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On Spectral Subspaces and Inner Endomorphisms of Some Semigroup Crossed Products

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Abstract. This note provides a look into some of the abstract properties of the semigroup crossed product of a unital C^*-algebra by an action consisting of endomorphisms for which there is a left inverse. The objective is to describe in general terms some of the relations between spectral subspaces for the canonical coaction on the crossed product and certain eigenspaces of a time-evolution on the crossed product. The present analysis is inspired by certain constructions due to Cornelissen and Marcolli [2].

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Introduction

Semigroup crossed products as models for C^*-algebras arising in number theory were first considered by Laca and Raeburn in [8], with motivation provided by the work in [1]. In employing techniques of the theory of semigroup crossed products they were able to offer simplifications of the presentation of the reduced Hecke C^*-algebra in [1] and show that this algebra possessed a universal property. Similar classes of semigroup crossed products were studied later by many authors, and recently they appeared in work of Cornelissen and Marcolli [2].

The starting point in [2], from the point of view of semigroup crossed products, is common to many of the previous studies, and consists of a dynamical system (A, S, α) where A is a unital C^*-algebra, S is a semigroup with nice properties (typically part of a lattice ordered group (G, S)), and α is an action of S by injective endomorphisms of A for which there is a left inverse. The interest in [2] is in the crossed product $\mathcal{A} = A \rtimes_\alpha S$ as part of a quantum statistical mechanical system $(\mathcal{A}, \sigma)$ arising from a time-evolution σ_t on $\mathcal{A}$ for $t \in \mathbb{R}$. Cornelissen and Marcolli's study of isomorphism of two quantum statistical mechanical systems $(\mathcal{A},\sigma)$ and $(\mathcal{A}',\sigma')$ associated to two number fields is heavily dependent on delicate number theoretic facts and arguments. However some of the ingredients introduced along the way pertain to the semigroup crossed product and may be looked at in general terms. This is the objective of the present note. We study here relations between the spectral subspaces of $A\rtimes_\alpha S$ associated to the canonical coaction of G and the eigenspaces of a time evolution σ on the crossed product. As application we identify inner endomorphisms (in the terminology of Cornelissen–Marcolli) of $(\mathcal{A},\sigma)$ that are dagger inner endomorphisms of $A\rtimes_\alpha S$, where the dagger subalgebra is another new ingredient introduced in [2]. A crucial aspect in [2] is preservation of the dagger subalgebra under inner endomorphisms, and one of their results relies on characterizing dagger inner endomorphisms as those inner endomorphisms preserving the dagger subalgebra. It is a natural question whether it is possible to identify in greater generality elements among the inner endomorphisms that are dagger inner isomorphisms. We shall provide an affirmative answer to this question under some suitable conditions.

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1. Semigroup crossed products

Suppose that S is a subsemigroup of a (discrete) group G containing the identity element e and $\alpha : S \to \mathrm{End}(A)$ is an action of S by endomorphisms of a unital C^*-algebra A. The semigroup crossed product $A\rtimes_\alpha S$ is the unital C^*-algebra generated by (the image of) a pair (i_A, i_S) in which i_A a $*$-homomorphism of A and i_S is a semigroup homomorphism of S satisfying the *covariance condition*

$$i_S(s)i_A(a)i_S(s)^* = i_A(\alpha_s(a)) \tag{1.1}$$

for all $s\in S$ and $a\in A$; the pair (i_A, i_S) is universal in the sense that any pair (π, V) (into $B(H)$ for some Hilbert space H) satisfying the analogue of (1.1) factors through (i_A, i_S). If the system admits a pair (π, V) satisfying the covariance condition with π injective, then i_A is injective. We shall assume henceforth that i_A is injective.

We identify g in G with its image as a generating unitary inside $C^*(G)$. It is known that there is a homomorphism i_G from $C^*(G)$ into $C^*(G)\otimes C^*(G)$, with $\otimes$ denoting the minimal tensor product, such that $i_G : g\mapsto g\otimes g$. Recall that a coaction δ of G on a unital C^*-algebra $\mathcal{A}$ is an injective homomorphism $\delta : \mathcal{A}\to \mathcal{A}\rtimes C^*(G)$ which satisfies $(\delta\circ\mathrm{id}_{C^*(G)})\circ\delta = (\mathrm{id}_{\mathcal{A}}\otimes i_G)\circ\delta$ and $\mathrm{span}(\delta(\mathcal{A})(1\otimes C^*(G)))$ is dense in $\mathcal{A}\otimes C^*(G)$. We refer to [10] for basic properties of coactions of discrete groups on C^*-algebras.

With the notation of the first paragraph of this section, [7, Proposition 6.1] (see also their Remark 6.2 which says that one can work with the minimal tensor product), shows that there is a coaction δ of G on $A\rtimes_\alpha S$ such that $\delta(i_A(a)) = i_A(a)\otimes e$ and $\delta(i_S(s)) = i_S(s)\otimes s$ for $a\in A$ and $s\in S$.

Proposition 1.1. *The cosystem $(A \rtimes_\alpha S, G, \delta)$ is a maximal coaction in the sense of* [4].

Proof. Let $\mathcal{A}$ be the Fell bundle associated to $(A \rtimes_\alpha S, G, \delta)$, and let $C^*(\mathcal{A})$ be its full sectional C^*-algebra. Then the universal properties of both algebras imply that $A \rtimes_\alpha S \cong C^*(\mathcal{A})$, and the result follows because $C^*(\mathcal{A})$ with its canonical coaction of G is a maximal coaction system. □

Let S be a cancellative semigroup with identity e. We say that S is right-reversible (or S is an Ore semigroup) provided that $Ss \cap Ss' \neq \emptyset$ for any pair $s, s' \in S$. It follows that S embeds in its group of left quotients $G = S^{-1}S$. If $S \cap S^{-1} = \{e\}$, then $x \leq_r y \iff yx^{-1} \in S$ is a right-invariant order on G. Note that the right-reversibility condition ensures that any two elements $s, s' \in S$ have a common upper bound in S. In the next section we shall assume that there exists a least upper bound for any two elements in S (and in G).

It is known that $A \rtimes_\alpha S$ is the closed span of monomials $i_S(s)^* i_A(a) i_S(s')$ for $a \in A$ and $s, s' \in S$, see [6, 9]. The algebraic crossed product is

$$A \rtimes_\alpha^{\mathrm{alg}} S = \mathrm{span}\{i_S(s)^* i_A(a) i_S(s') \mid s, s' \in S, a \in A\}.$$

By extending the terminology from [2] for a certain class of semigroup crossed products, the dagger subalgebra $(A \rtimes_\alpha S)^\dagger$ of $A \rtimes_\alpha S$ is the algebra generated by $i_A(A)$ and the isometries $i_S(s)$ for $s \in S$. In this case, $(A \rtimes_\alpha S)^\dagger = \mathrm{span}\{i_A(a) i_S(s) \mid a \in A, s \in S\}$.

The next result was claimed in [9] without proof. We include some details of the proof here because they will be useful later. Recall that the spectral subspace at $g \in G$ for the coaction δ on $A \rtimes_\alpha S$ is the space $(A \rtimes_\alpha S)_g$ consisting of $x \in A \rtimes_\alpha S$ such that $\delta(x) = x \otimes g$.

Lemma 1.2. *The spectral subspaces corresponding to δ are given by*

$$(A \rtimes_\alpha S)_g^\delta = \overline{\mathrm{span}}\,\{i_S(s)^* i_A(a) i_S(s') : g = s^{-1}s', s, s' \in S, a \in A\}, \tag{1.2}$$

for $g \in G$. In particular, the fixed point algebra $(A \rtimes_\alpha S)_e^\delta$ for δ is

$$(A \rtimes_\alpha S)_e^\delta = \overline{\mathrm{span}}\,\{i_S(s)^* i_A(a) i_S(s) : s \in S, a \in A\}. \tag{1.3}$$

Proof. We prove first (1.3). Clearly $\overline{\mathrm{span}}\,\{i_S(s)^* i_A(a) i_S(s) : s \in S, a \in A\}$ is contained in $(A \rtimes_\alpha S)_e$. For the other inclusion, let $c \in (A \rtimes_\alpha S)_e$. We may assume $c = \sum_{j=1}^n i_S(s_j)^* i_A(a_j) i_S(s'_j)$ for $s_j, s'_j \in S$ and $a_j \in A$. Let λ_G be the regular representation of G (extended to $C^*(G)$), and let i_G^r denote the embedding of G as unitaries in $C_r^*(G)$. The canonical trace τ on $C_r^*(G)$ carries $i_G^r(s_j^{-1}s'_j)$ to 1 if $s_j^{-1}s'_j = e$ and to zero otherwise. Since $\delta(c) - c \otimes e = 0$ in $(A \rtimes_\alpha S) \otimes C^*(G)$, by applying $(\mathrm{id} \otimes \tau) \circ (\mathrm{id} \otimes \lambda_G)$ to the difference we get that the sum of terms $i_S(s_j)^* i_A(a_j) i_S(s'_j)$ in c in which $s_j^{-1}s'_j \neq e$ is zero. This proves (1.3).

To prove the general case note that the right to left inclusion in (1.2) follows from the definition of δ. For the other inclusion let $c \in (A \rtimes_\alpha S)_g$ for some $g \in G$. Then $c^*c \in (A \rtimes_\alpha S)_g^*(A \rtimes_\alpha S)_g$, which is included in $(A \rtimes_\alpha S)_e$ by [10, Lemma

1.3]. Assume $c = \sum_{j=1}^{n} i_S(s_j)^* i_A(a_j) i_S(s'_j)$ for $s_j, s'_j \in S$ and $a_j \in A$. To simplify notation, let $M_j = i_S(s_j)^* i_A(a_j) i_S(s'_j)$. Then

$$\begin{aligned} c^*c &= \Big(\sum_j M_j\Big)^* \Big(\sum_k M_k\Big) \\ &= \sum_j M_j^* M_j + \sum_{j \neq k} M_j^* M_k \\ &= \sum_j i_S(s'_j)^* i_A(a_j^* \alpha_{s_j}(1) a_j) i_S(s'_j) + \sum_{j \neq k} M_j^* M_k; \end{aligned}$$

we compute the second sum separately. For each pair (j,k) with $j,k = 1,\dots,n$ and $j \neq k$ we choose $u_{(j,k)}, v_{(j,k)} \in S$ such that $s_j s_k^{-1} = u_{(j,k)}^{-1} v_{(j,k)}$. Then $w_{(j,k)} := u_{(j,k)} s_j = v_{(j,k)} s_k$. Letting

$$b_{(j,k)} := \alpha_{u_{(j,k)}}(a_j^*) \alpha_{w_{(j,k)}}(1) \alpha_{v_{(j,k)}}(a_k)$$

for $j \neq k$ we obtain

$$\begin{aligned} c^*c &= \sum_j i_S(s'_j)^* i_A(a_j^* \alpha_{s_j}(1) a_j) i_S(s'_j) + \sum_{j \neq k} M_j^* M_k \\ &= \sum_j i_S(s'_j)^* i_A(a_j^* \alpha_{s_j}(1) a_j) i_S(s'_j) + \sum_{j \neq k} i_S(u_{(j,k)} s'_j)^* i_A(b_{j,k}) i_S(v_{(j,k)} s'_k). \end{aligned}$$

Since $c^*c \in (A \rtimes_\alpha S)_e$, we obtain from two applications of (1.3) that

$$(u_{(j,k)} s'_j)^{-1} v_{(j,k)} s'_k = e$$

for all j,k with $j \neq k$. Thus we must have $(s'_j)^{-1} s_j s_k^{-1} s'_k = e$ or, equivalently, $s_j^{-1} s'_j = s_k^{-1} s'_k$ for all j,k with $j \neq k$. This means that

$$g' := s_j^{-1} s'_j \text{ for all } j = 1,\dots,n.$$

Then $\delta(c) = c \otimes g = c \otimes g'$, so $g = g' = s_j^{-1} s'_j$ for all j. □

2. Semigroup dynamical systems with left inverses

Semigroup dynamical systems of the form (A, S, α) with S a right-reversible Ore semigroup and α an action by injective endomorphisms of a unital C^*-algebra A are useful models of some C^*-algebras arising in number theory. In such examples, the action α has a left inverse. To formalise this, a *left inverse for* α is an action $\alpha' : S \to \mathrm{End}(A)$ such that $\alpha'_s \circ \alpha_s = \mathrm{id}$ and $\alpha_s \circ \alpha'_s$ is multiplication by the projection $\alpha_s(1)$ for all $s \in S$. We shall assume that $\alpha_e(1) = 1$ and $\alpha_s(1) \neq 1$ for $s \in S \setminus \{e\}$. If (A, S, α, α') is such a system, then i_A induces an isomorphism $i_A : A \xrightarrow{\cong} (A \rtimes_\alpha S)_e$, see, e.g., [9, Proposition 3.1(1)] (note that the properties (iii)–(iv) are not needed for assertion (1) in that proposition). Note also that $\alpha'_s(a) = i_S(s)^* i_A(a) i_S(s)$ for all $s \in S$ and $a \in A$. We next note that such systems have a gauge-invariant uniqueness property.

Proposition 2.1 (The gauge-invariant uniqueness property). *Let S be an Ore semigroup with enveloping group $G = S^{-1}S$ and $\alpha : S \to \mathrm{End}(A)$ an action of S on a unital C^*-algebra A for which there is a left inverse $\alpha' : S \to \mathrm{End}(A)$. A surjective homomorphism $\varphi : A \rtimes_\alpha S \to C$ is injective if and only if $\varphi|_{i_A(A)}$ is injective and there is a maximal coaction η of G on C such that φ is δ–η-equivariant.*

Proof. Since δ is maximal, this follows from the gauge-invariant uniqueness theorem for maximal coactions from [5, Corollary A.2]. $\square$

Given a right-reversible semigroup S such that $S \cap S^{-1} = \{e\}$, assume moreover that for any two elements $s, s' \in S$ there is $\tilde{s} \in S$ satisfying $Ss \cap Ss' = S\tilde{s}$. Then the element $\tilde{s}$ is the least upper bound of s, s' with respect to the order $\leq_r$. We denote $\tilde{s} = s \vee_r s'$, and refer to S as being right-lattice ordered whenever any two elements admit a $\vee_r$.

Suppose that S is right-lattice ordered. Let $G = S^{-1}S$ be its group of left quotients. If $x = s^{-1}t \in G$, then $x \leq_r t$, so every element in G admits a right upper bound in S. It is not true in general that any $x \in G$ admits a least upper bound in S. Assume that every element in G admits a least upper bound in S with respect to $\leq_r$, in which case we refer to (G, S) as being right-lattice ordered. Then similar to the argument of [3, Lemma 7], which deals with the case of left-invariant orders, it follows that (G, S) is right-lattice ordered precisely when for each $g \in G = S^{-1}S$, there is a unique pair of elements $s, s' \in S$ such that:

1. $s \wedge_l s' = e$ (where $\wedge_l$ denotes greatest lower bound in G with respect to the left-invariant order $x \leq_l y \iff x^{-1}y \in S$),
2. $g = s^{-1}s'$, and
3. for any decomposition $g = z^{-1}z'$ with $z, z' \in S$ we have $s \leq_r z$ and $s' \leq_r z'$.

We denote $g_- := s$ and $g_+ := s'$ and refer to (g_-, g_+) as the minimal pair in $S \times S$ associated to g.

Under these assumptions, the spectral subspaces admit a particularly nice description.

Corollary 2.2. *Assume that (A, S, α) is a dynamical system where A is unital, (G, S) is right-lattice ordered, and α is an action by endomorphisms for which there is a left-inverse α'. Then for each $g \in G$ with associated minimal pair $(g_-, g_+) \in S \times S$ we have*

$$(A \rtimes_\alpha S)_g = \overline{\mathrm{span}}\,\{i_S(g_-)^* i_A(a) i_S(g_+) : a \in A\}.$$

Proof. By (1.2), it suffices to prove the left to right inclusion. Let $z, z' \in S$ such that $g = z^{-1}z'$. Then $g_- \leq_r z$ and $g_+ \leq_r z'$ with $z(g_-)^{-1} = z'(g_+)^{-1} \in S$, and the claim follows from the calculations

$$\begin{aligned} i_S(z)^* i_A(a) i_S(z') &= i_S(g_-)^* i_S(z(g_-)^{-1})^* i_A(a) i_S(z'(g_+)^{-1}) i_S(g_+) \\ &= i_S(g_-)^* i_A(\alpha'_{z(g_-)^{-1}}(a)) i_S(g_+). \end{aligned}$$ $\square$

Let (G, S) be a right-lattice ordered group and (A, S, α) a semigroup dynamical system with injective endomorphisms of the unital C^*-algebra A. Let δ

be the canonical coaction of G on $A \rtimes_\alpha S$. Suppose that $N_G : G \to (0,\infty)$ is a homomorphism, where $(0,\infty)$ has its multiplicative structure. The universal property of $A \rtimes_\alpha S$ implies that there is a one-parameter group of automorphisms $\sigma : \mathbb{R} \to \mathrm{Aut}(A \rtimes_\alpha S)$ such that

$$\sigma_t(i_S(s)^* i_A(a) i_S(s')) = N_G(s^{-1}s')^{it} i_S(s)^* i_A(a) i_S(s'). \tag{2.1}$$

We need to recall some terminology from [2]. A quantum statistical mechanical system $(\mathcal{A},\sigma)$ consists of a C^*-algebra $\mathcal{A}$ with a one-parameter group of automorphisms (a time evolution) σ. An element $c \in \mathcal{A}$ is an eigenvector of σ if there is $\lambda \in (0,\infty)$ such that $\sigma_t(c) = \lambda^{it} c$ for all $t \in \mathbb{R}$. An endomorphism of $(\mathcal{A},\sigma)$ is a $*$-homomorphism $\varphi : \mathcal{A} \to \mathcal{A}$ such that $\varphi \circ \sigma_t = \sigma_t \circ \varphi$ for all $t \in \mathbb{R}$.

Definition 2.3 ([2, Definition 1.8]). Suppose that $(\mathcal{A},\sigma)$ is a quantum statistical mechanical system. An *inner endomorphism* of $(\mathcal{A},\sigma)$ is an endomorphism φ of $(\mathcal{A},\sigma)$ such that there exists u an isometry in $\mathcal{A}$ and an eigenvector of σ for which $\varphi(x) = uxu^*$ for all $x \in \mathcal{A}$.

To simplify notation, we let $\mathcal{A}^\sigma_\lambda$ denote the space of eigenvectors c of σ such that $\sigma_t(c) = \lambda^{it} c$ for all $t \in \mathbb{R}$. By (2.1), $(A \rtimes_\alpha S)_g \subseteq (A \rtimes_\alpha S)^\sigma_{N_G(g)}$ for all $g \in G$. The next couple of results present partial converses to this inclusion.

Remark 2.4. Suppose (G,S) is a right-lattice ordered group, (A,S,α) is a semigroup dynamical system with a left inverse α', and σ is a time-evolution on $A \rtimes_\alpha S$ given as in (2.1). Let $(A \rtimes_\alpha S)^\dagger$ be the dagger subalgebra of $A \rtimes_\alpha S$. Suppose that S is abelian. We claim that every inner endomorphism of $(A \rtimes_\alpha S, \sigma)$ preserves the closure of the dagger subalgebra. To see this, assume first that $u = i_S(s)^* i_A(b) i_S(s')$ and $x = i_A(a) i_S(p) \in (A \rtimes_\alpha S)^\dagger$. Then

$$uxu^* = i_A\big(\alpha'_s(b\alpha_{s'}(a)\alpha_p(\alpha_{s'}(1)b^*))\big) i_S(p) \in (A \rtimes_\alpha S)^\dagger.$$

By continuity, we have $u\overline{(A \rtimes_\alpha S)^\dagger}u^* \subseteq \overline{(A \rtimes_\alpha S)^\dagger}$ for any isometry $u \in A \rtimes_\alpha S$, showing the claim.

There is no reason to expect in general that an endomorphism of $(A \rtimes_\alpha S, \sigma)$ that preserves the dagger subalgebra necessarily must be an inner endomorphism. In [2, Definition 1.8], a *dagger inner endomorphism* of $(\mathcal{A},\sigma)$ is an inner endomorphism of the form $\varphi(x) = uxu^*$ for all $x \in \mathcal{A}$, where $u \in (A \rtimes_\alpha S)^\dagger$ is an isometry and an eigenvector of σ. We make the following slight modification of this notion.

Definition 2.5. A *dagger inner endomorphism* of $(\mathcal{A},\sigma)$ is an inner endomorphism φ such that for some $u \in \overline{(A \rtimes_\alpha S)^\dagger}$ which is an eigenvector of σ we have $\varphi(x) = uxu^*$ for all $x \in \mathcal{A}$.

In the proof of [2, Proposition 10.1], it is observed that for the particular systems under consideration, inner endomorphisms that preserve the dagger subalgebra coincide with the dagger inner endomorphisms. Thus a natural question is whether it is possible to identify in greater generality elements among the inner endomorphisms that are dagger inner isomorphisms. We shall provide an affirmative answer to this question under some suitable conditions.

Assume that the homomorphism N_G which induces the time evolution in (2.1) satisfies the following two conditions:

Hom1. $N_G(S) \subseteq \mathbb{N} \setminus \{0\}$, and
Hom2. $N_G(s)$ and $N_G(s')$ are co-prime whenever $s, s' \in S \setminus \{e\}$ and $s \wedge_l s' = e$.

Lemma 2.6. *Suppose (G, S) is a right-lattice ordered group, (A, S, α) is a semigroup dynamical system with a left inverse α', and σ is the time-evolution on $A \rtimes_\alpha S$ associated to a homomorphism N_G that satisfies* (Hom1) *and* (Hom2).

Let $c \in (A \rtimes_\alpha S) \cap (A \rtimes_\alpha S)^\sigma_{m/n}$, where m, n are positive integers such that m, n are coprime. If $c^*c \in (A \rtimes_\alpha S)_e$, then there is $g \in G$ with $N_G(g_+) = m$ and $N_G(g_-) = n$ such that $c \in (A \rtimes_\alpha S)_g$.

Proof. From the hypothesis we have $\sigma_t(c) = (\frac{m}{n})^{it} c$. Suppose first that $c = \sum_{j=1}^J i_S(s_j)^* i_A(a_j) i_S(s'_j)$ for $s_j, s'_j \in S$ and $a_j \in A$. As in the proof of (1.2), the assumption that $c^*c \in (A \rtimes_\alpha S)_e$ implies that there is $g \in G$ such that $g = s_j^{-1} s'_j$ for all $j = 1, \ldots, J$. By Corollary 2.2, we may assume $c \in i_S(g_-)^* i_A(A) i_S(g_+)$, where $g = (g_-)^{-1} g_+$ is the minimal decomposition of g such that $g_- \wedge_l g_+ = e$. Then $\sigma_t(c) = (N_G((g_-)^{-1} g_+))^{it} c$, and thus $N_G(g_-)^{-1} N_G(g_+) = m/n$. By (Hom2), we must have $N_G(g_-) = n$ and $N_G(g_+) = m$, and the lemma follows. $\square$

As a consequence of Lemma 2.6, it follows that the set of fixed-points of σ_t is exactly $i_A(A)$.

Corollary 2.7. *We have $i_A(A) = (A \rtimes_\alpha S)^\sigma_1$.*

Proof. Clearly the set of fixed-points of σ_t is a C^*-subalgebra of $A \rtimes_\alpha S$ that contains $i_A(A)$ by the definition of σ. Let $u \in A \rtimes_\alpha S$ be a unitary such that $\sigma_t(u) = u$. Lemma 2.6 implies that there are $s, s' \in S$ with $N_G(s) = N_G(s')$ and $s \wedge_l s' = e$ such that $u \in i_S(s)^* i_A(A) i_S(s')$. This in connection with (Hom2) forces $s = s' = e$ and the corollary follows. $\square$

In examples, a system of the form (A, S, α, α') often has an additional feature. Denoting $v_s := i_S(s)$ for $s \in S$, assume that

$$v_{s'}^* v_s = v_s v_{s'}^* \text{ when } s \wedge_l s' = e \text{ in } S. \tag{2.2}$$

Assuming (2.2), it follows that for every $a \in A$, we have

$$\begin{aligned} i_A(\alpha'_{s'}(\alpha_s(a))) &= v_{s'}^* i_A(\alpha_s(a)) v_{s'} = v_{s'}^* v_s i_A(a) v_s^* v_{s'} \\ &= v_s v_{s'}^* i_A(a) v_{s'} v_s^* = i_A(\alpha_s(\alpha'_{s'}(a))). \end{aligned}$$

Thus (2.2) says that the actions α and α' satisfy the identity

$$\alpha'_{s'} \circ \alpha_s = \alpha_s \circ \alpha'_{s'} \text{ whenever } s, s' \in S \text{ with } s \wedge_l s' = e. \tag{2.3}$$

Note that the converse is valid, too: applying (2.3) to $a = 1$ and writing out the identity using the isometries v gives (2.2).

Lemma 2.8. *Assume that the actions α and α' satisfy* (2.3) *and $\alpha_s(1)$ are central projections in A for all $s \in S$. Let $u \in A \rtimes_\alpha S$ be an isometry with $u \in (A \rtimes_\alpha S)^\sigma_{m/n}$ for positive integers m, n such that m, n are coprime. Then $n = 1$ and there exists $s' \in S$ such that $N_G(s') = m$ and $u \in (A \rtimes_\alpha S)_{s'}$. In particular, u belongs to the closure of $(A \rtimes_\alpha S)^\dagger$.*

Proof. By Lemma 2.6, we may assume that $u = i_S(s)^* i_A(a) i_S(s')$ for $s, s' \in S$ with $N_G(s) = n, N_G(s') = m$ and $s \wedge_l s' = e$. Then

$$\begin{aligned} u^*u &= i_A\big(\alpha'_{s'}(a^*\alpha_s(1)a)\big) \\ &= i_A\big(\alpha'_{s'}(\alpha_s(1)a^*a)\big) \text{ since } \alpha_s(1) \text{ is central} \\ &= i_A(\alpha'_{s'} \circ \alpha_s \circ \alpha'_s(a^*a)) \\ &= i_A(\alpha_s \circ \alpha'_{s'} \circ \alpha'_s(a^*a))) \text{ by (2.3).} \end{aligned}$$

The assumption $u^*u = 1$ implies that $\alpha_s(a') = 1$ for $a' = \alpha'_{s'} \circ \alpha'_s(a^*a)$, from which we infer that $a' = \alpha'_s(\alpha_s(a')) = \alpha'_s(1) = 1$. Thus $\alpha_s(1) = 1$, which necessarily implies that $s = e$, and in particular that $n = 1$. The lemma follows. □

Theorem 2.9. *Suppose (G, S) is a right-lattice ordered group, (A, S, α) is a semigroup dynamical system with a left inverse α', and σ is the time-evolution on $A \rtimes_\alpha S$ associated to a homomorphism N_G that satisfies* (Hom1) *and* (Hom2). *Assume further that α and α' satisfy* (2.3) *and $\alpha_s(1)$ are central projections in A for all $s \in S$.*

Then every inner endomorphism of $(A \rtimes_\alpha S, \sigma)$ corresponding to a positive rational eigenvalue of the time evolution is a dagger inner endomorphism.

Proof. Assume that φ is an inner endomorphism of the form $\varphi(x) = uxu^*$ for $x \in A \rtimes_\alpha S$, where $u \in (A \rtimes_\alpha S)^\sigma_q$ for some rational q. Writing $q = m/n$ for $m, n \in \mathbb{N}$, $n \neq 0$, m, n co-prime, it follows from Lemma 2.8 that u is in the closure of $(A \rtimes_\alpha S)^\dagger$, as claimed. □

Given systems (A, S, α) and (B, R, β) where (G, S) and (K, R) are right-lattice ordered, assume α' is a left-inverse for α such that $\alpha'_s(1) = 1$, $\alpha_s(1)$ is central in A for all $s \in S$ and α, α' satisfy (2.3), and similarly β' is a left-inverse for β such that $\beta'_r(1) = 1$, $\beta_r(1)$ are central in B for all $r \in R$ and β, β' satisfy (2.3). Let $N_G : G \to (0, \infty)$ and $N_K : K \to (0, \infty)$ be homomorphisms satisfying (Hom1)–(Hom2), and let σ and τ, respectively, be the time evolutions on $A \rtimes_\alpha S$ and $B \rtimes_\beta R$ given as in (2.1). Suppose that $\phi : A \rtimes_\alpha S \to B \rtimes_\beta R$ is an isomorphism such that $\phi \circ \sigma_t = \tau_t \circ \phi$ for all $t \in \mathbb{R}$. The first observation is that $\phi|_{i_A(A)} : i_A(A) \to i_B(B)$ is an isomorphism. Indeed, since $\phi(i_A(a))$ is a fixed point of τ_t, it follows that $\phi(i_A(a)) \in i_B(B)$ by Corollary 2.7. Hence $\phi(i_A(A)) \subseteq i_B(B)$, and the opposite direction is similar using the inverse ϕ^{-1}.

Let $y \in (A \rtimes_\alpha S)^\delta_s$ where $N_G(s) = m$ for some positive integer $m \geq 1$. The equivariance of ϕ with respect to σ and τ shows that $\phi(y) \in (B \rtimes^{\mathrm{alg}}_\beta R) \cap (B \rtimes_\beta R)^\tau_m$. Thus by Lemma 2.6 there is $r \in R$ with $N_K(r) = m$ such that $\phi(y) \in (B \rtimes_\beta R)^\varepsilon_r$. It is not clear in general that an isomorphism ϕ as above will preserve the dagger subalgebras. In the situation of [2], it is part of the assumptions that an isomorphism between $(\mathcal{A}, \sigma)$ and $(\mathcal{B}, \tau)$ preserves the dagger subalgebras. In the present setup, the following partial result holds true.

Corollary 2.10. *If ϕ is an isomorphism of quantum statistical mechanical systems $(A \rtimes_\alpha S, \sigma)$ and $(B \rtimes_\beta R, \tau)$ as above, then for every $m \geq 1$, ϕ is an isomorphism*

$$\phi : \overline{(A \rtimes_\alpha S)^\dagger} \cap (A \rtimes_\alpha S)^\sigma_m \to \overline{(R \rtimes_\beta R)^\dagger} \cap (B \rtimes_\beta R)^\varepsilon_m.$$

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