{2N} = 0_{2N},$$

where the mappings $\mathcal{P}_a^\pm$ are defined by (2.11) with a given by (3.14). Hence, by Theorem 2.2, the matrix function $\widetilde{G} = I_{2N} - a$ admits a canonical left factorization

$$\widetilde{G} = (I_{2N} + b_+)^{-1}(I_{2N} + b_-)^{-1} \tag{3.15}$$

where the matrix functions

$$\begin{aligned} b_+ &= a_+ + (a_+a)_+ + ((a_+a)_+a)_+ + \cdots + \underbrace{(\dots(a_+a)_+\dots a)_+}_{N-1 \text{ terms}}, \\ b_- &= a_- + (aa_-)_- + (a(aa_-)_-)_- + \cdots + \underbrace{(a\dots(aa_-)_-\dots)_-}_{N-1 \text{ terms}}, \end{aligned} \tag{3.16}$$

belong to $APW^+_{2N\times 2N}$ and $(APW_0^-)_{2N\times 2N}$, respectively, and $a_\pm := P_\pm a = \mathcal{P}_a^\pm I_{2N}$. Since each of $N \times N$ blocks in (3.14) is lower triangular matrix function with zero diagonal entries, it is easily seen from (3.16) that

$$(I_{2N} + b_+)^{-1} = I_{2N} + \sum_{k=1}^{N-1} (-1)^k b_+^k,$$

whence (3.15) takes the form

$$\widetilde{G} = \left(I_{2N} + \sum_{k=1}^{N-1} (-1)^k b_+^k \right) (I_{2N} + b_-)^{-1}. \tag{3.17}$$

Putting together (3.13) and (3.17), we arrive to the following conclusion.

Theorem 3.3. *Let G_F be the matrix function* (1.4) *with F given by* (1.5) *and* (3.4) *and satisfying* (3.5). *Then the multiples $G_\pm$ from its left canonical APW factorization* (2.1) *can be chosen as*

$$G_+ = \mathcal{G}_+ \left(I_{2N} + \sum_{k=1}^{N-1} (-1)^k b_+^k \right), \quad G_- = \mathcal{G}_-(I_{2N} + b_-), \tag{3.18}$$

with $\mathcal{G}_\pm$ and $b_\pm$ defined by (3.7)–(3.8) *and* (3.16) *respectively.*

It follows from Corollaries 2.6 and 2.8 that

$$\mathbf{M}(\mathcal{G}_\pm) = \begin{bmatrix} \operatorname{diag}\{\mathbf{M}(g_{s,1}^\pm)\}_{s=1}^N & \operatorname{diag}\{\mathbf{M}(\widetilde{g}_{s,1}^\pm)\}_{s=1}^N \\ \operatorname{diag}\{\mathbf{M}(g_{s,2}^\pm)\}_{s=1}^N & \operatorname{diag}\{\mathbf{M}(\widetilde{g}_{s,2}^\pm)\}_{s=1}^N \end{bmatrix}, \tag{3.19}$$

where

$$\mathbf{M}(g^+_{s,1}) = \begin{cases} -c_{-1,s}c^{-1}_{1,s}, \\ -c_{-2,s}c^{-1}_{2,s}, \end{cases} \qquad \mathbf{M}(\widetilde{g}^+_{s,1}) = \begin{cases} \sum_{n\in\mathbb{N}^+_{-\alpha}} \widetilde{X}_{n,s}, \\ \sum_{n\in\mathbb{N}^-_{-\alpha}} \widetilde{Y}_{n,s}, \end{cases}$$
$$\mathbf{M}(g^+_{s,2}) = \begin{cases} \sum_{n\in\mathbb{N}^+_\alpha} c_{1,s}X_{n,s} + \sum_{n\in\mathbb{N}^+_\beta} c_{2,s}X_{n,s}, \\ \sum_{n\in\mathbb{N}^-_\alpha} c_{1,s}Y_{n,s} + \sum_{n\in\mathbb{N}^-_\beta} c_{2,s}Y_{n,s}, \end{cases} \qquad \mathbf{M}(\widetilde{g}^+_{s,2}) = \begin{cases} c_{1,s}, \\ c_{2,s}, \end{cases}$$
$$\mathbf{M}(g^-_{s,1}) = 1, \quad \mathbf{M}(\widetilde{g}^-_{s,1}) = 0,$$
$$\mathbf{M}(g^-_{s,2}) = \begin{cases} -\sum_{n\in\mathbb{N}^+_\alpha} c_{-1,s}X_{n,s} - \sum_{n\in\mathbb{N}^+_\beta} c_{-2,s}X_{n,s}, \\ -\sum_{n\in\mathbb{N}^-_\alpha} c_{-1,s}Y_{n,s} - \sum_{n\in\mathbb{N}^-_\beta} c_{-2,s}Y_{n,s}, \end{cases} \tag{3.20}$$
$$\mathbf{M}(\widetilde{g}^-_{s,2}) = \begin{cases} -c_{-1,s}, \\ -c_{-2,s} \end{cases} \quad \text{if} \begin{cases} |c_{2,s}|^{1-\beta}|c_{-2,s}|^\beta < |c_{1,s}|^{1-\alpha}|c_{-1,s}|^\alpha, \\ |c_{2,s}|^{1-\beta}|c_{-2,s}|^\beta > |c_{1,s}|^{1-\alpha}|c_{-1,s}|^\alpha. \end{cases}$$

On the other hand, we infer from (3.18) and (3.16) that

$$\mathbf{M}(G_+) = \mathbf{M}(\mathcal{G}_+)\left(I_{2N} + \sum_{k=1}^{N-1}(-1)^k\mathbf{M}(b_+)^k\right), \quad \mathbf{M}(G_-) = \mathbf{M}(\mathcal{G}_-),$$

which in view of (2.8) implies the following

Corollary 3.4. *Under the conditions of Theorem* 3.2*, the geometric mean of the matrix function G given by* (1.4), (1.5) *and* (3.4) *is calculated by*

$$\mathbf{d}(G) = \mathbf{M}(\mathcal{G}_+)\left(I_{2N} + \sum_{k=1}^{N-1}(-1)^k\mathbf{M}(b_+)^k\right)\mathbf{M}(\mathcal{G}_-)^{-1}, \tag{3.21}$$

where $\mathbf{M}(\mathcal{G}_\pm)$ and b_+ are given by (3.19)–(3.20) *and* (3.16)*, respectively.*

4. The geometric mean in the case $N = 2$

Corollary 3.4 in principle allows to compute the geometric mean for any value of N. In practice the complexity of this computation grows with N substantially, in particular because each of the inequalities (3.5) can materialize in two different ways, and the resulting 2^N cases yield different formulas and thus have to be treated separately. We therefore restrict our attention to the case $N = 2$ which should suffice for illustrative purposes. Corollary 3.4 can then be restated as follows.

Theorem 4.1. *Under the conditions of Theorem* 3.2*, the geometric mean of the matrix function G_F given by* (1.4), (1.5) *and* (3.4) *for $N = 2$ is calculated by*

$$\mathbf{d}(G_F) = \mathbf{d}(G_{F_0}) + T_{\widetilde{F}}, \tag{4.1}$$

where $\mathbf{d}(G_{F_0}) = \mathbf{d}(\mathcal{G}) = \mathbf{M}(\mathcal{G}_+)\mathbf{M}(\mathcal{G}_-)^{-1}$,

$$T_{\widetilde{F}} = \mathbf{M}(\mathcal{G}_+) \begin{bmatrix} 0 & 0 & 0 & 0 \\ -k_2^{-1}\mathbf{M}(\widetilde{g}_{2,1}^+ f g_{1,1}^-) & 0 & -k_2^{-1}\mathbf{M}(\widetilde{g}_{2,1}^+ f \widetilde{g}_{1,1}^-) & 0 \\ 0 & 0 & 0 & 0 \\ k_2^{-1}\mathbf{M}(g_{2,1}^+ f g_{1,1}^-) & 0 & k_2^{-1}\mathbf{M}(g_{2,1}^+ f \widetilde{g}_{1,1}^-) & 0 \end{bmatrix} \mathbf{M}(\mathcal{G}_-)^{-1}, \quad (4.2)$$

$\mathbf{M}(\mathcal{G}_\pm)$ *are given by* (3.19)–(3.20), *the functions* $g_{2,1}^+, \widetilde{g}_{2,1}^+, g_{1,1}^-, \widetilde{g}_{1,1}^-$ *are given by* (3.8), k_2 *is given by* (3.10),

$$\widetilde{F} = \begin{bmatrix} 0 & 0 \\ f & 0 \end{bmatrix}, \quad f = \widetilde{c}_1 e_\alpha + \widetilde{c}_{-1} e_{\alpha-1} + \widetilde{c}_2 e_\beta + \widetilde{c}_{-2} e_{\beta-1} \quad (4.3)$$

and $\widetilde{c}_i := (c_i)_{2,1}$ *for all* $i = \pm 1, \pm 2$.

Proof. Since $N = 2$, we conclude from (3.16) that

$$\mathbf{M}(b_+) = \mathbf{M}(a_+) = \mathbf{M}(a), \quad (4.4)$$

where, by (3.14), (3.11) and (4.3),

$$\mathbf{M}(a) = \begin{bmatrix} K_2\mathbf{M}(\widetilde{\mathcal{G}}_1^+ \widetilde{F} \mathcal{G}_1^-) & K_2\mathbf{M}(\widetilde{\mathcal{G}}_1^+ \widetilde{F} \widetilde{\mathcal{G}}_1^-) \\ -K_2\mathbf{M}(\mathcal{G}_1^+ \widetilde{F} \mathcal{G}_1^-) & -K_2\mathbf{M}(\mathcal{G}_1^+ \widetilde{F} \widetilde{\mathcal{G}}_1^-) \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ k_2^{-1}\mathbf{M}(\widetilde{g}_{2,1}^+ f g_{1,1}^-) & 0 & k_2^{-1}\mathbf{M}(\widetilde{g}_{2,1}^+ f \widetilde{g}_{1,1}^-) & 0 \\ 0 & 0 & 0 & 0 \\ -k_2^{-1}\mathbf{M}(g_{2,1}^+ f g_{1,1}^-) & 0 & -k_2^{-1}\mathbf{M}(g_{2,1}^+ f \widetilde{g}_{1,1}^-) & 0 \end{bmatrix}. \quad (4.5)$$

Hence we infer from (3.21), (4.4) and (4.5) that

$$\mathbf{d}(G) = \mathbf{M}(\mathcal{G}_+)\big(I_4 - \mathbf{M}(a)\big)\mathbf{M}(\mathcal{G}_-)^{-1} = \mathbf{d}(\mathcal{G}) + T_{\widetilde{F}},$$

where $T_{\widetilde{F}}$ is given by (4.2). □

The explicit formulas for the mean values involved in (4.2) depend on which of the four possible cases

$$|c_{2,s}|^{1-\beta}|c_{-2,s}|^\beta < |c_{1,s}|^{1-\alpha}|c_{-1,s}|^\alpha \quad \text{for} \quad s = 1, 2; \quad (4.6)$$

$$|c_{2,s}|^{1-\beta}|c_{-2,s}|^\beta > |c_{1,s}|^{1-\alpha}|c_{-1,s}|^\alpha \quad \text{for} \quad s = 1, 2; \quad (4.7)$$

$$\begin{aligned} |c_{2,1}|^{1-\beta}|c_{-2,1}|^\beta &< |c_{1,1}|^{1-\alpha}|c_{-1,1}|^\alpha, \\ |c_{2,2}|^{1-\beta}|c_{-2,2}|^\beta &> |c_{1,2}|^{1-\alpha}|c_{-1,2}|^\alpha; \end{aligned} \quad (4.8)$$

$$\begin{aligned} |c_{2,1}|^{1-\beta}|c_{-2,1}|^\beta &> |c_{1,1}|^{1-\alpha}|c_{-1,1}|^\alpha, \\ |c_{2,2}|^{1-\beta}|c_{-2,2}|^\beta &< |c_{1,2}|^{1-\alpha}|c_{-1,2}|^\alpha \end{aligned} \quad (4.9)$$

takes place, and are delegated to the Appendix. Here is the final result under a simplifying condition on α, β, with a proof also delegated to the Appendix.

Theorem 4.2. *Let $G = G_F$ be given by* (1.4), (1.5) *and* (3.4), *where $N = 2$, $0 < \alpha < \beta < 1$ and*

$$m\alpha + n\beta \notin \{0, 1\} \quad \text{for all rational } \ m, n. \tag{4.10}$$

Then G admits a canonical left AP factorization $G = G_+ G_-^{-1}$ if and only if

$$|c_{2,s}|^{1-\beta} |c_{-2,s}|^{\beta} \neq |c_{1,s}|^{1-\alpha} |c_{-1,s}|^{\alpha} \quad \text{for all } \ s = 1, 2, \tag{4.11}$$

where $c_{i,s} := (c_i)_{s,s}$ for all $i = \pm 1, \pm 2$ and all $s = 1, 2$ are the diagonal entries of matrix coefficients (3.4) *in* (1.5). *If* (4.11) *holds, then*

$$\mathbf{d}(G) = \operatorname{diag}[T_1, T_2], \tag{4.12}$$

where

$$\begin{aligned} T_1 &= \begin{bmatrix} -c_{-1,1} c_{1,1}^{-1} & 0 \\ -(c_{1,1} \widetilde{c}_{-1} - c_{-1,1} \widetilde{c}_1) c_{1,1}^{-1} c_{1,2}^{-1} & -c_{-1,2} c_{1,2}^{-1} \end{bmatrix}, \\ T_2 &= \begin{bmatrix} -c_{1,1} c_{-1,1}^{-1} & 0 \\ (c_{1,2} \widetilde{c}_{-1} - c_{-1,2} \widetilde{c}_1) c_{-1,1}^{-1} c_{-1,2}^{-1} & -c_{1,2} c_{-1,2}^{-1} \end{bmatrix}, \end{aligned} \tag{4.13}$$

if (4.6) *holds;*

$$\begin{aligned} T_1 &= \begin{bmatrix} -c_{-2,1} c_{2,1}^{-1} & 0 \\ -(c_{2,1} \widetilde{c}_{-2} - c_{-2,1} \widetilde{c}_2) c_{2,1}^{-1} c_{2,2}^{-1} & -c_{-2,2} c_{2,2}^{-1} \end{bmatrix}, \\ T_2 &= \begin{bmatrix} -c_{2,1} c_{-2,1}^{-1} & 0 \\ (c_{2,2} \widetilde{c}_{-2} - c_{-2,2} \widetilde{c}_2) c_{-2,1}^{-1} c_{-2,2}^{-1} & -c_{2,2} c_{-2,2}^{-1} \end{bmatrix}, \end{aligned} \tag{4.14}$$

if (4.7) *holds;*

$$\begin{aligned} T_1 &= \begin{bmatrix} -c_{-1,1} c_{1,1}^{-1} & 0 \\ -(c_{1,1} \widetilde{c}_{-2} - c_{-1,1} \widetilde{c}_2) c_{1,1}^{-1} c_{2,2}^{-1} & -c_{-2,2} c_{2,2}^{-1} \end{bmatrix}, \\ T_2 &= \begin{bmatrix} -c_{1,1} c_{-1,1}^{-1} & 0 \\ (c_{2,2} \widetilde{c}_{-1} - c_{-2,2} \widetilde{c}_1) c_{-1,1}^{-1} c_{-2,2}^{-1} & -c_{2,2} c_{-2,2}^{-1} \end{bmatrix}, \end{aligned} \tag{4.15}$$

if (4.8) *holds;*

$$\begin{aligned} T_1 &= \begin{bmatrix} -c_{-2,1} c_{2,1}^{-1} & 0 \\ -(c_{2,1} \widetilde{c}_{-1} - c_{-2,1} \widetilde{c}_1) c_{2,1}^{-1} c_{1,2}^{-1} & -c_{-1,2} c_{1,2}^{-1} \end{bmatrix}, \\ T_2 &= \begin{bmatrix} -c_{2,1} c_{-2,1}^{-1} & 0 \\ (c_{1,2} \widetilde{c}_{-2} - c_{-1,2} \widetilde{c}_2) c_{-2,1}^{-1} c_{-1,2}^{-1} & -c_{1,2} c_{-1,2}^{-1} \end{bmatrix}, \end{aligned} \tag{4.16}$$

if (4.9) *holds.*

5. Appendix

5.1. Computation of $\mathbf{d}(\mathcal{G})$ and $T_{\widetilde{F}}$

Applying Corollaries 2.6 and 2.8 with notation (2.25), we infer that

$$\begin{aligned}
&\mathbf{d}(\mathcal{G}) = -\operatorname{diag}\left[c_{-1,1}c_{1,1}^{-1},\ c_{-1,2}c_{1,2}^{-1},\ c_{1,1}c_{-1,1}^{-1},\ c_{1,2}c_{-1,2}^{-1}\right] + \begin{bmatrix} 0 & -B_{1,2} \\ B_{2,1} & 0 \end{bmatrix},\\
&B_{1,2} = \operatorname{diag}\left[\sum_{n\in\mathbb{N}_{-\alpha}^{+}} c_{-1,1}^{-1}\widetilde{X}_{n,1},\ \sum_{n\in\mathbb{N}_{-\alpha}^{+}} c_{-1,2}^{-1}\widetilde{X}_{n,2}\right],\\
&B_{2,1} = \operatorname{diag}\left[\sum_{n\in\mathbb{N}_{\beta}^{+}}(c_{2,1}-c_{1,1}c_{-1,1}^{-1}c_{-2,1})X_{n,1},\ \sum_{n\in\mathbb{N}_{\beta}^{+}}(c_{2,2}-c_{1,2}c_{-1,2}^{-1}c_{-2,2})X_{n,2}\right],
\end{aligned} \tag{5.1}$$

if (4.6) holds;

$$\begin{aligned}
&\mathbf{d}(\mathcal{G}) = -\operatorname{diag}\left[c_{-2,1}c_{2,1}^{-1},\ c_{-2,2}c_{2,2}^{-1},\ c_{2,1}c_{-2,1}^{-1},\ c_{2,2}c_{-2,2}^{-1}\right] + \begin{bmatrix} 0 & -B_{1,2} \\ B_{2,1} & 0 \end{bmatrix},\\
&B_{1,2} = \operatorname{diag}\left[\sum_{n\in\mathbb{N}_{-\alpha}^{-}} c_{-2,1}^{-1}\widetilde{Y}_{n,1},\ \sum_{n\in\mathbb{N}_{-\alpha}^{-}} c_{-2,2}^{-1}\widetilde{Y}_{n,2}\right],\\
&B_{2,1} = \operatorname{diag}\left[\sum_{n\in\mathbb{N}_{\alpha}^{-}}(c_{1,1}-c_{2,1}c_{-2,1}^{-1}c_{-1,1})Y_{n,1},\ \sum_{n\in\mathbb{N}_{\alpha}^{-}}(c_{1,2}-c_{2,2}c_{-2,2}^{-1}c_{-1,2})Y_{n,2}\right],
\end{aligned} \tag{5.2}$$

if (4.7) holds;

$$\begin{aligned}
&\mathbf{d}(\mathcal{G}) = -\operatorname{diag}\left[c_{-1,1}c_{1,1}^{-1},\ c_{-2,2}c_{2,2}^{-1},\ c_{1,1}c_{-1,1}^{-1},\ c_{2,2}c_{-2,2}^{-1}\right] + \begin{bmatrix} 0 & -B_{1,2} \\ B_{2,1} & 0 \end{bmatrix},\\
&B_{1,2} = \operatorname{diag}\left[\sum_{n\in\mathbb{N}_{-\alpha}^{+}} c_{-1,1}^{-1}\widetilde{X}_{n,1},\ \sum_{n\in\mathbb{N}_{-\alpha}^{-}} c_{-2,2}^{-1}\widetilde{Y}_{n,2}\right],\\
&B_{2,1} = \operatorname{diag}\left[\sum_{n\in\mathbb{N}_{\beta}^{+}}(c_{2,1}-c_{1,1}c_{-1,1}^{-1}c_{-2,1})X_{n,1},\ \sum_{n\in\mathbb{N}_{\alpha}^{-}}(c_{1,2}-c_{2,2}c_{-2,2}^{-1}c_{-1,2})Y_{n,2}\right],
\end{aligned} \tag{5.3}$$

if (4.8) holds;

$$\begin{aligned}
&\mathbf{d}(\mathcal{G}) = -\operatorname{diag}\left[c_{-2,1}c_{2,1}^{-1},\ c_{-1,2}c_{1,2}^{-1},\ c_{2,1}c_{-2,1}^{-1},\ c_{1,2}c_{-1,2}^{-1}\right] + \begin{bmatrix} 0 & -B_{1,2} \\ B_{2,1} & 0 \end{bmatrix},\\
&B_{1,2} = \operatorname{diag}\left[\sum_{n\in\mathbb{N}_{-\alpha}^{-}} c_{-2,1}^{-1}\widetilde{Y}_{n,1},\ \sum_{n\in\mathbb{N}_{-\alpha}^{+}} c_{-1,2}^{-1}\widetilde{X}_{n,2}\right],\\
&B_{2,1} = \operatorname{diag}\left[\sum_{n\in\mathbb{N}_{\alpha}^{-}}(c_{1,1}-c_{2,1}c_{-2,1}^{-1}c_{-1,1})Y_{n,1},\ \sum_{n\in\mathbb{N}_{\beta}^{+}}(c_{2,2}-c_{1,2}c_{-1,2}^{-1}c_{-2,2})X_{n,2}\right],
\end{aligned} \tag{5.4}$$

if (4.9) holds.

On the other hand, by (4.2) and (3.19), we obtain

$$T_{\widetilde{F}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ t_{2,1} & 0 & t_{2,3} & 0 \\ 0 & 0 & 0 & 0 \\ t_{4,1} & 0 & t_{4,3} & 0 \end{bmatrix}, \tag{5.5}$$

where

$$\begin{aligned} t_{2,1} &= \big[-\mathbf{M}(g_{2,1}^{+})\mathbf{M}(\widetilde{g}_{2,1}^{+} f g_{1,1}^{-}) + \mathbf{M}(\widetilde{g}_{2,1}^{+})\mathbf{M}(g_{2,1}^{+} f g_{1,1}^{-})\big] k_2^{-1} \\ &\quad + \big[\mathbf{M}(g_{2,1}^{+})\mathbf{M}(\widetilde{g}_{2,1}^{+} f \widetilde{g}_{1,1}^{-}) - \mathbf{M}(\widetilde{g}_{2,1}^{+})\mathbf{M}(g_{2,1}^{+} f \widetilde{g}_{1,1}^{-})\big] \mathbf{M}(g_{1,2}^{-}) k_1^{-1} k_2^{-1}, \\ t_{2,3} &= \big[-\mathbf{M}(g_{2,1}^{+})\mathbf{M}(\widetilde{g}_{2,1}^{+} f \widetilde{g}_{1,1}^{-}) + \mathbf{M}(\widetilde{g}_{2,1}^{+})\mathbf{M}(g_{2,1}^{+} f \widetilde{g}_{1,1}^{-})\big] k_1^{-1} k_2^{-1}, \\ t_{4,1} &= \big[-\mathbf{M}(g_{2,2}^{+})\mathbf{M}(\widetilde{g}_{2,1}^{+} f g_{1,1}^{-}) + \mathbf{M}(\widetilde{g}_{2,2}^{+})\mathbf{M}(g_{2,1}^{+} f g_{1,1}^{-})\big] k_2^{-1} \\ &\quad + \big[\mathbf{M}(g_{2,2}^{+})\mathbf{M}(\widetilde{g}_{2,1}^{+} f \widetilde{g}_{1,1}^{-}) - \mathbf{M}(\widetilde{g}_{2,2}^{+})\mathbf{M}(g_{2,1}^{+} f \widetilde{g}_{1,1}^{-})\big] \mathbf{M}(g_{1,2}^{-}) k_1^{-1} k_2^{-1}, \\ t_{4,3} &= \big[-\mathbf{M}(g_{2,2}^{+})\mathbf{M}(\widetilde{g}_{2,1}^{+} f \widetilde{g}_{1,1}^{-}) + \mathbf{M}(\widetilde{g}_{2,2}^{+})\mathbf{M}(g_{2,1}^{+} f \widetilde{g}_{1,1}^{-})\big] k_1^{-1} k_2^{-1}, \end{aligned} \tag{5.6}$$

and $\mathbf{M}(g_{2,j}^{+}), \mathbf{M}(\widetilde{g}_{2,j}^{+}), \mathbf{M}(g_{1,2}^{-})$ for $j = 1, 2$ are given by (3.20).

5.2. Computation of $\mathbf{M}(\widetilde{g}_{2,1}^{+} f g_{1,1}^{-})$, $\mathbf{M}(\widetilde{g}_{2,1}^{+} f \widetilde{g}_{1,1}^{-})$, $\mathbf{M}(g_{2,1}^{+} f g_{1,1}^{-})$, $\mathbf{M}(g_{2,1}^{+} f \widetilde{g}_{1,1}^{-})$

Below, along with $\mathbb{N}_{\gamma}^{\pm}$ given by (2.25), we use the following notation for $\gamma = \alpha, \beta$ and $l = 1, 2$:

$$\begin{aligned} \widetilde{\mathbb{N}}_{\gamma}^{\pm} &:= \{n \in \mathbb{N}_{\pm} : \{n(\beta - \alpha)\} + \gamma = 1\}, \\ (\mathbb{Z}_{\pm} \times \mathbb{Z}_{\pm})_{\gamma,l} &:= \{(n,k) \in \mathbb{Z}_{\pm} \times \mathbb{Z}_{\pm} : \{n(\beta - \alpha) - \alpha\} \\ &\qquad + \{k(\beta - \alpha) - \alpha\} + \gamma = l\}, \\ (\mathbb{Z}_{\pm} \times \mathbb{Z}_{\pm})_{\gamma,l}^{\sim} &:= \{(n,k) \in \mathbb{Z}_{\pm} \times \mathbb{Z}_{\pm} : \{n(\beta - \alpha)\} + \{k(\beta - \alpha)\} + \gamma = l\}. \end{aligned} \tag{5.7}$$

If (4.6) holds, then

$$\begin{aligned} \widetilde{g}_{2,1}^{+} g_{1,1}^{-} &= \Big(\sum_{n=0}^{\infty} \widetilde{X}_{n,2} e_{\{n(\beta-\alpha)-\alpha\}}\Big)\Big(1 + \sum_{k=0}^{\infty} X_{k,1} e_{\{k(\beta-\alpha)\}-1}\Big), \\ \widetilde{g}_{2,1}^{+} \widetilde{g}_{1,1}^{-} &= \Big(\sum_{n=0}^{\infty} \widetilde{X}_{n,2} e_{\{n(\beta-\alpha)-\alpha\}}\Big)\Big(\sum_{k=0}^{\infty} \widetilde{X}_{k,1} e_{\{k(\beta-\alpha)-\alpha\}-1}\Big), \\ g_{2,1}^{+} g_{1,1}^{-} &= \Big(e_1 + \sum_{n=0}^{\infty} X_{n,2} e_{\{n(\beta-\alpha)\}}\Big)\Big(1 + \sum_{k=0}^{\infty} X_{k,1} e_{\{k(\beta-\alpha)\}-1}\Big), \\ g_{2,1}^{+} \widetilde{g}_{1,1}^{-} &= \Big(e_1 + \sum_{n=0}^{\infty} X_{n,2} e_{\{n(\beta-\alpha)\}}\Big)\Big(\sum_{k=0}^{\infty} \widetilde{X}_{k,1} e_{\{k(\beta-\alpha)-\alpha\}-1}\Big), \end{aligned}$$

which with f given by (4.3) implies, respectively, that

$$\begin{aligned}
\mathbf{M}(\widetilde{g}_{2,1}^{+} f g_{1,1}^{-}) &= \widetilde{c}_{-1}\widetilde{X}_{0,2} + \widetilde{c}_{1}\widetilde{X}_{0,2}X_{0,1},\\
\mathbf{M}(\widetilde{g}_{2,1}^{+} f \widetilde{g}_{1,1}^{-}) &= \sum_{(n,k)\in(\mathbb{Z}_+\times\mathbb{Z}_+)_{\alpha,1}} \widetilde{c}_{1}\widetilde{X}_{n,2}\widetilde{X}_{k,1} + \sum_{(n,k)\in(\mathbb{Z}_+\times\mathbb{Z}_+)_{\alpha,2}} \widetilde{c}_{-1}\widetilde{X}_{n,2}\widetilde{X}_{k,1}\\
&\quad + \sum_{(n,k)\in(\mathbb{Z}_+\times\mathbb{Z}_+)_{\beta,1}} \widetilde{c}_{2}\widetilde{X}_{n,2}\widetilde{X}_{k,1} + \sum_{(n,k)\in(\mathbb{Z}_+\times\mathbb{Z}_+)_{\beta,2}} \widetilde{c}_{-2}\widetilde{X}_{n,2}\widetilde{X}_{k,1},\\
\mathbf{M}(g_{2,1}^{+} f g_{1,1}^{-}) &= \sum_{n\in\widetilde{\mathbb{N}}_{\alpha}^{+}} \widetilde{c}_{-1}(X_{n,1}+X_{n,2}) + \sum_{n\in\widetilde{\mathbb{N}}_{\beta}^{+}} \widetilde{c}_{-2}(X_{n,1}+X_{n,2}) \qquad (5.8)\\
&\quad + \sum_{(n,k)\in(\mathbb{Z}_+\times\mathbb{Z}_+)_{\widetilde{\alpha},1}} \widetilde{c}_{1}X_{n,2}X_{k,1} + \sum_{(n,k)\in(\mathbb{Z}_+\times\mathbb{Z}_+)_{\widetilde{\alpha},2}} \widetilde{c}_{-1}X_{n,2}X_{k,1}\\
&\quad + \sum_{(n,k)\in(\mathbb{Z}_+\times\mathbb{Z}_+)_{\widetilde{\beta},1}} \widetilde{c}_{2}X_{n,2}X_{k,1} + \sum_{(n,k)\in(\mathbb{Z}_+\times\mathbb{Z}_+)_{\widetilde{\beta},2}} \widetilde{c}_{-2}X_{n,2}X_{k,1},\\
\mathbf{M}(g_{2,1}^{+} f \widetilde{g}_{1,1}^{-}) &= \widetilde{c}_{-1}\widetilde{X}_{0,1} + \widetilde{c}_{1}X_{0,2}\widetilde{X}_{0,1}.
\end{aligned}$$

If (4.7) holds, then

$$\widetilde{g}_{2,1}^{+} g_{1,1}^{-} = \left(\sum_{n=-\infty}^{-1} \widetilde{Y}_{n,2} e_{\{n(\beta-\alpha)-\alpha\}}\right)\left(1 + \sum_{k=-\infty}^{0} Y_{k,1} e_{\{k(\beta-\alpha)\}-1}\right),$$

$$\widetilde{g}_{2,1}^{+} \widetilde{g}_{1,1}^{-} = \left(\sum_{n=-\infty}^{-1} \widetilde{Y}_{n,2} e_{\{n(\beta-\alpha)-\alpha\}}\right)\left(\sum_{k=-\infty}^{-1} \widetilde{Y}_{k,1} e_{\{k(\beta-\alpha)-\alpha\}-1}\right),$$

$$g_{2,1}^{+} g_{1,1}^{-} = \left(e_1 + \sum_{n=-\infty}^{0} Y_{n,2} e_{\{n(\beta-\alpha)\}}\right)\left(1 + \sum_{k=-\infty}^{0} Y_{k,1} e_{\{k(\beta-\alpha)\}-1}\right),$$

$$g_{2,1}^{+} \widetilde{g}_{1,1}^{-} = \left(e_1 + \sum_{n=-\infty}^{0} Y_{n,2} e_{\{n(\beta-\alpha)\}}\right)\left(\sum_{k=-\infty}^{-1} \widetilde{Y}_{k,1} e_{\{k(\beta-\alpha)-\alpha\}-1}\right),$$

which with f given by (4.3) implies, respectively, that

$$\begin{aligned}
\mathbf{M}(\widetilde{g}_{2,1}^{+} f g_{1,1}^{-}) &= \widetilde{c}_{-2}\widetilde{Y}_{-1,2} + \widetilde{c}_{2}\widetilde{Y}_{-1,2}Y_{0,1},\\
\mathbf{M}(\widetilde{g}_{2,1}^{+} f \widetilde{g}_{1,1}^{-}) &= \sum_{(n+1,k+1)\in(\mathbb{Z}_-\times\mathbb{Z}_-)_{\alpha,1}} \widetilde{c}_{1}\widetilde{Y}_{n,2}\widetilde{Y}_{k,1} + \sum_{(n+1,k+1)\in(\mathbb{Z}_-\times\mathbb{Z}_-)_{\alpha,2}} \widetilde{c}_{-1}\widetilde{Y}_{n,2}\widetilde{Y}_{k,1}\\
&\quad + \sum_{(n+1,k+1)\in(\mathbb{Z}_-\times\mathbb{Z}_-)_{\beta,1}} \widetilde{c}_{2}\widetilde{Y}_{n,2}\widetilde{Y}_{k,1} + \sum_{(n+1,k+1)\in(\mathbb{Z}_-\times\mathbb{Z}_-)_{\beta,2}} \widetilde{c}_{-2}\widetilde{Y}_{n,2}\widetilde{Y}_{k,1},
\end{aligned}$$

$$\begin{aligned}\mathbf{M}(g_{2,1}^{+}fg_{1,1}^{-}) = &\sum_{n\in\widetilde{\mathbb{N}}_{\alpha}^{-}}\widetilde{c}_{-1}(Y_{n,1}+Y_{n,2}) + \sum_{n\in\widetilde{\mathbb{N}}_{\beta}^{-}}\widetilde{c}_{-2}(Y_{n,1}+Y_{n,2})\\ &+\sum_{(n,k)\in(\mathbb{Z}_-\times\mathbb{Z}_-)_{\alpha,1}^{\sim}}\widetilde{c}_{1}Y_{n,2}Y_{k,1} + \sum_{(n,k)\in(\mathbb{Z}_-\times\mathbb{Z}_-)_{\alpha,2}^{\sim}}\widetilde{c}_{-1}Y_{n,2}Y_{k,1}\\ &+\sum_{(n,k)\in(\mathbb{Z}_-\times\mathbb{Z}_-)_{\beta,1}^{\sim}}\widetilde{c}_{2}Y_{n,2}Y_{k,1} + \sum_{(n,k)\in(\mathbb{Z}_-\times\mathbb{Z}_-)_{\beta,2}^{\sim}}\widetilde{c}_{-2}Y_{n,2}Y_{k,1},\\ \mathbf{M}(g_{2,1}^{+}f\widetilde{g}_{1,1}^{-}) = &\,\widetilde{c}_{-2}\widetilde{Y}_{-1,1}+\widetilde{c}_{2}Y_{0,2}\widetilde{Y}_{-1,1}.\end{aligned} \tag{5.9}$$

If (4.8) holds, then

$$\widetilde{g}_{2,1}^{+}g_{1,1}^{-} = \left(\sum_{n=-\infty}^{-1}\widetilde{Y}_{n,2}e_{\{n(\beta-\alpha)-\alpha\}}\right)\left(1+\sum_{k=0}^{\infty}X_{k,1}e_{\{k(\beta-\alpha)\}-1}\right),$$

$$\widetilde{g}_{2,1}^{+}\widetilde{g}_{1,1}^{-} = \left(\sum_{n=-\infty}^{-1}\widetilde{Y}_{n,2}e_{\{n(\beta-\alpha)-\alpha\}}\right)\left(\sum_{k=0}^{\infty}\widetilde{X}_{k,1}e_{\{k(\beta-\alpha)-\alpha\}-1}\right),$$

$$g_{2,1}^{+}g_{1,1}^{-} = \left(e_1+\sum_{n=-\infty}^{0}Y_{n,2}e_{\{n(\beta-\alpha)\}}\right)\left(1+\sum_{k=0}^{\infty}X_{k,1}e_{\{k(\beta-\alpha)\}-1}\right),$$

$$g_{2,1}^{+}\widetilde{g}_{1,1}^{-} = \left(e_1+\sum_{n=-\infty}^{0}Y_{n,2}e_{\{n(\beta-\alpha)\}}\right)\left(\sum_{k=0}^{\infty}\widetilde{X}_{k,1}e_{\{k(\beta-\alpha)-\alpha\}-1}\right),$$

which with f given by (4.3) implies, respectively, that

$$\begin{aligned}\mathbf{M}(\widetilde{g}_{2,1}^{+}fg_{1,1}^{-}) = &\,\widetilde{c}_{-2}\widetilde{Y}_{-1,2}+\widetilde{c}_{2}\widetilde{Y}_{-1,2}X_{0,1},\\ \mathbf{M}(\widetilde{g}_{2,1}^{+}f\widetilde{g}_{1,1}^{-}) = &\sum_{(n+1,k)\in(\mathbb{Z}_-\times\mathbb{Z}_+)_{\alpha,1}}\widetilde{c}_{1}\widetilde{Y}_{n,2}\widetilde{X}_{k,1} + \sum_{(n+1,k)\in(\mathbb{Z}_-\times\mathbb{Z}_+)_{\alpha,2}}\widetilde{c}_{-1}\widetilde{Y}_{n,2}\widetilde{X}_{k,1}\\ &+\sum_{(n+1,k)\in(\mathbb{Z}_-\times\mathbb{Z}_+)_{\beta,1}}\widetilde{c}_{2}\widetilde{Y}_{n,2}\widetilde{X}_{k,1} + \sum_{(n+1,k)\in(\mathbb{Z}_-\times\mathbb{Z}_+)_{\beta,2}}\widetilde{c}_{-2}\widetilde{Y}_{n,2}\widetilde{X}_{k,1},\\ \mathbf{M}(g_{2,1}^{+}fg_{1,1}^{-}) = &\sum_{k\in\widetilde{\mathbb{N}}_{\alpha}^{+}}\widetilde{c}_{-1}X_{k,1} + \sum_{k\in\widetilde{\mathbb{N}}_{\beta}^{+}}\widetilde{c}_{-2}X_{k,1} + \sum_{n\in\widetilde{\mathbb{N}}_{\alpha}^{-}}\widetilde{c}_{-1}Y_{n,2} + \sum_{n\in\widetilde{\mathbb{N}}_{\beta}^{-}}\widetilde{c}_{-2}Y_{n,2}\\ &+\sum_{(n,k)\in(\mathbb{Z}_-\times\mathbb{Z}_+)_{\alpha,1}^{\sim}}\widetilde{c}_{1}Y_{n,2}X_{k,1} + \sum_{(n,k)\in(\mathbb{Z}_-\times\mathbb{Z}_+)_{\alpha,2}^{\sim}}\widetilde{c}_{-1}Y_{n,2}X_{k,1}\\ &+\sum_{(n,k)\in(\mathbb{Z}_-\times\mathbb{Z}_+)_{\beta,1}^{\sim}}\widetilde{c}_{2}Y_{n,2}X_{k,1} + \sum_{(n,k)\in(\mathbb{Z}_-\times\mathbb{Z}_+)_{\beta,2}^{\sim}}\widetilde{c}_{-2}Y_{n,2}X_{k,1},\\ \mathbf{M}(g_{2,1}^{+}f\widetilde{g}_{1,1}^{-}) = &\,\widetilde{c}_{-1}\widetilde{X}_{0,1}+\widetilde{c}_{1}Y_{0,2}\widetilde{X}_{0,1}.\end{aligned} \tag{5.10}$$

If (4.9) holds, then

$$\widetilde{g}^+_{2,1} g^-_{1,1} = \Big(\sum_{n=0}^{\infty} \widetilde{X}_{n,2} e_{\{n(\beta-\alpha)-\alpha\}} \Big) \Big(1 + \sum_{k=-\infty}^{0} Y_{k,1} e_{\{k(\beta-\alpha)\}-1} \Big),$$

$$\widetilde{g}^+_{2,1} \widetilde{g}^-_{1,1} = \Big(\sum_{n=0}^{\infty} \widetilde{X}_{n,2} e_{\{n(\beta-\alpha)-\alpha\}} \Big) \Big(\sum_{k=-\infty}^{-1} \widetilde{Y}_{k,1} e_{\{k(\beta-\alpha)-\alpha\}-1} \Big),$$

$$g^+_{2,1} g^-_{1,1} = \Big(e_1 + \sum_{n=0}^{\infty} X_{n,2} e_{\{n(\beta-\alpha)\}} \Big) \Big(1 + \sum_{k=-\infty}^{0} Y_{k,1} e_{\{k(\beta-\alpha)\}-1} \Big),$$

$$g^+_{2,1} \widetilde{g}^-_{1,1} = \Big(e_1 + \sum_{n=0}^{\infty} X_{n,2} e_{\{n(\beta-\alpha)\}} \Big) \Big(\sum_{k=-\infty}^{-1} \widetilde{Y}_{k,1} e_{\{k(\beta-\alpha)-\alpha\}-1} \Big),$$

which with f given by (4.3) implies, respectively, that

$$\begin{aligned}
\mathbf{M}(\widetilde{g}^+_{2,1} f g^-_{1,1}) &= \widetilde{c}_{-1} \widetilde{X}_{0,2} + \widetilde{c}_1 \widetilde{X}_{0,2} Y_{0,1}, \\
\mathbf{M}(\widetilde{g}^+_{2,1} f \widetilde{g}^-_{1,1}) &= \sum_{(n,k+1)\in(\mathbb{Z}_+\times\mathbb{Z}_-)_{\alpha,1}} \widetilde{c}_1 \widetilde{X}_{n,2} \widetilde{Y}_{k,1} + \sum_{(n,k+1)\in(\mathbb{Z}_+\times\mathbb{Z}_-)_{\alpha,2}} \widetilde{c}_{-1} \widetilde{X}_{n,2} \widetilde{Y}_{k,1} \\
&\quad + \sum_{(n,k+1)\in(\mathbb{Z}_+\times\mathbb{Z}_-)_{\beta,1}} \widetilde{c}_2 \widetilde{X}_{n,2} \widetilde{Y}_{k,1} + \sum_{(n,k+1)\in(\mathbb{Z}_+\times\mathbb{Z}_-)_{\beta,2}} \widetilde{c}_{-2} \widetilde{X}_{n,2} \widetilde{Y}_{k,1}, \\
\mathbf{M}(g^+_{2,1} f g^-_{1,1}) &= \sum_{n\in\widetilde{\mathbb{N}}^+_\alpha} \widetilde{c}_{-1} X_{n,2} + \sum_{n\in\widetilde{\mathbb{N}}^+_\beta} \widetilde{c}_{-2} X_{n,2} + \sum_{k\in\widetilde{\mathbb{N}}^-_\alpha} \widetilde{c}_{-1} Y_{k,1} + \sum_{k\in\widetilde{\mathbb{N}}^-_\beta} \widetilde{c}_{-2} Y_{k,1} \\
&\quad + \sum_{(n,k)\in(\mathbb{Z}_+\times\mathbb{Z}_-)_{\widetilde{\alpha},1}} \widetilde{c}_1 X_{n,2} Y_{k,1} + \sum_{(n,k)\in(\mathbb{Z}_+\times\mathbb{Z}_-)_{\widetilde{\alpha},2}} \widetilde{c}_{-1} X_{n,2} Y_{k,1} \\
&\quad + \sum_{(n,k)\in(\mathbb{Z}_+\times\mathbb{Z}_-)_{\widetilde{\beta},1}} \widetilde{c}_2 X_{n,2} Y_{k,1} + \sum_{(n,k)\in(\mathbb{Z}_+\times\mathbb{Z}_-)_{\widetilde{\beta},2}} \widetilde{c}_{-2} X_{n,2} Y_{k,1}, \\
\mathbf{M}(g^+_{2,1} f \widetilde{g}^-_{1,1}) &= \widetilde{c}_{-2} \widetilde{Y}_{-1,1} + \widetilde{c}_2 X_{0,2} \widetilde{Y}_{-1,1}.
\end{aligned} \tag{5.11}$$

5.3. Computation of $\mathrm{M}(\widetilde{g}^+_{2,1} f g^-_{1,1})$ and $\mathrm{M}(g^+_{2,1} f \widetilde{g}^-_{1,1})$

Since

$$\widetilde{X}_{0,1} = \widetilde{X}_{0,2} = 1, \quad \widetilde{Y}_{-1,1} = \widetilde{Y}_{-1,2} = 1$$

and

$$X_{0,1} = -\frac{c_{-1,1}}{c_{1,1}}, \quad X_{0,2} = -\frac{c_{-1,2}}{c_{1,2}}, \quad Y_{0,1} = -\frac{c_{-2,1}}{c_{2,1}}, \quad Y_{0,2} = -\frac{c_{-2,2}}{c_{2,2}},$$

we deduce from (5.8), (5.9), (5.10) and (5.11) that

$$\mathbf{M}(\widetilde{g}_{2,1}^{+} f g_{1,1}^{-}) = \begin{cases} \widetilde{c}_{-1}\widetilde{X}_{0,2} + \widetilde{c}_1 \widetilde{X}_{0,2} X_{0,1}, \\ \widetilde{c}_{-2}\widetilde{Y}_{-1,2} + \widetilde{c}_2 \widetilde{Y}_{-1,2} Y_{0,1}, \\ \widetilde{c}_{-2}\widetilde{Y}_{-1,2} + \widetilde{c}_2 \widetilde{Y}_{-1,2} X_{0,1}, \\ \widetilde{c}_{-1}\widetilde{X}_{0,2} + \widetilde{c}_1 \widetilde{X}_{0,2} Y_{0,1}, \end{cases} = \begin{cases} \widetilde{c}_{-1} - \widetilde{c}_1 (c_{-1,1}/c_{1,1}), \\ \widetilde{c}_{-2} - \widetilde{c}_2 (c_{-2,1}/c_{2,1}), \\ \widetilde{c}_{-2} - \widetilde{c}_2 (c_{-1,1}/c_{1,1}), \\ \widetilde{c}_{-1} - \widetilde{c}_1 (c_{-2,1}/c_{2,1}), \end{cases} \tag{5.12}$$

$$\mathbf{M}(g_{2,1}^{+} f \widetilde{g}_{1,1}^{-}) = \begin{cases} \widetilde{c}_{-1}\widetilde{X}_{0,1} + \widetilde{c}_1 X_{0,2} \widetilde{X}_{0,1}, \\ \widetilde{c}_{-2}\widetilde{Y}_{-1,1} + \widetilde{c}_2 Y_{0,2} \widetilde{Y}_{-1,1}, \\ \widetilde{c}_{-1}\widetilde{X}_{0,1} + \widetilde{c}_1 Y_{0,2} \widetilde{X}_{0,1}, \\ \widetilde{c}_{-2}\widetilde{Y}_{-1,1} + \widetilde{c}_2 X_{0,2} \widetilde{Y}_{-1,1}, \end{cases} = \begin{cases} \widetilde{c}_{-1} - \widetilde{c}_1 (c_{-1,2}/c_{1,2}), \\ \widetilde{c}_{-2} - \widetilde{c}_2 (c_{-2,2}/c_{2,2}), \\ \widetilde{c}_{-1} - \widetilde{c}_1 (c_{-2,2}/c_{2,2}), \\ \widetilde{c}_{-2} - \widetilde{c}_2 (c_{-1,2}/c_{1,2}), \end{cases} \tag{5.13}$$

if, respectively, (4.6), (4.7), (4.8) and (4.9) holds.

Substituting $\mathbf{M}(\widetilde{g}_{2,1}^{+} f g_{1,1}^{-})$ and $\mathbf{M}(g_{2,1}^{+} f \widetilde{g}_{1,1}^{-})$ given by (5.12) and (5.13), respectively, and $\mathbf{M}(\widetilde{g}_{2,1}^{+} f \widetilde{g}_{1,1}^{-})$ and $\mathbf{M}(g_{2,1}^{+} f g_{1,1}^{-})$ given by (5.8), (5.9), (5.10) and (5.11) into (5.6) and applying (3.20), we obtain the entries of the matrix $T_{\widetilde{F}}$ (see (5.5)), which together with $\mathbf{d}(\mathcal{G}) = \mathbf{d}(G_{F_0})$ obtained in (5.1)–(5.4) gives $\mathbf{d}(G_F)$ due to (4.1).

5.4. Proof of Theorem 4.2

Under condition (4.10), all the sets

$$\mathbb{N}_\gamma^{\pm},\ \widetilde{\mathbb{N}}_\gamma^{\pm},\ (\mathbb{Z}_\pm \times \mathbb{Z}_\pm)_{\gamma,l},\ (\mathbb{Z}_\pm \times \mathbb{Z}_\pm)_{\gamma,l}^{\sim} \tag{5.14}$$

given by (2.25) and (5.7) are empty. Hence, by (5.1)–(5.4), we infer that

$$\mathbf{d}(\mathcal{G}) = \begin{cases} -\operatorname{diag}\left[c_{-1,1}c_{1,1}^{-1},\ c_{-1,2}c_{1,2}^{-1},\ c_{1,1}c_{-1,1}^{-1},\ c_{1,2}c_{-1,2}^{-1}\right], \\ -\operatorname{diag}\left[c_{-2,1}c_{2,1}^{-1},\ c_{-2,2}c_{2,2}^{-1},\ c_{2,1}c_{-2,1}^{-1},\ c_{2,2}c_{-2,2}^{-1}\right], \\ -\operatorname{diag}\left[c_{-1,1}c_{1,1}^{-1},\ c_{-2,2}c_{2,2}^{-1},\ c_{1,1}c_{-1,1}^{-1},\ c_{2,2}c_{-2,2}^{-1}\right], \\ -\operatorname{diag}\left[c_{-2,1}c_{2,1}^{-1},\ c_{-1,2}c_{1,2}^{-1},\ c_{2,1}c_{-2,1}^{-1},\ c_{1,2}c_{-1,2}^{-1}\right], \end{cases} \tag{5.15}$$

if, respectively, conditions (4.6), (4.7), (4.8) or (4.9) hold. Since the sets (5.14) are empty, it follows from (3.20) and (5.8)–(5.11) that

$$\mathbf{M}(\widetilde{g}_{2,1}^{+}) = \mathbf{M}(g_{2,2}^{+}) = \mathbf{M}(g_{1,2}^{-}) = 0$$

and

$$\mathbf{M}(\widetilde{g}_{2,1}^{+} f \widetilde{g}_{1,1}^{-}) = \mathbf{M}(g_{2,1}^{+} f g_{1,1}^{-}) = 0.$$

Then we deduce from (5.6) that

$$\begin{aligned} t_{2,1} &= -\mathbf{M}(g_{2,1}^{+})\mathbf{M}(\widetilde{g}_{2,1}^{+} f g_{1,1}^{-}) k_2^{-1}, \qquad & t_{2,3} &= 0, \\ t_{4,3} &= \mathbf{M}(\widetilde{g}_{2,2}^{+})\mathbf{M}(g_{2,1}^{+} f \widetilde{g}_{1,1}^{-}) k_1^{-1} k_2^{-1}, & t_{4,1} &= 0. \end{aligned} \tag{5.16}$$

Taking into account the relations $k_s = \mathbf{M}(\widetilde{g}_{s,2}^{-})$ for $s = 1, 2$, applying (3.20) for $\mathbf{M}(g_{2,1}^{+})$, $\mathbf{M}(\widetilde{g}_{2,2}^{+})$ and $\mathbf{M}(\widetilde{g}_{s,2}^{-})$, and using (5.12) and (5.13) for $\mathbf{M}(\widetilde{g}_{2,1}^{+} f g_{1,1}^{-})$ and

$\mathbf{M}(g^+_{2,1} f \widetilde{g}^-_{1,1})$, we infer from (5.16) that

$$t_{2,1} = \begin{cases} -(c_{1,1}\widetilde{c}_{-1} - c_{-1,1}\widetilde{c}_1)c^{-1}_{1,1}c^{-1}_{1,2}, \\ -(c_{2,1}\widetilde{c}_{-2} - c_{-2,1}\widetilde{c}_2)c^{-1}_{2,1}c^{-1}_{2,2}, \\ -(c_{1,1}\widetilde{c}_{-2} - c_{-1,1}\widetilde{c}_2)c^{-1}_{1,1}c^{-1}_{2,2}, \\ -(c_{2,1}\widetilde{c}_{-1} - c_{-2,1}\widetilde{c}_1)c^{-1}_{2,1}c^{-1}_{1,2}, \end{cases} \quad t_{4,3} = \begin{cases} (c_{1,2}\widetilde{c}_{-1} - c_{-1,2}\widetilde{c}_1)c^{-1}_{-1,1}c^{-1}_{-1,2}, \\ (c_{2,2}\widetilde{c}_{-2} - c_{-2,2}\widetilde{c}_2)c^{-1}_{-2,1}c^{-1}_{-2,2}, \\ (c_{2,2}\widetilde{c}_{-1} - c_{-2,2}\widetilde{c}_1)c^{-1}_{-1,1}c^{-1}_{-2,2}, \\ (c_{1,2}\widetilde{c}_{-2} - c_{-1,2}\widetilde{c}_2)c^{-1}_{-2,1}c^{-1}_{-1,2} \end{cases} \tag{5.17}$$

in the cases (4.6), (4.7), (4.8) and (4.9), respectively.

Finally, substituting $t_{2,3} = 0$, $t_{4,1} = 0$ and also $t_{2,1}$ and $t_{4,3}$ given by (5.17) into (5.5) and applying (5.15) and (4.1), we immediately obtain (4.12) with triangular 2×2 matrices T_1 and T_2 given by (4.13)–(4.16).

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