Operator Theory:
Advances and Applications, Vol. 242, 279–306

Inequalities and Convexity

Lars-Erik Persson and Natasha Samko

Dedicated to Professor António Ferreira dos Santos

Abstract. It is a close connection between various kinds of inequalities and the concept of convexity. The main aim of this paper is to illustrate this fact in a unified way as an introduction of this area. In particular, a number of variants of classical inequalities, but also some new ones, are derived and discussed in this general frame.

Mathematics Subject Classification (2010). Primary 26D10, 26D15, 26A51, 39B62, 46B70.

Keywords. Inequalities, convexity, Jensen's inequality, interpolation theory.

1. Introduction

Different kinds of inequalities are very important in various areas of mathematics and its applications. Today the knowledge about inequalities has been developed to be an independent area with many papers, Journals, conferences and books (see, e.g., [6], [13], [18], [20], [21], and the references given there). Moreover, there are also some books fully or partly devoted to convexity techniques (see, e.g., [14], [22], [28], and the references given there). These areas are of independent interest but there are also a huge numbers of examples how these subjects have supported each other in the further developments of these areas but also of other areas within mathematical sciences and even in other more applied areas.

Already G.H. Hardy, J.E. Littlewood and G. Polya in their classical book [13] clearly understood the crucial role of convexity to develop the theory of inequalities. Our intention with this paper is to complement and on some points further develop the content of this book. The main idea is to further explain and use the crucial role of convexity (Jensen's inequality), to further develop and explain the rich area of inequalities in an elementary and unified way.

The work was partially supported by CEAF under FCT project PEst-OE/MAT/UI4032/2011.

As an example of the importance of inequalities we mention the following classical sentence by G.H. Hardy in his Presidental address at the meeting of the London Mathematical Society in November 8, 1928: “All analysts spend half their time hunting through the literature for inequalities which they want to use and cannot prove”. Our hope and main aim is that this paper can help these researchers to find what they are looking for, e.g., by directly finding the inequality and if not to give powerful ideas and hints to be able to derive even inequalities not explicitly stated in our paper.

It is maybe a matter of fate that Hardy himself never discovered that also his famous inequality (6.1) (after a simple substitution) in fact follows directly from Jensen’s inequality. Moreover, also his first powerweighted versions of his inequality is a consequence of the same simple technique. And maybe even more remarkable is that “all” powerweighted Hardy inequalities (for $p = q$) are, in fact, equivalent to the simple inequality

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(x)\right)^p \frac{dx}{x} \le \int_0^\infty f^p(x)\frac{dx}{x}, p \ge 1, \tag{1.1}$$

which easily follows from Jensen’s inequality. Moreover, also a number of other classical inequalities (including those by Carleman, Pólya–Knopp etc.) follows directly from this fact.

We pronounce that all inequalities derived in this paper are sharp.

The content is organized as follows: In Section 2 we point out a number of elementary inequalities which follow more or less directly from convexity (Young’s inequality. Clarcson’s discrete inequality, two fundamental inequalities [6], etc.). In Section 3 we reformulate Jensen’s inequality as an equivalence theorem connected to the concept of convexity. In the next sections the following inequalities are derived and analyzed by using convexity arguments from Sections 2–3.

Section 4: Classical Hölder’s inequality and various variants of this inequality (including a version for infinite many Lebesgue spaces).

Section 5: Classical Minkowski’s inequality and various variants of this inequality (including an integral version of Fubini type).

Section 6: Some classical inequalities (by Hardy, Carleman and Pólya–Knopp).

It is also pointed out that all these inequalities (via substitutions and limit arguments) can be derived from the same basic inequality (1.1), which, in turn, follows from Jensen’s inequality. In particular, these calculations show that “all” powerweighted Hardy inequalities are in fact equivalent because they are equivalent to this basic inequality.

Section 7: Some more Hardy type inequalities including variants with finite intervals involved (a precise equivalence result is proved, and thus improving and making the statements in Section 6 more clear).

Section 8 is reserved for some further results and final remarks. It is shortly mentioned how also interpolation theory is closely related to the concept of convexity. This fact is further explained and developed in [25]. As an example there we just derive Young’s integral inequality (including a limit case) via interpolation and convexity. We also mention a fairly new idea that Jensen's inequality can be "refined" if convex functions are replaced by superquadratic functions. We point out that in particular our technique and this fact implies a new refined Hardy type inequality with "breaking point" $p = 2$ (for $p = 2$ we even get a new integral identity). This is in contrast to usual Hardy type inequalities where the "breaking point" is $p = 1$.

2. Convexity – some elementary inequalities

Let I denote a finite or infinite interval on $\mathbb{R}_+$. We say that a function f is convex on I if, for $0 < \lambda < 1$, and all $x, y \in I$,

$$f((1-\lambda)x + \lambda y) \leq (1-\lambda)f(x) + \lambda f(y).$$

If the inequality holds in the reversed direction, then we say that the function f is concave.

Moreover, we say that the function f is midpoint convex on I if, for all $x, y \in I$,

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2}.$$

There are many well-known facts concerning convex functions, see, e.g., the book [22] by C.P. Niculescu and L.E. Persson. Here we just mention a few introductory but useful facts:

* It follows directly from the definition of convexity that if f is convex on $I = [a, b]$, then, for all $x \in [a, b]$,

$$f(x) \leq f(a) + \frac{f(b) - f(a)}{b - a}(x - a). \tag{2.1}$$

* Assume that $f(x)$ is continuous on I. Then
  a) f is convex if and only if it is midpoint convex,
  b) f is convex if and only if

$$f(x+h) + f(x-h) - 2f(x) \geq 0.$$

Example 1. The function $f(x) = x^p, p \geq 1, x \geq 1$, is an elementary example of a convex function and as we will see later on this simple fact implies, e.g., the Hardy inequality (see (6.1)). And since this function is convex also when $p < 0$ this inequality holds also for $p < 0$, a fact which was also not noted by Hardy himself. Another elementary example of convex function is $f(x) = e^x$ and by using this function in a similar way we obtain a "trivial" proof of the Pólya–Knopp inequality (6.4), see Examples 23 and 25, and Remark 6.4.

But there are also many non-trivial examples of convex functions which have been important for the development. We will here only present one example which we will use later, namely the following one by M. Riesz which was crucial when he proved his convexity theorem, which was very important when interpolation theory was initiated via the famous Riesz–Thorin interpolation theorem, see, e.g., the book [7] by J. Bergh and J. Löfström.

Example 2. Let a and b be complex numbers. Then the function

$$f(\alpha,\beta)=\log\max\frac{\left(|a+b|^{1/\alpha}+|a-b|^{1/\alpha}\right)^{\alpha}}{\left(|a|^{1/\beta}+|b|^{1/\beta}\right)^{\beta}}$$

is convex on the triangle $T: 0\le\alpha\le\beta\le 1$.

We shall now present some useful elementary inequalities, which follow directly from convexity, sometimes combined by some other argument from calculus.

Example 3. Let $a,b>0$. Then

$$a^p+b^p\le(a+b)^p\le 2^{p-1}(a^p+b^p),\ \ p\ge 1,$$
$$2^{p-1}(a^p+b^p)\le(a+b)^p\le a^p+b^p, 0<p\le 1.$$

Proof. The function $f(u)=u^p$ is convex when $p\ge1$ and concave when $0<p<1$. Hence,

$$\text{if } p\ge 1, \text{then} \left(\frac{a+b}{2}\right)^p\le\frac{a^p+b^p}{2},\ \text{ i.e., } (a+b)^p\le 2^{p-1}(a^p+b^p),$$

and

$$\text{if } 0<p<1, \text{then} \left(\frac{a+b}{2}\right)^p\ge\frac{a^p+b^p}{2},\ \text{ i.e., } (a+b)^p\ge 2^{p-1}(a^p+b^p).$$

We may without loss of generality assume that $b\le a$. Consider the function

$$f(t)=(1+t)^p-t^p, [t=b/a]$$

and note that

$$f'(t)=p(1+t)^{p-1}-pt^{p-1}.$$

Hence $f'(t)\ge 0\ for\ p\ge 1$ and $f'(t)\le 0$ for $0<p<1$. Moreover, $f(0)=1$ and we conclude that

$$\begin{aligned} f(t)\ge 0 \text{ for } p\ge 1 &\Leftrightarrow \left(1+\frac{b}{a}\right)^p-1-\left(\frac{b}{a}\right)^p\ge 0 \text{ for } p\ge 1\\ &\Leftrightarrow a^p+b^p\le(a+b)^p\ for\ p\ge 1,\end{aligned}$$

and

$$\begin{aligned} f(t)\le 0 \text{ for } 0<p\le 1 &\Leftrightarrow \left(1+\frac{b}{a}\right)^p-1-\left(\frac{b}{a}\right)^p\le 0\ for\ 0<p<1\\ &\Leftrightarrow (a+b)^p\le a^p+b^p\ for\ 0<p<1.\end{aligned}$$

□

Remark 2.1. The inequalities in Example 3 can be unified as follows

$$c_1(a^p+b^p)\le(a+b)^p\le c_2(a^p+b^p), p>0, \tag{2.2}$$

where $c_1=\min\{2^{p-1},1\}$ and $c_2=\max\{2^{p-1},1\}$. When (2.2) holds for some positive numbers c_1 and c_2, we also write $(a+b)^p\approx(a^p+b^p)$. This equivalence notion can be generalized to more general situations in a natural way.

By using induction we can generalize Example 3 as follows:

Example 4. Let $a_1, a_2, \dots, a_n$ be positive numbers. Then

$$\text{(a)} \qquad \sum_{i=1}^{n} a_i^p \leq \left(\sum_{i=1}^{n} a_i\right)^p \leq n^{p-1} \sum_{i=1}^{n} a_i^p, \quad p \geq 1,$$

$$\text{(b)} \qquad n^{p-1} \sum_{i=1}^{n} a_i^p \leq \left(\sum_{i=1}^{n} a_i\right)^p \leq \sum_{i=1}^{n} a_i^p, \quad 0 < p \leq 1.$$

Example 5. (Two fundamental inequalities). If $x > 0$ and $\alpha \in \mathbb{R}$, then

$$\begin{cases} x^\alpha - \alpha x + \alpha - 1 \geq 0 & \text{for} \quad \alpha > 1 \text{ and } \alpha < 0 \\ x^\alpha - \alpha x + \alpha - 1 \leq 0 & \text{for} \quad 0 < \alpha < 1. \end{cases} \tag{2.3}$$

Remark 2.2. In the book [6] E.F. Beckenbach and R. Bellman called (2.3) "A fundamental relationship" (see page 12). In particular, they showed later in the book that several well-known inequalities follow directly from (2.3), e.g., the AG-inequality, Hölder's inequality, Minkowski's inequality, etc. In [6] it was given two different proofs of (2.3) but in view of the main argument in this paper we mention another "proof" namely that (2.3) follows directly from the fact that the function $f(x) = x^\alpha$ is convex for $\alpha > 1$ and $\alpha < 0$ and concave for $0 < \alpha < 1$. In fact, if $f(x) = x^\alpha$, then the equation for the tangent at $x = 1$ is equal to $y = \alpha(x-1)+1$ and (2.3) follows directly. Moreover, this proof shows that we have equality in both inequalities in (2.3) if and only if $x = 1$ for all α.

Example 6 (Discrete Young inequality). For any $a, b > 0, p, q \in \mathbb{R} \setminus \{0\}, \frac{1}{p} + \frac{1}{q} = 1$, it yields that

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \text{ if } p > 1 \tag{2.4}$$

and

$$ab \geq \frac{a^p}{p} + \frac{b^q}{q}, \text{ if } p < 1, p \neq \{0\}. \tag{2.5}$$

Proof. In fact, (2.4) follows directly from (2.3) applied with $x = \frac{a}{b}$ and $\alpha = \frac{1}{p}$ (the case $0 < \alpha < 1$) and (2.5) follows from (2.3) in the same way by instead applying (2.3) in the cases $\alpha > 1$ and $\alpha < 0$. □

Remark 2.3. Another proof of (2.4) is obtained by directly using the fact that $f(x) = e^x$ is convex:

$$ab = e^{\ln ab} = e^{\frac{1}{p}\ln a^p + \frac{1}{q}\ln b^q} \leq \frac{1}{p}e^{\ln a^p} + \frac{1}{q}e^{\ln b^q} = \frac{1}{p}a^p + \frac{1}{q}b^q.$$

By using the same argument and induction (cf. also Proposition 3.1) we obtain the following generalization of Young's inequality (2.4): Let $a_i > 0, p_i > 1, n =$

$1, 2, \ldots, n, n \in \mathbb{Z}_+, n \geq 2, \sum_{i=1}^{n} \frac{1}{p_i} = 1$. Then

$$\prod_{i=1}^{n} a_i \leq \sum_{i=1}^{n} \frac{1}{p_i} a_i^{p_i}.$$

It seems not to be possible to derive a similar generalization of (2.5).

Example 7 (A generalization of (2.3)). Our simple proof of (2.3) gives directly the following more general result: Let $\Phi(x)$ be a convex function on $\mathbb{R}_+$, which is differentiable at $x = 1$. Then

$$\Phi(x) - \Phi'(1)x + \Phi'(1) - \Phi(1) \geq 0. \tag{2.6}$$

If instead $\Phi(x)$ is concave, then (2.6) holds in the reverse direction.

Another way to understand and generalize (2.4) is as follows:

Example 8 (Generalized discrete Young inequality). Let $\Phi(x)$ be a continuous and strictly increasing function for $x \geq 0$ and $\Phi(0) = 0$. The inverse of Φ is denoted Ψ (draw the figure of the situation). By examining the areas in this figure we see that

$$ab \leq \int_0^a \Phi(x)dx + \int_0^b \Psi(y)dy. \tag{2.7}$$

Inequality (2.4) is obtained by applying (2.7) with $\Phi(x) = x^{p-1}, p > 1$. This argument also shows that we have equality in (2.7) exactly when $b = \Phi(a)$, in particular we have equality in (2.4) exactly when $b = a^{p-1}$. It seems not to be possible to have some similar generalization of inequality (2.5).

We finish this section by showing that also another useful inequality follows from convexity via Example 2.

Example 9 (Discrete Clarkson inequality). Let $a, b \in \mathbb{R}, 1 < p \leq 2$ and $q = \frac{p}{p-1}$. Then

$$\left(|a+b|^q + |a-b|^q\right)^{1/q} \leq 2^{1/q} \left(|a|^p + |b|^p\right)^{1/p}. \tag{2.8}$$

The inequality is sharp, i.e., $2^{1/q}$ can not be replaced by any smaller number.

Proof. Consider the convex function $f(\alpha, \beta)$ defined in Example 2. By using the parallelogram law

$$|a+b|^2 + |a-b|^2 = 2(|a|^2 + |b|^2) \tag{2.9}$$

we see that $f(1/2, 1/2) = \frac{1}{2} \log 2$. Moreover, we easily find that $f(0+, 0+) = \log 2$ and $f(0+, 1-) = 0$.

The linear function $g(\alpha, \beta)$ that coincides with $f(\alpha, \beta)$ at the points $(1/2, 1/2)$, $(0, 0)$ and $(0, 1)$ is $g(\alpha, \beta) = (1 - \beta) \log 2$. Moreover, the convexity implies that $f(\alpha, \beta) \leq g(\alpha, \beta) = (1 - \beta) \log 2$, and by choosing $\beta = 1/p, 1 \leq p \leq 2$, and $\alpha = 1 - \beta = 1/q$ we obtain that

$$\log \frac{\left(|a+b|^q + |a-b|^q\right)^{1/q}}{\left(|a|^p + |b|^p\right)^{1/p}} \leq \frac{1}{q} \log 2 = \log 2^{1/q}$$

and (2.8) is proved. Finally, we note that by putting $a = b$ in (2.8) we have equality in (2.8). The proof is complete. $\square$

Remark 2.4. We see that in the case $p = q = 2$, we have even equality in (2.8) via the parallelogram law (2.9) and in the other extreme case when $p \to 1$ we have

$$\max(|a+b|, |a-b|) \leq (|a| + |b|), \tag{2.10}$$

which is just the triangle inequality. Hence, (2.8) is just some "interpolated" inequality between these two extreme cases. This argument can be formalized to a formal proof by considering the operator $T : (a, b) \to (a + b, a - b)$ and note that it maps $\ell_2^2 \to \ell_2^2$ with norm $\sqrt{2}$ (see (2.9)) and $\ell_1^2 \to \ell_\infty^2$ with norm 1 (see (2.10)) and the usual Riesz–Thorin interpolation theorem (see [7]) gives the result.

Remark 2.5. Inequality (2.8) is fundamental for proving Clarkson type inequalities and its applications to uniformly convex spaces. Moreover, by combining (2.8) with other convexity inequalities we obtain more general inequalities, which are also important for applications.

3. Convexity = Jensen's inequality

Proposition 3.1. (*Discrete Jensen inequality*). *Let $n \in \{2, 3, \dots\}$ and let $a = \{a_k\}_1^n$ be a sequence of positive real numbers. If $\Phi(x)$ is convex on an interval including a, then, for $\lambda_k > 0, \sum_{k=1}^n \lambda_k = 1$, it yields that*

$$\Phi\left(\sum_{k=1}^n \lambda_k a_k\right) \leq \sum_{k=1}^n \lambda_k \Phi(a_k). \tag{3.1}$$

Proof. In fact, for $n = 2$ it is just the definition of convexity and for $n = 3$ it follows by using the definition two times:

$$\begin{aligned}
\Phi(\lambda_1 a_1 + \lambda_2 a_2 + \lambda_1 a_3) &= \Phi\left(\lambda_1 a_1 + (\lambda_1 + \lambda_2)\left[\frac{\lambda_2}{\lambda_2 + \lambda_3} a_2 + \frac{\lambda_3}{\lambda_2 + \lambda_3} a_3\right]\right) \\
&\leq \lambda_1 \Phi(a_1) + (\lambda_2 + \lambda_3)\Phi\left(\frac{\lambda_2}{\lambda_2 + \lambda_3} a_2 + \frac{\lambda_3}{\lambda_2 + \lambda_3} a_3\right) \\
&\leq \lambda_1 \Phi(a_1) + \lambda_2 \Phi(a_2) + \lambda_3 \Phi(a_3).
\end{aligned}$$

The proof follows by repeating this argument and formalize it via induction. $\square$

Of course the above argument shows that in fact the discrete Jensen inequality is equivalent to the definition of convexity. We shall now continue by reformulating the classical Jensen inequality

$$\Phi\left(\int_\Omega f d\mu\right) \leq \int_\Omega \Phi(f) d\mu, \tag{3.2}$$

where $\mu(\Omega) = 1$, as a more general form of such an equivalence statement.

Here and in the sequel we let μ denote a positive measure on a σ-algebra in a set Ω.

Theorem 3.2. *Let f be a real μ-measurable function on Ω such that $-\infty \leq a < f(x) < b \leq +\infty$ for all $x \in \Omega$ and Φ be a function on $I = (a, b)$. Then the following conditions are equivalent:*

(i) Φ *is convex,*

(ii) *the inequality*

$$\Phi\left(\frac{1}{\mu(\Omega)}\int_\Omega f d\mu\right) \leq \frac{1}{\mu(\Omega)}\int_\Omega \Phi(f) d\mu \tag{3.3}$$

holds for all measures such that $0 < \mu(\Omega) < \infty$.

Proof. (ii)⇒(i): Apply (3.3) with the measure μ defined as the point mass $1 - \lambda, (0 < \lambda < 1)$ at x and λ at y for $x, y \in I$ and we find by (3.3) that

$$\Phi((1-\lambda)x + \lambda y) \leq (1-\lambda)\Phi(x) + \lambda\Phi(y),$$

i.e., Φ is convex.
(i)⇒(ii): It is obviously sufficient to prove (3.3) with the restriction $\mu(\Omega) = 1$, i.e., that (3.2) holds. First we note that since Φ is convex (cf. (2.1)) it yields that

$$\frac{\Phi(t) - \Phi(s)}{t - s} \leq \frac{\Phi(u) - \Phi(t)}{u - t} \tag{3.4}$$

whenever $a < s < t < u < b$.

Let $t = \int_\Omega f d\mu$. Then $a < t < b$.

Put $\beta =$ supremum of all quotients to the left in (3.4) for fixed $u \in (t, b)$.

Hence $\frac{\Phi(t)-\Phi(s)}{t-s} \leq \beta$ so that $\Phi(s) \geq \Phi(t) + \beta(s - t)$.

Thus, for all $x \in \Omega$ (with $s = f(x)$) it yields that

$$\Phi(f(x)) - \Phi(t) - \beta(f(x) - t) \geq 0.$$

We integrate and get that

$$\begin{aligned}
&\int_\Omega \Phi(f(x))d\mu - \int_\Omega \Phi(t)d\mu - \beta\int_\Omega (f(x) - t)d\mu \\
&\quad = \int_\Omega \Phi(f(x))d\mu - \Phi(t) - \beta\int_\Omega f(x)d\mu + \beta\int_\Omega t d\mu \\
&\quad = \int_\Omega \Phi(f(x))d\mu - \Phi\left(\int_\Omega f d\mu\right) - \beta\int_\Omega f(x)d\mu + \beta\int_\Omega f(x)d\mu \\
&\quad = \int_\Omega \Phi(f(x))d\mu - \Phi\left(\int_\Omega f d\mu\right) \geq 0.
\end{aligned}$$

The proof is complete. □

Remark 3.3. The arguments in the proof of (i)⇒(ii) are the same as those in most Functional Analysis books but the formulation of Theorem 3.2 as an equivalence theorem is important and done for our further purposes.

Our proof of Theorem 3.2 shows that we also have the following characterization of concave functions:

Theorem 3.4. *Let f and Φ be defined as in Theorem* 3.2. *Then the following conditions are equivalent:*

(iii) Φ *is concave,*

(iv) *the inequality* (3.3) *holds in the reversed direction for all measures μ such that* $0 < \mu(\Omega) < \infty$.

Remark 3.5. If $\Omega = \mathbb{R}_+, n = 2, 3, \ldots,\ \mu = \sum_{k=1}^{n} \lambda_k \delta_k$ (δ_k is the unity mass at $t = k$), $\lambda_k > 0$ and $\sum_{k=1}^{n} \lambda_k = 1$, then Jensen's inequality (3.2) coincides with the discrete Jensen inequality (3.1), with $f(k) = a_k$. Moreover, if Φ is concave, then Theorem 3.4 shows that (3.1) holds in the reversed direction.

The original forms of Jensen's inequality traces back to his original papers [15] and [16] from 1905–06.

4. Various variants of Hölder's inequality via convexity

As usual, the space $L_p = L_p(\mu), 0 < p \leq \infty$, consists of all functions f on Ω such that

$$\|f\|_{L_p} := \left(\int_\Omega |f|^p d\mu \right)^{1/p} < \infty, \text{ if } 0 < p < \infty,$$

and

$$\|f\|_{L_\infty} := \operatorname*{ess\,sup}_{x \in \Omega} |f(x)| \quad < \infty, \text{ if } p = \infty.$$

Example 10 (Hölder's inequality). Let $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$\|fg\|_{L_1} \leq \|f\|_{L_p} \|g\|_{L_q},$$

i.e.,

$$\int_\Omega |fg| d\mu \leq \left(\int_\Omega |f|^p d\mu \right)^{1/p} \left(\int_\Omega |g|^q d\mu \right)^{1/q}. \tag{4.1}$$

Proof 1. First we let $\|f_0\|_{L_p} = \|g_0\|_{L_q} = 1$ and use convexity of the exponential function via Young's inequality (2.4) and find that

$$\int_\Omega |f_0 g_0| d\mu \leq \frac{1}{p} \int_\Omega |f_0|^p d\mu + \frac{1}{q} \int_\Omega |g_0|^q d\mu = \frac{1}{p} \cdot 1 + \frac{1}{q} \cdot 1 = 1.$$

Apply this inequality with $f_0 = \frac{f}{\|f\|_{L_p}}$ and $g_0 = \frac{g}{\|g\|_{L_q}}$ and (4.1) is proved. □

Another even more direct convexity proof is the following one:

Proof 2. We may without loss of generality assume that $0 < \int_\Omega |g| d\mu < \infty$ and apply Jensen's inequality (3.2) with the convex function $\Phi(u) = u^p$ to obtain that

$$\left(\frac{1}{\int_\Omega |g| d\mu} \int_\Omega |fg| d\mu \right)^p \leq \left(\int_\Omega |g| d\mu \right)^{-1} \int_\Omega |f|^p |g| d\mu,$$

i.e., that

$$\int_\Omega |fg|d\mu \leq \left(\int_\Omega |g|d\mu\right)^{1-1/p} \left(\int_\Omega |f|^p|g|d\mu\right)^{1/p}.$$

Put $|f||g|^{1/p} = |f_1|$ and $|g|^{1/q} = |g_1|$ and we find that

$$\int_\Omega |f_1g_1|d\mu \leq \left(\int_\Omega |f_1|^p d\mu\right)^{1/p} \left(\int_\Omega |g_1|^q d\mu\right)^{1/q}.$$

We just change notation and (4.1) is proved. □

Remark 4.1. It is easy to use the first proof to find (all) cases of equality in Hölder's inequality namely when $g(x) = (f(x))^{p-1}$ (see Example 8). In particular, this means that the following important relation

$$\left(\int_\Omega |f|^p d\mu\right)^{1/p} = \sup \int_\Omega |f|\varphi d\mu, \tag{4.2}$$

yields for each $p > 1$, where supremum is taken over all $\varphi \geq 0$ such that $\int_\Omega \varphi^q d\mu = 1$. This technique is example of a technique called quasi-linearization.

Example 11 (Hölder's inequality – the reversed form). Let $\frac{1}{p} + \frac{1}{q} = 1, 0 < p < 1$. Then

$$\int_\Omega |fg|d\mu \geq \left(\int_\Omega |f|^p d\mu\right)^{1/p} \left(\int_\Omega |g|^q d\mu\right)^{1/q}. \tag{4.3}$$

Proof. Note that the function $\Phi(u) = u^p$ is convex also for $p < 0$. Therefore as in the second proof of Hölder's inequality we find that (with the same notation)

$$\left(\int_\Omega |f_1g_1|d\mu\right)^p \leq \int_\Omega |f_1|^p d\mu \left(\int_\Omega |g_1|^q d\mu\right)^{p-1}.$$

Hence

$$\int_\Omega |f_1g_1|d\mu \geq \left(\int_\Omega |f_1|^p d\mu\right)^{1/p} \left(\int_\Omega |g_1|^q d\mu\right)^{1/q}$$

for $p < 0$ (and $0 < q < 1$) and, hence, by interchanging the completely symmetric roles of p and q and change notation we obtain (4.3) and the proof is complete. □

We shall now formulate a more general result, which includes Examples 10 and 11 as special cases.

Example 12 (Hölder's inequality – completely symmetric form). Let p, q and r be real numbers $\neq 0$ such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. Then

$$\text{(a)} \qquad \left(\int_\Omega |fg|^r d\mu\right)^{\frac{1}{r}} \leq \left(\int_\Omega |f|^p d\mu\right)^{1/p} \left(\int_\Omega |g|^q d\mu\right)^{1/q}$$

if $p > 0, q > 0, r > 0$ or $p < 0, q > 0, r < 0$ or $p > 0, q < 0, r < 0$, and

$$\text{(b)} \qquad \left(\int_\Omega |fg|^r d\mu\right)^{\frac{1}{r}} \geq \left(\int_\Omega |f|^p d\mu\right)^{1/p} \left(\int_\Omega |g|^q d\mu\right)^{1/q}$$

if $p > 0, q < 0, r > 0$ or $p < 0, q > 0, r > 0$ or $p < 0, q < 0, r < 0$. (Whenever some parameter is negative we assume that the involved function is strictly positive.)

Proof. The case $\underline{p > 0, q > 0, r > 0.}$ First we note that the convexity of the function $f(u) = e^u$ implies that

$$\begin{aligned}|f(t)g(t)|^r &= \exp(r(\ln|f(t)| + \ln|g(t)|)) \\ &= \exp\left(\frac{r}{p}\ln|f(t)|^p + \frac{r}{q}\ln|g(t)|^q\right) \le \frac{r}{p}|f(t)|^p + \frac{r}{q}|g(t)|^q.\end{aligned}$$

By now integrating and discussing as in the proof of the special case $r = 1$ (see Proof 1 of Example 10) we obtain (a).

The case $\underline{p > 0, q < 0, r > 0.}$ By using the estimate we just have proved we find that

$$\begin{aligned}\left(\int_\Omega |f(t)|^p d\mu\right)^{1/p} &= \left(\int_\Omega |f(t)g(t)|^p \frac{1}{|g(t)|^p} d\mu\right) \\ &\le \left(\int_\Omega |f(t)g(t)|^r d\mu\right)^{1/r} \left(\int_\Omega \left|\frac{1}{g(t)}\right|^{-q} d\mu\right)^{1/-q},\end{aligned}$$

i.e.,

$$\left(\int_\Omega |f(t)|^p d\mu\right)^{1/p} \left(\int_\Omega |g(t)|^q d\mu\right)^{1/q} \le \left(\int_\Omega |f(t)g(t)|^r d\mu\right)^{1/r},$$

which means that (b) holds.

By symmetry we see that (b) holds also for the case $\underline{p < 0,\ q > 0,\ r > 0.}$

For the cases $\underline{p < 0, q > 0, r < 0.}$ and $\underline{p > 0, q < 0, r < 0,}$ we use the obtained results with r, p, q replaced by $-r, -p, -q$, respectively, and obtain that

$$\begin{aligned}\left(\int_\Omega |f(t)g(t)|^r d\mu\right)^{1/r} &= \left(\int_\Omega \left|\frac{1}{f(t)g(t)}\right|^{-r} d\mu\right)^{1/-r} \\ &\le \left(\int_\Omega \left|\frac{1}{f(t)}\right|^{-p} d\mu\right)^{1/-p} \Big/ \left(\int_\Omega \left|\frac{1}{g(t)}\right|^{-q} d\mu\right)^{1/-q},\end{aligned}$$

which means that (a) holds.

The proof of the case $\underline{p < 0, q < 0, r < 0}$ is similar. □

Another well-known generalization of Example 10 is the following:

Example 13 (Hölder's inequality for $n-L^p$ spaces). Let $p_1, p_2, \ldots, p_n, n = 2, 3, \ldots,$ be positive numbers such that $\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_n} = 1$. Then

$$\int_\Omega |f_1 f_2 \cdots f_n| d\mu \le \left(\int_\Omega |f_1|^{p_1} d\mu\right)^{1/p_1} \cdots \left(\int_\Omega |f_n|^{p_n} d\mu\right)^{1/p_n}. \tag{4.4}$$

The proof follows by just using (4.1) and induction or by using directly the discrete Jensen inequality (3.1) and discussing as in Proof 1 of (4.1).

Remark 4.2. Note that if we put $1/p = \theta, 0 < \theta < 1$, and replace $|f|$ by $|f|^\theta$ and $|g|$ by $|g|^{1-\theta}$, then Hölder's inequality (4.1) can be written

$$\int_\Omega |f|^\theta |g|^{1-\theta} d\mu \le \left(\int_\Omega |f| d\mu\right)^\theta \left(\int_\Omega |g| d\mu\right)^{1-\theta}, \tag{4.5}$$

where $0 < \theta < 1$.

Remark 4.3. If a and b are positive numbers, then the number $a^{1-\theta} b^\theta, 0 < \theta < 1$, is a geometric type mean of the numbers a and b (for $\theta = 1/2$ we have the usual geometric mean). Moreover, the geometric mean of a positive function f over an interval $[0, b]$ is defined as follows

$$G_f := \exp\left(\frac{1}{b}\int_0^x \ln f(t) dt\right).$$

Accordingly to the Remarks 4.2 and 4.3 it is tempting to think that Example 14 can be generalized to the case with infinite many L^p spaces (cf. (4.7) below) and in fact this is also true. The reader shall here think of that the functions $f_t(x), t \in (0, b)$, belongs to the space $L_{p(t)}$, where $p(t)$ is sufficiently "smooth" so the involved integrals make sense.

Example 14 (A Hölder inequality for infinite many functions involved). Let $p(t)$ be positive on $[0, b]$ and let p be defined by

$$\frac{1}{p} = \frac{1}{b}\int_0^b \frac{1}{p(t)} dt. \tag{4.6}$$

Then

$$\begin{aligned} &\left(\int_\Omega \left(\exp \frac{1}{b}\int_0^b \log|f_t(x)| dt\right)^p d\mu\right)^{1/p} \\ &\le \exp \frac{1}{b}\int_0^b \log\left(\int_\Omega |f_t(x)|^{p(t)} d\mu\right)^{1/p(t)} dt. \end{aligned} \tag{4.7}$$

Remark 4.4. If we put $b = 1, 0 = a_0 < a_1 < a_2 < \cdots < a_n = 1, \alpha_i = a_{i+1} - a_i, i = 1, 2, \ldots, n, f_t(x) = f_i(x)$ for $a_i < t \le a_{i+1}, i = 1, 2, \ldots, n$, then $p(t) = \frac{1}{a_i}, i = 1, 2, \ldots, n$, so (4.7) reads

$$\int_\Omega \left(\prod_{i=1}^n |f_i(x)|^{\alpha_i}\right) d\mu \le \prod_{i=1}^n \left(\int_\Omega |f_i(x)| d\mu\right)^{\alpha_i},$$

where $\sum_{i=1}^n \alpha_i = 1$, which is a generalization of (4.5) and equivalent to (4.4).

Remark 4.5. Inequality (4.7) was stated and proved in a little different form in the paper [23] by L. Nikolova and L.E. Persson, where they used the theory of interpolation between infinite many Banach spaces. However, here we shall finish this section by presenting another proof, which shows that also (4.7) follows essentially from Jensen's inequality (convexity).

Proof. To prove (4.7) first we note that it is sufficient to prove that

$$I_0 := \int_\Omega \left(\exp \frac{1}{b} \int_0^b \log |g_t(x)| dt \right)^p d\mu \le 1, \tag{4.8}$$

where

$$g_t = g_t(x) = \frac{f_t(x)}{\left(\int_\Omega |f_t(x)|^{p(t)} d\mu\right)^{1/p(t)}}.$$

Since

$$\left(\exp \frac{1}{b} \int_0^b \log |g_t(x)| dt \right)^p = \exp \int_0^b \log |g_t(x)|^{p(t)} \frac{p}{p(t)} \frac{1}{b} dx$$

the function $\Phi(u) = e^u$ is convex and, by (4.6), $\int_0^b \frac{p}{p(t)} \frac{1}{b} dt = 1$, we can use Jensen's inequality to obtain that

$$\left(\exp \frac{1}{b} \int_0^b \log |g_t(x)| dt \right)^p \le \frac{1}{b} \int_0^b |g_t(x)|^{p(t)} \frac{p}{p(t)} dt.$$

Hence, by integrating and using Fubini's theorem and (4.6), we find that

$$\begin{aligned} I_0 \le \int_\Omega \left(\frac{p}{b} \int_0^b |g_t(x)|^p(t) \frac{1}{p(t)} dt \right) d\mu &= \int_0^b \frac{p}{b} \frac{1}{p(t)} \int_\Omega \left(\frac{|f_t(x)|^p(t)}{\int_\Omega |f_t(x)|^{p(t)} d\mu} \right) d\mu \\ &= \int_0^b \frac{p}{b} \frac{1}{p(t)} dt = 1, \end{aligned}$$

and (4.8) is proved. The proof is complete. □

5. Various variants of Minkowski's inequality via convexity

The standard variant of Minkowski's inequality reads:

Example 15 (Minkowski's inequality). If $p \ge 1$, then

$$\left(\int_\Omega |f+g|^p d\mu \right)^{1/p} \le \left(\int_\Omega |f|^p d\mu \right)^{1/p} + \left(\int_\Omega |g|^p d\mu \right)^{1/p}. \tag{5.1}$$

Remark 5.1. The inequality (5.1) can be written

$$\|f+g\|_{L_p(\Omega)} \le \|f\|_{L_p(\Omega)} + \|g\|_{L_p(\Omega)},$$

which is the triangle inequality in $L_p(\Omega)$-spaces. This is the crucial property that the spaces $L_p(\Omega)$ are normed spaces, even Banach spaces, for $p \ge 1$.

Proof 1. By the triangle inequality and Hölder's inequality we have that

$$\begin{aligned} \int_\Omega |f+g|^p d\mu = \int_\Omega |f+g|^{p-1} |f+g| d\mu &\le \int_\Omega |f+g|^{p-1} (|f|+|g|) d\mu \\ = \int_\Omega |f+g|^{p-1} |f| d\mu + \int_\Omega |f+g|^{p-1} |g| d\mu \end{aligned}$$

$$\le \left(\int_\Omega |f|^p d\mu\right)^{1/p} \left(\int_\Omega |f+g|^p d\mu\right)^{1/q} + \left(\int_\Omega |g|^p d\mu\right)^{1/p} \left(\int_\Omega |f+g|^p d\mu\right)^{1/q}$$
$$= \left(\int_\Omega |f+g|^p d\mu\right)^{1/q} \left[\left(\int_\Omega |f|^p d\mu\right)^{1/p} + \left(\int_\Omega |g|^p d\mu\right)^{1/p}\right].$$

Hence

$$\left(\int_\Omega |f+g|^p d\mu\right)^{1-1/q} \le \left(\int_\Omega |f|^p d\mu\right)^{1/p} + \left(\int_\Omega |g|^p d\mu\right)^{1/p}$$

and since $1 - 1/q = 1/p$ we obtain (5.1). □

Proof 1 is the most common proof in Functional Analysis books but we present here also another proof of (5.1), which is easier to generalize and which is a special case of a general technique called quasi-linearization. In our case we do this linearization by using (4.2).

Proof 2. The fact that we have equality in Hölder's inequality means that

$$\left(\int_\Omega |f|^p d\mu\right)^{1/p} = \sup_{\varphi \ge 0} \int_\Omega |f| \varphi d\mu,$$

(see (4.2)), where supremum is taken over all φ such that for $q = p/(p-1)$

$$\left(\int_\Omega \varphi^q d\mu\right)^{1/q} \equiv 1.$$

Hence, by the usual triangle inequality for numbers and an obvious estimate, we have that

$$\left(\int_\Omega |f+g|^p d\mu\right)^{1/p} = \sup_{\varphi \ge 0} \int_\Omega |f+g| \varphi d\mu \le \sup_{\varphi \ge 0} \int_\Omega (|f|\varphi + |g|\varphi) d\mu$$
$$\le \sup_{\varphi \ge 0} \int_\Omega |f| \varphi d\mu + \sup_{\varphi \ge 0} \int_\Omega |g| \varphi d\mu = \left(\int_\Omega |f|^p d\mu\right)^{1/p} + \left(\int_\Omega |g|^p d\mu\right)^{1/p}. \quad \square$$

A generalization of (5.1) reads:

Example 16 (Minkowski's inequality for n functions $f_1, f_2, \dots, f_n$). If $p \ge 1$, $n = 2, 3, \dots$, then

$$\left(\int_\Omega |f_1 + f_2 + \cdots + f_n|^p d\mu\right)^{1/p} \le \left(\int_\Omega |f_1|^p d\mu\right)^{1/p} + \cdots + \left(\int_\Omega |f_n|^p d\mu\right)^{1/p}.$$

The proof of this inequality follows by generalizing Proof 2 above of (5.1) in an obvious way or simply by using induction and (5.1).

Next we shall present Minkowski's inequality for infinite many functions $K_y(x) = K(x, y)$, which usually is called Minkowski's integral inequality.

Example 17 (Minkowski's integral inequality). Let $-\infty \leq a \leq b \leq \infty, -\infty \leq c \leq d \leq \infty$, and let $K(x,y)$ be measurable on $[a,b] \times [c,d]$.
If $p \geq 1$, then

$$\left(\int_a^b \left(\int_c^d K(x,y)dy\right)^p dx\right)^{1/p} \leq \int_c^d \left(\int_a^b K^p(x,y)dx\right)^{1/p} dy. \tag{5.2}$$

Proof. Let $p > 1$. We use again the quasi-linearization idea from (4.2) and obtain that

$$I_0 := \left(\int_a^b \left(\int_c^d K(x,y)dy\right)^p dx\right)^{1/p} = \sup_{\varphi \geq 0} \int_a^b \varphi(x) \int_c^d K(x,y)dydx,$$

where the supremum is taken over all measurable φ such that $\int_a^b \varphi^q(x)dx = 1, q = p/(p-1)$. Hence, by using the Fubini theorem and an obvious estimate, we have that

$$\begin{aligned} I_0 = \sup_{\varphi \geq 0} \int_c^d \int_a^b K(x,y)\varphi(x)dx\ dy &\leq \int_c^d \left(\sup_{\varphi \geq 0} \int_a^b K(x,y)\varphi(x)dx\right) dy \\ &= \int_c^d \left(\int_a^b K^p(x,y)dx\right)^{1/p} dy. \end{aligned}$$

For $p = 1$ we have even equality in (5.2) because of the Fubini theorem, so the proof is complete. $\square$

Next we shall consider a special case of Example 17, which is useful, e.g., when working with mixed-norm L_p spaces and we need some estimate replacing the Fubini theorem. More exactly, we put

$$K(x,y) = \begin{cases} k(x,y)\Psi(y)\Psi_0^{1/p}(x), & a \leq y \leq x, \\ 0 & , \ x < y \leq b, \end{cases}$$

where $k(x,y), \Psi(y)$ and $\Psi_0(x)$ are measurable so that Minkowski's integral inequality (5.2) can be used. Under this assumption we have the following:

Example 18 (Minkowski's integral inequality of Fubini type). If $p \geq 1, -\infty \leq a < b \leq \infty$, then

$$\begin{aligned} &\left(\int_a^b \left(\int_a^x k(x,y)\Psi(y)dy\right)^p \Psi_0(x)dx\right)^{1/p} \\ &\leq \int_a^b \left(\int_y^b k^p(x,y)\Psi_0(x)dx\right)^{1/p} \Psi(y)dy. \end{aligned} \tag{5.3}$$

Remark 5.2. With the same proof as above we can also formulate more general forms of the estimates (5.2) and (5.3) by replacing the measures dx and dy by general measures $d\mu(x)$ and $d\mu(y)$, respectively, and thus, e.g., also cover cases with double sums instead of double integrals.

In particular, we have the following discrete variant of (5.3):

Example 19. Let $p \geq 1$ and let $\{a_{k\ell}\}_{k,\ell=1}^{\infty,\infty}$, $\{b_k\}_{k=1}^{\infty}$ and $\{c_k\}_{k=1}^{\infty}$ be positive sequences. Then

$$\left(\sum_{k=1}^{\infty}\left(\sum_{\ell=1}^{k} a_{k\ell}b_\ell\right)^p c_k\right)^{1/p} \leq \sum_{\ell=1}^{\infty}\left(\sum_{k=\ell}^{\infty} a_{k\ell}^p c_k\right)^{1/p} b_\ell. \tag{5.4}$$

Remark 5.3. In the same way we can prove the following associate variants of (5.3) and (5.4):

$$\left(\int_a^b \left(\int_x^b k(x,y)\Psi(y)dy\right)^p \Psi_0(x)dx\right)^{1/p}$$
$$\leq \int_a^b \left(\int_a^y k^p(x,y)\Psi_0(x)dx\right)^{1/p} \Psi(y)dy,$$

respectively,

$$\left(\sum_{k=1}^{\infty}\left(\sum_{\ell=k}^{\infty} a_{k\ell}b_\ell\right)^p c_k\right)^{1/p} \leq \sum_{\ell=1}^{\infty}\left(\sum_{1}^{\ell} a_{k\ell}^p c_k\right)^{1/p} b_\ell.$$

6. Some classical inequalities (by Hardy, Carleman and Pólya–Knopp) via convexity

The main information in this and the next section is mainly taken from the recent paper [27] (cf. also [26]) by L.E. Persson and N. Samko. But the formulation of some crucial results are different and put to this more general frame.

Example 20 (Hardy's inequality (continuous form)). If f is non-negative and p-integrable over $(0,\infty)$, then

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(y)dy\right)^p dx \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)dx, \quad p>1. \tag{6.1}$$

Example 21 (Hardy's inequality (discrete form)). If $\{a_n\}_1^\infty$ is a sequence of non-negative numbers, then

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{n=1}^{n} a_i\right)^p \leq \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{\infty} a_n^p, \quad p>1. \tag{6.2}$$

Remark 6.1. The dramatic more than 10 years period of research until Hardy stated in 1920 (see [10]) and proved in 1925 (see [11]) his inequality (6.1) was recently described in [19]. It is historically clear that Hardy's original motivation when he discovered his inequalities was to find a simple proof of Hilbert's double series inequality, so first he even only considered the case $p=2$.

Remark 6.2. It is clear that (6.1) $\Rightarrow$ (6.2), which can be seen by applying (6.1) with step functions. This was pointed out to Hardy in a private letter from F. Landau already in 1921 and here Landau even included a proof of (6.2).

Example 22 (Carleman's inequality). If $\{a_n\}_1^\infty$ is a sequence of positive numbers, then

$$\sum_{n=1}^{\infty} \sqrt[n]{a_1 \cdots a_n} \quad \le \quad e\sum_{n=1}^{\infty} a_n. \tag{6.3}$$

Remark 6.3. This inequality was proved by T. Carleman in 1922 (see [8]) in connection to this important work on quasianalytical functions. Carleman's idea of proof was to find maximum of $\sum_{i=1}^{n}(a_1 \cdots a_i)^{1/i}$ under the constraint $\sum_{i=1}^{n} a_i = 1, n \in Z_+$. However, (6.3) is in fact a limit inequality (as $p \to \infty$) of the inequalities (6.2) according to the following:

Replace a_i with $a_i^{1/p}$ in the Hardy discrete inequality (6.2) and we obtain that

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{n=1}^{n} a_i^{1/p}\right)^p \le \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{\infty} a_n, \ p > 1.$$

Moreover, when $p \to \infty$ we have that

$$\left(\frac{1}{n}\sum_{i=1}^{n} a_i^{1/p}\right)^p \to \left(\prod_{i=1}^{n} a_i\right)^{1/n} \qquad \text{and} \qquad \left(\frac{p}{p-1}\right)^p \to e.$$

In view of the fact that Carleman and Hardy had a direct cooperation at that time (see, e.g., [9]) it is maybe a surprise that Carleman did not mention this fact and simpler proof in his paper.

Example 23 (The Pólya–Knopp inequality). If f is a positive and integrable function on $(0, \infty)$, then

$$\int_0^\infty \exp\left(\frac{1}{x}\int_0^x \ln f(y)dy\right) dx \le e\int_0^\infty f(x)dx. \tag{6.4}$$

Remark 6.4. Sometimes (6.4) is referred to as the Knopp inequality with reference to his 1928 paper [17]. But it is clear that it was known before and in his 1925 paper [11] Hardy informed that G. Pólya had pointed out the fact that (6.4) is in fact a limit inequality (as $p \to \infty$) of the inequality (6.1) and the proof is literally the same as that above that (6.2) implies (6.3), see Remark 6.3. Accordingly, nowadays (6.4) is many times referred to as the Pólya–Knopp inequality and we have adopted this terminology.

All inequalities above are sharp, i.e., the constants in the inequalities can not be replaced by any smaller constants.

In particular, the discussion above shows that indeed all the inequalities (6.1)–(6.4) are proved as soon as (6.1) is proved. Our next aim is to present a really simple ("miracle") proof of this inequality via convexity, but first we need the following:

Basic observation 6.5. We note that for $p > 1$ it yields that

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(y)dy\right)^p dx \le \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)dx, \tag{6.5}$$

$$\Leftrightarrow$$

$$\int_0^\infty \left(\frac{1}{x}\int_0^x g(y)dy\right)^p \frac{dx}{x} \le 1 \cdot \int_0^\infty g^p(x)\frac{dx}{x}, \tag{6.6}$$

where $f(x) = g(x^{1-1/p})x^{-1/p}$. In fact, consider (6.5) (= (6.1)) and we find that:

$$\begin{aligned}\left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)dx &= \left(\frac{p}{p-1}\right)^p \int_0^\infty g^p(x^{1-1/p})\frac{dx}{x}\\ &= \left(\frac{p}{p-1}\right)^{p+1} \int_0^\infty g^p(y)\frac{dy}{y},\end{aligned}$$

and

$$\begin{aligned}\int_0^\infty \left(\frac{1}{x}\int_0^x g(t^{1-1/p})t^{-1/p}dt\right)^p dx &= \left(\frac{p}{p-1}\right)^p \int_0^\infty \left(\frac{1}{x}\int_0^{x^{1-1/p}} g(s)ds\right)^p dx\\ &= \left(\frac{p}{p-1}\right)^{p+1} \int_0^\infty \left(\frac{1}{y}\int_0^y g(s)ds\right)^p \frac{dy}{y},\end{aligned}$$

which proves this statement.

According to the Basic observation 6.5 we have proved (6.1) (and thus also (6.2)–(6.4)) as soon as (6.6) is proved and here is the ("miracle") proof of (6.1): By Jensen's inequality and Fubini's theorem we have that

$$\begin{aligned}\int_0^\infty \left(\frac{1}{x}\int_0^x g(y)dy\right)^p \frac{dx}{x} &\le \int_0^\infty \left(\frac{1}{x}\int_0^x g^p(y)dy\right)\frac{dx}{x}\\ &= \int_0^\infty g^p(y)\int_y^\infty \frac{dx}{x^2}dy = \int_0^\infty g^p(y)\frac{dy}{y}.\end{aligned} \tag{6.7}$$

Remark 6.6. Since the function $\Phi(u) = u^p$ is convex also when $p < 0$ this simple proof shows that (6.1) in fact also holds for $p < 0$, a fact which was not noted by Hardy himself.

Remark 6.7. In 1927 G.H. Hardy himself (see [12]) proved the first weighted version of his inequality (6.1) namely the following: The inequality

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(y)dy\right)^p x^a dx \le \left(\frac{p}{p-1-a}\right)^p \int_0^\infty f^p(x)x^a dx \tag{6.8}$$

holds for all measurable and non-negative functions f on $(0,\infty)$ whenever $a < p-1, p \ge 1$.

Hardy obviously believed that this was a generalization of (6.1) but, in fact, by making the substitution

$$f(t) = g(t^{\frac{p-1-\alpha}{p}})t^{-\frac{1+\alpha}{p}}$$

and calculations like in the Basic observation 6.5 we see that also (6.8) for any considered α is equivalent to (6.6).

Remark 6.8. There exists also an associate variant of (6.8), namely the following:

$$\int_0^\infty \left(\frac{1}{x}\int_x^\infty f(y)dy\right)^p x^{\alpha_0}dx \leq \left(\frac{p}{\alpha_0+1-p}\right)^p \int_0^\infty f^p(x)x^{\alpha_0}dx, \tag{6.9}$$

which holds for all measurable and non-negative functions on $(0,\infty)$ whenever $\alpha_0 > p-1, p \geq 1$. In fact, also this inequality is equivalent to the basic inequality (6.6) so, in particular, (6.8) and (6.9) are equivalent (with the relation $\alpha_0 = -\alpha - 2 + 2p$ as we will see later on). Moreover, since the function $\Phi(u) = u^p$ is convex also for $p < 0$ it yields that

a) (6.8) holds also in the case $p < 0, \alpha > p-1$,
b) (6.9) holds also in the case $p < 0, \alpha_0 < p-1$,

and these inequalities are equivalent also then.

Finally, we note that since the function $\Phi(u) = u^p$ is concave for $0 < p < 1$ it yields that (6.8) and (6.9) hold in the reversed direction for $0 < p < 1$ with the same restrictions on α and α_0.

This important remark is a special case of a more general statement (Proposition 7.3) proved and discussed in detail in our next section.

7. More Hardy type inequalities via convexity

The same convexity argument as that in the proof (see (6.7)) of the basic inequality (6.6) shows that we have the following more general statement:

Example 24. Let f be a measurable function on $\mathbb{R}_+$ and let Φ be a convex function on $D_f = \{f(x)\}$. Then

$$\int_0^\infty \Phi\left(\frac{1}{x}\int_0^x f(y)dy\right)\frac{dx}{x} \leq \int_0^\infty \Phi\big(f(x)\big)\frac{dx}{x}. \tag{7.1}$$

If Φ instead is positive and concave, then the reversed inequality holds.

In fact, by Jensen's inequality and Fubini's theorems we have that

$$\begin{aligned}\int_0^\infty \Phi\left(\frac{1}{x}\int_0^x f(y)dy\right)\frac{dx}{x} &\leq \int_0^\infty \int_0^x \Phi(f(y))dy\frac{dx}{x^2} \\ &= \int_0^\infty \Phi(f(y))\int_y^\infty \frac{1}{x^2}dx\, dy = \int_0^\infty \Phi(f(y))\frac{dy}{y}.\end{aligned}$$

If Φ is concave, then the only inequality holds in the reverse direction.

Example 25. Consider the convex function $\Phi(u) = e^u$ and replace $f(y)$ with $\ln\ f(y)$. Then (7.1) reads

$$\int_0^\infty \exp\left(\frac{1}{x}\int_0^x \ln f(y)dy\right)\frac{dx}{x} \leq \int_0^\infty f(x)\frac{dx}{x}. \tag{7.2}$$

By now making the substitution $f(x) = xg(x)$ we transform (7.2) to the inequality

$$\int_0^\infty \exp\left(\frac{1}{x}\int_0^x \ln g(y)dy\right)dx \le e\int_0^\infty g(x)dx,$$

i.e., we obtain another proof of the Pólya–Knopp inequality (6.4) without going via the limit argument mentioned in the Remark 6.4.

It is also known that the Hardy inequality (6.1) holds for finite intervals, e.g., that

$$\int_0^\ell \left(\frac{1}{x}\int_0^x f(y)dy\right)^p dx \le \left(\frac{p}{p-1}\right)^p \int_0^\ell f^p(x)dx, \quad p > 1, \tag{7.3}$$

holds for any $\ell, 0 < \ell \le \infty$, and the constant $\left(\frac{p}{p-1}\right)^p$ is still sharp also for $\ell < \infty$.

But the inequality (7.3) can be improved to the following:

$$\int_0^\ell \left(\frac{1}{x}\int_0^x f(y)dy\right)^p dx \le \left(\frac{p}{p-1}\right)^p \int_0^\ell f^p(x)\left[1-\left(\frac{x}{\ell}\right)^{\frac{p-1}{p}}\right]dx, \tag{7.4}$$

where $p > 1$ or $p < 0$.

This fact is a special case of the more general result (Proposition 7.3) we next aim to prove and discuss. As a preparation of independent interest we first state the following generalization of our previous basic inequality (6.6):

Lemma 7.1. *Let g be a non-negative and measurable function on* $(0,\ell), 0 < \ell \le \infty$.

a) *If $p < 0$ or $p \ge 1$, then*

$$\int_0^\ell \left(\frac{1}{x}\int_0^x g(y)dy\right)^p \frac{dx}{x} \le 1\cdot \int_0^\ell g^p(x)\left(1-\frac{x}{\ell}\right)\frac{dx}{x}. \tag{7.5}$$

(*In the case $p < 0$ we assume that $g(x) > 0, 0 < x \le \ell$.*)

b) *If* $0 < p \le 1$, *then* (7.5) *holds in the reversed direction.*

c) *The constant $C = 1$ is sharp in both* a) *and* b)*.*

Proof. By using Jensen's inequality with the convex function $\Phi(u) = u^p, p \ge 1$ or $p < 0$, and reversing the order of integration, we find that

$$\int_0^\ell \left(\frac{1}{x}\int_0^x g(y)dy\right)^p \frac{dx}{x} \le \int_0^\ell \frac{1}{x}\int_0^x g^p(y)dy\frac{dx}{x} = \int_0^\ell g^p(y)\left(\int_y^\ell \frac{1}{x^2}dx\right)dy$$

$$= \int_0^\ell g^p(y)\left(\frac{1}{y}-\frac{1}{\ell}\right)dy = \int_0^\ell g^p(y)\left(1-\frac{y}{\ell}\right)\frac{dy}{y}.$$

The only inequality in this proof holds in the reversed direction when $0 < p \le 1$ so the proof of b) follows in the same way.

Concerning the sharpness of the inequality (7.5) we first let $\ell < \infty$ and assume that

$$\int_0^\ell \left(\frac{1}{x}\int_0^x g(y)dy\right)^p \frac{dx}{x} \le C\cdot \int_0^\ell g^p(x)\left(1-\frac{x}{\ell}\right)\frac{dx}{x} \tag{7.6}$$

for all non-negative and measurable functions g on $(0,\ell)$ with some constant $C, 0 < C < 1$. Let $p \geq 1$ and $\varepsilon > 0$ and consider $g_\varepsilon(x) = x^\varepsilon$ (for the case $p < 0$ we assume that $-1 < \varepsilon < 0$). By inserting this function into (7.6) we obtain that

$$C \geq (\varepsilon p + 1)(\varepsilon + 1)^{-p},$$

so that, by letting $\varepsilon \to 0_+$ we have that $C \geq 1$. This contradiction shows that the best constant in (7.5) is $C = 1$. In the same way we can prove that the constant $C = 1$ is sharp also in the case b). For the case $\ell = \infty$ the sharpness follows by just making a limit procedure with the result above in mind. The proof is complete. □

Remark 7.2. For the case $\ell = \infty$ (7.5) coincides with the basic inequality (6.6) and, thus, the constant $C = 1$ is sharp, which, in its turn, implies the well-known fact that the constant $C = \left(\frac{p}{p-1}\right)^p$ in Hardy's inequality (6.1) is sharp for $p > 1$ and as we see above this holds also for $p < 0$.

Moreover, since also the weighted variants (6.8) and (6.9) are equivalent to the basic inequality (6.6) via substitutions we conclude that also these constants are sharp in all considered cases.

We are now ready to formulate our main result in this section.

Proposition 7.3. *Let $0 < \ell \leq \infty$, $\ell_0 = 1/\ell$, $a, a_0 \in \mathbb{R}$ and let $p \in \mathbb{R} \setminus \{0\}$ be a fixed number and f be a non-negative function.*

a) *Let f be a measurable function on $(0,\ell]$. Then*

$$\begin{aligned}\int_0^\ell \left(\frac{1}{x}\int_0^x f(y)dy\right)^p x^a dx \\ \leq \left(\frac{p}{p-1-a}\right)^p \int_0^\ell f^p(x) x^a \left[1 - \left(\frac{x}{\ell}\right)^{\frac{p-a-1}{p}}\right] dx\end{aligned} \tag{7.7}$$

holds for the following cases:

a$_1$) $p \geq 1$, $a < p - 1$,

a$_2$) $p < 0$, $a > p - 1$.

b) *For the case $0 < p < 1, a < p - 1$, inequality (7.7) holds in the reversed direction in both cases* a$_1$) *and* a$_2$).

c) *Let f be a measurable function on $[\ell,\infty)$. Then*

$$\begin{aligned}\int_{\ell_0}^\infty \left(\frac{1}{x}\int_x^\infty f(y)dy\right)^p x^{a_0} dx \\ \leq \left(\frac{p}{a_0+1-p}\right)^p \int_{\ell_0}^\infty f^p(x) x^{a_0} \left[1 - \left(\frac{\ell_0}{x}\right)^{\frac{a_0+1-p}{p}}\right] dx\end{aligned} \tag{7.8}$$

holds for the following cases:

c$_1$) $p \geq 1$, $a_0 > p - 1$,

c$_2$) $p < 0$, $a_0 < p - 1$.

d) *For the case* $0 < p \le 1, a_0 > p-1$ *inequality* (7.8) *holds in the reversed direction in both cases* c_1) *and* c_2).
e) *All inequalities above are sharp.*
f) *Let* $p \ge 1$ *or* $p < 0$. *Then, the inequalities* (7.7) *and* (7.8) *are equivalent for all permitted* a *and* a_0 *because they are in all cases equivalent to* (7.5) *via substitutions.*
g) *Let* $0 < p < 1$. *Then, the reversed inequalities* (7.7) *and* (7.8) *are equivalent for all permitted* a *and* a_0.

Remark 7.4. The formal relation between the parameters a and a_0 is that $a = 2p - a_0 - 2$, but this is not important according to f) and g).

Proof. First we prove that (7.7) in the case (a_1) in fact is equivalent to (7.5) via the relation

$$f(x) = g\left(x^{\frac{p-a-1}{p}}\right) x^{-\frac{a+1}{p}}.$$

In fact, with $f(x) = g\left(x^{\frac{p-a-1}{p}}\right) x^{-\frac{a+1}{p}}$ and $\ell_0 = \ell^{\frac{p}{p-a-1}}$, in (7.7) we get that

$$\begin{aligned}
&\left(\frac{p}{p-1-a}\right)^p \int_0^{\ell_0} g^p\left(x^{\frac{p-a-1}{p}}\right)\left[1-\left(\frac{x}{\ell_0}\right)^{\frac{p-1-a}{p}}\right]\frac{dx}{x} \\
&= \left(\frac{p}{p-1-a}\right)^{p+1} \int_0^{\ell^{\frac{p-a-1}{p}}} g^p(y)\left[1-\frac{y}{\ell_0^{\frac{p-1-a}{p}}}\right]\frac{dy}{y} \\
&= \left(\frac{p}{p-1-a}\right)^{p+1} \int_0^{\ell} g^p(y)\left[1-\frac{y}{\ell}\right]\frac{dy}{y},
\end{aligned}$$

where $y = x^{\frac{p-a-1}{p}}$, $dy = x^{-\frac{a+1}{p}}\left(\frac{p-1-a}{p}\right)dx$, and

$$\begin{aligned}
&\int_0^{\ell_0}\left(\frac{1}{x}\int_0^x g\left(y^{\frac{p-a-1}{p}}\right) y^{-\frac{a+1}{p}} dy\right)^p x^a dx \\
&= \left(\frac{p}{p-1-a}\right)^p \int_0^{\ell_0}\left(\frac{1}{x^{\frac{p-a-1}{p}}}\int_0^{x^{\frac{p-a-1}{p}}} g(s)ds\right)^p \frac{dx}{x} \\
&= \left(\frac{p}{p-1-a}\right)^{p+1} \int_0^{\ell}\left(\frac{1}{y}\int_0^y g(s)ds\right)^p \frac{dy}{y}.
\end{aligned}$$

Since we have only equalities in the calculations above we conclude that (7.5) and (7.7) are equivalent and, thus, by Lemma 7.1, a) is proved for the case (a_1).

For the case (a_2) all calculations above are still valid and, according to Lemma 7.1, (7.5) holds also in this case and a) is proved also for the case (a_2).

For the case $0 < p \le 1, a < p-1$, all calculations above are still true and both (7.5) and (7.7) hold in the reversed direction according to Lemma 7.1. Hence also b) is proved.

For the proof of c) we consider (7.7) with $f(x)$ replaced by $f(1/x)$, with a replaced by a_0 and with ℓ replaced by $\ell_0 = 1/\ell$:

$$\begin{aligned}
&\int_0^{\ell_0} \left(\frac{1}{x}\int_0^x f(1/y)dy\right)^p x^{a_0}dx \\
&\quad\leq \left(\frac{p}{p-1-a_0}\right)^p \int_0^{\ell_0} f^p(1/x)x^{a_0}\left[1-\left(\frac{x}{\ell_0}\right)^{\frac{p-a_0-1}{p}}\right]dx.
\end{aligned} \tag{7.9}$$

Moreover, by making first the variable substitution $x = 1/s$ and after that $y = 1/x$, we find that left-hand side of (7.9) is equal to

$$\begin{aligned}
\int_0^{\ell_0} \left(\frac{1}{x}\int_{1/x}^\infty \frac{f(s)}{s^2}ds\right)^p x^{a_0}dx &= \int_\ell^\infty \left(y\int_y^\infty \frac{f(s)}{s^2}ds\right)^p y^{-a_0-2}dy \\
&= \int_\ell^\infty \left(\frac{1}{y}\int_y^\infty \frac{f(s)}{s^2}ds\right)^p y^{-a_0-2+2p}dy
\end{aligned}$$

[Put $\frac{f(s)}{s^2} = g(s)$]

$$= \int_\ell^\infty \left(\frac{1}{y}\int_y^\infty g^p(y)\right)^p y^{2p-a_0-2}dy,$$

and, by using the substitution $y = 1/x$, we obtain that right-hand side of (7.9) is equal to

$$\begin{aligned}
&\left(\frac{p}{p-1-a_0}\right)^p \int_\ell^\infty f^p(y)y^{-a_0}\left[1-\left(\frac{\ell}{y}\right)^{\frac{p-a_0-1}{p}}\right] y^{-2}dy \\
&= \left(\frac{p}{p-1-a_0}\right)^p \int_\ell^\infty g^p(y)y^{2p-a_0-2}\left[1-\left(\frac{\ell}{y}\right)^{\frac{p-a_0-1}{p}}\right] dy.
\end{aligned}$$

Now replace $2p-a_0-2$ by a and g by f and we have that $a_0 = 2p-a-2, p-1-a_0 = a+1-p$. Hence, it yields that

$$\int_\ell^\infty \left(\frac{1}{x}\int_x^\infty f(s)ds\right)^p x^a dx \leq \left(\frac{p}{a+1-p}\right)^p \int_\ell^\infty f^p(x)x^a\left[1-\left(\frac{\ell}{x}\right)^{\frac{a+1-p}{p}}\right]dx$$

and, moreover,

$$a_0 < p-1 \Leftrightarrow 2p-a-2 < p-1 \Leftrightarrow a > p-1.$$

By changing notation and using the symmetry between the parameters, we find that c) with the conditions (c_1) and (c_2) are in fact equivalent to a) with the conditions (a_1) and (a_2), respectively, and also c) is proved. (The formal relation between the parameters is $a = 2p - a_0 - 2$ and $\ell_0 = 1/\ell$.)

The calculations above hold also in the case d) and the only inequality holds in the reversed direction in this case so also d) is proved.

Finally, we note that the proof above only consists of suitable substitutions and equalities to reduce all inequalities to the sharp inequality (7.5), or the reversed inequality (7.5), and we obtain a proof also of the statements e) and f) according to Lemma 7.1. The proof is complete. □

8. Some further results and final remarks

Remark 8.1. Our presented simple convexity technique to prove powerweighted Hardy inequalities can be useful for several generalizations. We only present here the following generalization of our fundamental inequality (7.5) for the case of piecewise-constant $p(x)$:

Example 26. Let

$$p(x) = \begin{cases} p_0, & 0 \le x \le 1, \\ p_1, & x > 1, \end{cases} \tag{8.1}$$

where $p_0, p_1 \in \mathbb{R} \setminus \{0\}$. Let $0 < \ell \le \infty$, and let $p(x)$ be defined by (8.1). Then, for every non-negative and measurable function f,

$$\int_0^\ell \left(\frac{1}{x}\int_0^x f(t)dt\right)^{p(x)} \frac{dx}{x} \le 1 \cdot \int_0^\ell (f(x))^{p(x)} \left(1 - \frac{x}{\ell}\right) \frac{dx}{x} \tag{8.2}$$
$$+ \max\left\{0, 1 - \frac{1}{\ell}\right\} \int_0^1 \left[(f(x))^{p_1} - (f(x))^{p_0}\right] dx,$$

whenever $p(x) \ge 1$ or $p(x) < 0$ (for the case $p(x) < 0$ we also assume that $f(x) > 0$.)

For the case $0 < p(x) < 1$ inequality (8.2) holds in the reversed direction.

The constant $C = 1$ in front of the first integral is sharp.

Remark 8.2. The proof of this and more general statements of this type can be found in [27]. Note that for the case $p_0 = p_1 = p$ (8.2) coincides with (7.5). Hence, our inequality (8.2) is a genuine generalization not only of (7.5) but also of all Hardy type inequalities we have derived from (7.5) in this paper (see, e.g., Proposition 7.3).

Remark 8.3. We have already mentioned that convexity was very important when modern interpolation theory was initiated (see Example 2 and [7]). Hence, it is not surprising that interpolation theory is also very important tool when proving inequalities. The main aim of [25] is to illustrate and develop this close connection between Convexity, Interpolation and Inequalities. Here we just mention the following example:

Example 27 (Young's integral inequality). Consider the convolution operator T defined by

$$Tf(x) = \int_{-\infty}^{+\infty} k(x-y)f(y)dy = k * f(x).$$

If $k \in L_r = L_r(-\infty, +\infty)$ and $f \in L_p = L_p(-\infty, +\infty)$, where $1 < p < r' = r/(r-1)$, then $k * f \in L_q$, where $1/q + 1/p - 1/r'$ and

$$\|k * f\|_{L_q} \le \|k\|_{L_r} \|f\|_{L_p}. \tag{8.3}$$

Proof. By our variant of Minkowski's inequality we have that

$$\|Tf\|_{L_r} \leq \|k\|_{L_r}\|f\|_{L_1}$$

and, by Hölder's inequality,

$$\|Tf\|_{L_\infty} \leq \|k\|_{L_r}\|f\|_{L_{r'}}.$$

This means that $T : L_1 \to L_r$ and $T : L_{r'} \to L_\infty$ with norm $\|k\|_{L_r}$ in both cases. By interpolating between these two situations with the usual relation for the parameters in intermediate spaces we obtain (8.3) and the proof is complete (cf. [7], p. 6). □

Remark 8.4. Note that this argument of proof does not work for the case $r = 1$ (so that $p = q$) but this limit case holds also, which can be seen by just using a direct convexity argument.

Example 28. If $f \in L_p, 1 \leq p \leq \infty$, and $g \in L_1$, then $f * g \in L^p$ and, moreover,

$$\|f * g\|_{L_p} \leq \|g\|_{L_1}\|f\|_{L_p}.$$

Proof. We shall prove that

$$\left(\int_{-\infty}^{+\infty}\left|\int_{-\infty}^{+\infty} f(y)g(x-y)dy\right|^p dx\right)^{1/p} \leq \int_{-\infty}^{+\infty}|g(x)|dx\left(\int_{-\infty}^{+\infty}|f(x)|^p dx\right)^{1/p}.$$

The cases $p = 1$ and $p = \infty$ are trivial so we assume that $1 < p < \infty$. First we note that, by Hölder's inequality, for each $x \in \mathbb{R}$ we have that

$$\begin{aligned}|f * g(x)| &= \left|\int_{-\infty}^{+\infty} f(y)g(x-y)dy\right| \leq \int_{-\infty}^{+\infty}|f(y)||g(x-y)|^{1/p}|g(x-y)|^{1/p'}dy\\ &\leq \left(\int_{-\infty}^{+\infty}|f(y)|^p|g(x-y)|dy\right)^{1/p}\left(\int_{-\infty}^{+\infty}|g(x-y)|dy\right)^{1/p'}.\end{aligned}$$

We now take the L_p norm of both sides and use Fubini's theorem to obtain that

$$\begin{aligned}\|f * g\|_{L_p} &\leq (\|g\|_{L_1})^{1/p'}\left(\int_{-\infty}^{+\infty}|f(y)|^p\left(\int_{-\infty}^{+\infty}|g(x-y)|dx\right)dy\right)^{1/p}\\ &= (\|g\|_{L_1})^{1/p'}(\|g\|_{L_1})^{1/p}\left(\int_{-\infty}^{+\infty}|f(y)|^p dy\right)^{1/p} = \|g\|_{L_1}\|f\|_{L_p}.\end{aligned}$$ □

Remark 8.5. We claim that also a number of other classical and new inequalities can be derived and understood in this uniform way via convexity and interpolation. For further information concerning this we refer to [25].

We shall finish this section by shortly discussing the possibility to change the concept of convexity a little and thus be able to prove some refined versions of classical inequalities.

In this connection we mention that the following concept of super-quadratic (sub-quadratic) function was introduced in 2004 by S.Abramovich et al. in [2]:

Definition 8.6. [2, Definition 2.1]. A function $\varphi : [0, \infty) \to \mathbb{R}$ is superquadratic provided that for all $x \geq 0$ there exists a constant $C_x \in \mathbb{R}$ such that

$$\varphi(y) - \varphi(x) - \varphi(|y - x|) \geq C_x (y - x)$$

for all $y \geq 0$.

We say that f is subquadratic if $-f$ is superquadratic.

Remark 8.7. It is easy to see that the function $f(u) = u^p$ is super-quadratic for $p \geq 2$ and sub-quadratic for $1 < p \leq 2$.

In the paper [2] the authors proved the following remarkable refinement of Jensen's inequality for super-quadratic functions:

Theorem 8.8. *Let (Ω, μ) be a measure space with $\mu(\Omega) = 1$. The inequality*

$$\varphi\left(\int_\Omega f(s)d\mu(s)\right) \leq \int_\Omega \varphi(f(s))d\mu(s) - \int_\Omega \varphi\left(\left|f(s) - \int_\Omega f(s)d\mu(s)\right|\right) d\mu(s) \quad (8.4)$$

holds for all probability measures μ and all nonnegative μ-integrable functions f if and only if φ is super-quadratic. Moreover, (8.4) *holds in the reversed direction if and only if φ is sub-quadratic.*

In view of Remark 8.7 we have the following important special case of Theorem 8.8:

Example 29. Let (Ω, μ) be a measure space with $\mu(\Omega) = 1$. If $p \geq 2$, then

$$\left(\int_\Omega f(s)d\mu(s)\right)^p \leq \int_\Omega (f(s))^p d\mu(s) - \int_\Omega \left|f(s) - \int_\Omega f(s)d\mu(s)\right|^p d\mu(s) \qquad (8.5)$$

holds and the reversed inequality holds when $1 < p \leq 2$ (see also [1, Example 1, p. 1448]).

By now using the same technique as in our previous sections but with this refined Jensen inequality (see Example 29) we can obtain for example the following refined Hardy type inequalities (the details in the calculations can be found in the paper [24] by J. Oguntuase and L.E. Persson).

Example 30. Let $p > 1$, $k > 1$, $0 < b \leq \infty$, and let the function f be locally integrable on $(0, b)$ such that $0 < \int_0^b x^{p-k} f^p(x)dx < \infty$.

(i) If $p \geq 2$, then

$$\begin{aligned} &\int_0^b x^{-k} \left(\int_0^x f(t)dt\right)^p dx + \frac{k-1}{p} \int_0^b \int_t^b \left|\frac{p}{k-1} \left(\frac{t}{x}\right)^{1-\frac{k-1}{p}} f(t) \right. \\ &\qquad \left. - \frac{1}{x} \int_0^x f(t)dt\right|^p \cdot x^{p-k-\frac{k-1}{p}} dx t^{\frac{k-1}{p}-1} dt \qquad (8.6) \\ &\quad \leq \left(\frac{p}{k-1}\right)^p \int_0^b \left[1 - \left(\frac{x}{b}\right)^{\frac{k-1}{p}}\right] x^{p-k} f^p(x)dx. \end{aligned}$$

(ii) If $1 < p \leq 2$, then inequality (8.6) holds in the reversed direction.

Remark 8.9. Note that (8.6) with $b = \infty$ means that if $p \geq 2$, then the classical Hardy inequality for $k > 1$ can be refined by adding a second term on the left-hand side. In fact, this factor is so big that the inequality holds in the reversed direction for $1 < p \leq 2$ so that, in particular, for $p = 2$ we have the following identity:

$$\int_0^\infty x^{-k}\left(\int_0^x f(t)dt\right)^2 dx + \frac{k-1}{2}\int_0^\infty \int_t^\infty \left(\frac{2}{k-1}\left(\frac{t}{x}\right)^{1-\frac{k-1}{2}} f(t) - \frac{1}{x}\int_0^x f(t)dt\right)^2 \cdot x^{2-k-\frac{k-1}{2}} dx t^{\frac{k-1}{2}-1} dt = \left(\frac{2}{k-1}\right)^2 \int_0^\infty x^{2-k} f^2(x)dx.$$

Remark 8.10. As we have seen the "normal" behaviour in Hardy type inequalities is that the natural "breaking point" (the point where it reverses) is $p = 1$ but in the refined Hardy inequality (8.6) the "breaking point" is $p = 2$. Further research in this direction can be found in recent papers by S. Abramovich and the present authors (see [3], [4] and [5]).

Acknowledgment

We thank the careful referee for some useful comments and suggestions, which have improved the final version of this paper.

References

[1] S. Abramovich, S. Banić, and M. Matić, *Superquadratic functions in several variables.* J. Math. Anal. Appl. **327**(2) (2007), 1444–1460.

[2] S. Abramovich, G. Jameson, and G Sinnamon, *Refining of Jensen's inequality.* Bull. Math. Soc. Sci. Math. Roumanie (N.S) 47(95) (2004), 3–14.

[3] S. Abramovich and L.E. Persson, *Some new scales of refined Hardy type inequalities via functions related to superquadraticity.* Math. Inequal Appl., to appear.

[4] S. Abramovich and L.E. Persson, *Hardy type inequalities with breaking points $p = 2$ and $p = 3$.* Volume 236 of *Operator Theory: Advances and Applications*, Birkhäuser Verlag, Basel, 2013, to appear.

[5] S. Abramovich, L.E. Persson, and N. Samko, *On some new developments of Hardy-type inequalities.* AIP (American Institute of Physics), 1493 (2012), 739–746.

[6] E.F. Beckenbach and R. Bellman, *Inequalities.* Volume 30 of *Ergebnisse der Mathematik und ihrer Grenzgebiete.* Springer-Verlag, Berlin-Heidelberg-New York-Tokyo, 1983, forth ed.

[7] J. Bergh and J. Löfström, *Interpolation spaces: An Introduction.* Grundlehren der Mathematischen Wissenschaften, N 223, Springer-Verlag, Berlin-New York, 1976.

[8] T. Carleman, *Sur les functions quasi-analytiques.* In Comptes Rendus du V Congrès des Mathématiciens Scandinaves, Helsinki, pages 181–196, 1922.

[9] T. Carleman and G.H. Hardy, *Fourier series and analytic functions.* Proc. R. Soc. Lond. Ser. A **101** (1922), 124–133.

[10] G.H. Hardy, *Note on a theorem of Hilbert.* Math. Z. **6** (1920), 314–317.

[11] G.H. Hardy, *Notes on some points in the integral calculus, LX. An inequality between integrals.* Messenger of Math. **54** (1925), 150–156.

[12] G.H. Hardy. *Notes on some points in the integral calculus, LXIV. Further inequalities between integrals.* Messenger of Math. **57** (1927), 12–16.

[13] G.H. Hardy, J.E. Littlewood, and G. Pólya, *Inequalities.* Cambridge University Press, Cambridge, 1934.

[14] L. Hörmander, *Notions of Convexity.* Birkhäuser Verlag Boston, Inc., Boston, 1994.

[15] J.L.W.V. Jensen, *Om konvexe funktioner og uligheter mellom middelvaerdier* (in Norwegian). Nyt. Tidsskr. Math. **16**B (1905), 49–69.

[16] J.L.W.V. Jensen, *Sur les fonctions convexes et les inégalités entre les valeurs moyennes.* Acta Math. **30** (1906), 175–193.

[17] K. Knopp, *Über Reihen mit positiven Gliedern.* J. London Math. Soc. **3** (1928), 205–212.

[18] V. Kokilashvili, A. Meskhi, and L.E. Persson, *Weighted norm inequalities for integral transforms with product kernels.* Mathematics Research Developments Series, Nova Science Publishers, New York, 2010.

[19] A. Kufner, L. Maligranda, and L.E. Persson, *The prehistory of the Hardy inequality.* Amer. Math. Monthly **113**(8) (2006), 715–732.

[20] A. Kufner, L. Maligranda, and L.E. Persson, *The Hardy Inequality: About its History and Some Related Results.* Vydavatelsky Servis Publishing House, Pilsen, 2007.

[21] A. Kufner and L.E. Persson, *Weighted inequalities of Hardy type.* World Scientific Publishing Co. Inc., River Edge, NJ, 2003.

[22] C.P. Niculescu and L.E. Persson, *Convex Functions and their Applications – A Contemporary Approach.* CMS Books in Mathematics, Springer, New York, 2006.

[23] L.N. Nikolova and L.E. Persson, *Some properties of X^p-spaces.* Function Spaces (Poznan, 1989) Teubner-Texte Math., 120 (1991), 174–185.

[24] J.A. Oguntuase and L.E. Persson, *Refinement of Hardy's inequalities via superquadratic and subquadratic functions.* J. Math. Anal. Appl. **339** (2008), 1305–1312.

[25] L.E. Persson and N. Samko, *Inequalities via Convexity and Interpolation.* Book manuscript in preparation.

[26] L.E. Persson and N. Samko, *Some remarks and new developments concerning Hardy-type inequalities.* Rend. Circ. Mat. Palermo, serie II, 82 (2010), 1–29.

[27] L.E. Persson and N. Samko, *What should have happened if Hardy had discovered this?* J. Inequal. Appl., SpringerOpen, 2012:29, 2012.

[28] R.T. Rockafellar, *Convex Analysis.* Princeton University Press, Princeton, N.J., 1970.

Lars-Erik Persson and Natasha Samko
Luleå University of Technology, SE 971 87, Sweden

and

Narvik University College, P.O. Box 385, N-8505, Norway
e-mail: larserik@ltu.se
Natasha.Samko@ltu.se