Operator Theory:
Advances and Applications, Vol. 242, 307–310

On a Question by M. Seidel and the Answer by D. Dragičević et al.

Steffen Roch

Dedicated to Prof. António Ferreira dos Santos

Abstract. According to [1], Markus Seidel asked whether certain homomorphisms which identify local algebras can be also viewed as lifting homomorphisms. The authors of [1] give an affirmative answer in the context of concrete Banach algebras. The purpose of this short note is to show that this question *always* has an affirmative answer, with the meaning of "always" explained below.

Mathematics Subject Classification (2010). Primary 65R20; Secondary 46L99, 47N40.

Keywords. Lifting theorem, central localization, strong limit homomorphism.

Introduction. To keep the paper simple and short, we consider the Hilbert space setting only. In principle, all notions can be adapted to the setting of separable reflexive Banach spaces as well, and the basic arguments remain valid in this context.

Given a sequence of separable Hilbert spaces H_n with identity operators I_n, write $\mathcal{F}$ for the set of all bounded sequences $(A_n)_{n\in\mathbb{N}}$ of operators $A_n \in L(H_n)$. Provided with pointwise defined operations and the supremum norm, $\mathcal{F}$ becomes a C^*-algebra, and the set $\mathcal{G}$ of all sequences $(G_n) \in \mathcal{F}$ with $\|G_n\| \to 0$ is a closed ideal of $\mathcal{F}$. A basic task of numerical analysis is, for a given unital C^*-subalgebra $\mathcal{A}$ of $\mathcal{F}$ which contains $\mathcal{G}$, to examine the stability of sequences (A_n) in $\mathcal{A}$ or, equivalently, the invertibility of the coset $(A_n) + \mathcal{G}$ in $\mathcal{F}/\mathcal{G}$.

This task is usually performed in two steps: a lifting step, which provides us with a C^*-subalgebra $\mathcal{F}^{\mathcal{J}}$ of $\mathcal{F}$ which contains $\mathcal{A}$, and with a closed ideal $\mathcal{J}$ of $\mathcal{F}^{\mathcal{J}}$ which contains $\mathcal{G}$, such that the invertibility of a coset $(A_n) + \mathcal{J}$ can be lifted by a family of lifting homomorphisms and such that the quotient $\mathcal{F}^{\mathcal{J}}/\mathcal{J}$ possesses a sufficiently large C^*-algebra $\mathcal{C}$ in its center, and a second, localization, step where localization over $\mathcal{C}$ is employed in order to study invertibility in $\mathcal{F}^{\mathcal{J}}/\mathcal{J}$. The basic tools of these steps, the lifting theorem and the local principle by Allan–Douglas, are briefly described in the following paragraphs.

A basic technical ingredient employed in both steps are *strong limit homomorphisms*. These are defined in terms of a sequence $E := (E_n)$ of isometries E_n on H_n with values in an Hilbert space H^E, i.e., it is $E_n^* E_n = I_n$, and we assume that the projections $E_n E_n^*$ converge strongly to the identity operator I^E on H^E. Note that the latter requirement ensures the separability of H^E. Let $\mathcal{F}^E$ be the set of all sequences $\mathbf{A} = (A_n)$ in $\mathcal{F}$ for which the limits

$$W^E(\mathbf{A}) := \text{s-lim}\, E_n A_n E_n^* \quad \text{and} \quad \text{s-lim}\, E_n A_n^* E_n^*$$

exist in the strong operator topology (in which case we call $(E_n A_n E_n^*)$ a **-strongly convergent* sequence). Then $\mathcal{F}^E$ is a closed unital subalgebra of $\mathcal{F}$, W^E is a unital *-homomorphism from $\mathcal{F}^E$ to $L(H^E)$, and the set

$$\mathcal{J}^E := \{(E_n^* K E_n + G_n) : K \in L(H^E) \text{ compact, } (G_n) \in \mathcal{G}\}$$

is a closed ideal of $\mathcal{F}^E$.

Lifting. Let now $\{E^t\}_{t\in T}$ be a family of sequences $E^t = (E_n^t)$ of isometries as above and with the additional property that

$$E_n^s (E_n^t)^* \to 0 \text{ weakly as } n \to \infty \text{ whenever } s \neq t. \tag{1}$$

For $t \in T$, we put $W_t := W^{E^t}$ and $\mathcal{J}_t := \mathcal{J}^{E^t}$, We further set $\mathcal{F}_T := \cap_{t\in T} \mathcal{F}^{E^t}$ and write $\mathcal{J}_T$ for the smallest closed ideal of $\mathcal{F}_T$ which contains all ideals $\mathcal{J}_t$ with $t \in T$. The condition (1) implies that $\mathcal{J}_s \cap \mathcal{J}_t = \mathcal{G}$ whenever $s \neq t$ (use Theorem 5.51 in [2]). We therefore refer to (1) as the *ideal separation condition.*

Theorem 1 (Lifting theorem). (*Theorems* 5.37 *and* 5.51 *in* [2]). *Let the family* $\{E^t\}_{t\in T}$ *satisfy the ideal separation condition* (1). *Then a sequence* $\mathbf{A} \in \mathcal{F}_T$ *is stable if and only if the operators* $W_t(\mathbf{A})$ *are invertible for every* $t \in T$ *and if the coset* $\mathbf{A} + \mathcal{J}_T$ *is invertible in* $\mathcal{F}_T/\mathcal{J}_T$.

Localization. Let $\mathcal{A}$ be a closed unital C^*-subalgebra of $\mathcal{F}$. In many circumstances, one is able to find a family $\{E^t\}_{t\in T}$ of sequences of isometries which satisfy the ideal separation condition (1) and for which

- $\mathcal{A} \subseteq \mathcal{F}_T$ and
- the quotient algebra $(\mathcal{A} + \mathcal{J}_T)/\mathcal{J}_T$ has a non-trivial center $\mathcal{C}$.

In this setting, it is an evident idea to use central localization in order to study the invertibility of a coset $\mathbf{A} + \mathcal{J}_T$ in $(\mathcal{A} + \mathcal{J}_T)/\mathcal{J}_T$ (equivalently, in $\mathcal{F}_T/\mathcal{J}_T$). The context of the general central localization theorem by Allan and Douglas is as follows. We are given a unital C^*-algebra $\mathcal{B}$ and a C^*-subalgebra $\mathcal{C}$ of the center of $\mathcal{B}$ which contains the unit element. For every maximal ideal s of the commutative C^*-algebra $\mathcal{C}$, let I_s denote the smallest closed ideal of $\mathcal{B}$ which contains s, and write Φ_s for the canonical homomorphism from $\mathcal{B}$ to $\mathcal{B}/I_s$. In this context the following holds.

Theorem 2 (Allan–Douglas). *An element $b \in \mathcal{B}$ is invertible in $\mathcal{B}$ if and only if $\Phi_s(b)$ is invertible in $\mathcal{B}/I_s$ for every maximal ideal s of $\mathcal{C}$.*

The question. Applying the local principle in order to study invertibility in $(\mathcal{A} + \mathcal{J}_T)/\mathcal{J}_T$ requires to study the local algebras $((\mathcal{A} + \mathcal{J}_T)/\mathcal{J}_T)/I_s$, where $s \in S$, the maximal ideal space of $\mathcal{C}$. We suppose that this can be done again by a family of strong limit homomorphisms $\{F^s\}_{s \in S}$, i.e., for every $s \in S$, there is a sequence $F^s = (F_n^s)$ of isometries such that

- $\mathcal{A} + \mathcal{J}_T \subseteq \mathcal{F}^{F^s}$, $\mathcal{J}_T \subseteq \ker W^{F^s}$, and
- the quotient mapping $W^{F^s}/\mathcal{J}_T : \mathbf{A} + \mathcal{J}_T \mapsto W^{F^s}(\mathbf{A})$ has the ideal I_s in its kernel, and the quotient mapping $(W^{F^s}/\mathcal{J}_T)/I_s$ is injective on $((\mathcal{A} + \mathcal{J}_T)/\mathcal{J}_T)/I_s$ (thus, a coset $(\mathbf{A} + \mathcal{J}_T) + I_s$ in $((\mathcal{A} + \mathcal{J}_T)/\mathcal{J}_T)/I_s$ is invertible if and only if the operator $W^{F^s}(\mathbf{A})$ in invertible on H^{F^s}).

We thus have two families of strong limit homomorphisms: one which provides us with the lifting mechanism, and one which identifies local algebras. Markus Seidel's question was if one can include both families into a large family which still satisfies the conditions of the lifting theorem, and how this would affect the structure of the local algebras.

The answer. We formulate the first part of Seidel's question in a slightly more general way. Suppose we are given two families of strong limit homomorphisms: one defined by a family $\{E^t\}_{t \in T}$ of sequences of isometries which satisfies the ideal separation condition (1), and one by a family $\{F^s\}_{s \in S}$ of sequences of isometries with $\mathcal{J}_T \subseteq \ker W^{F^s}$ for all s which satisfies the following *point separation condition*

- for every pair of distinct points s_1, $s_2 \in S$, there is a sequence $\mathbf{A} \in \mathcal{A}$ such that $W^{F^{s1}}(\mathbf{A}) = I$ and $W^{F^{s2}}(\mathbf{A}) = 0$.

It is clear that a family which allows identification of the local algebras satisfies this condition (the algebra of the Gelfand transforms of a commutative C^*-algebra separates the points of the maximal ideal space).

The only thing we have to check is if the family $\{E^t\}_{t \in T} \cup \{F^s\}_{s \in S}$ satisfies the ideal separation condition (1). We employ the following elementary observation for sequences (A_n), (B_n) of bounded linear operators:

1. *If $A_n \to A$ strongly and $B_n \to B$ weakly, then $B_n A_n \to BA$ weakly.*
2. *If $A_n^* \to A^*$ strongly and $B_n \to B$ weakly, then $A_n B_n \to AB$ weakly.*
3. *If $A_n \to 0$ strongly and (B_n) is bounded, then $B_n A_n \to 0$ strongly.*

Let $E = (E_n)$ and $F = (F_n)$ be sequences in $\{E^t\}_{t \in T} \cup \{F^s\}_{s \in S}$.

We distinguish between three cases.

Case 1: $\boldsymbol{E}, \boldsymbol{F} \in \{\boldsymbol{E^t}\}_{\boldsymbol{t \in T}}$. Then (1) holds by assumption.

Case 2: $\boldsymbol{E} \in \{\boldsymbol{E^t}\}_{\boldsymbol{t \in T}}$ and $\boldsymbol{F} \in \{\boldsymbol{F^s}\}_{\boldsymbol{s \in S}}$. Then the ideal $\mathcal{J}^E$ lies in $\ker W^F$ by assumption. Hence, $F_n E_n^* K E_n F_n^* \to 0$ strongly for every compact operator K. This implies the weak convergence of $E_n F_n^*$ to zero as in the proof of Theorem 5.51 in [2]. Taking adjoints we get the weak convergence of $F_n E_n^*$ to zero as well.

Case 3: $\boldsymbol{E}, \boldsymbol{F} \in \{\boldsymbol{F_s}\}_{\boldsymbol{s \in S}}$. By the point separation condition, there is a sequence $\mathbf{A} = (A_n)$ such that $E_n A_n E_n^* \to I$ and $F_n A_n F_n^* \to 0$ *-strongly. In particular, with $Y_n := F_n E_n^*$,

$$E_n A_n E_n^* = E_n (F_n^* F_n) A_n (F_n^* F_n) E_n^* = Y_n^* F_n A_n F_n^* Y_n \to I \tag{2}$$

*-strongly. Since $Y_n Y_n^* = F_n E_n^* E_n F_n^* = F_n F_n^* \to I$ and $F_n A_n F_n^* \to 0$ *-strongly, we conclude that $Y_n Y_n^* F_n A_n F_n^* \to 0$ *-strongly.

Suppose that $Y_n \to Y$ weakly for some operator Y. From Observation 2 and (2) we then conclude that $Y_n Y_n^* F_n A_n F_n^* Y_n \to 0$ weakly. On the other hand, we can use Observation 3 to conclude from (2) that $Y_n Y_n^* F_n A_n F_n^* Y_n - Y_n \to 0$ strongly. Hence, $Y_n \to 0$ weakly. The same argument shows that whenever a subsequence of (Y_n) converges weakly, then it converges weakly to zero.

It remains to show that the sequence (Y_n) indeed converges weakly to 0. Suppose it does not. Then there are vectors $x \in H^E$, $y \in H^F$ and an $\varepsilon > 0$ such that $|\langle Y_{n_k} x, y \rangle| \geq \varepsilon$ for all elements in an (infinite) subsequence (Y_{n_k}) of (Y_n). A standard diagonal argument yields that this subsequence has a weakly convergent subsequence[1], which then converges weakly to zero as shown before. Contradiction.

In particular, we have seen that the point separation property implies the ideal separation property. Thus, whenever the combination of lifting theorem and local principle makes sense, the first part of Seidel's question has an affirmative answer (the general answer to the second part is already in [1]).

References

[1] D. Dragičević, P.A. Santos, and M. Szamotulski, *On a Question by Markus Seidel.* This volume, pp. 159–172.

[2] R. Hagen, S. Roch, and B. Silbermann, *C^*-Algebras and Numerical Analysis.* Marcel Dekker, Inc., New York, Basel 2001.

Steffen Roch
Fachbereich Mathematik
Technische Universität Darmstadt
Schlossgartenstrasse 7
D-64289 Darmstadt, Germany
e-mail: roch@mathematik.tu-darmstadt.de

[1]Choose dense countable subsets $H_E \subseteq H^E$ and $H_F \subseteq H^F$ and let $n \mapsto (x(n), y(n))$ be a bijection from $\mathbb{N}$ onto $H_E \times H_F$. Set $Y_k^{(0)} := Y_{n_k}$. For every $n \geq 1$, choose a subsequence $(Y_k^{(n)})_{k \geq 1}$ of $(Y_k^{(n-1)})_{k \geq 1}$ such that $\langle Y_k^{(n)} x(n), y(n) \rangle$ converges as $k \to \infty$. Set $Z_k := Y_k^{(k)}$. Then (Z_k) is the desired subsequence.