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Editorial Introduction

This volume is devoted to the Workshop on Operator Theory, Operator Algebras and Applications, WOAT 2012, held at Instituto Superior Técnico, Universidade de Lisboa, Portugal, from September 11th to September 14th of 2012. The main scientific goal of WOAT 2012 was to present developments in Operator Theory, Operator Algebras and their applications and to promote research exchanges in the areas of Operator Theory and Operator Algebras. WOAT 2012 continued a series of conferences organized by the Center for Functional Analysis and Applications since 2006 and was dedicated to Professor António Ferreira dos Santos on the occasion of his seventieth birthday.

This volume consists of 22 peer-reviewed papers contributing to the main topics of WOAT 2012. More specifically, it contains the articles on Operator Theory and Harmonic Analysis (in particular, singular integral operators with shifts, pseudodifferential operators, factorization of almost periodic functions, Riesz potential operators, inequalities), on Operators Algebras (namely, Fredholm and invertibility theory in C^*-algebras, Følner sequences, sequences related to the finite section method), and on Mathematical Physics (operator approach to diffraction problems, Poisson structures).

The organizers acknowledge the support of the WOAT 2012 sponsors: The Portuguese Foundation for Science and Technology and the Center for Functional Analysis and Applications. The organizers would also like to thank the Associação de Turismo de Lisboa for providing publicity material on Lisbon.

The editors of this volume express their deep gratitude to Birkhäuser's editorial team, Sylvia Lotrovsky and Thomas Hempfling, for their kind and prompt help during the preparation of the volume

Lisbon, 20th October 2013 The Editorial Board

António Ferreira dos Santos

Scientific Life of António Ferreira dos Santos

(Communication of M. Amélia Bastos in the WOAT 2012 opening session)

Professor Ferreira dos Santos was born in 1939 in the town of Almeirim. Determination and passion for knowledge are some of the personal characteristics which were typical for him since his early youth. At that time he used to cross the bridge over Tejo river every day by bike to get to Santarém where he attended his secondary school. This bridge, pictured on WOAT 2012 poster, symbolizes precisely this determination to cross the barriers in order to reach knowledge.

Despite his family's economic difficulties, he managed to complete his university studies with the help of a grant from Gulbenkian Foundation which was awarded to him as the best student of Liceu de Santarém in 1957. With this support he finished his studies at Instituto Superior Técnico graduating with Electrical Engineering degree in 1963. In 1966, also with the support of Gulbenkian Foundation, he went to London where he received his PhD degree in Electrical Engineering from the Imperial College, for his work on wave propagation. Professor Ferreira do Santos returned to Instituto Superior Técnico in 1971 as an assistant professor, thus starting his university career. In the beginning he joined the Centro de Análise e Processamento de Sinais, where he continued his research on Mathematical Methods in Wave Propagation, publishing papers and supervising two PhD students. At the end of the 80s he decided to move to Pure Mathematics. He did the habilitation in Mathematics in 1989 and has been full professor in Mathematics since that time.

Professor Ferreira dos Santos is the founder of the research group in Operator Theory in Portugal. I vividly recall the first meetings under his direction, when four young researchers, Francisco Teixeira, Amarino Lebre, Cristina Câmara and I, were actively searching for areas of Operator Theory in which it was possible to make substantial contributions. I also acknowledge the importance of the international support of Professor Gohberg, Professor Meister and Professor Kaashoek which our group enjoyed at the time.

Professor Ferreira dos Santos was also the main founder of the Center of Applied Mathematics and later of the Center for Functional Analysis and Applications, the president of which he had been until his retirement in 2009. During his research career he published more than forty papers and made important contribution to several topics in Operator Theory, in particular in the factorization theory and the theory of singular integral operators. To name a few: Factorization of matrix functions of Daniele–Khrapkov type and factorization on a Riemann surface; Fredholm theory for equations on compact intervals; Factorization of almost periodic matrix functions; Relations between the factorization theory and corona problems; Applications of the factorization theory to integrable systems. His research activity also includes the supervision of six PhD students who all now are professors at the Mathematics Department of IST.

Professor Ferreira dos Santos was also very active as an educator. He introduced and was responsible for a number of courses, both at an undergraduate and master levels: Applied Mathematics for Electrical Engineering; Complex Analysis; Functional Analysis; Integral Equations: Linear Operators; Operators Algebras; Mathematical Analysis. For some of these courses he wrote lecture notes. Professor Ferreira dos Santos played a crucial role in the IST management. He was a member of the Management Board of IST in 1974–75 and President of the Management Board in 1978–1979. He was also President of the Pedagogical Board of IST in 1979–80. In the periods of 1983–1984 and 1991–1992 he was President of the Mathematics Department. Since the beginning of 1983 until 1991 he was the coordinator of the Master Course in Applied Mathematics. In the period of 2001–2002 and 2003–2004 he was President of the Scientific Board of IST.

In all aspects of his activities, Professor Ferreira dos Santos always acted with determination and passion, and he conveyed this approach to all his collaborators. From myself, and on behalf of the Center for Functional Analysis and Applications, I would like to thank Professor Ferreira dos Santos for his outstanding contribution to the research in Operator Theory in Portugal and the development of Mathematics.

Professor Ferreira dos Santos, I wish you to continue crossing different bridges to extend boundaries of knowledge with the same determination, passion and joy for many years to come!

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Følner Sequences in Operator Theory and Operator Algebras

Pere Ara, Fernando Lledó and Dmitry V. Yakubovich

Abstract. The present article is a review of recent developments concerning the notion of Følner sequences both in operator theory and operator algebras. We also give a new direct proof that any essentially normal operator has an increasing Følner sequence $\{P_n\}$ of non-zero finite rank projections that strongly converges to $\mathbb{1}$. The proof is based on Brown–Douglas–Fillmore theory. We use Følner sequences to analyze the class of finite operators introduced by Williams in 1970. In the second part of this article we examine a procedure of approximating any amenable trace on a unital and separable C^*-algebra by tracial states $\mathrm{Tr}(\cdot P_n)/\mathrm{Tr}(P_n)$ corresponding to a Følner sequence and apply this method to improve spectral approximation results due to Arveson and Bédos. The article concludes with the analysis of C^*-algebras admitting a non-degenerate representation which has a Følner sequence or, equivalently, an amenable trace. We give an abstract characterization of these algebras in terms of unital completely positive maps and define Følner C^*-algebras as those unital separable C^*-algebras that satisfy these equivalent conditions. This is analogous to Voiculescu's abstract characterization of quasidiagonal C^*-algebras.

Mathematics Subject Classification (2010). Primary 43A07; Secondary 47B20, 47A65, 46L05.

Keywords. Følner sequences, non-normal operators, essentially normal operators, C^*-algebra, amenable trace, spectral approximation.

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1. Introduction

In their beginnings the single operator theory and the theory of operator algebras were a common subject and shared many techniques. As an example recall that von Neumann algebras were introduced (as *rings of operators*) in 1929 by von Neumann in his second paper on spectral theory [39]. In recent times, however, each of these theories has developed own elaborated techniques which in many cases remain unknown to experts of the other area. Nevertheless single operator theory and the theory of operator algebras have also had fruitful and important interactions ever since. Brown, Douglas and Fillmore's theory was motivated by the classification of essentially normal operators and ended with the introduction of the *Ext*-group as a fundamental invariant for operator algebras. Finally, Voiculescu's work on quasidiagonality also shows the importance of the dialog between these communities (cf. [26, 49, 50, 48]).

In more recent times operator algebra techniques, in particular exact C^*-algebras, have also been used to solve Herrero's approximation problem for quasidiagonal operators (cf. [15]). Moreover, operator algebras have shown to be a useful tool in order to address problems in spectral approximation: given a sequence of linear operators $\{T_n\}_{n\in\mathbb{N}}$ in a complex separable Hilbert space $\mathcal{H}$ that approximates an operator T in a suitable sense, a natural question is how do the spectral characteristics of T (the spectrum, spectral measures, numerical ranges, pseudospectra etc.) relate with those of T_n as n grows. (Excellent books that include a large number of examples and references are, e.g., [21, 2]. See also [11, 29] for the application of C^*-algebra techniques in numerical analysis.) Arveson's seminal series of articles [3, 4, 5] on this topic were directly inspired by Szegő's classical approximation theorem for Toeplitz operators. Among other interesting results, Arveson gave conditions that guarantee that the essential spectrum of a large class of selfadjoint operators T may be recovered from the sequence of eigenvalues of certain finite-dimensional compressions T_n. These results were then refined by Bédos who systematically applied the concept of Følner sequence of non-zero finite rank projections to spectral approximation problems (see [8, 7, 6] as well as [29, 37]; for a precise definition of Følner sequence and additional results we refer to Section 2). It is stated in Section 7.2 of [29] that SeLegue also considered Szegő-type theorems for Toeplitz operators in the context of C^*-algebras. Hansen extends some of the mentioned results to the case of unbounded operators (cf. [33, § 7]; see also [34] for recent developments in the non-selfadjoint case). Brown shows in [16] that abstract results in C^*-algebra theory can be applied to compute spectra of important operators in mathematical physics like almost Mathieu operators or periodic magnetic Schrödinger operators on graphs.

In the last two decades, the relation between spectral approximation problems and Følner sequences for non-selfadjoint and non-normal operators has been also explored, see, for instance, [52, 45, 12, 43].

The aim of this article is to present in a single publication recent operator theoretic and operators algebraic results that involve the notion of Følner sequences for operator. Følner sequences were introduced in the context of operator algebras by Connes in Section V of his seminal paper [22] (see also [23, Section 2]). This notion is an algebraic analogue of Følner's characterization of amenable discrete groups and was used by Connes as an essential tool in the classification of injective type II_1 factors. Part of the material of this paper is taken from [1, 38]. There is though a new and complete proof that any essentially normal operator has a proper Følner sequence (cf. Subsection 3.1) which, in our opinion, is interesting in its own right. The proof is based on the absorbing property for direct sums stated in Proposition 2.3 and pure operator theoretic arguments including Brown–Douglas–Fillmore theory.

In this article we will present (with the exception of Subsection 3.1) only short proofs that improve the comprehension of the statement or that contain useful techniques. For more difficult and elaborate arguments we will refer to the original publications. Section 3 is completed with the analysis of the relations between the class of finite operators (introduced by Williams in [53]) and the notion of Følner sequence. It shown that Følner sequences for operators provide a very useful and natural tool to analyze this class of operators. In the last section we will study the role of Følner sequences in operator algebras. First we review the relation between Følner sequences for a unital and separable C^*-algebra $\mathcal{A}$ and amenable traces. In particular, we present an approximation procedure for amenable traces in terms of Følner sequences of projections [41, Theorem 6.1] (see also [17, Theorem 6.2.7]). We apply this method in Theorem 4.2 to extend a spectral approximation result for scalar spectral measures in the spirit of Arveson and Bédos. In Subsection 4.2 we give finally an abstract characterization of unital separable C^*-algebras $\mathcal{A}$ admitting a non-degenerate representation π on a Hilbert space such that there is a Følner sequence for $\pi(\mathcal{A})$ or, equivalently, such that $\pi(\mathcal{A})$ has an amenable trace (see Theorem 4.7). We conclude with a brief discussion of C^*-algebras that can also be related to a given Følner sequence and that appear naturally in the context of spectral approximation problems. In the last section we summarize some of the main relations and differences in the analysis of Følner sequences for single operators and for abstract C^*-algebras.

Notation: We will denote by $\mathcal{L}(\mathcal{H})$ the C^*-algebra of bounded and linear operators on the complex separable Hilbert space $\mathcal{H}$, and by $\mathcal{K}(\mathcal{H})$ the ideal of compact operators on $\mathcal{H}$. Next, $\mathcal{P}_{\mathrm{fin}}(\mathcal{H})$ is the set of all non-zero finite rank orthogonal projections on $\mathcal{H}$ and $[A,B] := AB - BA$ stands for the commutator of two operators $A, B \in \mathcal{L}(\mathcal{H})$. We denote by $\mathrm{Tr}(\cdot)$ the standard trace on $\mathcal{L}(\mathcal{H})$ and by $\mathrm{tr}(\cdot)$ the unique tracial state on a matrix algebra $M_n(\mathbb{C})$, $n \in \mathbb{N}$.

2. Basic properties of Følner sequences for operators

The notion of Følner sequences for operators has its origins in group theory. Recall that a discrete countable group Γ is said to be *amenable* if it has an invariant mean, i.e., there is a positive linear functional ψ on the von Neumann algebra[1] $\ell^\infty(\Gamma)$ with norm one such that

$$\psi(\gamma f) = \psi(f) \;, \quad \gamma \in \Gamma \;, \quad f \in \ell^\infty(\Gamma) \;,$$

where $(\gamma f)(\gamma_0) := f(\gamma^{-1}\gamma_0)$. Abelian groups, finite groups and their extensions are amenable. A Følner sequence for the group Γ is a sequence of non-empty finite subsets $\Gamma_i \subset \Gamma$ that satisfy

$$\lim_i \frac{|(\gamma\Gamma_i)\triangle\Gamma_i|}{|\Gamma_i|} = 0 \qquad \text{for all} \quad \gamma \in \Gamma \;, \tag{2.1}$$

where $\triangle$ denotes the symmetric difference and $|X|$ is the cardinality of a set X. Then, Γ has a Følner sequence if and only if Γ is amenable (see, e.g., Chapter 4 in [42]). An analysis of different properties of approximability of a group by finite groups and their relation to amenability has been undertaken in the review [47].

The counterpart of the preceding definition in the context of operators is given next. First we need to recall that if $T \in \mathcal{L}(\mathcal{H})$, then $\|T\|_p$, $p = 1, 2, \ldots$, is its norm in the Schatten–von Neumann class.

Definition 2.1. Let $\mathcal{T} \subset \mathcal{L}(\mathcal{H})$ be a set of operators. A sequence of non-zero finite rank orthogonal projections $\{P_n\}_{n\in\mathbb{N}} \subset \mathcal{P}_{\rm fin}(\mathcal{H})$ on $\mathcal{H}$ is called a *Følner sequence for $\mathcal{T}$* if

$$\lim_{n\to\infty} \frac{\|TP_n - P_nT\|_2}{\|P_n\|_2} = 0 \;\;, \quad T \in \mathcal{T} \;. \tag{2.2}$$

If the Følner sequence $\{P_n\}_{n\in\mathbb{N}}$ satisfies, in addition, that it is increasing and converges strongly to $\mathbb{1}$, then we say it is a *proper Følner sequence for $\mathcal{T}$*.

The existence of a Følner sequence has already important structural consequences, see, for instance, Proposition 2.5 and Corollary 3.9 below. Notice, however, that proper Følner sequences are important in the context of spectral approximation in the spirit of works [52, 45, 43] and others.

In the preceding definition we have not specified any structure on the set of operators $\mathcal{T}$. Typically, $\mathcal{T}$ will be a single operator or a concrete C^*-algebra, realized in a Hilbert space $\mathcal{H}$. The next result collects some immediate consequences of the definition of a Følner sequence for operators. Part (ii) is shown in Lemma 1 of [6] (see also [38, Proposition 2.1]).

Proposition 2.2. *Let $\mathcal{T} \subset \mathcal{L}(\mathcal{H})$ be a set of operators and $\{P_n\}_{n\in\mathbb{N}} \subset \mathcal{P}_{fin}(\mathcal{H})$ a sequence of non-zero finite rank orthogonal projections. Then we have*

(i) *$\{P_n\}_{n\in\mathbb{N}}$ is a Følner sequence for $\mathcal{T}$ if and only if it is a Følner sequence for $C^*(\mathcal{T}, \mathbb{1})$, where $C^*(\cdot)$ is the C^*-algebra generated by its argument. Moreover, $\{P_n\}_{n\in\mathbb{N}}$ is a proper Følner sequence for $\mathcal{T}$ if and only if it is a proper Følner sequence for*

$$C^*(\mathcal{T}, \mathcal{K}(\mathcal{H}), \mathbb{1}) \;.$$

[1] We identify here each $f \in \ell^\infty(\Gamma)$ with the multiplication operator with f on the Hilbert space $\ell_2(\Gamma)$.

(ii) $\{P_n\}_{n\in\mathbb{N}}$ *is a Følner sequence for* $\mathcal{T}$ *if and only if the following condition holds:*

$$\lim_{n\to\infty} \frac{\|TP_n - P_nT\|_1}{\|P_n\|_1} = 0 \,, \quad T \in \mathcal{T} \,. \tag{2.3}$$

If $\mathcal{T}$ *is a self-adjoint set (i.e.,* $\mathcal{T}^* = \mathcal{T}$*), then* $\{P_n\}_{n\in\mathbb{N}}$ *is a Følner sequence for* $\mathcal{T}$ *if and only if for all* $T \in \mathcal{T}$*:*

$$\lim_{n\to\infty} \frac{\|(I-P_n)TP_n\|_p}{\|P_n\|_p} = 0, \qquad p \in \{1,2\} \,. \tag{2.4}$$

(iii) *Let* $T \in \mathcal{L}(\mathcal{H})$ *and* $\{Q_n\}_n \subset \mathcal{P}_{\mathit{fin}}(\mathcal{H})$ *be such that the sequence* $\{\dim Q_n\}$ *is unbounded and*

$$\lim_{n\to\infty} \frac{\|TQ_n - Q_nT\|_2}{\|Q_n\|_2} = 0 \,.$$

Then there exists a proper Følner sequence for T*.*

Proper Følner sequences for operators have the following characteristic absorbing property for direct sums:

Proposition 2.3. *Let* $\mathcal{H}$ *and* $\mathcal{H}'$ *be separable Hilbert spaces with* $\dim \mathcal{H} = \infty$*. If* T *has a proper Følner sequence, then* $T \oplus X \in \mathcal{L}(\mathcal{H} \oplus \mathcal{H}')$ *has a proper Følner sequence for* any $X \in \mathcal{L}(\mathcal{H}')$*.*

Proof. Let $\{P_n\}_{n\in\mathbb{N}}$ be a proper Følner sequence for T and since the sequence of projections is increasing we may assume that $\dim P_n\mathcal{H} \geq n^2$. Let $\{e_i \mid i \in \mathbb{N}\}$ be an orthonormal basis of $\mathcal{H}'$ and denote by Q_n the orthogonal projection onto $\mathrm{span}\{e_1, \ldots, e_n\} \subset \mathcal{H}'$. Then the following calculation shows that $\{P_n \oplus Q_n\}_n$ is a proper Følner sequence for $T \oplus X$, $X \in \mathcal{L}(\mathcal{H}')$:

$$\begin{aligned}
\frac{\big\|\,[T\oplus X, P_n \oplus Q_n]\,\big\|_2^2}{\|P_n \oplus Q_n\|_2^2} &= \frac{\|[T,P_n]\|_2^2 + \|[X,Q_n]\|_2^2}{\|P_n\|_2^2 + n} \\
&\leq \frac{\|[T,P_n]\|_2^2}{\|P_n\|_2^2} + \frac{4\,\|Q_n\|_2^2\;\|X\|^2}{n^2+n} \\
&= \frac{\|[T,P_n]\|_2^2}{\|P_n\|_2^2} + 4\,\|X\|\,\frac{n}{n^2+n} \quad \underset{n\to\infty}{\longrightarrow} \quad 0\,.
\end{aligned}$$

□

Next, we mention some first operator algebraic consequences related to the existence of Følner sequences. For this we need to recall the following notion:

Definition 2.4. A state τ on the unital C^*-algebra $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ (i.e., a positive and normalized linear functional on $\mathcal{A}$) is called an *amenable trace* if there exists a state ψ on $\mathcal{L}(\mathcal{H})$ such that $\psi \restriction \mathcal{A} = \tau$ and

$$\psi(XA) = \psi(AX) \,, \quad X \in \mathcal{L}(\mathcal{H}) \,, \; A \in \mathcal{A} \,.$$

The state ψ is also referred in the literature as a *hypertrace for* $\mathcal{A}$.

Note that an amenable trace is really a trace on $\mathcal{A}$ (i.e., $\tau(AB) = \tau(BA)$, $A, B \in \mathcal{A}$). We also refer to [13, 41] for a thorough description of the relations of amenable traces and Følner sequences to other important areas like, e.g., Connes' embedding problem. Hypertraces are the algebraic analogue of the invariant mean on groups mentioned at the beginning of this section. Later we will need the following standard result. (See [22, 23] for the original statement and more results in the context of operator algebras; see also [7, 1] for additional results in the context of C^*-algebras related to the existence of a hypertrace.)

Proposition 2.5. *Let $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ be a separable unital C^*-algebra. Then $\mathcal{A}$ has a Følner sequence if and only if $\mathcal{A}$ has an amenable trace.*

Finally, we also mention the following useful results in the context of single operator theory. We need to introduce first the following definition.

Definition 2.6. We say that $T \in \mathcal{L}(\mathcal{H})$ is *finite block reducible* if T has a non-trivial finite-dimensional reducing subspace, i.e., there is an orthogonal decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ which reduces T and where $\mathcal{H}_0$ is finite dimensional and non-zero.

The following two propositions are technical and we refer to Section 3 in [38] for a complete proof.

Proposition 2.7. *Let $T = T_0 \oplus \widetilde{T}$ on $\mathcal{H} = \mathcal{H}_0 \oplus \widetilde{\mathcal{H}}$, where $\dim \mathcal{H}_0 < \infty$. Then, T has a proper Følner sequence if and only if $\widetilde{T}$ has a proper Følner sequence.*

Note that in the reverse implication of Proposition 2.5 the sequence of projections does not have to be a proper Følner sequence in the sense of Definition 2.1. In fact, one can easily construct the following counterexample: consider a finite block reducible operator $T = T_0 \oplus T_1$ on the Hilbert space $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$, with $1 \leq \dim \mathcal{H}_0 < \infty$ and where T_1 has no Følner sequence (examples of these type of operators will be given in Section 3.3). Then, one can show that $C^*(T, \mathbb{1})$ has a hypertrace (see Williams' theorem in Subsection 3.2) and by Proposition 2.5 it has a Følner sequence also. The obvious choice of Følner sequence is the constant sequence $P_n = \mathbb{1}_{\mathcal{H}_0} \oplus 0$, $n \in \mathbb{N}$, which trivially satisfies (2.2) for T. But T cannot have a proper Følner sequence, because T_1 has no Følner sequence by Proposition 2.7.

The following proposition clarifies the relation between Følner sequences and proper Følner sequences in the context of operator theory. In a sense the difference between Følner sequence and proper Følner sequence can only appear if the operator is finite block reducible.

Proposition 2.8. *Let $T \in \mathcal{L}(\mathcal{H})$ and suppose that $TP - PT \neq 0$ for all $P \in \mathcal{P}_{fin}(\mathcal{H})$. If there is a Følner sequence of projections $\{P_n\}_n \subset \mathcal{P}_{fin}(\mathcal{H})$ of a constant rank, then T has a proper Følner sequence.*

3. Følner sequences in operator theory

Using a classical result by Berg that states that any normal operator can be expressed as a sum of a diagonal operator and a compact operator (cf. [26, Section II.4]) it is immediate that any normal operator has a proper Følner sequence. In the next subsection we will address the question of existence of proper Følner sequences for an important class of non-normal operators. We will also explore the structure of operators that have no proper Følner sequence. Finally we will show the strong link between the class of finite operators (in the sense of Williams [53]) and the notion of Følner sequence.

3.1. Essentially normal operators

In this subsection we give a proof of the fact that any essentially normal operator has a proper Følner sequence. The proof below is from an earlier version of [38]. In this reference we present a stronger statement, namely that any essentially hyponormal operator has a proper Følner sequence by using different techniques (see Theorem 5.1 in [38])[2].

Nevertheless in our opinion the present direct proof is interesting in itself and the reasoning is completely different from that in [38]. The proof below is based on the absorbing property for direct sums given in Proposition 2.3 and pure operator theoretic arguments including Brown–Douglas–Fillmore theory.

We begin showing that the unilateral shift S has a canonical proper Følner sequence. In fact, define S on $\mathcal{H} := \ell^2(\mathbb{N}_0)$ by $Se_i := e_{i+1}$, where $\{e_i \mid i = 0, 1, 2, \dots\}$ is the canonical basis of $\mathcal{H}$ and consider for any n the orthogonal projection P_n onto $\mathrm{span}\{e_i \mid i = 0, 1, 2, \dots, n\}$. Then

$$\big\|[P_n, S]\big\|_2^2 = \sum\nolimits_{i=1}^{\infty} \Big\|[P_n, S]e_i\Big\|^2 = \|e_{n+1}\|^2 = 1$$

and

$$\frac{\big\|[P_n, S]\big\|_2}{\|P_n\|_2} = \frac{1}{\sqrt{n+1}} \underset{n\to\infty}{\longrightarrow} 0\,.$$

Next we recall some definitions and facts concerning essentially normal operators. Details and additional references can be found, e.g., in [19, 28]; see also [27] for an excellent brief up-to day account of essential normality and the Brown–Douglas–Fillmore theory. An operator $T \in \mathcal{L}(\mathcal{H})$ is called *essentially normal* if its self-commutator is a compact operator, i.e., if $[T, T^*] \in \mathcal{K}(\mathcal{H})$. If ρ is the quotient map from $\mathcal{L}(\mathcal{H})$ onto the Calkin algebra $\mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$, then T is essentially normal if and only if $\rho(T)$ is normal in the Calkin algebra. The unilateral shift S mentioned above is a standard example of an essentially normal operator, since its self-commutator is a rank 1 projection. We recall that an operator $F \in \mathcal{L}(\mathcal{H})$ is called *Fredholm* if its range $\operatorname{ran} F$ is closed and both $\ker F$ and $(\operatorname{ran} F)^\perp$ are finite

[2] An operator $T \in \mathcal{L}(\mathcal{H})$ is called *hyponormal* if its self-commutator $[T^*, T]$ is nonnegative. T is called *essentially hyponormal* if the image in the Calkin algebra $\mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ of $[T^*, T]$ is a nonnegative element. Any essentially normal operator is essentially hyponormal (see, e.g., [24, Chapter 4] or the review [51] for additional results).

dimensional. The index of a Fredholm operator F is defined as

$$\operatorname{ind}(F) = \dim\ker F - \dim(\operatorname{ran} F)^{\perp} .$$

The essential spectrum of an operator T is

$$\sigma_{\mathrm{ess}}(T) := \{\lambda \in \mathbb{C} \mid T - \lambda\mathbb{1} \ \text{ is not Fredholm}\} .$$

If F is Fredholm and K is compact, then $F + K$ is Fredholm and $\operatorname{ind}(F + K) = \operatorname{ind}(F)$. Finally, $F \in \mathcal{L}(\mathcal{H})$ is Fredholm if and only if $\rho(F)$ is invertible in the Calkin algebra. Therefore, the essential spectrum of any $T \in \mathcal{L}(\mathcal{H})$ coincides with the spectrum of $\rho(T)$. We refer to Section I.8 in [44] for an accessible exposition of Fredholm operators.

We will need later the following standard facts:

Proposition 3.1. *Let $\{T_n\}_{n\in\mathbb{N}}$ be a sequence of bounded operators in $\mathcal{L}(\mathcal{H}_n)$.*

(i) *Assume* $\sup_n \{\|T_n\|\} < \infty$ *and define the bounded operator $\widehat{T} = \oplus_n T_n$ on $\oplus_n \mathcal{H}_n$. Then, $\widehat{T}$ is invertible if and only if each T_n is invertible and*

$$\sup_n \left\{\|T_n^{-1}\|\right\} < \infty .$$

(ii) $\sigma_{\mathrm{ess}}(T_1 \oplus T_2) = \sigma_{\mathrm{ess}}(T_1) \cup \sigma_{\mathrm{ess}}(T_2)$.

(iii) *If T_1, T_2 are Fredholm operators, then* $\operatorname{ind}(T_1 \oplus T_2) = \operatorname{ind}(T_1) + \operatorname{ind}(T_2)$.

The proof of the main result of this subsection is based on the existence of operators having specific spectral properties. In what follows, for a subset Ω of the complex plane, we denote by $\Omega^{\,\mathrm{cl}}$ its closure and put

$$\overline{\Omega} = \{\bar{z} \mid z \in \Omega\} .$$

We will use the space $R^2(\Omega)$, defined as the closure in $L^2(\Omega)$ (with the Lebesgue measure) of the set of rational functions with poles off $\Omega^{\,\mathrm{cl}}$ (see, e.g., Chapter 1 in [24]).

We need to recall here also some other standard notions in operator theory. An operator $T \in \mathcal{L}(\mathcal{H})$ is called *finitely multicyclic* if there are finitely many vectors $g_1, \ldots, g_m \in \mathcal{H}$ such that the span of the set

$$\{u(T)g_i \mid 1 \le i \le m\ ,\ \ u \ \text{ rational function with poles off } \sigma(T)\}$$

is dense in $\mathcal{H}$. The vectors $g_1, \ldots, g_m$ are called a cyclic set of vectors. If T is finitely multicyclic and m is the smallest number of cyclic vectors, then T is called m-multicyclic.

For the reader's convenience, we recall the following classical result due to Berger and Shaw and which we will use several times. For details we refer to the original article [9] or to Section IV.2 in [24].

Theorem 3.2 (Berger–Shaw). *Suppose T is an m-multicyclic hyponormal operator. Then its self-commutator $[T^*, T]$ is of trace class and the canonical trace satisfies*

$$\operatorname{Tr}\left([T^*, T]\right) \le \frac{m}{\pi}\,\operatorname{area}(\sigma(T)) ,$$

where $\sigma(T)$ denotes the spectrum of T.

Lemma 3.3. *Let Ω be an open, bounded and connected subset of $\mathbb{C}$. Then, the multiplication operator on $R^2(\Omega)$ given by*

$$(M_\Omega f)(z) := z\, f(z)\,, \quad f \in R^2(\Omega)\,,$$

satisfies the following properties:

(i) $\sigma(M_\Omega) = \Omega^{\,cl}$ *and* $\|M_\Omega\| = \max_{z\in\Omega^{\,cl}}\{|z|\}$.
(ii) $\sigma_{\mathrm{ess}}(M_\Omega) \subset \partial\Omega$ *and*

$$\operatorname{ind}(M_\Omega - \lambda \mathbb{1}) = \begin{cases} 0\,, & \lambda \notin \Omega^{\,cl} \\ -1\,, & \lambda \in \Omega\,. \end{cases}$$

(iii) $\|(M_\Omega - \lambda)^{-1}\| = \left(\operatorname{dist}\left(\lambda, \Omega^{\,cl}\right)\right)^{-1}\,,\ \lambda \notin \Omega^{\,cl}$.
(iv) M_Ω *is a hyponormal operator.*[3]
(v) *The self-commutator* $[M_\Omega^*, M_\Omega]$ *is a trace-class operator and*

$$\operatorname{Tr}\left([M_\Omega^*, M_\Omega]\right) \le \frac{1}{\pi}\operatorname{area}(\Omega)\,. \tag{3.1}$$

Proof. It is a standard fact that $R^2(\Omega)$ consists of analytic functions on Ω and that for any $\lambda \in \Omega$, the evaluation functional $f \mapsto f(\lambda)$ is bounded on $R^2(\Omega)$ (see, e.g., Section II.7 in [24]). Parts (i) and (iii) follow from standard properties of the multiplication operator. (Note that for $\lambda \notin \Omega^{\,\mathrm{cl}}$ the function $(z-\lambda)^{-1}$ is bounded and analytic in Ω.) To prove (ii), it suffices to observe that $\ker(M_\Omega - \lambda) = \{0\}$ for any $\lambda \in \mathbb{C}$, that $\operatorname{Ran}(M_\Omega - \lambda) = R^2(\Omega)$ for $\lambda \notin \Omega^{\,\mathrm{cl}}$ and that

$$\operatorname{Ran}(M_\Omega - \lambda) = \left\{f \in R^2(\Omega) \mid f(\lambda) = 0\right\} \quad \text{for } \lambda \in \Omega\,.$$

This gives the formula for the index stated above.

To prove (iv) note that $M_\Omega^* f = Q_R(\overline{z}\, f)$, $f \in R^2(\Omega)$, where Q_R denotes the orthogonal projection from $L^2(\Omega)$ onto $R^2(\Omega)$. Therefore $\|M_\Omega^* f\| \le \|M_\Omega f\|$, $f \in R^2(\Omega)$, which implies that M_Ω is a hyponormal. Finally, by the definition of $R^2(\Omega)$, the constant function 1 is cyclic for M_Ω, so that M_Ω is 1-multicyclic. Hence we can apply the Berger–Shaw Theorem to conclude that $[M_\Omega^*, M_\Omega]$ is a trace-class operator and that the inequality stated above holds. ☐

Definition 3.4. Let $T, R \in \mathcal{L}(\mathcal{H})$ be essentially normal operators. We say that T and R have the same spectral picture the following two conditions hold:

(i) $\sigma_{\mathrm{ess}}(T) = \sigma_{\mathrm{ess}}(R) =: X$
(ii) $\operatorname{ind}(T - \lambda\mathbb{1}) = \operatorname{ind}(R - \lambda\mathbb{1})\,,\ \lambda \notin X$.

Theorem 3.5. *Any essentially normal operator $T \in \mathcal{L}(\mathcal{H})$ has a proper Følner sequence.*

[3]In fact a stronger property holds: M_Ω is a subnormal operator (i.e., the restriction of a normal operator to an invariant subspace); see [24].

Proof. (i) The first step of the proof uses the following classical result of the Brown–Douglas–Fillmore theory (see [18, Section V] or [19, Theorem 11.1]). Let $T, R \in \mathcal{L}(\mathcal{H})$ be essentially normal, then we have that $T = U(R+K)U^*$ for some compact operator K and some unitary U if and only if operators T and R have the same spectral picture (cf. Definition 3.4). Therefore to prove that T has a proper Følner sequence it will be enough to construct an essentially normal operator R with a proper Følner sequence and having the same spectral picture as T. Indeed, if $\{P_n\}_n$ is a proper Følner sequence for R, then by Proposition 2.2 (i) it is also a proper Følner sequence for $R+K$ for any compact operator K and therefore $\widehat{P}_n := UP_nU^*$ is a proper Følner sequence for $T = U(R+K)U^*$.

(ii) Given the essentially normal operator T, the construction of an essentially normal operator R with the same spectral picture as T and having a proper Følner sequence goes as follows. The set $X := \sigma_{\mathrm{ess}}(T)$ is a closed and bounded subset of $\mathbb{C}$, so that we consider its decomposition

$$\mathbb{C} \setminus X := \mathop{\cup}_{j\in J} \Omega_j$$

into open, connected and disjoint sets; here $J \subset \mathbb{N}$ is a set of indices. The index function $\cup_j \Omega_j \ni \lambda \mapsto \mathrm{ind}\,(T - \lambda \mathbb{1})$ is continuous and therefore constant on each connected component Ω_j.

We denote for $\lambda \in \Omega_j$ the index by $n_j := \mathrm{ind}\,(T-\lambda\mathbb{1}) \in \mathbb{Z}$, and put

$$J_- := \{j \in J \mid n_j < 0\},\ \ J_+ := \{j \in J \mid n_j > 0\},\ \ J_\cup = J_- \cup J_+ .$$

These sets of indices may be finite or infinite.

To construct R, first take any normal operator N on an infinite-dimensional Hilbert space $\mathcal{K}$ such that $\sigma_{\mathrm{ess}}(N) = X$. (A concrete example can be constructed as follows: put $\mathcal{H} = \ell^2(\mathbb{N})$ and let $\{d_n\}_{n\in\mathbb{N}}$ be a dense sequence of points in X. Any isolated point in X is repeated infinitely many times. Then the diagonal operator $N := \mathrm{diag}\,(\{d_n\}_n)$ is normal and $\sigma_{\mathrm{ess}}(N) = X$.) Since N is normal we have $\mathrm{ind}\,(N - \lambda\mathbb{1}) = 0$, $\lambda \notin X$.

Second, for any bounded Ω_j, $j \in J_+$ (i.e., $n_j > 0$) we consider the operator $M_j := M_{\Omega_j}$ on $R^2(\Omega_j)$ as in Lemma 3.3 and that satisfies the properties (i)–(iv). If $j \in J_-$, then we put $M_j := M_{\overline{\Omega}_j}$ on $R^2(\overline{\Omega}_j)$. Define the Hilbert spaces $\mathcal{K}_j := \oplus^{n_j} R^2(\Omega_j)$ for $n_j > 0$ and $\mathcal{K}_j := \oplus^{|n_j|} R^2(\overline{\Omega}_j)$ for $n_j < 0$. Next, we construct on $\mathcal{K}_j$ the operator

$$S_j := \begin{cases} \mathop{\oplus}\limits_{1}^{|n_j|} M_j^* \ , & \text{if} \quad n_j < 0 \ , \\ \mathop{\oplus}\limits_{1}^{n_j} M_j \ , & \text{if} \quad n_j > 0 \ . \end{cases}$$

From Proposition 3.1 (iii) and Lemma 3.3 (ii) we have $\mathrm{ind}\,(S_j - \lambda\mathbb{1}) = n_j$ for any $\lambda \in \Omega_j$. Then we consider the operator

$$\widehat{S} := \Big(\mathop{\oplus}_{j\in J_\cup} S_j \Big) \quad \text{on} \quad \widehat{\mathcal{K}} := \mathop{\oplus}_{j\in J_\cup} \mathcal{K}_j$$

and, finally, we put

$$R := N \oplus \widehat{S} \in \mathcal{L}(\mathcal{K} \oplus \widehat{\mathcal{K}}) \ .$$

(iii) The last part of the proof consists in showing that R satisfies all the required properties.

Since N is normal it has a proper Følner sequence. By the absorbing property of proper Følner sequences for direct sums stated in Proposition 2.3 we conclude that R has a proper Følner sequence too.

Next we show that R has the same spectral picture as the given operator T. For this purpose we prove first that $\sigma_{\mathrm{ess}}(\widehat{S}) \subset X$ and that for λ in Ω_j, $\mathrm{ind}\,(\widehat{S}-\lambda\mathbb{1}) = n_j$. Assume that $\lambda \notin X$. Then $\lambda \in \Omega_k$ for some index $k \in J$. If $k \notin J_\cup$, put $d := \inf_{j\in J_\cup} \left\{\mathrm{dist}\left(\lambda, \Omega_j^{\mathrm{cl}}\right)\right\}$. In this case $d > 0$. From Lemma 3.3 (iii) we obtain

$$\left\|(S_j - \lambda\mathbb{1})^{-1}\right\| = \frac{1}{\mathrm{dist}\left(\lambda, \Omega_j^{\mathrm{cl}}\right)} \leq \frac{1}{d}\,, \qquad j \in J_\cup \,.$$

We conclude that the operator $\widehat{S} - \lambda\mathbb{1}$ is invertible (recall Proposition 3.1 (i)), hence it is Fredholm of index 0 and $\lambda \notin \sigma_{\mathrm{ess}}(\widehat{S})$.

Now consider the case when $\lambda \in \Omega_k$, where $k \in J_\cup$. Then we may consider the decomposition

$$\widehat{S} - \lambda\mathbb{1} = (S_k - \lambda\mathbb{1}) \oplus \Big(\bigoplus_{j\neq k} S_j - \lambda\mathbb{1}\Big) \ .$$

The same argument as before shows that $\bigoplus_{j\neq k} (S_j - \lambda\mathbb{1})$ is invertible, hence Fredholm of index 0. By construction of S_k (see Lemma 3.3 (ii)) and by Proposition 3.1 (iii) we conclude that $\lambda \notin \sigma_{\mathrm{ess}}(\widehat{S})$ and that $\mathrm{ind}\,(\widehat{S}-\lambda\mathbb{1}) = n_k$, for any $\lambda \in \Omega_k$. Therefore we have that $\sigma_{\mathrm{ess}}(\widehat{S}) \subset X$.

From the properties of the normal operator N constructed in step (ii), we have $\sigma_{\mathrm{ess}}(N) = X$. Using now Proposition 3.1 (ii) we conclude that

$$\sigma_{\mathrm{ess}}(R) = \sigma_{\mathrm{ess}}(N) \cup \sigma_{\mathrm{ess}}(\widehat{S}) = X \ .$$

Moreover, we have for any $\lambda \in \Omega_j$

$$\mathrm{ind}\,(R - \lambda\mathbb{1}) = 0 + n_j = \mathrm{ind}\,(T - \lambda\mathbb{1}) \ ,$$

and we have shown that T and R have the same spectral picture.

Finally, we still have to show that R is essentially normal, i.e., that the self-commutator of R is compact. For this note that

$$[R^*, R] = 0 \oplus [\widehat{S}^*, \widehat{S}] \ .$$

We need to consider two cases: if the index set $J_\cup$ is finite, then by Lemma 3.3 (v) the operator $\widehat{\mathcal{S}}$ is trace class, hence R is essentially normal. Note that $\bigcup_{j\in J_\cup} \Omega_j$ is bounded. Therefore, if the set $J_\cup$ has infinite cardinality, then we have, in addition,

$$\lim_{J_\cup \ni j \to \infty} \mathrm{area}(\Omega_j) = 0 \ . \qquad (3.2)$$

Consider the partial direct sum $\widehat{S}_N := \bigoplus_{j\in J_\cup,\, j\le N} S_j$. Applying again Lemma 3.3 (v) we get

$$\left\| [\widehat{S}^*, \widehat{S}] - [\widehat{S}_N^*, \widehat{S}_N] \right\| = \left\| \bigoplus_{j\in J_\cup,\, j>N} [S_j^*, S_j] \right\| = \sup_{j\in J_\cup,\, j>N} \left\| [M_j^*, M_j] \right\|$$
$$\le \frac{1}{\pi} \sup_{j\in J_\cup,\, j>N} \operatorname{area}(\Omega_j) \to 0 \qquad \text{as } N\to\infty$$

(see Eq. (3.2)). Since $[\widehat{S}_N^*, \widehat{S}_N]$ is a trace-class operator, it follows that the self-commutator $[R^*, R]$ can be approximated in norm by trace-class operators, hence it is compact and we conclude that R is essentially normal. □

Corollary 3.6. *If $T \in \mathcal{L}(\mathcal{H})$ is an m-multicyclic hyponormal operator, then T has a proper Følner sequence.*

Proof. By the Berger–Shaw Theorem it follows that the self-commutator $[T^*, T]$ is trace-class and, therefore, T is essentially normal and the assertion follows from Theorem 3.5. □

We conclude this subsection mentioning that any quasinormal operator (i.e., any operator Q that commutes with Q^*Q) has a proper Følner sequence. Recall also that an operator T on $\mathcal{H}$ is called subnormal if there is a normal operator N acting on a Hilbert space $\widetilde{\mathcal{H}}$ containing $\mathcal{H}$ such that $\mathcal{H}$ is invariant for N and T is the restriction of N to $\mathcal{H}$. It can also be shown that any subnormal operator has a proper Følner sequence. See [38] for details and also Chapter II in [24] for the relations between these classes of operators.

3.2. Finite operators

In this subsection we study the class of finite operators introduced by Williams in [53] and their relation to proper Følner sequences. (See also [36].)

We begin recalling the main definition and known results.

Definition 3.7. $T \in \mathcal{L}(\mathcal{H})$ is called a *finite operator* if

$$0 \in \Big(W\left([T, X]\right) \Big)^{\mathrm{cl}} \quad \text{for all} \quad X \in \mathcal{L}(\mathcal{H}) ,$$

where $W(T)$ denotes the numerical range of the operator T, i.e.,

$$W(T) = \{ \langle Tx, x\rangle \mid x \in \mathcal{H} \ , \ \|x\| = 1 \} ,$$

and where the $(\cdot)^{\mathrm{cl}}$ means the closure of the corresponding subset in $\mathbb{C}$.

We collect in the following theorem some standard results due to Williams about the class of finite operators (cf. [53]).

Theorem 3.8 (Williams). *An operator $T \in \mathcal{L}(\mathcal{H})$ is finite if and only if $C^*(T, \mathbb{1})$ has an amenable trace. The class of finite operators is closed in the operator norm and contains all finite block reducible operators.*

It follows that the norm closure of the set of all finite block reducible operators is contained in the class of finite operators. Combining Williams' Theorem with Proposition 2.5, we get the following fact.

Corollary 3.9. *For any operator $T \in \mathcal{L}(H)$, the following properties are equivalent:*

(i) *T is finite;*
(ii) *T has a Følner sequence;*
(iii) *$C^*(T, \mathbb{1})$ has an amenable trace.*

The next result shows the strong link between finite operators and proper Følner sequences. We include the proof, because it is short and illustrative (cf. [38, Theorem 4.1]).

Theorem 3.10. *Let $T \in \mathcal{L}(\mathcal{H})$. Then, T is a finite operator if and only if T is finite block reducible or T has a proper Følner sequence.*

Proof. (i) If T is finite block reducible, then T is a finite operator (cf. [53]). Moreover, if T has a proper Følner sequence, then the C^*-algebra $C^*(T, \mathbb{1})$ has the same proper Følner sequence and, by Proposition 2.5, it also has an amenable trace. Then, by Williams' theorem (see also Theorem 4 in [53]) we conclude that T is finite.

(ii) To prove the other implication, assume T is a finite operator. We consider several cases. If there exists a (non-zero) $P \in \mathcal{P}_{\rm fin}(\mathcal{H})$ such that $[T, P] = 0$, then T is finite block reducible. Consider next the situation where $[T, P] \neq 0$ for all $P \in \mathcal{P}_{\rm fin}(\mathcal{H})$. Since T is finite we can use Williams' Theorem to conclude that $C^*(T, \mathbb{1})$ has an amenable trace. Applying Proposition 2.5 (see also Theorem 1.1 in [7]) we conclude that there exists a Følner sequence of non-zero finite rank projections $\{P_n\}_n$, i.e., we have

$$\lim_{n\to\infty} \frac{\|[T, P_n]\|_2}{\|P_n\|_2} = 0 \; .$$

(Note that P_n is not necessarily a proper Følner sequence in the sense of Definition 2.1.) Two cases may appear: if $\dim P_n\mathcal{H} \leq m$ for some $m \in \mathbb{N}$, then choose a subsequence with constant rank and by Proposition 2.8 we conclude that T has a proper Følner sequence. If the dimensions of $P_n\mathcal{H}$ are not bounded, then from Proposition 2.2 (iii) we also have that T has a proper Følner sequence. □

3.3. Strongly non-Følner operators

In the present subsection we study the operators with no Følner sequence. For this we introduce the following notion of operator that is far from having a non-trivial finite-dimensional reducing subspace.

Definition 3.11. Let $\mathcal{H}$ be an infinite-dimensional Hilbert space and T an operator on $\mathcal{H}$. We will say that T is *strongly non-Følner* if there exists an $\varepsilon > 0$ such that all projections $P \in \mathcal{P}_{\rm fin}(\mathcal{H})$ satisfy

$$\frac{\|TP - PT\|_2}{\|P\|_2} \geq \varepsilon \; .$$

The following result shows the structure of operators with no proper Følner sequence. Its proof is long and technical and we refer to Section 3 in [38] for details.

Theorem 3.12. *Let $T \in \mathcal{L}(\mathcal{H})$ with $\dim \mathcal{H} = \infty$. Then T has no proper Følner sequence if and only if T has an orthogonal sum representation $T = T_0 \oplus \widetilde{T}$ on $\mathcal{H} = \mathcal{H}_0 \oplus \widetilde{\mathcal{H}}$, where $\dim \mathcal{H}_0 < \infty$ and $\widetilde{T}$ is strongly non-Følner.*

Next we mention some concrete examples of strongly non-Følner operators. We will use the amenable trace that appears in Proposition 2.5 as an obstruction. Recall the definition of the Cuntz algebra $\mathcal{O}_n$ (cf. [25, 26]): it is the universal C^*-algebra generated by $n \geq 2$ non-unitary isometries $S_1, \ldots, S_n$ with the property that their final projections add up to the identity, i.e.,

$$\sum_{k=1}^{n} S_k S_k^* = \mathbb{1} \,. \tag{3.3}$$

This condition implies in particular that the range projections are pairwise orthogonal, i.e.,

$$S_l^* S_k = \delta_{lk} \mathbb{1} \,. \tag{3.4}$$

It is easy to realize the Cuntz algebra on the complex Hilbert space ℓ_2 of square summable sequences.

Proposition 3.13. *The Cuntz algebra $\mathcal{O}_n$, $n \geq 2$, is singly generated and its generator is strongly non-Følner.*

Proof. By Corollary 4 (or Theorem 9) in [40] any Cuntz algebra $\mathcal{O}_n$, $n \geq 2$, has a single generator C_n, i.e., $\mathcal{O}_n = C^*(C_n)$. We assert that C_n is strongly non-Følner. Indeed, assume that, to the contrary, it is not; then by Corollary 3.14 (ii), C_n is finite. By Corollary 3.9, it would follow that $\mathcal{O}_n = C^*(C_n)$ has an amenable trace τ. But this gives a contradiction since applying τ to the equations (3.3) and (3.4) we obtain $n = 1$. □

Other examples of a strongly non-Følner operators can be obtained from the proof of Theorem 5 in [30]. It is also worth mentioning that Corollary 4 in [20] gives an example of a strongly non-Følner operator generating a type II_1 factor.

Theorem 3.10 allows to divide the class of bounded linear operators into the following mutually disjoint subclasses summarized in the following table:

	Operators with a proper Følner sequence	Operators with no proper Følner sequence
Finite block reducible	$\mathcal{W}_{0+}$	$\mathcal{W}_{0-}$
Non finite block reducible	$\mathcal{W}_{1+}$	$\mathcal{S}$

Table 1

Finally, we conclude this analysis with the following immediate consequences:

Corollary 3.14. *Let* $T \in \mathcal{L}(\mathcal{H})$*. Then*

(i) *T is a finite operator if and only if T is in one of the following mutually disjoint classes:* $\mathcal{W}_{0+}$, $\mathcal{W}_{0-}$, $\mathcal{W}_{1+}$.

(ii) *T is not a finite operator (i.e., it is of class $\mathcal{S}$) if and only if T is strongly non-Følner.*

(iii) *The class of strongly non-Følner operators is open and dense in* $\mathcal{L}(\mathcal{H})$.

Proof. The characterization of finite operators and its complement stated in (i) and (ii) follows from Theorem 3.10 and Williams' theorem. To prove part (iii) we use that the class of finite operators is closed and nowhere dense (cf. [35]). Therefore the set of strongly non-Følner operators is an open and dense subset of $\mathcal{L}(\mathcal{H})$. □

As a summary let us mention that proper Følner sequences for operators provide a useful and natural tool to analyze the class of finite operators. To illustrate this with an example note that the preceding corollary already implies that the class of finite operators is closed in $\mathcal{L}(H)$.

4. Følner sequences in operator algebras

We start the analysis of Følner sequences in the context of operator algebras stating some approximation results for amenable traces. We will apply them to spectral approximation problems of scalar spectral measures. In the final part of this section we will give an abstract characterization in terms of unital completely positive maps of C^*-algebras admitting a faithful essential representation which has a Følner sequence or, equivalently, an amenable trace.

4.1. Approximations of amenable traces

Part (i) of the following result is a standard weak*-compactness argument. Part (ii) is known to experts (see, e.g., Exercise 6.2.6 in [17]) or [1] for a complete proof).

Proposition 4.1. *Let $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ be a unital separable C^*-algebra.*

(i) *If $\mathcal{A}$ has a Følner sequence $\{P_n\}_n$, then $\mathcal{A}$ has an amenable trace.*

(ii) *Assume that $\mathcal{A} \cap \mathcal{K}(\mathcal{H}) = \{0\}$, and let τ be an amenable trace on $\mathcal{A}$. Then $\mathcal{A}$ has a Følner sequence $\{P_n\}_n$ satisfying*

$$\tau(A) = \lim_{n\to\infty} \frac{\mathrm{Tr}(AP_n)}{\mathrm{Tr}(P_n)}\,, \quad A \in \mathcal{A}\,, \tag{4.1}$$

where Tr *denotes the canonical trace on* $\mathcal{L}(\mathcal{H})$.

We will now present an application of Proposition 4.1 (ii) to obtain an approximation result for scalar spectral measures. For this we need to recall from [6] the definition of Szegő pairs for a concrete C^*-algebra $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$. This notion incorporates the good spectral approximation behavior of scalar spectral measures of selfadjoint elements in $\mathcal{A}$ and is motivated by Szegő's classical approximation results mentioned in the introduction.

Let $\mathcal{A}$ be a unital C^*-algebra acting on $\mathcal{H}$ and let τ be a tracial state on $\mathcal{A}$. For any selfadjoint element $T \in \mathcal{A}$ we denote by μ_T the spectral measure associated with the trace τ of $\mathcal{A}$. Consider a sequence $\{P_n\}_n$ of non-zero finite rank projections on $\mathcal{H}$ and write the corresponding (selfadjoint) compressions as $T_n := P_n T P_n$. Denote by μ_T^n the probability measure on $\mathbb{R}$ supported on the spectrum of T_n, i.e., for any $T = T^* \in \mathcal{A}$ we have

$$\mu_T^n(\Delta) := \frac{N_T^n(\Delta)}{\|P_n\|_1} , \quad \Delta \subset \mathbb{R} \quad \text{Borel} ,$$

where $N_T^n(\Delta)$ is the number of eigenvalues of T_n (multiplicities counted) contained in Δ. We say that $(\{P_n\}_n , \tau)$ is a *Szegő pair* for $\mathcal{A}$ if $\mu_T^n \to \mu_T$ weakly for all selfadjoint elements $T \in \mathcal{A}$, i.e.,

$$\lim_{n\to\infty} \frac{1}{d_n}\Big(f(\lambda_{1,n}) + \cdots + f(\lambda_{d_n,n})\Big) = \int f(\lambda)\, d\mu_T(\lambda) , \quad f \in C_0(\mathbb{R}) ,$$

where $d_n = \|P_n\|_1$ is the dimension of the $P_n\mathcal{H}$ and $\{\lambda_{1,n}, \dots, \lambda_{d_n,n}\}$ are the eigenvalues (repeated according to multiplicity) of T_n.

By [6, Theorem 6 (i), (ii)], if $(\{P_n\}_n , \tau)$ is a Szegő pair for $\mathcal{A}$, then $\{P_n\}_n$ must be a Følner sequence for $\mathcal{A}$, τ must be an amenable trace, and equation (4.1) must hold for every $A \in \mathcal{A}$. Proposition 4.1 (ii) allows one to complete *any* amenable trace τ on $\mathcal{A}$ with a Følner sequence so that the pair $(\{P_n\}_n , \tau)$ is a *Szegő pair* for $\mathcal{A}$, as follows. The proof of the following result requires the construction of an increasing sequence of operators that approximates simultaneously the corresponding commutator and the amenable trace. We refer to Theorem 3.2 in [1] for details

Theorem 4.2. *Let $\mathcal{A}$ be a unital, separable C^*-algebra acting on a separable Hilbert space $\mathcal{H}$, and assume that $\mathcal{A} \cap \mathcal{K}(\mathcal{H}) = \{0\}$. If τ is an amenable trace on $\mathcal{A}$, then there exists a proper Følner sequence $\{P_n\}_n$ such that $(\{P_n\}_n , \tau)$ is a Szegő pair for $\mathcal{A}$.*

Remark 4.3. We conclude this subsection recalling that an important step in the proof of the Arveson–Bédos spectral approximation results mentioned in the introduction is the compatibility between the choice of the Følner sequence in the Hilbert space and the amenable trace. In fact, if a unital and separable concrete C^*-algebra $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ has an amenable trace τ and $\{P_n\}_n$ is a Følner sequence of non-zero finite rank projections for $\mathcal{A}$ it is needed that the projections approximate the amenable trace in the following natural sense

$$\tau(A) = \lim_{n\to\infty} \frac{\mathrm{Tr}(AP_n)}{\mathrm{Tr}(P_n)} , \quad A \in \mathcal{A} . \tag{4.2}$$

Now given $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ with an amenable trace τ it is possible to construct a Følner sequence in different ways. As observed by Bédos in [7] one way to obtain a Følner sequence $\{P_n\}$ for $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ is essentially contained in [22, 23]. In these articles Connes adapts the group theoretic methods by Day and Namioka to the context of operators. Using this technique one loses track of the initial amenable trace τ, in the sense that the sequence $\{P_n\}$ does not necessarily satisfy (4.2). To avoid this problem one may assume in addition that $\mathcal{A}$ has a unique tracial state. This is sufficient to guarantee a good spectral approximation behavior of relevant examples like almost Mathieu operators, which are contained in the irrational rotation algebra (cf. [10]).

In contrast with the previous method, the construction of a Følner sequence given in [41, Theorem 6.1] (see also [17, Theorem 6.2.7]) allows one to approximate the original trace as in Eq. (4.2). In the precedent theorem it was crucial to use this method to prove a spectral approximation result in the spirit of Arveson and Bédos, but removing the hypothesis of a unique trace (compare Theorem 4.2 with [7, Theorem 1.3] or [6, Theorem 6 (iii)] and the formulation in p. 354 of [4]).

4.2. Følner C^*-algebras

The existence of a Følner sequence for a set of operators $\mathcal{T}$ is a weaker notion than quasidiagonality. Recall that a set of operators $\mathcal{T} \subset \mathcal{L}(\mathcal{H})$ is said to be quasidiagonal if there exists an increasing sequence of finite-rank projections $\{P_n\}_{n\in\mathbb{N}}$ converging strongly to $\mathbb{1}$ and such that

$$\lim_n \|TP_n - P_nT\| = 0\,, \quad T \in \mathcal{T}\,. \tag{4.3}$$

(See, e.g., [32, 49] or Chapter 16 in [17].) The existence of proper Følner sequences can be understood as a quasidiagonality condition, but relative to the growth of the dimension of the underlying spaces. It can be easily shown that if $\{P_n\}_n$ quasidiagonalizes a family of operators $\mathcal{T}$, then this sequence of non-zero finite rank orthogonal projections is also a proper Følner sequence for $\mathcal{T}$. The unilateral shift is a basic example that shows the difference between the notions of proper Følner sequences and quasidiagonality. It is a well-known fact that the unilateral shift S is not a quasidiagonal operator. (This was shown by Halmos in [31]; in fact, in this reference it is shown that S is not even quasitriangular.) In the setting of abstract C^*-algebras it can also be shown that a C^*-algebra containing a non-unitary isometry is not quasidiagonal (see, e.g., [14, 17]).

In [50], Voiculescu characterized abstractly quasidiagonality for unital separable C^*-algebras in terms of unital completely positive (u.c.p.) maps[4] (see also [49]). This has become by now the standard definition of quasidiagonality for operator algebras (see, for example, [17, Definition 7.1.1]):

Definition 4.4. A unital separable C^*-algebra $\mathcal{A}$ is called *quasidiagonal* if there exists a sequence of u.c.p. maps $\varphi_n\colon \mathcal{A} \to M_{k(n)}(\mathbb{C})$ which is both asymptotically multiplicative (i.e., $\|\varphi_n(AB) - \varphi_n(A)\varphi_n(B)\| \to 0$ for all $A, B \in \mathcal{A}$) and asymptotically isometric (i.e., $\|A\| = \lim_{n\to\infty} \|\varphi_n(A)\|$ for all $A \in \mathcal{A}$).

[4]Recall that in this context a linear map $\varphi\colon \mathcal{A} \to \mathcal{B}$ between unital C^*-algebras $\mathcal{A}$, $\mathcal{B}$ is called *unital completely positive* (*u.c.p.*), if $\varphi(\mathbb{1}) = \mathbb{1}$ and if the inflations $\varphi_n := \varphi \otimes \mathrm{id}_n\colon \mathcal{A} \otimes M_n(\mathbb{C}) \to \mathcal{B} \otimes M_n(\mathbb{C})$ are positive for all $n \geq 1$.

Inspired by Voiculescu's work on quasidiagonality we introduce in this section an abstract definition of a Følner C^*-algebra and formulate our main result characterizing Følner C^*-algebras in terms of Følner sequences and also of amenable traces.

Recall that $\mathrm{tr}(\cdot)$ denotes the unique tracial state on a matrix algebra $M_n(\mathbb{C})$.

Definition 4.5. Let $\mathcal{A}$ be a unital, separable C^*-algebra.

(i) We say that $\mathcal{A}$ is a *Følner C^*-algebra* if there exists a sequence of u.c.p. maps $\varphi_n \colon \mathcal{A} \to M_{k(n)}(\mathbb{C})$ such that

$$\lim_n \|\varphi_n(AB) - \varphi_n(A)\varphi_n(B)\|_{2,\mathrm{tr}} = 0 \;, \quad A, B \in \mathcal{A} \;, \tag{4.4}$$

where $\|F\|_{2,\mathrm{tr}} := \sqrt{\mathrm{tr}(F^*F)}$, $F \in M_n(\mathbb{C})$.

(ii) We say that $\mathcal{A}$ is a *proper Følner C^*-algebra* if there exists a sequence of u.c.p. maps $\varphi_n \colon \mathcal{A} \to M_{k(n)}(\mathbb{C})$ satisfying (4.4) and which, in addition, are asymptotically isometric, i.e.,

$$\|A\| = \lim_n \|\varphi_n(A)\| \;, \quad A \in \mathcal{A} \;. \tag{4.5}$$

It is clear that if $\mathcal{A}$ is a separable, unital and quasidiagonal C^*-algebra (cf. Definition 4.4), then $\mathcal{A}$ is a proper Følner algebra. The Toeplitz algebra serves as a counter-example to the reverse implication.

Although, in principle, the two concepts – Følner and proper Følner – seem to be different for C^*-algebras, we can show that they indeed define the same class of unital, separable C^*-algebras. The proof of the next proposition includes a useful trick so that we will include it here (cf. [1, Proposition 3.2]).

Proposition 4.6. *Let $\mathcal{A}$ be a unital separable C^*-algebra. Then $\mathcal{A}$ is a Følner C^*-algebra if and only if $\mathcal{A}$ is a proper Følner C^*-algebra.*

Proof. Assume that $\mathcal{A}$ is a Følner C^*-algebra, and let $\varphi_n \colon \mathcal{A} \to M_{k(n)}(\mathbb{C})$ be a sequence of u.c.p. maps such that (4.4) holds. Considering the direct sum of a sufficiently large number of copies of φ_n, for each n, we may assume that

$$\lim_{n\to\infty} \frac{n}{k(n)} = 0. \tag{4.6}$$

Let $\pi \colon \mathcal{A} \to \mathcal{L}(\mathcal{H})$ be a faithful representation of $\mathcal{A}$ on a separable Hilbert space $\mathcal{H}$. Let $\{P_n\}_n$ be an increasing sequence of orthogonal projections on $\mathcal{H}$, converging to $\mathbb{1}$ in the strong operator topology and such that $\dim P_n(\mathcal{H}) = n$ for all n. Then for all $A \in \mathcal{A}$ we have $\|A\| = \lim_n \|P_n\pi(A)P_n\|$. Let $\psi_n \colon \mathcal{A} \to M_{k(n)+n}(\mathbb{C})$ be given by:

$$\psi_n(A) = \varphi_n(A) \oplus P_n\pi(A)P_n, \qquad A \in \mathcal{A}.$$

Then ψ_n is a u.c.p. map. For $A, B \in \mathcal{A}$, set $X_n = P_n\pi(A)(1 - P_n)\pi(B)P_n$. Then we have

$$\|\psi_n(AB) - \psi_n(A)\psi_n(B)\|^2_{2,\mathrm{tr}} \le \|\varphi_n(AB) - \varphi_n(A)\varphi_n(B)\|^2_{2,\mathrm{tr}} + \frac{\mathrm{Tr}(X_n^* X_n)}{k(n) + n}$$

$$\le \|\varphi_n(AB) - \varphi_n(A)\varphi_n(B)\|^2_{2,\mathrm{tr}} + \frac{n\|A\|^2\|B\|^2}{k(n) + n}\,.$$

Using (4.6) we get

$$\lim_n \|\psi_n(AB) - \psi_n(A)\psi_n(B)\|_{2,\mathrm{tr}} = 0.$$

On the other hand, for $A \in \mathcal{A}$, we have

$$\|A\| - \|\psi_n(A)\| \le \|A\| - \|P_n\pi(A)P_n\| \to 0$$

so that (4.5) holds for the sequence (ψ_n). This concludes the proof. □

For the next result recall that a representation π of an abstract C^*-algebra $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ is called *essential* if $\pi(\mathcal{A})$ contains no nonzero compact operators. The proof uses the same approximation technique as the proof of Theorem 4.2 (see Theorem 3.4 in [1] for details).

Theorem 4.7. *Let $\mathcal{A}$ be a unital separable C^*-algebra. Then the following conditions are equivalent:*

(i) *There exists a faithful representation $\pi\colon \mathcal{A} \to \mathcal{L}(\mathcal{H})$ such that $\pi(\mathcal{A})$ has a Følner sequence.*
(ii) *There exists a faithful essential representation $\pi\colon \mathcal{A} \to \mathcal{L}(\mathcal{H})$ such that $\pi(\mathcal{A})$ has a Følner sequence.*
(iii) *Every faithful essential representation $\pi\colon \mathcal{A} \to \mathcal{L}(\mathcal{H})$ satisfies that $\pi(\mathcal{A})$ has a proper Følner sequence.*
(iv) *There exists a non-zero representation $\pi\colon \mathcal{A} \to \mathcal{L}(\mathcal{H})$ such that $\pi(\mathcal{A})$ has an amenable trace.*
(v) *Every faithful representation $\pi\colon \mathcal{A} \to \mathcal{L}(\mathcal{H})$ satisfies that $\pi(\mathcal{A})$ has an amenable trace.*
(vi) *$\mathcal{A}$ is a Følner C^*-algebra.*

Remark 4.8.

(i) The class of C^*-algebras introduced in this section has been considered before by Bédos. In [7] the author defines a C^*-algebra $\mathcal{A}$ to be *weakly hypertracial* if $\mathcal{A}$ has a non-degenerate representation π such that $\pi(\mathcal{A})$ has a hypertrace. In this sense, the preceding theorem gives a new characterization of weakly hypertracial C^*-algebras in terms of u.c.p. maps.
(ii) Note also that the equivalences between (i), (iv) and (v) in Theorem 4.7 are basically known (see [7]).

We conclude mentioning that in the study of growth properties of C^*-algebras (and motivated by previous work done by Arveson and Bédos) Vaillant defined the following natural unital C^*-algebra (see Section 3 in [46]): given an increasing sequence $\mathcal{P} := \{P_n\}_n \subset \mathcal{P}_{\rm fin}(\mathcal{H})$ of orthogonal finite rank projections strongly converging to $\mathbb{1}$, consider the set of all bounded linear operators in $\mathcal{H}$ that have $\mathcal{P}$ as a proper Følner sequence, i.e.,

$$\mathcal{F}_{\mathcal{P}}(\mathcal{H}) := \left\{ X \in \mathcal{L}(\mathcal{H}) \mid \lim_{n\to\infty} \frac{\|XP_n - P_nX\|_2}{\|P_n\|_2} = 0 \right\} .$$

This unital C^*-subalgebra of $\mathcal{L}(\mathcal{H})$ (called Følner algebra by Hagen, Roch and Silbermann in Section 7.2.1 of [29]) has shown to be very useful in the analysis of the classical Szegő limit theorems for Toeplitz operators and some generalizations of them (see, e.g., Section 7.2 of [29] and [12]).

The C^*-algebra $\mathcal{F}_{\mathcal{P}}$ is always non-separable for the operator norm. Indeed, consider the ℓ^∞-direct sum of matrix algebras $\mathcal{A} = \prod_i M_{n_i}(\mathbb{C})$, where n_i are the ranks of the orthogonal projections $P_{i+1} - P_i$, $i \in \mathbb{N}$, with norm given by $\|(a_i)\| = \sup_i \|a_i\|$. It is clear that $\mathcal{A}$ is not separable, and the elements of $\mathcal{A}$ can be seen inside $\mathcal{F}_{\mathcal{P}}$ as block-diagonal operators, so the algebra $\mathcal{F}_{\mathcal{P}}$ is also non-separable.

5. Final remarks: Følner versus proper Følner

As was mentioned at the beginning of Section 2, Følner sequences appeared first in the context of groups. Note that if countable discrete group Γ has a Følner sequence one can always find another Følner sequence which, in addition to Eq. (2.1), is also proper, i.e., $\Gamma_i \subset \Gamma_j$ if $i \leq j$ and $\Gamma = \cup_i \Gamma_i$. In the context of operators and due to the linear structure of the underlying Hilbert spaces the difference between Følner sequence and proper Følner sequence is relevant. As was mentioned after Proposition 2.7 if $T = T_0 \oplus T_1$ is a finite block reducible operator on the Hilbert space $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$, with $1 \leq \dim \mathcal{H}_0 < \infty$, and T_1 strongly non-Følner (cf. Subsection 3.3), then T has an obvious constant Følner sequence but can not have a proper Følner sequence. Moreover, Proposition 2.8 shows that the difference between Følner and proper Følner sequence for single operators can only appear in the case when there is a non-trivial finite-dimensional invariant subspace.

At the level of abstract C^*-algebras Proposition 4.6 shows that Følner C^*-algebras and proper Følner C^*-algebras define the same class of unital separable C^*-algebras. Note that by Theorem 4.7 (i) the direct sum of a matrix algebra and the Cuntz algebra

$$\mathcal{A} := M_n(\mathbb{C}) \oplus \mathcal{O}_n$$

is a Følner (hence proper Følner) C^*-algebra. But in its natural representation on $\mathcal{H} := \mathbb{C}^n \oplus \ell_2$ this algebra can not have a proper Følner sequence because the representation is not essential (see Theorem 4.7 (iii)).

Finally, if $\mathcal{B}$ is a *unital* C^*-subalgebra of a Følner C^*-algebra $\mathcal{A}$, then one can restrict the u.c.p. maps of $\mathcal{A}$ to $\mathcal{B}$ to show that $\mathcal{B}$ is also a Følner C^*-algebra. This is not true if $\mathcal{B}$ is a non-unital C^*-subalgebra (i.e., if $\mathbb{1}_{\mathcal{A}} \notin \mathcal{B}$). Consider, for example, the concrete C^*-algebra on $\mathcal{K} := \ell_2 \oplus \mathcal{H}$ given by

$$\mathcal{A} := C^*(S) \oplus C^*(T_1) ,$$

where S is the unilateral shift and T_1 is a strongly non-Følner operator. Then, again, $\mathcal{A}$ is a Følner (hence proper Følner) C^*-algebra, but the non-unital C^*-subalgebra $\mathcal{B} := 0 \oplus C^*(T_1)$ is not a Følner C^*-algebra.

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References

[1] P. Ara and F. Lledó, *Amenable traces and Følner C^*-algebras.* To appear in Expo. Math., preprint 2013, arXiv:math.OA/1206.1488v3.

[2] M. Ahues, A. Largillier and B.V. Limaye, *Spectral Computations for Bounded Operators.* Chapman & Hall/CRC, Boca Raton, 2001.

[3] W. Arveson, *Improper filtrations for C^*-algebras: spectra of unilateral tridiagonal operators.* Acta Sci. Math. (Szeged) **57** (1993), 11–24.

[4] W. Arveson, *C^*-algebras and numerical linear algebra.* J. Funct. Anal. **122** (1994), 333–360.

[5] W. Arveson, *The role of C^*-algebras in infinite-dimensional numerical linear algebra.* In *C^*-Algebras:* 1943–1993. *A Fifty Year Celebration*, R.S. Doran (ed.), Contemporary Mathematics Vol. 167, American Mathematical Society, Providence, Rhode Island, 1994; pp. 115–129.

[6] E. Bédos, *On Følner nets, Szegő's theorem and other eigenvalue distribution theorems.* Expo. Math. **15** (1997), 193–228. Erratum: Expo. Math. **15** (1997), 384.

[7] E. Bédos, *Notes on hypertraces and C^*-algebras.* J. Operator Theory **34** (1995), 285–306.

[8] E. Bédos, *On filtrations for C^*-algebras.* Houston J. Math. **20** (1994), 63–74.

[9] C.A. Berger, B.I. Shaw, *Selfcommutators of multicyclic hyponormal operators are always trace class.* Bull. Amer. Math. Soc. **79** (1973), 1193–1199.

[10] F.P. Boca, *Rotation C^*-algebras and Almost Mathieu Operators.* The Theta Foundation, Bucarest, 2001.

[11] A. Böttcher, *C^*-algebras in numerical analysis.* Irish Math. Soc. Bull. **45** (2000), 57–133.

[12] A. Böttcher, P. Otte, *The first Szegő limit theorem for non-selfadjoint operators in the Fölner algebra.* Math. Scand. **97** (2005), 115–126.

[13] N.P. Brown, *Invariant means and finite representation theory of C^*-algebras.* Mem. Am. Math. Soc. **184** (2006) no. 865, 1–105.

[14] N.P. Brown, *On quasidiagonal C^*-algebras.* In *Adanced Studies in Pure Mathematics* 38*: Operator Algebras and Applications*, H. Kosaki (ed.), Mathematical Society of Japan, 2004, pp. 19–64.

[15] N.P. Brown, *Herrero's approximation problem for quasidiagonal operators.* J. Funct. Anal. **186** (2001), 360–365.

[16] N.P. Brown, *AF embeddings and the numerical computation of spectra in the irrational rotation algebra.* Numer. Funct. Anal. Optim. **27** (2006), 517–528.

[17] N.P. Brown and N. Ozawa, *C^*-Algebras and Finite-Dimensional Approximations.* American Mathematical Society, Providence, Rhode Island, 2008.

[18] L.G. Brown, R.G. Douglas and P.A. Fillmore, *Extension of C^*-algebras, operators with compact self-commutators, and K-homology.* Bull. Amer. Math. Soc. **79** (1973), 973–978.

[19] L.G. Brown, R.G. Douglas and P.A. Fillmore, *Unitary equivalence modulo the compact operators and extensions of C^*-algebras.* In *Proceedings of a Conference on Operator Theory,* P.A. Fillmore (ed.), Lecture Notes in Mathematics **345**, Springer Verlag, Berlin, 1973; pp. 58–128.

[20] J.W. Bunce, *Finite operators and amenable C^*-algebras.* Proc. Amer. Math. Soc., **56** (1976), 145–151.

[21] F. Chatelin, *Spectral Approximation of Linear Operators.* Academic Press, New York, 1983.

[22] A. Connes, *Classification of injective factors. Cases II_1, II_∞, III_λ, $\lambda \neq 1$.* Ann. Math. **104** (1976), 73–115.

[23] A. Connes, *On the classification of von Neumann algebras and their automorphisms.* In Symposia Mathematica, Vol. XX (*Convegno sulle Algebre C^* e loro Applicazioni in Fisica Teorica, Convegno sulla Teoria degli Operatori Indice e Teoria K, INDAM, Rome,* 1975), Academic Press, London, 1976.

[24] J.B. Conway, *The Theory of Subnormal Operators.* American Mathematical Society, Providence, Rhode Island, 1991.

[25] J. Cuntz, *Simple C^*-algebras generated by isometries.* Commun. Math. Phys. **57** (1977), 173–185.

[26] K.R. Davidson, C^*-*Algebras by Example.* American Mathematical Society, Providence, Rhode Island, 1996.

[27] K.R. Davidson, *Essentially normal operators. A glimpse at Hilbert space operators.* Oper. Theory Adv. Appl. **207** (2010), 209–222.

[28] P.A. Fillmore, J.G. Stampfli, J.P. Williams, *On the essential numerical range, the essential spectrum, and a problem of Halmos.* Acta Sci. Math. (Szeged) **33** (1972), 179–192.

[29] R. Hagen, S. Roch and B. Silbermann, *C^*-Algebras and Numerical Analysis.* Marcel Dekker, Inc., New York, 2001.

[30] P.R. Halmos, *Commutators of operators, II.* Amer. J. Math. **76** (1954), 129–136.

[31] P.R. Halmos, *Quasitriangular operators.* Acta Sci. Math. (Szeged) **29** (1968), 283–293.

[32] P.R. Halmos, *Ten problems in Hilbert space.* Bull. Amer. Math. Society **76** (1970), 887–933.

[33] A.C. Hansen, *On the approximation of spectra of linear operators on Hilbert spaces.* J. Funct. Anal. **254** (2008), 2092–2126.

[34] A.C. Hansen, *On the Solvability Complexity Index, the n-pseudospectrum and approximations of spectra of operators.* J. Amer. Math. Soc. **24** (2011), 81–124.

[35] D. Herrero, *The set of finite operators is nowhere dense.* Canad. Math. Bull. **32** (1989), 320–326.

[36] D. Herrero, *What is a finite operator?* In *Linear and Complex Analysis Problem Book*, V.P. Havin and N.K. Nikolski (eds.), Springer Verlag, Berlin, 1994. pp. 226–228.

[37] F. Lledó, *On spectral approximation, Følner sequences and crossed products.* J. Approx. Theory **170** (2013) 155–171.

[38] F. Lledó and D. Yakubovich, *Følner sequences and finite operators.* J. Math. Anal. Appl. **403** (2013) 464–476.

[39] J.v. Neumann, *Zur Algebra der Funktionaloperationen und der Theorie der normalen Operatoren* **102** (1929), 307–427.

[40] C.L. Olsen and W.R. Zame, *Some C^*-algebras with a single generator.* Trans. Amer. Math. Soc. **215** (1976), 205–217.

[41] N. Ozawa, *About the QWEP conjecture.* Internat. J. Math. **15** (2004), 501–530.

[42] A.L. Paterson, *Amenability.* American Mathematical Society, Providence, Rhode Island, 1988.

[43] S. Roch and B. Silbermann, *Szegő limit theorems for operators with almost periodic diagonals.* Operators and Matrices **1** (2007) 1–29.

[44] M. A. Shubin, *Pseudodifferential operators and spectral theory.* Second edition. Springer Verlag, Berlin, 2001.

[45] P. Tilli, *Some Results on Complex Toeplitz Eigenvalues.* J. Math. Anal. Appl. **239** (1999), 390–401.

[46] G. Vaillant, *Følner conditions, nuclearity, and subexponential growth in C^*-algebras.* J. Funct. Anal. **141** (1996), 435–448.

[47] A. Vershik, *Amenability and approximation of infinite groups.* Sel. Math. Sov. **2** (1982), 311–330.

[48] D. Voiculescu, *A non-commutative Weyl–von Neumann theorem.* Rev. Roum. Math. Pures et Appl. **XXI** (1976), 97–113.

[49] D. Voiculescu, *Around quasidiagonal operators.* Integr. Equat. Oper. Theory **17** (1993), 137–149.

[50] D. Voiculescu, *A note on quasi-diagonal C^*-algebras and homotopy.* Duke Math. J. **62** (1991), 267–271.

[51] A.L. Vol'berg, V.V. Peller and D.V. Yakubovich, *A brief excursion to the theory of hyponormal operators.* Leningrad Math. J. **2** (1991), 211–243.

[52] H. Widom, *Eigenvalue distribution for nonselfadjoint Toeplitz matrices.* Oper. Theor. Adv. Appl. **71** (1994), 1–7.

[53] J.P. Williams, *Finite operators.* Proc. Amer. Math. Soc. **26** (1970), 129–136.

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On the Factorization of Some Block Triangular Almost Periodic Matrix Functions

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To Professor António Ferreira dos Santos, in celebration of his 70th birthday.

Abstract. Canonical factorization criterion is established for a class of block triangular almost periodic matrix functions. Explicit factorization formulas are also obtained, and the geometric mean of matrix functions in question is computed.

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1. Introduction

Factorization of matrix functions is a standard tool in solving systems of convolution type equations

$$k * \phi = f \tag{1.1}$$

on a half-line, going back to the classical paper [27] and known as the Wiener–Hopf technique; see, e.g., the monographs [6, 17, 16] for detailed presentation and further references. This technique was modified by Ganin [15] to allow for consideration of equations on intervals of finite length. Ganin's approach, however, gives rise to matrix functions

$$\begin{bmatrix} e_\lambda I_N & 0 \\ \widehat{k} & e_{-\lambda} I_N \end{bmatrix}, \tag{1.2}$$

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where λ is the length of the interval,

$$e_\lambda(x) = e^{i\lambda x}, \quad x \in \mathbb{R}, \tag{1.3}$$

and $\widehat{k}$ is the Fourier transform of the $N \times N$ kernel k. These matrix functions have a second kind discontinuity at ∞, even when $\widehat{k}$ behaves nicely. Besides, the size of the matrix doubles, so even scalar equations (1.1) yield a factorization problem for a 2×2 matrix function, more complicated than that for scalar functions.

It was shown by two of the authors [19] that for a rather wide class of kernels k the factorization of (1.2) reduces to that for matrix functions of the same block triangular structure in which the off-diagonal block is substituted by its so-called almost periodic representatives at $\pm\infty$. Thus emerged the factorization problem for almost periodic matrix functions G, with special interest in the case of

$$G_F = \begin{bmatrix} e_1 I_N & 0_N \\ F & e_{-1} I_N \end{bmatrix}. \tag{1.4}$$

(Note that the change of λ to 1 in (1.4) can be achieved by a simple change of variable and can therefore be adopted without any loss of generality.)

A systematic exposition of the factorization theory for such matrix functions can be found in [5], while some more recent results are in [2, 9, 10, 11, 12, 13, 18, 25]. Still, the theory is far from being complete, even for matrix functions (1.4).

The factorability criterion for matrix functions (1.4) in the case $N = 1$,

$$F = C_1 e_\alpha + C_{-1} e_{\alpha-1} + C_2 e_\beta + C_{-2} e_{\beta-1} \tag{1.5}$$

with $0 < \alpha < \beta < 1$ was established (in somewhat different terms) in [1], with an alternative approach and some generalization presented in [25]. In our previous paper [2], we provided explicit factorization formulas for this setting. For $N > 1$ the canonical factorability criterion and the factorization formulas are available if $C_1 = 0$ or $C_{-2} = 0$, see [24, Theorem 6.5]. If $C_2 = 0$ or $C_{-1} = 0$ then the respective results can be derived from [21, Theorem 6.1], but only under an additional assumption that the remaining matrix coefficients can be simultaneously put in a triangular form via the same equivalence transformation.

The goal of this paper is to establish respective results (that is, the canonical factorization criterion and explicit factorization formulas) under the similar "triangularizability" requirement on the coefficients C_j in (1.5) without supposing that either of them vanishes, thus extending the statements of [2]. This is done in Section 3. Section 2 contains necessary notation and background information, including a slight variation of a known result on factorability in decomposing algebras. Section 4 provides formulas for the so-called geometric mean of the matrices G_F when $N = 2$, with technical details delegated to the Appendix.

2. Preliminaries

2.1. Background information on AP factorization

For any algebra $\mathfrak{A}$, we denote by $\mathcal{G}\mathfrak{A}$ the group of its invertible elements, and by $\mathfrak{A}_{N\times N}$ the algebra of all $N \times N$ matrices with the entries in $\mathfrak{A}$.

Let APP be the algebra of almost periodic polynomials, that is, the set of all finite linear combinations of elements e_λ ($\lambda \in \mathbb{R}$), with e_λ defined by (1.3). The closure of APP with respect to the uniform norm is the C^*-algebra AP of almost periodic functions, and the closure of APP with respect to the stronger norm,

$$\|\textstyle\sum_\lambda c_\lambda e_\lambda\|_W = \sum_\lambda |c_\lambda|, \quad c_\lambda \in \mathbb{C},$$

is the Banach algebra APW.

The basic information about AP functions can be found in several monographs, including [4, 14] and [22]. For our purposes, the following will suffice.

For any $f \in AP$ there exists the Bohr mean value

$$\mathbf{M}(f) = \lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} f(x)\,dx.$$

The functions $f \in AP$ are defined uniquely by the Bohr–Fourier series

$$\sum_{\lambda\in\Omega(f)} \widehat{f}(\lambda) e_\lambda$$

where $\Omega(f) := \left\{\lambda \in \mathbb{R} : \ \widehat{f}(\lambda) \neq 0\right\}$ is the Bohr–Fourier spectrum of f and the numbers $\widehat{f}(\lambda) = \mathbf{M}(f e_{-\lambda})$ are referred to as the Bohr–Fourier coefficients of f.

Let

$$AP^\pm := \{f \in AP\colon \Omega(f) \subset \mathbb{R}_\pm\}, \quad APW^\pm := AP^\pm \cap APW,$$

$$APW_0^\pm := \{f \in APW^\pm\colon \widehat{f}(0) = 0\},$$

where, as usual, $\mathbb{R}_\pm = \{x \in \mathbb{R}\colon \pm\, x \geq 0\}$.

A function $f \in AP$ is invertible in AP if and only if it is invertible in $L_\infty(\mathbb{R})$, that is, if and only if $\inf_{x\in\mathbb{R}} |f(x)| > 0$. For every $f \in \mathcal{G}AP$, the following limits exist, are finite, equal and independent of the choice of a continuous branch of the argument of f:

$$\kappa(f) := \lim_{T\to+\infty} \frac{1}{2T} \Big\{ \arg f(x) \Big\}_{-T}^{T} = \lim_{T\to\pm\infty} \frac{1}{T} \Big\{ \arg f(x) \Big\}_{0}^{T}.$$

Their common value is called the *mean motion* (or the *AP index*) of f.

We say that $G \in AP_{N\times N}$ admits a canonical left AP factorization if

$$G = G_+ G_-^{-1} \tag{2.1}$$

with $G_\pm \in \mathcal{G}AP^\pm_{N\times N}$. If in fact $G_\pm \in \mathcal{G}APW^\pm_{N\times N}$, (2.1) is said to be a canonical left APW factorization of G. More generally, a left AP or APW factorization (not necessarily canonical) of G is a representation $G = G_+ D G_-^{-1}$ with $G_\pm$ as above and an extra middle factor $D = \operatorname{diag}[e_{\kappa_1}, \ldots, e_{\kappa_N}]$. The parameters $\kappa_j \in \mathbb{R}$ are defined by G uniquely up to a permutation whenever the factorization exists, and are called the (left) *partial AP indices* of G. Of course, condition $G \in \mathcal{G}AP_{N\times N}$ (resp., $G \in \mathcal{G}APW_{N\times N}$) is necessary in order for G to admit a left AP (resp., APW) factorization, and

$$\kappa_1 + \cdots + \kappa_N = \kappa(\det G).$$

A canonical AP factorization of $G \in APW_{N\times N}$ is automatically its (naturally, also canonical) APW factorization. For $N = 1$, any $G \in \mathcal{G}APW$ admits an APW factorization, and thus AP (and even APW) factorable functions form a dense subset of AP. As was discovered recently [8], this is not the case any more if $N > 1$.

However, for matrix functions of the form (1.4) with $N = 1$, that is,

$$G_f = \begin{bmatrix} e_\lambda & 0 \\ f & e_{-\lambda} \end{bmatrix}, \tag{2.2}$$

it is presently not known whether (and therefore still a priori possible that) the set of f for which G_f admits an AP factorization is dense in AP; see open problems in [7]. Let us denote by $\mathcal{E}$ the closure of this set, and say that $E \subset \mathbb{R}$ is *admissible* if

$$\Omega(f) \subset E \Longrightarrow f \in \mathcal{E}.$$

From previous work on the factorization theory it follows in particular that grids $-\nu + h\mathbb{Z}$ and sets E with a gap of length at least 1 inside $(-1, 1)$ are admissible.

The next result implies that the set of $f \in APW$ for which (2.2) admits a *canonical AP* factorization is dense in $\mathcal{E} \cap APW$.

Lemma 2.1. *Let G_f be APW factorable. Then in every neighborhood of f in APW metric there exist g for which G_g admit a canonical AP factorization.*

Proof. Step 1. It is a standard trick in AP factorization theory (see, e.g., [5, Proposition 13.4]) to consider along with G_f the matrix function

$$\begin{bmatrix} 1 & 0 \\ \phi_+ & 1 \end{bmatrix} G_f \begin{bmatrix} 1 & 0 \\ \phi_- & 1 \end{bmatrix} = G_{\widetilde{f}},$$

where $\widetilde{f} = f + e_\lambda \phi_+ + e_{-\lambda}\phi_-$. Obviously, G_f and $G_{\widetilde{f}}$ are APW factorable only simultaneously and have the same sets of partial AP indices, provided that $\phi_\pm \in APW^\pm$. Moreover, small perturbations of $\widetilde{f}$ are equivalent to small perturbations of f, for $\phi_\pm$ being fixed. So, choosing for $f \in APW$

$$\phi_\pm = -\sum_{\mu\in\Omega(f), \pm\mu\geq\lambda} \widehat{f}(\mu) e_{(\mu\mp\lambda)},$$

we reduce the general case to the situation when

$$\Omega(f) \subset (-\lambda, \lambda). \tag{2.3}$$

Step 2. Suppose that (2.3) holds and G_f admits an APW factorization $G_+DG_-^{-1}$. Then its partial AP indices are $\pm\nu$ for some $\nu \in [0, \lambda]$, $\Omega(G_+^{\pm 1}) \subset [0, \lambda]$ and

$\Omega(G_-^{\pm 1}) \subset [-\lambda, 0]$. Of course, only the case $\nu \neq 0$ is of interest. Thus, we may suppose that $D = \operatorname{diag}[e_\nu, e_{-\nu}]$ with $\nu \in (0, \lambda]$.

Since for $c \neq 0$

$$\begin{bmatrix} e_\nu & 0 \\ c & e_{-\nu} \end{bmatrix} = \begin{bmatrix} 1 & e_\nu \\ 0 & c \end{bmatrix} \begin{bmatrix} e_{-\nu} & 1 \\ -c & 0 \end{bmatrix}^{-1},$$

the matrix function

$$G_f + c\,G_+ \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} G_-^{-1} = G_+ \begin{bmatrix} e_\nu & 0 \\ c & e_{-\nu} \end{bmatrix} G_-^{-1}$$

admits a canonical *APW* factorization. So, there are arbitrarily small (in *APW* metric) perturbations of G_f by matrix functions with the Bohr–Fourier spectrum in $[-\lambda, \lambda]$ admitting a canonical *AP* factorization.

Step 3. Let $H \in APW_{2\times 2}$ be a small perturbation the existence of which was proved at Step 2, that is, $\Omega(H) \subset [-\lambda, \lambda]$ and $G_f + H$ admits a canonical *AP* factorization. Observe that

$$\begin{aligned} G_f + H &= \begin{bmatrix} e_\lambda(1 + e_{-\lambda}h_{11}) & h_{12} \\ f + h_{21} & e_{-\lambda}(1 + e_\lambda h_{22}) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 + e_\lambda h_{22} \end{bmatrix} \begin{bmatrix} e_\lambda & h_{12} \\ \widetilde{f} & e_{-\lambda} \end{bmatrix} \begin{bmatrix} 1 + e_{-\lambda}h_{11} & 0 \\ 0 & 1 \end{bmatrix}, \end{aligned} \tag{2.4}$$

where

$$\widetilde{f} = \frac{f + h_{21}}{(1 + e_{-\lambda}h_{11})(1 + e_\lambda h_{22})}. \tag{2.5}$$

Since $\Omega(e_\lambda h_{22}) \subset [0, 2\lambda]$, $1 + e_\lambda h_{22} \in \mathcal{G}APW^+$ provided that $\|H\|_W$ is sufficiently small. Similarly, $1 + e_{-\lambda}h_{11} \in \mathcal{G}APW^-$. From (2.4) we then conclude that the matrix function

$$\begin{bmatrix} e_\lambda & h_{12} \\ \widetilde{f} & e_{-\lambda} \end{bmatrix} \tag{2.6}$$

admits a canonical *AP* factorization, while (2.5) implies that $\widetilde{f}$ can be made arbitrarily close to f. In other words, the perturbation H can be made off-diagonal, with $\Omega(h_{12}) \subset [-\lambda, \lambda]$.

Step 4. Consider now a small perturbation of G_f the existence of which was established at Step 3, and represent it as

$$\begin{bmatrix} e_\lambda & h_{12} \\ f + h_{21} & e_{-\lambda} \end{bmatrix} = \begin{bmatrix} e_\lambda & 0 \\ f + h_{21} & e_{-\lambda}(1 - h_{12}(f + h_{21})) \end{bmatrix} \begin{bmatrix} 1 & e_{-\lambda}h_{12} \\ 0 & 1 \end{bmatrix}. \tag{2.7}$$

Since $\Omega(e_{-\lambda}h_{12}) \subset [-2\lambda, 0]$, the right factor in (2.7) belongs to $\mathcal{G}APW_{2\times 2}^-$. Thus, the left factor in the right-hand side of (2.7) admits a canonical *AP* factorization along with its left-hand side. In its turn, $1 - h_{12}(f + h_{21})$ is a function close to 1 in *APW* and therefore admitting a canonical factorization $g_+g_-^{-1}$ with the multiples also close to 1. From here we conclude that the matrix function

$$\begin{bmatrix} e_\lambda & 0 \\ (f + h_{21})g_+^{-1} & e_{-\lambda} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & g_+^{-1} \end{bmatrix} \begin{bmatrix} e_\lambda & 0 \\ f + h_{21} & e_{-\lambda}(1 - h_{12}(f + h_{21})) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & g_- \end{bmatrix}$$

also admits a canonical AP factorization. It remains to observe that $(f+h_{21})g_+^{-1}$ can be made arbitrarily close to f by choosing $\|H\|_W$ small enough. □

If $G \in AP_{N\times N}$ has a canonical left AP factorization then the matrix

$$\mathbf{d}(G) := \mathbf{M}(G_+)\mathbf{M}(G_-)^{-1}, \tag{2.8}$$

with $\mathbf{M}(G_\pm)$ understood entry-wise, does not depend on the particular choice of such a factorization and is called the *geometric mean* of G.

The value of $\mathbf{d}(G)$ plays an important role in the Fredholmness criteria for the related convolution type equations. It is known to depend continuously on G ([26], see also [5]).

2.2. Factorization in decomposing algebras

Let $\mathcal{B}$ be a decomposing unital Banach algebra with unit e, that is, $\mathcal{B}$ admits a direct sum decomposition

$$\mathcal{B} = \mathcal{B}_+ \oplus \mathcal{B}_- \tag{2.9}$$

with $\mathcal{B}_\pm$ being subalgebras of $\mathcal{B}$, and let $P_\pm$ be the complementary projections associated with this decomposition, $P_\pm \colon \mathcal{B} \to \mathcal{B}_\pm$.

We say that $b = e - a \in \mathcal{B}$ admits a canonical left factorization if

$$e - a = (e + b_+)\,(e + b_-), \tag{2.10}$$

where $e + b_\pm \in \mathcal{G}\mathcal{B}$, $b_\pm \in \mathcal{B}_\pm$ and $(e+b_\pm)^{-1} - e \in \mathcal{B}_\pm$.

The existence of such a factorization when $\|a\|$ is sufficiently small is well known, see, e.g., [16, Chapter I, Lemma 5.1] or [17, Chapter XXIX, Theorem 9.1]. For our purposes we need a variation of this result presented below.

Consider the linear mappings

$$\begin{aligned} \mathcal{P}_a^+ &\colon \mathcal{B} \to \mathcal{B}_+, \quad x \mapsto P_+(xa), \\ \mathcal{P}_a^- &\colon \mathcal{B} \to \mathcal{B}_-, \quad x \mapsto P_-(ax). \end{aligned} \tag{2.11}$$

Theorem 2.2. *Let $\mathcal{B}$ be a decomposing unital Banach algebra with associated direct sum decomposition* (2.9) *and let $b = e - a \in \mathcal{B}$ be invertible in $\mathcal{B}$. If*

$$(\mathcal{P}_a^+)^{n_+} e = 0 \quad \textit{and} \quad (\mathcal{P}_a^-)^{n_-} e = 0 \quad \textit{for some} \quad n_\pm \in \mathbb{N}, \tag{2.12}$$

then $e - a$ admits the canonical left factorization (2.10)*, where*

$$e + b_+ = \left(\sum_{n=0}^{n_+-1} (\mathcal{P}_a^+)^n e \right)^{-1}, \quad e + b_- = \left(\sum_{n=0}^{n_--1} (\mathcal{P}_a^-)^n e \right)^{-1}. \tag{2.13}$$

Proof. Setting

$$\begin{aligned} e + c_- &:= e - P_-\left(\left(\sum_{n=0}^{n_+-1} (\mathcal{P}_a^+)^n e \right) a \right), \\ e + c_+ &:= e - P_+\left(a \left(\sum_{n=0}^{n_--1} (\mathcal{P}_a^-)^n e \right) \right), \end{aligned}$$

it is easily seen from (2.12) that

$$\left(\sum_{n=0}^{n_+-1} (\mathcal{P}_a^+)^n e\right)(e-a) = e + c_-, \tag{2.14}$$

$$(e-a)\left(\sum_{n=0}^{n_--1} (\mathcal{P}_a^-)^n e\right) = e + c_+. \tag{2.15}$$

In view of the invertibility of $e-a$, equation (2.15) is equivalent to

$$(e-a)^{-1}(e+c_+) = \sum_{n=0}^{n_--1} (\mathcal{P}_a^-)^n e. \tag{2.16}$$

Multiplying (2.14) and (2.16), we obtain

$$\left(\sum_{n=0}^{n_+-1} (\mathcal{P}_a^+)^n e\right)(e+c_+) = (e+c_-)\left(\sum_{n=0}^{n_--1} (\mathcal{P}_a^-)^n e\right) \tag{2.17}$$

or, equivalently,

$$c_+ + \left(\sum_{n=1}^{n_+-1} (\mathcal{P}_a^+)^n e\right)(e+c_+) = c_- + (e+c_-)\left(\sum_{n=1}^{n_--1} (\mathcal{P}_a^-)^n e\right), \tag{2.18}$$

where the expression on the left of (2.18) belongs to $\mathcal{B}_+$ and on the right of (2.18) belongs to $\mathcal{B}_-$. Since $\mathcal{B}_+ \cap \mathcal{B}_- = \{0\}$, both sides of (2.18) equal zero. Hence, (2.17) can be rewritten in the form

$$\left(\sum_{n=0}^{n_+-1} (\mathcal{P}_a^+)^n e\right)(e+c_+) = (e+c_-)\left(\sum_{n=0}^{n_--1} (\mathcal{P}_a^-)^n e\right) = e, \tag{2.19}$$

which means that the elements $\sum_{n=0}^{n_\pm-1} (\mathcal{P}_a^\pm)^n e$ are one-sided inverses for the elements $e + c_\pm$, respectively.

Replacing a by λa, where $\lambda \in [0,1]$, and following the proof of [16, Chapter I, Lemma 5.1], we infer that all the multiples in (2.19) are two-sided invertible. Then (2.14) and (2.19) imply the canonical left factorization (2.10) with $e + b_\pm$ given by (2.13). □

2.3. *APW* factorization in the scalar quadrinomial case

In what follows we use the notation $\lfloor x \rfloor$ and $\lceil x \rceil$ for the best integer approximation to $x \in \mathbb{R}$ from below and above, respectively; $\{x\}$ denotes the fractional part of $x \in \mathbb{R}$: $\{x\} = x - \lfloor x \rfloor$. Also, as usual,

$$\mathbb{N} := \{1,2,\dots\}, \quad \mathbb{N}_- := \{-1,-2,\dots\}, \quad \mathbb{Z}_+ := \mathbb{N} \cup \{0\}, \quad \mathbb{Z}_- := \mathbb{N}_- \cup \{0\}.$$

The results of this subsection are not new, and are listed here for convenience of reference.

Theorem 2.3 ([1, 25]). *Let in* (2.2)

$$\lambda = 1 \quad \textit{and} \quad f = C_1 e_\alpha + C_{-1} e_{\alpha-1} + C_2 e_\beta + C_{-2} e_{\beta-1}, \; C_{\pm 1}, C_{\pm 2} \in \mathbb{C}, \tag{2.20}$$

where $0 < \alpha < \beta < 1$ *and the number* $\beta - \alpha$ *is irrational. Then* G_f *admits a canonical left AP factorization if and only if*

$$|C_2|^{1-\beta} |C_{-2}|^{\beta} \neq |C_1|^{1-\alpha} |C_{-1}|^{\alpha}.$$

Corollary 2.4. *Functions* f *of the form* (2.20) *belong to* $\mathcal{E}$, *with say, the sets* $\{\alpha, \beta, \alpha - 1, \beta - 1\}$ *are admissible.*

Note that the matrix function (2.2) with

$$|C_2|^{1-\beta} |C_{-2}|^{\beta} = |C_1|^{1-\alpha} |C_{-1}|^{\alpha} \neq 0$$

in (2.20) is not AP factorable [25] while for $C_2 C_{-2} = C_1 C_{-1} = 0$ its APW factorization exists but it is not canonical. Also, only the case of irrational $\beta - \alpha$ is of interest, since otherwise the distances between all the points in $\Omega(f)$ are commensurable. The latter situation, with an arbitrary number of terms in f, was covered earlier in [20] (see also [5, Section 14.4]).

The remaining portion of this subsection is a restatement of the results from [2] in a form convenient for our current purposes.

Theorem 2.5. *Suppose that* G *is given by* (2.20) *where* $0 < \alpha < \beta < 1$, *the number* $\beta - \alpha$ *is irrational, and*

$$|C_2|^{1-\beta} |C_{-2}|^{\beta} < |C_1|^{1-\alpha} |C_{-1}|^{\alpha}.$$

Then G *admits a canonical left APW factorization* (2.1) *where the matrix functions* $G_\pm, G_\pm^{-1} \in APW^\pm_{2\times 2}$ *are given by*

$$G_\pm = \begin{bmatrix} \varphi_1^\pm & \widetilde{\varphi}_1^\pm \\ \varphi_2^\pm & \widetilde{\varphi}_2^\pm \end{bmatrix}, \quad G_\pm^{-1} = \frac{1}{\det G_\pm} \begin{bmatrix} \widetilde{\varphi}_2^\pm & -\widetilde{\varphi}_1^\pm \\ -\varphi_2^\pm & \varphi_1^\pm \end{bmatrix},$$

$$\begin{aligned}
\varphi_1^+ &= e_1 + \sum_{n=0}^{\infty} X_n e_{\{n(\beta-\alpha)\}}, \\
\varphi_1^- &= 1 + \sum_{n=0}^{\infty} X_n e_{\{n(\beta-\alpha)\}-1}, \\
\varphi_2^+ &= C_1 e_\alpha + C_2 e_\beta + \sum_{\{n \in \mathbb{Z}_+ :\, 0 \le \{\alpha + n(\beta-\alpha)\} < \alpha\}} C_1 X_n e_{\{\alpha + n(\beta-\alpha)\}} \\
&\quad + \sum_{\{n \in \mathbb{Z}_+ :\, 0 \le \{\beta + n(\beta-\alpha)\} < \beta\}} C_2 X_n e_{\{\beta + n(\beta-\alpha)\}}, \\
\varphi_2^- &= - \sum_{\{n \in \mathbb{Z}_+ :\, \{\alpha + n(\beta-\alpha)\} = 0\}} C_{-1} X_n - \sum_{\{n \in \mathbb{Z}_+ :\, \alpha \le \{\alpha + n(\beta-\alpha)\} < 1\}} C_{-1} X_n e_{\{\alpha + n(\beta-\alpha)\}-1} \\
&\quad - \sum_{\{n \in \mathbb{Z}_+ :\, \{\beta + n(\beta-\alpha)\} = 0\}} C_{-2} X_n - \sum_{\{n \in \mathbb{Z}_+ :\, \beta \le \{\beta + n(\beta-\alpha)\} < 1\}} C_{-2} X_n e_{\{\beta + n(\beta-\alpha)\}-1},
\end{aligned} \tag{2.21}$$

$$\begin{aligned}
\widetilde{\varphi}_1^+ &= \sum\nolimits_{n=0}^{\infty} \widetilde{X}_n e_{\{n(\beta-\alpha)-\alpha\}},\\
\widetilde{\varphi}_1^- &= \sum\nolimits_{n=0}^{\infty} \widetilde{X}_n e_{\{n(\beta-\alpha)-\alpha\}-1},\\
\widetilde{\varphi}_2^+ &= \sum_{\{n\in\mathbb{Z}_+:\, 0\le\{n(\beta-\alpha)\}<\alpha\}} C_1\widetilde{X}_n e_{\{n(\beta-\alpha)\}} + \sum_{\{n\in\mathbb{Z}_+:\, 0\le\{(n+1)(\beta-\alpha)\}<\beta\}} C_2\widetilde{X}_n e_{\{(n+1)(\beta-\alpha)\}},\\
\widetilde{\varphi}_2^- &= -\sum_{\{n\in\mathbb{Z}_+:\, \alpha\le\{n(\beta-\alpha)\}<1\}} C_{-1}\widetilde{X}_n e_{\{n(\beta-\alpha)\}-1}\\
&\quad -\sum_{\{n\in\mathbb{Z}_+:\, \beta\le\{(n+1)(\beta-\alpha)\}<1\}} C_{-2}\widetilde{X}_n e_{\{(n+1)(\beta-\alpha)\}-1} - C_{-1}\widetilde{X}_0.
\end{aligned} \tag{2.22}$$

The coefficients X_n *and* $\widetilde{X}_n$ $(n\in\mathbb{Z}_+)$ *here are given by*

$$X_n = \begin{cases}
(-1)^{n-1}\left(\frac{C_{-2}}{C_1}\right)^k \left(\frac{C_2}{C_1}\right)^{-1+\left(n-\lceil \frac{k}{\beta-\alpha}\rceil\right)+\sum_{s=1}^{k}\left(\lceil\frac{s-\alpha}{\beta-\alpha}\rceil-\lceil\frac{s-1}{\beta-\alpha}\rceil-1\right)}\\
\times\left(\frac{C_{-2}}{C_{-1}}\right)^{\sum_{s=1}^{k}\left(\lceil\frac{s}{\beta-\alpha}\rceil-\lceil\frac{s-\alpha}{\beta-\alpha}\rceil\right)} X_1 \quad \textit{if}\ \ n=\lceil\frac{k}{\beta-\alpha}\rceil+1,\dots,\lceil\frac{k+1-\alpha}{\beta-\alpha}\rceil-1,\\
(-1)^{n-1}\left(\frac{C_{-2}}{C_1}\right)^k \left(\frac{C_{-2}}{C_{-1}}\right)^{\left(n-\lceil \frac{k+1-\alpha}{\beta-\alpha}\rceil+1\right)+\sum_{s=1}^{k}\left(\lceil\frac{s}{\beta-\alpha}\rceil-\lceil\frac{s-\alpha}{\beta-\alpha}\rceil\right)}\\
\times\left(\frac{C_2}{C_1}\right)^{-1+\sum_{s=1}^{k+1}\left(\lceil\frac{s-\alpha}{\beta-\alpha}\rceil-\lceil\frac{s-1}{\beta-\alpha}\rceil-1\right)} X_1 \quad \textit{if}\ \ n=\lceil\frac{k+1-\alpha}{\beta-\alpha}\rceil,\dots,\lceil\frac{k+1}{\beta-\alpha}\rceil-1,\\
(-1)^{n-1}\left(\frac{C_{-2}}{C_1}\right)^{k+1} \left(\frac{C_{-2}}{C_{-1}}\right)^{\sum_{s=1}^{k+1}\left(\lceil\frac{s}{\beta-\alpha}\rceil-\lceil\frac{s-\alpha}{\beta-\alpha}\rceil\right)}\\
\times\left(\frac{C_2}{C_1}\right)^{-1+\sum_{s=1}^{k+1}\left(\lceil\frac{s-\alpha}{\beta-\alpha}\rceil-\lceil\frac{s-1}{\beta-\alpha}\rceil-1\right)} X_1 \quad \textit{if}\ \ n=\lceil\frac{k+1}{\beta-\alpha}\rceil
\end{cases} \tag{2.23}$$

and

$$\widetilde{X}_n = \begin{cases}
(-1)^{n}\left(\frac{C_{-2}}{C_{-1}}\right)^{\left(n-\lceil \frac{k}{\beta-\alpha}\rceil\right)+\sum_{s=0}^{k-1}\left(\lceil\frac{s+\alpha}{\beta-\alpha}\rceil-\lceil\frac{s}{\beta-\alpha}\rceil\right)} \left(\frac{C_{-2}}{C_1}\right)^k\\
\times\left(\frac{C_2}{C_1}\right)^{\sum_{s=1}^{k}\left(\lceil\frac{s}{\beta-\alpha}\rceil-\lceil\frac{s-1+\beta}{\beta-\alpha}\rceil\right)} \widetilde{X}_0 \quad \textit{if}\ \ n=\lceil\frac{k}{\beta-\alpha}\rceil,\dots,\lceil\frac{k+\alpha}{\beta-\alpha}\rceil-1,\\
(-1)^{n}\left(\frac{C_{-2}}{C_{-1}}\right)^{-1+\sum_{s=0}^{k}\left(\lceil\frac{s+\alpha}{\beta-\alpha}\rceil-\lceil\frac{s}{\beta-\alpha}\rceil\right)} \left(\frac{C_{-2}}{C_1}\right)^{k+1}\\
\times\left(\frac{C_2}{C_1}\right)^{\sum_{s=1}^{k}\left(\lceil\frac{s}{\beta-\alpha}\rceil-\lceil\frac{s-1+\beta}{\beta-\alpha}\rceil\right)} \widetilde{X}_0 \quad \textit{if}\ \ n=\lceil\frac{k+\alpha}{\beta-\alpha}\rceil=\lceil\frac{k+\beta}{\beta-\alpha}\rceil-1,\\
(-1)^{n}\left(\frac{C_{-2}}{C_1}\right)^{k+1} \left(\frac{C_2}{C_1}\right)^{\left(n-\lceil \frac{k+\beta}{\beta-\alpha}\rceil+1\right)+\sum_{s=1}^{k}\left(\lceil\frac{s}{\beta-\alpha}\rceil-\lceil\frac{s-1+\beta}{\beta-\alpha}\rceil\right)}\\
\times\left(\frac{C_{-2}}{C_{-1}}\right)^{-1+\sum_{s=0}^{k}\left(\lceil\frac{s+\alpha}{\beta-\alpha}\rceil-\lceil\frac{s}{\beta-\alpha}\rceil\right)} \widetilde{X}_0 \quad \textit{if}\ \ n=\lceil\frac{k+\beta}{\beta-\alpha}\rceil,\dots,\lceil\frac{k+1}{\beta-\alpha}\rceil-1
\end{cases} \tag{2.24}$$

for $k=0,1,2,\dots$, *with the initial conditions* $\widetilde{X}_0=1$,

$$X_0 = -\frac{C_{-1}}{C_1}, \quad X_1 = -\frac{C_{-2}}{C_1}+\frac{C_2C_{-1}}{C_1^2}.$$

To simplify the forthcoming formulas, we let

$$\mathbb{N}_\gamma^\pm := \{n\colon\ \pm n \in \mathbb{N} \text{ and } \gamma + n(\beta-\alpha) \in \mathbb{Z}\} \quad \text{for} \quad \gamma = \pm\alpha, \beta. \tag{2.25}$$

Corollary 2.6. *In the setting of Theorem* 2.5 *we have*

$$\mathbf{M}(G_-) = \begin{bmatrix} 1 & 0 \\ -\sum_{n\in\mathbb{N}_\alpha^+} C_{-1}X_n - \sum_{n\in\mathbb{N}_\beta^+} C_{-2}X_n & -C_{-1} \end{bmatrix},$$

$$\mathbf{M}(G_+) = \begin{bmatrix} -C_{-1}C_1^{-1} & \sum_{n\in\mathbb{N}_{-\alpha}^+} \widetilde{X}_n \\ \sum_{n\in\mathbb{N}_\alpha^+} C_1X_n + \sum_{n\in\mathbb{N}_\beta^+} C_2X_n & C_1 \end{bmatrix},$$

and hence the geometric mean of G *is given by*

$$\mathbf{d}(G) = \begin{bmatrix} -C_{-1}C_1^{-1} & -\sum_{n\in\mathbb{N}_{-\alpha}^+} C_{-1}^{-1}\widetilde{X}_n \\ \sum_{n\in\mathbb{N}_\beta^+} (C_2 - C_1C_{-1}^{-1}C_{-2})X_n & -C_1C_{-1}^{-1} \end{bmatrix}.$$

Theorem 2.7. *Suppose that* G *is given by* (2.20) *where* $0 < \alpha < \beta < 1$, *the number* $\beta - \alpha$ *is irrational, and*

$$|C_2|^{1-\beta}|C_{-2}|^\beta > |C_1|^{1-\alpha}|C_{-1}|^\alpha.$$

Then G *admits a canonical left APW factorization* (2.1) *where the matrix functions* $G_\pm, G_\pm^{-1} \in APW_{2\times 2}^\pm$ *are given by*

$$G_\pm = \begin{bmatrix} \psi_1^\pm & \widetilde{\psi}_1^\pm \\ \psi_2^\pm & \widetilde{\psi}_2^\pm \end{bmatrix}, \quad G_\pm^{-1} = \frac{1}{\det G_\pm} \begin{bmatrix} \widetilde{\psi}_2^\pm & -\widetilde{\psi}_1^\pm \\ -\psi_2^\pm & \psi_1^\pm \end{bmatrix}, \tag{2.26}$$

with

$$\begin{aligned}
\psi_1^+ &= e_1 + \sum\nolimits_{n=-\infty}^0 Y_n e_{\{n(\beta-\alpha)\}}, \\
\psi_1^- &= 1 + \sum\nolimits_{n=-\infty}^0 Y_n e_{\{n(\beta-\alpha)\}-1}, \\
\psi_2^+ &= C_1e_\alpha + C_2e_\beta + \sum_{\{n\in\mathbb{Z}_-\colon 0\le\{\alpha+n(\beta-\alpha)\}<\alpha\}} C_1Y_ne_{\{\alpha+n(\beta-\alpha)\}} \\
&\quad + \sum_{\{n\in\mathbb{Z}_-\colon 0\le\{\beta+n(\beta-\alpha)\}<\beta\}} C_2Y_ne_{\{\beta+n(\beta-\alpha)\}}, \\
\psi_2^- &= -\sum_{\{n\in\mathbb{Z}_-\colon \{\alpha+n(\beta-\alpha)\}=0\}} C_{-1}Y_n - \sum_{\{n\in\mathbb{Z}_-\colon \alpha\le\{\alpha+n(\beta-\alpha)\}<1\}} C_{-1}Y_ne_{\{\alpha+n(\beta-\alpha)\}-1} \\
&\quad - \sum_{\{n\in\mathbb{Z}_-\colon \{\beta+n(\beta-\alpha)\}=0\}} C_{-2}Y_n - \sum_{\{n\in\mathbb{Z}_-\colon \beta\le\{\beta+n(\beta-\alpha)\}<1\}} C_{-2}Y_ne_{\{\beta+n(\beta-\alpha)\}-1},
\end{aligned} \tag{2.27}$$

$$
\begin{aligned}
\widetilde{\psi}_1^+ &= \sum\nolimits_{n=-\infty}^{-1} \widetilde{Y}_n e_{\{n(\beta-\alpha)-\alpha\}}, \\
\widetilde{\psi}_1^- &= \sum\nolimits_{n=-\infty}^{-1} \widetilde{Y}_n e_{\{n(\beta-\alpha)-\alpha\}-1}, \\
\widetilde{\psi}_2^+ &= \sum_{\{n\in\mathbb{N}_-:\, 0\le\{n(\beta-\alpha)\}<\alpha\}} C_1\widetilde{Y}_n e_{\{n(\beta-\alpha)\}} + \sum_{\{n\in\mathbb{N}_-:\, 0\le\{(n+1)(\beta-\alpha)\}<\beta\}} C_2\widetilde{Y}_n e_{\{(n+1)(\beta-\alpha)\}}, \\
\widetilde{\psi}_2^- &= -\sum_{\{n\in\mathbb{N}_-:\, \alpha\le\{n(\beta-\alpha)\}<1\}} C_{-1}\widetilde{Y}_n e_{\{n(\beta-\alpha)\}-1} \\
&\quad - \sum_{\{n\in\mathbb{N}_-:\, \beta\le\{(n+1)(\beta-\alpha)\}<1\}} C_{-2}\widetilde{Y}_n e_{\{(n+1)(\beta-\alpha)\}-1} - C_{-2}\widetilde{Y}_{-1}.
\end{aligned} \tag{2.28}
$$

The coefficients Y_n $(n\in\mathbb{Z}_-)$ and $\widetilde{Y}_n$ $(n\in\mathbb{N}_-)$ here are defined by

$$
Y_{n-1} = \begin{cases}
(-1)^{|n|}\left(\frac{C_{-1}}{C_{-2}}\right)^{\left(|n|-\left\lceil\frac{k}{\beta-\alpha}\right\rceil\right)+\sum_{s=0}^{k-1}\left(\left\lceil\frac{s+\alpha}{\beta-\alpha}\right\rceil-\left\lceil\frac{s}{\beta-\alpha}\right\rceil\right)}\left(\frac{C_1}{C_{-2}}\right)^k \\
\times\left(\frac{C_1}{C_2}\right)^{\sum_{s=1}^{k}\left(\left\lceil\frac{s}{\beta-\alpha}\right\rceil-\left\lceil\frac{s-1+\alpha}{\beta-\alpha}\right\rceil-1\right)}Y_{-1} \quad \textit{if} \ -n=\left\lceil\frac{k}{\beta-\alpha}\right\rceil,\dots,\left\lceil\frac{k+\alpha}{\beta-\alpha}\right\rceil-1, \\
(-1)^{|n|}\left(\frac{C_1}{C_2}\right)^{\left(|n|-\left\lceil\frac{k+\alpha}{\beta-\alpha}\right\rceil+1\right)+\sum_{s=1}^{k}\left(\left\lceil\frac{s}{\beta-\alpha}\right\rceil-\left\lceil\frac{s-1+\alpha}{\beta-\alpha}\right\rceil-1\right)}\left(\frac{C_1}{C_{-2}}\right)^k \\
\times\left(\frac{C_{-1}}{C_{-2}}\right)^{-1+\sum_{s=0}^{k}\left(\left\lceil\frac{s+\alpha}{\beta-\alpha}\right\rceil-\left\lceil\frac{s}{\beta-\alpha}\right\rceil\right)}Y_{-1} \ \textit{if} \ -n=\left\lceil\frac{k+\alpha}{\beta-\alpha}\right\rceil,\dots,\left\lceil\frac{k+1}{\beta-\alpha}\right\rceil-2, \\
(-1)^{|n|}\left(\frac{C_{-1}}{C_{-2}}\right)^{-1+\sum_{s=0}^{k}\left(\left\lceil\frac{s+\alpha}{\beta-\alpha}\right\rceil-\left\lceil\frac{s}{\beta-\alpha}\right\rceil\right)}\left(\frac{C_1}{C_{-2}}\right)^{k+1} \\
\times\left(\frac{C_1}{C_2}\right)^{\sum_{s=1}^{k+1}\left(\left\lceil\frac{s}{\beta-\alpha}\right\rceil-\left\lceil\frac{s-1+\alpha}{\beta-\alpha}\right\rceil-1\right)} Y_{-1} \quad \textit{if} \quad -n=\left\lceil\frac{k+1}{\beta-\alpha}\right\rceil-1
\end{cases} \tag{2.29}
$$

and

$$
\widetilde{Y}_{n-1} = \begin{cases}
(-1)^{|n|}\left(\frac{C_1}{C_2}\right)^{\left(|n|-\left\lfloor\frac{k}{\beta-\alpha}\right\rfloor\right)+\sum_{s=1}^{k}\left(\left\lfloor\frac{s-\beta}{\beta-\alpha}\right\rfloor-\left\lfloor\frac{s-1}{\beta-\alpha}\right\rfloor\right)}\left(\frac{C_1}{C_{-2}}\right)^k \\
\times\left(\frac{C_{-1}}{C_{-2}}\right)^{\sum_{s=1}^{k}\left(\left\lfloor\frac{s}{\beta-\alpha}\right\rfloor-\left\lfloor\frac{s-\beta}{\beta-\alpha}\right\rfloor-1\right)}\widetilde{Y}_{-1} \quad \textit{if} \quad -n=\left\lceil\frac{k}{\beta-\alpha}\right\rceil,\dots,\left\lfloor\frac{k+1-\beta}{\beta-\alpha}\right\rfloor, \\
(-1)^{|n|}\left(\frac{C_1}{C_2}\right)^{\sum_{s=1}^{k+1}\left(\left\lfloor\frac{s-\beta}{\beta-\alpha}\right\rfloor-\left\lfloor\frac{s-1}{\beta-\alpha}\right\rfloor\right)}\left(\frac{C_1}{C_{-2}}\right)^{k+1} \\
\times\left(\frac{C_{-1}}{C_{-2}}\right)^{\sum_{s=1}^{k}\left(\left\lfloor\frac{s}{\beta-\alpha}\right\rfloor-\left\lfloor\frac{s-\beta}{\beta-\alpha}\right\rfloor-1\right)}\widetilde{Y}_{-1} \quad \textit{if} \quad -n=\left\lfloor\frac{k+1-\beta}{\beta-\alpha}\right\rfloor+1, \\
(-1)^{|n|}\left(\frac{C_1}{C_{-2}}\right)^{k+1}\left(\frac{C_{-1}}{C_{-2}}\right)^{\left(|n|-\left\lfloor\frac{k+1-\beta}{\beta-\alpha}\right\rfloor-1\right)+\sum_{s=1}^{k}\left(\left\lfloor\frac{s}{\beta-\alpha}\right\rfloor-\left\lfloor\frac{s-\beta}{\beta-\alpha}\right\rfloor-1\right)} \\
\times\left(\frac{C_1}{C_2}\right)^{\sum_{s=1}^{k+1}\left(\left\lfloor\frac{s-\beta}{\beta-\alpha}\right\rfloor-\left\lfloor\frac{s-1}{\beta-\alpha}\right\rfloor\right)}\widetilde{Y}_{-1} \quad \textit{if} \quad -n=\left\lfloor\frac{k+1-\beta}{\beta-\alpha}\right\rfloor+2,\dots,\left\lfloor\frac{k+1}{\beta-\alpha}\right\rfloor
\end{cases} \tag{2.30}
$$

for $k=0,1,2,\dots,$ with the initial conditions $\widetilde{Y}_{-1}=1$,

$$
Y_0 = -\frac{C_{-2}}{C_2}, \quad Y_{-1} = -\frac{C_{-1}}{C_{-2}} + \frac{C_1}{C_2}.
$$

In the notation (2.25), we have the following.

Corollary 2.8. *In the setting of Theorem* 2.7 *we have*

$$\mathbf{M}(G_-) = \begin{bmatrix} 1 & 0 \\ -\sum_{n\in\mathbb{N}_\alpha^-} C_{-1}Y_n - \sum_{n\in\mathbb{N}_\beta^-} C_{-2}Y_n & -C_{-2} \end{bmatrix},$$

$$\mathbf{M}(G_+) = \begin{bmatrix} -C_{-2}C_2^{-1} & \sum_{n\in\mathbb{N}_{-\alpha}^-} \widetilde{Y}_n \\ \sum_{n\in\mathbb{N}_\alpha^-} C_1 Y_n + \sum_{n\in\mathbb{N}_\beta^-} C_2 Y_n & C_2 \end{bmatrix},$$

and hence the geometric mean of G *is given by*

$$\mathbf{d}(G) = \begin{bmatrix} -C_{-2}C_2^{-1} & -\sum_{n\in\mathbb{N}_{-\alpha}^-} C_{-2}^{-1}\widetilde{Y}_n \\ \sum_{n\in\mathbb{N}_\alpha^-}(C_1 - C_2C_{-1}C_{-2}^{-1})Y_n & -C_2C_{-2}^{-1} \end{bmatrix}.$$

3. Factorization of some block triangular matrix functions

3.1. A conditional criterion of *AP* factorability

Factorability properties of $G \in AP_{N\times N}$ obviously do not change under multiplication on the left and on the right by matrices from $\mathcal{G}\mathbb{C}_{N\times N}$. In particular, G of the form (1.4) admits a left AP or APW factorization only simultaneously with

$$\begin{bmatrix} Q^{-1} & 0 \\ 0 & P \end{bmatrix} G \begin{bmatrix} Q & 0 \\ 0 & P^{-1} \end{bmatrix} = \begin{bmatrix} e_1 I_N & 0 \\ PFQ & e_{-1}I_N \end{bmatrix} \tag{3.1}$$

for any $P, Q \in \mathcal{G}\mathbb{C}_{N\times N}$, and the partial AP indices of G_F and G_{PFQ} coincide.

Proposition 3.1. *Let* $F \in APW_{N\times N}$ *be a triangular matrix function with the diagonal entries* f_j. *Then in order for* G_F *to admit a canonical AP factorization it is sufficient, and if* $f_j \in \mathcal{E}$ *for* $j = 1, \dots, N$ *also necessary, that all* 2×2 *matrix functions* G_{f_j} *admit such a factorization.*

Proof. Choosing $P = Q = [\delta_{j,N-j+1}]$ in (3.1), we can switch between lower and upper triangular F. So, without loss of generality we may suppose that F is lower triangular.

Sufficiency. Observe that

$$F = F_0 + \widetilde{F}, \tag{3.2}$$

where $F_0 = \mathrm{diag}[f_1, \dots, f_N]$ and $\widetilde{F}$ is lower triangular with zero diagonal. Letting now

$$P = Q^{-1} = \mathrm{diag}[1, \epsilon, \dots, \epsilon^{N-1}],$$

we can make the difference $PFQ - F_0$ arbitrarily small by an appropriate choice of ϵ. Since canonical AP factorable matrices form an open set, it suffices to show that G_{F_0} lies there. But the latter matrix is permutationally similar to

$$\mathrm{diag}[G_{f_1}, \dots, G_{f_N}],$$

and thus admits a left canonical AP factorization along with its diagonal 2×2 blocks.

Necessity. Suppose G_F admits a left canonical AP factorization. Consider h_j so close to f_j $(j = 2, \dots, N)$ that the matrix G_H with

$$H = \widetilde{F} + \operatorname{diag}[f_1, h_2, \dots, h_N]$$

still admits a left canonical AP factorization, while the matrices G_{h_j} also are AP factorable with zero partial AP indices, $j = 2, \dots, N$. (This is possible due to Lemma 2.1 since $f_j \in \mathcal{E} \cap APW$.) Via a permutational similarity corresponding to the permutation $\{1, N+1, 2, \dots, 2N\}$, the matrix G_H can be put in a block triangular form

$$\begin{bmatrix} G_{f_1} & 0 \\ * & G_{H_1} \end{bmatrix}, \tag{3.3}$$

where $H_1 \in APW_{(N-1)\times(N-1)}$ is lower triangular with the diagonal entries $h_2, \dots, h_N$. By the already proven sufficiency, G_{H_1} admits a canonical AP factorization.

So, a block triangular matrix (3.3) and one of its diagonal blocks both admit a left canonical AP factorization. Since canonical AP factorability of APW matrices is equivalent to the invertibility of the respective Toeplitz operators, from here it follows that the other diagonal block of (3.3), that is, G_{f_1}, must admit a canonical AP factorization.

In its turn, the same permutational similarity can be used to rewrite the unperturbed matrix G_F in a block triangular form, with the diagonal blocks being G_{f_1} and G_{F_1}, where $F_1 \in APW_{(N-1)\times(N-1)}$ is simply F with the first row and column deleted. From the canonical AP factorability of G_F and G_{f_1} we now conclude that G_{F_1} also admits a canonical AP factorization. Since the statement is trivially correct for $N = 1$, the induction argument thus completes the proof. □

3.2. Quadrinomial case: Existence

We now pass to the case of matrix functions (1.4) with $N > 1$ and the off-diagonal block (1.5) such that its coefficients $C_j \in \mathbb{C}_{N\times N}$ can be put in a triangular form by the same transformation $C_j \mapsto PC_jQ$ with some $P, Q \in \mathcal{G}\mathbb{C}_{N\times N}$. This condition is satisfied, in particular, if C_j pairwise commute, in which case it is possible to choose $Q = P^{-1}$ (see, e.g., [23, Lemma 4.3]).

Since the matrix functions G_F given by (1.4) and G_{PFQ} admit a canonical factorization only simultaneously, we may without loss of generality suppose that C_i are themselves (lower) triangular:

$$C_i = \begin{bmatrix} (c_i)_{1,1} & 0 & 0 & \dots & 0 & 0 \\ (c_i)_{2,1} & (c_i)_{2,2} & 0 & \dots & 0 & 0 \\ (c_i)_{3,1} & (c_i)_{3,2} & (c_i)_{3,3} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ (c_i)_{N-1,1} & (c_i)_{N-1,2} & (c_i)_{N-1,3} & \dots & (c_i)_{N-1,N-1} & 0 \\ (c_i)_{N,1} & (c_i)_{N,2} & (c_i)_{N,3} & \dots & (c_i)_{N,N-1} & (c_i)_{N,N} \end{bmatrix} \tag{3.4}$$

$(i = \pm1, \pm2)$. In what follows, we will relabel the diagonal entries $(c_i)_{s,s}$ of the matrices (3.4) by $c_{i,s}$. Note that in the case of pairwise commuting (but a priori not necessarily triangular) matrices C_i, $c_{i,s}$ are their so-called *bonded eigenvalues*, in the terminology of [3].

Theorem 3.2. *Let $G = G_F$ be given by* (1.4), (1.5) *and* (3.4)*, where* $0 < \alpha < \beta < 1$ *and the number* $\beta - \alpha$ *is irrational. Then G admits a canonical left AP factorization* $G = G_+ G_-^{-1}$ *if and only if*

$$|c_{2,s}|^{1-\beta}|c_{-2,s}|^{\beta} \neq |c_{1,s}|^{1-\alpha}|c_{-1,s}|^{\alpha} \quad \textit{for all} \quad s = 1, 2, \ldots, N, \tag{3.5}$$

where $c_{i,s} := (c_i)_{s,s}$ *for all* $i = \pm 1, \pm 2$ *and all* $s = 1, 2, \ldots, N$ *are the diagonal entries of matrix coefficients* (3.4) *in* (1.5).

Proof. Follows directly by combining Corollary 2.4 with Proposition 3.1. □

3.3. Quadrinomial case: Explicit factorization

We now turn to the explicit factorization construction of matrix functions G_F with F given by (1.5), (3.4) when its canonical factorization exists, that is, conditions (3.5) hold. Decomposition (3.2) in our setting yields matrices F_0 and $\widetilde{F}$ of the same structure (1.5) as F but with C_i replaced by $\operatorname{diag}[c_{i,1}, \ldots, c_{i,N}]$ for F_0 and by

$$\widetilde{C}_i = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ (c_i)_{2,1} & 0 & 0 & \cdots & 0 & 0 \\ (c_i)_{3,1} & (c_i)_{3,2} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ (c_i)_{N-1,1} & (c_i)_{N-1,2} & (c_i)_{N-1,3} & \cdots & 0 & 0 \\ (c_i)_{N,1} & (c_i)_{N,2} & (c_i)_{N,3} & \cdots & (c_i)_{N,N-1} & 0 \end{bmatrix} \tag{3.6}$$

for $\widetilde{F}$. Then $G_F = \mathcal{G} + K$, where $\mathcal{G}$ is an abbreviated notation for G_{F_0} and

$$K = \begin{bmatrix} 0_N & 0_N \\ \widetilde{F} & 0_N \end{bmatrix}.$$

Consider the matrix functions

$$\mathcal{G}_\pm = \begin{bmatrix} \mathcal{G}_1^\pm & \widetilde{\mathcal{G}}_1^\pm \\ \mathcal{G}_2^\pm & \widetilde{\mathcal{G}}_2^\pm \end{bmatrix}, \tag{3.7}$$

$$\begin{aligned} \mathcal{G}_1^\pm &:= \operatorname{diag}\{g_{s,1}^\pm\}_{s=1}^N, & \widetilde{\mathcal{G}}_1^\pm &:= \operatorname{diag}\{\widetilde{g}_{s,1}^\pm\}_{s=1}^N, \\ \mathcal{G}_2^\pm &:= \operatorname{diag}\{g_{s,2}^\pm\}_{s=1}^N, & \widetilde{\mathcal{G}}_2^\pm &:= \operatorname{diag}\{\widetilde{g}_{s,2}^\pm\}_{s=1}^N, \end{aligned}$$

where for $s = 1, 2, \ldots, N$ and $j = 1, 2$,

$$g_{s,j}^\pm := \left\{ \begin{array}{l} \varphi_{s,j}^\pm \\ \psi_{s,j}^\pm \end{array} \right., \quad \widetilde{g}_{s,j}^\pm := \left\{ \begin{array}{l} \widetilde{\varphi}_{s,j}^\pm \\ \widetilde{\psi}_{s,j}^\pm \end{array} \right. \quad \text{if} \quad \left\{ \begin{array}{l} |c_{2,s}|^{1-\beta}|c_{-2,s}|^{\beta} < |c_{1,s}|^{1-\alpha}|c_{-1,s}|^{\alpha}, \\ |c_{2,s}|^{1-\beta}|c_{-2,s}|^{\beta} > |c_{1,s}|^{1-\alpha}|c_{-1,s}|^{\alpha}, \end{array} \right. \tag{3.8}$$

$\varphi^{\pm}_{s,j}$ and $\widetilde{\varphi}^{\pm}_{s,j}$ are given by (2.21) and (2.22), respectively, with C_i replaced by $c_{i,s}$ for $i = \pm 1, \pm 2$, and with X_n and $\widetilde{X}_n$ $(n \in \mathbb{Z}_+)$ calculated by formulas (2.23), (2.24) (again with C_i replaced by $c_{i,s}$ for $i = \pm 1, \pm 2$), where

$$X_0 = -\frac{c_{-1,s}}{c_{1,s}}, \quad X_1 = -\frac{c_{-2,s}}{c_{1,s}} + \frac{c_{2,s}c_{-1,s}}{c_{1,s}^2}, \quad \widetilde{X}_0 = 1;$$

$\psi^{\pm}_{s,j}$ and $\widetilde{\psi}^{\pm}_{s,j}$ are given by (2.27) and (2.28), respectively, with C_i replaced by $c_{i,s}$ for $i = \pm 1, \pm 2$, and with Y_n $(n \in \mathbb{Z}_-)$ and $\widetilde{Y}_n$ $(n \in \mathbb{N}_-)$ calculated by formulas (2.29), (2.30) (once again, with C_i replaced by $c_{i,s}$ for $i = \pm 1, \pm 2$), where

$$Y_0 = -\frac{c_{-2,s}}{c_{2,s}}, \quad Y_{-1} = -\frac{c_{-1,s}}{c_{-2,s}} + \frac{c_{1,s}}{c_{2,s}}, \quad \widetilde{Y}_{-1} = 1.$$

Below we will denote these X_n, $\widetilde{X}_n$, Y_n and $\widetilde{Y}_n$ as $X_{n,s}$, $\widetilde{X}_{n,s}$, $Y_{n,s}$ and $\widetilde{Y}_{n,s}$.

For $s = 1, 2, \ldots, N$, we put

$$G^{\pm}_s = \begin{bmatrix} g^{\pm}_{s,1} & \widetilde{g}^{\pm}_{s,1} \\ g^{\pm}_{s,2} & \widetilde{g}^{\pm}_{s,2} \end{bmatrix}, \tag{3.9}$$

where $g^{\pm}_{s,j}$ and $\widetilde{g}^{\pm}_{s,j}$ for $j = 1, 2$ are given by (3.8). Note that

$$\det G^+_s = \det G^-_s = k_s \quad \text{where} \quad k_s = \mathbf{M}(\det G^{\pm}_s). \tag{3.10}$$

We also define the matrix

$$K_N = \operatorname{diag}\left[k_1^{-1}, \ldots, k_N^{-1}\right]. \tag{3.11}$$

By [2], the matrix function $\mathcal{G} = G_{F_0}$ admits the canonical left APW factorization

$$\mathcal{G} = \mathcal{G}_+ \mathcal{G}_-^{-1}. \tag{3.12}$$

It follows from (3.12) that

$$G = \mathcal{G}_+ \widetilde{G} \mathcal{G}_-^{-1},$$

where

$$\begin{aligned} \widetilde{G} = \mathcal{G}_+^{-1} G \mathcal{G}_- &= \begin{bmatrix} K_N \widetilde{\mathcal{G}}_2^+ & -K_N \widetilde{\mathcal{G}}_1^+ \\ -K_N \mathcal{G}_2^+ & K_N \mathcal{G}_1^+ \end{bmatrix} \begin{bmatrix} e_1 I_N & 0_N \\ F & e_{-1} I_N \end{bmatrix} \begin{bmatrix} \mathcal{G}_1^- & \widetilde{\mathcal{G}}_1^- \\ \mathcal{G}_2^- & \widetilde{\mathcal{G}}_2^- \end{bmatrix} \\ &= \begin{bmatrix} I_N & 0_N \\ 0_N & I_N \end{bmatrix} + \begin{bmatrix} -K_N \widetilde{\mathcal{G}}_1^+ \widetilde{F} \mathcal{G}_1^- & -K_N \widetilde{\mathcal{G}}_1^+ \widetilde{F} \widetilde{\mathcal{G}}_1^- \\ K_N \mathcal{G}_1^+ \widetilde{F} \mathcal{G}_1^- & K_N \mathcal{G}_1^+ \widetilde{F} \widetilde{\mathcal{G}}_1^- \end{bmatrix} \end{aligned} \tag{3.13}$$

and K_N is given by (3.11).

Consider now $APW_{N\times N}$ as the decomposing Banach algebra $\mathcal{B}$, with $\mathcal{B}_+ = APW^+_{N\times N}$ and $\mathcal{B}_- = (APW_0^-)_{N\times N}$. Letting

$$a := \begin{bmatrix} K_N \widetilde{\mathcal{G}}_1^+ \widetilde{F} \mathcal{G}_1^- & K_N \widetilde{\mathcal{G}}_1^+ \widetilde{F} \widetilde{\mathcal{G}}_1^- \\ -K_N \mathcal{G}_1^+ \widetilde{F} \mathcal{G}_1^- & -K_N \mathcal{G}_1^+ \widetilde{F} \widetilde{\mathcal{G}}_1^- \end{bmatrix} \tag{3.14}$$

and observing that each of its blocks is lower triangular with zero diagonal entries, we conclude that

$$(\mathcal{P}_a^+)^N I_{2N} = 0_{2N} \quad \text{and} \quad (\mathcal{P}_a^-)^N I_{2N} = 0_{2N},$$

where the mappings $\mathcal{P}_a^\pm$ are defined by (2.11) with a given by (3.14). Hence, by Theorem 2.2, the matrix function $\widetilde{G} = I_{2N} - a$ admits a canonical left factorization

$$\widetilde{G} = (I_{2N} + b_+)^{-1}(I_{2N} + b_-)^{-1} \tag{3.15}$$

where the matrix functions

$$\begin{aligned} b_+ &= a_+ + (a_+a)_+ + ((a_+a)_+a)_+ + \cdots + \underbrace{(\ldots(a_+a)_+\ldots a)_+}_{N-1 \text{ terms}}, \\ b_- &= a_- + (aa_-)_- + (a(aa_-)_-)_- + \cdots + \underbrace{(a\ldots(aa_-)_-\ldots)_-}_{N-1 \text{ terms}}, \end{aligned} \tag{3.16}$$

belong to $APW^+_{2N\times 2N}$ and $(APW_0^-)_{2N\times 2N}$, respectively, and $a_\pm := P_\pm a = \mathcal{P}_a^\pm I_{2N}$. Since each of $N \times N$ blocks in (3.14) is lower triangular matrix function with zero diagonal entries, it is easily seen from (3.16) that

$$(I_{2N} + b_+)^{-1} = I_{2N} + \sum_{k=1}^{N-1} (-1)^k b_+^k,$$

whence (3.15) takes the form

$$\widetilde{G} = \left(I_{2N} + \sum_{k=1}^{N-1} (-1)^k b_+^k \right) (I_{2N} + b_-)^{-1}. \tag{3.17}$$

Putting together (3.13) and (3.17), we arrive to the following conclusion.

Theorem 3.3. *Let* G_F *be the matrix function* (1.4) *with* F *given by* (1.5) *and* (3.4) *and satisfying* (3.5). *Then the multiples* $G_\pm$ *from its left canonical APW factorization* (2.1) *can be chosen as*

$$G_+ = \mathcal{G}_+ \left(I_{2N} + \sum_{k=1}^{N-1} (-1)^k b_+^k \right), \quad G_- = \mathcal{G}_-(I_{2N} + b_-), \tag{3.18}$$

with $\mathcal{G}_\pm$ *and* $b_\pm$ *defined by* (3.7)–(3.8) *and* (3.16) *respectively.*

It follows from Corollaries 2.6 and 2.8 that

$$\mathbf{M}(\mathcal{G}_\pm) = \begin{bmatrix} \operatorname{diag}\{\mathbf{M}(g_{s,1}^\pm)\}_{s=1}^N & \operatorname{diag}\{\mathbf{M}(\widetilde{g}_{s,1}^\pm)\}_{s=1}^N \\ \operatorname{diag}\{\mathbf{M}(g_{s,2}^\pm)\}_{s=1}^N & \operatorname{diag}\{\mathbf{M}(\widetilde{g}_{s,2}^\pm)\}_{s=1}^N \end{bmatrix}, \tag{3.19}$$

where

$$\mathbf{M}(g_{s,1}^{+}) = \begin{cases} -c_{-1,s}c_{1,s}^{-1}, \\ -c_{-2,s}c_{2,s}^{-1}, \end{cases} \qquad \mathbf{M}(\widetilde{g}_{s,1}^{+}) = \begin{cases} \sum_{n\in\mathbb{N}_{-\alpha}^{+}} \widetilde{X}_{n,s}, \\ \sum_{n\in\mathbb{N}_{-\alpha}^{-}} \widetilde{Y}_{n,s}, \end{cases}$$

$$\mathbf{M}(g_{s,2}^{+}) = \begin{cases} \sum_{n\in\mathbb{N}_{\alpha}^{+}} c_{1,s}X_{n,s} + \sum_{n\in\mathbb{N}_{\beta}^{+}} c_{2,s}X_{n,s}, \\ \sum_{n\in\mathbb{N}_{\alpha}^{-}} c_{1,s}Y_{n,s} + \sum_{n\in\mathbb{N}_{\beta}^{-}} c_{2,s}Y_{n,s}, \end{cases} \qquad \mathbf{M}(\widetilde{g}_{s,2}^{+}) = \begin{cases} c_{1,s}, \\ c_{2,s}, \end{cases}$$

$$\mathbf{M}(g_{s,1}^{-}) = 1, \quad \mathbf{M}(\widetilde{g}_{s,1}^{-}) = 0,$$

$$\mathbf{M}(g_{s,2}^{-}) = \begin{cases} -\sum_{n\in\mathbb{N}_{\alpha}^{+}} c_{-1,s}X_{n,s} - \sum_{n\in\mathbb{N}_{\beta}^{+}} c_{-2,s}X_{n,s}, \\ -\sum_{n\in\mathbb{N}_{\alpha}^{-}} c_{-1,s}Y_{n,s} - \sum_{n\in\mathbb{N}_{\beta}^{-}} c_{-2,s}Y_{n,s}, \end{cases} \tag{3.20}$$

$$\mathbf{M}(\widetilde{g}_{s,2}^{-}) = \begin{cases} -c_{-1,s}, \\ -c_{-2,s} \end{cases} \quad \text{if} \begin{cases} |c_{2,s}|^{1-\beta}|c_{-2,s}|^{\beta} < |c_{1,s}|^{1-\alpha}|c_{-1,s}|^{\alpha}, \\ |c_{2,s}|^{1-\beta}|c_{-2,s}|^{\beta} > |c_{1,s}|^{1-\alpha}|c_{-1,s}|^{\alpha}. \end{cases}$$

On the other hand, we infer from (3.18) and (3.16) that

$$\mathbf{M}(G_+) = \mathbf{M}(\mathcal{G}_+)\left(I_{2N} + \sum_{k=1}^{N-1}(-1)^k\mathbf{M}(b_+)^k\right), \quad \mathbf{M}(G_-) = \mathbf{M}(\mathcal{G}_-),$$

which in view of (2.8) implies the following

Corollary 3.4. *Under the conditions of Theorem* 3.2*, the geometric mean of the matrix function G given by* (1.4), (1.5) *and* (3.4) *is calculated by*

$$\mathbf{d}(G) = \mathbf{M}(\mathcal{G}_+)\left(I_{2N} + \sum_{k=1}^{N-1}(-1)^k\mathbf{M}(b_+)^k\right)\mathbf{M}(\mathcal{G}_-)^{-1}, \tag{3.21}$$

where $\mathbf{M}(\mathcal{G}_\pm)$ and b_+ are given by (3.19)–(3.20) *and* (3.16)*, respectively.*

4. The geometric mean in the case $N = 2$

Corollary 3.4 in principle allows to compute the geometric mean for any value of N. In practice the complexity of this computation grows with N substantially, in particular because each of the inequalities (3.5) can materialize in two different ways, and the resulting 2^N cases yield different formulas and thus have to be treated separately. We therefore restrict our attention to the case $N = 2$ which should suffice for illustrative purposes. Corollary 3.4 can then be restated as follows.

Theorem 4.1. *Under the conditions of Theorem* 3.2*, the geometric mean of the matrix function G_F given by* (1.4), (1.5) *and* (3.4) *for $N = 2$ is calculated by*

$$\mathbf{d}(G_F) = \mathbf{d}(G_{F_0}) + T_{\widetilde{F}}, \tag{4.1}$$

where $\mathbf{d}(G_{F_0}) = \mathbf{d}(\mathcal{G}) = \mathbf{M}(\mathcal{G}_+)\mathbf{M}(\mathcal{G}_-)^{-1}$,

$$T_{\widetilde{F}} = \mathbf{M}(\mathcal{G}_+)\begin{bmatrix} 0 & 0 & 0 & 0 \\ -k_2^{-1}\mathbf{M}(\widetilde{g}_{2,1}^+ f g_{1,1}^-) & 0 & -k_2^{-1}\mathbf{M}(\widetilde{g}_{2,1}^+ f \widetilde{g}_{1,1}^-) & 0 \\ 0 & 0 & 0 & 0 \\ k_2^{-1}\mathbf{M}(g_{2,1}^+ f g_{1,1}^-) & 0 & k_2^{-1}\mathbf{M}(g_{2,1}^+ f \widetilde{g}_{1,1}^-) & 0 \end{bmatrix}\mathbf{M}(\mathcal{G}_-)^{-1}, \quad (4.2)$$

$\mathbf{M}(\mathcal{G}_\pm)$ *are given by* (3.19)–(3.20), *the functions* $g_{2,1}^+, \widetilde{g}_{2,1}^+, g_{1,1}^-, \widetilde{g}_{1,1}^-$ *are given by* (3.8), k_2 *is given by* (3.10),

$$\widetilde{F} = \begin{bmatrix} 0 & 0 \\ f & 0 \end{bmatrix}, \quad f = \widetilde{c}_1 e_\alpha + \widetilde{c}_{-1} e_{\alpha-1} + \widetilde{c}_2 e_\beta + \widetilde{c}_{-2} e_{\beta-1} \quad (4.3)$$

and $\widetilde{c}_i := (c_i)_{2,1}$ *for all* $i = \pm 1, \pm 2$.

Proof. Since $N = 2$, we conclude from (3.16) that

$$\mathbf{M}(b_+) = \mathbf{M}(a_+) = \mathbf{M}(a), \quad (4.4)$$

where, by (3.14), (3.11) and (4.3),

$$\mathbf{M}(a) = \begin{bmatrix} K_2\mathbf{M}(\widetilde{\mathcal{G}}_1^+ \widetilde{F} \mathcal{G}_1^-) & K_2\mathbf{M}(\widetilde{\mathcal{G}}_1^+ \widetilde{F} \widetilde{\mathcal{G}}_1^-) \\ -K_2\mathbf{M}(\mathcal{G}_1^+ \widetilde{F} \mathcal{G}_1^-) & -K_2\mathbf{M}(\mathcal{G}_1^+ \widetilde{F} \widetilde{\mathcal{G}}_1^-) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ k_2^{-1}\mathbf{M}(\widetilde{g}_{2,1}^+ f g_{1,1}^-) & 0 & k_2^{-1}\mathbf{M}(\widetilde{g}_{2,1}^+ f \widetilde{g}_{1,1}^-) & 0 \\ 0 & 0 & 0 & 0 \\ -k_2^{-1}\mathbf{M}(g_{2,1}^+ f g_{1,1}^-) & 0 & -k_2^{-1}\mathbf{M}(g_{2,1}^+ f \widetilde{g}_{1,1}^-) & 0 \end{bmatrix}. \quad (4.5)$$

Hence we infer from (3.21), (4.4) and (4.5) that

$$\mathbf{d}(G) = \mathbf{M}(\mathcal{G}_+)\big(I_4 - \mathbf{M}(a)\big)\mathbf{M}(\mathcal{G}_-)^{-1} = \mathbf{d}(\mathcal{G}) + T_{\widetilde{F}},$$

where $T_{\widetilde{F}}$ is given by (4.2). □

The explicit formulas for the mean values involved in (4.2) depend on which of the four possible cases

$$|c_{2,s}|^{1-\beta}|c_{-2,s}|^\beta < |c_{1,s}|^{1-\alpha}|c_{-1,s}|^\alpha \quad \text{for} \quad s = 1, 2; \quad (4.6)$$

$$|c_{2,s}|^{1-\beta}|c_{-2,s}|^\beta > |c_{1,s}|^{1-\alpha}|c_{-1,s}|^\alpha \quad \text{for} \quad s = 1, 2; \quad (4.7)$$

$$\begin{aligned} |c_{2,1}|^{1-\beta}|c_{-2,1}|^\beta &< |c_{1,1}|^{1-\alpha}|c_{-1,1}|^\alpha, \\ |c_{2,2}|^{1-\beta}|c_{-2,2}|^\beta &> |c_{1,2}|^{1-\alpha}|c_{-1,2}|^\alpha; \end{aligned} \quad (4.8)$$

$$\begin{aligned} |c_{2,1}|^{1-\beta}|c_{-2,1}|^\beta &> |c_{1,1}|^{1-\alpha}|c_{-1,1}|^\alpha, \\ |c_{2,2}|^{1-\beta}|c_{-2,2}|^\beta &< |c_{1,2}|^{1-\alpha}|c_{-1,2}|^\alpha \end{aligned} \quad (4.9)$$

takes place, and are delegated to the Appendix. Here is the final result under a simplifying condition on α, β, with a proof also delegated to the Appendix.

Theorem 4.2. *Let $G = G_F$ be given by* (1.4), (1.5) *and* (3.4), *where $N = 2$, $0 < \alpha < \beta < 1$ and*

$$m\alpha + n\beta \notin \{0, 1\} \quad \text{for all rational} \;\; m, n. \tag{4.10}$$

Then G admits a canonical left AP factorization $G = G_+ G_-^{-1}$ if and only if

$$|c_{2,s}|^{1-\beta} |c_{-2,s}|^{\beta} \neq |c_{1,s}|^{1-\alpha} |c_{-1,s}|^{\alpha} \quad \text{for all} \;\; s = 1, 2, \tag{4.11}$$

where $c_{i,s} := (c_i)_{s,s}$ for all $i = \pm 1, \pm 2$ and all $s = 1, 2$ are the diagonal entries of matrix coefficients (3.4) *in* (1.5). *If* (4.11) *holds, then*

$$\mathbf{d}(G) = \operatorname{diag}[T_1, T_2], \tag{4.12}$$

where

$$\begin{aligned} T_1 &= \begin{bmatrix} -c_{-1,1} c_{1,1}^{-1} & 0 \\ -(c_{1,1} \widetilde{c}_{-1} - c_{-1,1} \widetilde{c}_1) c_{1,1}^{-1} c_{1,2}^{-1} & -c_{-1,2} c_{1,2}^{-1} \end{bmatrix}, \\ T_2 &= \begin{bmatrix} -c_{1,1} c_{-1,1}^{-1} & 0 \\ (c_{1,2} \widetilde{c}_{-1} - c_{-1,2} \widetilde{c}_1) c_{-1,1}^{-1} c_{-1,2}^{-1} & -c_{1,2} c_{-1,2}^{-1} \end{bmatrix}, \end{aligned} \tag{4.13}$$

if (4.6) *holds;*

$$\begin{aligned} T_1 &= \begin{bmatrix} -c_{-2,1} c_{2,1}^{-1} & 0 \\ -(c_{2,1} \widetilde{c}_{-2} - c_{-2,1} \widetilde{c}_2) c_{2,1}^{-1} c_{2,2}^{-1} & -c_{-2,2} c_{2,2}^{-1} \end{bmatrix}, \\ T_2 &= \begin{bmatrix} -c_{2,1} c_{-2,1}^{-1} & 0 \\ (c_{2,2} \widetilde{c}_{-2} - c_{-2,2} \widetilde{c}_2) c_{-2,1}^{-1} c_{-2,2}^{-1} & -c_{2,2} c_{-2,2}^{-1} \end{bmatrix}, \end{aligned} \tag{4.14}$$

if (4.7) *holds;*

$$\begin{aligned} T_1 &= \begin{bmatrix} -c_{-1,1} c_{1,1}^{-1} & 0 \\ -(c_{1,1} \widetilde{c}_{-2} - c_{-1,1} \widetilde{c}_2) c_{1,1}^{-1} c_{2,2}^{-1} & -c_{-2,2} c_{2,2}^{-1} \end{bmatrix}, \\ T_2 &= \begin{bmatrix} -c_{1,1} c_{-1,1}^{-1} & 0 \\ (c_{2,2} \widetilde{c}_{-1} - c_{-2,2} \widetilde{c}_1) c_{-1,1}^{-1} c_{-2,2}^{-1} & -c_{2,2} c_{-2,2}^{-1} \end{bmatrix}, \end{aligned} \tag{4.15}$$

if (4.8) *holds;*

$$\begin{aligned} T_1 &= \begin{bmatrix} -c_{-2,1} c_{2,1}^{-1} & 0 \\ -(c_{2,1} \widetilde{c}_{-1} - c_{-2,1} \widetilde{c}_1) c_{2,1}^{-1} c_{1,2}^{-1} & -c_{-1,2} c_{1,2}^{-1} \end{bmatrix}, \\ T_2 &= \begin{bmatrix} -c_{2,1} c_{-2,1}^{-1} & 0 \\ (c_{1,2} \widetilde{c}_{-2} - c_{-1,2} \widetilde{c}_2) c_{-2,1}^{-1} c_{-1,2}^{-1} & -c_{1,2} c_{-1,2}^{-1} \end{bmatrix}, \end{aligned} \tag{4.16}$$

if (4.9) *holds.*

5. Appendix

5.1. Computation of $\mathbf{d}(\mathcal{G})$ and $T_{\widetilde{F}}$

Applying Corollaries 2.6 and 2.8 with notation (2.25), we infer that

$$\begin{aligned}
\mathbf{d}(\mathcal{G}) &= -\operatorname{diag}\left[c_{-1,1}c_{1,1}^{-1},\ c_{-1,2}c_{1,2}^{-1},\ c_{1,1}c_{-1,1}^{-1},\ c_{1,2}c_{-1,2}^{-1}\right] + \begin{bmatrix} 0 & -B_{1,2} \\ B_{2,1} & 0 \end{bmatrix}, \\
B_{1,2} &= \operatorname{diag}\left[\sum_{n\in\mathbb{N}^+_{-\alpha}} c_{-1,1}^{-1}\widetilde{X}_{n,1},\ \sum_{n\in\mathbb{N}^+_{-\alpha}} c_{-1,2}^{-1}\widetilde{X}_{n,2}\right], \\
B_{2,1} &= \operatorname{diag}\left[\sum_{n\in\mathbb{N}^+_{\beta}}(c_{2,1}-c_{1,1}c_{-1,1}^{-1}c_{-2,1})X_{n,1},\ \sum_{n\in\mathbb{N}^+_{\beta}}(c_{2,2}-c_{1,2}c_{-1,2}^{-1}c_{-2,2})X_{n,2}\right],
\end{aligned} \tag{5.1}$$

if (4.6) holds;

$$\begin{aligned}
\mathbf{d}(\mathcal{G}) &= -\operatorname{diag}\left[c_{-2,1}c_{2,1}^{-1},\ c_{-2,2}c_{2,2}^{-1},\ c_{2,1}c_{-2,1}^{-1},\ c_{2,2}c_{-2,2}^{-1}\right] + \begin{bmatrix} 0 & -B_{1,2} \\ B_{2,1} & 0 \end{bmatrix}, \\
B_{1,2} &= \operatorname{diag}\left[\sum_{n\in\mathbb{N}^-_{-\alpha}} c_{-2,1}^{-1}\widetilde{Y}_{n,1},\ \sum_{n\in\mathbb{N}^-_{-\alpha}} c_{-2,2}^{-1}\widetilde{Y}_{n,2}\right], \\
B_{2,1} &= \operatorname{diag}\left[\sum_{n\in\mathbb{N}^-_{\alpha}}(c_{1,1}-c_{2,1}c_{-2,1}^{-1}c_{-1,1})Y_{n,1},\ \sum_{n\in\mathbb{N}^-_{\alpha}}(c_{1,2}-c_{2,2}c_{-2,2}^{-1}c_{-1,2})Y_{n,2}\right],
\end{aligned} \tag{5.2}$$

if (4.7) holds;

$$\begin{aligned}
\mathbf{d}(\mathcal{G}) &= -\operatorname{diag}\left[c_{-1,1}c_{1,1}^{-1},\ c_{-2,2}c_{2,2}^{-1},\ c_{1,1}c_{-1,1}^{-1},\ c_{2,2}c_{-2,2}^{-1}\right] + \begin{bmatrix} 0 & -B_{1,2} \\ B_{2,1} & 0 \end{bmatrix}, \\
B_{1,2} &= \operatorname{diag}\left[\sum_{n\in\mathbb{N}^+_{-\alpha}} c_{-1,1}^{-1}\widetilde{X}_{n,1},\ \sum_{n\in\mathbb{N}^-_{-\alpha}} c_{-2,2}^{-1}\widetilde{Y}_{n,2}\right], \\
B_{2,1} &= \operatorname{diag}\left[\sum_{n\in\mathbb{N}^+_{\beta}}(c_{2,1}-c_{1,1}c_{-1,1}^{-1}c_{-2,1})X_{n,1},\ \sum_{n\in\mathbb{N}^-_{\alpha}}(c_{1,2}-c_{2,2}c_{-2,2}^{-1}c_{-1,2})Y_{n,2}\right],
\end{aligned} \tag{5.3}$$

if (4.8) holds;

$$\begin{aligned}
\mathbf{d}(\mathcal{G}) &= -\operatorname{diag}\left[c_{-2,1}c_{2,1}^{-1},\ c_{-1,2}c_{1,2}^{-1},\ c_{2,1}c_{-2,1}^{-1},\ c_{1,2}c_{-1,2}^{-1}\right] + \begin{bmatrix} 0 & -B_{1,2} \\ B_{2,1} & 0 \end{bmatrix}, \\
B_{1,2} &= \operatorname{diag}\left[\sum_{n\in\mathbb{N}^-_{-\alpha}} c_{-2,1}^{-1}\widetilde{Y}_{n,1},\ \sum_{n\in\mathbb{N}^+_{-\alpha}} c_{-1,2}^{-1}\widetilde{X}_{n,2}\right], \\
B_{2,1} &= \operatorname{diag}\left[\sum_{n\in\mathbb{N}^-_{\alpha}}(c_{1,1}-c_{2,1}c_{-2,1}^{-1}c_{-1,1})Y_{n,1},\ \sum_{n\in\mathbb{N}^+_{\beta}}(c_{2,2}-c_{1,2}c_{-1,2}^{-1}c_{-2,2})X_{n,2}\right],
\end{aligned} \tag{5.4}$$

if (4.9) holds.

On the other hand, by (4.2) and (3.19), we obtain

$$T_{\widetilde{F}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ t_{2,1} & 0 & t_{2,3} & 0 \\ 0 & 0 & 0 & 0 \\ t_{4,1} & 0 & t_{4,3} & 0 \end{bmatrix}, \tag{5.5}$$

where

$$\begin{aligned} t_{2,1} &= \big[-\mathbf{M}(g^+_{2,1})\mathbf{M}(\widetilde{g}^+_{2,1} f g^-_{1,1}) + \mathbf{M}(\widetilde{g}^+_{2,1})\mathbf{M}(g^+_{2,1} f g^-_{1,1})\big] k_2^{-1} \\ &\quad + \big[\mathbf{M}(g^+_{2,1})\mathbf{M}(\widetilde{g}^+_{2,1} f \widetilde{g}^-_{1,1}) - \mathbf{M}(\widetilde{g}^+_{2,1})\mathbf{M}(g^+_{2,1} f \widetilde{g}^-_{1,1})\big]\mathbf{M}(g^-_{1,2}) k_1^{-1} k_2^{-1}, \\ t_{2,3} &= \big[-\mathbf{M}(g^+_{2,1})\mathbf{M}(\widetilde{g}^+_{2,1} f \widetilde{g}^-_{1,1}) + \mathbf{M}(\widetilde{g}^+_{2,1})\mathbf{M}(g^+_{2,1} f \widetilde{g}^-_{1,1})\big] k_1^{-1} k_2^{-1}, \\ t_{4,1} &= \big[-\mathbf{M}(g^+_{2,2})\mathbf{M}(\widetilde{g}^+_{2,1} f g^-_{1,1}) + \mathbf{M}(\widetilde{g}^+_{2,2})\mathbf{M}(g^+_{2,1} f g^-_{1,1})\big] k_2^{-1} \\ &\quad + \big[\mathbf{M}(g^+_{2,2})\mathbf{M}(\widetilde{g}^+_{2,1} f \widetilde{g}^-_{1,1}) - \mathbf{M}(\widetilde{g}^+_{2,2})\mathbf{M}(g^+_{2,1} f \widetilde{g}^-_{1,1})\big]\mathbf{M}(g^-_{1,2}) k_1^{-1} k_2^{-1}, \\ t_{4,3} &= \big[-\mathbf{M}(g^+_{2,2})\mathbf{M}(\widetilde{g}^+_{2,1} f \widetilde{g}^-_{1,1}) + \mathbf{M}(\widetilde{g}^+_{2,2})\mathbf{M}(g^+_{2,1} f \widetilde{g}^-_{1,1})\big] k_1^{-1} k_2^{-1}, \end{aligned} \tag{5.6}$$

and $\mathbf{M}(g^+_{2,j}), \mathbf{M}(\widetilde{g}^+_{2,j}), \mathbf{M}(g^-_{1,2})$ for $j = 1, 2$ are given by (3.20).

5.2. Computation of $\mathbf{M}(\widetilde{g}^+_{2,1} f g^-_{1,1})$, $\mathbf{M}(\widetilde{g}^+_{2,1} f \widetilde{g}^-_{1,1})$, $\mathbf{M}(g^+_{2,1} f g^-_{1,1})$, $\mathbf{M}(g^+_{2,1} f \widetilde{g}^-_{1,1})$

Below, along with $\mathbb{N}^\pm_\gamma$ given by (2.25), we use the following notation for $\gamma = \alpha, \beta$ and $l = 1, 2$:

$$\begin{aligned} \widetilde{\mathbb{N}}^\pm_\gamma &:= \{n \in \mathbb{N}_\pm : \{n(\beta-\alpha)\} + \gamma = 1\}, \\ (\mathbb{Z}_\pm \times \mathbb{Z}_\pm)_{\gamma,l} &:= \{(n,k) \in \mathbb{Z}_\pm \times \mathbb{Z}_\pm : \{n(\beta-\alpha) - \alpha\} \\ &\qquad + \{k(\beta-\alpha) - \alpha\} + \gamma = l\}, \\ (\mathbb{Z}_\pm \times \mathbb{Z}_\pm)^\sim_{\gamma,l} &:= \{(n,k) \in \mathbb{Z}_\pm \times \mathbb{Z}_\pm : \{n(\beta-\alpha)\} + \{k(\beta-\alpha)\} + \gamma = l\}. \end{aligned} \tag{5.7}$$

If (4.6) holds, then

$$\begin{aligned} \widetilde{g}^+_{2,1} g^-_{1,1} &= \Big(\sum_{n=0}^\infty \widetilde{X}_{n,2} e_{\{n(\beta-\alpha)-\alpha\}}\Big)\Big(1 + \sum_{k=0}^\infty X_{k,1} e_{\{k(\beta-\alpha)\}-1}\Big), \\ \widetilde{g}^+_{2,1} \widetilde{g}^-_{1,1} &= \Big(\sum_{n=0}^\infty \widetilde{X}_{n,2} e_{\{n(\beta-\alpha)-\alpha\}}\Big)\Big(\sum_{k=0}^\infty \widetilde{X}_{k,1} e_{\{k(\beta-\alpha)-\alpha\}-1}\Big), \\ g^+_{2,1} g^-_{1,1} &= \Big(e_1 + \sum_{n=0}^\infty X_{n,2} e_{\{n(\beta-\alpha)\}}\Big)\Big(1 + \sum_{k=0}^\infty X_{k,1} e_{\{k(\beta-\alpha)\}-1}\Big), \\ g^+_{2,1} \widetilde{g}^-_{1,1} &= \Big(e_1 + \sum_{n=0}^\infty X_{n,2} e_{\{n(\beta-\alpha)\}}\Big)\Big(\sum_{k=0}^\infty \widetilde{X}_{k,1} e_{\{k(\beta-\alpha)-\alpha\}-1}\Big), \end{aligned}$$

which with f given by (4.3) implies, respectively, that

$$
\begin{aligned}
\mathbf{M}(\widetilde{g}_{2,1}^{+} f g_{1,1}^{-}) &= \widetilde{c}_{-1}\widetilde{X}_{0,2} + \widetilde{c}_{1}\widetilde{X}_{0,2}X_{0,1},\\
\mathbf{M}(\widetilde{g}_{2,1}^{+} f \widetilde{g}_{1,1}^{-}) &= \sum_{(n,k)\in(\mathbb{Z}_+\times\mathbb{Z}_+)_{\alpha,1}} \widetilde{c}_{1}\widetilde{X}_{n,2}\widetilde{X}_{k,1} + \sum_{(n,k)\in(\mathbb{Z}_+\times\mathbb{Z}_+)_{\alpha,2}} \widetilde{c}_{-1}\widetilde{X}_{n,2}\widetilde{X}_{k,1}\\
&\quad + \sum_{(n,k)\in(\mathbb{Z}_+\times\mathbb{Z}_+)_{\beta,1}} \widetilde{c}_{2}\widetilde{X}_{n,2}\widetilde{X}_{k,1} + \sum_{(n,k)\in(\mathbb{Z}_+\times\mathbb{Z}_+)_{\beta,2}} \widetilde{c}_{-2}\widetilde{X}_{n,2}\widetilde{X}_{k,1},\\
\mathbf{M}(g_{2,1}^{+} f g_{1,1}^{-}) &= \sum_{n\in\widetilde{\mathbb{N}}_{\alpha}^{+}} \widetilde{c}_{-1}(X_{n,1}+X_{n,2}) + \sum_{n\in\widetilde{\mathbb{N}}_{\beta}^{+}} \widetilde{c}_{-2}(X_{n,1}+X_{n,2})\\
&\quad + \sum_{(n,k)\in(\mathbb{Z}_+\times\mathbb{Z}_+)_{\widetilde{\alpha},1}} \widetilde{c}_{1}X_{n,2}X_{k,1} + \sum_{(n,k)\in(\mathbb{Z}_+\times\mathbb{Z}_+)_{\widetilde{\alpha},2}} \widetilde{c}_{-1}X_{n,2}X_{k,1}\\
&\quad + \sum_{(n,k)\in(\mathbb{Z}_+\times\mathbb{Z}_+)_{\widetilde{\beta},1}} \widetilde{c}_{2}X_{n,2}X_{k,1} + \sum_{(n,k)\in(\mathbb{Z}_+\times\mathbb{Z}_+)_{\widetilde{\beta},2}} \widetilde{c}_{-2}X_{n,2}X_{k,1},\\
\mathbf{M}(g_{2,1}^{+} f \widetilde{g}_{1,1}^{-}) &= \widetilde{c}_{-1}\widetilde{X}_{0,1} + \widetilde{c}_{1}X_{0,2}\widetilde{X}_{0,1}.
\end{aligned} \tag{5.8}
$$

If (4.7) holds, then

$$
\begin{aligned}
\widetilde{g}_{2,1}^{+} g_{1,1}^{-} &= \Bigg(\sum_{n=-\infty}^{-1} \widetilde{Y}_{n,2} e_{\{n(\beta-\alpha)-\alpha\}}\Bigg)\Bigg(1 + \sum_{k=-\infty}^{0} Y_{k,1} e_{\{k(\beta-\alpha)\}-1}\Bigg),\\
\widetilde{g}_{2,1}^{+} \widetilde{g}_{1,1}^{-} &= \Bigg(\sum_{n=-\infty}^{-1} \widetilde{Y}_{n,2} e_{\{n(\beta-\alpha)-\alpha\}}\Bigg)\Bigg(\sum_{k=-\infty}^{-1} \widetilde{Y}_{k,1} e_{\{k(\beta-\alpha)-\alpha\}-1}\Bigg),\\
g_{2,1}^{+} g_{1,1}^{-} &= \Bigg(e_1 + \sum_{n=-\infty}^{0} Y_{n,2} e_{\{n(\beta-\alpha)\}}\Bigg)\Bigg(1 + \sum_{k=-\infty}^{0} Y_{k,1} e_{\{k(\beta-\alpha)\}-1}\Bigg),\\
g_{2,1}^{+} \widetilde{g}_{1,1}^{-} &= \Bigg(e_1 + \sum_{n=-\infty}^{0} Y_{n,2} e_{\{n(\beta-\alpha)\}}\Bigg)\Bigg(\sum_{k=-\infty}^{-1} \widetilde{Y}_{k,1} e_{\{k(\beta-\alpha)-\alpha\}-1}\Bigg),
\end{aligned}
$$

which with f given by (4.3) implies, respectively, that

$$
\begin{aligned}
\mathbf{M}(\widetilde{g}_{2,1}^{+} f g_{1,1}^{-}) &= \widetilde{c}_{-2}\widetilde{Y}_{-1,2} + \widetilde{c}_{2}\widetilde{Y}_{-1,2}Y_{0,1},\\
\mathbf{M}(\widetilde{g}_{2,1}^{+} f \widetilde{g}_{1,1}^{-}) &= \sum_{(n+1,k+1)\in(\mathbb{Z}_-\times\mathbb{Z}_-)_{\alpha,1}} \widetilde{c}_{1}\widetilde{Y}_{n,2}\widetilde{Y}_{k,1} + \sum_{(n+1,k+1)\in(\mathbb{Z}_-\times\mathbb{Z}_-)_{\alpha,2}} \widetilde{c}_{-1}\widetilde{Y}_{n,2}\widetilde{Y}_{k,1}\\
&\quad + \sum_{(n+1,k+1)\in(\mathbb{Z}_-\times\mathbb{Z}_-)_{\beta,1}} \widetilde{c}_{2}\widetilde{Y}_{n,2}\widetilde{Y}_{k,1} + \sum_{(n+1,k+1)\in(\mathbb{Z}_-\times\mathbb{Z}_-)_{\beta,2}} \widetilde{c}_{-2}\widetilde{Y}_{n,2}\widetilde{Y}_{k,1},
\end{aligned}
$$

$$\begin{aligned}
\mathbf{M}(g_{2,1}^{+}fg_{1,1}^{-}) = & \sum_{n\in\widetilde{\mathbb{N}}_{\alpha}^{-}} \widetilde{c}_{-1}(Y_{n,1}+Y_{n,2}) + \sum_{n\in\widetilde{\mathbb{N}}_{\beta}^{-}} \widetilde{c}_{-2}(Y_{n,1}+Y_{n,2}) \\
& + \sum_{(n,k)\in(\mathbb{Z}_-\times\mathbb{Z}_-)_{\alpha,1}^{\sim}} \widetilde{c}_{1}Y_{n,2}Y_{k,1} + \sum_{(n,k)\in(\mathbb{Z}_-\times\mathbb{Z}_-)_{\alpha,2}^{\sim}} \widetilde{c}_{-1}Y_{n,2}Y_{k,1} \\
& + \sum_{(n,k)\in(\mathbb{Z}_-\times\mathbb{Z}_-)_{\beta,1}^{\sim}} \widetilde{c}_{2}Y_{n,2}Y_{k,1} + \sum_{(n,k)\in(\mathbb{Z}_-\times\mathbb{Z}_-)_{\beta,2}^{\sim}} \widetilde{c}_{-2}Y_{n,2}Y_{k,1}, \\
\mathbf{M}(g_{2,1}^{+}f\widetilde{g}_{1,1}^{-}) = & \, \widetilde{c}_{-2}\widetilde{Y}_{-1,1} + \widetilde{c}_{2}Y_{0,2}\widetilde{Y}_{-1,1}.
\end{aligned} \tag{5.9}$$

If (4.8) holds, then

$$\widetilde{g}_{2,1}^{+}g_{1,1}^{-} = \left(\sum_{n=-\infty}^{-1} \widetilde{Y}_{n,2}e_{\{n(\beta-\alpha)-\alpha\}}\right)\left(1+\sum_{k=0}^{\infty} X_{k,1}e_{\{k(\beta-\alpha)\}-1}\right),$$

$$\widetilde{g}_{2,1}^{+}\widetilde{g}_{1,1}^{-} = \left(\sum_{n=-\infty}^{-1} \widetilde{Y}_{n,2}e_{\{n(\beta-\alpha)-\alpha\}}\right)\left(\sum_{k=0}^{\infty} \widetilde{X}_{k,1}e_{\{k(\beta-\alpha)-\alpha\}-1}\right),$$

$$g_{2,1}^{+}g_{1,1}^{-} = \left(e_1+\sum_{n=-\infty}^{0} Y_{n,2}e_{\{n(\beta-\alpha)\}}\right)\left(1+\sum_{k=0}^{\infty} X_{k,1}e_{\{k(\beta-\alpha)\}-1}\right),$$

$$g_{2,1}^{+}\widetilde{g}_{1,1}^{-} = \left(e_1+\sum_{n=-\infty}^{0} Y_{n,2}e_{\{n(\beta-\alpha)\}}\right)\left(\sum_{k=0}^{\infty} \widetilde{X}_{k,1}e_{\{k(\beta-\alpha)-\alpha\}-1}\right),$$

which with f given by (4.3) implies, respectively, that

$$\begin{aligned}
\mathbf{M}(\widetilde{g}_{2,1}^{+}fg_{1,1}^{-}) = & \, \widetilde{c}_{-2}\widetilde{Y}_{-1,2} + \widetilde{c}_{2}\widetilde{Y}_{-1,2}X_{0,1}, \\
\mathbf{M}(\widetilde{g}_{2,1}^{+}f\widetilde{g}_{1,1}^{-}) = & \sum_{(n+1,k)\in(\mathbb{Z}_-\times\mathbb{Z}_+)_{\alpha,1}} \widetilde{c}_{1}\widetilde{Y}_{n,2}\widetilde{X}_{k,1} + \sum_{(n+1,k)\in(\mathbb{Z}_-\times\mathbb{Z}_+)_{\alpha,2}} \widetilde{c}_{-1}\widetilde{Y}_{n,2}\widetilde{X}_{k,1} \\
& + \sum_{(n+1,k)\in(\mathbb{Z}_-\times\mathbb{Z}_+)_{\beta,1}} \widetilde{c}_{2}\widetilde{Y}_{n,2}\widetilde{X}_{k,1} + \sum_{(n+1,k)\in(\mathbb{Z}_-\times\mathbb{Z}_+)_{\beta,2}} \widetilde{c}_{-2}\widetilde{Y}_{n,2}\widetilde{X}_{k,1}, \\
\mathbf{M}(g_{2,1}^{+}fg_{1,1}^{-}) = & \sum_{k\in\widetilde{\mathbb{N}}_{\alpha}^{+}} \widetilde{c}_{-1}X_{k,1} + \sum_{k\in\widetilde{\mathbb{N}}_{\beta}^{+}} \widetilde{c}_{-2}X_{k,1} + \sum_{n\in\widetilde{\mathbb{N}}_{\alpha}^{-}} \widetilde{c}_{-1}Y_{n,2} + \sum_{n\in\widetilde{\mathbb{N}}_{\beta}^{-}} \widetilde{c}_{-2}Y_{n,2} \\
& + \sum_{(n,k)\in(\mathbb{Z}_-\times\mathbb{Z}_+)_{\alpha,1}^{\sim}} \widetilde{c}_{1}Y_{n,2}X_{k,1} + \sum_{(n,k)\in(\mathbb{Z}_-\times\mathbb{Z}_+)_{\alpha,2}^{\sim}} \widetilde{c}_{-1}Y_{n,2}X_{k,1} \\
& + \sum_{(n,k)\in(\mathbb{Z}_-\times\mathbb{Z}_+)_{\beta,1}^{\sim}} \widetilde{c}_{2}Y_{n,2}X_{k,1} + \sum_{(n,k)\in(\mathbb{Z}_-\times\mathbb{Z}_+)_{\beta,2}^{\sim}} \widetilde{c}_{-2}Y_{n,2}X_{k,1}, \\
\mathbf{M}(g_{2,1}^{+}f\widetilde{g}_{1,1}^{-}) = & \, \widetilde{c}_{-1}\widetilde{X}_{0,1} + \widetilde{c}_{1}Y_{0,2}\widetilde{X}_{0,1}.
\end{aligned} \tag{5.10}$$

If (4.9) holds, then

$$\widetilde{g}^{+}_{2,1}g^{-}_{1,1} = \Big(\sum_{n=0}^{\infty}\widetilde{X}_{n,2}e_{\{n(\beta-\alpha)-\alpha\}}\Big)\Big(1+\sum_{k=-\infty}^{0}Y_{k,1}e_{\{k(\beta-\alpha)\}-1}\Big),$$

$$\widetilde{g}^{+}_{2,1}\widetilde{g}^{-}_{1,1} = \Big(\sum_{n=0}^{\infty}\widetilde{X}_{n,2}e_{\{n(\beta-\alpha)-\alpha\}}\Big)\Big(\sum_{k=-\infty}^{-1}\widetilde{Y}_{k,1}e_{\{k(\beta-\alpha)-\alpha\}-1}\Big),$$

$$g^{+}_{2,1}g^{-}_{1,1} = \Big(e_1+\sum_{n=0}^{\infty}X_{n,2}e_{\{n(\beta-\alpha)\}}\Big)\Big(1+\sum_{k=-\infty}^{0}Y_{k,1}e_{\{k(\beta-\alpha)\}-1}\Big),$$

$$g^{+}_{2,1}\widetilde{g}^{-}_{1,1} = \Big(e_1+\sum_{n=0}^{\infty}X_{n,2}e_{\{n(\beta-\alpha)\}}\Big)\Big(\sum_{k=-\infty}^{-1}\widetilde{Y}_{k,1}e_{\{k(\beta-\alpha)-\alpha\}-1}\Big),$$

which with f given by (4.3) implies, respectively, that

$$\begin{aligned}
\mathbf{M}(\widetilde{g}^{+}_{2,1}fg^{-}_{1,1}) &= \widetilde{c}_{-1}\widetilde{X}_{0,2}+\widetilde{c}_{1}\widetilde{X}_{0,2}Y_{0,1},\\
\mathbf{M}(\widetilde{g}^{+}_{2,1}f\widetilde{g}^{-}_{1,1}) &= \sum_{(n,k+1)\in(\mathbb{Z}_+\times\mathbb{Z}_-)_{\alpha,1}}\widetilde{c}_{1}\widetilde{X}_{n,2}\widetilde{Y}_{k,1}+\sum_{(n,k+1)\in(\mathbb{Z}_+\times\mathbb{Z}_-)_{\alpha,2}}\widetilde{c}_{-1}\widetilde{X}_{n,2}\widetilde{Y}_{k,1}\\
&\quad+\sum_{(n,k+1)\in(\mathbb{Z}_+\times\mathbb{Z}_-)_{\beta,1}}\widetilde{c}_{2}\widetilde{X}_{n,2}\widetilde{Y}_{k,1}+\sum_{(n,k+1)\in(\mathbb{Z}_+\times\mathbb{Z}_-)_{\beta,2}}\widetilde{c}_{-2}\widetilde{X}_{n,2}\widetilde{Y}_{k,1},\\
\mathbf{M}(g^{+}_{2,1}fg^{-}_{1,1}) &= \sum_{n\in\widetilde{\mathbb{N}}^{+}_{\alpha}}\widetilde{c}_{-1}X_{n,2}+\sum_{n\in\widetilde{\mathbb{N}}^{+}_{\beta}}\widetilde{c}_{-2}X_{n,2}+\sum_{k\in\widetilde{\mathbb{N}}^{-}_{\alpha}}\widetilde{c}_{-1}Y_{k,1}+\sum_{k\in\widetilde{\mathbb{N}}^{-}_{\beta}}\widetilde{c}_{-2}Y_{k,1}\\
&\quad+\sum_{(n,k)\in(\mathbb{Z}_+\times\mathbb{Z}_-)_{\widetilde{\alpha},1}}\widetilde{c}_{1}X_{n,2}Y_{k,1}+\sum_{(n,k)\in(\mathbb{Z}_+\times\mathbb{Z}_-)_{\widetilde{\alpha},2}}\widetilde{c}_{-1}X_{n,2}Y_{k,1}\\
&\quad+\sum_{(n,k)\in(\mathbb{Z}_+\times\mathbb{Z}_-)_{\widetilde{\beta},1}}\widetilde{c}_{2}X_{n,2}Y_{k,1}+\sum_{(n,k)\in(\mathbb{Z}_+\times\mathbb{Z}_-)_{\widetilde{\beta},2}}\widetilde{c}_{-2}X_{n,2}Y_{k,1},\\
\mathbf{M}(g^{+}_{2,1}f\widetilde{g}^{-}_{1,1}) &= \widetilde{c}_{-2}\widetilde{Y}_{-1,1}+\widetilde{c}_{2}X_{0,2}\widetilde{Y}_{-1,1}.
\end{aligned}\tag{5.11}$$

5.3. Computation of $\mathrm{M}(\widetilde{g}^{+}_{2,1}fg^{-}_{1,1})$ and $\mathrm{M}(g^{+}_{2,1}f\widetilde{g}^{-}_{1,1})$

Since

$$\widetilde{X}_{0,1}=\widetilde{X}_{0,2}=1,\quad \widetilde{Y}_{-1,1}=\widetilde{Y}_{-1,2}=1$$

and

$$X_{0,1}=-\frac{c_{-1,1}}{c_{1,1}},\quad X_{0,2}=-\frac{c_{-1,2}}{c_{1,2}},\quad Y_{0,1}=-\frac{c_{-2,1}}{c_{2,1}},\quad Y_{0,2}=-\frac{c_{-2,2}}{c_{2,2}},$$

we deduce from (5.8), (5.9), (5.10) and (5.11) that

$$\mathbf{M}(\widetilde{g}_{2,1}^{+} f g_{1,1}^{-}) = \begin{cases} \widetilde{c}_{-1}\widetilde{X}_{0,2} + \widetilde{c}_{1}\widetilde{X}_{0,2}X_{0,1}, \\ \widetilde{c}_{-2}\widetilde{Y}_{-1,2} + \widetilde{c}_{2}\widetilde{Y}_{-1,2}Y_{0,1}, \\ \widetilde{c}_{-2}\widetilde{Y}_{-1,2} + \widetilde{c}_{2}\widetilde{Y}_{-1,2}X_{0,1}, \\ \widetilde{c}_{-1}\widetilde{X}_{0,2} + \widetilde{c}_{1}\widetilde{X}_{0,2}Y_{0,1}, \end{cases} = \begin{cases} \widetilde{c}_{-1} - \widetilde{c}_{1}(c_{-1,1}/c_{1,1}), \\ \widetilde{c}_{-2} - \widetilde{c}_{2}(c_{-2,1}/c_{2,1}), \\ \widetilde{c}_{-2} - \widetilde{c}_{2}(c_{-1,1}/c_{1,1}), \\ \widetilde{c}_{-1} - \widetilde{c}_{1}(c_{-2,1}/c_{2,1}), \end{cases} \tag{5.12}$$

$$\mathbf{M}(g_{2,1}^{+} f \widetilde{g}_{1,1}^{-}) = \begin{cases} \widetilde{c}_{-1}\widetilde{X}_{0,1} + \widetilde{c}_{1}X_{0,2}\widetilde{X}_{0,1}, \\ \widetilde{c}_{-2}\widetilde{Y}_{-1,1} + \widetilde{c}_{2}Y_{0,2}\widetilde{Y}_{-1,1}, \\ \widetilde{c}_{-1}\widetilde{X}_{0,1} + \widetilde{c}_{1}Y_{0,2}\widetilde{X}_{0,1}, \\ \widetilde{c}_{-2}\widetilde{Y}_{-1,1} + \widetilde{c}_{2}X_{0,2}\widetilde{Y}_{-1,1}, \end{cases} = \begin{cases} \widetilde{c}_{-1} - \widetilde{c}_{1}(c_{-1,2}/c_{1,2}), \\ \widetilde{c}_{-2} - \widetilde{c}_{2}(c_{-2,2}/c_{2,2}), \\ \widetilde{c}_{-1} - \widetilde{c}_{1}(c_{-2,2}/c_{2,2}), \\ \widetilde{c}_{-2} - \widetilde{c}_{2}(c_{-1,2}/c_{1,2}), \end{cases} \tag{5.13}$$

if, respectively, (4.6), (4.7), (4.8) and (4.9) holds.

Substituting $\mathbf{M}(\widetilde{g}_{2,1}^{+} f g_{1,1}^{-})$ and $\mathbf{M}(g_{2,1}^{+} f \widetilde{g}_{1,1}^{-})$ given by (5.12) and (5.13), respectively, and $\mathbf{M}(\widetilde{g}_{2,1}^{+} f \widetilde{g}_{1,1}^{-})$ and $\mathbf{M}(g_{2,1}^{+} f g_{1,1}^{-})$ given by (5.8), (5.9), (5.10) and (5.11) into (5.6) and applying (3.20), we obtain the entries of the matrix $T_{\widetilde{F}}$ (see (5.5)), which together with $\mathbf{d}(\mathcal{G}) = \mathbf{d}(G_{F_0})$ obtained in (5.1)–(5.4) gives $\mathbf{d}(G_F)$ due to (4.1).

5.4. Proof of Theorem 4.2

Under condition (4.10), all the sets

$$\mathbb{N}_{\gamma}^{\pm},\ \widetilde{\mathbb{N}}_{\gamma}^{\pm},\ (\mathbb{Z}_{\pm} \times \mathbb{Z}_{\pm})_{\gamma,l},\ (\mathbb{Z}_{\pm} \times \mathbb{Z}_{\pm})_{\gamma,l}^{\sim} \tag{5.14}$$

given by (2.25) and (5.7) are empty. Hence, by (5.1)–(5.4), we infer that

$$\mathbf{d}(\mathcal{G}) = \begin{cases} -\operatorname{diag}\left[c_{-1,1}c_{1,1}^{-1},\ c_{-1,2}c_{1,2}^{-1},\ c_{1,1}c_{-1,1}^{-1},\ c_{1,2}c_{-1,2}^{-1}\right], \\ -\operatorname{diag}\left[c_{-2,1}c_{2,1}^{-1},\ c_{-2,2}c_{2,2}^{-1},\ c_{2,1}c_{-2,1}^{-1},\ c_{2,2}c_{-2,2}^{-1}\right], \\ -\operatorname{diag}\left[c_{-1,1}c_{1,1}^{-1},\ c_{-2,2}c_{2,2}^{-1},\ c_{1,1}c_{-1,1}^{-1},\ c_{2,2}c_{-2,2}^{-1}\right], \\ -\operatorname{diag}\left[c_{-2,1}c_{2,1}^{-1},\ c_{-1,2}c_{1,2}^{-1},\ c_{2,1}c_{-2,1}^{-1},\ c_{1,2}c_{-1,2}^{-1}\right], \end{cases} \tag{5.15}$$

if, respectively, conditions (4.6), (4.7), (4.8) or (4.9) hold. Since the sets (5.14) are empty, it follows from (3.20) and (5.8)–(5.11) that

$$\mathbf{M}(\widetilde{g}_{2,1}^{+}) = \mathbf{M}(g_{2,2}^{+}) = \mathbf{M}(g_{1,2}^{-}) = 0$$

and

$$\mathbf{M}(\widetilde{g}_{2,1}^{+} f \widetilde{g}_{1,1}^{-}) = \mathbf{M}(g_{2,1}^{+} f g_{1,1}^{-}) = 0.$$

Then we deduce from (5.6) that

$$\begin{aligned} t_{2,1} &= -\mathbf{M}(g_{2,1}^{+})\mathbf{M}(\widetilde{g}_{2,1}^{+} f g_{1,1}^{-})k_2^{-1}, \qquad & t_{2,3} &= 0, \\ t_{4,3} &= \mathbf{M}(\widetilde{g}_{2,2}^{+})\mathbf{M}(g_{2,1}^{+} f \widetilde{g}_{1,1}^{-})k_1^{-1}k_2^{-1}, \qquad & t_{4,1} &= 0. \end{aligned} \tag{5.16}$$

Taking into account the relations $k_s = \mathbf{M}(\widetilde{g}_{s,2}^{-})$ for $s = 1, 2$, applying (3.20) for $\mathbf{M}(g_{2,1}^{+})$, $\mathbf{M}(\widetilde{g}_{2,2}^{+})$ and $\mathbf{M}(\widetilde{g}_{s,2}^{-})$, and using (5.12) and (5.13) for $\mathbf{M}(\widetilde{g}_{2,1}^{+} f g_{1,1}^{-})$ and

$\mathbf{M}(g^+_{2,1} f \widetilde{g}^-_{1,1})$, we infer from (5.16) that

$$t_{2,1} = \begin{cases} -(c_{1,1}\widetilde{c}_{-1} - c_{-1,1}\widetilde{c}_1)c^{-1}_{1,1}c^{-1}_{1,2}, \\ -(c_{2,1}\widetilde{c}_{-2} - c_{-2,1}\widetilde{c}_2)c^{-1}_{2,1}c^{-1}_{2,2}, \\ -(c_{1,1}\widetilde{c}_{-2} - c_{-1,1}\widetilde{c}_2)c^{-1}_{1,1}c^{-1}_{2,2}, \\ -(c_{2,1}\widetilde{c}_{-1} - c_{-2,1}\widetilde{c}_1)c^{-1}_{2,1}c^{-1}_{1,2}, \end{cases} \quad t_{4,3} = \begin{cases} (c_{1,2}\widetilde{c}_{-1} - c_{-1,2}\widetilde{c}_1)c^{-1}_{-1,1}c^{-1}_{-1,2}, \\ (c_{2,2}\widetilde{c}_{-2} - c_{-2,2}\widetilde{c}_2)c^{-1}_{-2,1}c^{-1}_{-2,2}, \\ (c_{2,2}\widetilde{c}_{-1} - c_{-2,2}\widetilde{c}_1)c^{-1}_{-1,1}c^{-1}_{-2,2}, \\ (c_{1,2}\widetilde{c}_{-2} - c_{-1,2}\widetilde{c}_2)c^{-1}_{-2,1}c^{-1}_{-1,2} \end{cases} \tag{5.17}$$

in the cases (4.6), (4.7), (4.8) and (4.9), respectively.

Finally, substituting $t_{2,3} = 0$, $t_{4,1} = 0$ and also $t_{2,1}$ and $t_{4,3}$ given by (5.17) into (5.5) and applying (5.15) and (4.1), we immediately obtain (4.12) with triangular 2×2 matrices T_1 and T_2 given by (4.13)–(4.16).

References

[1] S. Avdonin, A. Bulanova, and W. Moran, *Construction of sampling and interpolating sequences for multi-band signals. The two-band case*, Int. J. Appl. Math. Comput. Sci. **17** (2007), no. 2, 143–156.

[2] M.A. Bastos, A. Bravo, Yu.I. Karlovich, and I.M. Spitkovsky, *Constructive factorization of some almost periodic triangular matrix functions with a quadrinomial off diagonal entry*, J. Math. Anal. Appl. **376** (2011), 625–640.

[3] M.A. Bastos, Yu.I. Karlovich, I.M. Spitkovsky, and P.M. Tishin, *On a new algorithm for almost periodic factorization*, Recent Progress in Operator Theory (Regensburg, 1995) (I. Gohberg, R. Mennicken, and C. Tretter, eds.), Operator Theory: Advances and Applications, vol. 103, Birkhäuser Verlag, Basel and Boston, 1998, pp. 53–74.

[4] A.S. Besicovitch, *Almost periodic functions*, Dover Publications Inc., New York, 1955.

[5] A. Böttcher, Yu.I. Karlovich, and I.M. Spitkovsky, *Convolution operators and factorization of almost periodic matrix functions*, Operator Theory: Advances and Applications, vol. 131, Birkhäuser Verlag, Basel and Boston, 2002.

[6] A. Böttcher and B. Silbermann, *Analysis of Toeplitz operators*, second ed., Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2006, Prepared jointly with A. Karlovich.

[7] A. Böttcher and I.M. Spitkovsky, *The factorization problem: Some known results and open questions*, Advances in Harmonic Analysis and Operator Theory. The Stefan Samko Anniversary Volume (A. Almeida, L. Castro, and F.-O. Speck, eds.), Operator Theory: Advances and Applications, vol. 129, Birkhäuser/Springer, Basel, 2013, pp. 101–122.

[8] A. Brudnyi, L. Rodman, and I.M. Spitkovsky, *Non-denseness of factorable matrix functions*, J. Functional Analysis **261** (2011), 1969–1991.

[9] M. C. Câmara, C. Diogo, Yu.I. Karlovich, and I.M. Spitkovsky, *Factorizations, Riemann–Hilbert problems and the corona theorem*, J. London Math. Soc. **86** (2012), 852–878.

[10] M.C. Câmara, A.F. dos Santos, and M.C. Martins, *A new approach to factorization of a class of almost-periodic triangular symbols and related Riemann–Hilbert problems*, J. Funct. Anal. **235** (2006), no. 2, 559–592.

[11] M.C. Câmara, Yu.I. Karlovich, and I.M. Spitkovsky, *Almost periodic factorization of some triangular matrix functions*, Modern Analysis and Applications. The Mark Krein Centenary Conference (V. Adamyan, Y. Berezansky, I. Gohberg, M. Gorbachuk, A. Kochubei, H. Langer, and G. Popov, eds.), Operator Theory: Advances and Applications, vol. 190, Birkhäuser Verlag, Basel and Boston, 2009, pp. 171–190.

[12] ______, *Constructive almost periodic factorization of some triangular matrix functions*, J. Math. Anal. Appl. **367** (2010), 416–433.

[13] M.C. Câmara and M.C. Martins, *Explicit almost-periodic factorization for a class of triangular matrix functions*, J. Anal. Math. **103** (2007), 221–260.

[14] C. Corduneanu, *Almost periodic functions*, J. Wiley & Sons, 1968.

[15] M.P. Ganin, *On a Fredholm integral equation whose kernel depends on the difference of the arguments*, Izv. Vys. Uchebn. Zaved. Matematika (1963), no. 2 (33), 31–43.

[16] I.C. Gohberg and I.A. Feldman, *Convolution equations and projection methods for their solution*, Nauka, Moscow, 1971 (in Russian), English translation *Amer. Math. Soc. Transl. of Math. Monographs* **41**, Providence, R.I. 1974.

[17] I. Gohberg, S. Goldberg, and M.A. Kaashoek, *Classes of linear operators. Vol. II*, Birkhäuser Verlag, Basel and Boston, 1993.

[18] Yu.I. Karlovich, *Approximation approach to canonical APW factorability*, Izv. Vuzov., Sev.-Kavk. Region, 2005, pp. 143–151 (in Russian).

[19] Yu.I. Karlovich and I.M. Spitkovsky, *Factorization of almost periodic matrix-valued functions and the Noether theory for certain classes of equations of convolution type*, Mathematics of the USSR, Izvestiya **34** (1990), 281–316.

[20] ______, *Almost periodic factorization: An analogue of Chebotarev's algorithm*, Contemporary Math. **189** (1995), 327–352.

[21] ______, *Factorization of almost periodic matrix functions*, J. Math. Anal. Appl. **193** (1995), 209–232.

[22] B.M. Levitan and V.V. Zhikov, *Almost periodic functions and differential equations*, Cambridge University Press, 1982.

[23] G.S. Litvinchuk and I.M. Spitkovskii, *Factorization of measurable matrix functions*, Operator Theory: Advances and Applications, vol. 25, Birkhäuser Verlag, Basel, 1987, translated from the Russian by B. Luderer, with a foreword by B. Silbermann.

[24] D. Quint, L. Rodman, and I.M. Spitkovsky, *New cases of almost periodic factorization of triangular matrix functions*, Michigan Math. J. **45** (1998), 73–102.

[25] A. Rastogi, L. Rodman, and I.M. Spitkovsky, *Almost periodic factorization of* 2×2 *matrix functions: New cases of off diagonal spectrum*, Recent Advances and New Directions in Applied and Pure Operator Theory (Williamsburg, 2008) (J.A. Ball, V. Bolotnikov, J.W. Helton, L. Rodman, and I.M. Spitkovsky, eds.), Operator Theory: Advances and Applications, vol. 202, Birkhäuser, Basel, 2010, pp. 469–487.

[26] I.M. Spitkovsky, *On the factorization of almost periodic matrix functions*, Math. Notes **45** (1989), no. 5-6, 482–488.

[27] N. Wiener and E. Hopf, *Über eine Klasse singulärer Integralgleichungen*, Sitzungsber. Preuss. Akad. Wiss. Berlin, Phys.-Math. Kl. **30/32** (1931), 696–706.

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A C^*-algebra of Singular Integral Operators with Shifts Similar to Affine Mappings

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Dedicated to Professor António Ferreira dos Santos

Abstract. Representations on Hilbert spaces for the nonlocal C^*-algebra $\mathfrak{B}$ of singular integral operators with piecewise slowly oscillating coefficients, which is extended by the unitary shift operators U_g associated with the solvable discrete group G of diffeomorphisms $g : \mathbb{T} \to \mathbb{T}$ that are similar to affine mappings on the real line, are constructed. Such shifts may change or preserve the orientation on $\mathbb{T}$ and have both common fixed points for all $g \in G$ and distinct fixed points for different shifts. Using the theory developed for C^*-algebras of singular integral operators with shifts preserving the orientation of a contour, a Fredholm symbol calculus for the C^*-algebra $\mathfrak{B}$ is constructed and a Fredholm criterion for the operators $B \in \mathfrak{B}$ is established.

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1. Introduction

In this paper we deal with the nonlocal C^*-algebra $\mathfrak{B}$ generated by the C^*-algebra of singular integral operators with piecewise slowly oscillating coefficients and by the discrete group of unitary operators U_G associated with the group G of all diffeomorphisms (shifts) g of the unit circle $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ onto itself,

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given by

$$g = \alpha \circ \widehat{g} \circ \alpha^{-1}, \tag{1.1}$$

where

$$\alpha : \mathbb{R} \to \mathbb{T}, \quad \alpha(x) = \frac{x-i}{x+i} \quad \text{for} \quad x \in \mathbb{R}, \tag{1.2}$$

and

$$\widehat{g} : \mathbb{R} \to \mathbb{R}, \quad \widehat{g}(x) = k_g x + h_g \quad \text{for} \quad x \in \mathbb{R}, \tag{1.3}$$

with $k_g \in \mathbb{R} \setminus \{0\}$, $h_g \in \mathbb{R}$.

Let $\mathcal{B} := \mathcal{B}(L^2(\mathbb{T}))$ be the C^*-algebra of all bounded linear operators on the space $L^2(\mathbb{T})$ and $\mathcal{K} := \mathcal{K}(L^2(\mathbb{T}))$ be the ideal of compact operators in $\mathcal{B}$. An operator $B \in \mathcal{B}$ is said to be a *Fredholm operator* if and only if the coset $B^\pi := B+\mathcal{K}$ is invertible in the Calkin algebra $\mathcal{B}/\mathcal{K}$. A representation $\Psi_{\mathfrak{B}} : \mathfrak{B} \to \mathcal{B}(\mathcal{H}_{\mathfrak{B}})$, where $\mathcal{H}_{\mathfrak{B}}$ is a Hilbert space, is called a *Fredholm symbol map* for the C^*-algebra $\mathfrak{B}$ if the Fredholmness of each operator $B \in \mathfrak{B}$ is equivalent to the invertibility of $\Psi_{\mathfrak{B}}(B)$ on $\mathcal{H}_{\mathfrak{B}}$.

C^*-algebras of singular integral operators with piecewise slowly oscillating coefficients and different classes of groups G of preserving-orientation shifts were studied in [4]–[8]. The aim in this paper is to construct, using similar ideas to those used in [8], a *Fredholm symbol map* for the C^*-algebra

$$\mathfrak{B} := \text{alg}\,(PSO(\mathbb{T}), S_{\mathbb{T}}, U_G) \subset \mathcal{B}(L^2(\mathbb{T})), \tag{1.4}$$

where the shifts $g \in G$ may also change the orientation of $\mathbb{T}$. Thus, the C^*-algebra $\mathfrak{B}$ is generated by all multiplication operators aI with $a \in PSO(\mathbb{T})$, by the Cauchy singular integral operator $S_{\mathbb{T}}$ defined by

$$(S_{\mathbb{T}}\varphi)(t) := \lim_{\varepsilon \to 0} \frac{1}{\pi i} \int_{\mathbb{T} \setminus \mathbb{T}(t,\varepsilon)} \frac{\varphi(\tau)}{\tau - t}\, d\tau, \quad \mathbb{T}(t,\varepsilon) = \{\tau \in \mathbb{T} : |\tau - t| < \varepsilon\}, \quad t \in \mathbb{T},$$

and by the group $U_G := \{U_g : g \in G\}$ of unitary weighted shift operators U_g given by

$$(U_g\varphi)(t) := |g'(t)|^{1/2}\varphi(g(t)) \quad \text{for} \quad t \in \mathbb{T},$$

where G is the group of shifts defined by (1.1)–(1.3) and acting on $\mathbb{T}$ from the right: $(g_1 g_2)(t) = g_2(g_1(t))$ for all $t \in \mathbb{T}$ and all g_1, $g_2 \in G$.

Notice that different shifts $g \in G$ have both common and distinct fixed points. Indeed, denoting by $\mathbb{T}_g$ the set of fixed points of the shift $g = \alpha \circ \widehat{g} \circ \alpha^{-1}$, where $\widehat{g}(x) = k_g x + h_g$, we have

$$\mathbb{T}_g = \begin{cases} \mathbb{T} & \text{if} \ \ k_g = 1, \ h_g = 0; \\ \{1\} & \text{if} \ \ k_g = 1, \ h_g \neq 0; \\ \left\{1, \frac{h_g + i(k_g-1)}{h_g - i(k_g-1)}\right\} & \text{if} \ \ k_g \neq 1. \end{cases} \tag{1.5}$$

Moreover, in contrast to [8], where all shifts preserve the orientation of $\mathbb{T}$, the shifts g in the present group G preserve or change the orientation of $\mathbb{T}$ if, respectively, $k_g > 0$ or $k_g < 0$.

In view of (1.1), the reflection $\widehat{\gamma}(x) = -x$ on $\mathbb{R}$ generates on $\mathbb{T}$ the reflection $\gamma \in G$ given by

$$\gamma(t) = \overline{t} \quad \text{for} \quad t \in \mathbb{T}. \tag{1.6}$$

Denoting by G_0 the normal subgroup of G consisting of the shifts that preserve the orientation of $\mathbb{T}$, we easily conclude that for each shift $g \in G \setminus G_0$ there exist $\widetilde{g}_1, \widetilde{g}_2 \in G_0$ such that $g = \widetilde{g}_1\gamma = \gamma\widetilde{g}_2$. Group G is then given by the union $G = G_0 \cup G_0\gamma$, where $G_0\gamma = \{\widetilde{g}\gamma : \widetilde{g} \in G_0\}$.

The paper is organized as follows. Section 2 contains preliminaries: descriptions of the C^*-algebra $PSO(\mathbb{T})$ of piecewise slowly oscillating functions on $\mathbb{T}$ and its maximal ideal space $M(PSO(\mathbb{T}))$, a Fredholm symbol map (Theorem 2.2) for the C^*-algebra

$$\mathfrak{A} := \operatorname{alg}(PSO(\mathbb{T}), S_{\mathbb{T}}) \subset \mathfrak{B}, \tag{1.7}$$

generated by the operator $S_{\mathbb{T}}$ and all operators aI with $a \in PSO(\mathbb{T})$, and the description of a spectral measure associated with a central subalgebra of the C^*-algebra $\mathfrak{B}/\mathcal{K}$. Section 2 also contains the main tools for studying the C^*-algebra $\mathfrak{B}$: a C^*-algebra version of the lifting theorem (Theorem 2.6) and a suitable version of the local-trajectory method (Theorem 2.7).

Section 3 contains the main results of the paper: a representation $\Psi_{\mathfrak{B}}$ of the C^*-algebra $\mathfrak{B}$ on a Hilbert space $\mathcal{H}_{\mathfrak{B}} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$, where

$$\Psi_{\mathfrak{B}} = \Psi_1 \oplus \Psi_2 \oplus \Psi_3 \tag{1.8}$$

is the direct sum of representations $\Psi_i : \mathfrak{B} \to \mathcal{B}(\mathcal{H}_i)$ $(i = 1, 2, 3)$, such that $\operatorname{Ker} \Psi_{\mathfrak{B}} = \mathcal{K}$. As a result, a Fredholm symbol map for $\mathfrak{B}$ is obtained (Theorem 3.1) and a Fredholm criterion for the operators $B \in \mathfrak{B}$ in terms of their Fredholm symbols is established (Theorem 3.2).

In Section 4 we study the invertibility in the quotient C^*-algebra $\widehat{\mathfrak{B}} := \mathfrak{B}/\mathfrak{H}$ where $\mathfrak{H}$ is the closed two-sided ideal in $\mathfrak{B}$ generated by all commutators $[aI, S_{\mathbb{T}}]$, with $a \in PSO(\mathbb{T})$. The C^*-algebra $\widehat{\mathfrak{B}}$ can be viewed as

$$\widehat{\mathfrak{B}} = \operatorname{alg}(\widehat{\mathfrak{A}}, \widehat{U}_G), \tag{1.9}$$

the C^*-algebra generated by the commutative C^*-subalgebra $\widehat{\mathfrak{A}} := (\mathfrak{A} + \mathfrak{H})/\mathfrak{H}$ and by the group of the unitary cosets $\widehat{U}_G = \{U_g + \mathfrak{H} : g \in G\}$. Using the local-trajectory method, the Gelfand transform of the C^*-algebra $\widehat{\mathfrak{A}}$ is extended to a faithful representation of the C^*-algebra $\widehat{\mathfrak{B}}$. As a consequence, an invertibility symbol map for the C^*-algebra $\widehat{\mathfrak{B}}$ is constructed, an invertibility criterion for its elements is obtained (Corollary 4.4 and Theorem 4.3), and the *-homomorphism $\Psi_3 : \mathfrak{B} \to \mathcal{B}(\mathcal{H}_3)$ is defined (Theorem 4.5 and Corollary 4.6).

The continuity of mappings $\Psi_1 : \mathfrak{B} \to \mathcal{B}(\mathcal{H}_1)$ and $\Psi_2 : \mathfrak{B} \to \mathcal{B}(\mathcal{H}_2)$ is established in Section 5. These mappings are associated, respectively, with two G-orbits: $\omega_1 = \{1\}$ and $\omega_2 = \mathbb{T} \setminus \{1\}$ resulting from the action of G on $\mathbb{T}$. Using ideas from [8] and [16] we establish the continuity of mappings Ψ_1 and Ψ_2 for the considered group G that admits shifts changing the orientation of $\mathbb{T}$ (Theorem 5.4).

Relations between the mappings Ψ_1, Ψ_2 and the ideals $\mathfrak{H}_1, \mathfrak{H}_2$ generating $\mathfrak{H}$ are also investigated (Theorem 5.5).

Finally, in Section 6, applying (1.8) and collecting results from Sections 4 and 5, and also using the lifting theorem presented in Subsection 2.4, we prove Theorems 3.1 and 3.2, the main results of the paper.

2. Preliminaries

2.1. The C^*-algebra of $PSO(\mathbb{T})$ functions

Let $L^\infty(\mathbb{T})$ be the C^*-algebra of all bounded measurable functions on the unit circle $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$. Let $C(\mathbb{T})$, $PC(\mathbb{T})$ and $SO(\mathbb{T})$ denote the C^*-subalgebras of $L^\infty(\mathbb{T})$ consisting, respectively, of all continuous functions on $\mathbb{T}$, all piecewise continuous functions on $\mathbb{T}$, that is, the functions having one-sided limits at each point $t \in \mathbb{T}$, and all slowly oscillating functions on $\mathbb{T}$, that is, the functions f that are slowly oscillating at each point $\lambda \in \mathbb{T}$:

$$\lim_{\varepsilon\to 0} \operatorname{ess\,sup} \big\{ |f(z_1) - f(z_2)| : z_1, z_2 \in \mathbb{T}_\varepsilon(\lambda) \big\} = 0,$$

where $\mathbb{T}_\varepsilon(\lambda) := \{z \in \mathbb{T} : \varepsilon/2 \le |z - \lambda| \le \varepsilon\}$. Denoting by $SO_\lambda(\mathbb{T})$ the C^*-subalgebra of $L^\infty(\mathbb{T})$ consisting of the continuous functions on $\mathbb{T} \setminus \{\lambda\}$ that are slowly oscillating at $\lambda \in \mathbb{T}$, we deduce that $SO(\mathbb{T}) := \operatorname{alg} \{SO_\lambda(\mathbb{T}) : \lambda \in \mathbb{T}\}$ is the smallest C^*-subalgebra of $L^\infty(\mathbb{T})$ containing all C^*-algebras $SO_\lambda(\mathbb{T})$, $\lambda \in \mathbb{T}$.

Denoting by $PC^0(\mathbb{T})$ and $SO^0(\mathbb{T})$ the non-closed subalgebras of $L^\infty(\mathbb{T})$ consisting of all bounded functions that are continuous at all points of $\mathbb{T}$ except perhaps a finite number of points where these functions are, respectively, piecewise continuous or slowly oscillating, we conclude that $PC(\mathbb{T}) = \overline{PC^0(\mathbb{T})}$ and $SO(\mathbb{T}) = \overline{SO^0(\mathbb{T})}$, with closures taken in $L^\infty(\mathbb{T})$.

Let $PSO(\mathbb{T}) := \operatorname{alg}(SO(\mathbb{T}), PC(\mathbb{T}))$ be the C^*-subalgebra of $L^\infty(\mathbb{T})$ generated by the C^*-algebras $SO(\mathbb{T})$ and $PC(\mathbb{T})$. Obviously, $PSO(\mathbb{T})$ is the closure in $L^\infty(\mathbb{T})$ of the set $PSO^0(\mathbb{T})$ consisting of all bounded functions on $\mathbb{T}$ admitting piecewise slowly oscillating discontinuities at finite subsets of $\mathbb{T}$ and being continuous at all other points of $\mathbb{T}$.

As usual, we do not distinguish the non-zero multiplicative linear functionals on $\mathcal{A}$ and their kernels which are the maximal ideals of $\mathcal{A}$.

It is known that the maximal ideal space of $C(\mathbb{T})$ and $PC(\mathbb{T})$ can be identified, respectively, with $\mathbb{T}$ and $\mathbb{T} \times \{0, 1\}$,

$$M(C(\mathbb{T})) = \mathbb{T}, \quad M(PC(\mathbb{T})) = \mathbb{T} \times \{0, 1\},$$

where the points $t \in \mathbb{T}$ are identified with the evaluation functionals δ_t given by $\delta_t(f) = f(t)$ for $f \in C(\mathbb{T})$, and the pairs $(t, 0)$ and $(t, 1)$ are the multiplicative linear functionals defined for $a \in PC(\mathbb{T})$ by $(t, 0)a = a(t-0)$ and $(t, 1)a = a(t+0)$, where $a(t-0)$ and $a(t+0)$ are the left and right one-sided limits of a at the point $t \in \mathbb{T}$. The base of open sets on $\mathbb{T} \times \{0, 1\}$ consists of all sets of the form $(t, \tau) \times \{0, 1\}$,

$((t,\tau] \times \{0\}) \cup ((t,\tau) \times \{1\})$, $((t,\tau) \times \{0\}) \cup ([t,\tau) \times \{1\})$, where $t,\tau \in \mathbb{T}$. Since $C(\mathbb{T}) \subset SO(\mathbb{T}) \subset PSO(\mathbb{T})$, it follows from [4] that

$$M(SO(\mathbb{T})) = \bigcup_{t\in\mathbb{T}} M_t(SO(\mathbb{T})), \quad M(PSO(\mathbb{T})) = \bigcup_{\xi\in M(SO(\mathbb{T}))} M_\xi(PSO(\mathbb{T})), \tag{2.1}$$

where the corresponding fibers are given for $t \in \mathbb{T}$ and $\xi \in M(SO(\mathbb{T}))$ by

$$M_t(SO(\mathbb{T})) = \big\{\xi \in M(SO(\mathbb{T})) : \xi|_{C(\mathbb{T})} = t\big\},$$
$$M_\xi(PSO(\mathbb{T})) = \big\{y \in M(PSO(\mathbb{T})) : y|_{SO(\mathbb{T})} = \xi\big\}.$$

Given $\xi \in M_t(SO(\mathbb{T}))$ for $t \in \mathbb{T}$, for any sequence $\{a_i\} \subset SO(\mathbb{T})$ according to [4, Corollary 4.4] there exists a sequence $\{t_n\} \subset \mathbb{T} \setminus \{t\}$ such that

$$\xi(a_i) = \lim_{n\to\infty} a_i(t_n) \quad \text{for all} \quad i \in \mathbb{N}.$$

The fibers $M_\xi(PSO(\mathbb{T}))$ for $\xi \in M(SO(\mathbb{T}))$ can be characterized as follows.

Theorem 2.1. [4, Theorem 4.6] *If $\xi \in M_t(SO(\mathbb{T}))$ with $t \in \mathbb{T}$, then*

$$M_\xi(PSO(\mathbb{T})) = \{(\xi,0),\, (\xi,1)\}, \tag{2.2}$$

where, for $\mu \in \{0,1\}$, $(\xi,\mu)|_{SO(\mathbb{T})} = \xi$, $(\xi,\mu)|_{C(\mathbb{T})} = t$, $(\xi,\mu)|_{PC(\mathbb{T})} = (t,\mu)$.

By (2.1) and (2.2) we have $M(PSO(\mathbb{T})) = M(SO(\mathbb{T})) \times \{0,1\}$. With the Gelfand topology described in [5], $M(PSO(\mathbb{T}))$ becomes a compact Hausdorff space.

2.2. The C^*-algebra $\mathfrak{A}$

Consider the C^*-algebra $\mathfrak{A}$ of singular integral operators on $L^2(\mathbb{T})$ with $PSO(\mathbb{T})$ coefficients, which is given by (1.7). With $\mathfrak{A}$ we associate the set

$$\mathfrak{M} := M(SO(\mathbb{T})) \times \overline{\mathbb{R}}, \tag{2.3}$$

where $\overline{\mathbb{R}} = [-\infty, +\infty]$. Let $B(\mathfrak{M}, \mathbb{C}^{2\times 2})$ be the C^*-algebra of all bounded matrix functions $f : \mathfrak{M} \to \mathbb{C}^{2\times 2}$. According to [9, Section 7] and [5, Theorem 5.1] we have the following symbol calculus for the C^*-algebra $\mathfrak{A}$.

Theorem 2.2. *The map* $\mathrm{Sym} : \{aI : a \in PSO(\mathbb{T})\} \cup \{S_\mathbb{T}\} \to B(\mathfrak{M}, \mathbb{C}^{2\times 2})$ *given by the matrix functions*

$$[\mathrm{Sym}\,(aI)](\xi,x) = \begin{pmatrix} a(\xi,1) & 0 \\ 0 & a(\xi,0) \end{pmatrix}, \quad [\mathrm{Sym}\,S_\mathbb{T}](\xi,x) = \begin{pmatrix} u(x) & -v(x) \\ v(x) & -u(x) \end{pmatrix}, \tag{2.4}$$

where $a(\xi,\mu)$ is the Gelfand transform of a function $a \in PSO(\mathbb{T})$ at the point $(\xi,\mu) \in M(PSO(\mathbb{T}))$ and

$$u(x) := \tanh(\pi x), \quad v(x) := -i/\cosh(\pi x) \quad \textit{for} \quad x \in \overline{\mathbb{R}}, \tag{2.5}$$

extends to a C^*-algebra homomorphism $\mathrm{Sym} : \mathfrak{A} \to B(\mathfrak{M}, \mathbb{C}^{2\times 2})$ *whose kernel consists of all compact operators on $L^2(\mathbb{T})$. An operator $A \in \mathfrak{A}$ is Fredholm on the space $L^2(\mathbb{T})$ if and only if* $\det([\mathrm{Sym}\,A](\xi,x)) \neq 0$ *for all $(\xi,x) \in \mathfrak{M}$.*

To each point $t \in \mathbb{T}$ we assign the operator $V_t \in \mathcal{B}(L^2(\mathbb{T}))$ with fixed singularity at t, which is given for $z \in \mathbb{T}$ by

$$(V_t\varphi)(z) := \frac{\chi_t^+(z)}{\pi i}\int_{\mathbb{T}} \frac{\varphi(y)\chi_t^+(y)}{y+z-2t}\,dy - \frac{\chi_t^-(z)}{\pi i}\int_{\mathbb{T}} \frac{\varphi(y)\chi_t^-(y)}{y+z-2t}\,dy, \tag{2.6}$$

where $\chi_t^\pm$ are the characteristic functions of arcs $\gamma_t^\pm$ such that $\gamma_t := \gamma_t^+ \cup \gamma_t^-$ is a neighborhood of t separated from $-t$, $\gamma_t^+ \cap \gamma_t^- = \{t\}$, and $\gamma_t^+ \cap (-t,t) = \varnothing$, $\gamma_t^- \cap (t,-t) = \varnothing$. The operators V_t for all $t \in \mathbb{T}$ belong to the C^*-algebra $\mathfrak{A}$ (see, e.g., [5, Lemma 5.3]).

Let $\mathcal{P}$ consist of all polynomials $\sum_{k=0}^n a_k u^k$ ($a_k \in \mathbb{C}$, $n = 0,1,\dots$), and

$$\mathcal{Z} := \operatorname{alg}\big\{aI,\ H_{P,t} :\ a \in SO(\mathbb{T}),\ P \in \mathcal{P},\ t \in \mathbb{T}\big\} \subset \mathcal{B}(L^2(\mathbb{T})) \tag{2.7}$$

be the C^*-subalgebra of $\mathfrak{A}$ generated by the operators aI ($a \in SO(\mathbb{T})$) and

$$H_{P,t} := P(\chi_t^+ S_{\mathbb{T}} \chi_t^+ I - \chi_t^- S_{\mathbb{T}} \chi_t^- I)V_t \in \mathfrak{A} \quad (P \in \mathcal{P},\ t \in \mathbb{T}).$$

By [5, (4.10)–(4.11)] and [16, (5.24)], we get

$$aH_{P,t} \simeq H_{P,t}aI, \quad S_{\mathbb{T}}H_{P,t} \simeq H_{P,t}S_{\mathbb{T}}, \quad U_gH_{P,t} \simeq H_{P,g^{-1}(t)}U_g \tag{2.8}$$

for all $a \in PSO(\mathbb{T})$, $t \in \mathbb{T}$ and $g \in G$. Moreover, because

$$bS_{\mathbb{T}} \simeq S_{\mathbb{T}}bI \quad \text{for all} \quad b \in SO(\mathbb{T}),$$

we conclude that $\mathcal{Z}^\pi := (\mathcal{Z} + \mathcal{K})/\mathcal{K}$ is a central C^*-subalgebra of the C^*-algebra $\mathfrak{A}^\pi := \mathfrak{A}/\mathcal{K}$, where $\mathcal{K} := \mathcal{K}(L^2(\mathbb{T}))$. Given the set

$$\dot{\mathfrak{M}} := M(SO(\mathbb{T})) \times \dot{\mathbb{R}} \tag{2.9}$$

with $\dot{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$, we infer from [5, Theorem 6.3] that $\mathcal{Z}^\pi \cong C(\dot{\mathfrak{M}})$, where $\dot{\mathfrak{M}}$ is the compact Hausdorff space equipped with the Gelfand topology.

By analogy with [5, Lemma 5.4], we easily obtain its generalization for shifts changing the orientation of $\mathbb{T}$ from (2.6) and (2.4).

Lemma 2.3. *Let g be an orientation-preserving diffeomorphism on $\mathbb{T}$, $t \in \mathbb{T}$, $v(x) = -i/\cosh(\pi x)$ for $x \in \overline{\mathbb{R}}$, and let $\mathfrak{M}$ be the set* (2.3).

(i) *If $g(t) = t$, then $U_gV_t \in \mathfrak{A}$ and*

$$[\operatorname{Sym}(U_gV_t)](\xi,x) := \begin{cases} \operatorname{diag}\big\{e^{ix\ln g'(t)}v(x), e^{ix\ln g'(t)}v(x)\big\} & \textit{if}\ \ (\xi,x) \in M_t(SO(\mathbb{T})) \times \overline{\mathbb{R}}, \\ 0_{2\times 2} & \textit{if}\ \ (\xi,x) \in \mathfrak{M} \setminus (M_t(SO(\mathbb{T})) \times \overline{\mathbb{R}}). \end{cases}$$

(ii) *If $\overline{g(t)} = t$, then $U_gU_\gamma V_t \in \mathfrak{A}$ and*

$$[\operatorname{Sym}(U_gU_\gamma V_t)](\xi,x) := \begin{cases} \begin{pmatrix} 0 & e^{ix\ln|g'(t)|}v(x) \\ e^{ix\ln|g'(t)|}v(x) & 0 \end{pmatrix} & \textit{if}\ \ (\xi,x) \in M_t(SO(\mathbb{T})) \times \overline{\mathbb{R}}, \\ 0_{2\times 2} & \textit{if}\ \ (\xi,x) \in \mathfrak{M} \setminus (M_t(SO(\mathbb{T})) \times \overline{\mathbb{R}}). \end{cases}$$

2.3. The spectral measure associated with the C^*-algebra $\mathfrak{B}^\pi$

Consider an isometric representation

$$\varphi : \mathfrak{B}^\pi \to \mathcal{B}(\mathcal{H}_\varphi), \quad B^\pi \mapsto \varphi(B^\pi) \tag{2.10}$$

of the C^*-algebra $\mathfrak{B}^\pi := \mathfrak{B}/\mathcal{K}$ on an abstract Hilbert space $\mathcal{H}_\varphi$. Let $\mathfrak{R}(\dot{\mathfrak{M}})$ be the σ-algebra of all Borel subsets of the compact set $\dot{\mathfrak{M}}$ given by (2.9), and let

$$P_\varphi : \mathfrak{R}(\dot{\mathfrak{M}}) \to \mathcal{B}(\mathcal{H}_\varphi) \tag{2.11}$$

be the unique spectral measure associated to the representation (2.10) restricted to the commutative unital C^*-algebra $\mathcal{Z}^\pi$, where $\mathcal{Z}$ is defined by (2.7). Since all shifts $g \in G$ have continuous derivative on $\mathbb{T}$, for the generators of $\mathfrak{B}$ we have the relations

$$U_g a U_g^{-1} = (a \circ g)I, \quad a \in PSO(\mathbb{T}), \quad g \in G, \tag{2.12}$$

and

$$U_g S_{\mathbb{T}} U_g^{-1} \simeq S_{\mathbb{T}}, \quad U_{g\gamma} S_{\mathbb{T}} U_{g\gamma}^{-1} \simeq -S_{\mathbb{T}}, \quad g \in G_0, \tag{2.13}$$

where $a \circ g,\ a \circ \gamma \in PSO(\mathbb{T})$ (cf. [4, Lemma 4.2]).

As a consequence, for each $g \in G$ the mapping

$$\alpha_g : A^\pi \mapsto U_g^\pi A^\pi (U_g^\pi)^{-1}$$

is a *-automorphism of the C^*-algebra $\mathfrak{A}^\pi$ and its central C^*-subalgebra $\mathcal{Z}^\pi$. These *-automorphisms in view of (2.12) and (2.8) induce on $M(\mathcal{Z}^\pi) = \dot{\mathfrak{M}}$ the group of homeomorphisms

$$\beta_g : \dot{\mathfrak{M}} \to \dot{\mathfrak{M}}, \quad (\xi, x) \mapsto (g(\xi), x) \quad \text{if} \;\; g \in G,$$

where $\xi \mapsto g(\xi)$ is the homeomorphism on $M(SO(\mathbb{T}))$ given by

$$a(g(\xi)) = (a \circ g)(\xi) \quad \text{for all} \;\; a \in SO(\mathbb{T}) \;\; \text{and all} \;\; \xi \in M(SO(\mathbb{T})) \tag{2.14}$$

(as usual $a(\xi) := \xi(a)$).

Denoting by $\mathfrak{R}_G(\dot{\mathfrak{M}})$ the subset of $\mathfrak{R}(\dot{\mathfrak{M}})$ given by

$$\mathfrak{R}_G(\dot{\mathfrak{M}}) := \big\{ Q \in \mathfrak{R}(\dot{\mathfrak{M}}) : \beta_g(Q) = Q \;\; \text{for all} \;\; g \in G \big\},$$

we conclude from [14] that, for each $Q \in \mathfrak{R}_G(\dot{\mathfrak{M}})$ and each operator $B \in \mathfrak{B}$,

$$P_\varphi(Q)\varphi(B^\pi) = \varphi(B^\pi)P_\varphi(Q).$$

For every point $t \in \mathbb{T}$ we introduce the open subset of $\dot{\mathfrak{M}}$ given by

$$\mathfrak{M}_t^\circ := M_t(SO(\mathbb{T})) \times \mathbb{R}. \tag{2.15}$$

For each $g \in G$, the homeomorphism $\xi \mapsto g(\xi)$ defined by (2.14) sends the fibers $M_t(SO(\mathbb{T}))$ onto the fibers $M_{g(t)}(SO(\mathbb{T}))$. Since 1 is a fixed point for any shift $g \in G$, for every function $a \in SO(\mathbb{T})$ it follows that $a(g(\xi)) = a(\xi)$ for all $\xi \in M_1(SO(\mathbb{T}))$ (see, e.g., [5, Theorem 6.4]), and therefore $\mathfrak{M}_1^\circ$ is a set of fixed points for all homeomorphisms β_g $(g \in G)$. Consequently, $\mathfrak{M}_1^\circ \in \mathfrak{R}_G(\dot{\mathfrak{M}})$, while for all $t \in \mathbb{T} \setminus \{1\}$ the sets $\mathfrak{M}_t^\circ$ do not belong to $\mathfrak{R}_G(\dot{\mathfrak{M}})$. Similarly to [7, Lemma 4.2] we obtain the following.

Lemma 2.4. *For every $t \in \mathbb{T}$ and every $g \in G$,*

$$P_\varphi(\mathfrak{M}_t^\circ)\varphi(U_g^\pi) = \varphi(U_g^\pi)P_\varphi(\mathfrak{M}_{g(t)}^\circ). \tag{2.16}$$

In particular, from Lemma 2.4 it follows for the reflection $\gamma \in G$ given by (1.6) and for every $t \in \mathbb{T}$ that

$$P_\varphi(\mathfrak{M}_t^\circ)\varphi(U_\gamma^\pi) = \varphi(U_\gamma^\pi)P_\varphi(\mathfrak{M}_{\bar{t}}^\circ).$$

For each $t \in \mathbb{T}$, let $\mathfrak{A}_t^\circ$ be the abstract C^*-subalgebra of $\mathcal{B}(\mathcal{H}_\varphi)$ given by

$$\mathfrak{A}_t^\circ := P_\varphi(\mathfrak{M}_t^\circ)\varphi(\mathfrak{A}^\pi).$$

Using the fact that $\mathfrak{M}_t^\circ$ is an open subset of $\dot{\mathfrak{M}}$, we obtained in [5, Corollary 9.3] the following invertibility criterion for the operators in $\mathfrak{A}_t^\circ$.

Lemma 2.5. *For any $t \in \mathbb{T}$, the map $\mathrm{Sym}_t^\circ : \mathfrak{A}_t^\circ \to \mathcal{B}(l^2(\mathfrak{M}_t^\circ, \mathbb{C}^2))$ defined by*

$$\mathrm{Sym}_t^\circ : \ P_\varphi(\mathfrak{M}_t^\circ)\varphi(\mathfrak{A}^\pi) \to \mathcal{B}(l^2(\mathfrak{M}_t^\circ, \mathbb{C}^2)), \quad P_\varphi(\mathfrak{M}_t^\circ)\varphi(A^\pi) \mapsto (\mathrm{Sym}\, A)|_{\mathfrak{M}_t^\circ} I$$

is an isometric C^*-algebra homomorphism. An operator $P_\varphi(\mathfrak{M}_t^\circ)\varphi(A^\pi)$ for $A \in \mathfrak{A}$ is invertible on the space $P_\varphi(\mathfrak{M}_t^\circ)\mathcal{H}_\varphi$ if and only if

$$\det([\mathrm{Sym}\, A](\xi, x)) \neq 0 \quad \textit{for all} \quad (\xi, x) \in M_t(SO(\mathbb{T})) \times \overline{\mathbb{R}}.$$

2.4. Lifting theorem and the local-trajectory method

Let $\mathcal{B} := \mathcal{B}(\mathcal{H})$ be the C^*-algebra of all bounded linear operators on a Hilbert space $\mathcal{H}$ and let $\mathfrak{B}$ be a C^*-subalgebra of $\mathcal{B}$ containing the identity operator $I \in \mathcal{B}$. Suppose the ideal $\mathcal{K} := \mathcal{K}(\mathcal{H})$ is contained in $\mathfrak{B}$. Given $B \in \mathcal{B}$, let $|B|$ denote the essential norm of B, that is,

$$|B| = \|B^\pi\| = \inf\{\|B + K\| : K \in \mathcal{K}\}.$$

To investigate the Fredholmness of operators $B \in \mathfrak{B}$, the central result is the following analogue of the lifting theorem from [12, Theorem 1.8] (see also [18, Section 6.3]), which is a C^*-algebra modification of [15, Theorem 3.3]. For reader's convenience we give its proof.

Theorem 2.6. *Let Λ be an index set and suppose that, for each $\lambda \in \Lambda$, we are given a unital C^*-algebra $\mathcal{L}_\lambda$, a *-homomorphism $\Psi_\lambda : \mathfrak{B} \to \mathcal{L}_\lambda$, and a closed two-sided ideal $\mathfrak{H}_\lambda \subset \mathfrak{B}$ such that:*

(i) $\mathcal{K} \subset \mathfrak{H}_\lambda \cap \mathrm{Ker}\, \Psi_\lambda$ *and* $\mathfrak{H}_\mu \subset \mathrm{Ker}\, \Psi_\lambda$ *for all* $\mu \in \Lambda \setminus \{\lambda\}$;

(ii) *the restriction of the quotient homomorphism*

$$\mathfrak{B}/\mathcal{K} \to \mathcal{L}_\lambda, \quad B + \mathcal{K} \mapsto \Psi_\lambda(B)$$

onto the ideal $\mathfrak{H}_\lambda/\mathcal{K}$ is a *-isomorphism of $\mathfrak{H}_\lambda/\mathcal{K}$ onto the closed two-sided ideal $\mathcal{R}_\lambda := \Psi_\lambda(\mathfrak{H}_\lambda)$ of the C^*-algebra $\mathfrak{B}_\lambda := \Psi_\lambda(\mathfrak{B}) \subset \mathcal{L}_\lambda$.

Let $\mathfrak{H}$ be the smallest closed two-sided ideal of $\mathfrak{B}$ containing all ideals $\mathfrak{H}_\lambda$ $(\lambda \in \Lambda)$. Then an operator $B \in \mathfrak{B}$ is Fredholm if and only if the coset $B + \mathfrak{H}$ is invertible in $\mathfrak{B}/\mathfrak{H}$ and for every $\lambda \in \Lambda$ the element $\Psi_\lambda(B)$ is invertible in $\mathcal{L}_\lambda$.

Proof. Necessity. Let $B \in \mathfrak{B}$ be a Fredholm operator. Then there is an operator $B' \in \mathfrak{B}$ such that $BB' = I + K$ and $B'B = I + \widetilde{K}$ with $K, \widetilde{K} \in \mathcal{K}$. Since the quotient C^*-algebra $\mathfrak{B}^\pi = \mathfrak{B}/\mathcal{K}$ is inverse closed in the C^*-algebra $\mathcal{B}^\pi = \mathcal{B}/\mathcal{K}$, that is, any coset $B^\pi \in \mathfrak{B}^\pi$ invertible in $\mathcal{B}^\pi$ is invertible in $\mathfrak{B}^\pi$, and since $\mathcal{K} \subset \mathfrak{H}$, we infer that the coset $B + \mathfrak{H}$ is invertible in $\mathfrak{B}/\mathfrak{H}$ and its inverse is the coset $B' + \mathfrak{H}$. Because the homomorphisms $\Psi_\lambda : \mathfrak{B}/\mathcal{K} \to \mathcal{L}_\lambda$ preserve the invertibility of elements, for every $\lambda \in \Lambda$ the element $\Psi_\lambda(B) = \Psi_\lambda(B + \mathcal{K})$ is invertible in $\mathcal{L}_\lambda$.

Sufficiency. Suppose that the coset $B+\mathfrak{H}$ is invertible in the quotient algebra $\mathfrak{B}/\mathfrak{H}$. Then there are elements $D \in \mathfrak{B}$ and $H \in \mathfrak{H}$ such that $DB = I + H$. By definition of the ideal $\mathfrak{H}$ one can choose a set $\{\lambda_1, \lambda_2, \dots, \lambda_n\} \subset \Lambda$, elements $H_{\lambda_i} \in \mathfrak{H}_{\lambda_i}$ $(i = 1, 2, \dots, n)$ and $H' \in \mathfrak{H}$ such that

$$H' \simeq H_{\lambda_1} + \cdots + H_{\lambda_n} \quad \text{and} \quad |H - H'| < 1.$$

In addition, for each $\lambda \in \Lambda$, since $\Psi_\lambda(H_\mu) = 0$ for all $\mu \in \Lambda \setminus \{\lambda\}$ and all $H_\mu \in \mathfrak{H}_\mu$, it follows that $\Psi_\lambda(\mathfrak{H}) = \Psi_\lambda(\mathfrak{H}_\lambda) = \mathcal{R}_\lambda$, and therefore the invertibility of the coset $B + \mathfrak{H}$ in $\mathfrak{B}/\mathfrak{H}$ implies the invertibility of the coset $\Psi_\lambda(B) + \mathcal{R}_\lambda$ in the quotient C^*-algebra $\mathfrak{B}_\lambda/\mathcal{R}_\lambda$.

Let $\Psi_{\lambda_i}^{-1}(B)$ stand for the inverse of $\Psi_{\lambda_i}(B)$ in $\mathcal{L}_{\lambda_i}$, $i = 1, 2, \dots, n$. Because $\mathfrak{B}_{\lambda_i}$ is a C^*-algebra, we conclude that $\Psi_{\lambda_i}^{-1}(B) \in \mathfrak{B}_{\lambda_i}$, whence $\Psi_{\lambda_i}(H_{\lambda_i})\Psi_{\lambda_i}^{-1}(B)$ belongs to the ideal $\mathcal{R}_{\lambda_i}$ of the C^*-algebra $\mathfrak{B}_{\lambda_i}$. Then by (ii) there exists an element $D_{\lambda_i} \in \mathfrak{H}_{\lambda_i}$ such that

$$\Psi_{\lambda_i}(D_{\lambda_i}) = \Psi_{\lambda_i}(H_{\lambda_i})\Psi_{\lambda_i}^{-1}(B). \tag{2.17}$$

If we put $D' = D - D_{\lambda_1} - \cdots - D_{\lambda_n}$, then

$$\begin{aligned} D'B &= I + H - D_{\lambda_1}B - \cdots - D_{\lambda_n}B \\ &= I + (H - H') + (H_{\lambda_1} - D_{\lambda_1}B) + \cdots + (H_{\lambda_n} - D_{\lambda_n}B). \end{aligned} \tag{2.18}$$

It follows from (i) and (ii) that actually

$$\mathfrak{H}_\lambda \cap \operatorname{Ker} \Psi_\lambda = \mathcal{K}. \tag{2.19}$$

Indeed, if $H_\lambda \in \mathfrak{H}_\lambda$ with $\Psi_\lambda(H_\lambda) = 0$, then coset $H_\lambda + \mathcal{K}$ belongs to the kernel of the *-isomorphism mentioned in (ii). Thus, $H_\lambda + \mathcal{K} = 0 + \mathcal{K}$ and hence $H_\lambda \in \mathcal{K}$.

Since, by definition, $H_{\lambda_i} - D_{\lambda_i}B \in \mathfrak{H}_{\lambda_i}$ and, in view of (2.17), $\Psi_{\lambda_i}(H_{\lambda_i} - D_{\lambda_i}B) = 0$, we conclude from (2.19) that $H_{\lambda_i} - D_{\lambda_i}B \in \mathcal{K}$. Thus we infer from (2.18) that

$$D'B \simeq I + H - H'.$$

In view of the inequality $|H - H'| < 1$, the coset $I + H - H' + \mathcal{K}$ is invertible in $\mathfrak{B}/\mathcal{K}$, and $(I + H - H' + \mathcal{K})^{-1} = B_0 + \mathcal{K}$ with $B_0 \in \mathfrak{B}$. Then $B_0D'B \simeq I$, whence the operator $B_0D' \in \mathfrak{B}$ is a left regularizer of B.

Analogously one can prove the existence of a right regularizer of B, which together with the previous part implies the Fredholmness of B. □

In order to study the invertibility of cosets in the C^*-algebra $\mathfrak{B}/\mathfrak{H}$, we will use the following version of the local-trajectory method (cf. [13], [14], [5] and [1]–[3]).

Let $\mathcal{A}$ be a commutative C^*-algebra with unit I, G a discrete group with unit e, $u : g \mapsto u_g$ a homomorphism of the group G onto a group $u_G = \{u_g : g \in G\}$ of unitary elements such that $u_{g_1 g_2} = u_{g_1} u_{g_2}$ and $u_e = I$. Suppose $\mathcal{A}$ and u_G are contained in a C^*-algebra $\mathcal{Y}$ and assume that

(A1) *for every* $g \in G$, *the mapping* $\alpha_g : d \mapsto u_g \, d \, u_g^*$ *is a* **-automorphism of the commutative* C^**-algebra* $\mathcal{A}$;

(A2) G *is an amenable discrete group.*

By [11, § 1.2], a discrete group G is called *amenable* if the C^*-algebra $l^\infty(G)$ of all bounded complex-valued functions on G with sup-norm has an invariant mean, that is, a positive linear functional ρ of norm 1 such that

$$\rho(f) = \rho({}_s f) = \rho(f_s) \quad \text{for all} \;\; s \in G \quad \text{and all} \;\; f \in l^\infty(G),$$

where $({}_s f)(g) = f(s^{-1} g)$, $(f_s)(g) = f(gs)$, $g \in G$.

Let $\mathcal{D} := \operatorname{alg}(\mathcal{A}, u_G)$ be the minimal C^*-algebra containing the C^*-algebra $\mathcal{A}$ and the group u_G. By virtue of (A1), $\mathcal{D}$ is the closure of the set $\mathcal{D}^0$ of elements $d = \sum a_g u_g$, where $a_g \in \mathcal{A}$ and g runs through finite subsets of G.

Let $M := M(\mathcal{A})$ be the maximal ideal space of the commutative C^*-algebra $\mathcal{A}$. By the Gelfand–Naimark theorem [17, § 16], $\mathcal{A} \cong C(M)$. Under assumption (A1), identifying characters φ_m of the C^*-algebra $\mathcal{A}$ and the maximal ideals $m = \operatorname{Ker} \varphi_m \in M$, we obtain the homomorphism $g \mapsto \beta_g(\cdot)$ of the group G into the homeomorphism group of M according to the rule

$$a(\beta_g(m)) = (\alpha_g(a))(m), \quad a \in \mathcal{A}, \; m \in M, \; g \in G,$$

where $a(\cdot) \in C(M)$ is the Gelfand transform of the element $a \in \mathcal{A}$. The set $G(m) := \{\beta_g(m) : g \in G\}$ is called the G-orbit of a point $m \in M$.

Suppose that the next version of *topologically free* action of G holds (see [13], [5]):

(A3) *there is a set* $M_0 \subset M(\mathcal{A})$ *such that for every finite set* $G_0 \subset G \setminus \{e\}$ *and every nonempty open set* $V \subset M(\mathcal{A})$ *there exists a point* $m_0 \in V \cap G(M_0)$ *such that* $\beta_g(m_0) \neq m_0$ *for all* $g \in G_0$.

For each $m \in M$, we take the representation $\pi_m : \mathcal{A} \to \mathcal{B}(\mathbb{C})$, $a \mapsto a(m)$. Given $M_0 \subset M$, let $\Omega(M_0)$ be the set of G-orbits of all points $m \in M_0$. Fix a point $m = m_\omega$ in each G-orbit $\omega \in \Omega(M_0)$, and let $l^2(G)$ be the Hilbert space of all functions $f : G \mapsto \mathbb{C}$ such that $f(g) \neq 0$ for at most countable set of points $g \in G$ and $\|f\| := (\sum |f(g)|^2)^{1/2} < \infty$. For every $\omega \in \Omega(M_0)$ we consider the representation $\pi_\omega : \mathcal{D} \to \mathcal{B}(l^2(G))$ defined by

$$[\pi_\omega(a) f](g) = \pi_m(\alpha_g(a)) f(g), \quad [\pi_\omega(u_h) f](g) = f(gh)$$

for all $a \in \mathcal{A}$, all $g, h \in G$ and all $f \in l^2(G)$.

We infer the following nonlocal version of the Allan–Douglas local principle from [14, Theorems 4.1, 4.12] and [5, Theorem 3.1].

Theorem 2.7. *If assumptions* (A1)–(A3) *are satisfied, then an element $d \in \mathcal{D}$ is invertible in $\mathcal{D}$ if and only if for every orbit $\omega \in \Omega(M_0)$ the operator $\pi_\omega(d)$ is invertible on the space $l^2(G)$ and, in the case of infinite set $\Omega(M_0)$,*

$$\sup\left\{\|(\pi_\omega(d))^{-1}\|_{\mathcal{B}(l^2(G))} :\ \omega \in \Omega(M_0)\right\} < \infty.$$

3. Main results

Given $t, \tau \in \mathbb{T}$, we define the set

$$Y_{t,\tau} := \{g \in G : g(t) = \tau\}. \tag{3.1}$$

Since $t = 1$ is a common fixed point for all $g \in G$, we conclude that $Y_{1,1} = G$.

Fix $t_0 \in \mathbb{T} \setminus \{1\}$. Then its G-orbit $G(t_0) := \{g(t_0) : g \in G\}$ coincides with $\mathbb{T} \setminus \{1\}$. For each $\tau \in \mathbb{T} \setminus \{1\}$ let us fix a shift $g_\tau \in Y_{t_0,\tau}$ such that $g_{t_0} = e$, the unit of G. Observe that, for every $g \in Y_{t,\tau}$ with $t, \tau \in \mathbb{T} \setminus \{1\}$, we have

$$\widetilde{g}_{t,\tau} := g_t g g_\tau^{-1} = g_\tau^{-1} \circ g \circ g_t \in Y_{t_0,t_0}. \tag{3.2}$$

For each shift $g \in G$, we also define the function $\delta_g : \mathbb{T} \times \mathbb{T} \to \{0, 1\}$ by

$$\delta_g(t,\tau) := \begin{cases} 1 & \text{if } g \in Y_{t,\tau}, \\ 0 & \text{if } g \notin Y_{t,\tau}. \end{cases} \tag{3.3}$$

With the C^*-algebra $\mathfrak{B}$ we associate the Hilbert space

$$\mathcal{H}_{\mathfrak{B}} := \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$$

where

$$\begin{aligned} \mathcal{H}_1 &:= l^2(M_1(SO(\mathbb{T})) \times \mathbb{R},\ \mathbb{C}^2), \\ \mathcal{H}_2 &:= l^2(M_{t_0}(SO(\mathbb{T})) \times \mathbb{R},\ l^2(\mathbb{T} \setminus \{1\}, \mathbb{C}^2)), \\ \mathcal{H}_3 &:= l^2(M_{t_0}(SO(\mathbb{T})) \times \{0,1\},\ l^2(G)). \end{aligned} \tag{3.4}$$

Here $\mathcal{H}_1$ is the non-separable Hilbert space consisting of $\mathbb{C}^2$-valued functions defined on the set $M_1(SO(\mathbb{T})) \times \mathbb{R}$ and having at most countable sets of non-zero values. The norm of a vector function

$$\Phi :\ M_1(SO(\mathbb{T})) \times \mathbb{R} \to \mathbb{C}^2,\quad (\xi, x) \mapsto \Phi(\xi, x) = (\Phi_i(\xi, x))_{i=1}^2$$

in the Hilbert space $\mathcal{H}_1$ is given by

$$\|\Phi\| := \left(\sum_{(\xi,x)\in M_1(SO(\mathbb{T}))\times\mathbb{R}} \ \sum_{i=1}^{2} |\Phi_i(\xi,x)|^2 \right)^{1/2}.$$

Analogously, $\mathcal{H}_2$ and $\mathcal{H}_3$ are non-separable Hilbert spaces consisting, respectively, of $l^2(\mathbb{T}\setminus\{1\}, \mathbb{C}^2)$-valued functions defined on the set $M_{t_0}(SO(\mathbb{T})) \times \mathbb{R}$ and of $l^2(G)$-valued functions defined on the set $M_{t_0}(SO(\mathbb{T})) \times \{0, 1\}$, and all these functions have at most countable sets of non-zero values. In its turn, $l^2(\mathbb{T} \setminus \{1\}, \mathbb{C}^2)$ is the non-separable Hilbert space consisting of all vectors $f = (f_\tau)_{\tau\in\mathbb{T}\setminus\{1\}}$ with at most countable sets of non-zero entries $f_\tau = (f_{\tau,i})_{i=1}^2 \in \mathbb{C}^2$ and the norm

$$\|f\| := \left(\sum_{\tau\in\mathbb{T}\setminus\{1\}} \sum_{i=1}^{2} |f_{\tau,i}|^2 \right)^{1/2}.$$

Thus, the norm of vector functions

$$\Theta : \ M_{t_0}(SO(\mathbb{T})) \times \mathbb{R} \to l^2(\mathbb{T}\setminus\{1\}, \mathbb{C}^2), \quad (\xi, x) \mapsto \Theta(\xi, x) = (\Theta_\tau(\xi, x))_{\tau\in\mathbb{T}\setminus\{1\}}$$

and

$$\Delta : \ M_{t_0}(SO(\mathbb{T})) \times \{0,1\} \to l^2(G), \quad (\xi, \mu) \mapsto \Delta(\xi, \mu) = (\Delta_g(\xi, \mu))_{g\in G}$$

in the Hilbert spaces $\mathcal{H}_2$ and $\mathcal{H}_3$ are given, respectively, by

$$\begin{aligned} \|\Theta\| &:= \left(\sum_{(\xi,x)\in M_{t_0}(SO(\mathbb{T}))\times\mathbb{R}} \|\Theta(\xi, x)\|^2 \right)^{1/2} \\ &= \left(\sum_{(\xi,x)\in M_{t_0}(SO(\mathbb{T}))\times\mathbb{R}} \ \sum_{\tau\in\mathbb{T}\setminus\{1\}} \sum_{i=1}^{2} |\Theta_{\tau,i}(\xi, x)|^2 \right)^{1/2} \end{aligned}$$

and

$$\begin{aligned} \|\Delta\| &:= \left(\sum_{(\xi,\mu)\in M_{t_0}(SO(\mathbb{T}))\times\{0,1\}} \|\Delta(\xi, \mu)\|^2 \right)^{1/2} \\ &= \left(\sum_{(\xi,\mu)\in M_{t_0}(SO(\mathbb{T}))\times\{0,1\}} \ \sum_{g\in G} |\Delta_g(\xi, \mu)|^2 \right)^{1/2}. \end{aligned}$$

We now construct a representation

$$\Psi_{\mathfrak{B}} : \mathfrak{B} \to \mathcal{B}(\mathcal{H}_{\mathfrak{B}}), \quad B \mapsto \Psi_1(B) \oplus \Psi_2(B) \oplus \Psi_3(B) \tag{3.5}$$

of the C^*-algebra $\mathfrak{B}$ on the Hilbert space $\mathcal{H}_{\mathfrak{B}} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$. The values $\Psi_{\mathfrak{B}}(B)$ for $B \in \mathfrak{B}$ are bounded linear operators acting on the space $\mathcal{H}_{\mathfrak{B}}$. A Fredholm criterion for the operators $B \in \mathfrak{B}$ will be described in terms of invertibility of the operators $\Psi_{\mathfrak{B}}(B)$ on the space $\mathcal{H}_{\mathfrak{B}}$. Hence, the representation $\Psi_{\mathfrak{B}} : \mathfrak{B} \to \mathcal{B}(\mathcal{H}_{\mathfrak{B}})$ can be referred to as the *Fredholm symbol map* for the C^*-algebra $\mathfrak{B}$. The representation $\Psi_{\mathfrak{B}}$ can be considered as the direct sum of the following three C^*-algebra homomorphisms

$$\begin{aligned} \Psi_1 : \ \mathfrak{B} &\to \mathcal{B}(\mathcal{H}_1), \quad B \mapsto \Psi_1(B) = \mathrm{Sym}_1(B)I, \\ \Psi_2 : \ \mathfrak{B} &\to \mathcal{B}(\mathcal{H}_2), \quad B \mapsto \Psi_2(B) = \mathrm{Sym}_2(B)I, \\ \Psi_3 : \ \mathfrak{B} &\to \mathcal{B}(\mathcal{H}_3), \quad B \mapsto \Psi_3(B) = \mathrm{Sym}_3(B)I, \end{aligned} \tag{3.6}$$

defined initially on the generators of the C^*-algebra $\mathfrak{B}$.

In (3.6), $\Psi_1(B)$ are operators of multiplication by 2×2 matrix functions $\mathrm{Sym}_1(B): M_1(SO(\mathbb{T})) \times \mathbb{R} \to \mathbb{C}^{2\times 2}$ whose values at the points $(\xi,x) \in M_1(SO(\mathbb{T})) \times \mathbb{R}$ are defined on the generators of the C^*-algebra $\mathfrak{B}$ by

$$
\begin{aligned}
[\mathrm{Sym}_1(aI)](\xi,x) &:= \begin{pmatrix} a(\xi,1) & 0 \\ 0 & a(\xi,0) \end{pmatrix}, \\
[\mathrm{Sym}_1(S_{\mathbb{T}})](\xi,x) &:= \begin{pmatrix} \tanh(\pi x) & i/\cosh(\pi x) \\ -i/\cosh(\pi x) & -\tanh(\pi x) \end{pmatrix}, \\
[\mathrm{Sym}_1(U_g)](\xi,x) &:= \begin{pmatrix} e^{ixk_g} & 0 \\ 0 & e^{ixk_g} \end{pmatrix}, \\
[\mathrm{Sym}_1(U_\gamma)](\xi,x) &:= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\end{aligned} \tag{3.7}
$$

where $a \in PSO(\mathbb{T})$, $g \in G_0$, $k_g := \ln g'(1)$ with $g'(1) > 0$.

Further, $\Psi_2(B)$ are operators of multiplication by infinite matrix functions $\mathrm{Sym}_2(B)$ given on $M_{t_0}(SO(\mathbb{T})) \times \mathbb{R}$, where the values of these matrix functions at the points $(\xi,x) \in M_{t_0}(SO(\mathbb{T})) \times \mathbb{R}$ define bounded linear operators on the Hilbert space $l^2(\mathbb{T} \setminus \{1\}, \mathbb{C}^2)$ and are given on the generators of the C^*-algebra $\mathfrak{B}$ by

$$
\begin{aligned}
[\mathrm{Sym}_2(aI)](\xi,x) &:= \mathrm{diag}\left\{ \begin{pmatrix} (a\circ g_t)(\xi,1) & 0 \\ 0 & (a\circ g_t)(\xi,0) \end{pmatrix} \right\}_{t\in\mathbb{T}\setminus\{1\}}, \\
[\mathrm{Sym}_2(S_{\mathbb{T}})](\xi,x) &:= \mathrm{diag}\left\{ \begin{pmatrix} \tanh(\pi x) & i/\cosh(\pi x) \\ -i/\cosh(\pi x) & -\tanh(\pi x) \end{pmatrix} \right\}_{t\in\mathbb{T}\setminus\{1\}}, \\
[\mathrm{Sym}_2(U_g)](\xi,x) &:= \left(\begin{pmatrix} \delta_g(t,\tau)e^{ixk_{g,t,\tau}} & 0 \\ 0 & \delta_g(t,\tau)e^{ixk_{g,t,\tau}} \end{pmatrix} \right)_{t,\tau\in\mathbb{T}\setminus\{1\}}, \\
[\mathrm{Sym}_2(U_\gamma)](\xi,x) &:= \left(\begin{pmatrix} 0 & \delta_\gamma(t,\tau) \\ \delta_\gamma(t,\tau) & 0 \end{pmatrix} \right)_{t,\tau\in\mathbb{T}\setminus\{1\}},
\end{aligned} \tag{3.8}
$$

where $a \in PSO(\mathbb{T})$, $g \in G_0$, $k_{g,t,\tau} := \ln \widetilde{g}'_{t,\tau}(t_0)$ with $\widetilde{g}_{t,\tau}$ defined by (3.2) and $\widetilde{g}'_{t,\tau}(t_0) > 0$, and the function δ_h is given by (3.3) for $h \in G$.

Finally, $\Psi_3(B)$ are operators of multiplication by infinite matrix functions $\mathrm{Sym}_3(B)$ given on $M_{t_0}(SO(\mathbb{T})) \times \{0,1\}$, where the values of these matrix functions at the points $(\xi,\mu) \in M_{t_0}(SO(\mathbb{T})) \times \{0,1\}$ define bounded linear operators on the space $l^2(G)$ and are given on the generators of the C^*-algebra $\mathfrak{B}$ as follows:

$$
\begin{aligned}
[\mathrm{Sym}_3(aI)](\xi,\mu) &:= \mathrm{diag}\big\{(a\circ h)(\xi,\mu)\big\}_{h\in G}, \\
[\mathrm{Sym}_3(S_{\mathbb{T}})](\xi,\mu) &:= \mathrm{diag}\big\{\theta_h\big\}_{h\in G}, \\
[\mathrm{Sym}_3(U_g)](\xi,\mu) &:= \big(\delta_{hg,s}\big)_{h,s\in G},
\end{aligned} \tag{3.9}
$$

where $a \in PSO(\mathbb{T})$, $g \in G$, $\theta_h = 1$ if $h \in G$ preserves the orientation on $\mathbb{T}$, $\theta_h = -1$ if $h \in G$ changes the orientation on $\mathbb{T}$, and $\delta_{h,s}$ is the Kronecker symbol on G.

Given $a \in PSO(\mathbb{T})$, we also note that

$$(a \circ h)(\xi, \mu) = \begin{cases} a(h(\xi), \mu) & \text{if} \quad h \in G_0, \\ a(h(\xi), 1-\mu) & \text{if} \quad h \in G_0\gamma, \end{cases}$$

where the map $\xi \mapsto g(\xi)$ on $M(SO(\mathbb{T}))$ is given by (2.14).

We will prove below the following main results of the paper.

Theorem 3.1. *The map $\Psi_{\mathfrak{B}}$ defined on the generators of the C^*-algebra $\mathfrak{B}$ by formulas* (3.5)–(3.9) *extends to a C^*-algebra homomorphism of $\mathfrak{B}$ into the C^*-algebra $\mathcal{B}(\mathcal{H}_{\mathfrak{B}})$ such that $\|\Psi_{\mathfrak{B}}(B)\| \le |B|$ for all $B \in \mathfrak{B}$, and* $\operatorname{Ker}\Psi_{\mathfrak{B}}$ *coincides with the ideal $\mathcal{K}(L^2(\mathbb{T}))$ of all compact operators in the C^*-algebra $\mathcal{B}(L^2(\mathbb{T}))$.*

Theorem 3.2. *An operator $B \in \mathfrak{B}$ is Fredholm on the space $L^2(\mathbb{T})$ if and only if the operator $\Psi_{\mathfrak{B}}(B)$ is invertible on the space $\mathcal{H}_{\mathfrak{B}}$, that is, if the following three conditions hold:*

(i) *for every $(\xi, x) \in M_1(SO(\mathbb{T})) \times \mathbb{R}$ the matrix $[\mathrm{Sym}_1(B)](\xi, x)$ is invertible and*

$$\inf_{(\xi,x)\in M_1(SO(\mathbb{T}))\times\mathbb{R}} \big|\det\big([\mathrm{Sym}_1(B)](\xi,x)\big)\big| > 0;$$

(ii) *for every $(\xi, x) \in M_{t_0}(SO(\mathbb{T})) \times \mathbb{R}$ the operator $[\mathrm{Sym}_2(B)](\xi, x)I$ is invertible on the Hilbert space $l^2(\mathbb{T}\setminus\{1\}, \mathbb{C}^2)$ and*

$$\sup_{(\xi,x)\in M_{t_0}(SO(\mathbb{T}))\times\mathbb{R}} \big\|\big([\mathrm{Sym}_2(B)](\xi,x)I\big)^{-1}\big\|_{\mathcal{B}(l^2(\mathbb{T}\setminus\{1\},\mathbb{C}^2))} < \infty;$$

(iii) *for every $(\xi, \mu) \in M_{t_0}(SO(\mathbb{T})) \times \{0, 1\}$ the operator $[\mathrm{Sym}_3(B)](\xi, \mu)I$ is invertible on the Hilbert space $l^2(G)$ and*

$$\sup_{(\xi,\mu)\in M_{t_0}(SO(\mathbb{T}))\times\{0,1\}} \big\|\big([\mathrm{Sym}_3(B)](\xi,\mu)I\big)^{-1}\big\|_{\mathcal{B}(l^2(G))} < \infty.$$

By Theorems 3.1 and 3.2, the representation (3.5) is a map assigning a Fredholm symbol to every operator $B \in \mathfrak{B}$, and $\Psi_{\mathfrak{B}}(\mathfrak{B})$ is the C^*-algebra of the Fredholm symbols for the operators $B \in \mathfrak{B}$.

4. Invertibility in the C^*-algebra $\mathfrak{B}/\mathfrak{H}$. The homomorphism Ψ_3

Let $\mathfrak{A}^0$ and $\mathfrak{B}^0$ be the non-closed algebras consisting of operators of the form

$$\sum_{i=1}^{n} T_{i,1}T_{i,2}\dots T_{i,j_i} \quad (n, j_i \in \mathbb{N}) \tag{4.1}$$

where $T_{i,k}$ are, respectively, the generators aI $(a \in PSO^0(\mathbb{T}))$ and $S_{\mathbb{T}}$ of the C^*-algebras $\mathfrak{A}$ and the generators aI $(a \in PSO^0(\mathbb{T}))$, $S_{\mathbb{T}}$ and U_g $(g \in G)$ of the C^*-algebra $\mathfrak{B}$. Clearly, $\mathfrak{A}^0$ is a dense subalgebra of $\mathfrak{A}$, $\mathfrak{B}^0$ is a dense subalgebra of $\mathfrak{B}$.

Let $\mathfrak{H}$ be the closed two-sided ideal of $\mathfrak{B}$ generated by all commutators $[aI, S_{\mathbb{T}}] = aS_{\mathbb{T}} - S_{\mathbb{T}}aI$, where $a \in PSO^0(\mathbb{T})$. Then $\mathfrak{H}$ is the closure of the set

$$\mathfrak{H}^0 := \left\{ \sum_{i=1}^{n} B_i H_i C_i : B_i, C_i \in \mathfrak{B}^0,\ H_i = [a_i I, S_{\mathbb{T}}],\ a_i \in PSO^0(\mathbb{T}),\ n \in \mathbb{N} \right\}. \tag{4.2}$$

As is known (see, e.g., [10]), the ideal $\mathcal{K}$ of all compact operators on the space $L^2(\mathbb{T})$ is contained in $\mathfrak{H}$.

Fix $t_0 \in \mathbb{T} \setminus \{1\}$ and let $\mathfrak{H}_1$ and $\mathfrak{H}_2$ be the closed two-sided ideals of the C^*-algebra $\mathfrak{B}$ generated, respectively, by V_1 and $\mathcal{K}$ and by V_{t_0} and $\mathcal{K}$, where V_1 and V_{t_0} are the operators with fixed singularities defined by (2.6). Let $\mathfrak{H}_1^\pi := \mathfrak{H}_1/\mathcal{K}$ and $\mathfrak{H}_2^\pi := \mathfrak{H}_2/\mathcal{K}$. Similarly to [16, Lemma 5.4] and [5, Lemma 10.4] we have the following characterization of the ideal $\mathfrak{H}^\pi := \mathfrak{H}/\mathcal{K}$.

Lemma 4.1. *Every coset $H^\pi \in \mathfrak{H}^\pi$ can be written as $H^\pi = H_1^\pi + H_2^\pi$, where the cosets $H_1^\pi \in \mathfrak{H}_1^\pi$ and $H_2^\pi \in \mathfrak{H}_2^\pi$ have the form*

$$\begin{aligned} H_1^\pi &= \lim_{n\to\infty} (A_{n,0}^\pi V_1^\pi + A_{n,1}^\pi V_1^\pi U_\gamma^\pi), \\ H_2^\pi &= \lim_{n\to\infty} \sum_{t\in\mathbb{T}_n} \sum_{g\in F_n} \left(A_{t,g,n,0}^\pi V_t^\pi U_g^\pi + A_{t,g,n,1}^\pi V_t^\pi U_g^\pi U_\gamma^\pi \right), \end{aligned} \tag{4.3}$$

with $A_{n,0}$, $A_{n,1}$, $A_{t,g,n,0}$, $A_{t,g,n,1} \in \mathfrak{A}^0$, and $\mathbb{T}_n$ and F_n are finite subsets of $\mathbb{T}\setminus\{1\}$ and G_0, respectively.

Consider the quotient C^*-algebras $\widehat{\mathfrak{B}} := \mathfrak{B}/\mathfrak{H}$ and $\widehat{\mathfrak{A}} := (\mathfrak{A} + \mathfrak{H})/\mathfrak{H}$, where $\mathfrak{A}$ is the C^*-algebra given by (1.7). By [5, Theorem 5.2], the C^*-algebra $\widehat{\mathfrak{A}}$ consists of the cosets $\widehat{A} = a^+P_{\mathbb{T}}^+ + a^-P_{\mathbb{T}}^- + \mathfrak{H}$, where $a^\pm \in PSO(\mathbb{T})$, and is obviously commutative. Its maximal ideal space $M(\widehat{\mathfrak{A}})$ is homeomorphic to the compact set

$$\mathfrak{N} := M(SO(\mathbb{T})) \times \{0,1\} \times \{\pm\infty\}, \tag{4.4}$$

and hence $\widehat{\mathfrak{A}} \cong C(\mathfrak{N})$. The Gelfand topology on $\mathfrak{N}$ is defined as follows. If $t \in \mathbb{T}$ and $\xi \in M_t(SO(\mathbb{T}))$, a base of neighborhoods of the point $(\xi, \mu, x) \in \mathfrak{N}$ consists of all open sets

$$U_{(\xi,\mu,x)} = \begin{cases} \left(U_{\xi,t} \times \{0\} \times \{x\}\right) \cup \left(U_{\xi,t}^- \times \{0,1\} \times \{x\}\right) & \text{if } \mu = 0, \\ \left(U_{\xi,t} \times \{1\} \times \{x\}\right) \cup \left(U_{\xi,t}^+ \times \{0,1\} \times \{x\}\right) & \text{if } \mu = 1, \end{cases} \tag{4.5}$$

where $U_{\xi,t} = U_\xi \cap M_t(SO(\mathbb{T}))$, U_ξ is an open neighborhood of $\xi \in M(SO(\mathbb{T}))$, and $U_{\xi,t}^-$, $U_{\xi,t}^+$ consists of all $\zeta \in U_\xi$ such that $\tau = \zeta|_{C(\mathbb{T})}$ belong, respectively, to the open arcs $(te^{-i\varepsilon}, t)$ and $(t, te^{i\varepsilon})$ of $\mathbb{T}$ for some $\varepsilon \in (0, 2\pi)$. The Gelfand transform of cosets $\widehat{A} = a^+P_{\mathbb{T}}^+ + a^-P_{\mathbb{T}}^- + \mathfrak{H} \in \widehat{\mathfrak{A}}$ with $a^\pm \in PSO(\mathbb{T})$ is given for $(\xi, \mu, x) \in \mathfrak{N}$ by

$$\widehat{A}(\xi, \mu, x) := a^+(\xi, \mu)\mathcal{P}_+(x) + a^-(\xi, \mu)\mathcal{P}_-(x), \tag{4.6}$$

where

$$\mathcal{P}_\pm(x) = [1 \pm \tanh(\pi x)]/2 \quad \text{for all } x \in \overline{\mathbb{R}}.$$

Applying the local-trajectory method we will obtain here an invertibility criterion for the cosets $B+\mathfrak{H}$ $(B \in \mathfrak{B})$ in the C^*-algebra $\widehat{\mathfrak{B}}$. As a consequence we will construct the *-homomorphism Ψ_3 defined on the generators of $\mathfrak{B}$ by (3.9).

By (2.12) and (2.13), we conclude that for every $g \in G$ the map

$$\widehat{\alpha}_g : \widehat{\mathfrak{A}} \to \widehat{\mathfrak{A}}, \quad A+\mathfrak{H} \mapsto U_g(A+\mathfrak{H})U_g^{-1}$$

is a *-automorphism of the commutative C^*-algebra $\widehat{\mathfrak{A}}$. Indeed, taking $a^\pm \in PSO(\mathbb{T})$ we obtain

$$U_g(a^+P_{\mathbb{T}}^+ + a^-P_{\mathbb{T}}^- + \mathfrak{H})U_g^{-1} = (a^+\circ g)P_{\mathbb{T}}^+ + (a^-\circ g)P_{\mathbb{T}}^- + \mathfrak{H} \quad (g \in G_0),$$
$$U_g(a^+P_{\mathbb{T}}^+ + a^-P_{\mathbb{T}}^- + \mathfrak{H})U_g^{-1} = (a^+\circ g)P_{\mathbb{T}}^- + (a^-\circ g)P_{\mathbb{T}}^+ + \mathfrak{H} \quad (g \in G_0\gamma),$$

where $(a^\pm \circ g)(\xi,\mu) = a^\pm(g(\xi), 1-\mu)$ for all $(\xi,\mu) \in M(PSO(\mathbb{T}))$ and all $g \in G_0\gamma$. Hence the C^*-algebra $\widehat{\mathfrak{B}}$ is the closure of the algebra $\widehat{\mathfrak{B}}^0$ consisting of the cosets $\sum_{g\in F} A_gU_g + \mathfrak{H}$, where $A_g \in \mathfrak{A}^0$ and g runs through finite subsets $F \subset G$. For each $g \in G$, the *-automorphism $\widehat{\alpha}_g$ of the C^*-algebra $\widehat{\mathfrak{A}}$ induces on the maximal ideal space $\mathfrak{N}$ defined by (4.4) the homeomorphism given by

$$\begin{aligned} \widehat{\beta}_g &: (\xi,\mu,x) \mapsto (g(\xi),\mu,x) \quad \text{for all} \ \ g \in G_0, \\ \widehat{\beta}_g &: (\xi,\mu,x) \mapsto (g(\xi),1-\mu,-x) \quad \text{for all} \ \ g \in G_0\gamma, \end{aligned} \tag{4.7}$$

according to the rule

$$\widehat{A}\big[\widehat{\beta}_g(\xi,\mu,x)\big] = \big[\widehat{\alpha}_g(\widehat{A})\big](\xi,\mu,x), \quad \widehat{A} \in \widehat{\mathfrak{A}}, \quad (\xi,\mu,x) \in \mathfrak{N}, \quad g \in G,$$

where $\widehat{A}(\cdot,\cdot,\cdot) \in C(\mathfrak{N})$ is the Gelfand transform of a coset $\widehat{A} \in \widehat{\mathfrak{A}}$ (see (4.6)).

Since the homeomorphism $\xi \mapsto g(\xi)$ given by (2.14) sends the fibers $M_t(SO(\mathbb{T}))$ onto the fibers $M_{g(t)}(SO(\mathbb{T}))$ for all $t \in \mathbb{T}$, it follows from the proof of [5, Theorem 6.4] that $g(\xi) = \xi$ for every $\xi \in M_t(SO(\mathbb{T}))$ and every $t \in \mathbb{T}_g$, where $\mathbb{T}_g$ is the set (1.5) of all fixed points of g on $\mathbb{T}$. This in view of (4.7) gives the following.

Lemma 4.2. *For each $g \in G_0$, $\mathfrak{N}_g := \bigcup_{t\in\mathbb{T}_g} \big(M_t(SO(\mathbb{T})) \times \{0,1\} \times \{\pm\infty\}\big)$ is the set of all fixed points of $\widehat{\beta}_g$ on $\mathfrak{N}$. For all $g \in G_0\gamma$ the homeomorphisms $\widehat{\beta}_g$ do not have fixed points on $\mathfrak{N}$.*

Since G acts topologically freely on $\mathbb{T}$, we easily deduce from Lemma 4.2 and the Gelfand topology (4.5) on $\mathfrak{N}$ that the group G acts topologically freely on $\mathfrak{N}$ as well. Moreover, since the set

$$\mathfrak{N}_0 := \bigcup_{t\in\mathbb{T}\setminus\{1\}} \big(M_t(SO(\mathbb{T})) \times \{0,1\} \times \{\pm\infty\}\big)$$

is dense in $\mathfrak{N}$, we see that for every nonempty open set $W \subset \mathfrak{N}$ and every finite set $G_0 \subset G$ there exists a point $(\xi_0,\mu_0,x_0) \in W \cap \mathfrak{N}_0$ such that $\widehat{\beta}_g(\xi_0,\mu_0,x_0) \neq (\xi_0,\mu_0,x_0)$ for all $g \in G_0 \setminus \{e\}$. Due to this fact and the amenability of the solvable group G, we infer that all conditions of the local-trajectory method (see Subsection 2.4) for the C^*-algebra $\widehat{\mathfrak{B}}$ having the form (1.9) are fulfilled.

For $t_0 \in \mathbb{T} \setminus \{1\}$ the G-orbit $G(t_0) := \{g(t_0) : g \in G\}$ of t_0 on $\mathbb{T}$ coincides with $\mathbb{T} \setminus \{1\}$. From (2.14) and (4.7) it follows that the set

$$\Omega_{t_0} := M_{t_0}(SO(\mathbb{T})) \times \{0,1\} \times \{+\infty\} \tag{4.8}$$

contains exactly one point in each G-orbit of every point in $\mathfrak{N}_0$. Consider the Hilbert space $l^2(G)$ consisting of all complex-valued functions defined on G and having at most countable sets of non-zero values, and with every point $(\xi, \mu, x) \in \Omega_{t_0}$ we associate the representation

$$\Pi_{\xi,\mu,x} : \widehat{\mathfrak{B}} \to \mathcal{B}(l^2(G)),\ \widehat{B} := B + \mathfrak{H} \mapsto \widehat{B}_{\xi,\mu,x} \tag{4.9}$$

given on the coset $\widehat{B} = \sum_{g\in F} \widehat{A}_g \widehat{U}_g \in \widehat{\mathfrak{B}}^0$, where F is a finite subset of G, $\widehat{A}_g \in \widehat{\mathfrak{A}}^0$ and $\widehat{U}_g = U_g + \mathfrak{H}$, by

$$\big(\widehat{B}_{\xi,\mu,x} f\big)(h) = \sum_{g\in F} \Big(\big[\widehat{\alpha}_h(\widehat{A}_g)\big](\xi,\mu,x) \Big) f(hg) \tag{4.10}$$

for all $f \in l^2(G)$ and all $h \in G$. Then we immediately obtain the following invertibility criterion from Theorem 2.7.

Theorem 4.3. *A coset $\widehat{B} \in \widehat{\mathfrak{B}}$ is invertible in the quotient C^*-algebra $\widehat{\mathfrak{B}} = \mathfrak{B}/\mathfrak{H}$ if and only if the operators $\widehat{B}_{\xi,\mu,x}$ are invertible on the space $l^2(G)$ for all $(\xi,\mu,x) \in \Omega_{t_0}$ and*

$$\sup_{(\xi,\mu,x)\in\Omega_{t_0}} \Big\| \big(\widehat{B}_{\xi,\mu,x}\big)^{-1} \Big\|_{\mathcal{B}(l^2(G))} < \infty.$$

Applying Theorem 4.3 to the operator $\widehat{B}\widehat{B}^* \in \widehat{\mathfrak{B}}$ and using the relation $\|\widehat{B}\| = \|\widehat{B}\widehat{B}^*\|^{1/2} = \big[r\big(\widehat{B}\widehat{B}^*\big)\big]^{1/2}$, where $r\big(\widehat{B}\widehat{B}^*\big)$ is the spectral radius of the operator $\widehat{B}\widehat{B}^*$, we conclude that

$$\begin{aligned} \|\widehat{B}\| = \big[r\big(\widehat{B}\widehat{B}^*\big)\big]^{1/2} &= \sup_{(\xi,\mu,x)\in\Omega_{t_0}} \big[r\big(\widehat{B}_{\xi,\mu,x}\widehat{B}^*_{\xi,\mu,x}\big)\big]^{1/2}, \\ &= \sup_{(\xi,\mu,x)\in\Omega_{t_0}} \|\widehat{B}_{\xi,\mu,x}\|_{\mathcal{B}(l^2(G))}, \end{aligned}$$

whence we obtain the following.

Corollary 4.4. *The representation*

$$\bigoplus_{(\xi,\mu,x)\in\Omega_{t_0}} \Pi_{\xi,\mu,x} : \widehat{\mathfrak{B}} \to \mathcal{B}\Bigg(\bigoplus_{(\xi,\mu,x)\in\Omega_{t_0}} l^2(G) \Bigg), \tag{4.11}$$

where $\Pi_{\xi,\mu,x}$ and Ω_{t_0} are given by (4.9)–(4.10) *and* (4.8), *respectively, is an isometric C^*-algebra homomorphism.*

The Hilbert space $\mathcal{H}_3 := l^2(M_{t_0}(SO(\mathbb{T})) \times \{0,1\}, l^2(G))$ introduced in (3.4) is isometrically isomorphic to $\bigoplus_{(\xi,\mu,x)\in\Omega_{t_0}} l^2(G)$. Identifying these two Hilbert spaces, we easily conclude that the algebraic *-homomorphisms

$$\Psi_3 : \mathfrak{B}^0 \to \mathcal{B}(\mathcal{H}_3), \quad B \mapsto \Psi_3(B) = \mathrm{Sym}_3(B)I,$$

defined initially on the dense subalgebra $\mathfrak{B}^0$ of $\mathfrak{B}$ by (3.9), coincides with representation (4.11). Consequently, Ψ_3 admits a continuous extension to the C^*-algebra $\mathfrak{B}$ whose kernel is the ideal $\mathfrak{H}$. Thus, we get the following.

Theorem 4.5. *The algebraic *-homomorphisms $\Psi_3 : \mathfrak{B}^0 \to \mathcal{B}(\mathcal{H}_3)$ defined on the generators of the C^*-algebra $\mathfrak{B}$ by formulas* (3.6) *and* (3.9)*, extends by continuity to a representations $\Psi_3 : \mathfrak{B} \to \mathcal{B}(\mathcal{H}_3)$ such that*

$$\|\Psi_3(B)\| = \|\widehat{B}\| \le |B| \quad \textit{for all} \;\; B \in \mathfrak{B} \;\; \textit{and hence} \;\; \mathrm{Ker}\,\Psi_3 = \mathfrak{H} \supset \mathcal{K}.$$

For each operator $B \in \mathfrak{B}$, Corollary 4.4 gives the following invertibility criterion for the coset $\widehat{B}$ in terms of Ψ_3.

Corollary 4.6. *Given an operator $B \in \mathfrak{B}$, the coset $\widehat{B} := B + \mathfrak{H}$ is invertible in the quotient C^*-algebra $\widehat{\mathfrak{B}} := \mathfrak{B}/\mathfrak{H}$ if and only if the operator $\Psi_3(B)$ defined by* (3.9) *is invertible on the Hilbert space $\mathcal{H}_3$, that is, if for every $(\xi,\mu) \in M_{t_0}(SO(\mathbb{T})) \times \{0,1\}$ the operator $[\mathrm{Sym}_3(B)](\xi,\mu)I$ is invertible on the Hilbert space $l^2(G)$ and*

$$\sup_{(\xi,\mu)\in M_{t_0}(SO(\mathbb{T}))\times\{0,1\}} \big\|\big([\mathrm{Sym}_3(B)](\xi,\mu)I\big)^{-1}\big\|_{\mathcal{B}(l^2(G))} < \infty.$$

Let, for example, the coset $\widehat{B} \in \widehat{\mathfrak{B}}$ be given by

$$\widehat{B} = \sum_{g\in F} \big[(a_g^+ U_g + c_g^+ U_g U_\gamma)P_{\mathbb{T}}^+ + (a_g^- U_g + c_g^- U_g U_\gamma)P_{\mathbb{T}}^-\big] + \mathfrak{H},$$

where $a_g^\pm, c_g^\pm \in PSO(\mathbb{T})$ and F is a finite subset of the subgroup G_0 of G. Then $\Psi_3(B) = \mathrm{Sym}_3(B)I$, where the matrix function

$$[\mathrm{Sym}_3(B)](\xi,\mu) = \big(B_{h,s}(\xi,\mu)\big)_{h,s\in G} \quad \big((\xi,\mu) \in M_{t_0}(SO(\mathbb{T})) \times \{0,1\}\big),$$

whose values define the operator $\widehat{B}_{\xi,\mu,+\infty} \in \mathcal{B}(l^2(G))$, has the entries

$$\begin{aligned} B_{h,s}(\xi,\mu) = \sum_{g\in F} \Big(&\big[(a_g^+ \circ h)(\xi,\mu)\mathcal{P}_h^+(+\infty) + (a_g^- \circ h)(\xi,\mu)\mathcal{P}_h^-(+\infty)\big]\delta_{hg,s} \\ &+ \big[(c_g^+ \circ h)(\xi,\mu)\mathcal{P}_h^-(+\infty) + (c_g^- \circ h)(\xi,\mu)\mathcal{P}_h^+(+\infty)\big]\delta_{hg\gamma,s}\Big), \end{aligned}$$

and $\mathcal{P}_h^\pm(+\infty) = [1 \pm \theta_h]/2$. Consequently,

$$B_{h,s}(\xi,1) = \begin{cases} \sum_{g\in F} \Big(a_g^+(h(\xi),1)\delta_{hg,s} + c_g^-(h(\xi),1)\delta_{hg\gamma,s}\Big) & \text{if } \; h \in G_0, \\ \sum_{g\in F} \Big(a_g^-(h(\xi),0)\delta_{hg,s} + c_g^+(h(\xi),0)\delta_{hg\gamma,s}\Big) & \text{if } \; h \in G_0\gamma, \end{cases}$$

$$B_{h,s}(\xi,0) = \begin{cases} \sum_{g\in F} \Big(a_g^+(h(\xi),0)\delta_{hg,s} + c_g^-(h(\xi),0)\delta_{hg\gamma,s}\Big) & \text{if } \; h \in G_0, \\ \sum_{g\in F} \Big(a_g^-(h(\xi),1)\delta_{hg,s} + c_g^+(h(\xi),1)\delta_{hg\gamma,s}\Big) & \text{if } \; h \in G_0\gamma. \end{cases}$$

5. The homomorphisms Ψ_1 and Ψ_2

Fix $t_0 \in \mathbb{T} \setminus \{1\}$. Obviously, the set of G-orbits of points $t \in \mathbb{T}$ consists of only two G-orbits: the one-point orbit $G(1) = \{1\}$ and the non-countable orbit $\omega := G(t_0)$.

Consider the dense subalgebra $\mathfrak{B}^0$ of $\mathfrak{B}$ composed by operators of the form (4.1), where $T_{i,k}$ are the multiplication operators aI with $a \in PSO^0(\mathbb{T})$ or the Cauchy singular integral operator $S_\mathbb{T}$, or the unitary shift operators U_g with $g \in G$. Due to relations (2.12)–(2.13) and the equality $g = \widetilde{g}\gamma$ with $\widetilde{g} \in G_0$ for every $g \in G \setminus G_0$, the operators $N \in \mathfrak{B}^0$ can be written in the form

$$N = \sum_{g \in F_1} A_g^{(1)} U_g + \sum_{g \in F_2} A_g^{(2)} U_g U_\gamma, \tag{5.1}$$

where F_1, F_2 are finite subsets of G_0 and the operators $A_g^{(1)}, A_g^{(2)}$ belong to the dense subalgebra $\mathfrak{A}^0$ of $\mathfrak{A}$ generated by the multiplication operators aI with $a \in PSO^0(\mathbb{T})$ and by the Cauchy singular integral operator $S_\mathbb{T}$.

For the algebra $\mathfrak{B}^0$ we introduce the two algebraic *-homomorphisms

$$\begin{aligned} &\Psi_1 : \mathfrak{B}^0 \to \mathcal{B}(\mathcal{H}_1),\ \Psi_1(B) = \mathrm{Sym}_1(B)I, \\ &\Psi_2 : \mathfrak{B}^0 \to \mathcal{B}(\mathcal{H}_2),\ \Psi_2(B) = \mathrm{Sym}_2(B)I, \end{aligned} \tag{5.2}$$

where the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ are defined in (3.4) and the finite and infinite matrix-valued functions

$$(\xi, x) \mapsto [\mathrm{Sym}_1(B)](\xi, x), \quad (\xi, x) \mapsto [\mathrm{Sym}_2(B)](\xi, x)$$

are defined on the generators of the C^*-algebra $\mathfrak{B}$ by (3.7) and (3.8), respectively.

Given any set $\Gamma \subset \mathbb{T}$, we define the sets

$$M_\Gamma(SO(\mathbb{T})) := \bigcup_{t \in \Gamma} M_t(SO(\mathbb{T})), \quad \mathfrak{M}_\Gamma^\circ := M_\Gamma(SO(\mathbb{T})) \times \mathbb{R}. \tag{5.3}$$

Observe that from (5.3) and (2.15) it follows that $\mathfrak{M}_{\{t\}}^\circ = \mathfrak{M}_t^\circ$ for every $t \in \mathbb{T}$. For any finite set $\Gamma \subset \mathbb{T}$ we introduce the operator

$$V_\Gamma := \sum_{t \in \Gamma} V_t \in \mathfrak{H}, \tag{5.4}$$

where the operators V_t for $t \in \mathbb{T}$ are given by (2.6) and $\mathfrak{H}$ is the closed two-sided ideal being the closure of the set $\mathfrak{H}^0$ defined in (4.2).

For any set $Y \subset \mathbb{T} \setminus \{1\}$, let

$$\Pi_Y := \mathrm{diag}\,\{E_2 \chi_Y(t)\}_{t \in \mathbb{T} \setminus \{1\}}, \tag{5.5}$$

where $E_2 := \mathrm{diag}\{1, 1\}$ and χ_Y is the characteristic function of the set Y.

Lemma 5.1. *If $N \in \mathfrak{B}^0$, then*

$$|NV_1| = \|\Psi_1(N) vI\|_{\mathcal{B}(\mathcal{H}_1)}, \tag{5.6}$$

where the function v is given by (2.5).

Proof. Fix $N \in \mathfrak{B}^0$ of the form (5.1). Let φ be the isometric *-isomorphism of the quotient C^*-algebra $\mathfrak{B}^\pi = \mathfrak{B}/\mathcal{K}$ on a Hilbert space $\mathcal{H}_\varphi$ considered in (2.10). Then

$$|NV_1| = \|\varphi([NV_1]^\pi)\|_{\mathcal{B}(\mathcal{H}_\varphi)}. \tag{5.7}$$

We deduce from (5.1) and Lemma 2.3 that the operator

$$NV_1 = \sum_{g\in F_1} A_g^{(1)} U_g V_1 + \sum_{g\in F_2} A_g^{(2)} U_g U_\gamma V_1$$

belongs to the C^*-algebra $\mathfrak{A}$, and

$$[\mathrm{Sym}\,(NV_1)](\xi,x) = \sum_{g\in F_1}\big[\mathrm{Sym}\,A_g^{(1)}\big](\xi,x)e^{ixk_g}v(x) + \sum_{g\in F_2}\big[\mathrm{Sym}\,A_g^{(2)}\big](\xi,x)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}e^{ixk_g}v(x) \tag{5.8}$$

if $(\xi,x)\in M_1(SO(\mathbb{T}))\times\overline{\mathbb{R}}$, where $k_g = \ln g'(1)$, and

$$[\mathrm{Sym}\,(NV_1)](\xi,x) = 0_{2\times 2} \ \text{ if } \ (\xi,x)\in\mathfrak{M}\setminus(M_1(SO(\mathbb{T}))\times\overline{\mathbb{R}}). \tag{5.9}$$

Thus, it follows from (5.8) and (3.7) that

$$[\mathrm{Sym}\,(NV_1)](\xi,x) = [\Psi_1(N)](\xi,x)v(x), \quad (\xi,x)\in M_1(SO(\mathbb{T}))\times\overline{\mathbb{R}}.$$

Applying now the spectral projection given by (2.11), we infer from Lemma 2.5 that

$$\|P_\varphi(\mathfrak{M}_1^\circ)\varphi([NV_1]^\pi)\|_{\mathcal{B}(\mathcal{H}_\varphi)} = \|(\mathrm{Sym}\,(NV_1))|_{\mathfrak{M}_1^\circ}I\|_{\mathcal{B}(\mathcal{H}_1)} = \|\Psi_1(N)vI\|_{\mathcal{B}(\mathcal{H}_1)}. \tag{5.10}$$

Taking the open set $M_{\mathbb{T}\setminus\{1\}}(SO(\mathbb{T}))\times\dot{\mathbb{R}}$ and the closed set $M_1(SO(\mathbb{T}))\times\{\infty\}$ in $\dot{\mathfrak{M}}$, we infer from (5.9) by analogy with [5, Subsection 8.1] that

$$P_\varphi(M_{\mathbb{T}\setminus\{1\}}(SO(\mathbb{T}))\times\dot{\mathbb{R}})\,\varphi([NV_1]^\pi) = 0, \tag{5.11}$$

and, by [5, Lemma 10.5],

$$P_\varphi(M_1(SO(\mathbb{T}))\times\{\infty\})\,\varphi([NV_1]^\pi) = 0. \tag{5.12}$$

It follows from the partition

$$\dot{\mathfrak{M}} = \big(M_{\mathbb{T}\setminus\{1\}}(SO(\mathbb{T}))\times\dot{\mathbb{R}}\big)\cup\mathfrak{M}_1^\circ\cup\big(M_1(SO(\mathbb{T}))\times\{\infty\}\big)$$

that

$$I = P_\varphi(\dot{\mathfrak{M}}) = P_\varphi(M_{\mathbb{T}\setminus\{1\}}(SO(\mathbb{T}))\times\dot{\mathbb{R}}) + P_\varphi(\mathfrak{M}_1^\circ) + P_\varphi(M_1(SO(\mathbb{T}))\times\{\infty\}),$$

and hence we infer from (5.11)–(5.12) that

$$\|\varphi([NV_1]^\pi)\|_{\mathcal{B}(\mathcal{H}_\varphi)} = \|P_\varphi(\dot{\mathfrak{M}})\varphi([NV_1]^\pi)\|_{\mathcal{B}(\mathcal{H}_\varphi)} = \|P_\varphi(\mathfrak{M}_1^\circ)\varphi([NV_1]^\pi)\|_{\mathcal{B}(\mathcal{H}_\varphi)}. \tag{5.13}$$

Combining (5.7), (5.13) and (5.10), we obtain (5.6). □

Lemma 5.2. *If* $N\in\mathfrak{B}^0$ *and the operator* V_Γ *is given by* (5.4)*, then for every finite set* $\Gamma\subset\mathbb{T}\setminus\{1\}$*,*

$$|NV_\Gamma| = \|\Psi_2(N)\Pi_\Gamma vI\|_{\mathcal{B}(\mathcal{H}_2)}. \tag{5.14}$$

Proof. Fix a finite set $\Gamma \subset \mathbb{T} \setminus \{1\}$ and take an operator $N \in \mathfrak{B}^0$ written in the form (5.1), where $A_g^{(1)}, A_g^{(2)} \in \mathfrak{A}^0$ and $F_1,\ F_2$ are finite subsets of G_0. Consider the finite subset $\widetilde{\Gamma}$ of $\mathbb{T} \setminus \{1\}$ defined by

$$\widetilde{\Gamma} := \{g^{-1}(t) :\ t \in \Gamma,\ g \in F_1\} \cup \{g^{-1}(\overline{t}) :\ t \in \Gamma,\ g \in F_2\}.$$

Given the isometric *-isomorphism φ of the C^*-algebra $\mathfrak{B}^\pi = \mathfrak{B}/\mathcal{K}$ on a Hilbert space $\mathcal{H}_\varphi$ considered in (2.10), we conclude that

$$|NV_\Gamma| = \|\varphi([NV_\Gamma]^\pi)\|_{\mathcal{B}(\mathcal{H}_\varphi)}. \tag{5.15}$$

Applying (5.1) and taking into account the relations

$$U_gV_t \simeq V_{g^{-1}(t)}U_g,\quad U_\gamma V_t = V_{\overline{t}}U_\gamma,\quad A_g^{(i)}V_t \simeq V_tA_g^{(i)},\quad t \in \mathbb{T},\ g \in G_0,\ i = 1, 2,$$

we can represent the operator NV_Γ in the form

$$\begin{aligned} NV_\Gamma &= \sum_{g\in F_1}\sum_{t\in\Gamma} A_g^{(1)}U_gV_t + \sum_{g\in F_2}\sum_{t\in\Gamma} A_g^{(2)}U_gU_\gamma V_t \\ &= \sum_{g\in F_1}\sum_{t\in\Gamma} V_{g^{-1}(t)}A_g^{(1)}U_g + \sum_{g\in F_2}\sum_{t\in\Gamma} V_{g^{-1}(\overline{t})}A_g^{(2)}U_gU_\gamma + K, \end{aligned} \tag{5.16}$$

where $K \in \mathcal{K}$. Taking the symbol

$$[\mathrm{Sym}\, V_\Gamma](\xi, x) = \begin{cases} \mathrm{diag}\{v(x), v(x)\} & \text{if } (\xi, x) \in M_\Gamma(SO(\mathbb{T})) \times \overline{\mathbb{R}}, \\ 0 & \text{if } (\xi, x) \in \mathfrak{M} \setminus (M_\Gamma(SO(\mathbb{T})) \times \overline{\mathbb{R}}), \end{cases}$$

of the operator $V_\Gamma \in \mathfrak{A}$ (see Lemma 2.3), we infer from the second equality in (5.16), by analogy with (5.11) and (5.12), that

$$P_\varphi(M_{\mathbb{T}\setminus\widetilde{\Gamma}}(SO(\mathbb{T})) \times \dot{\mathbb{R}})\, \varphi([NV_\Gamma]^\pi) = 0$$

for the open set $M_{\mathbb{T}\setminus\widetilde{\Gamma}}(SO(\mathbb{T})) \times \dot{\mathbb{R}} \subset \dot{\mathfrak{M}}$, and

$$P_\varphi(M_{\widetilde{\Gamma}}(SO(\mathbb{T})) \times \{\infty\})\, \varphi([NV_\Gamma]^\pi) = 0$$

for the closed set $M_{\widetilde{\Gamma}}(SO(\mathbb{T})) \times \{\infty\} \subset \dot{\mathfrak{M}}$. Thus, since

$$\dot{\mathfrak{M}} = \mathfrak{M}_{\widetilde{\Gamma}}^\circ \cup \big(M_{\mathbb{T}\setminus\widetilde{\Gamma}}(SO(\mathbb{T})) \times \dot{\mathbb{R}}\big) \cup \big(M_{\widetilde{\Gamma}}(SO(\mathbb{T})) \times \{\infty\}\big)$$

and

$$P_\varphi(\mathfrak{M}_{\widetilde{\Gamma}}^\circ) + P_\varphi(M_{\mathbb{T}\setminus\widetilde{\Gamma}}(SO(\mathbb{T})) \times \dot{\mathbb{R}}) + P_\varphi(M_{\widetilde{\Gamma}}(SO(\mathbb{T})) \times \{\infty\}) = I,$$

we conclude, using (2.16), that

$$\begin{aligned} \|\varphi([NV_\Gamma]^\pi)\|_{\mathcal{B}(\mathcal{H}_\varphi)} &= \|P_\varphi(\mathfrak{M}_{\widetilde{\Gamma}}^\circ)\varphi([NV_\Gamma]^\pi)\|_{\mathcal{B}(\mathcal{H}_\varphi)} \\ &= \|P_\varphi(\mathfrak{M}_{\widetilde{\Gamma}}^\circ)\varphi([NV_\Gamma]^\pi)P_\varphi(\mathfrak{M}_{\Gamma}^\circ)\|_{\mathcal{B}(\mathcal{H}_\varphi)}. \end{aligned} \tag{5.17}$$

Let G_N be the subgroup of G generated by the finite set $F_1 \cup F_2\gamma$, with $F_2\gamma := \{g\gamma : g \in F_2\}$, and let $\Omega_{N,\Gamma}$ be the finite set of G_N-orbits ω of all points $t \in \Gamma$. Then $\Gamma_\omega := \Gamma \cap \omega$ is a finite subset of $\omega \in \Omega_{N,\Gamma}$. Since

$$V_\Gamma = \sum_{\omega \in \Omega_{N,\Gamma}} V_{\Gamma_\omega}, \quad V_{\Gamma_\omega} := \sum_{t \in \Gamma_\omega} V_t, \quad \Pi_\Gamma = \sum_{\omega \in \Omega_{N,\Gamma}} \Pi_{\Gamma_\omega},$$

and therefore

$$|NV_\Gamma| = \max_{\omega \in \Omega_{N,\Gamma}} |NV_{\Gamma_\omega}|, \quad \|\Psi_2(N)\Pi_\Gamma vI\|_{\mathcal{B}(\mathcal{H}_2)} = \max_{\omega \in \Omega_\Gamma} \|\Psi_2(N)\Pi_{\Gamma_\omega} vI\|_{\mathcal{B}(\mathcal{H}_2)},$$

we only need to prove (5.14) for V_Γ replaced by any V_{Γ_ω}. In what follows we will assume without loss of generality that $\Gamma \subset \omega$ and $\omega = G_N(t_0)$. Since the group G_N is at most countable, the same happens for the G_N-orbit ω.

Consider the Hilbert space $\mathcal{H}_{t_0} := \bigoplus_{t\in\omega} P_\varphi(\mathfrak{M}^\circ_{t_0})\mathcal{H}_\varphi$, which is isometrically isomorphic to $l^2(\mathfrak{M}^\circ_{t_0}, l^2(\omega, \mathbb{C}^2))$, and the isomorphism

$$\eta_\omega : P_\varphi(\mathfrak{M}^\circ_\omega)\mathcal{H}_\varphi \to \bigoplus_{t\in\omega} P_\varphi(\mathfrak{M}^\circ_{t_0})\mathcal{H}_\varphi, \;\; P_\varphi(\mathfrak{M}^\circ_\omega)f \mapsto \big(P_\varphi(\mathfrak{M}^\circ_{t_0})\varphi(U^\pi_{g_t})f\big)_{t\in\omega}, \tag{5.18}$$

where $f \in \mathcal{H}_\varphi$ and g_t for every $t \in \omega$ is a fixed shift in $Y_{t_0,t}$, that is, g_t possesses the property $g_t(t_0) = t$ (see (3.1)). Clearly, for $\widetilde{\Gamma} \subset \omega$ and every $f \in \mathcal{H}_\varphi$ we get

$$\eta_\omega(P_\varphi(\mathfrak{M}^\circ_{\widetilde{\Gamma}})f) = \Pi^{\widetilde{\Gamma}}_\omega \big(P_\varphi(\mathfrak{M}^\circ_{t_0})\varphi(U^\pi_{g_t})f\big)_{t\in\omega}, \tag{5.19}$$

where

$$\Pi^{\widetilde{\Gamma}}_\omega = \operatorname{diag}\{\chi_{\widetilde{\Gamma}}(t)\}_{t\in\omega} I. \tag{5.20}$$

Taking now the isometric C^*-algebra homomorphism

$$\Upsilon_\omega : \mathcal{B}(P_\varphi(\mathfrak{M}^\circ_\omega)\mathcal{H}_\varphi) \to \mathcal{B}\bigg(\bigoplus_{t\in\omega} P_\varphi(\mathfrak{M}^\circ_{t_0})\mathcal{H}_\varphi\bigg), \quad T \mapsto \eta_\omega T \eta_\omega^{-1}, \tag{5.21}$$

where η_ω is given by (5.18), and applying the relations

$$A^{(i)}_{g,t} := U_{g_t} A^{(i)}_g U^{-1}_{g_t} \in \mathfrak{A}, \quad U_{g_t} U_g U^{-1}_{g_\tau} = U_{\widetilde{g}_{t,\tau}} \quad (g \in G),$$
$$U_{g_\tau} V_s U^{-1}_{g_\tau} \simeq V_{g_\tau^{-1}(s)} \quad (s \in \Gamma)$$

for $t, \tau \in \omega$, where according to (3.2) we have $\widetilde{g}_{t,\tau} = g_t g g_\tau^{-1} \in Y_{t_0,t_0}$ if $g(t) = \tau$, we infer from (5.16), (5.19) and (5.20) that

$$\Upsilon_\omega\big(P_\varphi(\mathfrak{M}^\circ_{\widetilde{\Gamma}})\varphi([NV_\Gamma]^\pi)P_\varphi(\mathfrak{M}^\circ_\Gamma)\big) = \Upsilon_\omega\bigg(P_\varphi(\mathfrak{M}^\circ_{\widetilde{\Gamma}})\bigg(\sum_{g\in F_1}\sum_{s\in\Gamma}\varphi\big([A^{(1)}_g U_g V_s]^\pi\big)$$
$$+ \sum_{g\in F_2}\sum_{s\in\Gamma}\varphi\big([A^{(2)}_g U_{g\gamma} V_s]^\pi\big)\bigg)P_\varphi(\mathfrak{M}^\circ_\Gamma)\bigg), \tag{5.22}$$

where

$$\Upsilon_\omega\Big(P_\varphi(\mathfrak{M}^\circ_{\widetilde{\Gamma}})\sum_{g\in F_1}\sum_{s\in\Gamma}\varphi\big([A_g^{(1)}U_gV_s]^\pi\big)P_\varphi(\mathfrak{M}^\circ_\Gamma)\Big)$$
$$=\Pi_\omega^{\widetilde{\Gamma}}\Big(P_\varphi(\mathfrak{M}^\circ_{t_0})\sum_{g\in F_1}\sum_{s\in\Gamma}\varphi\big([U_{g_t}A_g^{(1)}U_{g_t}^{-1}]^\pi[U_{\widetilde{g}_{t,\tau}}]^\pi[U_{g_\tau}V_sU_{g_\tau}^{-1}]^\pi\big)P_\varphi(\mathfrak{M}^\circ_{t_0})\Big)_{t,\tau\in\omega}\Pi_\omega^\Gamma$$
$$=\Pi_\omega^{\widetilde{\Gamma}}\Big(\sum_{g\in F_1}\delta_g(t,\tau)P_\varphi(\mathfrak{M}^\circ_{t_0})\varphi\big([A_{g,t}^{(1)}U_{\widetilde{g}_{t,\tau}}V_{t_0}]^\pi\big)\Big)_{t,\tau\in\omega}\Pi_\omega^\Gamma \tag{5.23}$$

and analogously

$$\Upsilon_\omega\Big(P_\varphi(\mathfrak{M}^\circ_{\widetilde{\Gamma}})\sum_{g\in F_2}\sum_{s\in\Gamma}\varphi\big([A_g^{(2)}U_{g\gamma}V_s]^\pi\big)P_\varphi(\mathfrak{M}^\circ_\Gamma)\Big)$$
$$=\Pi_\omega^{\widetilde{\Gamma}}\Big(\sum_{g\in F_2}\delta_{g\gamma}(t,\tau)P_\varphi(\mathfrak{M}^\circ_{t_0})\varphi\big([A_{g,t}^{(2)}U_{(\widetilde{g\gamma})_{t,\tau}}V_{t_0}]^\pi\big)\Big)_{t,\tau\in\omega}\Pi_\omega^\Gamma$$
$$=\Pi_\omega^{\widetilde{\Gamma}}\Big(\sum_{g\in F_2}\delta_g(t,\overline{\tau})P_\varphi(\mathfrak{M}^\circ_{t_0})\varphi\big([A_{g,t}^{(2)}U_{(\widetilde{g\gamma})_{t,\tau}}V_{t_0}]^\pi\big)\Big)_{t,\tau\in\omega}\Pi_\omega^\Gamma. \tag{5.24}$$

From (5.21) and (5.22)–(5.24) it follows that

$$\big\|P_\varphi(\mathfrak{M}^\circ_{\widetilde{\Gamma}})\varphi([NV_\Gamma]^\pi)P_\varphi(\mathfrak{M}^\circ_\Gamma)\big\|_{\mathcal{B}(\mathcal{H}_\varphi)}$$
$$=\big\|\Upsilon_\omega\big(P_\varphi(\mathfrak{M}^\circ_{\widetilde{\Gamma}})\varphi([NV_\Gamma]^\pi)P_\varphi(\mathfrak{M}^\circ_\Gamma)\big)\big\|_{\mathcal{B}(\mathcal{H}_{t_0})}$$
$$=\Big\|\Pi_\omega^{\widetilde{\Gamma}}\Big(\sum_{g\in F_1}P^{(1)}_{g,t,\tau}+\sum_{g\in F_2}P^{(2)}_{g\gamma,t,\tau}\Big)_{t,\tau\in\omega}\Pi_\omega^\Gamma\Big\|_{\mathcal{B}(\mathcal{H}_{t_0})}, \tag{5.25}$$

where, for each $t,\tau\in\omega$,

$$P^{(1)}_{g,t,\tau}:=\delta_g(t,\tau)\,P_\varphi(\mathfrak{M}^\circ_{t_0})\varphi\big([A_{g,t}^{(1)}U_{\widetilde{g}_{t,\tau}}V_{t_0}]^\pi\big)\quad\text{if }\ g\in F_1,$$
$$P^{(2)}_{g\gamma,t,\tau}:=\delta_g(t,\overline{\tau})\,P_\varphi(\mathfrak{M}^\circ_{t_0})\varphi\big([A_{g,t}^{(2)}U_{(\widetilde{g\gamma})_{t,\tau}}V_{t_0}]^\pi\big)\quad\text{if }\ g\in F_2.$$

Since the cosets $[A_{g,t}^{(1)}U_{\widetilde{g}_{t,\tau}}V_{t_0}]^\pi$ and $[A_{g,t}^{(2)}U_{(\widetilde{g\gamma})_{t,\tau}}V_{t_0}]^\pi$ belong to the quotient C^*-algebra $\mathfrak{A}^\pi$, we deduce from Lemma 2.5 that for $t,\tau\in\omega$,

$$\Big\|\sum_{g\in F_1}P^{(1)}_{g,t,\tau}+\sum_{g\in F_2}P^{(2)}_{g,t,\tau}\Big\|_{\mathcal{B}(P_\varphi(\mathfrak{M}^\circ_{t_0})\mathcal{H}_\varphi)}$$
$$=\Big\|\sum_{g\in F_1}\delta_g(t,\tau)\big[\mathrm{Sym}\,N^{(1)}_{g,t,\tau}\big]\big|_{\mathfrak{M}^\circ_{t_0}}I+\sum_{g\in F_2}\delta_g(t,\overline{\tau})\big[\mathrm{Sym}\,N^{(2)}_{g\gamma,t,\tau}\big]\big|_{\mathfrak{M}^\circ_{t_0}}I\Big\|_{\mathcal{B}(l^2(\mathfrak{M}^\circ_{t_0},\mathbb{C}^2))}$$

where

$$N^{(1)}_{g,t,\tau}:=A_{g,t}^{(1)}U_{\widetilde{g}_{t,\tau}}V_{t_0}\quad\text{if }\ g\in F_1,$$
$$N^{(2)}_{g\gamma,t,\tau}:=A_{g,t}^{(2)}U_{(\widetilde{g\gamma})_{t,\tau}}V_{t_0}\quad\text{if }\ g\in F_2.$$

Hence, taking into account the finiteness of the sets $\Gamma, \widetilde{\Gamma} \subset \omega$ in (5.25), we infer from (3.8), (5.2), (5.5), (5.25) and Lemma 2.3 that

$$\begin{aligned}
&\Big\|\Pi_\omega^{\widetilde{\Gamma}}\Big(\sum_{g\in F_1} P^{(1)}_{g,t,\tau} + \sum_{g\in F_2} P^{(2)}_{g\gamma,t,\tau}\Big)_{t,\tau\in\omega}\Pi_\omega^{\Gamma}\Big\|_{\mathcal{B}(\mathcal{H}_{t_0})}\\
&= \Big\|\Pi_\omega^{\widetilde{\Gamma}}\Big(\sum_{g\in F_1}\delta_g(t,\tau)\big[\mathrm{Sym}\, N^{(1)}_{g,t,\tau}\big]\big|_{\mathfrak{M}^\circ_{t_0}} I\\
&\quad + \sum_{g\in F_2}\delta_g(t,\overline{\tau})\big[\mathrm{Sym}\, N^{(2)}_{g\gamma,t,\tau}\big]\big|_{\mathfrak{M}^\circ_{t_0}} I\Big)_{t,\tau\in\omega}\Pi_\omega^{\Gamma}\Big\|_{\mathcal{B}(l^2(\mathfrak{M}^\circ_{t_0},l^2(\omega,\mathbb{C}^2)))}\\
&= \Big\|\Pi_{\widetilde{\Gamma}}\Big(\sum_{g\in F_1}\delta_g(t,\tau)\big[\mathrm{Sym}\big(A^{(1)}_{g,t}U_{\widetilde{g}_{t,\tau}}V_{t_0}\big)\big]\big|_{\mathfrak{M}^\circ_{t_0}} I\\
&\quad + \sum_{g\in F_2}\delta_{g\gamma}(t,\tau)\big[\mathrm{Sym}\big(A^{(2)}_{g,t}U_{(\widetilde{g\gamma})_{t,\tau}}V_{t_0}\big)\big]\big|_{\mathfrak{M}^\circ_{t_0}} I\Big)_{t,\tau\in\mathbb{T}\setminus\{1\}}\Pi_\Gamma I\Big\|_{\mathcal{B}(\mathcal{H}_2)}\\
&= \|\Psi_2(N)\Pi_\Gamma vI\|_{\mathcal{B}(\mathcal{H}_2)}. \qquad (5.26)
\end{aligned}$$

Finally, combining (5.15), (5.17), (5.25) and (5.26), we obtain (5.14). $\square$

Lemmas 5.1 and 5.2 are the key to prove the continuity of the algebraic homomorphisms Ψ_1 and Ψ_2. Hence, following the proof of [8, Theorem 7.3] we get the following results.

Theorem 5.3. *If $N \in \mathfrak{B}^0$, then*

$$\|\Psi_1(N)\|_{\mathcal{B}(\mathcal{H}_1)} \le |N|, \quad \|\Psi_2(N)\|_{\mathcal{B}(\mathcal{H}_2)} \le |N|. \qquad (5.27)$$

Theorem 5.4. *The algebraic homomorphisms Ψ_1 and Ψ_2 given by* (3.7) *and* (3.8) *extend, respectively, to representations*

$$\Psi_1 : \mathfrak{B} \to \mathcal{B}(\mathcal{H}_1), \quad \Psi_2 : \mathfrak{B} \to \mathcal{B}(\mathcal{H}_2)$$

such that (5.27) *holds for every $N \in \mathfrak{B}$ and* $\mathrm{Ker}\,\Psi_\lambda \supset \mathcal{K}$ *for $\lambda = 1, 2$.*

For the representations Ψ_1 and Ψ_2 we also have the following result.

Theorem 5.5. *For every $\lambda \in \Lambda$, the restriction of the quotient homomorphism*

$$\Psi_\lambda : \ \mathfrak{B}/\mathcal{K} \to \mathcal{L}_\lambda := \mathcal{B}(\mathcal{H}_\lambda), \quad B + \mathcal{K} \mapsto \Psi_\lambda(B)$$

onto the ideal $\mathfrak{H}_\lambda/\mathcal{K}$ of $\mathfrak{B}/\mathcal{K}$ is an isometric *-isomorphism of $\mathfrak{H}_\lambda/\mathcal{K}$ onto the closed two-sided ideal $\Psi_\lambda(\mathfrak{H}_\lambda)$ of the C^*-algebra $\Psi_\lambda(\mathfrak{B}) \subset \mathcal{L}_\lambda$.

Proof. Since the set $\{NV_1 : N \in \mathfrak{B}^0\}$ is dense in the ideal $\mathfrak{H}_1$ and $\Psi_1(V_1) = vI$, from Lemma 5.1 it follows that the restriction $\Psi_1|\mathfrak{H}_1$ is a *-homomorphism of $\mathfrak{H}_1$ into $\mathcal{L}_1 = \mathcal{B}(\mathcal{H}_1)$ such that $\|\Psi_1(H_1)\| = |H_1|$ for every $H_1 \in \mathfrak{H}_1$. Consequently, $\mathrm{Ker}\,(\Psi_1|\mathfrak{H}_1) = \mathcal{K}$ and Ψ_1 is a C^*-algebra isomorphism of $\mathfrak{H}_1/\mathcal{K}$ onto $\Psi_1(\mathfrak{H}_1)$.

Analogously, because the set

$$\big\{NV_\Gamma : N \in \mathfrak{B}^0, \ \Gamma \text{ runs through finite subsets of } \mathbb{T}\setminus\{1\}\big\}$$

is dense in the ideal $\mathfrak{H}_2$ and since $\Psi_2(V_\Gamma) = \Pi_\Gamma vI$, we infer from Lemma 5.2 that the restriction $\Psi_2|\mathfrak{H}_2$ is a *-homomorphism of $\mathfrak{H}_2$ into $\mathcal{L}_2 = \mathcal{B}(\mathcal{H}_2)$ such that $\|\Psi_2(H_2)\| = |H_2|$ for every $H_2 \in \mathfrak{H}_2$. This implies that $\mathrm{Ker}\,(\Psi_2|\mathfrak{H}_2) = \mathcal{K}$, and therefore Ψ_2 is a C^*-algebra isomorphism of $\mathfrak{H}_2/\mathcal{K}$ onto $\Psi_2(\mathfrak{H}_2)$. □

6. Proofs of Theorems 3.1 and 3.2

In Sections 4 and 5 it was proved that the C^*-algebra homomorphisms Ψ_1, Ψ_2 and Ψ_3, given respectively by (3.7), (3.8) and (3.9), are well defined on $\mathfrak{B}$. Using the version of lifting theorem presented in Subsection 2.4, we give in this section the proofs of the main results of the paper.

6.1. Proof of Theorem 3.1

Theorems 4.5 and 5.4 imply that the map $\Psi_{\mathfrak{B}}$ defined on the generators of the C^*-algebra $\mathfrak{B}$ by formulas (3.5)–(3.9) extends to a C^*-algebra homomorphism of $\mathfrak{B}$ into the C^*-algebra $\mathcal{B}(\mathcal{H}_{\mathfrak{B}})$ such that

$$\|\Psi_{\mathfrak{B}}(B)\| = \max_{i=1,2,3} \|\Psi_i(B)\| \le |B| \quad \text{for all} \quad B \in \mathfrak{B}. \tag{6.1}$$

By (6.1), $\mathrm{Ker}\,\Psi_{\mathfrak{B}} \supset \mathcal{K}$. On the other hand, given an operator $B \in \mathfrak{B}$ such that $\Psi_{\mathfrak{B}}(B) = 0$ and hence $\Psi_3(B) = 0$, we infer from Theorem 4.5 that $B = H_B \in \mathfrak{H}$. Thus,

$$\Psi_\lambda(H_B) = \Psi_\lambda(B) = 0 \quad \text{for} \quad \lambda = 1, 2. \tag{6.2}$$

By Lemma 4.1, we have

$$H_B = H_1 + H_2, \quad H_1 \in \mathfrak{H}_1, \quad H_2 \in \mathfrak{H}_2. \tag{6.3}$$

Applying (5.2), (3.7), (3.8) and Lemma 2.3, we infer that

$$\Psi_1(V_t) = 0 \quad \text{for all} \quad t \in \mathbb{T} \setminus \{1\}, \quad \Psi_2(V_1) = 0.$$

This in view of (4.3) implies that

$$\Psi_1(\mathfrak{H}_2) = \{0\}, \quad \Psi_2(\mathfrak{H}_1) = \{0\}. \tag{6.4}$$

Hence, from (6.3), (6.4) and (6.2) it follows that

$$\Psi_1(H_1) = \Psi_1(H_B) = 0, \quad \Psi_2(H_2) = \Psi_2(H_B) = 0.$$

Thus, we conclude from Theorem 5.5 that $H_1, H_2 \in \mathcal{K}$. Consequently, $B = H_B = H_1 + H_2 \in \mathcal{K}$, and therefore $\mathrm{Ker}\Psi_{\mathfrak{B}} = \mathcal{K}$, which completes the proof of Theorem 3.1.

6.2. Proof of Theorem 3.2

Sufficiency. Let $\Lambda = \{1, 2\}$. It follows from Theorems 5.4, 5.5 and equalities (6.4) that all the conditions of Theorem 2.6 are fulfilled for the C^*-algebra $\mathfrak{B}$ defined by (1.4), for the ideal $\mathfrak{H} = \mathfrak{H}_1 + \mathfrak{H}_2$ generated by all commutators $[aI, S_{\mathbb{T}}]$ ($a \in PSO^0(\mathbb{T})$), and for the representations $\Psi_\lambda : \mathfrak{B} \to \mathcal{L}_\lambda = \mathcal{B}(\mathcal{H}_\lambda)$ given by (5.2) for $\lambda \in \Lambda$, where the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ are defined in (3.4). Hence, by Theorem 2.6, an operator $B \in \mathfrak{B}$ is Fredholm on the space $L^2(\mathbb{T})$ if the coset $\widehat{B} := B + \mathfrak{H}$ is invertible in the quotient C^*-algebra $\widehat{\mathfrak{B}} = \mathfrak{B}/\mathfrak{H}$ and for every $\lambda \in \Lambda$ the operator $\Psi_\lambda(B)$ is invertible in the C^*-algebra $\mathcal{L}_\lambda$.

Fix an operator $B \in \mathfrak{B}$. If conditions (iii) of Theorem 3.2 are fulfilled, then by Theorem 4.5 the coset $\widehat{B} := B + \mathfrak{H}$ is invertible in the C^*-algebra $\widehat{\mathfrak{B}}$. On the other hand, the fulfillment of conditions (i) and (ii) of Theorem 3.2 implies the invertibility of operators $\Psi_1(B) \in \mathcal{B}(\mathcal{H}_1)$ and $\Psi_2(B) \in \mathcal{B}(\mathcal{H}_2)$, respectively. Hence, we conclude from Theorem 2.6 that the operator $B \in \mathfrak{B}$ is Fredholm on the space $L^2(\mathbb{T})$.

Necessity. Let an operator $B \in \mathfrak{B}$ be Fredholm on the space $L^2(\mathbb{T})$ or, equivalently, the coset $B + \mathcal{K}$ be invertible in the quotient C^*-algebra $\mathfrak{B}/\mathcal{K}$. Since $\operatorname{Ker} \Psi_{\mathfrak{B}} = \mathcal{K}$ by Theorem 3.1, the quotient *-homomorphism

$$\mathfrak{B}/\mathcal{K} \to \mathcal{B}(\mathcal{H}_{\mathfrak{B}}), \quad N + \mathcal{K} \mapsto \Psi_{\mathfrak{B}}(N),$$

is a C^*-algebra isomorphism. Consequently, $\Psi_{\mathfrak{B}}(B)$ is invertible on the space $\mathcal{H}_{\mathfrak{B}}$ and then, according to (3.5), the operators $\Psi_\lambda(B)$ are invertible on the spaces $\mathcal{H}_\lambda$, with $\lambda = 1, 2, 3$, respectively. The invertibility of the operators $\Psi_\lambda(B)$, in view of (3.6)–(3.9), immediately implies parts (i)–(iii) of Theorem 3.2, which completes the proof.

References

[1] A. Antonevich, *Linear Functional Equations. Operator Approach.* Operator Theory: Advances and Applications **83**, Birkhäuser, Basel, 1996; Russian original: University Press, Minsk, 1988.

[2] A. Antonevich, M. Belousov, and A. Lebedev, *Functional Differential Equations: II. C^*-Applications. Part* 2 *Equations with Discontinuous Coefficients and Boundary Value Problems.* Pitman Monogr. Surveys Pure Appl. Math. **95**, Longman, Harlow, 1998.

[3] A. Antonevich and A. Lebedev, *Functional Differential Equations: I. C^*-Theory.* Pitman Monogr. Surveys Pure Appl. Math. **70**, Longman, Harlow, 1994.

[4] M.A. Bastos, C.A. Fernandes, and Yu.I. Karlovich, *C^*-algebras of integral operators with piecewise slowly oscillating coefficients and shifts acting freely.* Integral Equations and Operator Theory **55** (2006), 19–67.

[5] M.A. Bastos, C.A. Fernandes, and Yu.I. Karlovich, *Spectral measures in C^*-algebras of singular integral operators with shifts.* J. Funct. Anal. **242** (2007), 86–126.

[6] M.A. Bastos, C.A. Fernandes, and Yu.I. Karlovich, *C^*-algebras of singular integral operators with shifts having the same nonempty set of fixed points*. Compl. Anal. Oper. Theory **2** (2008), 241–272.

[7] M.A. Bastos, C.A. Fernandes, and Yu.I. Karlovich, *A nonlocal C^*-algebra of singular integral operators with shifts having periodic points*. Integral Equations and Operator Theory **71** (2011), 509–534.

[8] M.A. Bastos, C.A. Fernandes, and Yu.I. Karlovich, *A C^*-algebra of singular integral operators with shifts admitting distinct fixed points*, J. Math. Anal. Appl. **413** (2014), 502–524.

[9] A. Böttcher, S. Roch, B. Silbermann, and I.M. Spitkovsky, *A Gohberg–Krupnik–Sarason symbol calculus for algebras of Toeplitz, Hankel, Cauchy, and Carleman operators*. Operator Theory: Advances and Applications **48** (1990), 189–234.

[10] I. Gohberg and N. Krupnik, *On the algebra generated by the one-dimensional singular integral operators with piecewise continuous coefficients*. Funct. Anal. Appl. **4** (1970), 193–201.

[11] F.P. Greenleaf, *Invariant Means on Topological Groups and Their Representations*. Van Nostrand-Reinhold, New York, 1969.

[12] R. Hagen, S. Roch, and B. Silbermann, *Spectral Theory of Approximation Methods for Convolution Equations*. Birkhäuser, Basel, 1995.

[13] Yu.I. Karlovich, *The local-trajectory method of studying invertibility in C^*-algebras of operators with discrete groups of shifts*. Soviet Math. Dokl. **37** (1988), 407–411.

[14] Yu.I. Karlovich, *A local-trajectory method and isomorphism theorems for nonlocal C^*-algebras*. Operator Theory: Advances and Applications **170** (2007), 137–166.

[15] Yu.I. Karlovich and B. Silbermann, *Local method for nonlocal operators on Banach spaces*. Operator Theory: Advances and Applications **135** (2002), 235–247.

[16] Yu.I. Karlovich and B. Silbermann, *Fredholmness of singular integral operators with discrete subexponential groups of shifts on Lebesgue spaces*. Math. Nachr. **272** (2004), 55–94.

[17] M.A. Naimark, *Normed Algebras*. Wolters-Noordhoff, Groningen, 1972.

[18] S. Roch, P.A. Santos, and B. Silbermann, *Non-commutative Gelfand Theories. A Tool-kit for Operator Theorists and Numerical Analysts*. Springer, London, 2011.

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On Cauchy Type Integrals Related to the Cimmino System of Partial Differential Equations

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Abstract. In this paper, we established a one-to-one correspondence between quaternionic hyperholomorphic functions in $\mathbb{R}^4 \cong \mathbb{C}^2$ and solutions (pairs of complex-valued functions) for Cimmino system of partial differential equations written in complex form. This leads to a pair of Cauchy type integrals associated with Cimmino system. The topics of the paper concern theorems which cover basic properties of those Cauchy type integrals: the Sokhotski–Plemelj and Plemelj–Privalov type theorems for it as well as the necessary and sufficient condition for the possibility to extend a given pair of complex-valued Hölder-continuous functions from such a surface up to a solution of Cimmino system in a Jordan domain. Formulae for the square of the corresponding singular Cauchy type integrals are given. The proofs of all these facts are based on intimate relations between the theory of Cimmino system and some version of quaternionic analysis.

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1. Introduction

As is well known, the role of the complex Cauchy type integral in analytic function theory of one complex variable is very important. In the present work, motivated by [11], we make an attempt to start the construction of a function theory associated with the solutions of Cimmino system written in complex form, in the framework of exploiting hyperholomorphic function theory.

Let Ω be a domain in $\mathbb{R}^4$ and $f_i, i = 0, 1, 2, 3$, be $\mathbb{R}$-valued C^1-functions in Ω. The homogeneous system

$$\begin{cases} \dfrac{\partial f_0}{\partial x_0} + \dfrac{\partial f_2}{\partial x_2} - \dfrac{\partial f_1}{\partial x_1} - \dfrac{\partial f_3}{\partial x_3} = 0, \\ \dfrac{\partial f_0}{\partial x_1} + \dfrac{\partial f_1}{\partial x_0} - \dfrac{\partial f_2}{\partial x_3} - \dfrac{\partial f_3}{\partial x_2} = 0, \\ \dfrac{\partial f_0}{\partial x_2} + \dfrac{\partial f_3}{\partial x_1} - \dfrac{\partial f_1}{\partial x_3} - \dfrac{\partial f_2}{\partial x_0} = 0, \\ \dfrac{\partial f_0}{\partial x_3} + \dfrac{\partial f_1}{\partial x_2} + \dfrac{\partial f_2}{\partial x_1} + \dfrac{\partial f_3}{\partial x_0} = 0 \end{cases} \tag{1.1}$$

is called the Cimmino system, which was originally due to G. Cimmino [2] and greatly strengthened by S. Dragomir and E. Lanconelli [4]. This system offers a natural and elegant generalization to four-dimensional case of that of Cauchy–Riemann. Thus, the theory of solutions of the Cimmino system reduces, in some degenerate cases, to that of complex holomorphic functions. It is known that solutions of (1.1) satisfy the four-dimensional Laplace equation (cf. [11]). Hence, one may consider the former to be a refinement of harmonic analysis.

For our purpose we shall use the notation

$$\partial_{\bar{z}_1} := \frac{1}{2}\left(\frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1}\right), \quad \partial_{\bar{z}_2} := \frac{1}{2}\left(\frac{\partial}{\partial x_2} + i\frac{\partial}{\partial x_3}\right)$$
$$\partial_{z_1} := \frac{1}{2}\left(\frac{\partial}{\partial x_0} - i\frac{\partial}{\partial x_1}\right), \quad \partial_{z_2} := \frac{1}{2}\left(\frac{\partial}{\partial x_2} - i\frac{\partial}{\partial x_3}\right),$$

with $z_1 = x_0 + ix_1, z_2 = x_2 + ix_3$. Therefore (1.1) may be written now in a complex form as:

$$\begin{cases} \partial_{\bar{z}_1} u + \partial_{z_2} \bar{v} = 0, \\ \partial_{\bar{z}_2} u - \partial_{z_1} \bar{v} = 0, \end{cases} \tag{1.2}$$

where $u = f_0 + if_1$ and $v = f_2 + if_3$. Clearly, the set of solutions of system (1.2) contains all holomorphic functions of two complex variables.

In the present paper, we follow the approach presented in [9], where the third author studied an analogue of a Cauchy type integral for the theory of the Moisil–Theodoresco system of partial differential equations in the case of a piecewise Lyapunov surface of integration.

The paper is organized as follows: In Section 2, we formulate a series of theorems that cover basic properties of a pair of Cauchy type integrals for the theory of the Cimmino system of partial differential equations in the case of an Ahlfors–David regular surface of integration. The proofs of all of them can be found in Section 4 in the form of direct consequences of the corresponding facts valid for a slight modification of the hyperholomorphic function theory developed in [1]. Section 3 provides a detailed exposition of the Cimmino system in the terminology of hyperholomorphic function theory.

2. Cimmino system and Cauchy–Cimmino integrals

2.1. To facilitate access to the preliminary knowledge we collect here some definitions to be used in the sequel. Let E be a bounded subset of $\mathbb{R}^4 \cong \mathbb{C}^2 \cong \mathbb{C} \times \mathbb{C}$, $BC(E, \mathbb{R}^m), m \in \{2, 4\}$ be the class of $\mathbb{R}^m$-valued bounded continuous functions on E. For $\varphi \in BC(E, \mathbb{R}^m)$ we define the modulus of continuity of φ as the non-negative function $w_\varphi(t), t > 0$, by setting

$$w_\varphi(t) := \sup_{|x-y| \leq t} \{|\varphi(x) - \varphi(y)| : \ x, y \in E\}.$$

Let ν be a real number with $0 < \nu \leq 1$. If

$$\sup_{0<t\leq \operatorname{diam} E} \left\{ \frac{w_\varphi(t)}{t^\nu} \right\} < \infty,$$

then φ is Hölder continuous with exponent ν in E (Lipschitz continuous for $\nu = 1$). The collection of Hölder continuous function on E will be denoted by

$$C^{0,\nu}(E, \mathbb{R}^m) := \{\varphi \in BC(E, \mathbb{R}^m) : \sup_{0<t\leq \operatorname{diam} E} \left\{ \frac{w_\varphi(t)}{t^\nu} \right\} < \infty\}, \ 0 < \nu \leq 1,$$

and for $\varphi \in C^{0,\nu}(E, \mathbb{R}^m)$ let

$$||\varphi||_{C^{0,\nu}(E,\mathbb{R}^m)} := ||\varphi||_\infty + \sup_{0<t\leq \operatorname{diam} E} \left\{ \frac{w_\varphi(t)}{t^\nu} \right\},$$

where $||\varphi||_\infty$ is the sup norm.

We say (e.g., [3]) that a closed set E in $\mathbb{R}^4$ is an Ahlfors–David regular set (in short AD-regular) if there exists a constant $c > 0$ such that for all $x \in E$ and $0 < r < \operatorname{diam} E$ there holds

$$c^{-1} r^3 \leq \mathcal{H}^3(E \cap \mathbb{B}(x, r)) \leq c r^3,$$

where $\mathbb{B}(x, r)$ stands for the closed ball with center x and radius r and $\mathcal{H}^3$ is the 3-dimensional Hausdorff measure.

The AD-regularity condition implies a uniform positive and finite bound on E for the upper and lower density. Moreover, we notice that such a condition produces a very wide class of surfaces that contains the classes of surfaces classically considered in the literature: Liapunov surfaces, smooth surfaces and Lipschitz ones.

Finally we would like to remark that AD-regular sets are not always rectifiable in the sense of Federer [5] (see [8], Example 2 on p. 798), but if γ is a closed Jordan curve in the complex plane which is AD-regular, then it is automatically rectifiable.

2.2. In what follows, Ω stands for a bounded domain in $\mathbb{R}^4$ with an AD-regular boundary Γ, and introduce the temporary notation $\Omega^+ = \Omega$ and $\Omega^- = \mathbb{R}^4 \setminus \overline{\Omega^+}$, where both open sets are assumed to be connected.

Let $u, v : \Omega \longrightarrow \mathbb{C}$. We consider the following pair of integral operators

$$\mathcal{K}_1[u,v](z_1,z_2) = \int_\Gamma \frac{[(\bar{\zeta}_1 - \bar{z}_1)(n_0 + in_1) + (\bar{\zeta}_2 - \bar{z}_2)(n_2 + in_3)]u(\zeta_1,\zeta_2)}{2\pi^2(|\zeta_1 - z_1|^2 + |\zeta_2 - z_2|^2)^2} d\mathcal{H}^3 - \int_\Gamma \frac{[(\bar{\zeta}_2 - \bar{z}_2)\overline{(n_0 + in_1)} - (\bar{\zeta}_1 - \bar{z}_1)\overline{(n_2 + in_3)}]\bar{v}(\zeta_1,\zeta_2)}{2\pi^2(|\zeta_1 - z_1|^2 + |\zeta_2 - z_2|^2)^2} d\mathcal{H}^3,$$

$$\mathcal{K}_2[u,v](z_1,z_2) = \int_\Gamma \frac{[(\bar{\zeta}_1 - \bar{z}_1)(n_0 + in_1) + (\bar{\zeta}_2 - \bar{z}_2)(n_2 + in_3)]v(\zeta_1,\zeta_2)}{2\pi^2(|\zeta_1 - z_1|^2 + |\zeta_2 - z_2|^2)^2} d\mathcal{H}^3 + \int_\Gamma \frac{[(\bar{\zeta}_2 - \bar{z}_2)\overline{(n_0 + in_1)} - (\bar{\zeta}_1 - \bar{z}_1)\overline{(n_2 + in_3)}]\bar{u}(\zeta_1,\zeta_2)}{2\pi^2(|\zeta_1 - z_1|^2 + |\zeta_2 - z_2|^2)^2} d\mathcal{H}^3,$$

where (n_0, n_1, n_2, n_3) stands for the outward unit normal vector to the surface Γ due to Federer [5]. The pair $(\mathcal{K}_1, \mathcal{K}_2)$ of integrals for $(z_1, z_2) \in \mathbb{C}^2$ play the role of an analog of a Cauchy type integral in theory of the Cimmino system of partial differential equations. We call it the Cauchy–Cimmino type integrals.

Similarly the singular Cauchy–Cimmino integral operators are defined formally as the pair $(\mathcal{S}_1, \mathcal{S}_2)$, of the following singular integrals taken in the sense of Cauchy's principal value

$$\begin{aligned}\mathcal{S}_1[u,v](z_1,z_2) &= 2\int_\Gamma \frac{[(\bar{\zeta}_1 - \bar{z}_1)(n_0 + in_1) + (\bar{\zeta}_2 - \bar{z}_2)(n_2 + in_3)][u(\zeta_1,\zeta_2) - u(z_1,z_2)]}{2\pi^2(|\zeta_1 - z_1|^2 + |\zeta_2 - z_2|^2)^2} d\mathcal{H}^3\\ &\quad - 2\int_\Gamma \frac{[(\bar{\zeta}_2 - \bar{z}_2)\overline{(n_0 + in_1)} - (\bar{\zeta}_1 - \bar{z}_1)\overline{(n_2 + in_3)}][\bar{v}(\zeta_1,\zeta_2) - \bar{v}(z_1,z_2)]}{2\pi^2(|\zeta_1 - z_1|^2 + |\zeta_2 - z_2|^2)^2} d\mathcal{H}^3\\ &\quad + u(z_1,z_2),\end{aligned}$$

$$\begin{aligned}\mathcal{S}_2[u,v](z_1,z_2) &= 2\int_\Gamma \frac{[(\bar{\zeta}_1 - \bar{z}_1)(n_0 + in_1) + (\bar{\zeta}_2 - \bar{z}_2)(n_2 + in_3)][v(\zeta_1,\zeta_2) - v(z_1,z_2)]}{2\pi^2(|\zeta_1 - z_1|^2 + |\zeta_2 - z_2|^2)^2} d\mathcal{H}^3\\ &\quad + 2\int_\Gamma \frac{[(\bar{\zeta}_2 - \bar{z}_2)\overline{(n_0 + in_1)} - (\bar{\zeta}_1 - \bar{z}_1)\overline{(n_2 + in_3)}][\bar{u}(\zeta_1,\zeta_2) - \bar{u}(z_1,z_2)]}{2\pi^2(|\zeta_1 - z_1|^2 + |\zeta_2 - z_2|^2)^2} d\mathcal{H}^3\\ &\quad + v(z_1,z_2).\end{aligned}$$

We will now formulate the main results of the paper to be proved in the last section.

Theorem 2.1 (Sokhotski–Plemelj formulas for the Cauchy–Cimmino type integrals for Ahlfors–David regular surfaces). *Let Ω be a bounded domain in $\mathbb{C}^2$ with AD-regular boundary Γ. Let $(u,v) \in C^{0,\nu}(\Gamma,\mathbb{R}^2) \times C^{0,\nu}(\Gamma,\mathbb{R}^2)$. Then the following* *limits exist:*

$$\lim_{\Omega^{\pm}\ni(z_1,z_2)\to(\zeta_1,\zeta_2)\in\Gamma}(\mathcal{K}_1[u,v](z_1,z_2),\mathcal{K}_2[u,v](z_1,z_2))$$
$$=: (\mathcal{K}_1^{\pm}[u,v](\zeta_1,\zeta_2),\mathcal{K}_2^{\pm}[u,v](\zeta_1,\zeta_2)),$$

moreover the following identities hold:

$$(\mathcal{K}_1^{\pm}[u,v](\zeta_1,\zeta_2),\mathcal{K}_2^{\pm}[u,v](\zeta_1,\zeta_2))$$
$$= \frac{1}{2}[(\mathcal{S}_1[u,v](\zeta_1,\zeta_2),\mathcal{S}_2[u,v](\zeta_1,\zeta_2)) \pm (u(\zeta_1,\zeta_2),v(\zeta_1,\zeta_2))],$$

for all $(\zeta_1,\zeta_2)\in\Gamma$.

Theorem 2.2 (Plemelj–Privalov type theorem for the Cimmino system). *Let* Ω *be a bounded domain in* $\mathbb{C}^2$ *with AD-regular boundary* Γ. *Then*

$$(u,v)\in C^{0,\nu}(\Gamma,\mathbb{R}^2)\times C^{0,\nu}(\Gamma,\mathbb{R}^2) \Rightarrow (\mathcal{S}_1[u,v],\mathcal{S}_2[u,v])\in C^{0,\nu}(\Gamma,\mathbb{R}^2)\times C^{0,\nu}(\Gamma,\mathbb{R}^2),$$

for $0<\nu<1$.

Theorem 2.3 (Extension of a given pair of complex-valued Hölder continuous function on Γ up to a solution of the Cimmino system). *Let* Ω *be a bounded domain in* $\mathbb{C}^2$ *with AD-regular boundary* Γ. *Then we have:*

1. *In order to a pair* $(u,v)\in C^{0,\nu}(\Gamma,\mathbb{R}^2)\times C^{0,\nu}(\Gamma,\mathbb{R}^2)$ *be a boundary value of a solution of Cimmino system* (U,V) *into* Ω^+, *it is necessary and sufficient that*
$$(u(\zeta_1,\zeta_2),v(\zeta_1,\zeta_2)) = (\mathcal{S}_1[u,v](\zeta_1,\zeta_2),\mathcal{S}_2[u,v](\zeta_1,\zeta_2)),\ \ (\zeta_1,\zeta_2)\in\Gamma.$$
2. *In order to a pair* $(u,v)\in C^{0,\nu}(\Gamma,\mathbb{R}^2)\times C^{0,\nu}(\Gamma,\mathbb{R}^2)$ *be a boundary value of a solution of Cimmino system* (U,V) *into* Ω^-, *and vanishes at infinity, it is necessary and sufficient that*
$$(u(\zeta_1,\zeta_2),v(\zeta_1,\zeta_2)) = (-\mathcal{S}_1[u,v](\zeta_1,\zeta_2),-\mathcal{S}_2[u,v](\zeta_1,\zeta_2)),\ \ (\zeta_1,\zeta_2)\in\Gamma.$$

Theorem 2.4 (On the square of the singular Cauchy–Cimmino operators). *Let* Ω *be a bounded domain in* $\mathbb{C}^2$ *with AD-regular boundary* Γ, *then for* $(u,v)\in C^{0,\nu}(\Gamma,\mathbb{R}^2)\times C^{0,\nu}(\Gamma,\mathbb{R}^2)$, *the following formulas hold*

$$\mathcal{S}_1^2[u,v]-\mathcal{S}_2^2[u,v]=u,$$
$$\mathcal{S}_1[u,v]\mathcal{S}_2[u,v]+\mathcal{S}_2[u,v]\mathcal{S}_1[u,v]=-v.$$

3. Quaternionic function theory: general information

In this section, we provide some background on quaternionic analysis needed in this paper. For more information, we refer the reader to [6], [7].

3.1. Recall that the algebra $\mathbb{H}$ of real quaternions is an extension of the field $\mathbb{R}$ by the imaginary units $\mathbf{i},\mathbf{j},\mathbf{k}$ in Hamilton's classical notation. Throughout this paper, we denote the generators by $\mathbf{e_1} := \mathbf{i}$, $\mathbf{e_2} := \mathbf{j}$, $\mathbf{e_3} := \mathbf{k}$ and unit element of $\mathbb{H}$ by $\mathbf{e}_0$ for convenience. This means

$$\mathbb{H} := \{q = x_0 + x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3; \quad (x_0, x_1, x_2, x_3) \in \mathbb{R}^4\},$$

where $\mathbf{e}_i$ $(i = 1, 2, 3)$ are subject to the multiplication table in $\mathbb{H}$

$$\mathbf{e}_i\mathbf{e}_j + \mathbf{e}_j\mathbf{e}_i = -2\delta_{ij}, \quad \mathbf{e}_1\mathbf{e}_2 = \mathbf{e}_3,$$

and the usual component wise defined addition. Then $\mathbb{H} \cong \mathbb{R}^4$.

The quaternionic conjugation of $q = x_0 + x_1\mathbf{e}_1 + x_2\mathbf{e_2} + x_3\mathbf{e}_3$ is given by

$$\bar{q} := x_0 - x_1\mathbf{e_1} - x_2\mathbf{e_2} - x_3\mathbf{e_3}.$$

We use the Euclidean norm $|x|$ in $\mathbb{H}$, defined by

$$|q| := \sqrt{q\bar{q}} = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}.$$

We embed the usual complex linear space $\mathbb{C}^2$ into the skew-field $\mathbb{H}$ by means of the mapping that associates the pair

$$(z_1, z_2) = (x_0 + \mathbf{e_1}x_1, x_2 + \mathbf{e_1}x_3)$$

with the quaternion

$$q = z_1 + z_2\mathbf{e_2} = x_0 + \mathbf{e_1}x_1 + \mathbf{e_2}x_2 + \mathbf{e_3}x_3 \in \mathbb{H}.$$

The above embedding means that the set of elements of the form

$$q = z_1 + z_2 j, \quad z_1, z_2 \in \mathbb{C}, \quad j^2 = -1,$$

is endowed both with an obvious component-wise addition and with the associative multiplication. In particular, the commutation rule is then: $aj = j\bar{a}$ for every $a \in \mathbb{C}$, and the two quaternions $q = z_1 + z_2 j$ and $\xi = \zeta_1 + \zeta_2 j$ are multiplied according the rule:

$$q\,\xi = (z_1\zeta_1 - z_2\bar{\zeta}_2) + (z_1\zeta_2 + z_2\bar{\zeta}_1)j.$$

The quaternion conjugation gives:

$$\overline{z_1 + z_2 j} := \bar{z}_1 - z_2 j.$$

Note that the classical conjugation in $\mathbb{C}^2$ is

$$\overline{(z_1, z_2)} = (\bar{z}_1, \bar{z}_2).$$

In addition, $|q|^2 := |z_1|^2 + |z_2|^2$. For the properties of the quaternions taken in the form $q = z_1 + z_2 j$ we refer the reader to [7, Appendix 2, pp. 216–217]. Topology in $\mathbb{C}^2$ is determined by the metric $\operatorname{dist}(\xi, q) = |\xi - q|$.

3.2. Let the matrix

$$B_l(b) := \begin{pmatrix} b_0 & -b_1 & -b_2 & -b_3 \\ b_1 & b_0 & -b_3 & b_2 \\ b_2 & b_3 & b_0 & -b_1 \\ b_3 & -b_2 & b_1 & b_0 \end{pmatrix} \in \mathbb{R}^{4\times 4} \tag{3.1}$$

be the left regular representation of real quaternion b. Then $\mathbb{H}$ can be identified, as a skew-field, with $\mathcal{B}_l := \{B_l(b) \mid b \in \mathbb{H}\}$. Moreover, the left multiplication by the real quaternion b corresponds to the multiplication by the matrix $B_l(b)$, i.e.,

$$b \cdot q \leftrightarrow B_l(b) \cdot \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

3.3. For continuously real-differentiable function $f = \sum_{i=0}^{3} f_i \mathbf{e}_i : \ \Omega \to \mathbb{H}$, the operator

$${}^{\psi}D := \frac{\partial}{\partial x_0} + \psi^1 \frac{\partial}{\partial x_1} + \psi^2 \frac{\partial}{\partial x_2} + \psi^3 \frac{\partial}{\partial x_3}$$

associated to structural $\mathbb{H}$-vector ψ, i.e., $\psi = (\psi^0 \equiv 1, \psi^1, \psi^2, \psi^3)$, with the condition

$$\psi^\alpha \overline{\psi^\beta} + \psi^\beta \overline{\psi^\alpha} = 2\delta_{\alpha\,\beta},$$

($\delta_{\alpha\,\beta}$ is the Kronecker symbol) is called the *Cauchy–Riemann–Fueter operator*, see [12] for more details.

A function $f : \ \Omega \to \mathbb{H}$ is called left-ψ-hyperholomorphic if

$${}^{\psi}Df = 0. \tag{3.2}$$

For the particular case of the standard structural $\mathbb{H}$-vector

$$\psi = \psi_{st} := (1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3),$$

equation (3.2) is equivalent to the following system for $\mathbb{R}$-valued functions:

$$(CF) \begin{cases} \dfrac{\partial f_0}{\partial x_0} - \dfrac{\partial f_1}{\partial x_1} - \dfrac{\partial f_2}{\partial x_2} - \dfrac{\partial f_3}{\partial x_3} = 0, \\ \dfrac{\partial f_0}{\partial x_1} + \dfrac{\partial f_1}{\partial x_0} - \dfrac{\partial f_2}{\partial x_3} + \dfrac{\partial f_3}{\partial x_2} = 0, \\ \dfrac{\partial f_0}{\partial x_2} + \dfrac{\partial f_1}{\partial x_3} + \dfrac{\partial f_2}{\partial x_0} - \dfrac{\partial f_3}{\partial x_1} = 0, \\ \dfrac{\partial f_0}{\partial x_3} - \dfrac{\partial f_1}{\partial x_2} + \dfrac{\partial f_2}{\partial x_1} + \dfrac{\partial f_3}{\partial x_0} = 0. \end{cases}$$

We call both systems (3.2), (CF) the *Cauchy–Fueter system*. See [10] for a survey of the theory of ψ-hyperholomorphic functions along more classical lines.

3.4. Let $\theta(q) = \frac{1}{4\pi^2}\frac{1}{|q|^2}$ be a fundamental solution of the Laplace operator $\Delta_{\mathbb{C}^2} = \frac{1}{2}\Delta_{\mathbb{R}^4}$. Then a fundamental solution $\mathcal{K}_\psi$ of the operator ${}^{\psi}D$ is given by the formula

$$\mathcal{K}_\psi(q) := {}^{\overline{\psi}}D[\theta](q) = \frac{1}{2\pi^2|q|^4} \sum_{k=0}^{3} \overline{\psi^k} x_k = \frac{1}{2\pi^2|q|^4} B_l(V_{\overline{\psi}}^T \cdot q), \ \ q \neq 0,$$

where $\overline{\psi} := \{1, \overline{\psi^1}, \overline{\psi^2}, \overline{\psi^3}\}$ and $V_\psi := \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}$, with $a = (\psi_j^k)_{j,k=1}^3$.

Let $\sigma_\psi := dx_{[0]} - \psi^1 dx_{[1]} + \psi^2 dx_{[2]} - \psi^3 dx_{[3]}$, where $dx_{[k]}$ denotes, as usual the differential form $dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3$ with factor dx_k omitted.

The quaternionic Cauchy type integral is given by the formula,

$$^{\psi}C_\Gamma[f](q) = \int\limits_\Gamma \mathcal{K}_\psi(\xi - q)\, \sigma_\psi(\xi)\, f(\xi),\ \ q \notin \Gamma.$$

Besides the quaternionic Cauchy type integral we also need its singular version

$$^{\psi}S_\Gamma[f](q) = 2\int\limits_\Gamma \mathcal{K}_\psi(\xi - q)\, \sigma_\psi(\xi)\, (f(\xi) - f(q)) + f(q),\ \ q \in \Gamma.$$

The integral above has to be taken in the sense of Cauchy's principal value.

Note that if $(n_0(\xi), n_1(\xi), n_2(\xi), n_3(\xi))$ is the outward unit normal to the surface Γ at ξ due to Federer [5], one has the possibility to establish $(-1)^k n_k(\xi) d\mathcal{H}^3$ as an alternative definition of $dx_{[k]}$ for AD-regular surfaces.

According to the identity $\sigma_\psi(\xi) = n_\psi(\xi)\, d\mathcal{H}^3$, where $n_\psi = n_0 + \psi^1 n_1 + \psi^2 n_2 + \psi^3 n_3$, we are in a position to specify the appearance of the unit normal vector to the boundary in the definition of the quaternionic Cauchy type integrals. To this end one has

$$^{\psi}C_\Gamma[f](q) = \int_\Gamma \mathcal{K}_\psi(\xi - q)\, n_\psi(\xi) f(\xi)\, d\mathcal{H}^3,\ \ q \notin \Gamma.$$

In the same way, the singular Cauchy type integral, may be written as

$$^{\psi}S_\Gamma[f](q) = 2\int_\Gamma \mathcal{K}_\psi(\xi - q)\, n_\psi(\xi)(f(\xi) - f(q))\, d\mathcal{H}^3 + f(q),\ \ q \in \Gamma.$$

For a deeper discussion of the basic properties of these quaternionic integrals for the special structural $\mathbb{H}-$vector $\psi = (1, \mathbf{e}_1, \mathbf{e}_2, -\mathbf{e}_3)$ over rectifiable surfaces we refer the reader to [1, Section 3].

Investigation of the Cauchy–Cimmino type integrals requires us to consider, as a crucial fact, to replace ψ by $\hat{\psi} := \{1, \mathbf{e_1}, -\mathbf{e_2}, \mathbf{e_3}\}$. Following similar arguments to those in [1, Theorem 3.43.7], we can recover the same results.

For the convenience of the reader we state the corresponding aforementioned analogous results without proofs, thus making our exposition self-contained.

Theorem 3.1. *Let Ω be a bounded domain in $\mathbb{R}^4$ with AD-regular boundary Γ. Let $f \in C^{0,\nu}(\Gamma, \mathbb{R}^4)$. Then the following limits exist:*

$$\lim_{\Omega^\pm \ni q \to \xi \in \Gamma} ({}^{\hat{\psi}}C_\Gamma[f](q)) =: {}^{\hat{\psi}}C^\pm_\Gamma[f](\xi),$$

moreover the following identities hold:

$$^{\hat{\psi}}C^\pm_\Gamma[f](\xi) = \frac{1}{2}[{}^{\hat{\psi}}S_\Gamma[f](\xi) \pm f(\xi)],$$

for all $\xi \in \Gamma$.

Theorem 3.2. *Let Ω be a bounded domain in $\mathbb{R}^4$ with AD-regular boundary Γ. Then*

$$f \in C^{0,\nu}(\Gamma, \mathbb{R}^4) \Rightarrow {}^{\hat{\psi}}S_\Gamma[f] \in C^{0,\nu}(\Gamma, \mathbb{R}^4),$$

for $0 < \nu < 1$.

Theorem 3.3. *Let Ω be a bounded domain in $\mathbb{R}^4$ with AD-regular boundary Γ. Then we have:*

1. *In order to a function $f \in C^{0,\nu}(\Gamma, \mathbb{R}^4)$ be $\hat{\psi}$-hyperholomorphic extendable into Ω^+, it is necessary and sufficient that*

$$f(\xi) = {}^{\hat{\psi}}S_\Gamma[f](\xi), \ \xi \in \Gamma.$$

2. *In order to a function $f \in C^{0,\nu}(\Gamma, \mathbb{R}^4)$ be $\hat{\psi}$-hyperholomorphic extendable into Ω^-, and vanishes at infinity, it is necessary and sufficient that*

$$f(\xi) = -{}^{\hat{\psi}}S_\Gamma[f](\xi), \ \xi \in \Gamma.$$

Theorem 3.4. *If Γ is a AD-regular surface, then for $f \in C^{0,\nu}(\Gamma, \mathbb{R}^4)$, $0 < \mu < 1$ we have the following formula:*

$${}^{\hat{\psi}}S^2_\Gamma[f](\xi) = f(\xi), \ \xi \in \Gamma. \tag{3.3}$$

4. Proofs of the Theorems from Section 2

In this section, we prove all the theorems from Section 2 using the relations between the theory of the Cimmino system and that of $\hat{\psi}$-hyperholomorphic functions.

4.1. As it was pointed out by W. Tutschke (see [11]), the Cimmino system can be obtained by using the quaternionic analysis in the following way.

Note that the substitution $x_2 \mapsto -t$ transforms the Cauchy–Fueter system (CF) into the Cimmino system

$${}^{\hat{\psi}}Df = \left(\frac{\partial}{\partial x_0} + \mathbf{e_1}\frac{\partial}{\partial x_1} - \mathbf{e_2}\frac{\partial}{\partial t} + \mathbf{e_3}\frac{\partial}{\partial x_3} \right) f = 0. \tag{4.1}$$

Equation (4.1) is equivalent to the following Cimmino system (compare with (1.1)):

$$\begin{cases} \dfrac{\partial f_0}{\partial x_0} + \dfrac{\partial f_2}{\partial t} - \dfrac{\partial f_1}{\partial x_1} - \dfrac{\partial f_3}{\partial x_3} = 0, \\ \dfrac{\partial f_0}{\partial x_1} + \dfrac{\partial f_1}{\partial x_0} - \dfrac{\partial f_2}{\partial x_3} - \dfrac{\partial f_3}{\partial t} = 0, \\ \dfrac{\partial f_0}{\partial t} + \dfrac{\partial f_3}{\partial x_1} - \dfrac{\partial f_1}{\partial x_3} - \dfrac{\partial f_2}{\partial x_0} = 0, \\ \dfrac{\partial f_0}{\partial x_3} + \dfrac{\partial f_1}{\partial t} + \dfrac{\partial f_2}{\partial x_1} + \dfrac{\partial f_3}{\partial x_0} = 0. \end{cases} \tag{4.2}$$

The system (4.1) (or (4.2)) is deeply structural analogous to (CF), but the class of solutions of the first system, unlike the last, contains all holomorphic mappings of two complex variables. Nevertheless, (4.1) can be studied along lines parallel to the analysis of (3.2).

Using matrix (3.1), we can rewrite the equality ${}^{\hat{\psi}}Df = 0$ as follows:

$$B_l\left(\frac{\partial}{\partial x_0} + \mathbf{e_1}\frac{\partial}{\partial x_1} - \mathbf{e_2}\frac{\partial}{\partial t} + \mathbf{e_3}\frac{\partial}{\partial x_3}\right)\begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{pmatrix} = 0,$$

with

$$B_l(\frac{\partial}{\partial x_0} + \mathbf{e_1}\frac{\partial}{\partial x_1} - \mathbf{e_2}\frac{\partial}{\partial t} + \mathbf{e_3}\frac{\partial}{\partial x_3}) = \begin{pmatrix} \frac{\partial}{\partial x_0} & -\frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & -\frac{\partial}{\partial x_3} \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_0} & -\frac{\partial}{\partial x_3} & -\frac{\partial}{\partial x_2} \\ -\frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_0} & -\frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_0} \end{pmatrix}.$$

Thus

$${}^{\hat{\psi}}Df = 0 \Longleftrightarrow \begin{pmatrix} \frac{\partial}{\partial x_0} & -\frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & -\frac{\partial}{\partial x_3} \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_0} & -\frac{\partial}{\partial x_3} & -\frac{\partial}{\partial x_2} \\ -\frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_0} & -\frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_0} \end{pmatrix}\begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{pmatrix} = 0.$$

4.2. Let $q = z_1 + z_2 j \in \mathbb{H}$ and consider a complex form of the operator ${}^{\hat{\psi}}D$ associated to the structural vector $\hat{\psi} = (1, \mathbf{e_1}, -\mathbf{e_2}, \mathbf{e_3})$:

$${}^{\hat{\psi}}D = 2\left\{\frac{\partial}{\partial \bar{z}_1} - j\frac{\partial}{\partial \bar{z}_2}\right\}.$$

Note that

$${}^{\hat{\psi}}D\,{}^{\overline{\hat{\psi}}}D = {}^{\overline{\hat{\psi}}}D\,{}^{\hat{\psi}}D = \Delta_{\mathbb{R}^4},$$

where

$${}^{\overline{\hat{\psi}}}D = 2\left\{\frac{\partial}{\partial z_1} + j\frac{\partial}{\partial \bar{z}_2}\right\}.$$

We check at once that, if $f = u + vj$ with $u = f_0 + if_1$ and $v = f_2 + if_3$

$${}^{\hat{\psi}}Df = 0 \iff \begin{cases} \partial_{\bar{z}_1}u + \partial_{z_2}\bar{v} = 0 \\ \partial_{\bar{z}_2}u - \partial_{z_1}\bar{v} = 0. \end{cases} \tag{4.3}$$

From this it follows that the solution (u, v) of the Cimmino system (1.2) can be thought as the $\hat{\psi}$-hyperholomorphic function f.

The fundamental solution to the operator ${}^{\hat{\psi}}D$ may be written in complex form as

$$\begin{aligned} \mathcal{K}_{\hat{\psi}}(z_1 + z_2 j) &= {}^{\overline{\hat{\psi}}}D[\theta_4](z_1 + z_2 j) = \frac{1}{2\pi^2}\frac{\bar{z}_1 + \bar{z}_2 j}{(|z_1|^2 + |z_2|^2)^2} \\ &= \frac{1}{2\pi^2}\frac{B_l(V_{\hat{\psi}}^T \cdot (z_1 + z_2 j))}{(|z_1|^2 + |z_2|^2)^2}, \quad z_1, z_2 \neq 0. \end{aligned}$$

In this way $n_{\hat{\psi}}$ may be decomposed into $n_{\hat{\psi}} = (n_0 + in_1) - j(n_2 + in_3)$.

Using the presentation of the quaternionic Cauchy kernel $\mathcal{K}_{\hat{\psi}}$ and the normal vector $n_{\hat{\psi}}$ in the complex form, we have:

$$^{\hat{\psi}}C_\Gamma[u+vj] = \mathcal{K}_1[u,v] + \mathcal{K}_2[u,v]j, \tag{4.4}$$

$$^{\hat{\psi}}S_\Gamma[u+vj] = \mathcal{S}_1[u,v] + \mathcal{S}_2[u,v]j. \tag{4.5}$$

Having disposed of this preliminary preparation we are now in the position to proof our main results.

4.3. Proof of Theorem 2.1

Proof. Let $(u,v) \in C^{0,\nu}(\Gamma,\mathbb{R}^2) \times C^{0,\nu}(\Gamma,\mathbb{R}^2)$, consider $\mathcal{K}_1[u,v], \mathcal{K}_2[u,v]$. Then, applying Theorem 3.1 for $f = u+vj \in C^{0,\nu}(\Gamma,\mathbb{R}^4)$ we can assert that there exist $^{\hat{\psi}}C_\Gamma^\pm[f]$ and

$$^{\hat{\psi}}C_\Gamma^\pm[f](\xi) = \frac{1}{2}\left[^{\hat{\psi}}S_\Gamma[f](\xi) \pm f(\xi)\right],$$

for all $\xi = \zeta_1 + \zeta_2 j \in \Gamma$. Hence, using (4.4) it follows that there exist also $\mathcal{K}_1^\pm[u,v]$ and $\mathcal{K}_2^\pm[u,v]$. Therefore the complex decomposition given by formulae (4.5) yields that

$$\mathcal{K}_1^\pm[u,v](\zeta_1,\zeta_2) = \frac{1}{2}[\mathcal{S}_1[u,v](\zeta_1,\zeta_2) \pm u(\zeta_1,\zeta_2)],$$

$$\mathcal{K}_2^\pm[u,v](\zeta_1,\zeta_2) = \frac{1}{2}[\mathcal{S}_2[u,v](\zeta_1,\zeta_2) \pm v(\zeta_1,\zeta_2)],$$

for all $(\zeta_1,\zeta_2) \in \Gamma$. □

4.4. Proof of Theorem 2.2

Proof. If $(u,v) \in C^{0,\nu}(\Gamma,\mathbb{R}^2) \times C^{0,\nu}(\Gamma,\mathbb{R}^2)$, then $f = u+vj \in C^{0,\nu}(\Gamma,\mathbb{R}^4)$. By Theorem 3.2 we have $^{\hat{\psi}}S_\Gamma[f] \in C^{0,\nu}(\Gamma,\mathbb{R}^4)$. Now recalling the relation (4.5) we conclude that $(\mathcal{S}_1[u,v],\mathcal{S}_2[u,v]) \in C^{0,\nu}(\Gamma,\mathbb{R}^2) \times C^{0,\nu}(\Gamma,\mathbb{R}^2)$. □

4.5. Proof of Theorem 2.3

Proof. The proof is a direct consequence of Theorem 3.3. □

4.6. Proof of Theorem 2.4

Proof. The proof consists of taking into account Theorem 3.4 combined with a straightforward calculation. □

References

[1] R. Abreu Blaya, J. Bory Reyes and M. Shapiro, *On the Notion of the Bochner–Martinelli Integral for Domains with Rectifiable Boundary.* Compl. Anal. Oper. Theory **1** (2007), 143–168.

[2] G. Cimmino, *Su alcuni sistemi lineari omegeni di equazioni alle derivate parziali del primo ordine.* Rend. Sem. Mat. Univ. Padova **12** (1941), 89–113 (Italian).

[3] G. David and S. Semmes, *Analysis of and on uniformly rectifiable sets.* Mathematical Surveys and Monographs **38**, AMS, Providence, RI., (1993).

[4] S. Dragomir and E. Lanconelli, *On first-order linear PDF systems all of whose solutions are harmonic functions.* Tsukuba J. Math. **30**(1) (2006), 149–170.

[5] H. Federer, *Geometric Measure Theory.* Springer-Verlag, Heidelberg/New York, 1969.

[6] K. Gürlebeck and W. Sprössig, *Quaternionic and Clifford Calculus for Physicists and Engineers.* Wiley: New York, 1997.

[7] V. Kravchenko and M. Shapiro, *Integral Representations for Spatial Models of Mathematical Physics.* Pitman Res. Notes in Math. Ser. **351** (1996), Longman, Harlow.

[8] P. Mattila, *Singular integrals, analytic capacity and rectifiability.* In: Proceedings of the conference dedicated to Professor Miguel de Guzmán (El Escorial, 1996). J. Fourier Anal. Appl. **3** (1997)(Special Issue), 797–812.

[9] B. Schneider, *Some properties of a Cauchy type integral for the Moisil–Theodoresco system of partial differential equations.* Ukrainian Math. J. **58**(1) (2006), 18–126.

[10] M. Shapiro, *Some remarks on generalizations of the one-dimensional complex analysis: Hypercomplex approach.* Functional Analytic Methods in Complex Analysis and Applications to Partial Differential Equations (Trieste, 1993). World Scientific Publ., River Edge, N.J., (1995), 379–401.

[11] W. Tutschke, *Generalized analytic functions in higher dimensions.* Georgian Math. J. **14**(3) (2007), 581–595.

[12] N.L. Vasilevsky and M.V. Shapiro, *Some questions of hypercomplex analysis.* Complex Analysis and Applications '87 (Varna, 1987), 523–531 (1989), Publ. House Bulgar. Acad. Sci., Sofia.

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Singular Integral Operators with Linear Fractional Shifts on the Unit Circle

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Dedicated to Professor António Ferreira dos Santos

Abstract. In this paper we propose a classification to the linear-fractional shifts and consider a class of paired singular integral operators with a shift of that class. We show how the study of the this type of operators can be reduced to the study of paired operators with, what we call, a canonical shift. Some of the results obtained are used to construct explicit solutions for a class of singular integral equations with a non-Carleman shift.

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1. Introduction

Let $\mathbb{T}$ denote the unit circle with the positive (anticlockwise) orientation, and denote by $\mathbb{T}_+$ and $\mathbb{T}_-$ the interior and the exterior of $\mathbb{T}$, respectively. On the Lebesgue spaces $L_p := L_p(\mathbb{T})$, $p \in (1, \infty)$, we consider the singular operator with Cauchy kernel S, defined almost everywhere on $\mathbb{T}$ by

$$S\varphi(t) = (\pi i)^{-1} \int_{\mathbb{T}} \varphi(\tau)(\tau - t)^{-1} d\tau, \tag{1.1}$$

where the integral is understood in the sense of its principal value. The operator S is a bounded linear involutive operator ($S^2 = I$, where I is the identity operator on L_p). Then it is possible to define in L_p a pair of complementary projection operators in L_p by

$$P_{\pm} = \frac{1}{2}(I \pm S)$$

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and to decompose $L_p = L_p^+ \oplus \overset{\circ}{L_p^-}$, $L_p^+ = \operatorname{im} P_+$ and $\overset{\circ}{L_p^-} = \operatorname{im} P_-$. We also set $L_p^- = \overset{\circ}{L_p^-} \oplus \mathbb{C}$.

On the unit circle $\mathbb{T}$ we consider a linear-fractional shift α, i.e., the function $\alpha : \mathbb{T} \to \mathbb{T}$ is either of the form

$$\alpha(t) = e^{i\theta} \frac{t - \beta}{\overline{\beta} t - 1}, \quad \theta \in \mathbb{R}, \tag{1.2}$$

or

$$\alpha(t) = \frac{\tau}{t}, \qquad \tau \in \mathbb{T}. \tag{1.3}$$

The latter can be seen as the limit case of the first expression as $|\beta|$ tends to infinity and we will refer to it as a *flip* and the case $\tau = 1$ will be henceforth called the *canonical flip*.

It is well known that, if $|\beta| < 1$, α given by (1.2) preserves the given orientation of $\mathbb{T}$, we will call it a *forward shift*, and when $|\beta| > 1$, α reverses the given orientation of $\mathbb{T}$, this will be referred as a *backward shift*. For our purposes we also need to divide the linear-fractional shifts in two classes: Carleman and non-Carleman shifts.

The shift is said to be of *Carleman* type if it satisfies the Carleman condition for some $n \in \mathbb{N}$, that is

$$\alpha_n(t) \equiv t, \quad t \in \mathbb{T},$$

where $\alpha_1 = \alpha$, $\alpha_k = \alpha \circ \alpha_{k-1}, k = 2, \dots, n$. The least value of $n \in \mathbb{N}$ for which the above relation holds is then called the order of the shift. Otherwise, if α is not of Carleman type, it will be referred as a *non-Carleman* shift.

We consider as the shift operator associated with the shift α, $U_\alpha : L_p \to L_p$ one acting according to the rule

$$U_\alpha \varphi(t) = \mu^\alpha(t) \varphi(\alpha(t)), \quad t \in \mathbb{T}, \tag{1.4}$$

where the function μ^α is some function defined on $\mathbb{T}$ that will be specified later whenever it will be needed, otherwise $\mu^\alpha \equiv 1$. Its inclusion here is related to the fact that some known results about singular integral operators with shift are expressed in terms of shift operators satisfying some additional properties, for instance, if α is a Carleman shift of order $n \in \mathbb{N}$, then μ^α can be chosen so that $U_\alpha^n = I$ (see Sections 2 and 3 for details).

Let α be a linear-fractional shift and U_α the associated shift operator on L_p of the form (1.4). We consider singular integral operators with shift of the general form

$$T_{A_\alpha, B_\alpha} := A_\alpha P_+ + B_\alpha P_-, \tag{1.5}$$

with

$$A_\alpha := \sum_{k=0}^{m} a_k U_\alpha^k, \quad B_\alpha := \sum_{k=0}^{m} b_k U_\alpha^k, \quad m \in \mathbb{N}, \tag{1.6}$$

where $a_k, b_k \in L_\infty$, $k = \overline{0, m}$.

The paper is organized as follows. In Section 2 we propose a classification to the linear-fractional shifts based on the relative position of its fixed point on the plane, on preserving or reversing the orientation of $\mathbb{T}$ and on being or not a Carleman shift. In Section 3 we consider a class of paired singular integral operators with a linear-fractional shift. We show that the study of the this operator can be reduced to the study of a similar paired operator with, what we call, a canonical shift, by means of an anti-involutive operator. Some of the results obtained are used, in the forthcoming section, to construct explicit solutions for a class of singular integral equations with a non-Carleman shift.

2. Classification of linear-fractional shifts on $\mathbb{T}$

It is difficult to classify a linear-fractional shift as a Carleman or non-Carleman, when it is written in the form (1.2). For that reason we will use the idea of [4], Chapter 1. The linear-fractional shift α has either one or two fixed points on $\mathbb{C}$, given as solutions of the second-order equation

$$\overline{\beta}t^2 - (e^{i\theta}+1)t + \beta e^{i\theta} = 0. \tag{2.1}$$

If $|\beta| \neq |\cos(\frac{\theta}{2})|$, the shift α has two distinct fixed points on $\mathbb{C}$, henceforth denoted by $t_{\pm}$. When $|\beta| = |\cos(\frac{\theta}{2})|$ equation (2.1) has only one solution (although a double one, if $\beta \neq 0$):

$$t_+ = \frac{e^{i\theta}+1}{2\overline{\beta}}. \tag{2.2}$$

In the first place we will discuss the case when (1.2) has two distinct fixed points.

Proposition 2.1. *If α has two distinct fixed points $t_{\pm} \in \mathbb{C}\setminus\mathbb{T}$, then $\overline{t_{\pm}} = t_{\mp}^{-1}$.*

Proof. Suppose that $t_+ \in \mathbb{T}_+$ ($\mathbb{T}_+$ denotes the interior of $\mathbb{T}$) is one of the fixed points of the shift function (1.2) and let t be its symmetric point with respect to $\mathbb{T}$, $t' = \frac{1}{\overline{t_+}}$. Taking into account that $\alpha(t_+)$ and $\alpha(t')$ must be also symmetric with respect to $\mathbb{T}$, we get

$$\alpha(t') = \frac{1}{\overline{\alpha(t_+)}} = \frac{1}{\overline{t_+}} = t'.$$

So $t' = t_-$.

Clearly this argument also holds if the original fixed point belongs to $\mathbb{T}_-$, the exterior of $\mathbb{T}$, which completes the proof. □

In order to deal with α_n, $n \in \mathbb{N}$, the iterations of the shift function α, is very useful to introduce the so-called *multiplier* of the transformation (see, for instance, [4]):

$$\omega = \frac{e^{i\theta} - \overline{\beta}t_+}{e^{i\theta} - \overline{\beta}t_-}. \tag{2.3}$$

Proposition 2.2. *If the linear-fractional shift* (1.2) *has two distinct fixed points* $t_\pm$ *on* $\mathbb{C}$*, then it admits the following representation*

$$\alpha(t) = \frac{(t_+ - \omega t_-)t + (\omega - 1)t_+ t_-}{(1-\omega)t + \omega t_+ - t_-}. \tag{2.4}$$

Proof. By Viete's Theorem, from (2.1), it follows that

$$t_+ + t_- = \frac{e^{i\theta}+1}{\overline{\beta}}, \quad t_+ t_- = \frac{\beta e^{i\theta}}{\overline{\beta}}. \tag{2.5}$$

Using the first of these equalities and the definition of the multiplier ω, we conclude that

$$e^{i\theta} = \frac{\omega t_- - t_+}{\omega t_+ - t_-}. \tag{2.6}$$

On the other hand, by (2.6), the equality (2.3) is equivalent to

$$\overline{\beta} = \frac{\omega - 1}{\omega t_- - t_+} e^{i\theta} = \frac{\omega - 1}{\omega t_+ - t_-}. \tag{2.7}$$

Taking into account the second formula in (2.5), we get

$$\beta = \frac{(\omega - 1)t_+ t_-}{\omega t_- - t_+}. \tag{2.8}$$

Finally, from (2.6), (2.7) and (2.8) we can write the shift function (1.2) in the desired form (2.4). □

Note that the above proof permits to write the multiplier w of the shift α in the form

$$\omega = \frac{e^{i\theta} t_- - t_+}{e^{i\theta} t_+ - t_-}, \tag{2.9}$$

which can be useful. This results directly from (2.6), solving for ω.

The statement of the last proposition holds true for $\beta \neq 0$. If $\beta = 0$ the shift function α has the form

$$\alpha(t) = \omega t, \quad |\omega| = 1, \tag{2.10}$$

with one fixed point $t_+ = 0$. In this case, we may consider that α has two fixed points, the other one being the point at infinity: $t_- = \infty$. With this convention the shift α has two distinct fixed points on $\overline{\mathbb{C}}$ whenever $|\beta| \neq |\cos(\frac{\theta}{2})|$, $\theta \in (-\pi, \pi)$.

For the iterations of the shift, we have

$$\alpha_n(t) = \frac{(t_+ - \omega^n t_-)t + (\omega^n - 1)t_+ t_-}{(1-\omega^n)t + \omega^n t_+ - t_-}, \quad n \in \mathbb{N}.$$

Then, if (2.4) has two different fixed points, we conclude that α is a Carleman shift if and only if

$$\omega^n = 1,$$

where n is the order of the shift.

It must be also remarked that the flip (1.3) is a second-order backward shift, which admits the representation (2.4) with $\omega = -1$ and $t_\pm = \pm\sqrt{\tau}$.

Proposition 2.3. *Suppose that the linear-fractional shift* (1.2) *has two distinct fixed points on* $\mathbb{C}$. *Then the following holds for the multiplier* (2.3) *(or* (2.9)*):*

(i) *If* $t_\pm \notin \mathbb{T}$, *then* $|\omega| = 1$;

(ii) *If* $t_\pm \in \mathbb{T}$, *then* $\omega \in \mathbb{R}$.

Proof. From (2.6), we have

$$\frac{\omega t_- - t_+}{\omega t_+ - t_-} = \frac{\overline{\omega}\overline{t_+} - \overline{t_-}}{\overline{\omega}\overline{t_-} - \overline{t_+}},$$

which is equivalent to

$$|\omega|^2|t_-|^2 - \omega t_-\overline{t_+} - \overline{\omega}\overline{t_-}t_+ + |t_+|^2 = |\omega|^2|t_+|^2 - \omega t_+\overline{t_-} - \overline{\omega}\overline{t_+}t_- + |t_-|^2. \quad (2.11)$$

According to Proposition 2.1, if $t_\pm \notin \mathbb{T}$, then

$$\overline{t_\pm} = \frac{1}{t_\mp}$$

and (2.11) reduces to

$$(|\omega|^2 - 1)(|t_+|^2 - |t_-|^2) = 0,$$

which implies that $|\omega| = 1$.

On the other hand, from (2.11), if $t_\pm \in \mathbb{T}$, the following equality holds

$$(\overline{\omega} - \omega)(\overline{t_+}t_- - t_+\overline{t_-}) = 0.$$

Then either $t_+ = -t_-$ or $\overline{\omega} = \omega$. It is possible to show that always $\overline{\omega} = \omega$, that is $\omega \in \mathbb{R}$. In fact, If $t_+ = -t_-$, taking into account (2.5), we have

$$e^{i\theta} = -1, \quad t_+\overline{\beta} = \frac{\beta}{t_+},$$

and, from (2.3), it follows that

$$\omega = \frac{-1 - \overline{\beta}t_+}{-1 + \overline{\beta}t_+} = \frac{-1 - \frac{\beta}{t_+}}{-1 + \frac{\beta}{t_+}} = \overline{\omega}.$$ ◻

Proposition 2.4. *Under the condition* $|\beta| > |\cos\left(\frac{\theta}{2}\right)|$, *the linear-fractional shift* (1.2) *is a forward* (*backward*) *shift if and only if* $\omega > 0$ ($\omega < 0$), *where* ω *is the multiplier* (2.3).

Proof. According to [4], the multiplier ω is a solution of the equation

$$\omega + \frac{1}{\omega} = \frac{\left(e^{i\theta} - 1\right)^2}{e^{i\theta}\left(|\beta|^2 - 1\right)} - 2,$$

and, consequently,

$$\omega = \frac{1}{1 - |\beta|^2}\left(\sin\left(\frac{\theta}{2}\right) \pm \sqrt{|\beta|^2 - \cos^2\left(\frac{\theta}{2}\right)}\right)^2.$$

Taking into account that (1.2) is forward (backward) shift if and only if $|\beta| < 1$ ($|\beta| > 1$), the result follows. ◻

Proposition 2.5. *If $|\beta| > 1$, the linear-fractional shift* (1.2) *is a Carleman shift if and only if $\omega = -1$. In this case,*

$$\alpha(t) = \frac{t-\beta}{\overline{\beta}t - 1}. \tag{2.12}$$

Proof. According to the last proposition $\omega \in \mathbb{R}_-$, when $|\beta| > 1$. If α is a Carleman shift, since it is a backward shift, its order must be equal to two (see, for instance, [5]). Thus, $\omega^2 = 1$ together with $\omega \in \mathbb{R}_-$ implies that $\omega = -1$.

From (2.3) and the first equality of (2.5), is straightforward that $e^{i\theta} = 1$. □

Finally, we will find the general form of linear-fractional shifts with just one fixed point.

Proposition 2.6. *If the shift function* (1.2) *has only one fixed point t_+ on $\overline{\mathbb{C}}$, then*

$$\alpha(t) = \frac{2\omega t_+ t - (\omega+1)\,t_+^2}{(\omega+1)\,t - 2t_+}, \tag{2.13}$$

where $t_+ \in \mathbb{T}$ and $|\omega| = 1$.

Proof. As was already mentioned, if $|\beta| = \cos|\frac{\theta}{2}| \neq 0$, then the shift (1.2) has only one fixed point given by (2.2). Thus $|t_+| = 1$,

$$\overline{\beta} = \frac{e^{i\theta}+1}{2t_+}, \quad \text{and, consequently,} \quad \beta = \frac{\left(e^{-i\theta}+1\right)t_+}{2}.$$

Using the last two equations it is easy to see that α can be written in the form (2.13) with $\omega = e^{i\theta}$. □

We remark that the linear-functional (2.13) is a forward shift and thus, considering that a Carleman forward shift has not fixed points on $\mathbb{T}$ (see, for instance, [5]), we conclude that in such case (1.2) is a non-Carleman forward shift.

In appendix A we constructed a table containing the main features of linear-fractional shifts on $\mathbb{T}$.

3. Reduction to the canonical cases

In this section our objective is to show that any singular integral operator with a linear-fractional shift in L_p can be reduced (by means of an invertible bounded linear transformation) to a singular integral operator with a simpler shift, in the same space.

These kind of relations between linear-fractional shifts were used by Baturev, Kravchenko and Litvinchuk (see p. 8 in [2], see also p. 346 in [13]) and by Karelin in [6]. In [6] the author considered a reduction to the shift $\alpha(t) = -t$, on the real line. The main goal of this section is to give a complete solution of this problem in the case of the unit circle $\mathbb{T}$, when the shift under consideration is linear-fractional. Some of the results of the present section will be used in the next section to construct the solutions of a singular integral equation with a non-Carleman shift.

Another motivation to study this reduction problem is the following. It is well known that, in the theory of the singular integral equations with shift, the case where the shift is the canonical flip is the most important, either from the point of view of the applications as well as from point of view of the mathematical literature, where themes like Toeplitz plus Hankel operators are relatively well developed.

We will refer as *canonical shifts* one of the following linear-fractional shifts:

– A rotational shift (the fixed points are 0 and ∞),

$$f(t) = \omega t, \quad \omega \in \mathbb{T}, \tag{3.1}$$

– A shift having ± 1 as fixed points,

$$f(t) = \frac{(1+\omega)t + 1 - \omega}{(1-\omega)t + \omega + 1}, \quad \omega \in \mathbb{R} \setminus \{1\}, \tag{3.2}$$

– A shift having 1 as the unique fixed point,

$$f(t) = \frac{2\omega t - (\omega+1)}{(\omega+1)t - 2}. \tag{3.3}$$

Our goal is to show that, given a linear-fractional shift α there exist another linear-fractional shift η such that

$$\eta \circ \alpha = f \circ \eta, \tag{3.4}$$

where f is one of the canonical shifts mentioned above. The passage from α to f by means of η is called *reduction to the canonical form*, and η is then called the *reduction map.*

Let us stress that from the above relation one can immediately deduce the following assertions for the shifts α and f:

- Both are forward or backward shifts.
- Both are Carleman or non-Carleman shifts.
- When they are Carleman shifts, the order is the same.
- The number of fixed points is the same and $\eta(t_\pm)$ are the fixed points of the shift f.
- The fixed points t_+ and $\eta(t_\pm)$, simultaneously, are in $\mathbb{T}$ or do not belong to $\mathbb{T}$.

Keeping these ideas in mind, we obtain the reduction maps for each case. One can use the complex function theory to obtain deductively such maps, for instance, the very well-known result that states the following: Given two sets of three distinct points in the extended complex plane, say $\{z_1, z_2, z_3\}$ and $\{w_1, w_2, w_3\}$, there exists a unique linear-fractional map g such that $g(z_j) = w_j$, $j = 1, 2, 3$. However, in each of three the propositions below the proof can be made by straightforward verification of (3.4).

Proposition 3.1. *Let α be the linear-fractional shift* (2.4)*, with*

$$\overline{t_\pm} = t_\mp^{-1}, |t_+| < 1,$$

then α *can be reduced to* (3.1) *using the reduction map*

$$\eta(t) = \frac{t_-\,(t - t_+)}{t - t_-}, \tag{3.5}$$

where $t_\pm$ *and* $\omega \in \mathbb{T}$ *are, respectively, the fixed points and the multiplier of* α.

Proposition 3.2. *If* $\omega \in \mathbb{R} \setminus \{1\}$ *and* $t_\pm \in \mathbb{T}$ *are the fixed points of* (2.4), *then* α *can be reduced to* (3.2) *using the reduction map*

$$\eta(t) = \frac{(t_+ + t_-)\,t - 2t_+t_- + t_+ - t_-}{(2 + t_+ - t_-)\,t - t_+ - t_-}. \tag{3.6}$$

Under the conditions of the previous proposition, when α is a backward Carleman shift we have for the multiplier $\omega = -1$. Then, the canonical shit (3.2) becomes the canonical flip (1.3) ($\tau = 1$). So, according to the last proposition, any Carleman backward shift can be reduced to the canonical flip.

Proposition 3.3. *If the linear-fractional shift* α *has the form* (2.13), *where* $t_+ \in \mathbb{T}$ *is its unique fixed point, then* α *can be reduced to* (3.3) *using the reduction map*

$$\eta = \frac{2t_+t - t_+^2 - t_+}{(t_+ + 1)t - 2t_+}, \tag{3.7}$$

The next result show that the maps considered above exhaust all possible cases.

Lemma 3.4. *Let* α *be a linear-fractional of* $\mathbb{T}$. *Then* α *can be reduced to one of the canonical shifts* (3.1), (3.2) *or* (3.3), *using as reduction map the second-order Carleman shift* (3.5), (3.6) *or* (3.7), *respectively.*

Proof. From the results of Section 2 we conclude that, if α has not the form (2.10), then either

1. α is of the form (2.4) with $t\pm \notin \mathbb{T}$,
2. α is of the form (2.4) with $t\pm \in \mathbb{T}$, or
3. α is of the form (2.13) with $t_+ \in \mathbb{T}$.

Consequently, using the previous propositions, we can affirm that always exists a map η which reduces α to one of the canonical shifts. The map η has one of the forms (3.5), (3.6) or (3.7). In all cases η is a second-order Carleman shift.

Moreover, in the cases 1 and 3, η is a forward shift. In the case 2 the map η given by (3.6) is a forward shift if and only if

$$|2t_+t_- - t_+ + t_-| < |t_+ + t_-|.$$

□

Let α be a linear-fractional shift of the form (1.2) which is not a rotation and consider its reduction to the canonical form f by means of the reduction map η, according to Propositions 3.1 to 3.3. Since in every case η is linear-fractional, its derivative η' has no zeros and has a unique singular point, a double pole. Fix a square root of it, which is a rational function and that we will henceforth denote by ν, so $\nu(t) = (\eta'(t))^{\frac{1}{2}}$, $t \in \mathbb{T}$, and consider the linear operator $C_\eta : L_p \to L_p$ ($p \in (1,\infty)$), defined by the formula

$$C_\eta\varphi(t) = \nu(t)\,\varphi(\eta(t)), \quad t \in \mathbb{T}, \varphi \in L_p, \tag{3.8}$$

which clearly is bounded and invertible, with inverse given by

$$C_\eta^{-1}\varphi(t) = \breve{\nu}(t)\,\varphi(\zeta(t)), \quad t \in \mathbb{T}, \varphi \in L_p,$$

where $\zeta = \eta^{-1}$ is the inverse function to η and $\breve{\nu}(t) = 1/(\nu(\zeta(t))$, $t \in \mathbb{T}$. The transformation C_η will be called *reduction operator*. Note that in each of the Propositions 3.1 to 3.3 η is a second-order Carleman shift and thus $\zeta = \eta$ and C_η is an anti-involutive operator, $C_\eta^2 = -I$. However, we will keep this general notation, since other choices could be made for η.

We show that the reduction operator either commutes or anti-commutes with the singular integral operator with Cauchy kernel, depending on whether the reduction map is a forward or backward shift.

Proposition 3.5. *Let α be a shift of the form* (1.2) *with $\beta \neq 0$, η the map that reduces α to the canonical form, C_η the reduction operator* (3.8) *and S the Cauchy singular integral operator* (1.1) *in L_p, $p \in (1,\infty)$. Then*

$$SC_\eta = \pm C_\eta S.$$

where the plus or minus sign is taken depending on whether the reduction map η is a forward or backward shift, respectively.

Proof. As previously mentioned for each case of Propositions 3.1 to 3.3, the reduction map η is a Carleman shift of order 2 and can be given the form (1.2) with $\theta = 0$. Then η coincides with its inverse and a simple computation shows that

$$\frac{\nu(\eta(\xi))\eta'(\xi)}{\eta(\xi) - \eta(\xi')} = \frac{\nu(\eta(\xi'))}{\xi - \xi'}.$$

Then, for any $\varphi \in L_p$ ($p \in (1,\infty)$) and any $t \in \mathbb{T}$, we have

$$\begin{aligned} SC_\eta\varphi(t) &= \frac{1}{\pi i}\int_{\mathbb{T}} \frac{\nu(\tau)\varphi(\eta(\tau))}{\tau - t}d\tau = \pm\frac{1}{\pi i}\int_{\mathbb{T}} \frac{\nu(\eta(\xi))\eta'(\xi)\varphi(\xi)}{\eta(\xi) - \eta(\eta(t))}d\xi \\ &= \pm\frac{1}{\pi i}\int_{\mathbb{T}} \frac{\nu(t)\varphi(\xi)}{\xi - \eta(t)}d\xi = \pm C_\eta S\varphi(t), \end{aligned}$$

where the sign $+$ or $-$ is taken as stated in the proposition. □

Now we study the influence of the shift operator associated with the shift α on the reduction operator.

Proposition 3.6. *Let α be a shift of the form* (1.2), *U_α the associated shift operator* (1.4), *η a map that reduces α to the canonical form and C_η the corresponding reduction operator* (3.8). *Then*

$$U_\alpha C_\eta = M_w C_\eta U_f, \tag{3.9}$$

where M_w denotes the multiplication operator by the function w, given by

$$w(t) = \frac{\mu^\alpha(t)\nu(\alpha(t))}{\mu^f(\eta(t))\nu(t)}, \quad t \in \mathbb{T}. \tag{3.10}$$

Proof. The proof is straightforward using the definition of the operators C_η, U_α and U_f as well as the fundamental relation (3.4). For any $\varphi \in L_p$ and any $t \in \mathbb{T}$, we have

$$\begin{aligned} U_\alpha C_\eta \varphi(t) &= \mu^\alpha(t)\nu(\alpha(t))\varphi((\eta \circ \alpha)(t)) = \mu^\alpha(t)\nu(\alpha(t))\varphi((f \circ \eta)(t)) \\ &= \frac{\mu^\alpha(t)\nu(\alpha(t))}{\mu^f(\eta(t))}(U_f\varphi)(\eta(t)) = \frac{\mu^\alpha(t)\nu(\alpha(t))}{\mu^f(\eta(t))\nu(t)}(C_\eta U_f \varphi)(t) \\ &= w(t)(C_\eta U_f \varphi)(t). \end{aligned}$$

□

Using Propositions 3.5 and 3.6 we can reduce the study of the operator $T_{A,B}$, given by (1.5)–(1.6), to the study of a similar operator with a simpler shift, namely one of the canonical forms given above (see (3.1) to (3.3)). We first consider forward shifts, but we need to introduce some notation:

In what follows we shall use the following conventions of notation: for any $x \in L_\infty$, we put

$$\widehat{x} = x \circ \zeta,$$

where ζ is the inverse function to η. We put $w^{(0)} \equiv 1$, $\alpha_0 \equiv 1$, and for any $k \in \mathbb{N}$ we define

$$w^{(k)} = \prod_{j=1}^{k} w \circ \alpha_{j-1}.$$

Proposition 3.7. *Let α be a forward linear-fractional shift of the form* (1.2) *having either only one fixed point $t_+ \in \mathbb{T}$ or two fixed points $t_\pm \in \mathbb{C} \setminus \mathbb{T}$, η be a map that reduces α to the canonical form f and C_η the corresponding reduction operator on $L_p, p \in (1, \infty)$. Then C_η reduces the singular integral operator with shift α, T_{A_α,B_α}, to a singular integral operator with canonical shift f. More precisely, putting*

$$\widetilde{A}_f := \sum_{k=0}^{m} \widetilde{a_k} U_f^k, \quad \widetilde{B}_f := \sum_{k=0}^{m} \widetilde{b_k} U_f^k,$$

with

$$\widetilde{a_k} = \widehat{a_k}\widehat{w^{(k)}}, \quad \widetilde{b_k} = \widehat{b_k}\widehat{w^{(k)}},$$

we have

$$T_{\widetilde{A}_f, \widetilde{B}_f} := \widetilde{A}_f P_+ + \widetilde{B}_f P_- = C_\eta^{-1} T_{A_\alpha, B_\alpha} C_\eta.$$

Proof. With the convention of notation established before this proposition, for any $x \in L_\infty$, we have

$$C_\eta^{-1} x I C_\eta = \widehat{x} I.$$

As pointed out in the proof of Lemma 3.4, if α has only one fixed point $t_+ \in \mathbb{T}$ or two fixed points $t_\pm \notin \mathbb{T}$, then the reduction map η (which is given by (3.7) or by (3.5), respectively) is a Carleman forward shift.

From Proposition 3.5, it follows that

$$C_\eta^{-1} P_\pm C_\eta = P_\pm.$$

On the other hand, based on the equality (3.9), we have

$$U_\alpha^2 C_\eta = U_\alpha M_w C_\eta U_f = M_{w_\alpha} U_\alpha C_\eta U_f = M_{w_\alpha} M_w C_\eta U_f^2 = M_{w^{(2)}} C_\eta U_f^2.$$

By induction one easily establishes that, for any $k \in \mathbb{N}$, there holds

$$U_\alpha^k C_\eta = M_{w_{\alpha_{k-1}}} \cdots M_{w_\alpha} M_w C_\eta U_f^k = M_{w^{(k)}} C_\eta U_f^k$$

Then, for any $x \in L_\infty$, we have

$$C_\eta^{-1} x U_\alpha^k C_\eta = \widetilde{x} U_f^k$$

where $\widetilde{x} := \widehat{x}\, \widehat{w^{(k)}}$. Putting everything together, one obtains successively

$$\begin{aligned} C_\eta^{-1} T_{A_\alpha, B_\alpha} C_\eta &= C_\eta^{-1} A_\alpha C_\eta P_+ + C_\eta^{-1} B_\alpha C_\eta P_- \\ &= C_\eta^{-1} \sum_{k=0}^m a_k U_\alpha^k C_\eta P_+ + C_\eta^{-1} \sum_{k=0}^m b_k U_\alpha^k C_\eta P_- \\ &= C_\eta^{-1} \sum_{k=0}^m a_k w^{(k)} C_\eta U_f^k P_+ + C_\eta^{-1} \sum_{k=0}^m b_k w^{(k)} C_\eta U_f^k P_- \\ &= \sum_{k=0}^m \widetilde{a_k} U_f^k P_+ + \sum_{k=0}^m \widetilde{b_k} U_f^k P_- \end{aligned}$$

which completes the proof. □

As a direct consequence, we have the following.

Corollary 3.8. *Under the conditions of Proposition* 3.7*, if the function* μ^α *in the definition of the shift operator* U_α *in* (1.4) *is chosen so that* $w \equiv 1$*, then the coefficients* $\widetilde{A}_f$ *and* $\widetilde{B}_f$ *of the operator* $T_{\widetilde{A}_f, \widetilde{B}_f}$ *simplify to*

$$\widehat{A}_f := \sum_{k=0}^m \widehat{a_k} U_f^k, \quad \widehat{B}_f := \sum_{k=0}^m \widehat{b_k} U_f^k,$$

respectively.

Now we consider the remaining forward shifts and backward shifts.

Proposition 3.9. *Let* α *be a linear fractional shift of the form* (1.2) *having two fixed points* $t_\pm \in \mathbb{T}$*,* η *be a map that reduces* α *to the canonical form* f *and* C_η *the corresponding reduction operator on* $L_p, p \in (1, \infty)$*. Then* C_η *reduces the singular integral operator with shift* α*,* T_{A_α, B_α}*, to a singular integral operator with canonical shift* f*. More precisely, if*

$$|2t_+ t_- - t_+ + t_-| < |t_+ + t_-|, \quad \text{then} \quad T_{\widetilde{A}_f, \widetilde{B}_f} = C_\eta^{-1} T_{A_\alpha, B_\alpha} C_\eta,$$

and, if

$$|2t_+ t_- - t_+ + t_-| \geq |t_+ + t_-|, \quad \text{then} \quad T_{\widetilde{B}_f, \widetilde{A}_f} = C_\eta^{-1} T_{A_\alpha, B_\alpha} C_\eta.$$

Proof. The proof is similar to that of Proposition 3.7, but now we have two cases to consider because the reduction map η (which is given by (3.6)) can be either a Carleman forward or backward, depending on the position of the fixed points $t_\pm$, and as it follows from Proposition 3.5, we have

$$C_\eta^{-1} P_\pm C_\eta = \begin{cases} P_\pm & \text{if } \eta \text{ is a forward shift,} \\ P_\mp & \text{if } \eta \text{ is a backward shift.} \end{cases}$$

As was mentioned in the proof of Lemma 3.4, if $|2t_+t_- - t_+ + t_-| < |t_+ + t_-|$, then η is a Carleman forward shift and similarly to the proof of Proposition 3.7, we have

$$C_\eta^{-1} T_{A_\alpha, B_\alpha} C_\eta = T_{\widetilde{A}_f, \widetilde{B}_f}.$$

However, if $|2t_+t_- - t_+ + t_-| \geq |t_+ + t_-|$, then η is a Carleman backward shift and in such case, we have

$$\begin{aligned} C_\eta^{-1} T_{A_\alpha, B_\alpha} C_\eta &= C_\eta^{-1} A_\alpha C_\eta P_- + C_\eta^{-1} B_\alpha C_\eta P_+ \\ &= \sum_{k=0}^{m} \widetilde{a_k} U_f^k P_- + \sum_{k=0}^{m} \widetilde{b_k} U_f^k P_+ = T_{\widetilde{B}_f, \widetilde{A}_f}, \end{aligned}$$

which completes the proof. □

Again, as a direct consequence, we have the following.

Corollary 3.10. *Under the conditions of Proposition* 3.9*, if the function* μ^α *in the definition of the shift operator* U_α *in* (1.4) *is chosen so that* $w \equiv 1$*, then the coefficients* $\widetilde{A}_f$ *and* $\widetilde{B}_f$ *of the operator* $T_{\widetilde{A}_f, \widetilde{B}_f}$ *or of* $T_{\widetilde{B}_f, \widetilde{A}_f}$ *simplify to*

$$\widehat{A}_f := \sum_{k=0}^{m} \widehat{a_k} U_f^k, \quad \widehat{B}_f := \sum_{k=0}^{m} \widehat{b_k} U_f^k,$$

respectively.

As mentioned before in the case where the shift α in (1.2) is a Carleman shift of order n (recall that, if it is a backward shift, then necessarily $n = 2$) we may chose the weight μ^α in such a way that the associate shift operator satisfies $U_\alpha^n = I$ and, moreover, it commutes or anti commutes with the operator S according to whether it is forward or backward, respectively. According to the results in Section 2, since in such case to the shift α can be given the form (2.4), if we take, for some $c \in \mathbb{C}$:

$$\mu^\alpha(t) = c(\alpha'(t))^{\frac{1}{2}} = \frac{t_+ - t_-}{(1-\omega)t + \omega t_+ - t_-}, \quad t \in \mathbb{T}, \tag{3.11}$$

then the above-mentioned properties are satisfied, that is $U_\alpha^n = I$ and $U_\alpha S = \pm S U_\alpha$, the sign taken as previously explained. Notice that, according to Proposition 2.5, we have $\omega = -1$, if α is a Carleman backward shift.

As a consequence of the fundamental relation (3.4), if α has two fixed points $t_\pm \in \mathbb{C}$ and μ^α is given by (3.11), then the function w in (3.10) satisfies

$$w \equiv 1,$$

regardless of α being or not a Carleman shift. Therefore, either Corollary 3.8 or Corollary 3.10 applies.

Of course, if α is Carleman backward shift, since it is necessarily of order 2, it is enough to consider binomial coefficients ($m = 1$). In previous works of two of the authors of the present paper (see [8], [9]) it was shown that the Fredholm characteristics of a singular integral operator with a linear-fractional backward Carleman shift and binomial coefficients ($m = 1$) can be studied by means of a special factorization of a matrix function associated with it, which depends on the shift. The above proposition provides another way of analysis of such operators, namely one can first reduce the study of such operator to another one having as shift the canonical flip (1.3) ($\tau = 1$), these being usually called Toeplitz plus Hankel operators (see [3], [1]). The main advantage of this procedure is that the study of these last operators is more developed than the initial ones. Similar considerations are also valid, for instance, for the case of Carleman forward shifts (see [10]) and can also be used in the case of non-Carleman shifts, as we will see in the next section.

4. Explicit solutions for a non-Carleman shift

The main purpose of this section is to show that some results of the Fredholm theory of singular integral operators with a non-Carleman shift can be obtained in particular cases.

Specifically we have in mind those non-Carleman linear-fractional shifts α having two fixed points on $\mathbb{C}$, one in the interior and the other in the exterior of the unit circle. According to Proposition 3.1, singular integral operators with such shift can be reduced to singular integral operators with a rotational shift of the form (3.1). The non-Carleman condition means that the constant ω in (3.1) is not a primitive root of the identity, i.e., $\omega^n \neq 1, \forall n \in \mathbb{N}$. This particular case is sometimes called *irrotational shift* in the literature. The corresponding shift operator U_f is therefore defined as follows:

$$U_f\varphi(t) = \varphi(\omega t), \quad t \in \mathbb{T},\ \omega = e^{i\theta},\ \theta \in [0, 2\pi),\ \theta/2\pi \in \mathbb{R} \setminus \mathbb{Q}. \tag{4.1}$$

Associated with the shift α we consider the weighted shift operator U_α with weight given by (3.11). The reduction operator C_η takes the form

$$C_\eta\varphi(t) = \frac{\sqrt{t_+t_- - t_-^2}}{t - t_-}\varphi(\eta(t)), \quad t \in \mathbb{T}, \quad \varphi \in L_p, \tag{4.2}$$

where η is given by (3.5). As already said, with the above choices the function w defined by (3.10) is such that $w \equiv 1$, and so Corollary 3.8 applies, according to which we have

$$T_{\widehat{A}_f,\widehat{B}_f} = -C_\eta T_{A_\alpha,B_\alpha} C_\eta,$$

where

$$\widehat{A}_f = \sum_{k=0}^{m} \widehat{a}_k U_f^k, \quad \widehat{B}_f = \sum_{k=0}^{m} \widehat{b}_k U_f^k,$$

$\widehat{a}_k = a_k \circ \eta$, $\widehat{b}_k = b_k \circ \eta$, $k = \overline{0,m}$, and it was used the fact that C_η is an anti-involute operator.

We first analyze the simpler operator $T_{\widehat{A}_f, \widehat{B}_f}$.

4.1. The case of an irrotational shift

In this part we consider singular integral operators with shift of the form

$$T := T_{I,C} = P_+ + CP_- : L_p \to L_p, \qquad \text{with} \quad C = \sum_{j=0}^{m} c_j U^j,$$

where $U = U_f$ is the shift operator (4.1), $m \in \mathbb{N}$, and the coefficients c_j, $j = \overline{1,m}$ are continuous functions on the unit circle, $c_j \in C(\mathbb{T})$.

We note that the Fredholmness conditions for the operator T can be studied considering the matrix operator (see [11], [7], [12])

$$\widetilde{T} = P_+ + \widetilde{C} P_- : L_p^m \to L_p^m, \qquad \text{with} \quad \widetilde{C} = \widetilde{c}_0 I + \widetilde{c}_m U,$$

where

$$\widetilde{c}_0 = \begin{pmatrix} c_0 & O_{1\times(m-1)} \\ O_{(m-1)\times 1} & I_{m-1} \end{pmatrix},$$

$$\widetilde{c}_m = \begin{pmatrix} c_1 \quad c_2 \quad \cdots \quad c_{m-1} & c_m \\ -I_{m-1} & O_{(m-1)\times 1} \end{pmatrix},$$

I_m denotes the $m \times m$ identity matrix and we suppose that the operators U, $P_\pm$ act componentwise.

The following result holds.

Proposition 4.1. *The operator T is a Fredholm operator on L_p if and only if the operator $\widetilde{T}$ is a Fredholm operator on L_p^m. In the affirmative case,* $\dim\ker T = \dim\ker \widetilde{T}$ *and* $\dim\operatorname{coker} T = \dim\operatorname{coker} \widetilde{T}$.

Proof. As is known, the Fredholmness of a bounded linear operator T is preserved under its multiplication by invertible operators and so are the numbers $\dim\ker T$ and $\dim\operatorname{coker} T$.

We multiply $\widetilde{T}$ on the right by the invertible operator

$$N = \begin{pmatrix} 0 & 0 & \cdots & 0 & I \\ 0 & 0 & \cdots & I & UP_- \\ \vdots & & & & \vdots \\ 0 & I & \cdots & U^{m-3}P_- & U^{m-2}P_- \\ I & UP_- & \cdots & U^{m-2}P_- & U^{m-1}P_- \end{pmatrix}.$$

Using $UP_- = P_-U$, we obtain

$$\widetilde{T}N = \begin{pmatrix} D_1 & D_2 & \cdots & D_{m-1} & T \\ 0 & 0 & \cdots & I & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & I & \cdots & 0 & 0 \\ I & 0 & \cdots & 0 & 0 \end{pmatrix},$$

where

$$D_j = \sum_{k=0}^{j-1} c_{m-k} U^{j-k} P_-, \quad j = \overline{1, m-1}.$$

The operator $\widetilde{T}N$ is Fredholm if and only if the operator T is Fredholm. Moreover, the defect numbers of $\widetilde{T}N$ and T coincide. Since N is invertible the result follows. □

4.2. Some particular coefficients

Now let us consider the particular case of the operator T when the coefficients are first degree polynomials, which we rename as K, i.e.,

$$K = P_+ + \left(\sum_{j=0}^{m} p_j U^j\right) P_-. \tag{4.3}$$

where

$$p_j(t) = a_{j0} + a_{j1}t, \quad a_{j0}, a_{j1} \in \mathbb{C},\ j = \overline{0, m},\ m \in \mathbb{N}. \tag{4.4}$$

We will study the kernel of the operator K. To this end, suppose that $\varphi \in \ker K$ and decompose $\varphi = \varphi_+ + \varphi_-$, $\varphi_+ \in L_p^+$, $\varphi_- \in \overset{\circ}{L}{}_p^-$. Then

$$\varphi_+ + \left(\sum_{j=0}^{m} p_j U^j\right) \varphi_- = 0, \tag{4.5}$$

with

$$\varphi_+(t) = \sum_{k=0}^{\infty} \phi_k t^k, \qquad t \in \mathbb{T}_+, \quad \phi_k \in \mathbb{C};$$

$$\varphi_-(t) = \sum_{k=1}^{\infty} \varphi_k t^{-k}, \quad t \in \mathbb{T}_-, \quad \varphi_k \in \mathbb{C}.$$

Taking into account that, for any $s = \overline{0, n}$, there holds

$$p_s U^s \varphi_- = a_{s1}\varphi_1 \omega^{-s} + \sum_{k=1}^{\infty} a_{s1}\varphi_{k+1}\omega^{-(k+1)s} t^{-k} + \sum_{k=1}^{\infty} a_{s0}\varphi_k \omega^{-ks} t^{-k},$$

we arrive at the following equality

$$\varphi_+(t) + \varphi_1 \sum_{k=1}^{m} a_{k1}\omega^{-k} + \sum_{k=1}^{\infty} \left(a_k\varphi_{k+1} + b_k\varphi_k\right) t^{-k} = 0,$$

where, we recall, ω is given by (4.1) and

$$a_k = \sum_{j=0}^{m} a_{j1}\omega^{-j(k+1)}, \tag{4.6}$$

$$b_k = \sum_{j=0}^{m} a_{j0}\omega^{-jk}. \tag{4.7}$$

Then the functions $\varphi_\pm$ are a solution of (4.5) if and only if

$$\varphi_+(t) = -\varphi_1 \sum_{k=0}^{m} a_{k1}\omega^{-k}, \tag{4.8}$$

and

$$a_k\varphi_{k+1} + b_k\varphi_k = 0.$$

If $a_k \neq 0$, $k \in \mathbb{N}$, the general solution of the last equation is

$$\varphi_k = (-1)^{k-1}\varphi_1 \prod_{l=1}^{k-1} \frac{b_l}{a_l}, \quad k \in \mathbb{N}, \tag{4.9}$$

and, if $a_k = 0$ for some $k \in \mathbb{N}$, then $\varphi_k = 0$ for all $k \in \mathbb{N}$.

So, we can state the following result.

Proposition 4.2. *Let K be the operator defined by* (4.3)*. Then the function $\varphi = \varphi_+ + \varphi_- \in \ker K$ if and only if φ_+ is given by* (4.8) *and the series*

$$\varphi_-(t) = \sum_{k=1}^{\infty} \varphi_k t^{-k}, \quad t \in \mathbb{T}_-, \tag{4.10}$$

converges absolutely, where φ_k is the sequence (4.9)*. In the affirmative case,*

$$\dim\ker T = \dim\ker \widetilde{T} \quad \textit{and} \quad \dim\operatorname{coker} T = \dim\operatorname{coker}\widetilde{T}.$$

Further we will do a deeply study of the series (4.10). Namely we establish some sufficient conditions for its convergence or divergence. But first we need to introduce some useful notation.

We write the complex numbers $a_{\mu\nu}$ in the exponential form

$$a_{\mu\nu} = |a_{\mu\nu}|\, e^{i\theta_{\mu\nu}}, \quad \mu = \overline{1,m}, \quad \nu = 0,1,$$

and, using Euler's formula, we compute

$$\begin{aligned} |b_k|^2 &= \left(\sum_{j=0}^{m} a_{j0}e^{-ijk\theta}\right)\left(\sum_{j=0}^{m} \overline{a_{j0}}e^{ijk\theta}\right) \\ &= \sum_{t=0}^{m} |a_{t0}|^2 + 2\sum_{s=0}^{m-1}\sum_{r=s+1}^{m} |a_{s0}|\,|a_{r0}| \cos[\theta_{s0} - \theta_{r0} + (r-s)\theta k], \end{aligned}$$

and

$$|a_k|^2 = \left(\sum_{j=0}^{m} a_{j1} e^{-ij(k+1)\theta}\right)\left(\sum_{j=0}^{m} \overline{a_{j1}} e^{ijk\theta}\right)$$
$$= \sum_{t=0}^{m} |a_{t1}|^2 + 2\sum_{s=0}^{m-1}\sum_{r=s+1}^{m} |a_{s1}|\,|a_{r1}| \cos[\theta_{s1} - \theta_{r1} + (r-s)\theta(k+1)],$$

where a_k and b_k are given by (4.6) and (4.7).

Let us consider the functions of the real variable t

$$a(t) = \sum_{t=0}^{m} |a_{t1}|^2 + 2\sum_{s=0}^{m-1}\sum_{r=s+1}^{m} |a_{s1}|\,|a_{r1}| \cos[\theta_{s1} - \theta_{r1} + (r-s)\theta(t+1)], \quad (4.11)$$

and

$$b(t) = \sum_{t=0}^{m} |a_{t0}|^2 + 2\sum_{s=0}^{m-1}\sum_{r=s+1}^{m} |a_{s0}|\,|a_{r0}| \cos[\theta_{s0} - \theta_{r0} + (r-s)\theta t]. \quad (4.12)$$

Note that

$$a(k) = |a_k|^2, \quad b(k) = |b_k|^2, \quad k \in \mathbb{N}.$$

and it must be remarked that the functions $b(t)$ and $a(t)$ are periodic functions, with period equal to $\frac{2\pi}{\theta}$.

Lemma 4.3. *Let K be the operator defined by* (4.3)*, a and b be the continuous real periodic functions* (4.11) *and* (4.12)*, respectively. If the condition*

$$\max_{t\in\left[0,\frac{2\pi}{\theta}\right]} \left|\frac{b(t)}{a(t)}\right| < 1, \quad (4.13)$$

holds, then $\dim\ker K = 1$ *and*

$$\ker K = \{\varphi \in L_2 : \varphi = \varphi_+ + \varphi_-\}, \quad (4.14)$$

where $\varphi_\pm$ are given by (4.8) *and* (4.10).

Proof. From (4.13), it follows that $\frac{|b_k|}{|a_k|} < 1$. Therefore, for any $t \in \mathbb{T}_-$, we have

$$\left|\frac{\varphi_{k+1}}{\varphi_k\, t}\right| = \left|\frac{b_k}{a_k\, t}\right| < 1.$$

By the D'Alembert criterion, the series (4.10) converges absolutely in $\mathbb{T}_-$. Thus, according to the last proposition, $\dim\ker K = 1$ and the set $\ker K$ is given by (4.14). □

Lemma 4.4. *Let K be the operator defined by* (4.3)*, a and b be the continuous real periodic functions* (4.11) *and* (4.12)*, respectively. If the condition*

$$\min_{t\in\left[0,\frac{2\pi}{\theta}\right]} \left|\frac{b(t)}{a(t)}\right| > 1, \quad (4.15)$$

holds, then $\dim\ker K = 0$.

Proof. Let $c = \min_{t\in\left[0,\frac{2\pi}{\theta}\right]} \left|\frac{b(t)}{a(t)}\right|$. If $c > 1$, then

$$\left|\frac{\varphi_{k+1}}{\varphi_k\, t}\right| = \left|\frac{b_k}{a_k\, t}\right| \geq \frac{\sqrt{m}}{|t|}$$

and for $t \in \mathbb{T}_-$ such that $1 < |t| < \sqrt{m}$ it follows that

$$\left|\frac{\varphi_{k+1}}{\varphi_k\, t}\right| > 1.$$

Therefore, taking into account the D'Alembert criterion and Proposition 4.2, we can affirm that $\dim\ker K = 0$. □

Using Lemmas 4.3 and 4.4 we can state the following result.

Corollary 4.5. *Let $\widehat{K}$ be the paired operator on L_p*

$$\widehat{K} = P_+ + (r_0 I + r_1 U_\alpha + r_2 U_\alpha^2 + \cdots + r_m U_\alpha^m)P_-,$$

where U_α is the weighted operator (1.4)*, with non-Carleman shift* (2.4) *and*

$$\mu^\alpha = \frac{t_+ - t_-}{(1-\omega)\, t + \omega t_+ - t_-}.$$

The coefficients $r_j,\ \ j = \overline{0,m}$, are the rational scalar functions

$$r_j(t) = \frac{r_{j1} t + r_{j0}}{t - t_-},$$

and let

$$p_{j0} = \frac{t_+ r_{j1} + t_- r_{j0}}{t_+ - t_-^2}, \quad p_{j1} = \frac{t_- r_{j1} + r_{j0}}{t_-^2 - t_+}.$$

Additionally, let a and b be the scalar functions (4.11) *and* (4.12)*. Then the following statements hold true:*

(i) *If the condition* (4.13) *holds, then* $\dim\ker\widehat{K} = 1$ *and*

$$\ker\widehat{K} = \{\phi \in L_2 : \phi = C_\eta\varphi, \quad \varphi \in \ker K\},$$

where K is the operator (4.3), $\ker K$ *is the set* (4.14) *and C_η is the operator* (4.2)*.*

(ii) *If the condition* (4.15) *holds, then* $\dim\ker\widehat{K} = 0$.

Proof. The rational functions $r_j,\ \ j = \overline{0,m}$, satisfy the equality $r_j = p_j \circ \eta$, where p_j are the first degree polynomials (4.4) and η is the map (3.5). Then the proof can be completed taking into account that $\widehat{K} = -C_\eta K C_\eta$. □

Appendix

The table below contains the main features of linear-fractional shifts on $\mathbb{T}$.

Properties of β and θ	Properties of ω	Properties of $t_\pm$	Type of α	Classific. of α	General Form
$\beta = 0$	$\exists n \in \mathbb{N} : \omega^n = 1$	$t_+ = 0$, $t_- = \infty$	Carleman	Forward Shift	(2.10)
$\beta = 0$	$\lvert\omega\rvert = 1$, $\omega^n \neq 1, \forall n \in \mathbb{N}$	$t_+ = 0$, $t_- = \infty$	Non-Carleman	Forward Shift	(2.10)
$\lvert\beta\rvert < \lvert\cos\left(\frac{\theta}{2}\right)\rvert$	$\exists n \in \mathbb{N}$: $\omega^n = 1$	$\overline{t_\pm} = t_\mp^{-1}$	Carleman	Forward Shift	(2.4)
$\lvert\beta\rvert < \lvert\cos\left(\frac{\theta}{2}\right)\rvert$	$\lvert\omega\rvert = 1$, $\omega^n \neq 1, \forall n \in \mathbb{N}$	$\overline{t_\pm} = t_\mp^{-1}$	Non-Carleman	Forward Shift	(2.4)
$\lvert\beta\rvert = \lvert\cos\left(\frac{\theta}{2}\right)\rvert$	$\lvert\omega\rvert = 1$[1]	$t_+ \in \mathbb{T}$, $t_+ = t_-$	Non-Carleman	Forward Shift	(2.13)
$\lvert\cos\left(\frac{\theta}{2}\right)\rvert < \lvert\beta\rvert < 1$	$\omega \in \mathbb{R}_+$	$t_\pm \in \mathbb{T}$	Non-Carleman	Forward Shift	(2.4)
$\lvert\beta\rvert > 1$	$\omega = -1$	$t_\pm \in \mathbb{T}$	Carleman	Backward Shift	(2.4), (2.12)
$\lvert\beta\rvert > 1$	$\omega \in \mathbb{R}_- \backslash \{-1\}$	$t_\pm \in \mathbb{T}$	Non-Carleman	Backward Shift	(2.4)

References

[1] E.L. Basor and T. Ehrhardt, *Factorization theory for a class of Toeplitz + Hankel operators.* J. Operator Theory **51** (2004), no. 2, 411–433.

[2] A.A. Baturev, V.G. Kravchenko, and G.S. Litvinchuk, *Approximate methods for singular integral equations with a non-Carleman shift.* J. Integral Equations Appl. **8** (1996), no. 1, 1–17.

[3] T. Ehrhardt, *Invertibility theory for Toeplitz plus Hankel operators and singular integral operators with flip.* J. Funct. Anal. **208** (2004), no. 1, 64–106.

[4] L. Ford, *Automorphic Functions.* Chelsea Publishing Company, New York, 1957.

[5] N. Karapetiants and S. Samko, *Equations with Involutive Operators.* Birkhäuser, Boston, 2001.

[6] A. Karelin, *On a relation between singular integral operators with a Carleman linear-fractional shift and matrix characteristic operators without shifts.* Bol. Soc. Mat. Mexicana (3) **7** (2001), no. 2, 235–246.

[1]Here the constant ω do not denotes the multiplier of α. In the case when the shift function α has only one fixed point, the multiplier ω is equal to one (see [4]).

[7] V.G. Kravchenko and G.S. Litvinchuk, *Introduction to the Theory of Singular Integral Operators with Shift.* Mathematics and its Applications, vol. 289, Kluwer Academic Publishers, Dordrecht, 1994.

[8] V.G. Kravchenko, A.B. Lebre, and J.S. Rodríguez, *Factorization of singular integral operators with a Carleman shift via factorization of matrix functions: the anticommutative case.* Math. Nachr. **280** (2007), no. 9-10, 1157–1175.

[9] V.G. Kravchenko, A.B. Lebre, and J.S. Rodríguez, *Factorization of singular integral operators with a Carleman backward shift: the case of bounded measurable coefficients.* J. Anal. Math. **107** (2009), 1–37.

[10] V.G. Kravchenko, A.B. Lebre, and J.S. Rodríguez, *Factorization of singular integral operators with a Carleman shift via factorization of matrix functions.* In *Singular Integral Operators, Factorization and Applications*, Operator Theory: Advances and Applications, vol. 142, 189–211, Birkhäuser, Basel, 2003.

[11] V.G. Kravchenko and R.C. Marreiros, *On the kernel of some one-dimensional singular integral operators with shift.* In *The Extended Field of Operator Theory*, Operator Theory: Advances and Applications, vol. 171, 245–257, Birkhäuser, Basel, 2007.

[12] N. Krupnik, *Banach Algebras with Symbol and Singular Integral Operators.* Operator Theory: Advances and Applications, vol. 26, Birkhäuser, Basel, 1987.

[13] G.S. Litvinchuk, *Solvability Theory of Boundary Value Problems and Singular Integral Equations with Shift.* Mathematics and its Applications, vol. 523, Kluwer Academic Publishers, Dordrecht, 2000.

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Diffraction from Polygonal-conical Screens, an Operator Approach

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Abstract. The aim of this work is to construct explicitly resolvent operators for a class of boundary value problems in diffraction theory. These are formulated as boundary value problems for the three-dimensional Helmholtz equation with Dirichlet or Neumann conditions on a plane screen of polynomial-conical form (including unbounded and multiply-connected screens), in weak formulation. The method is based upon operator theoretical techniques in Hilbert spaces, such as the construction of matrical coupling relations and certain orthogonal projections, which represent new techniques in this area of applications. Various cross connections are exposed, particularly considering classical Wiener–Hopf operators in Sobolev spaces as general Wiener–Hopf operators in Hilbert spaces and studying relations between the crucial operators in game. Former results are extended, particularly to multiply-connected screens.

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1. Formulation of problems and main results

Given a proper open subset $\Sigma \subset \mathbb{R}^2$, we consider the domain Ω defined by

$$\begin{aligned} &\Omega = \mathbb{R}^3 \setminus \Gamma \\ &\Gamma = \overline{\Sigma} \times 0 = \{x = (x_1, x_2, 0) \in \mathbb{R}^3 : x' = (x_1, x_2) \in \overline{\Sigma}\}. \end{aligned} \tag{1.1}$$

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For convenience the boundary manifold Γ is closed (in $\mathbb{R}^3$) and the *screen* Σ is open (a domain in $\mathbb{R}^2$). However, the two sets will be identified in some sense (provided $\operatorname{int}\operatorname{clos}\Sigma = \Sigma$). As a rule we assume $\Sigma \neq \emptyset$, $\Sigma \neq \mathbb{R}^2$, if nothing else is said.

Problems of diffraction from a plane screen Γ are often formulated in terms of (or reduced to) the solution of the three-dimensional Helmholtz equation (HE) in Ω with Dirichlet or Neumann conditions on Γ, briefly written as

$$\begin{aligned} \left(\Delta + k^2\right) u &= 0 \quad \text{in } \Omega \\ Bu &= g \quad \text{on } \Gamma = \partial\Omega. \end{aligned} \tag{1.2}$$

Herein k is the wave number and we assume that $\Im m\, k > 0$ throughout this paper (some parts are restricted to $\Re e k = 0$). B stands for the boundary operator, taking the trace or normal derivative of u on Γ. We think of the weak formulation looking for $u \in L^2(\Omega)$ with restrictions $u^\pm = u|_{\Omega^\pm}$ to the upper and lower half-space $\Omega^\pm = \{x \in \mathbb{R}^3 : \pm x_3 > 0\}$ that satisfy $u^\pm \in H^1(\Omega^\pm)$ and the common transmission conditions across the complement of $\overline{\Sigma}$:

$$\Sigma' = \mathbb{R}^2 \setminus \overline{\Sigma},$$

namely

$$\begin{aligned} u_0^+ - u_0^- &= [u^+ - u^-]|_{x_3=0} &&= 0 \\ u_1^+ - u_1^- &= [\partial u^+/\partial x_3 - \partial u^-/\partial x_3]|_{x_3=0} &&= 0 \end{aligned} \qquad \text{on} \quad \Sigma' \tag{1.3}$$

according to the trace theorem and by help of representation formulas, see [20] and Section 2 for details. In a sense, this is equivalent to state that the HE holds across the complementary screen Σ' [26]. The boundary data g are arbitrarily given in the corresponding data space $H^{1/2}(\Sigma)$ or $H^{-1/2}(\Sigma)$, respectively (values of g in the boundary of Σ do not matter in this space setting).

For convenience we study the (homogeneous) HE, since boundary value problems for the inhomogeneous HE $Au = (\Delta + k^2)u = f$ can be "equivalently reduced" under the present assumptions, see [40]. Hence the operator associated with the boundary value problem (BVP) can be written as

$$B_0 = B|_{\ker A} \,:\, \mathcal{H}^1(\Omega) \to H^{\pm 1/2}(\Sigma) \tag{1.4}$$

where $\mathcal{H}^1(\Omega)$ denotes the space of weak solutions of the HE in Ω and B_0 denotes the restriction of B to this space. We are looking for the inverse B_0^{-1}, the so-called resolvent operator.

Sometimes different data $g^\pm$ are prescribed on the two banks $\Sigma^\pm$ of the screen corresponding to $x_3 = \pm 0$. This generalization is not very important from the physical point of view (where g denotes the trace of the "incoming field", e.g.), but useful for understanding the structure of the problems. In this case the BVPs can be briefly written in the form

$$\begin{aligned} & u \in \mathcal{H}^1(\Omega) \\ & B_0 u = \begin{pmatrix} B^+ \\ B^- \end{pmatrix} u = g = \begin{pmatrix} g^+ \\ g^- \end{pmatrix} \quad \text{on} \quad \Gamma = \partial\Omega \end{aligned} \tag{1.5}$$

where now

- in case of the Dirichlet problem: g is given in the space $H^{1/2}(\Sigma)^2 = H^{1/2}(\Sigma) \times H^{1/2}(\Sigma)$ with the compatibility condition that $g^+ - g^-$ is extensible by zero from Σ to the full plane (corresponding to $x_3 = 0$) within $H^{1/2}(\mathbb{R}^2)$,
- in case of the Neumann problem: g is given in the space $H^{-1/2}(\Sigma)^2 = H^{-1/2}(\Sigma) \times H^{-1/2}(\Sigma)$ with the compatibility condition that $g^+ - g^-$ be extensible by zero from Σ to the full plane (corresponding to $x_3 = 0$) within $H^{-1/2}(\mathbb{R}^2)$, see [20] for details.

We express these compatibility conditions briefly by writing

$$g \in H^{1/2}(\Sigma)^2_{\sim} \quad \text{respectively} \quad g \in H^{-1/2}(\Sigma)^2_{\sim}. \tag{1.6}$$

In several publications the second compatibility condition is written in the form $g^+ + g^- \in \widetilde{H}^{-1/2}(\Sigma)$ according to the convention that the normal derivative is always taken with respect to the outer (or inner) normal, i.e., $g^- = -\partial/\partial x_3 u$ on Σ^-, in contrast to the present situation.

The question of "low regularity", i.e., $u \in H^{1+\varepsilon}$, $\varepsilon \in]0, 1/2[$, could be included from the beginning, but will be answered only at the end of Section 5, to keep the notation short.

It is well known that all the above-mentioned BVPs are correctly posed provided Σ is a strong Lipschitz domain (bounded) or special Lipschitz domain (unbounded) [20, 42]. This results from the use of Green's formula (for uniqueness), reduction to boundary integral or pseudo-differential equations, their Fredholm property, an index formula and strong ellipticity (for existence). The fact that the associated operator $B_0 : \mathcal{H}^1(\Omega) \to H^{\pm 1/2}(\Sigma)$ is a bounded, linear and bijective operator acting in Hilbert spaces, implies (by the inverse mapping theorem) that B_0^{-1} is continuous.

The question is: Can we obtain an explicit formula of B_0^{-1}, not only for very special screens like half-planes and certain cones [25]? In the present article we shall admit solutions in closed analytical form or series expansion, as well as infinite operator products which are strongly convergent. In this sense, the answer will be positive for a surprisingly large class of problems and possibly not useless in view of the capacity of modern computers.

Let us briefly look at the classes of domains $\Sigma \subset \mathbb{R}^2$ under consideration. The following domain properties are crucial in what follows.

- First we shall assume the *strong extension property* [20, 23], i.e., for any $s \in \mathbb{R}$, there exists a continuous extension operator which is left invertible by restriction:

$$\begin{aligned} \ell^s_\Sigma \; &: H^s(\Sigma) \to H^s(\mathbb{R}^2) \\ r_\Sigma \ell^s_\Sigma &= I_{H^s(\Sigma)}. \end{aligned} \tag{1.7}$$

Lipschitz domains (that are bounded and characterized by fulfilling the uniform cone property [18, 20]) and (unbounded) special Lipschitz domains in the sense of [42] (of the form $\Sigma = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 > \varphi(x_1)\}$ where φ is uniformly Lipschitz continuous and rotations of this kind of domain) fulfil the strong extension property. The existence of continuous extension operators (1.7) guarantees the equivalence of B_0 to operators which have the form of a general Wiener–Hopf operator W (see Section 2), i.e., $B_0 = EWF$ where E and F are linear homeomorphisms.

- Second we shall confine our considerations to domains with the property $\operatorname{int}\operatorname{clos}\Sigma = \Sigma$, which is needed for a relaxed use of Sobolev spaces. Note that this excludes "cracks" in the screen (also called "slit domains") with discontinuities across the cracks, which could be considered using more complicated notation than $H^s(\Sigma)$ (in general, the notion of $H^s(\Sigma)$ with Lipschitz domains Σ is not suitable for that case, see [20], p. 110, and the introduction of [15], for instance).

A domain Σ with these two properties is said to be an *E-domain*. The properties are actually needed only for $s = \pm 1/2$ in the basic results.

Further we shall work with an algebra $\mathcal{A}_2$ of open subsets $\Sigma \subset \mathbb{R}^2$ which contains open half-planes, finite intersections and the interior of complements of elements of $\mathcal{A}_2$.

Therefore we introduce the following.

Definition 1.1. A *convex polygonal-conical domain* (convex PCD) in $\mathbb{R}^2$ is given by

$$\Sigma = \bigcap_{j=1,\dots,m} \Sigma_j \quad \text{where} \quad \Sigma_j \quad \text{are open half-planes.} \tag{1.8}$$

A *polygonal-conical domain* (PCD) in $\mathbb{R}^2$ is given by

$$\Sigma = \operatorname{int} \bigcup_{j=1,\dots,m} \operatorname{clos} \Sigma_j \tag{1.9}$$

where Σ_j are convex PCDs which do not meet in a corner.

Remark 1.2. The following observations are obvious:

1. Convex PCDs are simply connected, PCDs may be multiply connected, both possibly unbounded (cones are included).
2. PCDs are E-domains.
3. The set of PCDs (including $\Sigma = \emptyset$ and $\Sigma = \mathbb{R}^2$) coincides with the minimal set algebra $\mathcal{A}_2$ described above, since they allow a representation

$$\Sigma = \mathbb{R}^2 \setminus \Bigg(\bigcap_{j=1,\dots,m} (\mathbb{R}^2 \setminus \Sigma_j) \Bigg) \tag{1.10}$$

where Σ_j are convex PCDs. This results from the De Morgan formulas and some elementary topological consideration.

4. The set of Lipschitz domains does not form a set algebra, because the intersection of two Lipschitz domains does not necessarily have the strong extension property. Also special Lipschitz domains do not generate an algebra of sets which have the strong extension property, for the same reason.

In order to describe the spaces for the boundary data in more detail, we recall the definition of the usual Sobolev spaces $H^s = H^s(\mathbb{R}^n)$ (sometimes named Bessel potential or fractional Sobolev spaces) and of the Sobolev spaces $H^s(\Sigma)$, H^s_Σ, $\widetilde{H}^s(\Sigma)$, as well (see, e.g., [16, 20]). Thus let

$$H^s = H^s(\mathbb{R}^n) = \left\{ f \in \mathcal{S}' : \|f\| = \left(\int_{\mathbb{R}^n} \left| \hat{f}(\xi) \right|^2 (\xi^2+1)^s d\xi \right)^{1/2} < \infty \right\} \quad (1.11)$$

where ξ^2 stands for $|\xi|^2$ and $\mathcal{S}' = \mathcal{S}'(\mathbb{R}^n)$ denotes the Schwartz distribution space and $\hat{f}(\xi) = \mathcal{F}_{x\mapsto\xi} f(x) = \int_{\mathbb{R}^n} e^{ix\xi} f(x)dx$ the (n-dimensional) Fourier transform of $f \in \mathcal{S}$ extended to distributions $f \in \mathcal{S}'$. H^s is a Hilbert space with inner product

$$\langle \varphi, \psi \rangle_s = \int_{\mathbb{R}^n} \hat{\varphi}(\xi)\, \overline{\hat{\psi}(\xi)}\, (\xi^2+1)^s d\xi \quad , \quad \varphi, \psi \in H^s(\mathbb{R}^n). \quad (1.12)$$

The function $\lambda(\xi) = (\xi^2+1)^{1/2}$, $\xi \in \mathbb{R}^n$, will play a special role in what follows, since it can be considered as a particular case of the square root of the "Helmholtz symbol" $t(\xi) = \lambda_k(\xi) = (\xi^2 - k^2)^{1/2}$ for $k = i$ (the double notation has historical reasons). We shall always choose branches, continuous in $\mathbb{R}^n$, such that $\lambda_k(\xi) \to +\infty$ as $|\xi| \to +\infty$. It may be useful to consider the spaces H^s as the isometric images of the Bessel potential operators

$$\Lambda^{-s} = \mathcal{F}^{-1}\lambda^{-s} \cdot \mathcal{F} : L^2 \to H^s \quad , \quad s \in \mathbb{R}. \quad (1.13)$$

The restriction operator which restricts a function or distribution on $\mathbb{R}^n$ to an open subset Σ will be denoted by r_Σ. Thus $H^s(\Sigma) = r_\Sigma(H^s)$, and the norm in $H^s(\Sigma)$ is defined by

$$\|f\|_{H^s(\Sigma)} = \inf_{\ell} \|\ell f\|_{H^s}$$

where ℓf stands for any extension of f to a distribution in H^s. An equivalent norm can be defined via the Sobolev–Slobodetski norm for $s > 0$ and via a duality for $s < 0$. Furthermore, we denote by H^s_Σ the (closed) subspace of H^s which consists of all distributions with support in the closure of Σ. By $\widetilde{H}^s(\Sigma)$ we denote the space of all distributions which are the restrictions of distributions in H^s_Σ, i.e., $\widetilde{H}^s(\Sigma) = r_\Sigma(H^s_\Sigma)$. A norm is defined by

$$\|f\|_{\widetilde{H}^s(\Sigma)} = \inf_{\ell_0} \|\ell_0 f\|_{H^s}$$

where $\ell_0 f$ stands for any extension of f to a distribution in H^s_Σ (which is unique only for $s \geq -1/2$, see [14], pages 4–5, in which case the last infimum is redundant). Notice that while $\widetilde{H}^s(\Sigma)$ is always continuously embedded in $H^s(\Sigma)$, these two spaces coincide for $s \in]-1/2, 1/2[$. In various publications $\widetilde{H}^s(\Sigma)$ is defined as the set of $H^s(\Sigma)$ function(al)s that are extendible by zero to an element of $H^s(\mathbb{R}^n)$ [18, 20]. For E-domains this definition is equivalent to the present one.

Now we are in a position to summarize the main result.

Theorem 1.3 (Main Theorem). *Let Σ be a PCD. Then the resolvent operator B_0^{-1} (see (1.4), (1.5)) for the Dirichlet or Neumann problem is explicitly given in terms of infinite operator products (presented in Sections 4 and 5) which strongly converge in the common (Bessel potential) norm of $H^{\pm 1/2}(\mathbb{R}^2)$ for $k = i$ and in a modified equivalent norm for $k \in i\mathbb{R}_+$, respectively. In the remaining cases of $k \in \mathbb{C}$, $\Im m k > 0$ the resolvent operator can be explicitly represented by (additional) use of Neumann series.*

The principle steps are: (1) to show operator equivalence of B_0 with a boundary pseudo-differential operator that has the form of a general Wiener–Hopf operator (WHO), (2) to represent B_0^{-1} in terms of a certain projector acting in $H^{\pm 1/2}(\mathbb{R}^2)$, which depends heavily on the form of Σ, (3) for screens which are convex PCDs, to give an explicit formula for these kind of projectors in case of $k \in i\mathbb{R}_+$, choosing a topology where they are orthogonal and using a result of Halmos [19] for the representation of the orthogonal projector onto the intersection of closed Hilbert subspaces, (4) to reduce the case of arbitrary k with $\Im m k > 0$ to the previous by approximation, and finally (5) to reduce the case of non-convex screens to the case of convex screens by matrical coupling of associated WHOs and the so-called geometric perspective of Ferreira dos Santos [30, 31] for general WHOs, noting that not only complements of convex screens are admitted, but arbitrary PCDs.

Some similar ideas appeared already in special situations or different settings, see [15, 24, 25, 37, 39] and will be pointed out in the corresponding context. However, some of the cited results are presented here with a new, more compact proof, e.g., Theorem 3.5 and parts of Theorem 3.8 and of Theorem 3.10.

From the historical point of view, one can say, that the story started with the solution of Sommerfeld's half-plane problem [35] by modern Wiener–Hopf methods [26], contributions to the diffraction by a quarter-plane [11, 12, 25, 44] and the discovery of relations with general WHOs [31, 37]. The present paper could be regarded as an extension of [24] to non-convex, general polynomial-conical screens, however with several new techniques that provide a deeper insight into the structure of this kind of BVPs. Finally it should be noticed that the present screen problems are quite different from wedge diffraction problems in formulation and structure.

2. Reduction to boundary pseudo-differential equations and form of resolvent operators

For this step we need a precise notation of Wiener–Hopf operators in Sobolev spaces. We shall use only a scalar version (the matrix analogue is evident).

Definition 2.1. A Wiener–Hopf operator in Sobolev spaces (briefly referred to as *classical WHO*) is given by

$$W_{\phi,\Sigma} = r_\Sigma\, A_\phi \; : \; H^r_\Sigma \to H^s(\Sigma) \tag{2.1}$$

where $\Sigma \subset \mathbb{R}^n$ is a domain, $r, s \in \mathbb{R}$, $A_\phi = \mathcal{F}^{-1}\phi \cdot \mathcal{F}$ is a convolution (translation invariant) operator in $\mathbb{R}^n$ of order $r - s$, i.e., $\phi_0 = \phi\lambda^{r-s} \in L^\infty(\mathbb{R}^n)$.

Remark 2.2. Other popular notations of WHOs are the following. The "classical WHO" acting on $L^2(\mathbb{R}_+)$ (or on $L^p(\mathbb{R}_+)$ etc. [22]) is given by

$$Wf(x) \,=\, af(x) + \int_0^\infty k(x-y)f(y)dy \quad , \quad x > 0 \tag{2.2}$$

with $a \in \mathbb{C}, k \in L^1(\mathbb{R})$. It can be briefly written as

$$W = r_+ A_\phi \ell_0 : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+) \tag{2.3}$$

where $\phi = a + \mathcal{F}k$ is the Fourier symbol of W [17, 29] and ℓ_0 the zero extension from $L^2(\mathbb{R}_+)$ to $L^2(\mathbb{R})$. This is easily generalized to an operator acting on $L^2(\Sigma)$, $\Sigma \subset \mathbb{R}^n$ by writing

$$W = r_\Sigma A_\phi \ell_0 : L^2(\Sigma) \to L^2(\Sigma) \tag{2.4}$$

and makes sense already, if Σ is measurable. The direct generalization to Sobolev spaces makes sense if the extension $\ell_0 : H^s(\Sigma) \to H^s(\mathbb{R}^n)$ is well defined, e.g., for Lipschitz or special Lipschitz domains Σ. It can be written as

$$W = r_\Sigma A_\phi \ell_0 : H^s(\Sigma) \to H^s(\Sigma) \tag{2.5}$$

defined by restriction ($s > 0$) or by continuous extension ($s < 0$), if $s \in]-1/2, 1/2[$. In contrast, the "Eskin like notation" (2.1) makes sense for all $s \in \mathbb{R}$ and arbitrary domains Σ [16, 28].

Another generalization will be important for our purposes, the notion of "general WHOs". That will be discussed in Section 3.

Now we come to the point where these operators appear in reality.

Theorem 2.3 (Representation Theorem for the Dirichlet problem). *Assume that $\Sigma \subset \mathbb{R}^2$ be any (proper) open subset of $\mathbb{R}^2$, Ω be given by (1.1) and $\Omega^\pm = \{x \in \mathbb{R}^3 : \pm x_3 > 0\}$. Then the Dirichlet problem in Ω (see (1.5)) is well posed if and only if the following WHO is invertible:*

$$W_{t^{-1},\Sigma} = r_\Sigma\, A_{t^{-1}} : H^{-1/2}_\Sigma \to H^{1/2}(\Sigma). \tag{2.6}$$

In this case, the solution of the Dirichlet problem is given by the formulas

$$u = \mathcal{K}_{D,\Omega}(g_1, g_2) = \begin{cases} \mathcal{K}_{D,\Omega^+}\, u_0^+ & in \quad \Omega^+ \\ \mathcal{K}_{D,\Omega^-}\, u_0^- & in \quad \Omega^- \end{cases} \tag{2.7}$$

$$\mathcal{K}_{D,\Omega^+} u_0^+(x) = \mathcal{F}^{-1}_{\xi' \mapsto x'} e^{-t(\xi')x_3} \widehat{u_0^+}(\xi') = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{-i\xi' x' - t(\xi_1,\xi_2)x_3} \widehat{u_0^+}(\xi_1, \xi_2) d\xi'$$

$$\mathcal{K}_{D,\Omega^-} u_0^-(x) = \mathcal{F}^{-1}_{\xi' \mapsto x'} e^{t(\xi')x_3} \widehat{u_0^-}(\xi') = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{-i\xi' x' + t(\xi_1,\xi_2)x_3} \widehat{u_0^-}(\xi_1, \xi_2) d\xi'$$

$$\begin{pmatrix} u_0^+ \\ u_0^- \end{pmatrix} = \Upsilon_D^{-1} \begin{pmatrix} \ell_0 & 0 \\ 0 & A_{t^{-1}} W_{t^{-1},\Sigma}^{-1} \end{pmatrix} \Upsilon_D \begin{pmatrix} g^+ \\ g^- \end{pmatrix}, \quad \Upsilon_D = \begin{pmatrix} I & -I \\ I & I \end{pmatrix}$$

abbreviating $\xi' = (\xi_1, \xi_2) \in \mathbb{R}^2$, $d\xi' = d\xi_1 d\xi_2$, $t(\xi') = (\xi_1^2 + \xi_2^2 - k^2)^{1/2}$, $x' = (x_1, x_2) \in \mathbb{R}^2$ *and* $\xi' x' = \xi_1 x_1 + \xi_2 x_2$.

Proof. (Sketch) Based on ideas from [24] and [15] we see that a possible solution is represented by the trace data $u_0^\pm$ (in the entire plane $x_3 = 0$), where $u_0^+ - u_0^-$ is directly given and that $u_0^+ + u_0^- = A_t^{-1}(u_1^+ - u_1^-)$ where $u_1^+ - u_1^-$ satisfies a boundary pseudo-differential equation which has the form of a (generalized) Wiener–Hopf equation on Σ. If the WHO is invertible, the BVP is well posed, since one can verify that the formulas represent linear homeomorphisms between the data and the solution spaces. Conversely, if the BVP is well posed, the WHO must be bijective. As a bounded linear operator, it is necessarily a homeomorphism according to the inverse mapping theorem.

More details can be found in the context of Sommerfeld potentials (where Σ is a half-plane) [14, 15]. Also BVPs for the Lamé equation have been solved in a similar way already in [13]. □

Theorem 2.4 (Representation Theorem for the Neumann problem). *Assume that* $\Sigma \subset \mathbb{R}^2$ *be any (proper) open subset of* $\mathbb{R}^2$, Ω *be given by* (1.1) *and* $\Omega^\pm = \{x \in \mathbb{R}^3 : \pm x_3 > 0\}$. *Then the Neumann problem in* Ω *(see* (1.5)*) is well posed if and only if the following WHO is invertible:*

$$W_{t,\Sigma} = r_\Sigma A_t : H_\Sigma^{1/2} \to H^{-1/2}(\Sigma). \tag{2.8}$$

In this case, the solution of the Neumann problem is given by the formulas

$$u = \mathcal{K}_{N,\Omega}(g_1, g_2) = \begin{cases} \mathcal{K}_{N,\Omega^+} u_1^+ & in \quad \Omega^+ \\ \mathcal{K}_{N,\Omega^-} u_1^- & in \quad \Omega^- \end{cases} \tag{2.9}$$

$$\begin{aligned} \mathcal{K}_{N,\Omega^+} u_1^+(x) &= \mathcal{F}^{-1}_{\xi' \mapsto x'} e^{-t(\xi')x_3} \frac{-1}{t(\xi')} \widehat{u_1^+}(\xi') \\ &= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{-i\xi' x' - t(\xi_1,\xi_2)x_3} \frac{-1}{t(\xi_1,\xi_2)} \widehat{u_1^+}(\xi_1,\xi_2) d\xi_1 d\xi_2 \\ \mathcal{K}_{N,\Omega^-} u_1^-(x) &= \mathcal{F}^{-1}_{\xi' \mapsto x'} e^{t(\xi')x_3} \frac{1}{t(\xi')} \widehat{u_1^-}(\xi') \\ &= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{-i\xi' x' + t(\xi_1,\xi_2)x_3} \frac{1}{t(\xi_1,\xi_2)} \widehat{u_1^-}(\xi_1,\xi_2) d\xi_1 d\xi_2 \end{aligned}$$

$$\begin{pmatrix} u_1^+ \\ u_1^- \end{pmatrix} = \Upsilon_N^{-1} \begin{pmatrix} A_t W_{t,\Sigma}^{-1} & 0 \\ 0 & \ell_0 \end{pmatrix} \Upsilon_N \begin{pmatrix} g^+ \\ g^- \end{pmatrix}, \quad \Upsilon_N = \begin{pmatrix} I & I \\ I & -I \end{pmatrix}.$$

Proof. Conclusions are similar as before. □

Remark 2.5. Existence and uniqueness of a solution are known from [18, 43] in case of Lipschitz domains and from [25, 38] in case of half-planes. Hence uniqueness is trivial for convex PCDs and obvious for PCDs from (1.9).

The uniqueness result follows also directly from the Green formula, while existence can be shown with the Lax–Milgram Lemma (see, e.g., [9], Section 6.3).

On the other hand, by using the representation formulas of solutions with layer potentials and Plemelji–Sokhotskii formulas, one reduces both, the Dirichlet and the Neumann BVPs to boundary integral equations with positive definite symbol in the Bessel potential spaces where the operators act in the spaces $\widetilde{H}^{1/2}(\Sigma) \to H^{-1/2}(\Sigma)$ or $\widetilde{H}^{-1/2}(\Sigma) \to H^{1/2}(\Sigma)$ (cf. (2.6) and (2.8)). Those can also be studied with the help of results from [12] (lifting in the Bessel potential spaces and pseudo-differential operators with locally sectorial symbols).

We shall see later (in the proof of Theorem 5.1) that the crucial terms in these formulas $A_{t^{-1}} W^{-1}_{t^{-1},\Sigma}$ and $A_t W^{-1}_{t,\Sigma}$ can be interpreted as very particular extension operators or operators of the form of a composition $\Pi\ell$ where Π is a particular projector and ℓ an arbitrary extension of a functional from $H^{\pm 1/2}(\Sigma)$ to a functional in $H^{\pm 1/2}$.

3. Some results on general Wiener–Hopf operators

A *general Wiener–Hopf operator* (also abbreviated by WHO) is given by

$$W = P_2 A|_{P_1 X} \tag{3.1}$$

where $A : X \to Y$ is a bounded linear operator acting in Banach spaces and $P_1 \in \mathcal{L}(X), P_2 \in \mathcal{L}(Y)$ are projectors, i.e., $P_j^2 = P_j, j = 1, 2$. By convention, W is regarded as an operator from $P_1 X = \operatorname{im} P_1$ into $P_2 Y = \operatorname{im} P_2$, although, P_2 acts into Y, i.e., strictly speaking, not into $P_2 Y$ (cf. [27, Chapter III]). This *convention* will be applied and referred to in the sequel for convenience (and following the tradition). For practical reasons we enlarge the convention by identifying in some formulas W with $P_2\, A\, P_1$ as an operator acting between the full spaces. Occasionally we will also consider W^{-1} as acting on the full space, i.e., we consider $P_1 W^{-1} P_2 = P_1 (P_2 A|_{P_1 X})^{-1} P_2$. This makes some formulas more compact.

The notation (3.1) was introduced in [10, 33], first in a symmetric setting where $X = Y$ and $P_1 = P_2 = P$ for operators in Hilbert spaces, and later in the asymmetric setting of (3.1) [36, 37]. Main objective in those publications was the (generalized) inversion of W by an operator factorization of A (assuming that A is invertible). Here we study a completely different (abstract) idea, presented by A.F. dos Santos [30, 31], originally connected to more special applications [24, 25, 39].

3.1. Identification of general Wiener–Hopf operators

The connection between classical WHOs (2.1) and general WHOs (3.1) is given via a continuous extension operator

$$E^s_\Sigma : H^s(\Sigma) \to H^s(\mathbb{R}^n) \tag{3.2}$$

provided it exists for $\Sigma \subset \mathbb{R}^n$ (see Section 1), namely by the identification

$$\begin{aligned} X &= H^r &&, \quad Y = H^s \\ P_1 X &= H^r_\Sigma &&, \quad P_2 = E^s_\Sigma \, r_\Sigma \\ A &= \mathcal{F}^{-1}\,\phi \cdot \mathcal{F} &&: \quad H^r \to H^s. \end{aligned} \tag{3.3}$$

We observe, in the identification of a general WHO, that not the full definitions of P_1 and P_2 are relevant but only $\operatorname{im} P_1$ and $\ker P_2$. The domain $\operatorname{dom} W$ of W is a complemented subspace of X and can be seen as the image of any projector P with the same image $\operatorname{im} P = \operatorname{dom} W = \operatorname{im} P_1$ (and arbitrary complement for the kernel), i.e.,

$$PP_1 = P_1 \quad , \quad P_1 P = P. \tag{3.4}$$

Further, if P_2 and Π are projectors with the same kernel, then the following two WHOs are equivalent:

$$W = P_2 A|_{P_1 X} \quad \sim \quad \widetilde{W} = \Pi A|_{P_1 X} = \Pi A|_{PX} \tag{3.5}$$

because

$$\Pi P_2 = \Pi \quad , \quad P_2 \Pi = P_2. \tag{3.6}$$

In the classical case, Π reflects the variety of possible extension operators. In Hilbert spaces we conclude easily the following interesting result:

Proposition 3.1. *Let $W = P_2 A|_{P_1 X}$ be a general WHO (see (3.1)) where X, Y are Hilbert spaces. Then*

$$W \quad \sim \quad \widetilde{W} = \Pi A|_{PX} \tag{3.7}$$

where P and Π are orthogonal projectors.

Proof. It is well known that the orthogonal projectors P onto $P_1 X$ and Π along $(I - P_2)Y$ exist. Hence we have the equivalence relation

$$W = P_2 A|_{P_1 X} = P_2\, \Pi A|_{PX}\, P|_{P_1 X} = P_2|_{\Pi Y}\, \widetilde{W}\, P|_{P_1 X}$$

between W and $\widetilde{W}$ where the outer factors are bijective in the sense of the above-mentioned convention. □

The foregoing result will be used in Section 4 for the construction of certain non-orthogonal projectors (in the case $k \notin i\mathbb{R}$) based upon the knowledge of corresponding orthogonal projectors with the same image (resulting from Lemma 4.2), by an approximation argument.

Focusing on (generalized) inverses, we obtain in a similar way:

Proposition 3.2. *Let W, $\widetilde{W}$ be general WHOs related by (3.4)–(3.6) and let $\widetilde{W}^- : \Pi Y \to PX$ be a generalized inverse of $\widetilde{W}$. Then a generalized inverse of W is given by*

$$W^- = P_1|_{PX} \widetilde{W}^- \Pi|_{P_2 Y}. \tag{3.8}$$

Proof. By verification $WW^-W = W$. □

Clearly the statement includes the cases of one-sided invertibility, Fredholmness, and invertibility that is needed in this paper.

Now, the identification of the general WHO $W = P_2 A_\phi|_{P_1 X}$ with the Eskin type WHO is given by the equivalence relations

$$E^s_\Sigma W_{\phi,\Sigma} = W \quad , \quad W_{\phi,\Sigma} = r_\Sigma W.$$

Consequently, in case of invertible WHOs

$$W^{-1}_{\phi,\Sigma} = W^{-1} E^s_\Sigma \quad , \quad A_\phi W^{-1}_{\phi,\Sigma} = A_\phi W^{-1} E^s_\Sigma$$

which makes the connection with Theorems 2.3 and 2.4.

Another relationship between WHOs turns out to be very important in what follows. Given a general WHO (3.1), where now A is assumed to be boundedly invertible, and let the complemented projectors be denoted by $Q_1 = I_X - P_1$ and $Q_2 = I_Y - P_2$, respectively, we call

$$W_* = Q_1 A^{-1}|_{Q_2 Y} \;:\; Q_2 Y \to Q_1 X \tag{3.9}$$

the *WHO associated with* W. This notation was introduced in [10] for symmetric and in [36] for asymmetric setting, respectively. In a different context (realization theory, minimal factorization) it was called an "indicator" of W, thinking of various possibilities of extending W to an operator matrix

$$A = \begin{pmatrix} W & * \\ * & * \end{pmatrix}, \tag{3.10}$$

see [1, 2] for details.

3.2. A geometric perspective

Following an idea of A.F. dos Santos [30, 31] (which has roots in [10, 34] and [24]) we study a "geometric relation" between AP_1X and Q_2Y. In contrast to the existing literature we shall base this consideration upon the following result, which seems to be still unpublished [2] but very efficient.

Lemma 3.3. *Given two pairs of complementary projectors in Banach spaces,* P_1, $Q_1 = I - P_1 \in \mathcal{L}(X)$, P_2, $Q_2 = I - P_2 \in \mathcal{L}(Y)$ *and an invertible operator* $A \in \mathcal{L}(X, Y)$*, the following operator factorization is valid:*

$$\begin{aligned} \begin{pmatrix} P_2AP_1 & 0 \\ 0 & Q_2 \end{pmatrix} &= \begin{pmatrix} P_2AP_1 & 0 \\ Q_2AP_1 & Q_2 \end{pmatrix} \begin{pmatrix} P_1 & 0 \\ -Q_2AP_1 & Q_2 \end{pmatrix} \\ &= \begin{pmatrix} P_2AP_1 & P_2AQ_1 \\ Q_2AP_1 & Q_2AQ_1 \end{pmatrix} \begin{pmatrix} P_1 & 0 \\ 0 & Q_1A^{-1}Q_2 \end{pmatrix} \\ &\qquad \begin{pmatrix} P_1 & P_1A^{-1}Q_2 \\ 0 & Q_2 \end{pmatrix} \begin{pmatrix} P_1 & 0 \\ -Q_2AP_1 & Q_2 \end{pmatrix}. \end{aligned} \tag{3.11}$$

Proof. The formula can be verified easily. □

Remark 3.4. Note that the matrix operator on the left of (3.11) acts as

$$M : P_1X \oplus Q_2Y \to P_2Y \oplus Q_2Y \cong Y,$$

i.e., not from X into Y in contrast to the first factor of the second line. For the first line of (3.11) we do not need the invertibility of A. It would be more adequate to write $P_2A|_{P_1X}$ instead of P_2AP_1 etc. We avoided this just for cosmetic reasons (see the convention).

Theorem 3.5 (Ferreira dos Santos 1988). *Let W be a general WHO, given by* (3.1) *where A is injective. Then W is invertible if and only if AP_1X and Q_2Y are complemented subspaces of Y, in brief*

$$AP_1X \,\oplus\, Q_2Y = Y. \tag{3.12}$$

If A is invertible we equivalently have

$$P_1X \,\oplus\, A^{-1}Q_2Y = X. \tag{3.13}$$

Proof. Formula (3.12) is an interpretation of the first part of (3.11) if one takes into account that $P_2AP_1 + Q_2AP_1 = AP_1$ and that the last factor in the first line of (3.11) is invertible. The second conclusion (3.13) is then evident. □

Corollary 3.6. *Let $W = P_2A|_{P_1X} : P_1X \to P_2Y$ be a general WHO with $A : X \to Y$ being invertible. Assume that W is invertible* (*or, equivalently* (3.12) *holds*). *Then the inverse $W^{-1} : P_2Y \to P_1X$ can be represented by*

$$\begin{aligned} W^{-1} &= A^{-1}\Pi|_{P_2Y} \quad \textit{where } \Pi \textit{ projects onto } AP_1X \textit{ along } Q_2Y \\ &= PA^{-1}|_{P_2Y} \quad \textit{where } P \textit{ projects onto } P_1X \textit{ along } A^{-1}Q_2Y\,. \end{aligned} \tag{3.14}$$

Moreover, in this case these projectors are given by

$$\Pi \;=\; A\,W^{-1}\,P_2 \quad , \quad P \;=\; W^{-1}\,P_2\,A \tag{3.15}$$

as operators acting in Y and X, respectively.

Remark 3.7. For the symmetric setting ($X = Y$, $P_1 = P_2 = P$) Theorem 3.5 was proved in [10]. There are further generalizations to the case where W is Fredholm, e.g., see [30, 31]. Another generalization to the case where W is generalized invertible might be possible by ideas of [37], we suppose. However, they are not needed here.

The formulas (3.14) imply that $P = A^{-1}\Pi A$, i.e., it obviously suffices to construct one of the two projectors P and Π in order to invert W.

3.3. Matrical coupling

Let us recall two definitions and a few known results. Two bounded linear operators in Banach spaces $S \in \mathcal{L}(X_1, Y_1)$, $T \in \mathcal{L}(X_2, Y_2)$ are said to be *matrically coupled*, if there is an invertible operator matrix (with suitable entries $*$) such that

$$\begin{pmatrix} S & * \\ * & * \end{pmatrix} = \begin{pmatrix} * & * \\ * & T \end{pmatrix}^{-1}. \tag{3.16}$$

Two bounded linear operators in Banach spaces S and T are said to be *equivalent after extension*, in brief

$$S \overset{*}{\sim} T, \tag{3.17}$$

if there exist Banach spaces Z_1, Z_2 and linear homeomorphisms E, F such that

$$\begin{pmatrix} S & 0 \\ 0 & I_{Z_1} \end{pmatrix} = E \begin{pmatrix} T & 0 \\ 0 & I_{Z_2} \end{pmatrix} F. \tag{3.18}$$

Example. Looking at Lemma 3.3 and Remark 3.4 we can interpret the first line of (3.11) as

$$W \overset{*}{\sim} (AP_1|Q_2) \ : \ P_1X \oplus Q_2Y \to Y. \tag{3.19}$$

In fact, the relation presented in the first line of (3.11) may be viewed as a particular form of an equivalence after extension relation since the extension is made from one side only. Namely, we have there

$$W \oplus I_{Q_2} \sim (AP_1|Q_2), \tag{3.20}$$

which is a so-called *equivalence after one-sided extension*. The equivalence after one-sided extension concept, being stronger than the equivalence after extension, is intimately related with the *Schur coupling* notion [5, 32]. Schur coupled operators allow even more direct relationships between their null spaces and range spaces than in the equivalence after extension relation (cf. [3, §2–3] for extra details and still existing open problems within the Schur coupling theory).

We also would like to point out that in [46], Chapter 0 by S. Puntanen and G.P.H. Styan, we may find a very pleasant historical introduction about the Schur complement. There, the last formula of page 4 presents in fact an equivalence after one-sided extension which yields a very direct proof of the famous Schur (determinant) lemma [32].

Theorem 3.8 (Bart–Tsekanovsky 1991). *Let S and T be bounded linear operators in Banach spaces. Then $S \overset{*}{\sim} T$ if and only if S and T are matrically coupled.*

Remark 3.9. The importance of this theorem for us consists in the consequence, that an inverse of T can be computed from an inverse of S (and vice versa, if E^{-1} and F^{-1} are known). This is obvious from (3.18) but not from (3.16) – and was a celebrated fact in the 1980s [1].

The sufficiency ("if") part was already proved in [1], the necessity part ("only if") later in [4].

However it was observed in [2] that the sufficiency part in the symmetric case (where $X = Y, P_1 = P_2$) is an interpretation of a well-known formula, see, e.g., [17, 29]

$$\begin{aligned} PAP + Q \ &= \ (AP + Q)(I - QAP) \ = \ A(P + A^{-1}Q)(I - QAP) \\ &= \ A(P + QA^{-1}Q)(I + PA^{-1}Q)(I - QAP). \end{aligned} \tag{3.21}$$

In the asymmetric case, it is a consequence of the formula (3.11), which can be regarded as a direct generalization of (3.21).

At the end, the second (necessity) part of Theorem 3.8 is not so evident, particularly the construction of a coupling relation from an equivalence after extension relation, see [4]. However the conclusion that is most important in our applications can be proved independently and more directly as follows.

Theorem 3.10 (Speck 1985). *Let S and T be bounded linear operators in Banach spaces which are matrically coupled, i.e., $S = W = P_2A|_{P_1X}$ and $T = W_* = Q_1A^{-1}|_{Q_2Y}$ in the above notation. Further let V be a generalized inverse of W, i.e., $WVW = W$. Then a generalized inverse of W_* is given by*

$$V_* = Q_2(A - AP_1VP_2A)|_{Q_1X}. \tag{3.22}$$

Proof. Formula (3.22) is a consequence of the first part of Theorem 3.8 (matrically coupled operators are equivalent after extension) and formula (3.11) that implies: a generalized inverse of W yields a generalized inverse of the matrix on the left which yields a generalized inverse of the second factor of the second line which yields a generalized inverse of W_*. □

An earlier detailed and independent proof can be found in [37], pp. 21–22. Note that the present proof is constructive. In [37] the formula (3.22) was just guessed and verified. Clearly it includes the cases of one-sided invertibility, Fredholmness, and invertibility that is needed here.

4. Construction of the projectors P and Π onto/along $H_\Sigma^{\pm 1/2}$

In order to determine the WHO inverses needed in Theorem 2.3 and 2.4, we are now going to calculate the corresponding projectors (related by Corollary 3.6). Clearly all these operators exist and are unique as seen before in the introduction and in Theorem 3.5.

For convenience let us recall the relevant notation. Actually there appear various sceneries: (1) The abstract setting (with orthogonal and non-orthogonal projectors), (2) the concrete realizations (of Section 3.1) where Σ is an E-domain (with two cases concerning the Dirichlet and the Neumann problem, respectively), and (3) the special situations where Σ has particular form (half-plane, convex PCD, etc.). In the abstract setting we continue to consider the projectors

$$\begin{array}{llll} \Pi & \text{onto} & AP_1X & \text{along} \quad Q_2Y \\ P & \text{onto} & P_1X & \text{along} \quad A^{-1}Q_2Y \end{array} \tag{4.1}$$

where $A \in \mathcal{L}(X,Y)$ is boundedly invertible, $P_1 \in \mathcal{L}(X)$, $P_2 \in \mathcal{L}(Y)$ arbitrary projectors and $Q_1 = I_X - P_1$, $Q_2 = I_Y - P_2$. Using partly the same letters (for identification) we further consider the following realization where Σ is an E-domain,

$s \in \mathbb{R}$ and

$$\begin{aligned} &A = A_{t^{2s}} = \mathcal{F}^{-1}\, t^{2s} \cdot \mathcal{F} \,:\, H^s(\mathbb{R}^2) \,\to\, H^{-s}(\mathbb{R}^2) \\ &P_1 \quad \text{is any projector in} \quad H^s(\mathbb{R}^2) \quad \text{onto} \quad H^s_\Sigma \\ &P_2 \quad \text{is any projector in} \quad H^{-s}(\mathbb{R}^2) \quad \text{along} \quad H^{-s}_{\Sigma'}. \end{aligned} \tag{4.2}$$

For the Dirichlet problem we have $A = A_{t^{-1}}$, i.e., $s = -1/2$, for the Neumann problem $A = A_t$, i.e., $s = 1/2$. However, many of the following considerations work for general $s \in \mathbb{R}$.

The simplest case appears when Σ is a half-plane and $k = i$. Here we obtain formulas in closed analytical form and orthogonal projectors as follows:

Example. Consider the half-plane $\Sigma = \mathbb{R}^2_{1+} = \{x \in \mathbb{R}^2 : x_1 > 0\}$, the (orthogonal) projectors $P_+ = \ell_0 r_+ : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ onto $L^2_\Sigma = L^2_{\mathbb{R}^2_{1+}}$, $P_- = I - P_+$ and the Bessel potential operators [11, 12, 16] of order $s \in \mathbb{R}$:

$$\Lambda^s_+ = A_{\lambda^s_+} \quad , \quad \lambda^s_+(\xi) \;=\; \left(\xi_1 + i\sqrt{\xi_2^2 + 1}\right)^s \,,\, \xi \in \mathbb{R}^2$$

$$\Lambda^s_- = A_{\lambda^s_-} \quad , \quad \lambda^s_-(\xi) \;=\; \left(\xi_1 - i\sqrt{\xi_2^2 + 1}\right)^s \,,\, \xi \in \mathbb{R}^2.$$

For any $s \in \mathbb{R}$ we find the orthogonal projectors [7]

$$\begin{aligned} P^s_+ &= \Lambda^{-s}_+ P_+ \Lambda^s_+ \quad \text{onto} \quad H^s_\Sigma \\ P^s_- &= \Lambda^{-s}_- P_- \Lambda^s_- \quad \text{onto} \quad H^s_{\Sigma'} \\ \Pi^s_+ &= \Lambda^{-s}_- P_+ \Lambda^s_- \quad \text{along} \quad H^s_{\Sigma'} \\ \Pi^s_- &= \Lambda^{-s}_+ P_- \Lambda^s_+ \quad \text{along} \quad H^s_\Sigma. \end{aligned}$$

Hence $P^s_+ + \Pi^s_- = I_{H^s}$ and $P^s_- + \Pi^s_+ = I_{H^s}$.

If we specify $\Sigma = \mathbb{R}^2_{1+} = \{x \in \mathbb{R}^2 : x_1 > 0\}$ and $k = i$ in the second scenery, then $P_1 = P^s_+$ and $P_2 = \Pi^{-s}_+$ satisfy (4.2). If we specify moreover $A = A_{t^{2s}}$ in the first scenery, then $P = P_1 = P^s_+$ and $\Pi = P_2 = \Pi^{-s}_+$ satisfy (4.1).

The projector $\ell_0 r_+$ in $L^2(\mathbb{R}^2)$ coincides with the multiplication by the characteristic functions χ_+ of the half-space $\mathbb{R}^2_{1,+}$. This observation can be generalized to $\ell_0 r_\Sigma$ acting in the spaces $H^s(\mathbb{R}^2)$ if $|s| < 1/2$ provided Σ is a Lipschitz domain or an E-domain in $\mathbb{R}^2$, e.g., instead of the half-space $\mathbb{R}^2_{1,+}$.

4.1. Preliminaries

First we mention two facts which are independent of the choice of Σ. The first is a consequence of (1.12), the second is known from [19].

Lemma 4.1. *The projectors* Π, P *of* (4.1) *in the situation* (4.2) *are orthogonal with respect to the inner product* (1.12) *if* $k = i$.

Proof. Orthogonality of the two projectors is equivalent to the fact that

$$\langle A_t \varphi, \psi \rangle_{-s} = 0 \quad \text{for} \quad \varphi \in H^s_\Sigma \,,\, \psi \in H^{-s}_{\Sigma'}.$$

It suffices to consider smooth, rapidly decreasing functions, which are dense in these spaces, $\varphi \in \mathcal{S}_\Sigma = \{\varphi \in \mathcal{S} : \operatorname{supp}\varphi \subset \overline{\Sigma}\}$ and $\psi \in \mathcal{S}_{\Sigma'}$. These must satisfy

$$\int_{\mathbb{R}^2} A_\phi \varphi(\xi)\overline{\psi(\xi)} d\xi = 0 \quad \text{where} \quad \phi = \left(\frac{t(\xi)}{\lambda(\xi)}\right)^{2s} = \frac{(\xi^2 - k^2)^s}{(\xi^2+1)^s}.$$

Now $\operatorname{supp} A_\phi \varphi \subset \overline{\Sigma}$ for all $\varphi \in \mathcal{S}_\Sigma$ with $\operatorname{supp}\varphi \subset \overline{\Sigma}$ obviously holds if $k = i$, i.e., $t = \lambda$. □

Lemma 4.2 (Halmos 1982, Problem 96). *Given any Hilbert space H and orthogonal projectors $p_1, p_2, \ldots, p_m \in \mathcal{L}(H)$, the orthogonal projector onto $\operatorname{im} p_1 \cap \operatorname{im} p_2$ is given by the so-called infimum of the two projectors:*

$$p_1 \wedge p_2 = \prod_{j=1}^{\infty} (p_1 p_2)^j = \lim_{N\to\infty} \prod_{j=1}^{N} (p_1 p_2)^j \tag{4.3}$$

which converges strongly. The orthogonal projector p onto $\operatorname{im} p_1 \cap \cdots \cap \operatorname{im} p_m$ is given by

$$p = p_1 \wedge \cdots \wedge p_m = \wedge_{j=1}^m p_j \tag{4.4}$$

that is defined by iteration and represents an associative operation.

4.2. Case $k = i$, convex PCDs

In this section we assume $A_t = \Lambda = A_\lambda$, i.e., $k = i$ (see (1.13)) and use the following brief notation. For any open half-plane $\Sigma \subset \mathbb{R}^2$ let

$$M_\Sigma \ : \ \Sigma \ \to \ \mathbb{R}^2_{1+} \ = \ \{x \in \mathbb{R}^2 : x_1 > 0\} \tag{4.5}$$

be the *canonical linear transformation* that transforms Σ onto $\mathbb{R}^2_{1+}$, i.e., by a minimal dilation plus a rotation in the mathematical positive sense, say. Moreover let

$$J_\Sigma f(x) = f(M_\Sigma x) \quad , \quad x \in \Sigma \text{ or } x \in \mathbb{R}^2, \tag{4.6}$$

for functions and distributions defined on Σ or defined on $\mathbb{R}^2$, as well.

Theorem 4.3. *Let Σ be a convex PCD, i.e., $\Sigma = \cap_{j=1}^m \Sigma_j$ with half-planes $\Sigma_j \subset \mathbb{R}^2, j = 1, \ldots, m$ and $s \in \mathbb{R}$. Then the orthogonal projector P^s_Σ onto $P_1 X = H^s_\Sigma$ projects along $\Lambda^{-2s} H^{-s}_{\Sigma'}$ and is given by*

$$\begin{aligned} P^s_\Sigma &= \wedge_{j=1}^m P^s_{\Sigma_j} \\ P^s_{\Sigma_j} &= J^{-1}_{\Sigma_j} P^s_+ J_{\Sigma_j} \quad , \quad j = 1, \ldots, m. \end{aligned} \tag{4.7}$$

The orthogonal projector Π onto $\Lambda^{2s} H^s_\Sigma$ projects along $Q_2 Y = H^{-s}_{\Sigma'}$ and is given by

$$\Pi^{-s}_\Sigma = \Lambda^{2s} P^s_\Sigma \Lambda^{-2s}. \tag{4.8}$$

Proof. As H^s_Σ is a closed subspace of the Hilbert space H^s, it is complemented and the orthogonal projector onto H^s_Σ exists and is unique. Lemma 4.2 implies that the orthogonal projector onto H^s_Σ is given by formula (4.7).

Every projector $P^s_{\Sigma_j}$ projects along $\Lambda^{-2s} H^{-s}_{\Sigma'_j}$. Since $\Sigma = \bigcap \Sigma_j$, the orthogonal projector onto H^s_Σ projects along $\Lambda^{-2s} H^{-s}_{\Sigma'}$, because $\Sigma' = \operatorname{int} \bigcup \overline{\Sigma'_j}$.

The second part of the theorem with formula (4.8) is a consequence of the first part, exchanging the roles of Σ and Σ', of s and $-s$, and thinking of the complementary projector (exchanging "onto" and "along"). □

4.3. Case $k = i$, arbitrary PCDs

Theorem 4.4. *Let $\Sigma \subset \mathcal{A}_2$, i.e., $\Sigma = \operatorname{int} \bigcup_{j=1,\ldots,n} \operatorname{clos} \Sigma_j$ where Σ_j are convex PCDs, and assume that $s \in \mathbb{R}$. Then the orthogonal projector P^s_Σ onto H^s_Σ projects along $\Lambda^{-2s} H^{-s}_{\Sigma'}$, i.e., $P^s_\Sigma = I - \Pi^s_{\Sigma'}$ and is given by*

$$P^s_\Sigma = I - \wedge_{j=1}^m \Pi^s_{\Sigma'_j} = I - \wedge_{j=1}^m (I - P^s_{\Sigma_j}) \tag{4.9}$$

with $P^s_{\Sigma_j}$ taken from Theorem 4.3 (representing each Σ_j as intersection of half-planes).

Proof. The assumption $\Sigma \subset \mathcal{A}_2$ implies that $\Sigma' = \bigcap_{j=1}^m \Sigma'_j$ where $\Sigma'_j = \mathbb{R}^2 \setminus \overline{\Sigma_j}$. Looking at (4.9), $P^s_{\Sigma_j}$ projects onto $H^s_{\Sigma_j}$ along $\Lambda^{-2s} H^{-s}_{\Sigma'_j}$. Hence $I - P^s_{\Sigma_j}$ projects along $H^s_{\Sigma_j}$ onto $\Lambda^{-2s} H^{-s}_{\Sigma'_j}$ and $\wedge(I - P^s_{\Sigma_j})$ is the orthogonal projector onto $\bigcap(\Lambda^{-2s} H^{-s}_{\Sigma'_j}) = \Lambda^{-2s} \bigcap H^{-s}_{\Sigma'_j}$, thus projecting along H^s_Σ. This implies that P^s_Σ as given by (4.9) projects orthogonally onto H^s_Σ. □

4.4. Case $k \in i\mathbb{R}_+$, i.e., $\Re e k = 0$, $\Im m k > 0$

In this section we show that the previous results remain valid for $k \in i\mathbb{R}_+$ if we change the topology to another equivalent one. I.e., we remain in the same Hilbert spaces but infinite series and infinite products converge in a different sense, with respect to a modified norm.

Definition 4.5.

$$\begin{aligned} \langle \varphi, \psi \rangle_{s,k} &= \langle A^s_t \varphi \,,\, A^s_t \psi \rangle_0 \\ &= \int_{\mathbb{R}^n} A^s_t \varphi(x) \cdot \overline{A^s_t \psi(x)}\, dx \\ &= \int_{\mathbb{R}^n} \widehat{\varphi}(\xi)\, \widehat{\psi}(\xi)\, |\xi^2 - k^2|^s \, d\xi. \end{aligned} \tag{4.10}$$

Proposition 4.6. *For any $k \in \mathbb{C} \setminus \mathbb{R}$, $H^{s,k}(\mathbb{R}^n)$ is a Hilbert space with norm*

$$\|\varphi\|_{s,k} = \langle \varphi, \varphi \rangle^{1/2}_{s,k} \tag{4.11}$$

that is equivalent to the norm of $H^s(\mathbb{R}^n)$.

Proof. It is a consequence of the fact that $A^s_t : H^s(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a linear homeomorphism. □

Remark 4.7. Obviously we have $H^{s,i}(\mathbb{R}^n) = H^s(\mathbb{R}^n)$. The spaces $H^{s,k}(\Sigma)$, $\widetilde{H}^{s,k}(\Sigma)$, $H^{s,k}_\Sigma$ may be defined by analogy to the spaces where $k = i$. Evidently the analogue of Proposition 4.6 holds for these spaces, as well.

Remark 4.8. The spaces $H^s(\mathbb{R}^n) = H^{s,i}(\mathbb{R}^n)$ and $H^{s,k}(\mathbb{R}^n)$ are isomorphic for all $k \in i\mathbb{R}$ and the norms in these spaces are equivalent, because both of them are isomorphic to $L^2(\mathbb{R}^n)$. The isomorphism is performed by the Fourier convolution operator $A_{\omega_{s,k}} = \mathcal{F}^{-1}\omega_{s,k}\mathcal{F}$ with the symbol

$$\omega_{s,k}(\xi) := \left(\frac{\xi^2 - k^2}{\xi^2 + 1}\right)^s$$

which is invertible by $A_{\omega^{-1}_{s,k}}$.

Proposition 4.9. *For any domain Σ, any number $n = 2, 3, \dots$ and $s \in \mathbb{R}$ the subspaces $A^{2s}_t\, H^{s,k}_{\Sigma}$ and $H^{-s,k}_{\Sigma'}$ where $\Sigma' = \operatorname{int}(\mathbb{R}^2 \setminus \Sigma)$ are orthogonal to each other if $k \in i\mathbb{R}_+$.*

Proof. The algebraic decomposition

$$A^{2s}_t\, H^{s,k}_{\Sigma} \dot{+} H^{-s,k}_{\Sigma'} = H^{-s,k} \tag{4.12}$$

is clear from the case $k = i$, see Section 3.2. By definition $H^{s,k}_{\Sigma}$ and $H^{-s,k}_{\Sigma'}$ are closed subspaces and, for $\varphi \in H^{s,k}_{\Sigma}$, $\psi \in H^{-s,k}_{\Sigma'}$, we have

$$\begin{aligned} \langle A^{2s}_t \varphi\,,\, \psi\rangle_{-s,k} &= \langle A^s_t \varphi\,,\, A^{-s}_t \psi\rangle_0 \\ &= \int_{\mathbb{R}^n} \widehat{\varphi}(\xi)(\xi^2 - k^2)^{s/2}\, \overline{\widehat{\psi}(\xi)\,(\xi^2 - k^2)^{-s/2}}\, d\xi \end{aligned} \tag{4.13}$$

which disappears for any such pair φ, ψ if k^2 is real. □

Evidently, the analogue of Theorem 4.4 for $k \in i\mathbb{R}_+$ instead of $k = i$ is valid, as well.

Corollary 4.10. *Let Σ be a PCD, $k \in i\mathbb{R}_+$ and $s \in \mathbb{R}$. Then the following orthogonal projectors can be presented explicitly:*

$$\begin{aligned} P^s_{\Sigma} &= I - \wedge^m_{j=1}\Pi^s_{\Sigma'_j} = I - \wedge^m_{j=1}(I - P^s_{\Sigma_j}) \quad \textit{onto} \quad H^s_{\Sigma} \\ P^s_{\Sigma'} &= I \,-\, A^{-2s}_t\, P^{-s}_{\Sigma}\, A^{2s}_t \quad \textit{onto} \quad H^s_{\Sigma'} \\ \Pi^s_{\Sigma} &= A^{-2s}_t\, P^{-s}_{\Sigma}\, A^{2s}_t \quad \textit{along} \quad H^s_{\Sigma'} \\ \Pi^s_{\Sigma'} &= I \,-\, P^s_{\Sigma} \quad \textit{along} \quad H^s_{\Sigma} \end{aligned}$$

where the decomposition of Σ from Theorem 4.3 *is used and the infinite products converge in the sense of the norm* (4.11)*.*

4.5. Case $\Re e k \neq 0$, $\Im m k > 0$

Now a modification of the scalar product (4.10) does not help anymore. Thus we shall use a different idea coming up from [24] to present the (non-orthogonal) projectors $\Pi^{\pm 1/2}_{\Sigma}$ etc. by Neumann series approximation using the orthogonal projectors constructed before. Therefore we extend the notation by

$$\begin{aligned} &\Pi^s_{\Sigma,k} \quad - \quad \text{the projector onto} \quad A_{t^{-2s}} H^{-s}_{\Sigma} \quad \text{along} \quad H^s_{\Sigma'} \\ &\Pi^s_{\Sigma,i} \quad - \quad \text{the projector onto} \quad \Lambda^{-2s} H^{-s}_{\Sigma} \quad \text{along} \quad H^s_{\Sigma'} \end{aligned} \tag{4.14}$$

where the latter coincides with the first for $k = i$ and is orthogonal. Further projectors can be defined and treated by analogy. So we have:

$$\begin{array}{lcl} P^s_{\Sigma,k} & - & \text{the projector onto} \quad H^s_\Sigma \quad \text{along} \quad A_{t^{-2s}} H^{-s}_{\Sigma'} \\ P^s_{\Sigma,k} & = & I - \Pi^s_{\Sigma',k} \end{array} \tag{4.15}$$

which we employ basically for $s = \pm 1/2$ and call them briefly Π-*projectors* and *P-projectors.*

Proposition 4.11. *Abbreviate* $\Pi = \Pi^{1/2}_{\Sigma,i}$. *Then the projector* $\Pi^{1/2}_{\Sigma,k}$ *is given by*

$$\Pi^{1/2}_{\Sigma,k} = A_{t^{-1}}\, \Lambda\, W_0^{-1}\, \Pi \qquad , \qquad W_0 = \Pi\, A_{t^{-1}}\, \Lambda|_{\Pi H^{1/2}} \tag{4.16}$$

(using the convention) where the inverse W_0^{-1} is given by a Neumann series.

Proof. The operator $W_0 = \Pi\, A_{t^{-1}}\, \Lambda|_{\Pi H^{1/2}}$ is invertible by a Neumann series, since $\Re e A_{t^{-1}}\, \Lambda > 0$ and Π is orthogonal, see [10, 24, 37].

The operator $\Pi^{1/2}_{\Sigma,k}$ in (4.16) is obviously linear, bounded and idempotent, hence it is a projector. W_0^{-1} maps onto $\Pi H^{1/2} = \Lambda^{-1} H^{-1/2}_\Sigma$, thus $A_{t^{-1}}\, \Lambda\, W_0^{-1}\, \Pi$ maps onto $A_{t^{-1}} H^{-1/2}_\Sigma$. Finally its kernel is obviously $H^{-s}_{\Sigma'}$ and the proof is complete. □

The following result is analogous.

Proposition 4.12. *Abbreviate* $\Pi = \Pi^{-1/2}_{\Sigma,i}$. *Then the projector* $\Pi^{-1/2}_{\Sigma,k}$ *is given by*

$$\Pi^{-1/2}_{\Sigma,k} = A_t\, \Lambda^{-1}\, W_0^{-1}\, \Pi \qquad , \qquad W_0 = \Pi\, A_t\, \Lambda^{-1}|_{\Pi H^{-1/2}} \tag{4.17}$$

(using the convention) where the inverse W_0^{-1} is given by a Neumann series.

4.6. Equivalent constructions

Here we show that the construction of various involved operators is equivalent, i.e., they can be obtained easily from each other. As a matter of fact, this has nothing to do with the form of Σ nor with Wiener–Hopf factorization, but with the features exposed in Sections 3.2 and 3.3: the geometric perspective and matrical coupling.

Theorem 4.13. *Let* Σ *be an E-domain. Consider the Dirichlet problem for* Σ (*as described in* (1.3)–(1.5)) *and the Neumann problem for* Σ' (*by analogy*). *Then the two resolvent operators* (*see* (2.7), (2.9)), *the WHOs therein, and corresponding P-projectors and* Π-*projectors* (*see* (4.14), (4.15)) *can be computed from each other.*

Proof. Clearly each of the P-projectors yield the corresponding Π-projector by definition, see (4.15), and the corresponding WHO inverses by Corollary 3.6 which yield the corresponding resolvent operators, see Theorem 2.3 and Theorem 2.4. For clarity we summarize these very direct relations:

$$\begin{array}{lllll} P^{-1/2}_{\Sigma,k} = I - \Pi^{-1/2}_{\Sigma',k} & = W^{-1}_{t^{-1},\Sigma}\, r_\Sigma\, A_{t^{-1}} & \text{onto} \quad H^{-1/2}_\Sigma & \text{along} \quad A_t H^{1/2}_{\Sigma'} \\ P^{1/2}_{\Sigma',k} = I - \Pi^{1/2}_{\Sigma,k} & = W^{-1}_{t,\Sigma'}\, r_\Sigma\, A_t & \text{onto} \quad H^{1/2}_{\Sigma'} & \text{along} \quad A_{t^{-1}} H^{-1/2}_\Sigma \end{array}$$

according to the WHO inverses (– appearing in)

$$\begin{aligned} W^{-1}_{t^{-1},\Sigma} &= P^{-1/2}_{\Sigma,k}\, A_t\,\ell && - \quad \text{Dirichlet problem for} \quad \Sigma \\ W^{-1}_{t,\Sigma'} &= P^{1/2}_{\Sigma',k}\, A_{t^{-1}}\,\ell && - \quad \text{Neumann problem for} \quad \Sigma'. \end{aligned}$$

Herein ℓ denotes any extension operator from $H^{1/2}(\Sigma)$ into $H^{1/2}(\mathbb{R}^2)$ or from $H^{-1/2}(\Sigma)$ into $H^{-1/2}(\mathbb{R}^2)$, respectively.

Conversely, from the resolvent operators we obtain the corresponding projectors and WHO inverses by composition with trace and symmetrization operators, e.g., $W^{-1}_{t^{-1},\Sigma} : g \mapsto u \mapsto u_0^+ - u_0^-$ etc. (see Theorem 2.3 and Theorem 2.4).

Now the geometric perspective (see Theorem 3.5 and Corollary 3.6) implies that the projectors in the above list are related by

$$P^{-1/2}_{\Sigma,k} \;=\; A_t\, \Pi^{1/2}_{\Sigma,k}\, A_{t^{-1}}\,. \tag{4.18}$$

An alternative proof, instead of using (4.18), can be based upon the fact that the two WHOs in the above scheme, $W^{-1}_{t^{-1},\Sigma}$ and $W^{-1}_{t,\Sigma'}$, are matrically coupled (see Lemma 3.3). Again the relationship with the resolvent operators is evident from the representation formulas. □

Exchanging the roles of Σ and Σ' we observe that the operators corresponding to the Dirichlet problem for Σ' and the Neumann problem for Σ are related in a similar way.

5. Explicit solution of the BVPs

We come to the final results in concrete form presenting the details of the proof of Theorem 1.3.

Theorem 5.1. *Let $\Sigma \subset \mathbb{R}^2$ be a PCD and $k \in \mathbb{C}$, $\Im m k > 0$. The resolvent operators for the Dirichlet and Neumann problems are given by Theorem* 2.3 *and Theorem* 2.4*, respectively, where*

$$W^{-1}_{t^{-1},\Sigma} = A_t\, \Pi^{1/2}_{\Sigma,k}\, \ell \quad = P^{-1/2}_{\Sigma,k}\, A_t\, \ell \;:\; H^{1/2}(\Sigma) \to H^{-1/2}_{\Sigma} \tag{5.1}$$

$$W^{-1}_{t,\Sigma} = A_t^{-1}\, \Pi^{-1/2}_{\Sigma,k}\, \ell = P^{1/2}_{\Sigma,k}\, A_t^{-1}\, \ell \;:\; H^{-1/2}(\Sigma) \to H^{1/2}_{\Sigma} \tag{5.2}$$

with arbitrary extension operators ℓ into $H^{1/2}$ or $H^{-1/2}$, respectively, and the projectors are also explicitly given in

- *Theorem* 4.3 *for convex PCDs and $k = i$,*
- *Theorem* 4.4 *for arbitrary PCDs and $k = i$,*
- *Corollary* 4.9 *for arbitrary PCDs and $k \in i\mathbb{R}_+$,*
- *Proposition* 4.10 *for arbitrary PCDs, $\Im m k > 0$, the Dirichlet problem,*
- *Proposition* 4.11 *for arbitrary PCDs, $\Im m k > 0$, the Neumann problem.*

Proof. All that results directly from the previous as referred to. □

Remark 5.2. There are few cases where the resolvent operators can be obtained in closed analytical form (in a representation without infinite series and products). Half-plane screens represent one of them, see Section 4. The possibility of applying factorization methods to other screen problems with conical configurations is not positively answered till now in the authors opinion (in contrast to wedge problems, see [8, 14, 15, 41]).

Let us consider the special geometrical case where Σ is a cone, moreover connected and convex. V.B. Vasil'ev proposed in his book [44] to solve the diffraction problem by use of a so-called wave factorization of the function $t(\xi) = (\xi^2 - k^2)^{1/2}$ into two factors, holomorphic in certain tube domains. However, looking at the explicit form of the two factors, it turns out that they vanish within the corresponding tube domains, see [44], pages 28–29 and 38–39. This means that the given factorization is not a wave factorization in the sense of the author's own Definition 5.1 and therefore not helpful for the solution of the problem. The authors of the present article do not know any other example from mathematical physics where the method of wave factorization is applicable.

Other canonical screen problems such as the diffraction from a flat circular disc (see [21, 45], for instance) end up with Fredholm integral equations and series expansion, as well, but not with a solution in closed analytic form.

In view of the complexity of the derived formulas, some simplification can be obtained for a screen that is complementary to a convex PCD by the following corollaries. The idea is known as a sort of abstract Babinet principle [39].

Corollary 5.3. *Let Σ be an E-domain. Assume that the inverse of $W_{t,\Sigma}$ is known (which provides the resolvent to the Neumann problem for Σ by Theorem 2.4). Then the Dirichlet problem for Σ' is uniquely solved by Theorem 2.4 where $W^{-1}_{t^{-1},\Sigma'}$ is obtained from Theorem 3.10, substituting $V = W^{-1}_{t,\Sigma}$ and the corresponding other terms.*

Corollary 5.4. *Let Σ be an E-domain. Assume that the inverse of $W_{t^{-1},\Sigma}$ is known (which provides the resolvent to the Dirichlet problem for Σ by Theorem 2.3). Then the Neumann problem for Σ' is uniquely solved by Theorem 2.3 where $W^{-1}_{t,\Sigma'}$ is obtained from Theorem 3.10, substituting $V = W^{-1}_{t^{-1},\Sigma}$ and the corresponding other terms.*

We finish with a result on the low regularity of solutions to the BVPs.

Theorem 5.5. *Let $\Sigma \subset \mathbb{R}^2$ be a PCD, $\Im m k > 0$ and $\varepsilon \in]0, 1/2[$.*

I. *If $g \in H^{1/2+\varepsilon}(\Sigma)^2_\sim$ and $u \in \mathcal{H}^1(\Omega)$ is a solution of the Dirichlet problem for Σ, then $u \in \mathcal{H}^{1+\varepsilon}(\Omega)$, i.e., $u^\pm \in H^{1+\varepsilon}(\Omega^\pm)$, see (1.2)–(1.5). Moreover the resolvent operator (2.7) restricted to these spaces represents a linear homeomorphism:*

$$\mathcal{K}^\varepsilon_{D,\Omega} \; : \; H^{1/2+\varepsilon}(\Sigma)^2_\sim \to \mathcal{H}^{1+\varepsilon}(\Omega). \tag{5.3}$$

II. *If $g \in H^{-1/2+\varepsilon}(\Sigma)^2$ and $u \in \mathcal{H}^1(\Omega)$ is a solution of the Neumann problem for Σ, then $u \in \mathcal{H}^{1+\varepsilon}(\Omega)$, i.e., $u^\pm \in H^{1+\varepsilon}(\Omega^\pm)$, see (1.2)–(1.5). Moreover the resolvent* *operator* (2.7) *restricted to these spaces represents a linear homeomorphism:*

$$\mathcal{K}^{\varepsilon}_{N,\Omega} \,:\, H^{-1/2+\varepsilon}(\Sigma)^2 \to \mathcal{H}^{1+\varepsilon}(\Omega). \tag{5.4}$$

Proof. Following all the way long the foregoing construction of resolvent operators we realize that there is no problem to include the parameter $\varepsilon \in]0, 1/2[$. Notice that there is no compatibility condition in the second statement II since $\widetilde{H}^{-1/2+\varepsilon}(\Sigma) = H^{-1/2+\varepsilon}(\Sigma)$ for $\varepsilon \in]0, 1/2[$. □

6. Open problems

At the end we would like to formulate some unsolved problems that we found interesting.

Problem 6.1. How can we treat other boundary conditions like impedance, oblique derivative etc.? It is known that these BVPs lead to WHOs of the form $W_{\phi,\Sigma} = r_\Sigma A_\phi$ where the Fourier symbol ϕ is more complicated and a matrical coupling relation and orthogonal projectors can not be seen.

Problem 6.2. In non-Hilbert spaces $W^{s,p}$ we have no orthogonality. Is there any alternative approach, perhaps working with sesquilinear forms (as a generalization of the Halmos Theorem)?

Problem 6.3. Replacing the Helmholtz equation by the Lamé or Maxwell equations, can we obtain analogous results considering matrix WHOs?

Problem 6.4. Slit domains and cracks may be tackled by certain space modifications. Are there any interesting new results, techniques, applications?

Problem 6.5. Arbitrary convex screens could be formally treated by an infinite product $\wedge_{j=1}^{\infty} P_j$ considering Σ as an intersection of infinitely many half-planes. What about convergence?

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References

[1] H. Bart, I. Gohberg and M. Kaashoek, *The coupling method for solving integral equations.* Oper. Theory Adv. Appl. **2** (1984), 39–73.

[2] H. Bart, I. Gohberg and M. Kaashoek, *Addendum to: The coupling method for solving integral equations.* Integr. Equ. Oper. Theory **8** (1985), 890–891.

[3] H. Bart, I. Gohberg, M.A. Kaashoek, A.C.M. Ran, *Schur complements and state space realizations.* Linear Algebra Appl. **399** (2005), 203–224.

[4] H. Bart and V.E. Tsekanovskii, *Matricial coupling and equivalence after extension.* Oper. Theory Adv. Appl. **59** (1991), 143–160.

[5] H. Bart, V.E. Tsekanovskii, *Complementary Schur complements.* Linear Algebra Appl. **197** (1994), 651–658.

[6] L.P. Castro and F.-O. Speck, *Regularity properties and generalized inverses of delta-related operators.* Z. Anal. Anwend. **17** (1998), 577–598.

[7] L.P. Castro and F.-O. Speck, *Relations between convolution type operators on intervals and on the half-line.* Integr. Equ. Oper. Theory **37** (2000) 169–207.

[8] L.P. Castro, F.-O. Speck and F.S. Teixeira, *On a class of wedge diffraction problems posted by Erhard Meister.* Oper. Theory Adv. Appl. **147** (2004), 211–238.

[9] P.G. Ciarlet, *Mathematical Elasticity III: Theory of Shells.* Studies in Mathematics and Applications **29**. Elsevier, North-Holland, Amsterdam, 2000.

[10] A. Devinatz and M. Shinbrot, *General Wiener–Hopf operators.* Trans. AMS **145** (1969), 467–494.

[11] R. Duduchava and F.-O. Speck, *Bessel potential operators for the quarter-plane.* Appl. Anal. **45** (1992), 49-68.

[12] R. Duduchava and F.-O. Speck, *Pseudodifferential operators on compact manifolds with Lipschitz boundary.* Math. Nachr. **160** (1993), 149–191.

[13] R. Duduchava, W. Wendland, *The Wiener–Hopf method for systems of pseudodifferential equations with an application to crack problems.* Integr. Equ. Oper. Theory **23** (1995), 294–335.

[14] T. Ehrhardt, A.P. Nolasco and F.-O. Speck, *Boundary integral methods for wedge diffraction problems: the angle $2\pi/n$, Dirichlet and Neumann conditions.* Oper. Matrices **5** (2011), 1–40.

[15] T. Ehrhardt, A.P. Nolasco and F.-O. Speck, *A Riemann surface approach for diffraction from rational angles.* Oper. Matrices, to appear, 55 p.

[16] G.I. Eskin, *Boundary Value Problems for Elliptic Pseudodifferential Equations.* American Mathematical Society, Providence, Rhode Island, 1981.

[17] I.Z. Gochberg and I.A. Feldman, *Faltungsgleichungen und Projektionsverfahren zu ihrer Lösung* (in German). Birkhäuser and Akademie-Verlag, Berlin, 1974 (Russian edition in 1971).

[18] P. Grisvard, *Elliptic Problems in Non-Smooth Domains.* Pitman, London, 1985.

[19] P.R. Halmos, *A Hilbert Space Problem Book.* Second edition, Springer, New York, 1982.

[20] G.C. Hsiao and W.L. Wendland, *Boundary Integral Equations.* Springer, Berlin, 2008.

[21] D.S. Jones, *Diffraction at frequencies by a circular disc.* Proc. Camb. Philos. Soc. **61** (1965), 223–245.

[22] M.G. Krein, *Integral equations on a half-line with kernel depending upon the difference of the arguments.* AMS Transl., Ser. 2, **22** (1962), 163–288.

[23] V.G. Maz'ya and S.V. Poborchi, *Differentiable Functions in Bad Domains.* World Scientific, Singapure, 1997.

[24] E. Meister and F.-O. Speck, *Scalar diffraction problems for Lipschitz and polygonal screens.* Z. Angew. Math. Mech. **67** (1987), 434–435.

[25] E. Meister and F.-O. Speck, *A contribution to the quarter-plane problem in diffraction theory.* J. Math. Anal. Appl. **130** (1988), 223–236.

[26] E. Meister and F.-O. Speck, *Modern Wiener–Hopf methods in diffraction theory.* Ordinary and Partial Differential Equations **2**, Pitman Res. Notes Math. Ser. **216** (Longman, London, 1989), 130–171.

[27] S.G. Mikhlin and S. Prössdorf, *Singular Integral Operators.* Extended and partly modified translation from the German by A. Böttcher and R. Lehmann. Springer, Berlin, 1986 (German edition by Akademie-Verlag, Berlin, 1980).

[28] A. Moura Santos, F.-O. Speck and F.S. Teixeira, *Minimal normalization of Wiener–Hopf operators in spaces of Bessel potentials.* J. Math. Anal. Appl. **225** (1998), 501–531.

[29] S. Prössdorf, *Some Classes of Singular Equations.* North-Holland, Amsterdam, 1978 (German edition by Akademie-Verlag, Berlin, 1974).

[30] A.F. dos Santos, *Abstract Wiener–Hopf Operators – Geometric Perspective.* Thesis on the occasion of "Provas de Agregação" (in Portuguese), Departamento de Matemática, Instituto Superior Técnico, Universidade Técnica de Lisboa, Lisbon, 1988.

[31] A.F. dos Santos, *General Wiener–Hopf operators and representation of their generalized inverses.* Oper. Theory Adv. Appl. **41** (1989), 473–483.

[32] I. Schur, *Über Potenzreihen, die im Innern des Einheitskreises beschränkt sind. I, II.* J. für Math. **147** (1917), 205–232; **148** (1918), 122–145 (in German).

[33] M. Shinbrot, *On singular integral operators.* J. Math. Mech. **13** (1964), 395–406.

[34] M. Shinbrot, *On the range of general Wiener–Hopf operators.* J. Math. Mech. **18** (1969), 587–601.

[35] S.G. Sommerfeld, *Mathematische Theorie der Diffraction.* Math. Annalen **47** (1896), 317–374.

[36] F.-O. Speck, *On the generalized invertibility of Wiener–Hopf operators in Banach spaces.* Integr. Equ. Oper. Theory **6** (1983), 458–465.

[37] F.-O. Speck, *General Wiener–Hopf Factorization Methods.* Pitman, London, 1985.

[38] F.-O. Speck, *Mixed boundary value problems of the type of Sommerfeld's half-plane problem.* Proc. R. Soc. Edinb., Sect. A **104** (1986), 261–277.

[39] F.-O. Speck, *Diffraction from a three-quarter-plane using an abstract Babinet principle.* Z. Ang. Math. Mech. **93** (2013), 485–491.

[40] F.-O. Speck, *On the reduction of linear systems related to boundary value problems.* Oper. Theory Adv. Appl. **228** (2013), 391–406.

[41] F.-O. Speck and E. Stephan, *Boundary value problems for the Helmholtz equation in an octant.* Integr. Equ. Oper. Theory **62** (2008), 269–300.

[42] E. Stein, *Singular Integrals and Differentiability Properties of Functions.* Princeton University Press, Princeton, N.J., 1970.

[43] E. Stephan, *Boundary Integral Equations for Mixed Boundary Value Problems, Screen and Transmission Problems in* $\mathbb{R}^3$. Habilitation thesis, Preprint 848, Fachbereich Mathematik, Technische Hochschule Darmstadt, 1984.

[44] V.B. Vasil'ev, *Wave Factorization of Elliptic Symbols: Theory and Applications.* Kluver, Dordrecht, 2000.

[45] S.S. Vinogradov, *A method for the solution of a problem of diffraction on a thin disc.* Dokl. Akad. Nauk Ukr. SSR, Ser. A 1983, No.6 (1983), 37–40 (in Russian).

[46] F. Zhang (ed.), *The Schur Complement and its Applications.* Numerical Methods and Algorithms 4. Springer-Verlag, New York, 2005.

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Boundedness of the Maximal and Singular Operators on Generalized Orlicz–Morrey Spaces

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Dedicated to Professor António Ferreira dos Santos

Abstract. We consider generalized Orlicz–Morrey spaces $M_{\Phi,\varphi}(\mathbb{R}^n)$ including their weak versions. In these generalized spaces we prove the boundedness of the Hardy–Littlewood maximal operator and Calderón–Zygmund singular operators with standard kernel. In all the cases the conditions for the boundedness are given either in terms of Zygmund-type integral inequalities on $\varphi(r)$ without assuming any monotonicity property of $\varphi(r)$, or in terms of supremal operators, related to $\varphi(r)$.

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1. Introduction

Inequalities involving classical operators of harmonic analysis, such as maximal functions, fractional integrals and singular integrals of convolution type have been extensively investigated in various function spaces. Results on weak and strong type inequalities for operators of this kind in Lebesgue spaces are classical and can be found for example in [3, 41, 42, 44]. Generalizations of these results to Zygmund spaces are presented in [3]. An exhaustive treatment of the problem of boundedness of such operators in Lorentz and Lorentz–Zygmund spaces is given in [2]. See also [10, 11] for further extensions in the framework of generalized

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Lorentz–Zygmund spaces. As far as Orlicz spaces are concerned, we refer to the books [21, 23, 37] and note that a characterization of Young functions A for which the Hardy–Littlewood maximal operator or the Hilbert and Riesz transforms are of weak or strong type in Orlicz space L_A is known (see for example [5, 21]). In [33, 44] conditions on Young functions A and B are given for the fractional integral operator to be bounded from L_A into L_B under some restrictions involving the growth and certain monotonicity properties of A and B (see also [5]).

Orlicz spaces, introduced in [34, 35], are generalizations of Lebesgue spaces L_p. They are useful tools in harmonic analysis and its applications. For example, the Hardy–Littlewood maximal operator is bounded on L_p for $1 < p < \infty$, but not on L_1. Using Orlicz spaces, we can investigate the boundedness of the maximal operator near $p = 1$ more precisely (see [17, 18] and [5]).

In the study of local properties of solutions to of partial differential equations, together with weighted Lebesgue spaces, Morrey spaces $M_{p,\lambda}(\mathbb{R}^n)$ play an important role, see [12]. Introduced by C. Morrey [29] in 1938, they are defined by the norm

$$\|f\|_{M_{p,\lambda}} := \sup_{x,\, r>0} r^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,r))}, \tag{1.1}$$

where $0 \le \lambda \le n$, $1 \le p < \infty$. Here and everywhere in the sequel $B(x,r)$ stands for the ball in $\mathbb{R}^n$ of radius r centered at x. Let $|B(x,r)|$ be the Lebesgue measure of the ball $B(x,r)$ and $|B(x,r)| = v_n r^n$, where $v_n = |B(0,1)|$.

Note that $M_{p,0} = L_p(\mathbb{R}^n)$ and $M_{p,n} = L_\infty(\mathbb{R}^n)$. If $\lambda < 0$ or $\lambda > n$, then $M_{p,\lambda} = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.

We also denote by $WM_{p,\lambda} \equiv WM_{p,\lambda}(\mathbb{R}^n)$ the weak Morrey space of all functions $f \in WL_p^{\rm loc}(\mathbb{R}^n)$ for which

$$\|f\|_{WM_{p,\lambda}} = \sup_{x\in\mathbb{R}^n,\, r>0} r^{-\frac{\lambda}{p}} \|f\|_{WL_p(B(x,r))} < \infty.$$

We refer in particular to [24] for the classical Morrey spaces. Observe that Morrey spaces with r^λ replaced by a function $\varphi(r)$ first appeared in [9] and [46]; we also refer to the survey paper [36] for more various definitions of generalized Morrey spaces and note that study of classical operators of harmonic analysis in generalized Morrey spaces started in [13], [14], [30], up to authors' knowledge.

Last two decades there is an increasing interest to the study of variable exponent spaces and operators with variable parameters, in such spaces, we refer to the recent books [6], [8] and surveying papers [7], [20], [22], [38].

Orlicz–Morrey spaces and maximal and singular operators in such spaces were studied in [31], [32]. The most general spaces of such a type, Musielak–Orlicz–Morrey spaces, unifying the classical and variable exponent approaches, were studied in the recent paper [28], where potential operators were studied together with the corresponding Sobolev embeddings.

In this paper we study the maximal and singular operators in Orlicz–Morrey spaces, introduced in a less generality, but advance in the following two directions:

1) we make minimal assumptions on the functions defining the space avoiding any kind of monotonicity or growth condition, required, for instance, in [30], [31], [32],
2) we prove weak-type inequalities.

Our conditions for the boundedness are sufficient. We do not discuss their necessity in this paper but hope to do that in another paper.

We define the generalized Orlicz–Morrey space $M_{\Phi,\varphi}(\mathbb{R}^n)$ in question by the norm

$$\|f\|_{M_{\Phi,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} \, \|f\|_{L_\Phi(B(x,r))}.$$

where $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and Φ a Young function, but refer to Section 2 for all the precise definitions and comparison with other norms.

The main purpose of this paper is to find sufficient conditions on general Young function Φ and functions φ_1, φ_2 ensuring that the operators under consideration are of weak or strong type from generalized Orlicz–Morrey spaces $M_{\Phi,\varphi_1}(\mathbb{R}^n)$ into $M_{\Phi,\varphi_2}(\mathbb{R}^n)$. Our results for the maximal operator are presented in Section 4, while Section 5 deals with singular integrals.

1.1. Operators under consideration

We study the following operators: *the maximal operator*

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|dy,$$

Calderón–Zygmund type singular operators; by this we mean operators bounded in $L^2(\mathbb{R}^n)$ of the form

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y)f(y)dy,$$

where $K(x,y)$ is a "standard singular kernel", that is, a continuous function defined on $\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : x \neq y\}$ and satisfying the estimates

$$|K(x,y)| \leq C|x-y|^{-n} \qquad \text{for all } x \neq y,$$

$$|K(x,y) - K(x,z)| \leq C\frac{|y-z|^\sigma}{|x-y|^{n+\sigma}}, \ \sigma > 0, \ \text{ if } |x-y| > 2|y-z|,$$

$$|K(x,y) - K(\xi,y)| \leq C\frac{|x-\xi|^\sigma}{|x-y|^{n+\sigma}}, \ \sigma > 0, \ \text{ if } |x-y| > 2|x-\xi|.$$

Our main results are obtained in Theorems 4.6 and 5.5, where we use recent results presented in Theorems 2.11 and 2.12 to obtain a generalization of known conditions for the boundedness of maximal and singular operators in Orlicz–Morrey spaces, it is given in terms of conditions (4.8) and (5.7), respectively, without any assumption of monotonicity type on the functions φ_1 and φ_2 as, for instance, used in [28], [31] and other sources.

2. Some preliminaries on Orlicz and Orlicz–Morrey spaces

Definition 2.1. A function $\Phi : [0,+\infty] \to [0,\infty]$ is called a Young function if Φ is convex, left-continuous, $\lim\limits_{r\to+0} \Phi(r) = \Phi(0) = 0$ and $\lim\limits_{r\to+\infty} \Phi(r) = \Phi(\infty) = \infty$.

From the convexity and $\Phi(0) = 0$ it follows that any Young function is increasing. If there exists $s \in (0,+\infty)$ such that $\Phi(s) = +\infty$, then $\Phi(r) = +\infty$ for $r \geq s$.

We say that $\Phi \in \Delta_2$, if for any $a > 1$, there exists a constant $C_a > 0$ such that $\Phi(at) \leq C_a\Phi(t)$ for all $t > 0$. A Young function Φ is said to satisfy the ∇_2-condition, denoted also by $\Phi \in \nabla_2$, if

$$\Phi(r) \leq \frac{1}{2k}\Phi(kr), \qquad r \geq 0,$$

for some $k > 1$. The function $\Phi(r) = r$ satisfies the Δ_2-condition but does not satisfy the ∇_2-condition. If $1 < p < \infty$, then $\Phi(r) = r^p$ satisfies both the conditions. The function $\Phi(r) = e^r - r - 1$ satisfies the ∇_2-condition but does not satisfy the Δ_2-condition. The following two indices

$$q_\Phi = \inf_{t>0} \frac{t\varphi(t)}{\Phi(t)}, \qquad p_\Phi = \sup_{t>0} \frac{t\varphi(t)}{\Phi(t)},$$

of Φ, where $\varphi(t)$ is the right-continuous derivative of Φ, are well known in the theory of Orlicz spaces. As is well known,

$$p_\Phi < \infty \qquad \Longleftrightarrow \qquad \Phi \in \Delta_2,$$

and the function Φ is strictly convex if and only if $q_\Phi > 1$. If $0 < q_\Phi \leq p_\Phi < \infty$, then $\frac{\Phi(t)}{t^{q_\Phi}}$ is increasing and $\frac{\Phi(t)}{t^{p_\Phi}}$ is decreasing on $(0,\infty)$.

Lemma 2.2. ([21], *Lemma* 1.3.2) *Let $\Phi \in \Delta_2$. Then there exist $p > 1$ and $b > 1$ such that*

$$\frac{\Phi(t_2)}{t_2^p} \leq \frac{b\Phi(t_1)}{t_1^p}$$

for $0 < t_1 < t_2$.

Recall that a function Φ is said to be quasiconvex if there exist a convex function ω and a constant $c > 0$ such that

$$\omega(t) \leq \Phi(t) \leq c\omega(ct),\ t \in [0,\infty).$$

Let $\mathcal{Y}$ be the set of all Young functions Φ such that

$$0 < \Phi(r) < +\infty \qquad \text{for} \qquad 0 < r < +\infty \tag{2.1}$$

If $\Phi \in \mathcal{Y}$, then Φ is absolutely continuous on every closed interval in $[0,+\infty)$ and bijective from $[0,+\infty)$ to itself.

Definition 2.3 (Orlicz Space). For a Young function Φ, the set

$$L_\Phi(\mathbb{R}^n) = \left\{ f \in L_1^{loc}(\mathbb{R}^n) : \int_{\mathbb{R}^n} \Phi(k|f(x)|)dx < +\infty \text{ for some } k > 0 \right\}$$

is called Orlicz space. The space $L^{\rm loc}_{\Phi}(\mathbb{R}^n)$ endowed with the natural topology is defined as the set of all functions f such that $f\chi_{_B} \in L_\Phi(\mathbb{R}^n)$ for all balls $B \subset \mathbb{R}^n$.

Note that, $L_\Phi(\mathbb{R}^n)$ is a Banach space with respect to the norm

$$\|f\|_{L_\Phi} = \inf\Big\{\lambda > 0 : \int_{\mathbb{R}^n} \Phi\Big(\frac{|f(x)|}{\lambda}\Big) dx \le 1\Big\},$$

see, for example, [37], Section 3, Theorem 10, so that

$$\int_{\mathbb{R}^n} \Phi\Big(\frac{|f(x)|}{\|f\|_{L_\Phi}}\Big) dx \le 1.$$

For a measurable set $\Omega \subset \mathbb{R}^n$, a measurable function f and $t > 0$, let

$$m(\Omega,\ f,\ t) = |\{x \in \Omega : |f(x)| > t\}|.$$

In the case $\Omega = \mathbb{R}^n$, we shortly denote it by $m(f,\ t)$.

Definition 2.4. The weak Orlicz space

$$WL_\Phi(\mathbb{R}^n) = \{f \in L^{\rm loc}_1(\mathbb{R}^n) : \|f\|_{WL_\Phi} < +\infty\}$$

is defined by the norm

$$\|f\|_{WL_\Phi} = \inf\Big\{\lambda > 0\ :\ \sup_{t>0} \Phi(t) m\Big(\frac{f}{\lambda},\ t\Big)\ \le 1\Big\}.$$

For Young functions Φ and Ψ, we write $\Phi \approx \Psi$ if there exists a constant $C \ge 1$ such that

$$\Phi(C^{-1}r) \le \Psi(r) \le \Phi(Cr) \qquad \text{for all } r \ge 0$$

If $\Phi \approx \Psi$, then $L_\Phi(\mathbb{R}^n) = L_\Psi(\mathbb{R}^n)$ with equivalent norms. We note that, for Young functions Φ and Ψ, if there exist $C, R \ge 1$ such that

$$\Phi(C^{-1}r) \le \Psi(r) \le \Phi(Cr) \qquad \text{for } r \in (0, R^{-1}) \cup (R, \infty),$$

then $\Phi \approx \Psi$.

For a Young function Φ and $0 \le s \le +\infty$, let

$$\Phi^{-1}(s) = \inf\{r \ge 0 : \Phi(r) > s\} \qquad (\inf \emptyset = +\infty).$$

If $\Phi \in \mathcal{Y}$, then Φ^{-1} is the usual inverse function of Φ. We note that

$$\Phi(\Phi^{-1}(r)) \le r \le \Phi^{-1}(\Phi(r)) \quad \text{ for } 0 \le r < +\infty.$$

For a Young function Φ, the complementary function $\widetilde{\Phi}(r)$ is defined by

$$\widetilde{\Phi}(r) = \begin{cases} \sup\{rs - \Phi(s) : s \in [0, \infty)\} & , \quad r \in [0, \infty) \\ +\infty & , \quad r = +\infty. \end{cases} \tag{2.2}$$

The complementary function $\widetilde{\Phi}$ is also a Young function and $\widetilde{\widetilde{\Phi}} = \Phi$. If $\Phi(r) = r$, then $\widetilde{\Phi}(r) = 0$ for $0 \le r \le 1$ and $\widetilde{\Phi}(r) = +\infty$ for $r > 1$. If $1 < p < \infty$, $1/p + 1/p' = 1$ and $\Phi(r) = r^p/p$, then $\widetilde{\Phi}(r) = r^{p'}/p'$. If $\Phi(r) = e^r - r - 1$, then $\widetilde{\Phi}(r) = (1 + r)\log(1 + r) - r$.

Remark 2.5. Note that $\Phi \in \nabla_2$ if and only if $\widetilde{\Phi} \in \Delta_2$. Also, if Φ is a Young function, then $\Phi \in \nabla_2$ if and only if Φ^γ be quasiconvex for some $\gamma \in (0,1)$ (see, for example, [21], p. 15).

It is known that

$$r \le \Phi^{-1}(r)\widetilde{\Phi}^{-1}(r) \le 2r \qquad \text{for } r \ge 0. \tag{2.3}$$

The following analogue of the Hölder inequality is known, see [45].

Theorem 2.6 ([45]). *For a Young function Φ and its complementary function $\widetilde{\Phi}$, the following inequality is valid*

$$\|fg\|_{L_1(\mathbb{R}^n)} \le 2\|f\|_{L_\Phi}\|g\|_{L_{\widetilde{\Phi}}}.$$

Note that Young functions satisfy the property

$$\Phi(\alpha t) \le \alpha\Phi(t) \tag{2.4}$$

for all $0 < \alpha < 1$ and $0 \le t < \infty$, which is a consequence of the convexity: $\Phi(\alpha t) = \Phi(\alpha t + (1-\alpha)0) \le \alpha\Phi(t) + (1-\alpha)\Phi(0) = \alpha\Phi(t)$.

The following lemma is valid.

Lemma 2.7 ([3, 25]). *Let Φ be a Young function and B a set in $\mathbb{R}^n$ with finite Lebesgue measure. Then*

$$\|\chi_B\|_{WL_\Phi(\mathbb{R}^n)} = \|\chi_B\|_{L_\Phi(\mathbb{R}^n)} = \frac{1}{\Phi^{-1}\left(|B|^{-1}\right)}.$$

In the next sections where we prove our main estimates, we use the following lemma, which follows from Theorem 2.6 and Lemma 2.7.

Lemma 2.8. *For a Young function Φ and $B = B(x,r)$, the following inequality is valid*

$$\|f\|_{L_1(B)} \le 2|B|\Phi^{-1}\left(|B|^{-1}\right)\|f\|_{L_\Phi(B)}.$$

Definition 2.9. (Orlicz–Morrey space). For a Young function Φ and $0 \le \lambda \le n$, we denote by $M_{\Phi,\lambda}(\mathbb{R}^n)$ the Orlicz–Morrey space, defined as the space of all functions $f \in L_\Phi^{\rm loc}(\mathbb{R}^n)$ with finite quasinorm

$$\|f\|_{M_{\Phi,\lambda}} = \sup_{x\in\mathbb{R}^n,\ r>0} \Phi^{-1}\big(r^{-\lambda}\big)\|f\|_{L_\Phi(B(x,r))}.$$

Note that $M_{\Phi,\lambda}\big|_{\lambda=0} = L_\Phi(\mathbb{R}^n)$.

We also denote by $WM_{\Phi,\lambda}(\mathbb{R}^n)$ the weak Morrey space which consists of all functions $f \in WL_\Phi^{\rm loc}(\mathbb{R}^n)$ for which

$$\|f\|_{WM_{\Phi,\lambda}} = \sup_{x\in\mathbb{R}^n,\ r>0} \Phi^{-1}\big(r^{-\lambda}\big)\|f\|_{WL_\Phi(B(x,r))} < \infty,$$

where $WL_\Phi(B(x,r))$ denotes the weak L_Φ-space of measurable functions f for which

$$\|f\|_{WL_\Phi(B(x,r))} \equiv \|f\chi_{B(x,r)}\|_{WL_\Phi(\mathbb{R}^n)}.$$

Definition 2.10 (Generalized Orlicz–Morrey Space). Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and Φ any Young function. We denote by $M_{\Phi,\varphi}(\mathbb{R}^n)$ the generalized Morrey space, the space of all functions $f \in L^{\rm loc}_{\Phi}(\mathbb{R}^n)$ with finite quasinorm

$$\|f\|_{M_{\Phi,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} \, \|f\|_{L_\Phi(B(x,r))}.$$

It may be easily shown that $\|f\|_{M_{\Phi,\varphi}}$ is a norm and $M_{\Phi,\varphi}$ is a Banach space, for any Young function Φ.

By $WM_{\Phi,\varphi}(\mathbb{R}^n)$ we denote the weak generalized Morrey space of all functions $f \in WL^{\rm loc}_{\Phi}(\mathbb{R}^n)$ for which

$$\|f\|_{WM_{\Phi,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} \, \|f\|_{WL_\Phi(B(x,r))} < \infty.$$

If Φ satisfies the Δ_2-condition, then the norm $\|f\|_{M_{\Phi,\varphi}}$ is equivalent (see [28], p. 416) to the norm

$$\|f\|_{\overline{M}_{\Phi,\varphi}} = \inf\Big\{\lambda > 0 : \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} \int_{B(x,r)} \Phi\Big(\frac{|f(x)|}{\lambda}\Big) dx \le 1\Big\}.$$

The latter was used in [28, 31, 32, 39], see also references there.

Definition 2.10 recovers the spaces $M_{\Phi,\lambda}$ and $WM_{\Phi,\lambda}$ under the choice $\varphi(x,r) = 1/\Phi^{-1}\big(r^{-\lambda}\big)$ and the spaces $M_{p,\varphi}$ and $WM_{p,\varphi}$ under the choice $\Phi(r) = r^p$.

The following statement was proved in [1] (see also [4]).

Theorem 2.11. *Let $1 \le p < \infty$ and (φ_1, φ_2) satisfies the condition*

$$\sup_{r<t<\infty} \operatorname{ess\,sup} \inf_{t<s<\infty} \varphi_1(x,s)\, t^{-n/p} \le C\, \varphi_2(x,r)\, r^{-n/p}, \tag{2.5}$$

where C does not depend on x and r. Then the maximal operator M is bounded from M_{p,φ_1} to M_{p,φ_2} for $p > 1$ and from M_{1,φ_1} to WM_{1,φ_2}. Moreover, for $p > 1$

$$\|Mf\|_{M_{p,\varphi_2}} \lesssim \|f\|_{M_{p,\varphi_1}}, \quad \textit{and for } p = 1 \quad \|Mf\|_{WM_{1,\varphi_2}} \lesssim \|f\|_{M_{1,\varphi_1}}.$$

The following statement, containing results obtained in [13, 14, 15, 27, 30] was proved in [1] (see also [16]).

Theorem 2.12. *Let $1 \le p < \infty$ and (φ_1, φ_2) satisfies the condition*

$$\int_r^\infty \operatorname{ess\,sup} \inf_{t<s<\infty} \varphi_1(x,s)\, t^{-n/p} \frac{dt}{t} \le C\, \varphi_2(x,r)\, r^{-n/p},$$

where C does not depend on x and r. Then the singular operator T is bounded from M_{p,φ_1} to M_{p,φ_2} for $p > 1$ and from M_{1,φ_1} to WM_{1,φ_2}. Moreover, for $p > 1$

$$\|Tf\|_{M_{p,\varphi_2}} \lesssim \|f\|_{M_{p,\varphi_1}}, \quad \textit{and for } p = 1 \quad \|Tf\|_{WM_{1,\varphi_2}} \lesssim \|f\|_{M_{1,\varphi_1}}.$$

3. Some supremal and Hardy type inequalities

Let v be a weight. We denote by $L_{\infty,v}(0,\infty)$ the space of all functions $g(t)$, $t>0$ with finite norm

$$\|g\|_{L_{\infty,v}(0,\infty)} = \sup_{t>0} v(t)|g(t)|$$

and $L_\infty(0,\infty) \equiv L_{\infty,1}(0,\infty)$. Let $\mathfrak{M}(0,\infty)$ be the set of all Lebesgue-measurable functions on $(0,\infty)$ and $\mathfrak{M}^+(0,\infty)$ its subset of all nonnegative functions on $(0,\infty)$. We denote by $\mathfrak{M}^+(0,\infty;\uparrow)$ the cone of all functions in $\mathfrak{M}^+(0,\infty)$ which are non-decreasing on $(0,\infty)$ and

$$\mathcal{A} = \left\{ \varphi \in \mathfrak{M}^+(0,\infty;\uparrow) : \lim_{t\to 0+} \varphi(t) = 0 \right\}.$$

Let u be a continuous and non-negative function on $(0,\infty)$. We define the supremal operator $\overline{S}_u$ on $g \in \mathfrak{M}(0,\infty)$ by

$$(\overline{S}_u g)(t) := \|u\, g\|_{L_\infty(t,\infty)},\ t \in (0,\infty).$$

The following theorem was proved in [4].

Theorem 3.1. *Let v_1, v_2 be non-negative measurable functions satisfying $0 < \|v_1\|_{L_\infty(t,\infty)} < \infty$ for any $t>0$ and let u be a continuous non-negative function on $(0,\infty)$. Then the operator $\overline{S}_u$ is bounded from $L_{\infty,v_1}(0,\infty)$ to $L_{\infty,v_2}(0,\infty)$ on the cone $\mathcal{A}$ if and only if*

$$\left\| v_2 \overline{S}_u \left(\|v_1\|^{-1}_{L_\infty(\cdot,\infty)} \right) \right\|_{L_\infty(0,\infty)} < \infty. \tag{3.1}$$

We will use the following statement on the boundedness of the weighted Hardy operator

$$H_w^* g(t) := \int_t^\infty g(s) w(s) ds, \quad 0 < t < \infty,$$

where w is a weight.

The following theorem in the case $w=1$ was proved in [4].

Theorem 3.2. *Let v_1, v_2 and w be weights on $(0,\infty)$ and $v_1(t)$ be bounded outside a neighborhood of the origin. The inequality*

$$\sup_{t>0} v_2(t) H_w^* g(t) \le C \sup_{t>0} v_1(t) g(t) \tag{3.2}$$

holds for some $C>0$ for all non-negative and non-decreasing g on $(0,\infty)$ if and only if

$$B := \sup_{t>0} v_2(t) \int_t^\infty \frac{w(s)ds}{\sup_{s<\tau<\infty} v_1(\tau)} < \infty. \tag{3.3}$$

Moreover, the value $C = B$ is the best constant for (3.2).

Remark 3.3. In (3.2) and (3.3) it is assumed that $\frac{1}{\infty} = 0$ and $0 \cdot \infty = 0$.

Proof. Sufficiency. Suppose that (3.3) holds. Whenever F, G are non-negative functions on $(0,\infty)$ and F is non-decreasing, then

$$\sup_{t>0} F(t)G(t) = \sup_{t>0} F(t) \sup_{s>t} G(s), \ t>0. \tag{3.4}$$

By (3.4) we have

$$\begin{aligned}
\sup_{t>0} v_2(t) H_w^* g(t) &= \sup_{t>0} v_2(t) \int_t^\infty g(s)w(s) \frac{\sup_{s<\tau<\infty} v_1(\tau)}{\sup_{s<\tau<\infty} v_1(\tau)}\, ds \\
&\le \sup_{t>0} v_2(t) \int_t^\infty \frac{w(s)ds}{\sup_{s<\tau<\infty} v_1(\tau)} \sup_{t>0} g(t) \sup_{t<\tau<\infty} v_1(\tau) \\
&= \sup_{t>0} v_2(t) \int_t^\infty \frac{w(s)ds}{\sup_{s<\tau<\infty} v_1(\tau)} \sup_{t>0} g(t) v_1(t) \\
&\le B \sup_{t>0} g(t) v_1(t),
\end{aligned}$$

so that (3.2) holds with $C = B$.

Necessity. Suppose that the inequality (3.2) holds with some $C>0$. The function

$$g(t) = \frac{1}{\sup_{t<\tau<\infty} v_1(\tau)}, \ t>0$$

is nonnegative and non-decreasing on $(0,\infty)$. Thus

$$B = \sup_{t>0} v_2(t) \int_t^\infty \frac{w(s)ds}{\sup_{s<\tau<\infty} v_1(\tau)} \le C \sup_{t>0} \frac{v_1(t)}{\sup_{t<\tau<\infty} v_1(\tau)} \le C,$$

which competes the proof. □

4. Boundedness of the maximal operator in the spaces $M_{\Phi,\varphi}(\mathbb{R}^n)$

In this section sufficient conditions on φ for the boundedness of M in generalized Orlicz–Morrey spaces $M_{\Phi,\varphi}(\mathbb{R}^n)$ have been obtained.

Definition 4.1. The operator T is said to be of strong type (Φ,Ψ) if there exists a positive constant k such that

$$\|Tf\|_{L_\Psi} \le k\|f\|_{L_\Phi}$$

for all $f \in L_\Phi(\mathbb{R}^n)$.

The operator T is said to be of weak type (Φ,Ψ) if there exists a positive constant k such that

$$|\{y \in \mathbb{R}^n : |Tf(y)| > t\}| \le 1/\Psi\left(\frac{t}{k\|f\|_{L_\Phi}}\right)$$

for all $t>0$ and all $f \in L_\Phi(\mathbb{R}^n)$.

Necessary and sufficient conditions on Φ for the boundedness of M in Orlicz spaces $L_\Phi(\mathbb{R}^n)$ have been obtained in [19], Theorem 2.1 and [21], Theorem 1.2.1. With Remark 2.5 taken into account, the known boundedness statement runs as follows.

The strong estimate in the following theorem is well known, proved in fact in [21], [5], although not stated directly in the form we need (they may be also derived from the Lorentz–Shimogaki theorem (see [3], p. 154) on the boundedness of the maximal operator on rearrangement invariant spaces and Boyd's interpolation theorem. So we present the proof only of the weak estimate.

Theorem 4.2. *Let Φ be a Young function. Then the maximal operator M is bounded from $L_\Phi(\mathbb{R}^n)$ to $WL_\Phi(\mathbb{R}^n)$ and for $\Phi \in \nabla_2$ bounded in $L_\Phi(\mathbb{R}^n)$.*

Proof. To prove the weak estimate, we take $f \in L_\Phi$ satisfying $\|f\|_{L_\Phi} = 1$ so that $\rho_\Phi(f) := \int_{\mathbb{R}^n} \Phi(|f(x)|)dx \le 1$. By Jensen's inequality,

$$\Phi\left(\frac{1}{|B|}\int_B |f(y)|dy\right) \le \frac{1}{|B|}\int_B \Phi(|f(y)|)dy \tag{4.1}$$

for all balls B. Using (4.1) and definition of the maximal operator, we have

$$\Phi(Mf(x)) \le M[(\Phi\circ f)(x)]. \tag{4.2}$$

Then by (4.2) and the weak $(1,1)$-boundedness of the maximal operator we get

$$|\{x : Mf(x) > t\}| = |\{x : \Phi(Mf(x)) > \Phi(t)\}| \le |\{x : M(\Phi\circ f)(x) > \Phi(t)\}|$$
$$\le \frac{C}{\Phi(t)}\int_{\mathbb{R}^n}\Phi(|f(x)|)dx \le \frac{C}{\Phi(t)} \le \frac{1}{\Phi(\frac{t}{C\|f\|_{L_\Phi}})},$$

since $\|f\|_{L_\Phi} = 1$ and $\frac{1}{C}\Phi\left(t\right) \ge \Phi\left(\frac{t}{C}\right)$, $C \ge 1$. By the homogeneity of the norm $\|\cdot\|_{L_\Phi}$, we then have

$$|\{x : Mf(x) > t\}| \le \frac{1}{\Phi(\frac{t}{C\|f\|_{L_\Phi}})}$$

for every $f \in L_\Phi$, which completes the proof. □

The following lemma is valid.

Lemma 4.3. *Let $f \in L_\Phi^{\rm loc}(\mathbb{R}^n)$ and $B = B(x,r)$. Then*

$$\|Mf\|_{L_\Phi(B)} \lesssim \|f\|_{L_\Phi(B(x,2r))} + \frac{1}{\Phi^{-1}\big(r^{-n}\big)}\sup_{t>2r} t^{-n}\|f\|_{L_1(B(x,t))}, \tag{4.3}$$

for any Young function $\Phi \in \nabla_2$ and

$$\|Mf\|_{WL_\Phi(B)} \lesssim \|f\|_{L_\Phi(B(x,2r))} + \frac{1}{\Phi^{-1}\big(r^{-n}\big)}\sup_{t>2r} t^{-n}\|f\|_{L_1(B(x,t))} \tag{4.4}$$

for any Young function Φ.

Proof. Let $\Phi \in \nabla_2$. We put $f = f_1 + f_2$, where $f_1 = f\chi_{B(x,2r)}$ and $f_2 = f\chi_{\complement_{B(x,2r)}}$ and have

$$\|Mf\|_{L_\Phi(B)} \le \|Mf_1\|_{L_\Phi(B)} + \|Mf_2\|_{L_\Phi(B)}.$$

By the boundedness of the operator M on $L_\Phi(\mathbb{R}^n)$ provided by Theorem 4.2 we have

$$\|Mf_1\|_{L_\Phi(B)} \lesssim \|f\|_{L_\Phi(B(x,2r))}.$$

Let y be an arbitrary point from B. If $B(y,t) \cap {}^{\complement}(B(x,2r)) \neq \emptyset$, then $t > r$. Indeed, if $z \in B(y,t) \cap {}^{\complement}(B(x,2r))$, then $t > |y-z| \ge |x-z| - |x-y| > 2r - r = r$.

On the other hand, $B(y,t) \cap {}^{\complement}(B(x,2r)) \subset B(x,2t)$. Indeed, if $z \in B(y,t) \cap {}^{\complement}(B(x,2r))$, then we get $|x-z| \le |y-z| + |x-y| < t + r < 2t$.

Hence

$$\begin{aligned} Mf_2(y) &= \sup_{t>0} \frac{1}{|B(y,t)|} \int_{B(y,t)\cap {}^{\complement}(B(x,2r))} |f(z)|dz \\ &\le 2^n \sup_{t>r} \frac{1}{|B(x,2t)|} \int_{B(x,2t)} |f(z)|dz = 2^n \sup_{t>2r} \frac{1}{|B(x,t)|} \int_{B(x,t)} |f(z)|dz. \end{aligned}$$

Therefore, for all $y \in B$ we have

$$Mf_2(y) \le 2^n \sup_{t>2r} \frac{1}{|B(x,t)|} \int_{B(x,t)} |f(z)|dz. \tag{4.5}$$

Thus

$$\|Mf\|_{L_\Phi(B)} \lesssim \|f\|_{L_\Phi(B(x,2r))} + \frac{1}{\Phi^{-1}\big(r^{-n}\big)} \left(\sup_{t>2r} \frac{1}{|B(x,t)|} \int_{B(x,t)} |f(z)|dz \right).$$

Let now Φ be an arbitrary Young function. It is obvious that

$$\|Mf\|_{WL_\Phi(B)} \le \|Mf_1\|_{WL_\Phi(B)} + \|Mf_2\|_{WL_\Phi(B)}$$

for every ball $B = B(x,r)$.

By the boundedness of the operator M from $L_\Phi(\mathbb{R}^n)$ to $WL_\Phi(\mathbb{R}^n)$, provided by Theorem 4.2, we have

$$\|Mf_1\|_{WL_\Phi(B)} \lesssim \|f\|_{L_\Phi(B(x,2r))}.$$

Then by (4.5) we get the inequality (4.4). □

Lemma 4.4. *Let $f \in L_\Phi^{\mathrm{loc}}(\mathbb{R}^n)$ and $B = B(x,r)$. Then*

$$\|Mf\|_{L_\Phi(B)} \lesssim \frac{1}{\Phi^{-1}\big(r^{-n}\big)} \sup_{t>2r} \Phi^{-1}\big(t^{-n}\big) \|f\|_{L_\Phi(B(x,t))} \tag{4.6}$$

for any Young function $\Phi \in \nabla_2$ and

$$\|Mf\|_{WL_\Phi(B)} \lesssim \frac{1}{\Phi^{-1}\big(r^{-n}\big)} \sup_{t>2r} \Phi^{-1}\big(t^{-n}\big) \|f\|_{L_\Phi(B(x,t))} \tag{4.7}$$

for any Young function Φ.

Proof. Let $\Phi \in \nabla_2$. Denote

$$\mathcal{M}_1 := \frac{1}{\Phi^{-1}\big(r^{-n}\big)} \left(\sup_{t>2r} \frac{1}{|B(x,t)|} \int_{B(x,t)} |f(z)|dz \right),$$
$$\mathcal{M}_2 := \|f\|_{L_\Phi(B(x,2r))}.$$

Applying Hölder's inequality provided by Lemma 2.8, we get

$$\mathcal{M}_1 \lesssim \frac{1}{\Phi^{-1}\big(r^{-n}\big)} \sup_{t>2r} \frac{1}{|B(x,t)|} \|f\|_{L_\Phi(B(x,t))} \|1\|_{L_{\widetilde{\Phi}}(B(x,t))}$$
$$\lesssim \frac{1}{\Phi^{-1}\big(r^{-n}\big)} \sup_{t>2r} \Phi^{-1}\big(t^{-n}\big) \|f\|_{L_\Phi(B(x,t))}.$$

On the other hand,

$$\frac{1}{\Phi^{-1}\big(r^{-n}\big)} \sup_{t>2r} \Phi^{-1}\big(t^{-n}\big) \|f\|_{L_\Phi(B(x,t))}$$
$$\gtrsim \frac{1}{\Phi^{-1}\big(r^{-n}\big)} \sup_{t>2r} \Phi^{-1}\big(t^{-n}\big) \|f\|_{L_\Phi(B(x,2r))} \approx \mathcal{M}_2.$$

Since $\|Mf\|_{L_\Phi(B)} \leq \mathcal{M}_1 + \mathcal{M}_2$ by Lemma 4.3, we arrive at (4.6). Finally, when Φ is an arbitrary Young function. the inequality (4.7) directly follows from (4.4). □

Corollary 4.5. [1] *Let* $1 \leq p < \infty$ *and* $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$, $B = B(x_0, r)$, $x_0 \in \mathbb{R}^n, r > 0$. *Then, for* $1 < p < \infty$

$$\|Mf\|_{L_p(B(x_0,r))} \lesssim r^{\frac{n}{p}} \sup_{t>2r} t^{-\frac{n}{p}} \|f\|_{L_p(B(x_0,t))}$$

and for $p = 1$

$$\|Mf\|_{WL_1(B(x_0,r))} \lesssim r^n \sup_{t>2r} t^{-n} \|f\|_{L_1(B(x_0,t))}.$$

Theorem 4.6. *Let* Φ *be a Young function, the functions* φ_1, φ_2 *and* Φ *satisfy the condition*

$$\sup_{r<t<\infty} \operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s)\, \Phi^{-1}\big(t^{-n}\big) \leq C\, \varphi_2(x,r)\, \Phi^{-1}\big(r^{-n}\big), \tag{4.8}$$

where C *does not depend on* x *and* r. *Then the maximal operator* M *is bounded from* $M_{\Phi,\varphi_1}(\mathbb{R}^n)$ *to* $WM_{\Phi,\varphi_2}(\mathbb{R}^n)$ *and for* $\Phi \in \nabla_2$, *the operator* M *is bounded from* $M_{\Phi,\varphi_1}(\mathbb{R}^n)$ *to* $M_{\Phi,\varphi_2}(\mathbb{R}^n)$.

Proof. By Lemma 4.4 and Theorem 3.1 with $u(r) = \Phi^{-1}(r^{-n})$, $v_1(r) = \varphi_1(x,r)^{-1}$, $v_2(r) = \frac{1}{\varphi_2(x,r)\Phi^{-1}(r^{-n})}$ and $g(r) = \|f\|_{L_\Phi(B(x,r))}$ we get

$$\|Mf\|_{M_{\Phi,\varphi_2}} \lesssim \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{\varphi_2(x,r)\Phi^{-1}\big(r^{-n}\big)} \sup_{t>r} \Phi^{-1}\big(t^{-n}\big) \|f\|_{L_\Phi(B(x,t))}$$
$$\lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_1(x,r)^{-1} \|f\|_{L_\Phi(B(x,r))}$$
$$= \|f\|_{M_{\Phi,\varphi_1}},$$

if $\Phi \in \nabla_2$ and

$$\|Mf\|_{WM_{\Phi,\varphi_2}} \lesssim \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{\varphi_2(x,r)\Phi^{-1}(r^{-n})} \sup_{t>r} \Phi^{-1}(t^{-n}) \|f\|_{L_\Phi(B(x,t))}$$
$$\lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_1(x,r)^{-1} \|f\|_{L_\Phi(B(x,r))} = \|f\|_{M_{\Phi,\varphi_1}},$$

if Φ is an arbitrary Young function. □

Remark 4.7. Note that, in the case $\Phi(t) = t^p$ from Theorem 4.6 we get Theorem 2.11.

In the case $\varphi_1(x,r) = \frac{1}{\Phi^{-1}(r^{-\lambda_1})}$, $\varphi_2(x,r) = \frac{1}{\Phi^{-1}(r^{-\lambda_2})}$ of Orlicz–Morrey spaces from Theorem 4.6 we get

Corollary 4.8. *Let Φ be any Young function, $0 \le \lambda_1, \lambda_2 < n$ and*

$$\sup_{r<t<\infty} \frac{\Phi^{-1}(t^{-n})}{\Phi^{-1}(t^{-\lambda_1})} \le C \frac{\Phi^{-1}(r^{-n})}{\Phi^{-1}(r^{-\lambda_2})}. \tag{4.9}$$

Then the maximal operator M is bounded from $M_{\Phi,\lambda_1}(\mathbb{R}^n)$ to $WM_{\Phi,\lambda_2}(\mathbb{R}^n)$ and for $\Phi \in \nabla_2$ the operator M is bounded from $M_{\Phi,\lambda_1}(\mathbb{R}^n)$ to $M_{\Phi,\lambda_2}(\mathbb{R}^n)$.

5. Calderón–Zygmund operators in the spaces $M_{\Phi,\varphi}$

In this section, sufficient conditions on φ for the boundedness of the operator T in generalized Orlicz–Morrey spaces $M_{\Phi,\varphi}(\mathbb{R}^n)$ are obtained.

Sufficient conditions on Φ for the boundedness of the operator T in Orlicz spaces $L_\Phi(\mathbb{R}^n)$, as stated in the following theorem are known, see [21], Theorem 1.4.3 and [43], Theorem 3.3, and also [40]; in the next Theorem 5.2 we present the proof of the corresponding weak estimate.

Theorem 5.1. *Let Φ be a Young function and T a singular integral operator. If $\Phi \in \Delta_2 \bigcap \nabla_2$, then the operator T is bounded on $L_\Phi(\mathbb{R}^n)$.*

Theorem 5.2. *Let Φ be a Young function and T a singular integral operator. If $\Phi \in \Delta_2$, then the operator T is bounded from $L_\Phi(\mathbb{R}^n)$ to $WL_\Phi(\mathbb{R}^n)$.*

Proof. Let $\|f\|_{L_\Phi} = 1$. Fix $\lambda > 0$ and put $f = f_1 + f_2$, where $f_1 = \chi_{\{|f|>\lambda\}} \cdot f$ and $f_2 = \chi_{\{|f|\le\lambda\}} \cdot f$. We have

$$|\{|Tf| > \lambda\}| \le |\{|Tf_1| > \lambda/2\}| + |\{|Tf_2| > \lambda/2\}|$$

and

$$\Phi(\lambda)|\{|Tf| > \lambda\}| \le |\Phi(\lambda)\{|Tf_1| > \lambda/2\}| + \Phi(\lambda)|\{|Tf_2| > \lambda/2\}|.$$

By the weak (p,p)-boundedness of T, $p \geq 1$ we get

$$\{|T(\chi_{\{|f|>\lambda\}} \cdot f)| > \lambda\}| \lesssim \frac{1}{\lambda} \int_{\{|f|>\lambda\}} |f|,$$

$$\{|T(\chi_{\{|f|\leq\lambda\}} \cdot f)| > \lambda\}| \lesssim \frac{1}{\lambda^p} \int_{\{|f|\leq\lambda\}} |f|^p.$$

Since $f_1 \in WL_1(\mathbb{R}^n)$ and $\frac{\Phi(\lambda)}{\lambda}$ is increasing, we have

$$\begin{aligned}\Phi(\lambda)\big|\big\{x \in \mathbb{R}^n : |Tf_1(x)| > \frac{\lambda}{2}\big\}\big| &\lesssim \frac{\Phi(\lambda)}{\lambda} \int_{\mathbb{R}^n} |f_1(x)| dx \\ &= \frac{\Phi(\lambda)}{\lambda} \int_{\{x\in\mathbb{R}^n:|f(x)|>\lambda\}} |f(x)| dx \\ &\lesssim \int_{\mathbb{R}^n} |f(x)| \frac{\Phi(|f(x)|)}{|f(x)|} dx = \int_{\mathbb{R}^n} \Phi(|f(x)|) dx.\end{aligned}$$

By Lemma 2.2 we have

$$\begin{aligned}\Phi(\lambda)\,\big|\big\{x \in \mathbb{R}^n : |Tf_2(x)| > \frac{\lambda}{2}\big\}\big| &\lesssim \frac{\Phi(\lambda)}{\lambda^p} \int_{\mathbb{R}^n} |f_2(x)|^p dx \\ &= \frac{\Phi(\lambda)}{\lambda^p} \int_{\{x\in\mathbb{R}^n:|f(x)|\leq\lambda\}} |f(x)|^p dx \\ &\lesssim \int_{\mathbb{R}^n} |f(x)|^p \frac{\Phi(|f(x)|)}{|f(x)|^p} dx = \int_{\mathbb{R}^n} \Phi(|f(x)|) dx.\end{aligned}$$

Thus we get

$$|\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\}| \leq \frac{C}{\Phi(\lambda)} \int_{\mathbb{R}^n} \Phi(|f(x)|) dx \leq \frac{1}{\Phi\left(\frac{\lambda}{C\|f\|_{L_\Phi}}\right)}. \qquad \square$$

Lemma 5.3. *Let Φ be any Young function and $f \in L_\Phi^{\rm loc}(\mathbb{R}^n)$, $B = B(x_0, r)$, $x_0 \in \mathbb{R}^n, r > 0$ and T a singular integral operator. Then*

$$\|Tf\|_{L_\Phi(B)} \lesssim \frac{1}{\Phi^{-1}\big(r^{-n}\big)} \int_{2r}^{\infty} \|f\|_{L_\Phi(B(x_0,t))} \Phi^{-1}\big(t^{-n}\big) \frac{dt}{t},$$

when $\Phi \in \Delta_2 \bigcap \nabla_2$ and

$$\|Tf\|_{WL_\Phi(B)} \lesssim \frac{1}{\Phi^{-1}\big(r^{-n}\big)} \int_{2r}^{\infty} \|f\|_{L_\Phi(B(x_0,t))} \Phi^{-1}\big(t^{-n}\big) \frac{dt}{t}, \tag{5.1}$$

when $\Phi \in \Delta_2$.

Proof. Let $\Phi \in \Delta_2 \bigcap \nabla_2$ first. With the notation $2B = B(x_0, 2r)$, we represent f as

$$f = f_1 + f_2, \quad f_1(y) = f(y)\chi_{2B}(y), \quad f_2(y) = f(y)\chi_{\complement_{(2B)}}(y), \tag{5.2}$$

and then

$$\|Tf\|_{L_\Phi(B)} \leq \|Tf_1\|_{L_\Phi(B)} + \|Tf_2\|_{L_\Phi(B)}.$$

Since $f_1 \in L_\Phi(\mathbb{R}^n)$, by the boundedness of T in $L_\Phi(\mathbb{R}^n)$ provided by Theorem 5.2, it follows that

$$\|Tf_1\|_{L_\Phi(B)} \le \|Tf_1\|_{L_\Phi(\mathbb{R}^n)} \le C\|f_1\|_{L_\Phi(\mathbb{R}^n)} = C\|f\|_{L_\Phi(2B)}.$$

Next, observe that the inclusions $x \in B$, $y \in {}^{\complement}(2B)$ imply $\frac{1}{2}|x_0-y| \le |x-y| \le \frac{3}{2}|x_0-y|$. Then we get

$$|Tf_2(x)| \le C\int_{^{\complement}(2B)} \frac{|f(y)|}{|x_0-y|^n}dy.$$

By Fubini's theorem we have

$$\begin{aligned}\int_{^{\complement}(2B)} \frac{|f(y)|}{|x_0-y|^n}dy &\approx \int_{^{\complement}(2B)} |f(y)|\int_{|x_0-y|}^{\infty} \frac{dt}{t^{n+1}}dy \\ &\approx \int_{2r}^{\infty}\int_{2r\le|x_0-y|<t} |f(y)|dy\frac{dt}{t^{n+1}} \lesssim \int_{2r}^{\infty}\int_{B(x_0,t)} |f(y)|dy\frac{dt}{t^{n+1}}.\end{aligned}$$

Applying the Hölder's inequality (see, Lemma 2.8), we get

$$\begin{aligned}\int_{^{\complement}(2B)} \frac{|f(y)|}{|x_0-y|^n}dy &\lesssim \int_{2r}^{\infty} \|f\|_{L_\Phi(B(x_0,t))}\|1\|_{L_{\widetilde{\Phi}}(B(x_0,t))}\frac{dt}{t^{n+1}} \\ &= \int_{2r}^{\infty} \|f\|_{L_\Phi(B(x_0,t))}\frac{1}{\widetilde{\Phi}^{-1}(|B(x_0,t)|^{-1})}\frac{dt}{t^{n+1}} \\ &\approx \int_{2r}^{\infty} \|f\|_{L_\Phi(B(x_0,t))}\Phi^{-1}\big(t^{-n}\big)\frac{dt}{t}.\end{aligned} \tag{5.3}$$

Moreover,

$$\|Tf_2\|_{L_\Phi(B)} \lesssim \frac{1}{\Phi^{-1}\big(r^{-n}\big)}\int_{2r}^{\infty} \|f\|_{L_\Phi(B(x_0,t))}\Phi^{-1}\big(t^{-n}\big)\frac{dt}{t}. \tag{5.4}$$

Thus

$$\|Tf\|_{L_\Phi(B)} \lesssim \|f\|_{L_\Phi(2B)} + \frac{1}{\Phi^{-1}\big(r^{-n}\big)}\int_{2r}^{\infty} \|f\|_{L_\Phi(B(x_0,t))}\Phi^{-1}\big(t^{-n}\big)\frac{dt}{t}.$$

On the other hand, by (2.3) we get

$$\Phi^{-1}\big(r^{-n}\big) \approx \Phi^{-1}\big(r^{-n}\big)r^n\int_{2r}^{\infty}\frac{dt}{t^{n+1}} \lesssim \int_{2r}^{\infty}\Phi^{-1}\big(t^{-n}\big)\frac{dt}{t}$$

and then

$$\|f\|_{L_\Phi(2B)} \lesssim \frac{1}{\Phi^{-1}\big(r^{-n}\big)}\int_{2r}^{\infty} \|f\|_{L_\Phi(B(x_0,t))}\Phi^{-1}\big(t^{-n}\big)\frac{dt}{t}. \tag{5.5}$$

Thus

$$\|Tf\|_{L_\Phi(B)} \lesssim \frac{1}{\Phi^{-1}\big(r^{-n}\big)}\int_{2r}^{\infty} \|f\|_{L_\Phi(B(x_0,t))}\Phi^{-1}\big(t^{-n}\big)\frac{dt}{t}.$$

Let $\Phi \in \Delta_2$. By the weak boundedness of T on Orlicz space and (5.5) it follows that:

$$\begin{aligned}\|Tf_1\|_{WL_\Phi(B)} &\le \|Tf_1\|_{WL_\Phi(\mathbb{R}^n)} \lesssim \|f_1\|_{L_\Phi(\mathbb{R}^n)} \\ &= \|f\|_{L_\Phi(2B)} \lesssim \frac{1}{\Phi^{-1}\big(r^{-n}\big)} \int_{2r}^{\infty} \|f\|_{L_\Phi(B(x_0,t))}\Phi^{-1}\big(t^{-n}\big)\frac{dt}{t}.\end{aligned} \tag{5.6}$$

Then by (5.4) and (5.6) we get the inequality (5.1). □

Corollary 5.4. [13, 14, 15] *Let $1 \le p < \infty$ and $f \in L_p^{\rm loc}(\mathbb{R}^n)$, $B = B(x_0, r)$, $x_0 \in \mathbb{R}^n, r > 0$ and T a singular integral operator. Then, for $1 < p < \infty$*

$$\|Tf\|_{L_p(B(x_0,r))} \lesssim r^{\frac{n}{p}} \int_{2r}^{\infty} t^{-\frac{n}{p}-1}\|f\|_{L_p(B(x_0,t))}dt$$

and for $p = 1$

$$\|Tf\|_{WL_1(B(x_0,r))} \lesssim r^{n} \int_{2r}^{\infty} t^{-n-1}\|f\|_{L_1(B(x_0,t))}dt.$$

The following theorem contains Theorem 2.12 under the choice in the case $\Phi(t) = t^p$.

Theorem 5.5. *Let Φ any Young function, φ_1, φ_2 and Φ satisfy the condition*

$$\sup_{x\in\mathbb{R}^n,\, r>0} \frac{1}{\varphi_2(x,r)\Phi^{-1}\big(r^{-n}\big)} \int_r^{\infty} \operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s)\, \Phi^{-1}\big(t^{-n}\big)\frac{dt}{t} < \infty. \tag{5.7}$$

Then the operator T is bounded from $M_{\Phi,\varphi_1}(\mathbb{R}^n)$ to $M_{\Phi,\varphi_2}(\mathbb{R}^n)$ for $\Phi \in \Delta_2 \cap \nabla_2$ and from $M_{\Phi,\varphi_1}(\mathbb{R}^n)$ to $WM_{\Phi,\varphi_2}(\mathbb{R}^n)$ for $\Phi \in \Delta_2$.

Proof. By Lemma 5.3 and Theorem 3.2 with $w(r) = \frac{\Phi^{-1}(r^{-n})}{r}$, $v_1(r) = \varphi_1(x,r)^{-1}$, $v_2(r) = \frac{1}{\varphi_2(x,r)\Phi^{-1}(r^{-n})}$ and $g(r) = \|f\|_{L_\Phi(B(x,r))}$, we have

$$\begin{aligned}\|Tf\|_{M_{\Phi,\varphi_2}} &= \sup_{x\in\mathbb{R}^n,\, r>0} \varphi_2(x,r)^{-1}\|Tf\|_{L_\Phi(B(x,r))} \\ &\lesssim \sup_{x\in\mathbb{R}^n,\, r>0} \frac{1}{\varphi_2(x,r)\Phi^{-1}\big(r^{-n}\big)} \int_r^{\infty} \|f\|_{L_\Phi(B(x,t))}\Phi^{-1}\big(t^{-n}\big)\frac{dt}{t} \\ &\lesssim \sup_{x\in\mathbb{R}^n,\, r>0} \varphi_1(x,r)^{-1}\|f\|_{L_\Phi(B(x,r))} \lesssim \|f\|_{M_{\Phi,\varphi_1}}.\end{aligned}$$

if $\Phi \in \Delta_2 \cap \nabla_2$ and

$$\begin{aligned}\|Tf\|_{WM_{\Phi,\varphi_2}} &= \sup_{x\in\mathbb{R}^n,\, r>0} \varphi_2(x,r)^{-1}\|Tf\|_{WL_\Phi(B(x,r))} \\ &\lesssim \sup_{x\in\mathbb{R}^n,\, r>0} \frac{1}{\varphi_2(x,r)\Phi^{-1}\big(r^{-n}\big)} \int_r^{\infty} \|f\|_{L_\Phi(B(x,t))}\Phi^{-1}\big(t^{-n}\big)\frac{dt}{t} \\ &\lesssim \sup_{x\in\mathbb{R}^n,\, r>0} \varphi_1(x,r)^{-1}\|f\|_{L_\Phi(B(x,r))} \lesssim \|f\|_{M_{\Phi,\varphi_1}}.\end{aligned}$$

if $\Phi \in \nabla_2$. □

Remark 5.6. The condition (4.8) is weaker than (5.7). Indeed, (5.7) implies (4.8):

$$\begin{aligned}\varphi_2(x,r)\Phi^{-1}\big(r^{-n}\big) &\gtrsim \int_r^\infty \operatorname*{ess\,inf}_{t<\tau<\infty} \varphi_1(x,\tau)\,\Phi^{-1}\big(t^{-n}\big)\frac{dt}{t}\\ &\gtrsim \int_s^\infty \operatorname*{ess\,inf}_{t<\tau<\infty} \varphi_1(x,\tau)\,\Phi^{-1}\big(t^{-n}\big)\frac{dt}{t}\\ &\gtrsim \operatorname*{ess\,inf}_{s<\tau<\infty} \varphi_1(x,\tau)\int_s^\infty \Phi^{-1}\big(t^{-n}\big)\frac{dt}{t}\\ &\approx \operatorname*{ess\,inf}_{s<\tau<\infty} \varphi_1(x,\tau)\Phi^{-1}\big(s^{-n}\big),\end{aligned}$$

where we took $s \in (r,\infty)$, so that

$$\sup_{s>r} \operatorname*{ess\,inf}_{s<\tau<\infty} \varphi_1(x,\tau)\Phi^{-1}\big(s^{-n}\big) \lesssim \varphi_2(x,r)\Phi^{-1}\big(r^{-n}\big).$$

On the other hand the functions $\varphi_1(x,t) = \varphi_2(x,t) = \frac{1}{\Phi^{-1}\left(t^{-n}\right)}$ satisfy the condition (4.8), but do not satisfy the condition (5.7).

Corollary 5.7. *Let Φ be any Young function, $0 \le \lambda_1, \lambda_2 < n$ and*

$$\int_r^\infty \frac{\Phi^{-1}\big(t^{-n}\big)}{\Phi^{-1}\big(t^{-\lambda_1}\big)}\frac{dt}{t} \le C\,\frac{\Phi^{-1}\big(r^{-n}\big)}{\Phi^{-1}\big(r^{-\lambda_2}\big)}. \tag{5.8}$$

Then for $\Phi \in \Delta_2 \cap \nabla_2$, the singular operator T is bounded from $M_{\Phi,\lambda_1}(\mathbb{R}^n)$ to $M_{\Phi,\lambda_2}(\mathbb{R}^n)$ and for $\Phi \in \Delta_2$ is bounded from $M_{\Phi,\lambda_1}(\mathbb{R}^n)$ to $WM_{\Phi,\lambda_2}(\mathbb{R}^n)$.

Proof. Choose $\varphi_1(x,r) = \frac{1}{\Phi^{-1}\left(r^{-\lambda_1}\right)}$, $\varphi_2(x,r) = \frac{1}{\Phi^{-1}\left(r^{-\lambda_2}\right)}$ in Theorem 5.5. □

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References

[1] Ali Akbulut, V.S. Guliyev and R. Mustafayev, *On the boundedness of the maximal operator and singular integral operators in generalized Morrey spaces.* Math. Bohem. **137** (1) (2012), 27–43.

[2] C. Bennett and K. Rudnick, *On Lorentz–Zygmund spaces.* Dissertationes Math. 175 (1980).

[3] C. Bennett and R. Sharpley, *Interpolation of operators.* Academic Press, Boston, 1988.

[4] V. Burenkov, A. Gogatishvili, V.S. Guliyev, R. Mustafayev, *Boundedness of the fractional maximal operator in local Morrey-type spaces.* Complex Var. Elliptic Equ. **55** (8-10) (2010), 739–758.

[5] A. Cianchi, *Strong and weak type inequalities for some classical operators in Orlicz spaces.* J. London Math. Soc. **60** (2) (1999), no. 1, 187–202.

[6] D. Cruz-Uribe and A. Fiorenza, *Variable Lebesgue Spaces: Foundations and Harmonic Analysis.* Birkhäuser, 2013.

[7] L. Diening, P. Hästö and A. Nekvinda, *Open problems in variable exponent Lebesgue and Sobolev spaces.* Function Spaces, Differential Operators and Nonlinear Analysis, Proceedings of the Conference held in Milovy, Bohemian-Moravian Uplands, May 28–June 2, 2004, Math. Inst. Acad. Sci. Czech Republick, Praha, 2005, 38–58.

[8] L. Diening, P. Harjulehto, P. Hästö, and M. Ružička, *Lebesgue and Sobolev spaces with variable exponents.* Springer-Verlag, Lecture Notes in Mathematics, vol. 2017, Berlin, 2011.

[9] G.T. Dzhumakaeva and K.Zh. Nauryzbaev, *Lebesgue–Morrey spaces.* Izv. Akad. Nauk Kazakh. SSR Ser. Fiz.-Mat. **79** (1982), no. 5, 7–12.

[10] D.E. Edmunds, P. Gurka and B. Opic, *Double exponential integrability of convolution operators in generalized Lorentz–Zygmund spaces.* Indiana Univ. Math. J. **44** (1995), 19–43.

[11] D.E. Edmunds, P. Gurka and B. Opic, *On embeddings of logarithmic Bessel potential spaces.* J. Funct. Anal. **146** (1997) 116–150.

[12] M. Giaquinta, *Multiple integrals in the calculus of variations and nonlinear elliptic systems.* Princeton Univ. Press, Princeton, NJ, 1983.

[13] V.S. Guliyev, *Integral operators on function spaces on the homogeneous groups and on domains in* $\mathbb{R}^n$. Doctor's degree dissertation, Mat. Inst. Steklov, Moscow, 1994, 329 pp. (in Russian).

[14] V.S. Guliyev, *Function spaces, Integral Operators and Two Weighted Inequalities on Homogeneous Groups. Some Applications.* Casioglu, Baku, 1999, 332 pp. (in Russian).

[15] V.S. Guliyev, *Boundedness of the maximal, potential and singular operators in the generalized Morrey spaces.* J. Inequal. Appl. 2009, Art. ID 503948, 20 pp.

[16] V.S. Guliyev, S.S. Aliyev, T. Karaman, P.S. Shukurov, *Boundedness of sublinear operators and commutators on generalized Morrey Space.* Integr. Equat. Oper. Theory **71** (3) (2011), 327–355.

[17] H. Kita, *On maximal functions in Orlicz spaces.* Proc. Amer. Math. Soc. **124** (1996), 3019–3025.

[18] H. Kita, *On Hardy–Littlewood maximal functions in Orlicz spaces.* Math. Nachr. **183** (1997), 135–155.

[19] H. Kita, *Inequalities with weights for maximal functions in Orlicz spaces.* Acta Math. Hungar. **72** (4) (1996), 291–305.

[20] V. Kokilashvili, *On a progress in the theory of integral operators in weighted Banach function spaces.* Function Spaces, Differential Operators and Nonlinear Analysis, Proceedings of the Conference held in Milovy, Bohemian-Moravian Uplands, May 28–June 2, 2004, Math. Inst. Acad. Sci. Czech Republick, Praha, 2005, 152–175.

[21] V. Kokilashvili, M.M. Krbec, *Weighted Inequalities in Lorentz and Orlicz Spaces.* World Scientific, Singapore, 1991.

[22] V. Kokilashvili and S. Samko, *Weighted Boundedness of the Maximal, Singular and Potential Operators in Variable Exponent Spaces.* Analytic Methods of Analysis and Differential Equations, Cambridge Scientific Publishers, eds.: A.A. Kilbas and S.V. Rogosin, 139–164, 2008.

[23] M.A. Krasnoselskii and Ya.B. Rutickii, *Convex Functions and Orlicz Spaces*. English translation P. Noordhoff Ltd., Groningen, 1961.

[24] A. Kufner, O. John and S. Fuçik, *Function Spaces*. Noordhoff International Publishing: Leyden, Publishing House Czechoslovak Academy of Sciences: Prague, 1977.

[25] PeiDe Liu, YouLiang Hou, and MaoFa Wang, *Weak Orlicz space and its applications to the martingale theory*. Science China Mathematics **53** (4) (2010), 905–916.

[26] PeiDe Liu and MaoFa Wang, *Weak Orlicz spaces: Some basic properties and their applications to harmonic analysis*. Science China Mathematics **56** (4) (2013), 789–802.

[27] T. Mizuhara, *Boundedness of some classical operators on generalized Morrey spaces*. Harmonic Analysis (S. Igari, editor), ICM 90 Satellite Proceedings, Springer-Verlag, Tokyo (1991), 183–189.

[28] Y. Mizuta, E. Nakai, T. Ohno and T. Shimomura, *Maximal functions, Riesz potentials and Sobolev embeddings on Musielak–Orlicz–Morrey spaces of variable exponent in* R^n. Revista Matem. Complut. **25**:2 (2012), 413–434.

[29] C.B. Morrey, *On the solutions of quasi-linear elliptic partial differential equations*. Trans. Amer. Math. Soc. **43** (1938), 126–166.

[30] E. Nakai, *Hardy–Littlewood maximal operator, singular integral operators and Riesz potentials on generalized Morrey spaces*. Math. Nachr. **166** (1994), 95–103.

[31] E. Nakai, *Orlicz–Morrey spaces and the Hardy–Littlewood maximal function*. Studia Math. **188** (2008), no. 3, 193–221.

[32] E. Nakai, *Calderón–Zygmund operators on Orlicz–Morrey spaces and modular inequalities*. Banach and function spaces II, 393-410, Yokohama Publ., Yokohama, 2008.

[33] R. O'Neil, *Fractional integration in Orlicz spaces*. Trans. Amer. Math. Soc. **115** (1965), 300–328.

[34] W. Orlicz, *Über eine gewisse Klasse von Räumen vom Typus B*. Bull. Acad. Polon. A (1932), 207–220; reprinted in: Collected Papers, PWN, Warszawa, 1988, 217–230.

[35] W. Orlicz, *Über Räume* (L^M). Bull. Acad. Polon. A (1936), 93-107; reprinted in: Collected Papers, PWN, Warszawa, 1988, 345–359.

[36] H. Rafeiro, N. Samko, and S. Samko, *Morrey–Campanato Spaces: an Overview*. In Operator Theory, Pseudo-Differential Equations, and Mathematical Physics: The Vladimir Rabinovich Anniversary Volume, volume 228 of Operator Theory: Advances and Applications, Birkhäuser, 2013.

[37] M.M. Rao and Z.D. Ren, *Theory of Orlicz Spaces*. M. Dekker, Inc., New York, 1991.

[38] S. Samko, *On a progress in the theory of Lebesgue spaces with variable exponent: maximal and singular operators*. Integral Transforms Spec. Funct. **16** (5-6) (2005), 461–482.

[39] Y. Sawano, S. Sugano, H. Tanaka, *Orlicz–Morrey spaces and fractional operators*. Potential Anal. **36** (2012), no. 4, 517–556.

[40] Y. Sawano, *A Handbook of Harmonic Analysis*, Tokyo, 2011.

[41] E.M. Stein, *Singular integrals and differentiability of functions*. Princeton University Press, Princeton, NJ, 1970.

[42] E.M. Stein, *Harmonic Analysis: Real Variable Methods, Orthogonality and Oscillatory Integrals*. Princeton Univ. Press, Princeton NJ, 1993.

[43] R. Vodák, *The problem $\nabla \cdot v = f$ and singular integrals on Orlicz spaces*. Acta Univ. Palack. Olomuc. Fac. Rerum Natur. Math. **41** (2002), 161–173.

[44] A. Torchinsky, *Real Variable Methods in Harmonic Analysis*. Pure and Applied Math. 123, Academic Press, New York, 1986.

[45] G. Weiss, *A note on Orlicz spaces*. Port. Math. **15** (1956), 35–47.

[46] C.T. Zorko, *Morrey space*. Proc. Amer. Math. Soc. **98** (1986), no. 4, 586–592.

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On a Question by Markus Seidel

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Dedicated to Prof. António Ferreira dos Santos

Abstract. We give necessary and sufficient conditions for the applicability of the finite section method to an arbitrary operator in the Banach algebra generated by the operators of multiplication and the convolution operators with piecewise continuous generating functions on $L^p(\mathbb{R})$, $1 < p < \infty$ using a variation from the standard technique. We prove that it is possible to arrive to this result using only strong-limit homomorphism and with considerable simplification of the standard identification procedure for the local algebras.

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1. Introduction

Given $1 < p < \infty$, let $\mathcal{B} := \mathcal{B}(L^p(\mathbb{R}))$ denote the Banach algebra of all bounded linear operators on the Lebesgue space $L^p(\mathbb{R})$. Let PC be the smallest C^*-subalgebra of $L^\infty(\mathbb{R})$ containing all piecewise continuous functions on $\dot{\mathbb{R}}$, the one point compactification of the real line, and let PC_p stand for its Fourier multiplier analogue, which is a Banach subalgebra of $\mathcal{M}_p$, the Banach algebra of all Fourier multipliers on $L^p(\mathbb{R})$. The Fredholm theory for the smallest Banach subalgebra of $\mathcal{B}(L^p(\mathbb{R}))$ that contains all the convolution type operators $aF^{-1}bF$ where F is the Fourier transform and $a \in PC$, $b \in PC_p$ is well known (see, for instance, [11, Chapter 5]).

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Let $\mathbb{R}_+ := (0, \infty)$ and $\mathbb{R}_- := (-\infty, 0)$. For $\tau \in \mathbb{R}_+$, consider the operators

$$(P_\tau u)(t) := \begin{cases} u(t) & \text{if} \quad |t| < \tau, \\ 0 & \text{if} \quad |t| > \tau, \end{cases} \qquad Q_\tau := I - P_\tau$$

acting on $L^p(\mathbb{R})$ with norm 1. Clearly, $P_\tau \to I$ and $Q_\tau \to 0$ strongly as $\tau \to \infty$. One says that *the finite section method applies* to an operator $A \in \mathcal{B}(L^p(\mathbb{R}))$ if there exists a positive constant τ_0, such that for any $\tau > \tau_0$ and any $f \in L^p(\mathbb{R})$ there exists a unique solution u_τ of the equation

$$A_\tau u_\tau := (P_\tau A P_\tau + Q_\tau) u_\tau = f \tag{1.1}$$

and u_τ converges in the norm of $L^p(\mathbb{R})$ to a solution of the equation $Au = f$ as $\tau \to \infty$.

The above is equivalent to the *stability* of the sequence $(A_\tau)_{\tau>0}$ defined in (1.1). We say that a sequence is stable if there is a $\tau_0 > 0$ such that the elements A_τ are invertible for $\tau > \tau_0$ and the norms of the inverses are uniformly bounded.

We refer to the monographs by Gohberg and Feldman [2], Prössdorf and Silbermann [7], Hagen, Roch and Silbermann [3, 4], Roch, Santos and Silbermann [11] for a general theory of projection methods as well as for more specific issues of the finite section method for convolution type operators and algebras generated by them.

Roch, Silbermann, and one of the authors studied in [10] the applicability of the finite section method for an arbitrary operator in the Banach algebra generated by the operators of multiplication by piecewise continuous functions (PC) and by the convolution operators with piecewise continuous Fourier multipliers (PC_p).

The usual approach to analyze the applicability of the finite sections method follows a general scheme to treat approximation problems. This scheme goes back to Kozak [6] and Silbermann [13] and relates stability problems in numerical analysis with invertibility problems in suitably chosen Banach algebras. This scheme can be summarized as follows (see also [11, Sections 6.1–6.3]). Let $\mathcal{A}$ be a set of (generalized) sequences that contains all sequences of the form

$$(A_\tau) = (P_\tau A P_\tau + Q_\tau) \quad (\tau \in \mathbb{R}_+). \tag{1.2}$$

1. **Algebraization**: Find a unital Banach algebra $\mathcal{E}$ containing $\mathcal{A}$ and a closed ideal $\mathcal{G}$ of $\mathcal{E}$ such that the original problem becomes equivalent to an invertibility problem in the quotient algebra $\mathcal{E}/\mathcal{G}$.
2. **Essentialization**: Find a closed unital subalgebra $\mathcal{F}$ of $\mathcal{E}$ containing $\mathcal{A}$ such that $\mathcal{F}/\mathcal{G}$ is inverse closed in $\mathcal{E}/\mathcal{G}$ and a closed ideal $\mathcal{J}$ of $\mathcal{F}$ containing $\mathcal{G}$ such that $\mathcal{J}$ can be lifted. The latter means that one has full control about the difference between the invertibility of a coset of a sequence $(A_\tau) \in \mathcal{F}$ in the algebra $\mathcal{F}/\mathcal{G}$ and the invertibility of the coset of the same sequence in $\mathcal{F}/\mathcal{J}$. This control is usually guaranteed by a lifting theorem and involves the use of a collection W_t, t in some set $\mathbb{T}$, of homomorphisms from $\mathcal{F}$ to $\mathcal{B}$.
3. **Localization**: Find a unital subalgebra $\mathcal{L}$ of $\mathcal{F}$ such that
   (a) $\mathcal{A}, \mathcal{J} \subset \mathcal{L}$;

(b) $\mathcal{L}/\mathcal{J}$ is inverse closed in $\mathcal{F}/\mathcal{J}$;
(c) the quotient algebra $\mathcal{L}/\mathcal{J}$ has a large center.

Use a local principle to translate the invertibility problem in the algebra $\mathcal{L}/\mathcal{J}$ to a family of simpler invertibility problems in the local algebras.

4. **Identification**: Find conditions for the invertibility of the cosets of sequences in $\mathcal{A}$ in the local algebras. Here one uses another collection of homomorphisms H_η, indexed on the maximal ideal space used in the localization. In some cases, like the points at "infinity", the homomorphism was not known and other techniques are usually used, such as the two projections theorem.

During a break in a workshop in Altenberg in 2011, Markus Seidel posed the following question to one of the authors: "Would it be possible to use the second family of homomorphisms (the H_η above) in the essentialization step? What would happen then to the local algebras?"

The answer to his question is in this paper. It turns out that it is indeed possible to use all homomorphisms in the essentialization step, but several technical issues had to be resolved. This alternate procedure then simplifies considerably the identification step. For instance, there is no need to apply the two-projections theorem anymore. We will exemplify the use of the modified technique for the problem treated in [10], that is, finite section method applied to operators in the algebra of multiplication and convolution operators generated by piecewise continuous symbol acting on the Lebesgue spaces $L^p(\mathbb{R})$, $1 < p < \infty$. In this way we have a Banach algebra (and not the simpler C*-algebra) problem, but the operators are relatively simple (i.e., no slowly oscillating generating functions and no flip) so that the new techniques do not become obfuscated by technical detail. But in fact, with the necessary adaptations, it is possible to use this modified technique to treat larger algebras.

The paper is organized as follows. The next section are the basic definitions and several results on the limits of certain sequences of operators that will be needed later. Sections 3, 4 and 5 then follow the usual procedure regarding algebraization, essentialization and localization using the modified technique.

2. Notation and basic results

This paper follows closely the notation of [10]. As mentioned in the introduction, we will consider the operators acting on the Lebesgue space $L^p(\mathbb{R})$, $1 < p < \infty$ and write the *Fourier transform* F on the Schwartz space of rapidly decreasing infinite differentiable functions as

$$(Fu)(y) = \int_{-\infty}^{+\infty} e^{-2\pi \mathbf{i} yx} u(x)\,dx, \qquad y \in \mathbb{R}. \tag{2.1}$$

Then its inverse is given by

$$(F^{-1}v)(x) = \int_{-\infty}^{+\infty} e^{2\pi \mathbf{i} xy} v(y)\,dy, \qquad x \in \mathbb{R}. \tag{2.2}$$

It is well known that the operators F and F^{-1} can be extended continuously to bounded and unitary operators on the Hilbert space $L^2(\mathbb{R})$ and that F extends continuously to a bounded operator from $L^p(\mathbb{R})$ to $L^q(\mathbb{R})$ where $q := p/(p-1)$ if $1 < p \leq 2$ (see, for instance, [14, Theorem 74]).

Let $\mathcal{M}_p$ denote the set of all *Fourier multipliers*, i.e., the set of all functions $a \in L^\infty(\mathbb{R})$ with the following property: if $u \in L^2(\mathbb{R}) \cap L^p(\mathbb{R})$, then $F^{-1}aFu \in L^p(\mathbb{R})$, and there is a constant c_p independent of u such that $\|F^{-1}aFu\|_p \leq c_p\|u\|_p$. If $a \in \mathcal{M}_p$, then the operator $F^{-1}aF : (L^2(\mathbb{R}) \cap L^p(\mathbb{R})) \to L^p(\mathbb{R})$ extends continuously to a bounded operator on $L^p(\mathbb{R})$. This extension is called a (*Fourier*) *convolution operator* , and we denote it by $W^0(a)$. The function a is also called the *generating function* (or the symbol or presymbol) of $W^0(a)$. The set $\mathcal{M}_p$ of all Fourier multipliers forms a Banach algebra when equipped with the operations inherited from $L^\infty(\mathbb{R})$ and the norm

$$\|a\|_{\mathcal{M}_p} := \|W^0(a)\|_{\mathcal{L}(L^p(\mathbb{R}))}. \tag{2.3}$$

We call a function $a \in L^\infty(\mathbb{R})$ *piecewise constant* (resp. *piecewise linear*) if there is a partition $-\infty = t_0 < t_1 < \cdots < t_n = +\infty$ of the real line such that a is constant (resp. linear) on each interval $[t_k, t_{k+1}]$. Stetchkin's inequality (see, for instance, [1]) entails that the multiplier algebra $\mathcal{M}_p$ contains the (non-closed) algebras C_0 of all continuous and piecewise linear functions on $\dot{\mathbb{R}}$ and PC_0 of all piecewise constant functions on $\mathbb{R}$. Let C_p and PC_p denote the closures of C_0 and PC_0 in $\mathcal{M}_p$, respectively.

We introduce now the building blocks for the homomorphisms. They are three shifts.

For s, $t \in \mathbb{R}$ and $\tau \in (0, \infty)$, consider the operators

$$U_s : L^p(\mathbb{R}) \mapsto L^p(\mathbb{R}), \quad (U_s u)(x) = e^{-2\pi \mathbf{i} xs} u(x), \tag{2.4}$$

$$V_s : L^p(\mathbb{R}) \mapsto L^p(\mathbb{R}), \quad (V_s u)(x) = u(x - s), \tag{2.5}$$

$$Z_\tau : L^p(\mathbb{R}) \mapsto L^p(\mathbb{R}), \quad (Z_\tau u)(x) := \tau^{-1/p} u(x/\tau). \tag{2.6}$$

Clearly, $U_s^{-1} = U_{-s}$, $V_t^{-1} = V_{-t}$, and $Z_\tau^{-1} = Z_{\tau^{-1}}$, and these operators have norm 1. V_t is an additive shift in the normal space, U_s is the corresponding additive shift in the Fourier image space, and Z_τ is a multiplicative shift (in both the base space and the Fourier image space).

The following simple results can be proved by writing the operators explicitly.

Lemma 2.1. *If $a \in \mathcal{M}_p$ and $s \in \mathbb{R}$, then $U_{-s}W^0(a)U_s = W^0(V_s a V_{-s})$ and $V_s W^0(a) V_{-s} = W^0(a)$. Moreover, if $p = 2$ then*

$$U_s F^{-1} = F^{-1} V_{-s}, \quad FU_s = V_{-s} F, \quad V_s F^{-1} = F^{-1} U_s, \quad FV_s u = U_s F.$$

Recall that a sequence (A_τ) with $A_\tau \in \mathcal{B}(X)$ tends weakly to an operator $A \in \mathcal{B}(X)$ as $\tau \to \infty$ if

$$\langle v, (A_\tau - A)u \rangle \to 0, \quad \text{for any fixed } u \in X,\ v \in X^* \tag{2.7}$$

with $\langle \cdot\, , \cdot \rangle$ standing for the duality product. When X is $L^p(\mathbb{R})$ and the sequence is bounded, in order to check its weak convergence it is enough to show that, for arbitrary $a, b, c, d \in \mathbb{R}$, with $a < b$ and $c < d$, one has

$$\langle \chi_{[c,d]}, (A_\tau - A)\chi_{[a,b]} \rangle \to 0$$

since the linear combinations of characteristic functions form a dense subset of $L^p(\mathbb{R})$. This fact will be used in the proofs below.

Lemma 2.2.
(a) *The operators V_s and U_s converge weakly to zero as $s \to \pm\infty$.*
(b) *The operators $Z_\tau^{\pm 1}$ converge weakly to zero as $\tau \to \infty$.*

Proof. The result for Z_τ and V_s are in [10, Lemma 2.2]. In the case of U_s, consider the characteristic function $\chi_{[a,b]}$ of an interval $[a, b]$ of the real line, with $b > a$. We observe that $e^{-2\pi \mathbf{i} x s}$ is a periodic function with period $1/s$ and that the integral over a full period of the function gives 0. Then for s large, it is easy to see that there exist integers $k_{1,s}, k_{2,s}$ such that

$$\int_{\mathbb{R}} (U_s \chi_{[a,b]})(x)\, dx = \int_a^b e^{-2\pi \mathbf{i} x s}\, dx = \int_a^{k_{1,s}/s} e^{-2\pi \mathbf{i} x s}\, dx + \int_{k_{2,s}/s}^b e^{-2\pi \mathbf{i} x s}\, dx$$

with $|k_{1,s}/s - a| \to 0$, $|b - k_{2,s}/s| \to 0$, which means the value of the integral tends to zero as s goes to infinity. Thus

$$\langle \chi_{[c,d]}, U_s \chi_{[a,b]} \rangle \to 0 \quad \text{as} \quad s \to \infty$$

for any two characteristic functions of intervals of $\mathbb{R}$, which implies by a density argument that U_s converge weakly to zero. □

We will also need the following less trivial weak limits

Lemma 2.3. *The following sequences tend weakly to* 0 *as $\tau \to \infty$, for fixed $s, t \in \mathbb{R}$.*
(i) $Z_\tau^{-1} U_s$, $Z_\tau V_s$;
(ii) $Z_\tau^{-1} U_s V_{\pm\tau}$, $Z_\tau V_s V_{\pm\tau}$;
(iii) $Z_\tau^{-1} U_s V_t Z_\tau^{-1}$, $Z_\tau V_s U_t Z_\tau$;
(iv) $Z_\tau^{-1} U_s V_{\pm\tau} Z_\tau^{-1}$, $Z_\tau V_{\mp\tau} U_s Z_\tau$;
(v) $Z_\tau V_{-s} V_{\pm\tau} Z_\tau^{-1}$, $Z_\tau V_{\mp\tau} V_s Z_\tau^{-1}$;
(vi) $Z_\tau^{-1} U_s U_t Z_\tau$, $Z_\tau V_s V_t Z_\tau^{-1}$, *for $s \neq -t$.*

Proof. The weak limits of (i) come directly from Lemma 2.2 and the definition (2.7) because $U_s u$ and $V_s u$ are fixed functions in $L^p(\mathbb{R})$. Take now $a, b, c, d \in \mathbb{R}$, with $a < b$ and $c < d$. To prove the first part of (ii) note that

$$\begin{aligned}(Z_\tau^{-1} U_s V_{\pm\tau} \chi_{[a,b]})(x) &= \tau^{1/p} e^{-2\pi \mathbf{i} \tau x s} \chi_{[a,b]}(\tau x \mp \tau) \\ &= \tau^{1/p} e^{-2\pi \mathbf{i} \tau x s} \chi_{[\frac{a}{\tau} \pm 1, \frac{b}{\tau} \pm 1]}(x).\end{aligned}$$

Now,

$$\begin{aligned}\left|\langle \chi_{[c,d]}, Z_\tau^{-1} U_s V_{\pm\tau} \chi_{[a,b]}\rangle\right| &= \left|\int_c^d \tau^{1/p} e^{-2\pi \mathbf{i} \tau x s} \chi_{[\frac{a}{\tau}\pm 1, \frac{b}{\tau}\pm 1]}(x)\,dx\right| \\ &\leq \tau^{1/p}\int_c^d \chi_{[\frac{a}{\tau}\pm 1, \frac{b}{\tau}\pm 1]}(x)\,dx \\ &\leq \tau^{1/p}\frac{b-a}{\tau} = \tau^{1/p-1}(b-a) \to 0\end{aligned}$$

since $\chi_{[\frac{a}{\tau}\pm 1, \frac{b}{\tau}\pm 1]}$ is different from zero in a interval of length $(b-a)/\tau$ and $p>1$. For the second part of (ii) write

$$\begin{aligned}(Z_\tau V_s V_{\pm\tau}\chi_{[a,b]})(x) &= \tau^{-1/p}\chi_{[a,b]}(x/\tau - s \mp \tau) \\ &= \tau^{-1/p}\chi_{[\tau(a+s\pm\tau),\tau(b+s\pm\tau)]}(x)\end{aligned}$$

and

$$\begin{aligned}\left|\langle \chi_{[c,d]}, Z_\tau V_s V_{\pm\tau}\chi_{[a,b]}\rangle\right| &= \left|\int_c^d \tau^{-1/p}\chi_{[\tau(a+s\pm\tau),\tau(b+s\pm\tau)]}(x), dx\right| \\ &\leq \tau^{-1/p}(d-c)\to 0.\end{aligned}$$

The weak limits in (iii) are proved in the same way as those in (ii). Regarding those in (iv),

$$\begin{aligned}(Z_\tau^{-1} U_s V_{\pm\tau} Z_\tau^{-1}\chi_{[a,b]})(x) &= \tau^{2/p} e^{-2\pi\mathbf{i}\tau x s}\chi_{[a,b]}(\tau^2 x \mp \tau^2) \\ &= \tau^{2/p} e^{-2\pi\mathbf{i}\tau x s}\chi_{[\frac{a}{\tau^2}\pm 1, \frac{b}{\tau^2}\pm 1]}(x)\end{aligned}$$

and

$$\begin{aligned}\left|\langle \chi_{[c,d]}, Z_\tau^{-1} U_s V_{\pm\tau} Z_\tau^{-1}\chi_{[a,b]}\rangle\right| &= \left|\int_c^d \tau^{2/p} e^{-2\pi\mathbf{i}\tau x s}\chi_{[\frac{a}{\tau^2}\pm 1, \frac{b}{\tau^2}\pm 1]}(x)\,dx\right| \\ &\leq \tau^{2/p}\int_c^d \chi_{[\frac{a}{\tau^2}\pm 1, \frac{b}{\tau^2}\pm 1]}(x)\,dx \\ &\leq \tau^{2/p}\frac{b-a}{\tau^2} = \tau^{2/p-2}(b-a)\to 0\end{aligned}$$

since $\chi_{[\frac{a}{\tau^2}\pm 1, \frac{b}{\tau^2}\pm 1]}$ is different from zero in a interval of length $(b-a)/\tau^2$ and $p>1$. Finally,

$$\begin{aligned}(Z_\tau V_{\mp\tau} U_s Z_\tau \chi_{[a,b]})(x) &= \tau^{-2/p} e^{-2\pi\mathbf{i}(\frac{x}{\tau}\pm\tau)s}\chi_{[a,b]}(x/\tau^2 \pm 1) \\ &= \tau^{-2/p} e^{-2\pi\mathbf{i}(\frac{x}{\tau}\pm\tau)s}\chi_{[\tau^2 a\mp\tau^2, \tau^2 b\mp\tau^2]}(x)\end{aligned}$$

$$\begin{aligned}\left|\langle \chi_{[c,d]}, Z_\tau V_{\mp\tau} U_s Z_\tau \chi_{[a,b]}\rangle\right| &= \left|\int_c^d \tau^{-2/p} e^{-2\pi\mathbf{i}(\frac{x}{\tau}\pm\tau)s}\chi_{[\tau^2 a\mp\tau^2, \tau^2 b\mp\tau^2]}(x)\,dx\right| \\ &\leq \tau^{-2/p}\int_c^d \chi_{[\tau^2 a\mp\tau^2, \tau^2 b\mp\tau^2]}(x)\,dx \\ &\leq \tau^{-2/p}(d-c)\to 0.\end{aligned}$$

Assertion (v) is a direct consequence of Lemma 2.2 after writing

$$Z_\tau V_{-s} V_{\pm\tau} Z_\tau^{-1} = V_{\tau(-s\pm\tau)} \text{ and } Z_\tau V_{\mp\tau} V_s Z_\tau^{-1} = V_{\tau(s\mp\tau)}.$$

The weak limits in (vi) are proved in the same way as those in (v). Note that

$$Z_\tau^{-1} U_s U_t Z_\tau = U_{\tau(s+t)} \text{ and } Z_\tau V_s V_t Z_\tau^{-1} = V_{\tau(s+t)}. \qquad \square$$

3. Algebraization

The algebraization step is not changed from the standard technique. We let $\mathcal{E}$ be the set formed by all the sequences (A_τ) of operators $A_\tau \in \mathcal{B}$ such that

$$\sup_{\tau\in\mathbb{R}_+} \|A_\tau\|_{\mathcal{B}} < \infty.$$

The set $\mathcal{E}$ with the operations

$$\begin{aligned}(A_\tau) + (B_\tau) &:= (A_\tau + B_\tau),\\ (A_\tau)(B_\tau) &:= (A_\tau B_\tau),\\ \lambda(A_\tau) &:= (\lambda A_\tau) \quad (\lambda \in \mathbb{C}),\end{aligned}$$

the identity element (I), and the norm

$$\|(A_\tau)\|_{\mathcal{E}} := \sup_{\tau\in\mathbb{R}_+} \|A_\tau\|_{\mathcal{B}}$$

forms a unital Banach algebra. The set $\mathcal{G} \subset \mathcal{E}$ of sequences (G_τ) such that $\|G_\tau\| \to 0$ when $\tau \to \infty$ forms a closed two-sided ideal of $\mathcal{E}$.

The product of the algebraization step is the following result (see, for instance, [11, Theorem 6.2.2]:

Theorem 3.1. *The sequence (A_τ) defined in* (1.1) *is stable if and only if the coset $(A_\tau) + \mathcal{G}$ is invertible in $\mathcal{E}/\mathcal{G}$.*

4. Essentialization

We will say that a sequence of operators on a Banach space converges **-strongly* if it converges strongly and the sequence of the adjoint operators converges strongly on the dual space.

Let $\mathcal{F}$ denote the set of all sequences $\mathbf{A} = (A_\tau) \in \mathcal{E}$ for which there exist operators $\mathsf{W}_t(\mathbf{A})$, $t \in \{-1, 0, -1\}$, $\mathsf{H}_{(x,\infty)}(\mathbf{A})$, $\mathsf{H}_{(\infty,y)}(\mathbf{A})$, for $x, y \in \mathbb{R}$, and $\mathsf{H}_{(\infty,\infty)\pm}(\mathbf{A}) \in \mathcal{B}$ such that following limits exist in the *-strong sense for the sequences in the images of the algebra automorphisms (the limit operators that correspond to each homomorphism are indicated inside the square parenthesis):

$$
\begin{array}{lll}
\mathsf{I}_\bullet : & (A_\tau) \mapsto (A_\tau) & \big[\xrightarrow{*} \mathsf{W}_0(\mathbf{A}) \big]; \\
\mathsf{W}_{\pm 1,\bullet} : & (A_\tau) \mapsto (V_{\mp\tau} A_\tau V_{\pm\tau}) & \big[\xrightarrow{*} \mathsf{W}_{\pm 1}(\mathbf{A}) \big]; \\
\mathsf{H}_{(x,\infty),\bullet} : & (A_\tau) \mapsto (Z_\tau V_{-x} A_\tau V_x Z_\tau^{-1}) & \big[\xrightarrow{*} \mathsf{H}_{(x,\infty)}(\mathbf{A}) \big]; \\
\mathsf{H}_{(\infty,y),\bullet} : & (A_\tau) \mapsto (Z_\tau^{-1} U_y A_\tau U_{-y} Z_\tau) & \big[\xrightarrow{*} \mathsf{H}_{(\infty,y)}(\mathbf{A}) \big]; \\
\mathsf{H}_{(\infty,\infty)\pm,\bullet} : & (A_\tau) \mapsto (Z_\tau V_{\mp\tau} A_\tau V_{\pm\tau} Z_\tau^{-1}) & \big[\xrightarrow{*} \mathsf{H}_{\infty\pm}(\mathbf{A}) \big].
\end{array}
\tag{4.1}
$$

We represent the (operator) value of one of the above sequences at the point τ by substituting the symbol "$\bullet$" for τ.

The proof of the following result is the same as for Proposition 4.1 in [10]:

Proposition 4.1.

(i) *The set $\mathcal{F}$ is a unital closed subalgebra of $\mathcal{E}$. The strong limit mappings* (4.1) *act as bounded homomorphisms on $\mathcal{F}$, and the ideal $\mathcal{G}$ of $\mathcal{F}$ lies in the kernel of each these homomorphisms.*

(ii) *The algebra $\mathcal{F}$ is inverse-closed in $\mathcal{E}$, and the algebra $\mathcal{F}/\mathcal{G}$ is inverse-closed in $\mathcal{E}/\mathcal{G}$.*

The algebra $\mathcal{E}/\mathcal{G}$ is very large. The objective of the essentialization step is to change the invertibility problem in $\mathcal{E}/\mathcal{G}$ to a invertibility problem in the more amenable algebra $\mathcal{F}/\mathcal{J}$, with some homomorphisms controlling the difference. For that one uses a *lifting theorem*. We say that an homomorphism W_t lifts an ideal $\mathcal{J}_t$ if Ker $\mathsf{W}_t \cap \mathcal{J}_t$ is in the radical of $\mathcal{F}/\mathcal{G}$. The theorem reads as follows (see also [11, Section 6.3]):

Theorem 4.2 (Lifting). *For every element t of a certain set T, let $\mathcal{J}_t$ be an ideal of $\mathcal{F}$ which is lifted by a unital homomorphism W_t into $\mathcal{B}$. Suppose furthermore that $\mathsf{W}_t(\mathcal{J}_t)$ is an ideal of $\mathcal{B}$. Let $\mathcal{J}$ stand for the smallest ideal of $\mathcal{F}$ which contains all ideals $\mathcal{J}_t$. Then an element $a \in \mathcal{F}/\mathcal{G}$ is invertible if and only if the coset $a + \mathcal{J}$ is invertible in $\mathcal{F}/\mathcal{J}$ and if all elements $\mathsf{W}_t(a)$ are invertible in $\mathcal{B}$.*

To use this theorem, for each homomorphism W_t to be used it is necessary to find a suitable ideal $\mathcal{J}_t$. The known ideals are related the homomorphisms $\mathsf{W}_{-1,0,1}$ and give the ideal $\mathcal{J}$ used in [10]:

$$\mathcal{J} := \{(V_\tau K_1 V_{-\tau}) + (K_0) + (V_{-\tau} K_{-1} V_\tau) + \mathbf{G} : K_{-1}, K_0, K_1 \in \mathcal{K},\ \mathbf{G} \in \mathcal{G}\}.$$

In order to be able to use the homomorphisms H_* here, one needs to find the corresponding ideals. Analyzing the structure of the algebra automorphisms, the key observation is that it is possible to build such ideals using the inverse of the automorphisms applied to the constant sequence set $\mathcal{K} \subset \mathcal{F}$. Let $\eta \in \{(x,\infty), (\infty,y), (\infty,\infty)\pm :\ x, y \in \mathbb{R}\}$. Define

$$\mathcal{J}_\eta := \mathsf{H}_{\eta,\bullet}^{-1}(\mathcal{K}).$$

Specifying, we have

$$\mathcal{J}_{x,\infty} = \{(V_x Z_\tau^{-1} K Z_\tau V_{-x}) + \mathbf{G} : K \in \mathcal{K},\ \mathbf{G} \in \mathcal{G}\}; \tag{4.2}$$

$$\mathcal{J}_{\infty,y} = \{(U_{-y} Z_\tau K Z_\tau^{-1} U_y) + \mathbf{G} : K \in \mathcal{K},\ \mathbf{G} \in \mathcal{G}\}; \tag{4.3}$$

$$\mathcal{J}_{(\infty,\infty)\pm} = \{(V_{\pm\tau} Z_\tau^{-1} K Z_\tau V_{\mp\tau}) + \mathbf{G} : K \in \mathcal{K},\ \mathbf{G} \in \mathcal{G}\}. \tag{4.4}$$

Please note that the automorphisms and their inverses have norm 1. Thus $\mathsf{H}_{\eta,\bullet}^{-1}(\mathcal{G}) \subset \mathcal{G}$.

For $\xi \in \{(x,\infty), (\infty,y), (\infty,\infty)\pm :\ x,y \in \mathbb{R}\}$ we obtain the following result, usually known as the separation property of the strong limits.

Proposition 4.3. *For every compact operator K the sequence $\mathbf{J}_\eta = \mathsf{H}_{\eta,\bullet}^{-1}(K)$ belongs to $\mathcal{F}$ and*

$$\mathsf{W}_t(\mathbf{J}_\eta) = 0, \quad t \in \{-1,0,1\};$$

$$\mathsf{H}_\xi(\mathbf{J}_\eta) = \begin{cases} 0 & \text{if } \xi \neq \eta; \\ K & \text{if } \xi = \eta. \end{cases}$$

Proof. The case $\xi = \eta$ can be seen by direct computation. In the other cases, the resulting sequence is of the form $(Y_\tau K X_\tau)$, with Y_τ uniformly bounded, K a compact operator, and X_τ tending weakly to zero as seen in Lemma 2.3. The product $K X_\tau$ tends strongly to zero as a consequence of the uniform boundedness principle (see, for instance, [11, Lemma 1.4.6]) and the results follow. □

It is also necessary to show that the sets $\mathcal{J}_\eta \subset \mathcal{F}$ are ideals.

Proposition 4.4. *Let $\eta \in \{(x,\infty), (\infty,y), (\infty,\infty)\pm :\ x,y \in \mathbb{R}\}$. Then the sets $\mathcal{J}_\eta$ are closed two-sided ideals of $\mathcal{F}$.*

Proof. We will show that it is a left ideal. The proof for right ideal is similar, by taking adjoints. Let $\mathbf{A} = (A_\tau) \in \mathcal{F}$. We have

$$\begin{aligned}(A_\tau)\mathsf{H}_{\eta,\bullet}^{-1}(K) &= \mathsf{H}_{\eta,\bullet}^{-1}\left(\mathsf{H}_{\eta,\bullet}(A_\tau)K\right) \\ &= \mathsf{H}_{\eta,\bullet}^{-1}\left((\mathsf{H}_{\eta,\bullet}(A_\tau) - \mathsf{H}_\eta(\mathbf{A}) + \mathsf{H}_\eta(\mathbf{A}))K\right) \\ &= \mathsf{H}_{\eta,\bullet}^{-1}\left((\mathsf{H}_{\eta,\bullet}(A_\tau) - \mathsf{H}_\eta(\mathbf{A}))K\right) + \mathsf{H}_{\eta,\bullet}^{-1}\left(\mathsf{H}_\eta(\mathbf{A})K\right).\end{aligned}$$

The sequence $(\mathsf{H}_{\eta,\bullet}(A_\tau) - \mathsf{H}_\eta(\mathbf{A}))K$ is in $\mathcal{G}$, so the first summand is in $\mathcal{G}$. And because $\mathsf{H}_\eta(\mathbf{A})K$ is a compact operator, the coset corresponding to $\mathsf{H}_{\eta,\bullet}^{-1}\left(\mathsf{H}_\eta(\mathbf{A})K\right)$ belongs to $\mathcal{J}_\eta$. The proof of the closedness is standard (see, for instance, [11, Proposition 6.4.3]). □

Let now $\mathcal{J}$ be the smallest closed two-sided ideal of $\mathcal{F}$ containing the ideals $\mathcal{J}_t$, $t \in \{-1,0,1\}$ and the ideals $\mathcal{J}_\eta$. We are thus now in the conditions to apply the lifting theorem, and obtain:

Theorem 4.5. *Let $\mathbf{A} \in \mathcal{F}$. Then $\mathbf{A}$ is stable if and only if the operators $\mathsf{W}_t(\mathbf{A})$, $\mathsf{H}_\eta(\mathbf{A})$, with $t \in \{-1,0,1\}$, $\eta \in \{(x,\infty), (\infty,y), (\infty,\infty)\pm :\ x,y \in \mathbb{R}\}$ are invertible, and the coset $\mathbf{A} + \mathcal{J}$ is invertible in $\mathcal{F}/\mathcal{J}$.*

5. Localization

By including all homomorphisms in the previous step, there should be no new information about invertibility in the "rump" algebra $\mathcal{F}/\mathcal{J}$, at least in the case of sequences of finite section of convolution and multiplication operators. That is what we want to show now. For that we introduce an inverse-closed subalgebra of $\mathcal{F}$ which is a commutant modulo $\mathcal{J}$ with regards to constant sequences of convolution and multiplication operators with continuous generating function.

Let $\mathcal{L}$ be the set of all sequences in $\mathcal{F}$ that commute modulo $\mathcal{J}$ with the constant sequences (fI), $f \in C(\dot{\mathbb{R}})$, and $(W^0(g))$, $g \in C_p$. Then we have the following result, whose proof is also standard (see [5, Lemma 6.2]).

Proposition 5.1. *Let the set $\mathcal{L}$ be as above. Then:*

(i) *The set $\mathcal{L}$ is a closed and inverse-closed subalgebra of $\mathcal{F}$ that contains $\mathcal{J}$.*
(ii) *The algebra $\mathcal{L}/\mathcal{G}$ is inverse-closed in $\mathcal{F}/\mathcal{G}$.*

The cosets $(fI) + \mathcal{J}$, $f \in C(\dot{\mathbb{R}})$, and $(W^0(g)) + \mathcal{J}$, $g \in C_p$, form thus a central subalgebra of $\mathcal{L}/\mathcal{J}$. The maximal ideal space of this subalgebra is the subset $(\dot{\mathbb{R}} \times \{\infty\}) \cup (\{\infty\} \times \dot{\mathbb{R}})$ of the torus $\dot{\mathbb{R}} \times \dot{\mathbb{R}}$. Denote by $\mathcal{I}_{s,t}$ the smallest closed two-sided ideal of the quotient algebra $\mathcal{L}/\mathcal{J}$ that contains the maximal ideal corresponding to the point (s,t), and denote by $\Phi^{\mathcal{J}}_{s,t}$ the canonical homomorphism from $\mathcal{L}/\mathcal{J}$ onto the quotient algebra $\mathcal{L}^{\mathcal{J}}_{s,t} := (\mathcal{L}/\mathcal{J})/\mathcal{I}_{s,t}$. Then we use Allan's local principle (see, for instance, [11, Theorem 2.2.2]):

Theorem 5.2. *The element $\mathbf{A}+\mathcal{J}$ is invertible in $\mathcal{L}/\mathcal{J}$ if and only if for all $(s,t) \in (\dot{\mathbb{R}} \times \{\infty\}) \cup (\{\infty\} \times \dot{\mathbb{R}})$ the cosets $\Phi^{\mathcal{J}}_{s,t}(\mathbf{A} + \mathcal{J})$ are invertible in $\mathcal{L}^{\mathcal{J}}_{s,t}$.*

At the point (∞, ∞), it can be shown that the coset $\Phi^{\mathcal{J}}_{\infty,\infty}(\chi_+ + \mathcal{J})$ belongs to the center of the local algebra (see [10]), thus allowing a second localization into two other local algebras, corresponding to $(\infty,\infty)-$ and $(\infty,\infty)+$.

For $\eta \in \{(x,\infty), (\infty,y), (\infty,\infty)\pm\}$, the strong limit homomorphisms H_η are defined from $\mathcal{F}$ into $\mathcal{B}$, with $\mathsf{H}_\eta(\mathcal{J}) = \mathcal{K}$. The following theorem relates the invertibility (and Fredholm) property of the strong limits with invertibility in the local algebras. Let $\mathcal{D}_\eta \subset \mathcal{B}$ be the set such that if $A \in \mathcal{D}_\eta$, then $\mathsf{H}^{-1}_{\eta,\bullet}(A) \in \mathcal{L}$. Because $\mathcal{L}$ is inverse-closed in $\mathcal{E}$, it easy to see that the algebra $\mathcal{D}_\eta$ is inverse-closed in $\mathcal{B}$. In fact, if A is invertible, then $\mathsf{H}^{-1}_{\eta,\bullet}(A^{-1}) \in \mathcal{E}$, but $\mathsf{H}^{-1}_{\eta,\bullet}(A^{-1})$ is the inverse of $\mathsf{H}^{-1}_{\eta,\bullet}(A)$ and so must be in $\mathcal{L}$. But to obtain the result of Theorem 5.4 we need a stronger result, namely that also the regularizers of Fredholm operators in $\mathcal{D}_\eta$ can be lifted to a sequence in the sequence algebra $\mathcal{L}$.

Proposition 5.3. *Let $A \in \mathcal{D}_\eta$ be a Fredholm operator with regularizer $B \in \mathcal{B}$. Then $B \in \mathcal{D}_\eta$.*

Proof. If B is a regularizer for A, we have that there exist compact operators K and K' such that $BA - I = K$ and $AB - I = K'$.

Let $\mathbf{A}_\eta := \mathsf{H}^{-1}_{\eta,\bullet}(A)$, $\mathbf{B}_\eta := \mathsf{H}^{-1}_{\eta,\bullet}(B)$ and $\mathbf{K}_\eta := \mathsf{H}^{-1}_{\eta,\bullet}(K)$ for

$$\eta \in \{(x,\infty), (\infty,y), (\infty,\infty)\pm\}.$$

The above implies $\mathbf{B}_\eta \mathbf{A}_\eta - \mathbf{K}_\eta = \mathbf{I}$. Now let $\mathsf{O}_{\xi,\bullet}$, with

$$\xi \in \{-1, 0, 1, (x, \infty), (\infty, y), (\infty, \infty)\pm\}$$

represent one of the automorphisms defined in (4.1).

If $\eta = \xi$, then $\mathsf{O}_{\eta,\tau}(\mathbf{B}_\eta) = B$ is constant and there is nothing to prove. Consider $\eta \neq \xi$. Then $I = \mathsf{O}_{\xi,\tau}(\mathbf{B}_\eta)\mathsf{O}_{\xi,\tau}(\mathbf{A}_\eta) - \mathsf{O}_{\xi,\tau}(\mathbf{K}_\eta)$ for $\tau > 0$. Thus

$$\begin{aligned} \|u\|_p &= \left\|(\mathsf{O}_{\xi,\tau}(\mathbf{B}_\eta)\mathsf{O}_{\xi,\tau}(\mathbf{A}_\eta) - \mathsf{O}_{\xi,\tau}(\mathbf{K}_\eta))u\right\|_p \\ &\leq \left\|\mathsf{O}_{\xi,\tau}(\mathbf{B}_\eta)\mathsf{O}_{\xi,\tau}(\mathbf{A}_\eta)u\right\|_p + \left\|\mathsf{O}_{\xi,\tau}(\mathbf{K}_\eta)u\right\|_p \\ &\leq \|B\| \left\|\mathsf{O}_{\xi,\tau}(\mathbf{A}_\eta)u\right\|_p + \left\|\mathsf{O}_{\xi,\tau}(\mathbf{K}_\eta)u\right\|_p . \end{aligned}$$

Passing to the strong limits, we obtain that $\|u\|_p \leq \|B\|\|\mathsf{O}_\xi(\mathbf{A}_\eta)u\|$ hence $\mathsf{O}_\xi(\mathbf{A}_\eta)$ has trivial kernel and closed image. Applying the same arguments for the adjoint $\mathbf{A}_\eta^*$ one gets that $\mathsf{O}_\xi(\mathbf{A}_\eta^*) = \mathsf{O}_\xi(\mathbf{A}_\eta)^*$ has trivial kernel also. This implies that $\mathsf{O}_\xi(\mathbf{A}_\eta)$ is invertible. Now we show that $\mathsf{O}_{\xi,\tau}(\mathbf{B}_\eta) \xrightarrow{*} \mathsf{O}_\xi(\mathbf{A}_\eta)^{-1}$.

$$\begin{aligned} &\|(\mathsf{O}_{\xi,\tau}(\mathbf{B}_\eta) - \mathsf{O}_\xi(\mathbf{A}_\eta)^{-1})u\|_p \\ &\quad = \|\mathsf{O}_{\xi,\tau}(\mathbf{B}_\eta)u - (\mathsf{O}_{\xi,\tau}(\mathbf{B}_\eta)\mathsf{O}_{\xi,\tau}(\mathbf{A}_\eta) - \mathsf{O}_{\xi,\tau}(\mathbf{K}_\eta))\mathsf{O}_\xi(\mathbf{A}_\eta)^{-1})u\|_p \\ &\quad \leq \|\mathsf{O}_{\xi,\tau}(\mathbf{B}_\eta)(I - \mathsf{O}_{\xi,\tau}(\mathbf{A}_\eta)\mathsf{O}_{\xi,\tau}(\mathbf{A}_\eta)^{-1}))u\|_p + \|\mathsf{O}_{\xi,\tau}(\mathbf{K}_\eta))\mathsf{O}_\xi(\mathbf{A}_\eta)^{-1})u\|_p \\ &\quad \leq \|B\|\|(I - \mathsf{O}_{\xi,\tau}(\mathbf{A}_\eta)\mathsf{O}_{\xi,\tau}(\mathbf{A}_\eta)^{-1}))u\|_p + \|\mathsf{O}_{\xi,\tau}(\mathbf{K}_\eta))\mathsf{O}_\xi(\mathbf{A}_\eta)^{-1})u\|_p. \end{aligned}$$

Since both terms tend to 0 as $\tau \to \infty$ we get that $\mathsf{O}_{\xi,\tau}(\mathbf{B}_\eta) \to \mathsf{O}_\xi(\mathbf{A}_\eta)^{-1}$. The same argument when applied to B^* shows that $\mathsf{O}_{\xi,\tau}(\mathbf{B}_\eta)^* \to \mathsf{O}_\xi(\mathbf{A}_\eta^*)^{-1}$. This proves that $\mathsf{O}_\xi(\mathbf{B}_\eta) = \mathsf{O}_\xi(\mathbf{A}_\eta)^{-1}$, that is, the *-strong limits exist for $\eta \neq \xi$. We conclude that $\mathbf{B}_\eta \in \mathcal{F}$.

Now, let $\mathbf{C}$ be one of the sequences that commute with elements of $\mathcal{L}$ modulo $\mathcal{J}$. We have

$$\mathbf{C}\mathbf{A}_\eta - \mathbf{A}_\eta\mathbf{C} \in \mathcal{J} \Rightarrow \mathbf{B}_\eta\mathbf{C}\mathbf{A}_\eta\mathbf{B}_\eta - \mathbf{B}_\eta\mathbf{A}_\eta\mathbf{C}\mathbf{B}_\eta \in \mathcal{J} \Rightarrow \mathbf{B}_\eta\mathbf{C} - \mathbf{C}\mathbf{B}_\eta \in \mathcal{J},$$

which ends the proof. □

Theorem 5.4. *Let $\mathbf{A} \in \mathcal{L}$ such that $\mathsf{H}_\eta(\mathbf{A}) \in \mathcal{D}_\eta$ and $\Phi_\eta^{\mathcal{J}}\left(\mathsf{H}_{\eta,\bullet}^{-1}(\mathsf{H}_\eta(\mathbf{A}))\right) = \Phi_\eta^{\mathcal{J}}(\mathbf{A})$. Then*

(i) *if the coset $\Phi_\eta^{\mathcal{J}}(\mathbf{A})$ is invertible in $\mathcal{L}_\eta^{\mathcal{J}}$ then $\mathsf{H}_\eta(\mathbf{A})$ is Fredholm in $\mathcal{B}$;*
(ii) *If $\mathsf{H}_\eta(\mathbf{A})$ is Fredholm in $\mathcal{B}$, then the coset $\Phi_\eta^{\mathcal{J}}(\mathbf{A})$ is invertible in $\mathcal{L}_\eta^{\mathcal{J}}$.*

Proof. Part (i) is immediate because the homomorphism is unital. If $\mathsf{H}_\eta(\mathbf{A})$ is Fredholm, then there exist an operator $B \in \mathcal{B}$, K and K' compact operators such that $\mathsf{H}_\eta(\mathbf{A})B = I + K$ and $B\mathsf{H}_\eta(\mathbf{A}) = I + K'$. By the previous proposition, $B \in \mathcal{D}_\eta$. Considering, for instance, the first equation, it is then possible to apply the homomorphism $\mathsf{H}_{\eta,\bullet}^{-1}$ to both sides and obtain

$$\mathsf{H}_{\eta,\bullet}^{-1}(\mathsf{H}_\eta(\mathbf{A}))\mathsf{H}_{\eta,\bullet}^{-1}(B) = I + \mathbf{J} \qquad \text{in} \quad \mathcal{L},$$

with $\mathbf{J} \in \mathcal{J}$. Now, using the canonical homomorphism gives

$$\Phi_\eta^{\mathcal{J}}\left(\mathsf{H}_{\eta,\bullet}^{-1}\left(\mathsf{H}_\eta(\mathbf{A})\right)\right)\Phi_\eta^{\mathcal{J}}\left(\mathsf{H}_{\eta,\bullet}^{-1}(B)\right) = \Phi_\eta^{\mathcal{J}}(I)$$
$$\Phi_\eta^{\mathcal{J}}(\mathbf{A})\,\Phi_\eta^{\mathcal{J}}\left(\mathsf{H}_{\eta,\bullet}^{-1}(B)\right) = \Phi_\eta^{\mathcal{J}}(I)$$

which shows the right invertibility of $\Phi_\eta^{\mathcal{J}}(\mathbf{A})$. The same reasoning can be applied to obtain the left invertibility. □

The above result answers the second part of Markus Seidel's question. By factoring out the compact-like ideals at the essentialization step, the new local algebras can be thought as the Calkin counterpart to the usual local algebras. Thus, invertibility in the new local algebras of an element $\Phi_\eta^{\mathcal{J}}(\mathbf{A})$ is related to the Fredholm property of the corresponding homomorphism $\mathsf{H}_\eta(\mathbf{A})$, while the classical result for the old local algebras would relate to the invertibility of $\mathsf{H}_\eta(\mathbf{A})$.

It is possible now to give a final result on stability of sequences in $\mathcal{L}$.

Theorem 5.5. *Let $\mathbf{A} \in \mathcal{L}$ such that $\mathsf{H}_\eta(\mathbf{A}) \in \mathcal{D}_\eta$ and $\Phi_\eta^{\mathcal{J}}\left(\mathsf{H}_{\eta,\bullet}^{-1}\left(\mathsf{H}_\eta(\mathbf{A})\right)\right) = \Phi_\eta^{\mathcal{J}}(\mathbf{A})$. Then $\mathbf{A}$ is stable if and only if the operators $\mathsf{W}_t(\mathbf{A})$, $\mathsf{H}_\eta(\mathbf{A})$, with $t \in \{-1, 0, 1\}$, $\eta \in \{(x,\infty), (\infty, y), (\infty,\infty)\pm :\ x, y \in \mathbb{R}\}$ are invertible.*

As a particular case of the main result above, one can recover, for instance, the results in [10]–[12]:

Corollary 5.6. *Let $A : L^p(\mathbb{R}) \to L^p(\mathbb{R})$ be an operator belonging to the algebra generated by operators of multiplication by piecewise continuous functions and convolution operators with piecewise continuous multiplier generating functions. Then the sequence $\mathbf{A} = (A_\tau) := (P_\tau A P_\tau + Q_\tau)$ if and only if the following conditions are fulfilled:*

(i) *The operators A and $\chi_\mp \mathsf{W}_{\pm 1}(A)\chi_\mp I + \chi_\pm I$ are invertible;*
(ii) *The operators and $\chi_{[-1,1]}\mathsf{H}_{(\infty,x)}(A)\chi_{[-1,1]}$ are invertible in $L^p([-1,1])$ for $x \in \mathbb{R}$.*

Proof. One first has to check that the sequence (A_τ) belongs to $\mathcal{L}$. That the sequence is in $\mathcal{F}$ comes from [10, Propositions 4.5-4.7] and [12, Lemma 2]. The commutation modulo $\mathcal{J}$ with the sequences that define $\mathcal{L}$ is given in [10, Proposition 4.12]. On the other hand, the same propositions give that $\mathsf{H}_\eta(\mathbf{A})$, $\eta \in \{(x,\infty), (\infty, y), (\infty,\infty)\pm\}$ are elements of the algebra generated by convolution and multiplication operators, and so belong to $\mathcal{D}_\eta$. The last condition one needs to check in order to be able to apply Theorem 5.5 is that $\Phi_\eta^{\mathcal{J}}\left(\mathsf{H}_{\eta,\bullet}^{-1}\left(\mathsf{H}_\eta(\mathbf{A})\right)\right) = \Phi_\eta^{\mathcal{J}}(\mathbf{A})$. This was exactly the main arguments in the proofs of Theorems 4.13 and 4.16 in [10].

Now, the operators in (i) and (ii) result from the direct application of the homomorphims in Theorem 5.5 to the specific sequence $\mathbf{A}$. Note that because $\mathsf{H}_{(x,\infty)}((P_\tau)) = I$ for $x \in \mathbb{R}$, $\mathsf{H}_{(x,\infty)}(\mathbf{A}) = \mathsf{H}_{(x,\infty)}(A)$ and this operator is invertible if A is invertible. The homomorphisms $\mathsf{H}_{(\infty,\infty)\pm}(\mathbf{A})$ coincide in this case with $\mathsf{H}_{0,\infty} \circ \mathsf{W}_{\pm 1}$, as can be easily checked, and so the invertibility of $\mathsf{W}_{\pm 1}(\mathbf{A})$ imply automatically that of $\mathsf{H}_{(\infty,\infty)\pm}(\mathbf{A})$. □

6. Concluding remarks

With this work we have tried to gain a deeper understanding of the role and meaning of the strong limits that have been used since Silbermann's pioneering paper [13] and their relation to the "compact-like" sequence ideals to which they need to be connected. This attempt was primed by Markus Seidel question on the relation between the "lifting homomorphisms" and "localization homomorphisms" that until now appeared like mathematical objects of different kinds. We have succeeded in uniting them in a single concept, and, at the same time, understand the necessary properties they need to have for obtaining the usual stability results.

In this case, the strong limits are related to sequence algebra automorphisms, and the ideals result from the application of the inverses of these automorphisms to the set of (the constant sequences of) compact operators. In principle, a similar result can be obtained in the cases where the strong limits are given by matrix operators, as in [8] or [9], but there can arise some technical difficulties due to the fact that, in this case, the sequence algebra homomorphisms are not invertible in the large sequence algebra $\mathcal{E}$.

References

[1] R. Duduchava, *Integral equations with fixed singularities.* B.G. Teubner Verlagsgesellschaft, Leipzig, 1979.

[2] I. Gohberg and I.A. Feldman, *Convolution Equations and Projection Methods for their solution.* Amer. Math. Soc., Providence, RI., 1974. First published in Russian, Nauka, Moscow, 1971.

[3] R. Hagen, S. Roch, and B. Silbermann, *Spectral Theory of Approximation Methods for Convolution Equations.* Birkhäuser, Basel, 1995.

[4] R. Hagen, S. Roch, and B. Silbermann, *C^*-algebras and Numerical Analysis.* Marcel Dekker, Inc., New York, 2001.

[5] A. Karlovich, H. Mascarenhas, and P.A. Santos, *Finite section method for a Banach algebra of convolution type operators on $Lp(R)$ with symbols generated by PC and SO.* Integral Equations Operator Theory **67** (2010), no. 4, 559–600.

[6] A.V. Kozak, *A local principle in the theory of projection methods.* Dokl. Akad. Nauk SSSR **212** (1973), 1287–1289. (In Russian; English translation in Soviet Math. Dokl. **14** (1974), 1580–1583.)

[7] S. Prössdorf and B. Silbermann, *Numerical Analysis for Integral and Related Operator Equations.* Birkhäuser Verlag, Basel, 1991.

[8] S. Roch and P.A. Santos, *Finite section method in an algebra of convolution, multiplication and flip operators on L^p.* To appear, 2013.

[9] S. Roch, P.A. Santos, and B. Silbermann, *Finite section method in some algebras of multiplication and convolution operators and a flip.* Z. Anal. Anwendungen **16** (1997), no. 3, 575–606.

[10] S. Roch, P.A. Santos, and B. Silbermann, *A sequence algebra of finite sections, convolution and multiplication operators on $L^p(\mathbb{R})$.* Numer. Funct. Anal. Optim. **31** (2010), no. 1, 45–77.

[11] S. Roch, P.A. Santos, and B. Silbermann, *Non-commutative Gelfand Theories.* Springer-Verlag, 2011.

[12] S. Roch, P.A. Santos, and B. Silbermann, *Erratum to "A sequence algebra of finite sections, convolution and multiplication operators on $L^p(\mathbb{R})$"*. Numer. Funct. Anal. Optim. **34** (2013), no. 1, 113–116.

[13] B. Silbermann, *Lokale Theorie des Reduktionsverfahrens für Toeplitzoperatoren.* Math. Nachr. **104** (1981), 137–146.

[14] E.C. Titchmarsh, *Introduction to the Theory of Fourier Integrals.* Oxford University Press, Oxford, 1967.

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Invertibility in Groupoid C^*-algebras

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Abstract. Given a second-countable, Hausdorff, étale, amenable groupoid $\mathcal{G}$ with compact unit space, we show that an element a in $C^*(\mathcal{G})$ is invertible if and only if $\lambda_x(a)$ is invertible for every x in the unit space of $\mathcal{G}$, where λ_x refers to the *regular representation* of $C^*(\mathcal{G})$ on $\ell_2(\mathcal{G}_x)$. We also prove that, for every a in $C^*(\mathcal{G})$, there exists some $x \in \mathcal{G}^{(0)}$ such that $\|a\| = \|\lambda_x(a)\|$.

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1. Introduction

The structure of certain C^*-algebras is often best studied via large families of *-representations. According to this point of view, one tries to deduce the properties of any given element of the algebra by means of the properties of its images under the representations provided. Here we shall mostly be interested in *invertibility* questions, and thus on families of representations of a given C^*-algebra which are large enough to determine when an element is invertible.

One of the first, and arguably also the most influential such result is the Allan–Douglas local principle [1, Corollary 2.10], [4, Theorem 7.47], which asserts that an element in a unital Banach algebra is invertible if and only if it is invertible modulo certain ideals associated to the points of the spectrum of a given central subalgebra. This principle has been generalized to *nonlocal* algebras (see [7] and the references given there) and has successfully been applied to study Fredholm singular integral operators with semi-almost periodic coefficients [3].

The present paper is an attempt to transpose the *local-trajectory method* of [7] to the context of groupoid C^*-algebras. Since invertibility only makes sense on unital algebras, and since the C^*-algebra of a groupoid is unital only when the groupoid is étale and has a compact unit space, we restrict ourselves to this

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case (however our work suggests questions that might be relevant for more general groupoids). To be precise, our main result, Theorem 2.10 applies to second-countable, Hausdorff, étale, amenable groupoids with compact unit space. Given such a groupoid $\mathcal{G}$, we show that an element a in the groupoid C^*-algebra $C^*(\mathcal{G})$ is invertible if and only, for every x in the unit space of $\mathcal{G}$, one has that $\lambda_x(a)$ is invertible, where λ_x is the *regular representation* of $C^*(\mathcal{G})$ on $\ell_2(\mathcal{G}_x)$.

A crucial tool used to prove our main result is the theory of induced representations started by Renault in [9, Chap. II, §2] and improved by Ionescu and Williams in [5] and [6].

Since $\mathcal{G}$ is amenable, we have by [2, Theorem 6.1.4(iii)] that

$$\|a\| = \sup_{x\in\mathcal{G}^{(0)}} \|\lambda_x(a)\|, \quad \forall a \in C^*(\mathcal{G}). \tag{1.1}$$

As a byproduct of our work we have found a small improvement of this result, namely Corollary 3.3, below, which asserts that

$$\|a\| = \max_{x\in\mathcal{G}^{(0)}} \|\lambda_x(a)\|, \quad \forall a \in C^*(\mathcal{G}), \tag{1.2}$$

which is to say that the supremum in (1.1) is in fact *attained* for every a. The proof of this fact is a straightforward combination of Theorem 2.10 with a result of S. Roch [10], which we carefully describe below.

Even though the invertibility question treated in Theorem 2.10 only makes sense for groupoids with compact unit space, (1.2) applies to a wider context. A sensible question to be asked at this point is therefore whether or not (1.2) holds in the absence of the compactness hypothesis.

Dropping the assumption that $\mathcal{G}$ is amenable, it is well known that (1.1) holds as long as we replace the full by the reduced groupoid C^*-algebra. So it makes sense to ask whether or not

$$\|a\| = \max_{x\in\mathcal{G}^{(0)}} \|\lambda_x(a)\|, \quad \forall a \in C^*_r(\mathcal{G}) \ ? \tag{1.3}$$

Unfortunately we have not been able to answer any of these questions, which we are then forced to leave as open problems.

Attaining the supremum is a well-known property of continuous functions on compact spaces, so a proof of (1.3) could be obtained, at least in the case of a compact unit space, should we be able to prove that the function

$$x \mapsto \|\lambda_x(a)\|$$

is continuous for every $a \in C^*_r(\mathcal{G})$. However sensible this appears to be, we have not been able to determine its validity.

Last, but not least, I would like to thank Amélia Bastos and the members of "The Center for Functional Analysis and Applications – CEAF" of the "Instituto Superior Técnico de Lisboa" for bringing their work to my attention and also for their warm hospitality during two visits there where many interesting conversations on these topics took place and where the ideas for the present work developed. I would also like to thank Jean Renault for helpful e-mail exchanges.

2. Sufficient family of representations

Let A be a unital C^*-algebra. The following concept appears in [10, Section 5].

Definition 2.1. A family $\mathscr{F}$ of non-degenerated representations (always assumed to preserve the involution) of A is called *sufficient* if, for every a in A, one has that

$$a \text{ is invertible} \iff \pi(a) \text{ is invertible for all } \pi \in \mathscr{F}.$$

Observe that the implication "$\Rightarrow$" is always true, so the relevant property conveyed by this definition is the implication "$\Leftarrow$".

Proposition 2.2. *The set of all irreducible representations of A is a sufficient family of representations.*

Proof. If a is a non-invertible element of A, then either a^*a or aa^* are non-invertible. So we may assume, without loss of generality that a^*a is non-invertible. Let B be the closed *-subalgebra of A generated by a^*a and 1, and let X be the compact spectrum of B. Since a^*a is non-invertible, there exists some point x_0 in X such that $\widehat{a^*a}(x_0) = 0$, where the hat indicates the Gelfand transform.

The map

$$\phi : b \in B \mapsto \hat{b}(x_0) \in \mathbb{C}$$

is therefore a pure state of B, which may be extended to a pure state ψ on A. Let π be the GNS representation associated to ψ, so that π is an irreducible representation. If ξ is the associated cyclic vector we have

$$\|\pi(a)\xi\|^2 = \langle \pi(a)\xi, \pi(a)\xi\rangle = \langle \pi(a^*a)\xi, \xi\rangle = \psi(a^*a) = \phi(a^*a) = \widehat{a^*a}(x_0) = 0.$$

It follows that the operator $\pi(a)$ is not injective and hence non-invertible. $\square$

$\triangleright$ From now on we will be interested in the question of sufficiency for groupoid C^*-algebras. We therefore fix a second-countable, Hausdorff, étale groupoid $\mathcal{G}$, with source and range maps denoted by "s" and "r", respectively.

Given x in the unit space $\mathcal{G}^{(0)}$ of $\mathcal{G}$, we shall use the following standard notations:

$$\mathcal{G}_x = \{\gamma \in \mathcal{G} : s(\gamma) = x\},$$

$$\mathcal{G}^x = \{\gamma \in \mathcal{G} : r(\gamma) = x\}, \quad \text{and}$$

$$\mathcal{G}(x) = \mathcal{G}_x \cap \mathcal{G}^x.$$

Consider the Hilbert space $H_x = \ell_2(\mathcal{G}_x)$ and the *regular representation* λ_x of $C_c(\mathcal{G})$ on H_x, given by

$$\lambda_x(f)\xi|_\gamma = \sum_{\gamma'\gamma''=\gamma} f(\gamma')\xi(\gamma''), \quad \forall f \in C_c(\mathcal{G}), \quad \forall \xi \in H_x, \quad \forall \gamma \in \mathcal{G}_x,$$

which is well known to extend to $C^*(\mathcal{G})$. For each γ in $\mathcal{G}_x$, let e_γ be the basis vector of H_x corresponding to γ.

Proposition 2.3. *For every* γ_1 *and* γ_2 *in* $\mathcal{G}_x$*, and all* f *in* $C_c(\mathcal{G})$*, one has that*

$$\langle \lambda_x(f)e_{\gamma_1}, e_{\gamma_2}\rangle = f(\gamma_2\gamma_1^{-1}).$$

Proof. We have

$$\langle \lambda_x(f)e_{\gamma_1}, e_{\gamma_2}\rangle = \lambda_x(f)e_{\gamma_1}\big|_{\gamma_2} = \sum_{\gamma'\gamma''=\gamma_2} f(\gamma')e_{\gamma_1}(\gamma'')$$
$$= \sum_{\gamma'\gamma_1=\gamma_2} f(\gamma') = f(\gamma_2\gamma_1^{-1}). \qquad \square$$

Proposition 2.4. *Let* $\mathcal{H}$ *be a closed sub-groupoid of* $\mathcal{G}$*, viewed as a topological groupoid with the relative topology. Then the following are equivalent:*

(i) *the restriction of the range map* r *to* $\mathcal{H}$*, viewed as a mapping*

$$r|_{\mathcal{H}} : \mathcal{H} \to \mathcal{H}^{(0)},$$

is an open mapping,

(ii) $\mathcal{H}$ *is étale.*

Proof. Assuming (i), let $\gamma \in \mathcal{H}$ and choose an open set $U \subseteq \mathcal{G}$ such that r is a homeomorphism from U onto the open set $r(U) \subseteq \mathcal{G}^{(0)}$. Then $U \cap \mathcal{H}$ is open in the relative topology of $\mathcal{H}$ and, by (i), we have that $r(U\cap\mathcal{H})$ is open in $\mathcal{H}^{(0)}$. It is then clear that r is a homeomorphism from $U\cap\mathcal{H}$ to $r(U\cap\mathcal{H})$, showing that $r|_{\mathcal{H}}$ is a local homeomorphism and hence that $\mathcal{H}$ is étale. The converse is evident. $\square$

$\triangleright$ From now on we fix a closed sub-groupoid $\mathcal{H} \subseteq \mathcal{G}$, satisfying the equivalent conditions above. We will denote the unit spaces of $\mathcal{G}$ and $\mathcal{H}$ as follows

$$X := \mathcal{G}^{(0)}, \quad \text{and} \quad Y := \mathcal{H}^{(0)}.$$

Since H is closed in $\mathcal{G}$ and since $Y = \mathcal{H}\cap X$, we see that Y is a closed subspace of X.

Let us briefly describe the process of inducing representations from $C^*(\mathcal{H})$ to $C^*(\mathcal{G})$, cf. [9, Chap. II, §2] and [6, Section 2]. Given a representation L of $C^*(\mathcal{H})$ on a Hilbert space H_L, we want to produce a representation $\mathrm{Ind}_H^G L$ of $C^*(\mathcal{G})$ on a Hilbert space $H_{\mathrm{Ind}\,L}$. In order to do so, consider the closed subset of $\mathcal{G}$ given by

$$\mathcal{G}_Y = s^{-1}(Y) = \{\gamma \in \mathcal{G} : s(\gamma) \in Y\}.$$

For φ and ψ in $C_c(\mathcal{G}_Y)$, define $\langle \varphi, \psi\rangle_*$ in $C_c(\mathcal{H})$, by

$$\langle \varphi, \psi\rangle_*(\zeta) = \sum_{\gamma_1\gamma_2=\zeta} \overline{\varphi(\gamma_1^{-1})}\psi(\gamma_2), \quad \forall \zeta \in \mathcal{H}.$$

It should be noticed that the above sum ranges over all pairs of elements γ_1 and γ_2 in $\mathcal{G}$ (as opposed to $\mathcal{H}$), whose product equals ζ. In this case notice that both $r(\gamma_1)$ and $s(\gamma_2)$ lie in Y, so that γ_1^{-1} and γ_2 indeed belong to the domain of φ and ψ, respectively.

By [8, Theorem 2.8], one has that in fact $C_c(\mathcal{G}_Y)$ may be completed to a right $C^*(\mathcal{H})$-Hilbert module, which we will denote by M, the appropriate right multiplication being that which is described in [8, page 11]. It is therefore profitable to view $\langle\cdot\,,\cdot\rangle_*$ as a $C^*(\mathcal{H})$-valued map.

The space $H_{\operatorname{Ind} L}$, on which the induced representation will act, is then defined to be the completion of

$$C_c(\mathcal{G}_Y)\otimes H_L,$$

relative to the inner-product

$$\langle\varphi\otimes\xi,\psi\otimes\eta\rangle := \big\langle L\big(\langle\psi,\varphi\rangle_*\big)\xi,\eta\big\rangle,\quad \forall\varphi,\psi\in C_c(\mathcal{G}_Y),\quad \forall\xi,\eta\in H_L.$$

One next gives $C_c(\mathcal{G}_Y)$ the structure of a left $C_c(\mathcal{G})$–module by setting

$$(f*\varphi)(\gamma) := \sum_{\gamma_1\gamma_2=\gamma} f(\gamma_1)\varphi(\gamma_2),\quad \forall f\in C_c(\mathcal{G}),\quad \forall\varphi\in C_c(\mathcal{G}_Y),\quad \forall\gamma\in\mathcal{G}_Y.$$

Again by [8, Theorem 2.8], the above left-module structure may be extended to a bounded multiplication operation

$$(a,x)\in C^*(\mathcal{G})\times M\mapsto ax\in M.$$

In order to define the induced representation one may either work with the completion M described above or take the more pedestrian point of view of sticking to compactly supported functions. Taking the latter approach, for $f\in C_c(\mathcal{G})$ one initially defines $\operatorname{Ind}_{\mathcal{H}}^{\mathcal{G}} L(f)$ on the dense subspace $C_c(\mathcal{G}_Y)\otimes H_L\subseteq H_{\operatorname{Ind} L}$, by the formula

$$\operatorname{Ind}_{\mathcal{H}}^{\mathcal{G}} L(f)(\varphi\otimes\xi) := (f*\varphi)\otimes\xi,\quad \forall\varphi\in C_c(\mathcal{G}_Y),\quad \forall\xi\in H_L,$$

and then extend it by continuity to $H_{\operatorname{Ind} L}$. This provides a *-representation of $C_c(\mathcal{G})$ on $H_{\operatorname{Ind} L}$ which, in turn, may be extended to the whole of $C^*(\mathcal{G})$.

The resulting representation of $C^*(\mathcal{G})$ on $H_{\operatorname{Ind} L}$ is denoted by $\operatorname{Ind}_{\mathcal{H}}^{\mathcal{G}} L$, and is called the *representation induced by L from $\mathcal{H}$ up to $\mathcal{G}$*. For more details, see [9, Chap. II, §2] and [6, Section 2].

▷ Fix, for the time being, an element $x\in X$.

We would now like to consider the question of inducing representations from $\mathcal{H} := \mathcal{G}(x)$ up to $\mathcal{G}$. Observing that

$$Y = \mathcal{G}(x)^{(0)} = \{x\},$$

we have that $\mathcal{G}_Y = \mathcal{G}_x$, which is a discrete topological space. Consequently $C_c(\mathcal{G}_Y)$ is linearly generated by the set

$$\{e_\gamma : \gamma\in\mathcal{G}_x\},$$

where e_γ denotes the characteristic function of the singleton $\{\gamma\}$.

Proposition 2.5. *Given $\gamma,\gamma'\in\mathcal{G}_x$, we have that*

$$\langle e_\gamma,e_{\gamma'}\rangle_* = \begin{cases}\delta_{\gamma^{-1}\gamma'}, & \text{if } r(\gamma)=r(\gamma'),\\ 0, & \text{otherwise,}\end{cases}$$

where, for each $h\in\mathcal{G}(x)$, we denote by δ_h the characteristic function of the singleton $\{h\}$, viewed as an element of $C_c\big(\mathcal{G}(x)\big)\subseteq C^*\big(\mathcal{G}(x)\big)$.

Proof. We have, for every $\zeta \in \mathcal{G}(x)$, that

$$\langle e_\gamma, e_{\gamma'}\rangle_*(\zeta) = \sum_{\gamma_1\gamma_2=\zeta} \overline{e_\gamma(\gamma_1^{-1})} e_{\gamma'}(\gamma_2) = \big[\gamma^{-1}\gamma'=\zeta\big]$$

where the brackets denote the Boolean value of the expression inside, with the convention that a syntactically incorrect expression, e.g., when the multiplication $\gamma^{-1}\gamma'$ is illegal, the value is zero.

Thus, when $r(\gamma) = r(\gamma')$, we have that the product $\gamma^{-1}\gamma'$ is defined, evidently giving an element of $\mathcal{G}(x)$ and, in this case,

$$\langle e_\gamma, e_{\gamma'}\rangle_* = \delta_{\gamma^{-1}\gamma'},$$

On the other hand, when $r(\gamma) \neq r(\gamma')$, we clearly have that $\langle e_\gamma, e_{\gamma'}\rangle_* = 0$. ☐

The following elementary result is included in order to illustrate a simple example.

Proposition 2.6. *Let Λ be the left-regular representation of $C^*\big(\mathcal{G}(x)\big)$ on $\ell_2(\mathcal{G}(x))$. Then $\operatorname{Ind}_{\mathcal{G}(x)}^{\mathcal{G}} \Lambda$ is unitarily equivalent to λ_x.*

Proof. For each element $\gamma \in \mathcal{G}_x$, and each $g \in \mathcal{G}(x)$, consider the element

$$\varphi_{\gamma,g} = e_\gamma \otimes e_g \in C_c(\mathcal{G}_x) \otimes \ell_2(\mathcal{G}(x)) \subseteq H_{\operatorname{Ind}\Lambda}.$$

We first claim that

$$\langle \varphi_{\gamma,g}, \varphi_{\gamma',g'}\rangle = \big[\gamma g=\gamma' g'\big], \quad \forall \gamma, \gamma' \in \mathcal{G}_x, \quad \forall g, g' \in \mathcal{G}(x). \tag{2.1}$$

In fact, we have

$$\langle \varphi_{\gamma,g}, \varphi_{\gamma',g'}\rangle = \big\langle e_\gamma \otimes er(\gamma), e_{\gamma'} \otimes e_{r(\gamma')}\big\rangle = \big\langle \Lambda\big(\langle e_{\gamma'}, e_\gamma\rangle_*\big) e_{r(\gamma)}, e_{r(\gamma')}\big\rangle = (\dagger)$$

Consequently, if $r(\gamma) \neq r(\gamma')$ we have by Proposition 2.5 that $\langle \varphi_{\gamma,g}, \varphi_{\gamma',g'}\rangle = 0$, which proves (2.1) in this case. If $r(\gamma) = r(\gamma')$ then, again by Proposition 2.5, it follows that

$$(\dagger) = \big\langle \Lambda\big(\delta_{\gamma'^{-1}\gamma}\big) e_g, e_{g'}\big\rangle = \big\langle e_{\gamma'^{-1}\gamma g}, e_{g'}\big\rangle = \big[\gamma'^{-1}\gamma g=g'\big] = \big[\gamma g=\gamma' g'\big],$$

proving (2.1). In particular, this implies that

$$\langle \varphi_{\gamma,g}, \varphi_{\gamma',g'}\rangle = \langle \varphi_{\gamma g,x}, \varphi_{\gamma',g'}\rangle,$$

and since the collection of all $\varphi_{\gamma',g'}$ evidently spans $H_{\operatorname{Ind}\Lambda}$, we have that $\varphi_{\gamma,g} = \varphi_{\gamma g,x}$, and it is then clear that the mapping

$$e_\gamma \mapsto \varphi_{\gamma,x}$$

extends to a unitary operator $U : H_x \to H_{\operatorname{Ind}\Lambda}$. Given $f \in C_c(\mathcal{G})$, we claim that

$$\big\langle U^*(\operatorname{Ind}_H^G \Lambda(f)) U e_\gamma,\, e_{\gamma'}\big\rangle = \langle \lambda_x(f) e_\gamma, e_{\gamma'}\rangle, \quad \forall \gamma, \gamma' \in \mathcal{G}_x. \tag{2.2}$$

In order to verify it observe that the left-hand side equals

$$\begin{aligned}\big\langle \operatorname{Ind}_H^G \Lambda(f)(\varphi_{\gamma,x}),\, \varphi_{\gamma',x}\big\rangle &= \big\langle (f * e_\gamma) \otimes e_x,\, e_{\gamma'} \otimes e_{x'}\big\rangle \\ &= \big\langle \Lambda\big(\langle e_{\gamma'}, f * e_\gamma\rangle_*\big) e_x,\, e_x\big\rangle = (\diamondsuit)\end{aligned}$$

After checking that

$$f * e_\gamma = \sum_{\eta \in \mathcal{G}_x} f(\eta\gamma^{-1}) e_\eta,$$

we conclude that

$$\begin{aligned}(\diamondsuit) &= \sum_{\eta \in \mathcal{G}_x} f(\eta\gamma^{-1}) \big\langle \Lambda\big(\langle e_{\gamma'}, e_\eta\rangle_*\big) e_x,\, e_x\big\rangle \\ &= \sum_{\substack{\eta \in \mathcal{G}_x \\ r(\gamma') = r(\eta)}} f(\eta\gamma^{-1}) \big\langle \Lambda\big(\gamma_{\gamma'^{-1}\eta}\big) e_x,\, e_x\big\rangle \\ &= \sum_{\substack{\eta \in \mathcal{G}_x \\ r(\gamma') = r(\eta)}} f(\eta\gamma^{-1}) \big\langle e_{\gamma'^{-1}\eta},\, e_x\big\rangle = f(\gamma'\gamma^{-1}) \overset{\text{P.2.3}}{=} \langle \lambda_x(f) e_\gamma, e_{\gamma'}\rangle.\end{aligned}$$

This proves (2.2), and taking into account that γ and γ' are arbitrary, we conclude that $U^*(\mathrm{Ind}_H^G \Lambda(f))U = \lambda_x(f)$, finishing the proof. □

Notice that there are two completions of $C_c(\mathcal{G}_x)$ which are relevant to us. On the one hand M is the completion under the $C^*\big(\mathcal{G}(x)\big)$-valued inner-product $\langle \cdot\,, \cdot\rangle_*$, and, on the other, H_x is the completion for the 2-norm. These two spaces are related to each other by the following.

Proposition 2.7. *There is a bounded linear map*

$$j : M \to H_x,$$

such that $j(\varphi) = \varphi$, for every $\varphi \in C_c(\mathcal{G}_x)$.

Proof. Given $\varphi \in C_c(\mathcal{G}_x)$, notice that

$$\|\varphi\|_2^2 = \sum_{\gamma \in \mathcal{G}_x} \overline{\varphi(\gamma)}\varphi(\gamma) = \langle \varphi, \varphi\rangle_*(1) \le \|\langle \varphi, \varphi\rangle_*\|_{C^*(\mathcal{G}(x))} = \|\varphi\|_M^2.$$

This implies that the identity map on $C_c(\mathcal{G}_x)$ is continuous for $\|\cdot\|_M$ on its domain and the 2-norm on its codomain. The required map is then obtained by a continuous extension. □

If $\zeta \in \mathcal{G}(x)$, we have a well-defined bijective map

$$\gamma \in \mathcal{G}_x \mapsto \gamma\zeta \in \mathcal{G}_x,$$

and hence the map

$$R_\zeta : H_x \to H_x,$$

defined by

$$R_\zeta(\xi)_\gamma = \xi(\gamma\zeta), \quad \forall \xi \in H_x, \quad \forall \gamma \in \mathcal{G}_x,$$

is a unitary operator. It is also easy to see that $R_{\zeta_1} \circ R_{\zeta_2} = R_{\zeta_1\zeta_2}$, which is to say that R is a unitary representation of $\mathcal{G}(x)$ on H_x.

This representation will play an important role in our next result, but before stating it, we need to introduce a notation.

Given any discrete group G, and any $\zeta \in G$, the map

$$f \in C_c(G) \mapsto f(\zeta) \in \mathbb{C}$$

is well known to extend to a bounded linear functional on $C^*(G)$, which we will denote by

$$a \in C^*(G) \mapsto \hat{a}(\zeta) \in \mathbb{C}.$$

Proposition 2.8. *For every $a \in C^*(\mathcal{G})$, every $x, y \in M$, and every $\zeta \in \mathcal{G}(x)$, we have that*

$$\widehat{\langle x, ay\rangle_*}(\zeta) = \Big\langle \lambda_x(a) R_\zeta\big(j(y)\big), j(x)\Big\rangle.$$

Proof. Given $f \in C_c(\mathcal{G})$, and $\psi, \varphi \in C_c(\mathcal{G}_x)$, we have

$$\langle \varphi, f * \psi\rangle_*(\zeta) = \sum_{\gamma_1\gamma_2=\zeta} \overline{\varphi(\gamma_1^{-1})}(f*\psi)(\gamma_2) = \sum_{\gamma_1\gamma_2\gamma_3=\zeta} \overline{\varphi(\gamma_1^{-1})} f(\gamma_2)\psi(\gamma_3) = \cdots$$

With the change of variables "$\gamma_3' = \gamma_3\zeta^{-1}$" the above equals

$$\begin{aligned}\cdots &= \sum_{\gamma_1\gamma_2\gamma_3'=x} \overline{\varphi(\gamma_1^{-1})} f(\gamma_2)\psi(\gamma_3'\zeta)\\ &= \sum_{\gamma_1\gamma_2\gamma_3'=x} \overline{\varphi(\gamma_1^{-1})} f(\gamma_2)R_\zeta(\psi)(\gamma_3') = \big\langle f * R_\zeta(\psi), \varphi\big\rangle.\end{aligned}$$

This gives that

$$\langle \varphi, f*\psi\rangle_*(\zeta) = \big\langle f * R_\zeta(\psi), \varphi\big\rangle,$$

and the proof is concluded upon replacing

- f by the terms of a sequence $\{f_n\}_n$ converging to a in $C^*\big(\mathcal{G}(x)\big)$,
- φ by the terms of a sequence $\{\varphi_n\}_n$ converging to x in M, and finally
- ψ by the terms of a sequence $\{\psi_n\}_n$ converging to y in M. □

Corollary 2.9. *Given $x \in X$, suppose that a is an element of $C^*(\mathcal{G})$ such that $\lambda_x(a) = 0$. Then*

$$\operatorname{Ind}_{\mathcal{G}(x)}^{\mathcal{G}} L(a) = 0,$$

for any representation L of $C^*\big(\mathcal{G}(x)\big)$ which is weakly contained in Λ.

Proof. By Proposition 2.8, we deduce that

$$\widehat{\langle x, ay\rangle_*}(\zeta) = 0, \quad \forall \zeta \in \mathcal{G}(x), \quad \forall x, y \in M.$$

Temporarily fixing x and y, we then deduce that $\Lambda\big(\langle x, ay\rangle_*\big) = 0$, and hence that

$$L\big(\langle x, ay\rangle_*\big) = 0, \tag{2.3}$$

for any L as in the statement. Given $f \in C_c(\mathcal{G})$, $\varphi, \psi \in C_c(\mathcal{G}_x)$ and $\xi, \eta \in H_L$, we have that

$$\big\langle \operatorname{Ind}_{\mathcal{G}(x)}^{\mathcal{G}} L(f)(\varphi\otimes\xi), \psi\otimes\eta\big\rangle = \big\langle (f*\varphi)\otimes\xi, \psi\otimes\eta\big\rangle = \big\langle L\big(\langle \psi, f*\varphi\rangle_*\big)\xi, \eta\big\rangle.$$

Applying this for f ranging in a sequence $\{f_n\}_n$ converging to a in $C^*\big(\mathcal{G}(x)\big)$, we conclude that

$$\big\langle \mathrm{Ind}_{\mathcal{G}(x)}^{\mathcal{G}} L(a)(\varphi\otimes\xi), \psi\otimes\eta\big\rangle = \big\langle L\big(\langle\psi, a\varphi\rangle_*\big)\xi, \eta\big\rangle \overset{(2.3)}{=} 0,$$

from where the conclusion follows easily. □

We may now prove our main result:

Theorem 2.10. *Let $\mathcal{G}$ be a second-countable, Hausdorff, étale groupoid, such that $\mathcal{G}^{(0)}$ is compact. Suppose moreover that $\mathcal{G}$ is amenable. Then $\{\lambda_x\}_{x\in\mathcal{G}^{(0)}}$ is a sufficient family of representations for $C^*(\mathcal{G})$. In other words, if $a \in C^*(\mathcal{G})$ is such that $\lambda_x(a)$ is invertible for every x in the unit space of $\mathcal{G}$, then a is necessarily invertible.*

Proof. Suppose, by way of contradiction, that a is non-invertible. By Proposition 2.2 there exists an irreducible representation π of $C^*(\mathcal{G})$ such that $\pi(a)$ is non-invertible. Employing [5, Theorem 2.1] we have that, for some $x \in \mathcal{G}^{(0)}$, there exists an irreducible representation L of $C^*\big(\mathcal{G}(x)\big)$ such that π and $\mathrm{Ind}_{\mathcal{G}(x)}^{\mathcal{G}} L$ share null spaces.

Since $\mathcal{G}$ is amenable we have that $\mathcal{G}(x)$ is also amenable by [2, Proposition 5.1.1], and hence that L is weakly contained in the left-regular representation. We may therefore employ Corollary 2.9 to conclude that

$$\mathrm{Ker}\,(\lambda_x) \subseteq \mathrm{Ker}\,\big(\mathrm{Ind}_{\mathcal{G}(x)}^{\mathcal{G}} L\big) = \mathrm{Ker}\,(\pi).$$

By hypothesis a is invertible modulo $\mathrm{Ker}\,(\lambda_x)$, and hence it must also be invertible modulo $\mathrm{Ker}\,(\pi)$, a contradiction. □

3. Strictly norming family of representations

A family $\mathscr{F}$ of representations of a C^*-algebra A is often called *norming*, when

$$\|a\| = \sup_{\pi\in\mathscr{F}} \|\pi(a)\|, \quad \forall a \in A. \tag{3.1}$$

As an example, the family $\{\lambda_x\}_{x\in\mathcal{G}^{(0)}}$ is norming for the reduced groupoid C^*-algebra $C_r^*(\mathcal{G})$, for every (non-necessarily amenable) groupoid $\mathcal{G}$. Based on this concept, let us give the following:

Definition 3.1. A family $\mathscr{F}$ of representations of a C^*-algebra A will be called *strictly norming* when it is norming and, in addition, the supremum in (3.1) is *attained* for every a in A.

The next result, due to Roch, relates strictly norming and sufficient families in an interesting way. Its proof is included for the convenience of the reader and also because it is slightly simpler than the proof given by Roch in [10].

Theorem 3.2 ([10, Theorem 5.7]). *Let $\mathscr{F}$ be a family of non-degenerated representations of a unital C^*-algebra A. Then $\mathscr{F}$ is strictly norming if and only if it is sufficient.*

Proof. Arguing by contradiction, suppose that $\mathscr{F}$ is sufficient, but there exists $a \in A$ such that $\|\pi(a)\| < \|a\|$, for all π in $\mathscr{F}$. Replacing a by a^*a, we may assume that a is positive. For every π in $\mathscr{F}$, we then have that

$$\sigma\big(\pi(a)\big) \subseteq \big[0, \|\pi(a)\|\big] \subseteq \big[0, \|a\|\big).$$

Setting $b = a - \|a\|$, we then have by the spectral mapping theorem that

$$\sigma\big(\pi(b)\big) = \sigma\big(\pi(a) - \|a\|\big) = \sigma\big(\pi(a)\big) - \|a\| \subseteq \big[-\|a\|, 0\big).$$

It follows that $0 \notin \sigma\big(\pi(b)\big)$, and hence that $\pi(b)$ is invertible for every π in $\mathscr{F}$, but, since $\|a\|$ belongs to the spectrum of a, we see that b is not invertible, a contradiction.

To verify the "only if" part of the statement, let a be non-invertible. We thus need to find some $\pi \in \mathscr{F}$, such that $\pi(a)$ is non-invertible.

Since a is non-invertible, then either a^*a or aa^* is non-invertible. We suppose without loss of generality that the former is true, that is, that the element $c := a^*a$ is non-invertible. We then have that

$$0 \in \sigma(c) \subseteq \big[0, \|c\|\big].$$

With $b = \|c\| - c$, we conclude from the spectral mapping theorem that

$$\|c\| \in \sigma(b) \subseteq \|c\| - \big[0, \|c\|\big] = \big[0, \|c\|\big],$$

so $\|b\| = \|c\|$, and by hypothesis there exists $\pi \in \mathscr{F}$, such that $\|\pi(b)\| = \|c\|$. Since $\pi(b)$ is positive, this implies that $\|c\|$ lies in its spectrum, which is to say that $\|c\| - \pi(b)$ is non-invertible, but

$$\|c\| - \pi(b) = \pi(c),$$

so $\pi(c)$ is non-invertible which implies that $\pi(a)$ is non-invertible. $\square$

Putting Theorem 2.10 and Theorem 3.2 together, we therefore deduce the following important consequence:

Corollary 3.3. *Suppose we are given a second-countable, Hausdorff, étale, amenable groupoid $\mathcal{G}$ with $\mathcal{G}^{(0)}$ compact. Then, for every $a \in C^*(\mathcal{G})$, there exists $x \in \mathcal{G}^{(0)}$, such that*

$$\|a\| = \|\lambda_x(a)\|.$$

References

[1] G.R. Allan, *Ideals of vector-valued functions.* Proc. London Math. Soc. **18** (1968), 193–216.

[2] C. Anantharaman-Delaroche and J. Renault, *Amenable groupoids.* Monographies de L'Enseignement Mathématique, 36. L'Enseignement Mathématique, Geneva, 2000. 196 pp.

[3] M.A. Bastos, C.A. Fernandes, and Yu. I. Karlovich, *Spectral measures in C^*-algebras of singular integral operators with shifts.* J. Funct. Analysis **242** (2007), 86–126.

[4] R.G. Douglas, *Banach algebra techniques in operator theory.* Academic Press, 1972.

[5] M. Ionescu and D. Williams, *The generalized Effros-Hahn conjecture for groupoids.* Indiana Univ. Math. J. **58** (2009), no. 6, 2489–2508.

[6] M. Ionescu and D. Williams, *Irreducible representations of groupoid C^*-algebras.* Proc. Amer. Math. Soc. **137** (2009), 1323–1332.

[7] Yu.I. Karlovich, *A local-trajectory method and isomorphism theorems for nonlocal C^*-algebras.* In Operator Theory: Advances and Applications **170**, 137–166, Birkhäuser, Basel, 2006.

[8] P. Muhly, J. Renault, and D. Williams. *Equivalence and isomorphism for groupoid C^*-algebras.* J. Operator Theory **17** (1987), 3–22.

[9] J. Renault, *A groupoid approach to C^*-algebras.* Lecture Notes in Mathematics, vol. 793, Springer, 1980.

[10] S. Roch, *Algebras of approximation sequences: structure of fractal algebras.* In Operator Theory: Advances and Applications **142**, 287–310, Birkhäuser, Basel, 2003.

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Boundedness of Pseudodifferential Operators on Banach Function Spaces

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To Professor António Ferreira dos Santos

Abstract. We show that if the Hardy–Littlewood maximal operator is bounded on a separable Banach function space $X(\mathbb{R}^n)$ and on its associate space $X'(\mathbb{R}^n)$, then a pseudodifferential operator $\mathrm{Op}(a)$ is bounded on $X(\mathbb{R}^n)$ whenever the symbol a belongs to the Hörmander class $S^{n(\rho-1)}_{\rho,\delta}$ with $0 < \rho \le 1$, $0 \le \delta < 1$ or to the Miyachi class $S^{n(\rho-1)}_{\rho,\delta}(\varkappa, n)$ with $0 \le \delta \le \rho \le 1$, $0 \le \delta < 1$, and $\varkappa > 0$. This result is applied to the case of variable Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$.

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1. Introduction

We denote the usual operators of first-order partial differentiation on $\mathbb{R}^n$ by $\partial_{x_j} := \partial/\partial_{x_j}$. For every multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$ with non-negative integers α_j, we write $\partial^\alpha := \partial^{\alpha_1}_{x_1} \dots \partial^{\alpha_n}_{x_n}$. Further, put $|\alpha| := \alpha_1 + \dots + \alpha_n$, and for each vector $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$, define $\xi^\alpha := \xi_1^{\alpha_1} \dots \xi_n^{\alpha_n}$. Let $\langle \cdot, \cdot \rangle$ stand for the scalar product in $\mathbb{R}^n$ and $|\xi| := \sqrt{\langle \xi, \xi \rangle}$ for $\xi \in \mathbb{R}^n$.

Let $C_0^\infty(\mathbb{R}^n)$ denote the set of all infinitely differentiable functions with compact support. Recall that, given $u \in C_0^\infty(\mathbb{R}^n)$, a pseudodifferential operator $\mathrm{Op}(a)$ is formally defined by the formula

$$(\mathrm{Op}(a)u)(x) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} d\xi \int_{\mathbb{R}^n} a(x, \xi) u(y) e^{i\langle x-y, \xi \rangle} dy,$$

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where the symbol a is assumed to be bounded in both the spatial variable x and the frequency variable ξ, and satisfies certain regularity conditions.

An example of symbols one might consider is the Hörmander class $S^m_{\rho,\delta}$ introduced in [16] and consisting of $a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ with

$$|\partial_\xi^\alpha \partial_x^\beta a(x,\xi)| \le C_{\alpha,\beta}(1+|\xi|)^{m-\rho|\alpha|+\delta|\beta|} \quad (x,\xi \in \mathbb{R}^n),$$

where

$$m \in \mathbb{R}, \quad 0 \le \delta, \rho \le 1$$

and the positive constants $C_{\alpha,\beta}$ depend only on α and β. Along with the Hörmander class $S^m_{\rho,\delta}$, we will consider the generalized Hörmander class $S^m_{\rho,\delta}(\varkappa,\varkappa')$ introduced by Miyachi [29]. We will call $S^m_{\rho,\delta}(\varkappa,\varkappa')$ the Miyachi class of symbols. Its quite technical definition is postponed to Subsection 2.1. Here we only note that symbols in the Miyachi classes may lie beyond $C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ (that is, they are non-smooth, in general).

Let $f \in L^1_{\rm loc}(\mathbb{R}^n)$. For a cube $Q \subset \mathbb{R}^n$, put

$$f_Q := \frac{1}{|Q|}\int_Q f(x)dx.$$

Here, and throughout, cubes will be assumed to have their sides parallel to the coordinate axes and $|Q|$ will denote the volume of Q. The Fefferman–Stein sharp maximal operator $f \mapsto f^\#$ is defined by

$$f^\#(x) := \sup_{Q \ni x} \frac{1}{|Q|}\int_Q |f(x) - f_Q|dx \quad (x \in \mathbb{R}^n),$$

where the supremum is taken over all cubes Q containing x. Let $1 \le q < \infty$. Given $f \in L^q_{\rm loc}(\mathbb{R}^n)$, the qth maximal operator is defined by

$$(M_q f)(x) := \sup_{Q \ni x}\left(\frac{1}{|Q|}\int_Q |f(y)|^q dy\right)^{1/q} \quad (x \in \mathbb{R}^n),$$

where the supremum is taken over all cubes Q containing x. For $q = 1$ this is the usual Hardy–Littlewood maximal operator, which will be denoted by M.

The boundedness of pseudodifferential operators with smooth and non-smooth symbols on the classical Lebesgue spaces $L^p(\mathbb{R}^n)$ was studied by many authors. We refer to the monographs by Coifman and Meyer [8], Kumano-go [21], Journé [18], Taylor [37], Stein [36], Hörmander [17], Abels [1] and also to the papers by Miyachi [29] and Ashino, Nagase, and Vaillancourt [5] for corresponding results and further references.

Miller [27] proved the boundedness of pseudodifferential operators with symbols $a \in S^0_{1,0}$ on the weighted Lebesgue spaces $L^p(\mathbb{R}^n, w)$ with $1 < p < \infty$ and Muckenhoupt weights $w \in A_p(\mathbb{R}^n)$. One of the key ingredients in his proof was the pointwise estimate

$$(\mathrm{Op}(a)f)^\#(x) \le C_q(M_q f)(x) \quad (x \in \mathbb{R}^n), \tag{1}$$

where $q \in (1, \infty)$ and $C_q > 0$ is independent of $f \in C_0^\infty(\mathbb{R}^n)$. Another ingredients are the Fefferman–Stein inequality (see, e.g., [15, Theorem 5]) and self-improving properties of Muckenhoupt weights. Further, estimate (1) and the boundedness results for $\mathrm{Op}(a)$ on $L^p(\mathbb{R}^n, w)$ with $p \in (1, \infty)$ and $w \in A_p(\mathbb{R}^n)$ were extended to other classes of smooth and non-smooth symbols. We refer, for instance, to the works by Nishigaki [33], Yabuta [38, 39, 40, 41], Miyachi and Yabuta [30], Álvarez and Hounie [2], Álvarez, Hounie, and Pérez [3], Michalowski, Rule, and Staubach [26] and the references therein.

Rabinovich and Samko [35, Theorem 5.1] proved the boundedness of pseudo-differential operators with symbols $a \in S^0_{1,0}$ on so-called variable Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$ (see Subsection 3.1). Their proof did not rely on (1). Instead, they obtained another (more precise) pointwise estimate for $(\mathrm{Op}(a)f)^\#(x)$ in the spirit of [4]. Recently the author and Spitkovsky [19, Theorem 1.2] proved the boundedness of $\mathrm{Op}(a)$ on variable Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$ for the symbols $a \in S^{n(\rho-1)}_{\rho,\delta}$ with $0 < \rho \le 1$ and $0 \le \delta < 1$. That proof relies on (1) (obtained in [26]), on the Fefferman–Stein inequality for variable Lebesgue spaces, and on a certain self-improving property of the Hardy–Littlewood maximal function on $L^{p(\cdot)}(\mathbb{R}^n)$.

The aim of the present paper is to extend the results of [19, 35] to the case of so-called Banach function spaces. Our proof is based on estimate (1), on the Fefferman–Stein inequality for Banach function spaces proved recently by Lerner [23], and on a self-improving property of the Hardy–Littlewood maximal function on Banach function spaces proved by Lerner and Pérez [25]. Note that our results are true for all symbols classes admitting estimate (1). We choose here the classical Hörmander classes $S^m_{\rho,\delta}$ of smooth symbols and the Miyachi classes $S^m_{\rho,\delta}(\varkappa, \varkappa')$ of non-smooth symbols just as an illustration of the fact that the assumptions on smoothness of symbols imposed in [19, 35] can be essentially relaxed.

The set of all Lebesgue measurable complex-valued functions on $\mathbb{R}^n$ is denoted by $\mathcal{M}$. Let $\mathcal{M}^+$ be the subset of functions in $\mathcal{M}$ whose values lie in $[0, \infty]$. The characteristic function of a measurable set $E \subset \mathbb{R}^n$ is denoted by χ_E and the Lebesgue measure of E is denoted by $|E|$.

Definition 1.1 ([6, Chap. 1, Definition 1.1]). A mapping $\rho : \mathcal{M}^+ \to [0, \infty]$ is called a *Banach function norm* if, for all functions f, g, f_n $(n \in \mathbb{N})$ in $\mathcal{M}^+$, for all constants $a \ge 0$, and for all measurable subsets E of $\mathbb{R}^n$, the following properties hold:

(A1) $\rho(f) = 0 \Leftrightarrow f = 0$ a.e., $\rho(af) = a\rho(f)$, $\rho(f + g) \le \rho(f) + \rho(g)$,

(A2) $0 \le g \le f$ a.e. $\Rightarrow$ $\rho(g) \le \rho(f)$ (the lattice property),

(A3) $0 \le f_n \uparrow f$ a.e. $\Rightarrow$ $\rho(f_n) \uparrow \rho(f)$ (the Fatou property),

(A4) $|E| < \infty \Rightarrow \rho(\chi_E) < \infty$,

(A5) $|E| < \infty \Rightarrow \int_E f(x)\,dx \le C_E \rho(f)$

with $C_E \in (0, \infty)$ which may depend on E and ρ but is independent of f.

When functions differing only on a set of measure zero are identified, the set $X(\mathbb{R}^n)$ of all functions $f \in \mathcal{M}$ for which $\rho(|f|) < \infty$ is called a *Banach function* *space*. For each $f \in X(\mathbb{R}^n)$, the norm of f is defined by

$$\|f\|_{X(\mathbb{R}^n)} := \rho(|f|).$$

The set $X(\mathbb{R}^n)$ under the natural linear space operations and under this norm becomes a Banach space (see [6, Chap. 1, Theorems 1.4 and 1.6]).

If ρ is a Banach function norm, its associate norm ρ' is defined on $\mathcal{M}^+$ by

$$\rho'(g) := \sup\left\{ \int_{\mathbb{R}^n} f(x)g(x)\,dx \ : \ f \in \mathcal{M}^+, \ \rho(f) \le 1 \right\}, \quad g \in \mathcal{M}^+.$$

It is a Banach function norm itself [6, Chap. 1, Theorem 2.2]. The Banach function space $X'(\mathbb{R}^n)$ determined by the Banach function norm ρ' is called the *associate space* (*Köthe dual*) of $X(\mathbb{R}^n)$. The Lebesgue space $L^p(\mathbb{R}^n)$, $1 \le p \le \infty$, are the archetypical example of Banach function spaces. Other classical examples of Banach function spaces are Orlicz spaces, rearrangement-invariant spaces, and variable Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$.

Note that we do not assume that $X(\mathbb{R}^n)$ is rearrangement-invariant (see [6, Chap. 2]). Therefore, we are not allowed to use the interpolation theory to study the boundedness of $\mathrm{Op}(a)$ on $X(\mathbb{R}^n)$.

Theorem 1.2 (Main result). *Let $X(\mathbb{R}^n)$ be a separable Banach function space such that the Hardy–Littlewood maximal operator M is bounded on $X(\mathbb{R}^n)$ and on its associate space $X'(\mathbb{R}^n)$. If a belongs to one of the following symbol classes:*

(a) *the Hörmander class $S^{n(\rho-1)}_{\rho,\delta}$ with $0 < \rho \le 1$ and $0 \le \delta < 1$;*

(b) *the Miyachi class $S^{n(\rho-1)}_{\rho,\delta}(\varkappa, n)$ with $0 \le \delta \le \rho \le 1$, $0 \le \delta < 1$, and $\varkappa > 0$;*

then $\mathrm{Op}(a)$ *extends to a bounded operator on $X(\mathbb{R}^n)$.*

The paper is organized as follows. Section 2 is devoted to the proof of Theorem 1.2. First, we collect the main ingredients. We give the precise definition of the Miyachi class $S^m_{\rho,\delta}(\varkappa, \varkappa')$ in Subsection 2.1. The Fefferman–Stein inequality for Banach function spaces is stated in Section 2.3. A certain self-improving property of the Hardy–Littlewood maximal operator on Banach function spaces is discussed in Subsection 2.4. Precise assumptions on our symbols guaranteeing (1) are stated in Subsection 2.5. Finally, we assemble these ingredients in Subsection 2.6 and prove Theorem 1.2.

In Section 3 we apply Theorem 1.2 to the case of variable Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$. In Subsection 3.1 we recall the definition and some basic properties of variable Lebesgue spaces. In Subsection 3.2 we discuss the boundedness of the Hardy–Littlewood maximal operator on $L^{p(\cdot)}(\mathbb{R}^n)$. In particular, we recall that M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ if and only if M is bounded on its associate space. This allows us to simplify little bit the formulation of Theorem 1.2 for $L^{p(\cdot)}(\mathbb{R}^n)$ in Subsection 3.3.

2. Proof of the main result

2.1. The Miyachi class

The following class of symbols was introduced by Miyachi [29] (see also [28, 30]). If $h \in \mathbb{R}^n$ and f is a function on $\mathbb{R}^n$, then the first and the second differences are denoted by

$$\Delta_x(h)f(x) := f(x+h) - f(x),$$
$$\Delta_x^2(h)f(x) := f(x+2h) - 2f(x+h) + f(x).$$

Let

$$m \in \mathbb{R}, \quad 0 \le \delta, \rho \le 1, \quad \varkappa > 0, \quad \varkappa' > 0.$$

Let k and k' be nonnegative integers satisfying

$$k < \varkappa \le k+1, \quad k' < \varkappa' \le k'+1.$$

The Miyachi class $S^m_{\rho,\delta}(\varkappa, \varkappa')$ consists of all functions a on $\mathbb{R}^n \times \mathbb{R}^n$ such that the derivatives $\partial_x^\beta \partial_\xi^\alpha a(x,\xi)$ exist in the classical sense for $|\beta| \le k$ and $|\alpha| \le k'$ and the following four conditions are fulfilled:

(i) if $|\beta| \le k$ and $|\alpha| \le k'$, then

$$|\partial_x^\beta \partial_\xi^\alpha a(x,\xi)| \le A(1+|\xi|)^{m+\delta|\beta|-\rho|\alpha|};$$

(ii) if $|\beta| = k$ and $|\alpha| \le k'$, $h \in \mathbb{R}^n$, and $|h| \le (1+|\xi|)^{-\delta}$, then

$$|\Delta_x^2(h)\partial_x^\beta \partial_\xi^\alpha a(x,\xi)| \le A(1+|\xi|)^{m+\delta\varkappa-\rho|\alpha|}|h|^{\varkappa-k};$$

(iii) if $|\beta| \le k$ and $|\alpha| = k'$, $\eta \in \mathbb{R}^n$ and $|\eta| \le (1+|\xi|)^\rho/4$, then

$$|\Delta_\xi^2(\eta)\partial_x^\beta \partial_\xi^\alpha a(x,\xi)| \le A(1+|\xi|)^{m+\delta|\beta|-\rho\varkappa'}|\eta|^{\varkappa'-k'};$$

(iv) if $|\beta| = k$ and $|\alpha| = k'$, $h, \eta \in \mathbb{R}^n$, and $|h| \le (1+|\xi|)^{-\delta}$, $|\eta| \le (1+|\xi|)^\rho/4$, then

$$|\Delta_x^2(h)\Delta_\xi^2(\eta)\partial_x^\beta \partial_\xi^\alpha a(x,\xi)| \le A(1+|\xi|)^{m+\delta\varkappa-\rho\varkappa'}|h|^{\varkappa-k}|\eta|^{\varkappa'-k'}.$$

Here the constant A is independent of the multi-indices α, β and the variables $x, \xi, h, \eta \in \mathbb{R}^n$. The smallest such constant is denoted by $\|a\|_{m,\rho,\delta,\varkappa,\varkappa'}$.

It is not difficult to see that if $\varkappa_2 \le \varkappa_1$ and $\varkappa_2' \le \varkappa_1'$, then

$$S^m_{\rho,\delta} \subset S^m_{\rho,\delta}(\varkappa_1, \varkappa_1') \subset S^m_{\rho,\delta}(\varkappa_2, \varkappa_2') \quad \text{and} \quad \|a\|_{m,\rho,\delta,\varkappa_2,\varkappa_2'} \le \text{const}\|a\|_{m,\rho,\delta,\varkappa_1,\varkappa_1'}.$$

If $\varkappa$ (resp. $\varkappa'$) is not integer, then $\Delta_x^2(h)$ (resp. $\Delta_\xi^2(\eta)$) can be replaced by $\Delta_x^1(h)$ (resp. $\Delta_\xi^1(\eta)$). It should also be remarked that the assumptions $|h| \le (1+|\xi|)^{-\delta}$ and $|\eta| \le (1+|\xi|)^\rho/4$ can be replaced by $h \in \mathbb{R}^n$ and $|\eta| \le (1+|\xi|)/4$ if one modifies the constant A.

2.2. Density of smooth compactly supported functions

Lemma 2.1. *The set $C_0^\infty(\mathbb{R}^n)$ is dense in a separable Banach function space $X(\mathbb{R}^n)$.*

The proof is standard. For details, see [20, Lemma 2.10(b)], where this fact is proved for $n = 1$. The proof for arbitrary n is a minor modification of that one.

2.3. The Fefferman–Stein inequality for Banach function spaces

Let $S_0(\mathbb{R}^n)$ be the space of all measurable functions f on $\mathbb{R}^n$ such that

$$|\{x \in \mathbb{R}^n : |f(x)| > \lambda\}| < \infty$$

for any $\lambda > 0$. Chebyshev's inequality

$$|\{x \in \mathbb{R}^n : |f(x)| > \lambda\}| \le \frac{1}{\lambda^q} \int_{\mathbb{R}^n} |f(x)|^q \, dx$$

holds for every $q \in (0, \infty)$ and $\lambda > 0$. In particular, it implies that

$$\bigcup_{q \in (0,\infty)} L^q(\mathbb{R}^n) \subset S_0(\mathbb{R}^n).$$

It is obvious that $f^{\#}$ is pointwise dominated by Mf. Hence, by Axiom (A2),

$$\|f^{\#}\|_{X(\mathbb{R}^n)} \le \text{const}\|f\|_{X(\mathbb{R}^n)} \quad \text{for} \quad f \in X(\mathbb{R}^n)$$

whenever M is bounded on $X(\mathbb{R}^n)$. The converse inequality for Lebesgue spaces $L^p(\mathbb{R}^n)$, $1 < p < \infty$, was proved by Fefferman and Stein (see [15, Theorem 5] and also [36, Chap. IV, Section 2.2]). The following extension of the Fefferman–Stein inequality to Banach function spaces was proved in [23, Corollary 4.2].

Theorem 2.2 (Lerner). *Let M be bounded on a Banach function space $X(\mathbb{R}^n)$. Then M is bounded on its associate space $X'(\mathbb{R}^n)$ if and only if there exists a constant $C_{\#} > 0$ such that, for all $f \in S_0(\mathbb{R}^n)$,*

$$\|f\|_{X(\mathbb{R}^n)} \le C_{\#}\|f^{\#}\|_{X(\mathbb{R}^n)}.$$

2.4. Self-improving property of maximal operators on Banach function spaces

If $1 < q < \infty$, then from the Hölder inequality one can immediately get that

$$(Mf)(x) \le (M_q f)(x) \quad (x \in \mathbb{R}^n).$$

Thus, the boundedness of any M_q, $1 < q < \infty$, on a Banach function space $X(\mathbb{R}^n)$ immediately implies the boundedness of M. A partial converse of this fact, called a *self-improving property* of the Hardy–Littlewood maximal operator, is also true. It was proved in [25, Corollary 1.3] (see also [24] for another proof) in a more general setting of quasi-Banach function spaces.

Theorem 2.3 (Lerner–Pérez). *Let $X(\mathbb{R}^n)$ be a Banach function space. Then M is bounded on $X(\mathbb{R}^n)$ if and only if M_q is bounded on $X(\mathbb{R}^n)$ for some $q \in (1, \infty)$.*

2.5. The crucial pointwise estimate

Theorem 2.4. *If a belongs to one of the following symbol classes:*

(a) *the Hömander class $S^{n(\rho-1)}_{\rho,\delta}$ with $0 < \rho \le 1$ and $0 \le \delta < 1$;*

(b) *the Miyachi class $S^{n(\rho-1)}_{\rho,\delta}(\varkappa, n)$ with $0 \le \delta \le \rho \le 1$, $0 \le \delta < 1$, and $\varkappa > 0$;*

then for every $q \in (1, \infty)$ there exists a constant $C_q > 0$ such that

$$(\mathrm{Op}(a)f)^{\#}(x) \le C_q(M_q f)(x) \quad (x \in \mathbb{R}^n) \tag{2}$$

for all $f \in C_0^\infty(\mathbb{R}^n)$.

Part (a) was recently proved by Michalowski, Rule, and Staubach [26, Theorem 3.3]. Their estimate generalizes the pointwise estimate by Miller [27, Theorem 2.8] for $a \in S^0_{1,0}$ and by Álvarez and Hounie [2, Theorem 4.1] for $a \in S^m_{\rho,\delta}$ with the parameters satisfying $0 < \delta \le \rho \le 1/2$ and $m \le n(\rho - 1)$. Part (b) follows from the estimate by Miyachi and Yabuta [30, Theorem 2.4].

Corollary 2.5. *If the conditions of Theorem* 2.4 *are fulfilled, then* $\mathrm{Op}(a)f \in S_0(\mathbb{R}^n)$ *for every* $f \in C_0^\infty(\mathbb{R}^n)$.

Proof. By using the well-known L^p-estimates for the sharp maximal function (see [15, Theorem 5]) and for the maximal function M_q, one can show that if (2) holds for all $f \in C_0^\infty(\mathbb{R}^n)$, then $\mathrm{Op}(a)$ extends to a bounded operator on $L^p(\mathbb{R}^n)$ for $q < p < \infty$. In particular, this implies that $\mathrm{Op}(a)f \in S_0(\mathbb{R}^n)$ for every function $f \in C_0^\infty(\mathbb{R}^n)$. □

2.6. Proof of Theorem 1.2

The presented proof is an adaptation of the proof of [19, Theorem 1.2]. Its idea goes back to Miller [27]. Suppose $f \in C_0^\infty(\mathbb{R}^n)$. Then $\mathrm{Op}(a)f \in S_0(\mathbb{R}^n)$ in view of Corollary 2.5. By Lerner's theorem (Theorem 2.2), there exists a constant $C_\# > 0$ such that

$$\|\mathrm{Op}(a)f\|_{X(\mathbb{R}^n)} \le C_\# \|(\mathrm{Op}(a)f)^\#\|_{X(\mathbb{R}^n)} \tag{3}$$

Further, by the crucial pointwise estimate (Theorem 2.4), for every $q \in (1,\infty)$, there is a constant $C_q > 0$ such that

$$(\mathrm{Op}(a)f)^\#(x) \le C_q (M_q f)(x) \quad (x \in \mathbb{R}^n)$$

Hence, by Axioms (A1) and (A2),

$$\|(\mathrm{Op}(a)f)^\#\|_{X(\mathbb{R}^n)} \le C_q \|M_q f\|_{X(\mathbb{R}^n)}. \tag{4}$$

On the other hand, since M is bounded on $X(\mathbb{R}^n)$, by the Lerner–Pérez theorem (Theorem 2.3), there is a constant exponent $q_0 \in (1,\infty)$ and a constant $C'_{q_0} > 0$ such that

$$\|M_{q_0} f\|_{X(\mathbb{R}^n)} \le C'_{q_0} \|f\|_{X(\mathbb{R}^n)}. \tag{5}$$

Thus, combining (3)–(5), we arrive at

$$\|\mathrm{Op}(a)f\|_{X(\mathbb{R}^n)} \le C_\# C_{q_0} C'_{q_0} \|f\|_{X(\mathbb{R}^n)}$$

for all $f \in C_0^\infty(\mathbb{R}^n)$. It remains to recall that, in view of Lemma 2.1, $C_0^\infty(\mathbb{R}^n)$ is dense in $X(\mathbb{R}^n)$ whenever $X(\mathbb{R}^n)$ is separable. Thus, $\mathrm{Op}(a)$ extends to a bounded operator on the whole space $X(\mathbb{R}^n)$ by continuity. □

3. Pseudodifferential operators on variable Lebesgue spaces

3.1. Variable Lebesgue spaces

Let $p\colon \mathbb{R}^n \to [1,\infty]$ be a measurable a.e. finite function. By $L^{p(\cdot)}(\mathbb{R}^n)$ we denote the set of all complex-valued functions f on $\mathbb{R}^n$ such that

$$I_{p(\cdot)}(f/\lambda) := \int_{\mathbb{R}^n} |f(x)/\lambda|^{p(x)} dx < \infty$$

for some $\lambda > 0$. This set becomes a Banach function space when equipped with the norm

$$\|f\|_{p(\cdot)} := \inf\left\{\lambda > 0 : I_{p(\cdot)}(f/\lambda) \le 1\right\}.$$

It is easy to see that if p is constant, then $L^{p(\cdot)}(\mathbb{R}^n)$ is nothing but the standard Lebesgue space $L^p(\mathbb{R}^n)$. The space $L^{p(\cdot)}(\mathbb{R}^n)$ is referred to as a *variable Lebesgue space*.

We will always suppose that

$$1 < p_- := \operatorname*{ess\,inf}_{x\in\mathbb{R}^n} p(x), \quad \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} p(x) =: p_+ < \infty. \tag{6}$$

Under these conditions, the space $L^{p(\cdot)}(\mathbb{R}^n)$ is separable and reflexive, and its associate space is isomorphic to $L^{p'(\cdot)}(\mathbb{R}^n)$, where

$$1/p(x) + 1/p'(x) = 1 \quad (x \in \mathbb{R}^n)$$

(see, e.g., [9, Chap. 2] or [14, Chap. 3]).

3.2. The Hardy–Littlewood maximal function on variable Lebesgue spaces

By $\mathcal{M}(\mathbb{R}^n)$ denote the set of all measurable functions $p : \mathbb{R}^n \to [1,\infty]$ such that (6) holds and the Hardy–Littlewood maximal operator is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.

Assume that (6) is fulfilled. Diening [12] proved that if p satisfies

$$|p(x) - p(y)| \le \frac{c}{\log(e + 1/|x-y|)} \quad (x, y \in \mathbb{R}^n) \tag{7}$$

for some $c > 0$ independent of $x, y \in \mathbb{R}^n$ and p is constant outside some ball, then $p \in \mathcal{M}(\mathbb{R}^n)$. Further, the behavior of p at infinity was relaxed by Cruz-Uribe, Fiorenza, and Neugebauer [10, 11], where it was shown that if p satisfies (7) and there exists a $p_\infty > 1$ such that

$$|p(x) - p_\infty| \le \frac{c}{\log(e + |x|)} \quad (x \in \mathbb{R}^n) \tag{8}$$

with $c > 0$ independent of $x \in \mathbb{R}^n$, then $p \in \mathcal{M}(\mathbb{R}^n)$. Following [14, Section 4.1], we will say that if conditions (7)–(8) are fulfilled, then p is *globally log-Hölder continuous*. The class of all globally log-Hölder continuous exponents will be denoted by $\mathcal{P}^{\log}(\mathbb{R}^n)$.

Conditions (7) and (8) are optimal for the boundedness of M in the pointwise sense; the corresponding examples are contained in [34] and [10]. However, neither (7) nor (8) is necessary for $p \in \mathcal{M}(\mathbb{R}^n)$. Nekvinda [31] proved that if p satisfies

(6)–(7) and

$$\int_{\mathbb{R}^n} |p(x) - p_\infty| c^{1/|p(x)-p_\infty|}\, dx < \infty \tag{9}$$

for some $p_\infty > 1$ and $c > 0$, then $p \in \mathcal{M}(\mathbb{R}^n)$. One can show that (8) implies (9), but the converse, in general, is not true. The corresponding example is constructed in [7]. Nekvinda further relaxed condition (9) in [32]. Lerner [22] (see also [14, Example 5.1.8]) showed that there exist discontinuous at zero or/and at infinity exponents, which nevertheless belong to $\mathcal{M}(\mathbb{R}^n)$. Thus, the class of exponents in $\mathcal{P}^{\log}(\mathbb{R}^n)$ satisfying (6) is a proper subset of the class $\mathcal{M}(\mathbb{R}^n)$.

We will need the following remarkable result proved in [13, Theorem 8.1] (see also [14, Theorem 5.7.2]).

Theorem 3.1 (Diening). *We have $p \in \mathcal{M}(\mathbb{R}^n)$ if and only if $p' \in \mathcal{M}(\mathbb{R}^n)$.*

We refer to the recent monographs [9, 14] for further discussions concerning the class $\mathcal{M}(\mathbb{R}^n)$.

3.3. Boundedness of pseudodifferential operators on variable Lebesgue spaces

Combining Theorem 1.2 and Theorem 3.1, we immediately arrive at the following.

Theorem 3.2. *Suppose $p \in \mathcal{M}(\mathbb{R}^n)$. If a belongs to one of the following symbol classes:*

(a) *the Hörmander class $S^{n(\rho-1)}_{\rho,\delta}$ with $0 < \rho \le 1$ and $0 \le \delta < 1$;*

(b) *the Miyachi class $S^{n(\rho-1)}_{\rho,\delta}(\varkappa, n)$ with $0 \le \delta \le \rho \le 1$, $0 \le \delta < 1$, and $\varkappa > 0$;*

then $\operatorname{Op}(a)$ extends to a bounded operator on $L^{p(\cdot)}(\mathbb{R}^n)$.

Part (a) of the above theorem was obtained by the author and Spitkovsky in [19, Theorem 1.2]. Part (b) is new.

Corollary 3.3 (Rabinovich–Samko). *Let $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ satisfy (6). If $a \in S^0_{1,0}$, then $\operatorname{Op}(a)$ extends to a bounded operator on $L^{p(\cdot)}(\mathbb{R}^n)$.*

Proof. This statement immediately follows from part (a) of the previous result because $\mathcal{P}^{\log}(\mathbb{R}^n)$ is a (proper!) subset of $\mathcal{M}(\mathbb{R}^n)$. □

This result was proved in [35, Theorem 5.1].

References

[1] H. Abels, *Pseudodifferential and Singular Integral Operators. An Introduction with Applications.* De Gruyter, Berlin, 2012.

[2] J. Álvarez and J. Hounie, *Estimates for the kernel and continuity properties of pseudo-differential operators.* Ark. Mat. **28** (1990), 1–22.

[3] J. Álvarez, J. Hounie, and C. Pérez, *A pointwise estimate for the kernel of a pseudo-differential operator, with applications.* Rev. Union Mat. Argent. **37** (1991), 184–199.

[4] J. Álvarez and C. Pérez, *Estimates with A_∞ weights for various singular integral operators.* Boll. Un. Mat. Ital. A (7) **8** (1994), 123–133.

[5] R. Ashino, M. Nagase, and R. Vaillancourt, *Pseudodifferential operators in $L^p(\mathbb{R}^n)$ spaces.* Cubo **6** (2004), 91–129.

[6] C. Bennett and R. Sharpley, *Interpolation of Operators.* Academic Press, New York, 1988.

[7] C. Capone, D. Cruz-Uribe, and A. Fiorenza, *The fractional maximal operator and fractional integrals on variable L^p spaces.* Rev. Mat. Iberoamericana **23** (2007), 743–770.

[8] R. Coifman and Y. Meyer. *Au délà des Opérateurs Pseudo-Différentiels.* Astérisque **57**, Soc. Math. France, Paris, 1978 (in French).

[9] D. Cruz-Uribe and A. Fiorenza, *Variable Lebesgue Spaces.* Birkhäuser, Basel, 2013.

[10] D. Cruz-Uribe, A. Fiorenza, and C.J. Neugebauer, *The maximal function on variable L^p spaces.* Ann. Acad. Sci. Fenn. Math. **28** (2003), 223–238.

[11] D. Cruz-Uribe, A. Fiorenza, and C.J. Neugebauer, *Corrections to: "The maximal function on variable L^p spaces".* Ann. Acad. Sci. Fenn. Math. **29** (2004), 247–249.

[12] L. Diening, *Maximal function on generalized Lebesgue spaces $L^{p(\cdot)}$.* Math. Inequal. Appl. **7** (2004), 245–253.

[13] L. Diening, *Maximal function on Musielak-Orlicz spaces and generalized Lebesgue spaces.* Bull. Sci. Math. **129** (2005), 657–700.

[14] L. Diening, P. Harjulehto, P. Hästö, and M. Růžička, *Lebesgue and Sobolev Spaces with Variable Exponents.* Lecture Notes in Mathematics **2017**, Springer, Berlin, 2011.

[15] Ch. Fefferman and E.M. Stein, *H^p spaces of several variables.* Acta Math. **129** (1972), 137–193.

[16] L. Hörmander, *Pseudo-differential operators and hypoelliptic equations.* In: "Singular integrals (Proc. Sympos. Pure Math., Vol. X, Chicago, Ill., 1966)", pp. 138–183. Amer. Math. Soc., Providence, R.I., 1967.

[17] L. Hörmander, *The Analysis of Linear Partial Differential Operators. III: Pseudo-Differential Operators.* Reprint of the 1994 edition. Classics in Mathematics. Springer, Berlin, 2007.

[18] J.-L. Journe, *Calderón–Zygmund Operators, Pseudo-Differential Operators and the Cauchy Integral of Calderón.* Lecture Notes in Mathematics **994**. Springer, Berlin, 1983.

[19] A.Yu. Karlovich and I.M. Spitkovsky, *Pseudodifferential operators on variable Lebesgue spaces.* In: "Operator Theory, Pseudodifferential Equations and Mathematical Physics". Operator Theory: Advances and Applications **228** (2013), 173–184.

[20] A.Yu. Karlovich and I.M. Spitkovsky, *The Cauchy singular integral operator on weighted variable Lebesgue spaces.* In: "Concrete Operators, Spectral Theory, Operators in Harmonic Analysis and Approximation". Operator Theory: Advances and Applications **236** (2014), 275–291.

[21] H. Kumano-go, *Pseudo-Differential Operators.* The MIT Press. Cambridge, MA, 1982.

[22] A.K. Lerner, *Some remarks on the Hardy–Littlewood maximal function on variable L^p spaces.* Math. Z. **251** (2005), 509–521.

[23] A.K. Lerner, *Some remarks on the Fefferman–Stein inequality.* J. Anal. Math. **112** (2010), 329–349.

[24] A.K. Lerner and S. Ombrosi, *A boundedness criterion for general maximal operators.* Publ. Mat. **54** (2010), 53–71.

[25] A.K. Lerner and C. Pérez, *A new characterization of the Muckenhoupt A_p weights through an extension of the Lorentz–Shimogaki theorem.* Indiana Univ. Math. J. **56** (2007), 2697–2722.

[26] N. Michalowski, D. Rule, and W. Staubach, *Weighted L^p boundedness of pseudodifferential operators and applications.* Canad. Math. Bull. **55** (2012), 555–570.

[27] N. Miller, *Weighted Sobolev spaces and pseudodifferential operators with smooth symbols.* Trans. Amer. Math. Soc. **269** (1982), 91–109.

[28] A. Miyachi, *Estimates for pseudo-differential operators of class $S_{0,0}$.* Math. Nachr. **133** (1987), 135–154.

[29] A. Miyachi, *Estimates for pseudo-differential operators of class $S^m_{\rho,\delta}$ in L^p, h^p, and bmo.* In: "Analysis, Proc. Conf., Singapore 1986", North-Holland Math. Stud. **150** (1988), 177–187.

[30] A. Miyachi and K. Yabuta, *Sharp function estimates for pseudo-differential operators of class $S^m_{\rho,\delta}$.* Bull. Fac. Sci., Ibaraki Univ., Ser. A **19** (1987), 15–30.

[31] A. Nekvinda, *Hardy–Littlewood maximal operator on $L^{p(x)}(\mathbb{R}^n)$.* Math. Inequal. Appl. **7** (2004), 255–265.

[32] A. Nekvinda, *Maximal operator on variable Lebesgue spaces for almost monotone radial exponent.* J. Math. Anal. Appl. **337** (2008), 1345–1365.

[33] S. Nishigaki, *Weighted norm inequalities for certain pseudo-differential operators.* Tokyo J. Math. **7** (1984), 129–140.

[34] L. Pick and M. Růžička, *An example of a space of Lp(x) on which the Hardy–Littlewood maximal operator is not bounded.* Expo. Math. **19** (2001), 369–371.

[35] V.S. Rabinovich and S.G. Samko, *Boundedness and Fredholmness of pseudodifferential operators in variable exponent spaces.* Integral Equations Operator Theory **60** (2008), 507–537.

[36] E. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals.* Princeton University Press, Princeton, NJ, 1993.

[37] M. Taylor, *Pseudodifferential Operators and Nonlinear PDE.* Progress in Mathematics **100**. Birkhäuser Boston, Boston, MA, 1991.

[38] K. Yabuta, *Generalizations of Calderón–Zygmund operators.* Stud. Math. **82** (1985), 17–31.

[39] K. Yabuta, *Calderón–Zygmund operators and pseudo-differential operators.* Commun. Partial Differ. Equations **10** (1985), 1005–1022.

[40] K. Yabuta, *Weighted norm inequalities for pseudodifferential operators.* Osaka J. Math. **23** (1986), 703–723.

[41] K. Yabuta, *Sharp function estimates for a class of pseudodifferential operators.* Bull. Fac. Sci., Ibaraki Univ., Ser. A **21** (1989), 1–7.

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On the Dimension of the Kernel of a Singular Integral Operator with Shift

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Dedicated to Professor António Ferreira dos Santos

Abstract. Some estimates for the dimension of the kernel of the singular integral operator $I - cUP_+$: $L_2^n(\mathbb{T}) \to L_2^n(\mathbb{T})$, with a non-Carleman shift are obtained, where P_+ is the Cauchy projector, U is an isometric shift operator and $c(t)$ is a continuous matrix function. It is supposed that the shift has a finite set of fixed points and all the eigenvalues of the matrix $c(t)$ at the fixed points, simultaneously belong either to the interior of the unit circle $\mathbb{T}$ or to its exterior. Moreover, we show that some of the obtained results can be used in the case of an operator with a general shift.

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1. Introduction

Let $\mathbb{T}$ denote the unit circle and ω be a homeomorphism of $\mathbb{T}$ onto itself, which is differentiable on $\mathbb{T}$ and whose derivative does not vanish there. The function $\omega : \mathbb{T} \to \mathbb{T}$ is called a shift function or simply a shift on $\mathbb{T}$. By

$$\omega_k(t) \equiv \omega[\omega_{k-1}(t)], \quad \omega_1(t) \equiv \omega(t), \quad \omega_0(t) \equiv t, \quad t \in \mathbb{T},$$

we denote the kth iteration of the shift, $k \geq 2$, $k \in \mathbb{N}$.

A shift ω is called a (generalized) Carleman shift of order $n \in \mathbb{N} \setminus \{1\}$ if $\omega_n(t) \equiv t$, but $\omega_k(t) \not\equiv t$ for $k = \overline{1, n-1}$. Otherwise, if ω is not a Carleman shift, it is called a non-Carleman shift. In what follows we will consider six different shifts, i.e., $\omega = \zeta, \eta, \alpha, \beta, \gamma, \delta$: ζ, η and γ are general shifts, in the sense Carleman or non-Carleman shifts; α, β and δ are non-Carleman shifts having a finite set of

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fixed points $\{\tau_1, \tau_2, \ldots, \tau_s\}$, $s \geq 1$. Other properties of these shifts will be specified later on whenever necessary.

On the Lebesgue spaces $L_p^n(\mathbb{T})$, $p \in (1, \infty)$, we consider a shift operator U_ω, associated with a shift ω, defined by

$$(U_\omega \varphi)(t) = u_\omega(t)\varphi[\omega(t)], \quad t \in \mathbb{T},$$

where the function u_ω is chosen in such way that the following properties hold for U_ω[1]:

i) U_ω is isometric, i.e., $\|U_\omega \varphi\|_{L_p^n} = \|\varphi\|_{L_p^n}$, $\omega = \zeta, \eta, \alpha, \beta, \gamma, \delta$.

ii) $U_\omega S = S U_\omega$, where S is the operator of singular integration with Cauchy kernel (see bellow), $\omega = \eta, \beta, \gamma, \delta$.

In this paper, we consider the singular integral operator (SIO) with shift $T_\omega : L_p^n(\mathbb{T}) \to L_p^n(\mathbb{T})$, $\omega = \zeta, \eta, \alpha, \beta, \gamma, \delta$, defined by

$$T_\omega = I - cU_\omega P_+; \tag{1}$$

where I is the identity operator, $c \in C^{n\times n}(\mathbb{T})$ is a continuous matrix function, U_ω is the isometric shift operator defined above,

$$P_\pm = \frac{1}{2}(I \pm S)$$

are the mutually complementary projection operators and

$$(S\varphi)(t) = (\pi i)^{-1} \int_{\mathbb{T}} \varphi(\tau)(\tau - t)^{-1} d\tau$$

is the operator of singular integration with the Cauchy kernel.

We note that for the SIO with shift of the form

$$T(A_1, A_2) = A_1 P_+ + A_2 P_-, \tag{2}$$

where

$$A_1 = a_1 I + b_1 U_\omega, \quad A_2 = a_2 I + b_2 U_\omega$$

and $a_1, a_2, b_1, b_2 \in C^{n\times n}(\mathbb{T})$, the Fredholmness conditions and the index formulas are known [7]. The Fredholm criterion can be formulated as follows: the operator $T(A_1, A_2)$ is Fredholm in $L_p^n(\mathbb{T})$ if and only if the functional operators A_1 and A_2 are continuously invertible in $L_p^n(\mathbb{T})$. The spectral properties of the operator $T(A_1, A_2)$ have been less studied (see [3], [8], [9], [10] and [11]), even for the case of a Carleman shift. For the case of a non-Carleman shift, the only works known to the authors are [1], [12] and [13].

We can also write the operator T_ω defined by (1) in the form

$$T_\omega = (I - cU_\omega)P_+ + P_-.$$

So the question of Fredholmness of the operator T_ω is reduced to the question of continuous invertibility of the operator $I - cU_\omega$.

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[1] Given a shift ω, the property i) is always satisfied taking $u_\omega(t) = |\omega'(t)|^{\frac{1}{p}}$. To verify the property ii) the function u_ω has to be chosen depending on the concrete shift ω (see Section 3.2), which is not always possible.

We must say that, in general, in the case of a non-Carleman shift having a finite set of fixed points $\{\tau_1, \tau_2, \ldots, \tau_s\}$, $s \geq 1$, the shift α and the corresponding shift operator U_α considered in Section 3 of this paper, the necessary and sufficient conditions of invertibility for the operator $I - cU_\alpha$, can not be expressed in an explicit form. A specificity of the conditions is expressed by a particular choice of a, so-called, α-solutions of the homogeneous functional equation associated with the operator $I - cU_\alpha$ (see Sections 3.4.1–3.4.11, pp. 118–142, in [7]). The two extreme forms of these conditions are:

Case 1. $\sigma[c(\tau_j)] \subset \mathbb{T}_+$, $j = \overline{1, s}$;

Case 2. $\sigma[c(\tau_j)] \subset \mathbb{T}_-$, $j = \overline{1, s}$, and $\det c(t) \neq 0$ for all $t \in \mathbb{T}$;

here and below, $\mathbb{T}_+$ and $\mathbb{T}_-$ denote the interior and the exterior of the unit disk, $\sigma(g)$, $\rho(g)$ and $\|g\|_2$, denote the spectrum, the spectral radius and the spectral norm of a matrix $g \in \mathbb{C}^{n\times n}$, respectively.

In [1] (see also Chapter 9 in [16]), on the Hilbert space $L_2(\mathbb{T})$, an estimate for the defect number $\dim\ker T_\alpha$ was obtained for the operator $T_\alpha = I - cU_\alpha P_+$, satisfying the condition $|c(\tau_j)| < 1$ (the case 1, with $c \in C(\mathbb{T})$).

In [12], on the Hilbert space $L_2^n(\mathbb{T})$, we obtained estimates for $\dim\ker T_\omega$ for the operator $T_\omega = I - cU_\omega P_+$ ($\omega = \alpha, \beta$) with matrix coefficient, satisfying one of the two sets of Fredholmness conditions: the cases 1 ($\omega = \alpha$) and 2 ($\omega = \beta$), above. In [13], on the real line, analogous estimates were obtained, considering an operator with polynomial coefficient relative to the shift operator, under conditions corresponding to the matrix cases 1 and 2, above.

In the present paper we revisit the mentioned works [12] and [13]. We show that some of the obtained results can be used in the case of an operator with a general shift (Section 2). Then we review our results in [12] and [13]; we propose new proofs and make some minor corrections (Section 3). Finally we formulate the results on the scalar case and consider two examples (Section 4). The "auxiliary results" are formulated on the Banach spaces $L_p^n(\mathbb{T})$, $p \in (1, \infty)$, and the "main results", the estimates for $\dim\ker T_\omega$, on the Hilbert space $L_2^n(\mathbb{T})$. As far as we know this work is the "state of art" of the research on "the very difficult question related to the solvability theory of the SIO of type (2) with a non-Carleman shift" (G.S. Litvinchuk in [16], p. XVI).

2. A SIO with a general shift

2.1. Estimate one

Let us begin considering a general shift $\zeta : \mathbb{T} \to \mathbb{T}$, the associated isometric shift operator U_ζ, and the SIO with shift defined by (1) (with $\omega = \zeta$)

$$T_\zeta = I - cU_\zeta P_+. \tag{3}$$

The following results take place.

Proposition 2.1. *Let*

$$N = I - aU_\zeta P_+, \tag{4}$$

$$R = rP_- + (I - aU_\zeta)P_+, \tag{5}$$

$$M = I - rP_+r^{-1}P_-N^{-1}, \tag{6}$$

where N is an invertible operator; $a, r \in C^{n\times n}(\mathbb{T})$, the matrix function r is invertible and satisfies the condition

$$P_+r^{\pm 1}P_+ = r^{\pm 1}P_+. \tag{7}$$

Then the following equality holds

$$\dim\ker R = \dim\ker M. \tag{8}$$

Proof. Consider the invertible operator[2], whose kernel is trivial,

$$X = (P_+ + P_-r^{-1}P_-)N^{-1}.$$

We show that

$$RX = M.$$

Indeed we have

$$R = rP_- + I - P_- - aU_\zeta P_+ = (r - I)P_- + N$$

and so

$$\begin{aligned} RX &= [(r - I)P_- + N](P_+ + P_-r^{-1}P_-)N^{-1} \\ &= rP_-r^{-1}P_-N^{-1} - P_-r^{-1}P_-N^{-1} + NP_+N^{-1} + NP_-r^{-1}P_-N^{-1} \\ &= r(I - P_+)r^{-1}P_-N^{-1} + I - P_-N^{-1} = M. \end{aligned}$$

Thus, the equality (8) follows. □

Theorem 2.1. *Let $T_\zeta = I - cU_\zeta P_+$, $N = I - aU_\zeta P_+$ and $M = I - rP_+r^{-1}P_-N^{-1}$ be the operators defined by* (3), (4) *and* (6), *respectively, and r an invertible matrix satisfying the condition* (7). *Let $a(t) = r(t)c(t)r^{-1}[\zeta(t)]$. If the operator N is invertible, then the following equality holds*

$$\dim\ker T_\zeta = \dim\ker M. \tag{9}$$

Proof. We can write T_ζ as a product of the operators

$$T_\zeta = I - cU_\zeta P_+ = r^{-1}[rP_- + (I - aU_\zeta)P_+](P_- + rP_+).$$

Both operators $r^{-1}I$ and $P_- + rP_+$ are continuously invertible so that their kernels are trivial. Then

$$\dim\ker T_\zeta = \dim\ker R,$$

where R is the operator defined by (5). Now we apply Proposition 2.1. This implies (9). □

[2]The inverse of the operator $P_+ + P_-r^{-1}P_-$ has the form

$$(P_+ + P_-r^{-1}P_-)^{-1} = (r^{-1}P_+ + P_- + P_+r^{-1}P_-)rI.$$

Proposition 2.2. *Let M be the operator defined by* (6) *and r a $(n \times n)$ polynomial matrix satisfying the condition* (7)*; let*

$$l_1(r) = \sum_{i=1}^{n} \max_{j=\overline{1,n}} l_{i,j}, \tag{10}$$

where $l_{i,j}$ is the degree of the element $r_{i,j}$ of the polynomial matrix r. Then the following inequality holds

$$\dim \ker M \leq l_1(r). \tag{11}$$

Proof. We have

$$p \in \ker M \Leftrightarrow (I - rP_+r^{-1}P_-N^{-1})p = 0 \Leftrightarrow p = rP_+r^{-1}P_-N^{-1}p.$$

Making use of (7) and $P_+ = I - P_-$ we get

$$p = -P_+rP_-r^{-1}P_-N^{-1}p,$$

which means that p belongs to the image of the finite-dimensional operator P_+rP_-, i.e., $\ker M \subset \operatorname{im} P_+rP_-$. It is easy to see that

$$\dim \operatorname{im} P_+rP_- = l_1(r).$$

From this fact follows the inequality (11). □

We can state the following result.

Theorem 2.2. *Let $T_\zeta = I - cU_\zeta P_+$ be the operator defined by* (3) *on the Hilbert space $L_2^n(\mathbb{T})$, and r a polynomial matrix satisfying the conditions* (7) *and*

$$\max_{t\in\mathbb{T}} \left\| r(t)c(t)r^{-1}[\zeta(t)] \right\|_2 < 1. \tag{12}$$

Let R_c be the set of all such matrices r, $l_1(r)$ be the number defined by (10) *for each matrix r and*

$$l(c) = \min_{r\in R_c} \{l_1(r)\}. \tag{13}$$

If the set R_c is not empty, then the following estimate holds

$$\dim \ker T_\zeta \leq l(c).$$

Proof. We set $a(t) = r(t)c(t)r^{-1}[\zeta(t)]$; with (12) we can show that the operator defined by (4) is invertible. Indeed, since $\max_{t\in\mathbb{T}} \|a(t)\|_2 < 1$, $\|U_\zeta\|_{L_2} = 1$ and $\|P_+\|_{L_2} = 1$, it follows that $N = I - aU_\zeta P_+$ is an invertible operator whose inverse is given by the Neumann series

$$N^{-1} = I + aU_\zeta P_+ + (aU_\zeta P_+)^2 + \cdots .$$

Taking into account Theorem 2.1 and Proposition 2.2, the result follows. □

2.2. Estimate two

The following result takes place.

Proposition 2.3. *Let*

$$A : L_p^n(\mathbb{T}) \to L_p^n(\mathbb{T}),$$

be a linear bounded operator, and

$$D = \operatorname{diag}\{D_-, D_+\},$$

a diagonal matrix, where

$$D_- = \operatorname{diag}\{t^{k_1}, t^{k_2}, \dots, t^{k_p}\}, \quad D_+ = \operatorname{diag}\{t^{m_1}, t^{m_2}, \dots, t^{m_q}\},$$
$$k_j < 0, \quad j = \overline{1,p}, \quad m_i \geq 0, \quad i = \overline{1,q}, \quad p + q = n.$$

The following inequalities hold

$$\dim\ker(P_- + AP_+) + \sum_{j=1}^{p} k_j \leq \dim\ker(DP_- + AP_+), \tag{14}$$

$$\dim\ker(DP_- + AP_+) \leq \dim\ker(P_- + AP_+) + \sum_{i=1}^{q} m_i. \tag{15}$$

Proof. Let E_n denote the $(n \times n)$ identity matrix,

$$D_1 = \operatorname{diag}\{D_-, E_q\}, \quad D_2 = \operatorname{diag}\{E_p, D_+\};$$

and consider the left invertible operators

$$B_1 = D_1 P_- + P_+, \quad B_2 = D_2^{-1} P_- + P_+.$$

Then we have

$$\dim\ker B_1 = \dim\ker B_2 = 0,$$
$$\dim\operatorname{coker} B_1 = -\sum_{j=1}^{p} k_j, \quad \dim\operatorname{coker} B_2 = \sum_{i=1}^{q} m_i.$$

The following equality holds

$$\begin{aligned}(DP_- + AP_+)B_2 &= (P_- + AP_+)B_1 \\ &= B,\end{aligned} \tag{16}$$

where we denote

$$B = D_1 P_- + AP_+.$$

It follows from (16) that

$$\dim\ker B \leq \dim\ker(DP_- + AP_+)$$

and

$$\dim\ker(DP_- + AP_+) \leq \dim\ker B + \dim\operatorname{coker} B_2;$$

then

$$\dim\ker B \leq \dim\ker(DP_- + AP_+) \leq \dim\ker B + \sum_{i=1}^{q} m_i. \tag{17}$$

Moreover, also it follows from (16) that

$$\dim\ker B \le \dim\ker(P_- + AP_+)$$

and

$$\dim\ker(P_- + AP_+) \le \dim\ker B + \dim\operatorname{coker} B_1;$$

then

$$\dim\ker B \le \dim\ker(P_- + AP_+) \le \dim\ker B - \sum_{j=1}^{p} k_j. \tag{18}$$

Putting together (17) and (18) we obtain the inequalities (14) and (15). □

Consider now a shift η such that the corresponding shift operator U_η satisfies the additional property

$$U_\eta S = SU_\eta; \tag{19}$$

and the SIO with shift (1) (with $\omega = \eta$)

$$T_\eta = I - cU_\eta P_+. \tag{20}$$

Moreover we suppose that the matrix function $c \in C^{n\times n}(\mathbb{T})$ has the property

$$\det c(t) \ne 0, \quad \forall t \in \mathbb{T}. \tag{21}$$

Under condition (21) the continuous matrix function c admits the factorization in $L_p^{n\times n}(\mathbb{T})$ (see, for instance, Theorem 1.1, p. 165, in [2]; see also [17])

$$c = c_- \Lambda c_+, \tag{22}$$

where

$$c_-^{\pm 1} \in \left[L_p^-(\mathbb{T})\right]^{n\times n}, \quad c_+^{\pm 1} \in \left[L_p^+(\mathbb{T})\right]^{n\times n}, \quad \Lambda = \operatorname{diag}\{t^{\varkappa_j}\},$$

$\varkappa_j \in \mathbb{Z}$, $j = \overline{1,n}$, with $\varkappa_1 \ge \varkappa_2 \ge \cdots \ge \varkappa_n$, and, as usual, $L_p^+ = P_+L_p$ and $L_p^- = P_-L_p \oplus \mathbb{C}$. The integers $\varkappa_j$ are uniquely defined by the matrix function c and are called its partial indices. It is assumed that

$$c_\pm^{\pm 1} \in C^{n\times n}(\mathbb{T}). \tag{23}$$

We continue with the following result.

Theorem 2.3. *Let T_η be the operator defined by* (20)*, where $c \in C^{n\times n}(\mathbb{T})$ satisfies the conditions* (21)*,* (22) *and* (23)*; then the following estimate holds*

$$\dim\ker T_\eta \le \dim\ker(I - \widetilde{c}U_\eta^{-1}P_+) + \sum_{\varkappa_j<0} |\varkappa_j|, \tag{24}$$

where $\widetilde{c} = c_+c^{-1}c_+^{-1}(\eta_{-1})$.

Proof. We multiply T_η by the continuously invertible operator[3],

$$Y = c_-P_- - c_+^{-1}(\eta_{-1})U_\eta^{-1}P_+.$$

[3]The inverse of the operator Y has the form

$$c_-^{-1}P_- - c_+(\eta_{-1})U_\eta P_+.$$

We have

$$\begin{aligned} T_\eta Y &= (I - c_-\Lambda c_+ U_\eta P_+)(c_-P_- - c_+^{-1}(\eta_{-1})U_\eta^{-1}P_+) \\ &= c_-\Lambda(DP_- + AP_+), \end{aligned}$$

where we make use of (19) and had denoted

$$\begin{aligned} D &= \Lambda^{-1} = \operatorname{diag}\{t^{-\varkappa_j}\}, \\ A &= I - \widetilde{c}U_\eta^{-1}, \\ \widetilde{c} &= \Lambda^{-1}c_-^{-1}c_+^{-1}(\eta_{-1}) = c_+c^{-1}c_+^{-1}(\eta_{-1}). \end{aligned}$$

As $c_-\Lambda$ is a continuously invertible operator, we get

$$\dim\ker T_\eta = \dim\ker(DP_- + AP_+).$$

We have

$$DP_- + AP_+ = I - \widetilde{c}U_\eta^{-1}P_+.$$

The inequality (15) in Proposition 2.3, with $m_j = -\varkappa_j$, $\varkappa_j < 0$ ($m_j \geq 0$), yields the estimate (24). □

Now, supposing that the operator $I - \widetilde{c}U_\eta^{-1}P_+$ is under the conditions of Theorem 2.2, we can state the following result.

Theorem 2.4. *Let $T_\eta = I - cU_\eta P_+$ be the operator defined by* (20) *on the Hilbert space $L_2^n(\mathbb{T})$, where $c \in C^{n\times n}(\mathbb{T})$ satisfies the conditions* (21), (22) *and* (23)*; and r a polynomial matrix satisfying the conditions* (7) *and*

$$\max_{t\in\mathbb{T}} \left\| r(t)\widetilde{c}(t)r^{-1}[\eta(t)] \right\|_2 < 1,$$

where $\widetilde{c} = c_+c^{-1}c_+^{-1}(\eta_{-1})$. Let $R_{\widetilde{c}}$ be the set of all such matrices r and $l(\widetilde{c})$ the number defined by (13) *for the matrix $\widetilde{c}$.*

If the set $R_{\widetilde{c}}$ is not empty, then the following estimate holds

$$\dim\ker T_\eta \leq l(\widetilde{c}) + \sum_{\varkappa_j<0} |\varkappa_j|,$$

where $\varkappa_j \in \mathbb{Z}$, $j = \overline{1,n}$ are the partial indices of the matrix c.

Proof. Since the operators U_η and U_η^{-1} verify similar properties, the operator $I - \widetilde{c}U_\eta^{-1}P_+$ satisfies all the conditions of Theorem 2.2; thus

$$\dim\ker(I - \widetilde{c}U_\eta^{-1}P_+) \leq l(\widetilde{c}).$$

With (24) the result follows. □

2.3. An operator with polynomial coefficient relative to the shift operator

Now let us consider the SIO with shift of the form

$$K_\eta = A_\eta P_+ + P_- : L_p(\mathbb{T}) \to L_p(\mathbb{T}), \tag{25}$$

where

$$A_\eta = I + \sum_{i=1}^{n} a_i U_\eta^i,$$

$a_i \in C(\mathbb{T})$, $i = \overline{1, n}$, and U_η is the shift operator satisfying the property (19).

Consider also the matrix operator (see [13], [7], and [14])

$$\widetilde{T}_\eta = \widetilde{A}_\eta P_+ + P_- : L_p^n(\mathbb{T}) \to L_p^n(\mathbb{T}), \tag{26}$$

with

$$\widetilde{A}_\eta = I + aU_\eta,$$

where

$$a = \begin{pmatrix} a_1 & a_2 & \cdots & a_{n-1} & a_n \\ & & -E_{n-1} & & O_{(n-1)\times 1} \end{pmatrix}.$$

The following result holds

Proposition 2.4. *Let K_η and $\widetilde{T}_\eta$ be the operators defined by* (25) *and* (26)*, respectively. The operator K_η is a Fredholm operator on $L_p(\mathbb{T})$ if and only if the operator $\widetilde{T}_\eta$ is a Fredholm operator on $L_p^n(\mathbb{T})$. In the affirmative case,* $\dim\ker K_\eta = \dim\ker \widetilde{T}_\eta$ *and* $\dim\operatorname{coker} K_\eta = \dim\operatorname{coker} \widetilde{T}_\eta$.

Proof. This result is formulated in [13], Proposition 2.1, on the real line, considering the shift $\beta_r(t) = t + \mu$, $t \in \overset{\circ}{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$, μ is a fixed real number. Since $U_\eta P_+ = P_+ U_\eta$, we can prove this result on the unit circle, in a similar way. □

Obviously the operator $\widetilde{T}_\eta$ is a particular case of the operator T_η defined by (20). Then, taking into account Proposition 2.4, Theorems 2.2 and 2.4 can be used to study the operator K_η.

3. A SIO with a non-Carleman shift

The estimate of the dimension of the kernel of the operator T_ω, $\omega = \zeta, \eta, \alpha, \beta, \gamma, \delta$, is related with the construction of the polynomial matrix r (see Theorems 2.2 and 2.4); below we perform this task, in the case of a non-Carleman shift, $\omega = \alpha, \beta, \delta$, under certain conditions for the operator T_ω: cases 1 and 2 mentioned in the Introduction. Indeed, then we show that the set R_c ($R_{\widetilde{c}}$) introduced in Theorem 2.2 (Theorem 2.4) is not empty in case 1 (case 2).

3.1. Case 1

On the Hilbert space $L_2^n(\mathbb{T})$, we consider the SIO

$$T_\alpha = I - cU_\alpha P_+, \tag{27}$$

with a non-Carleman shift $\alpha : \mathbb{T} \to \mathbb{T}$, which has a finite set of fixed points $\{\tau_1, \tau_2, \dots, \tau_s\}$, $s \geq 1$; U_α is the associated isometric shift operator.

The following results take place[4].

Proposition 3.1. *For every continuous matrix function $d \in C^{n\times n}(\mathbb{T})$ such that*

$$\sigma[d(\tau_j)] \subset \mathbb{T}_+, \quad j = \overline{1,s}, \tag{28}$$

there exists a polynomial matrix r satisfying the conditions

$$\max_{t\in\mathbb{T}} \left\| r(t)d(t)r^{-1}[\alpha(t)] \right\|_2 < 1 \tag{29}$$

and

$$P_+ r^{\pm 1} P_+ = r^{\pm 1} P_+. \tag{30}$$

Proof. We consider only the case when $\max_{t\in\mathbb{T}} \|d(t)\|_2 > 1$, because otherwise we have simply $r = E_n$.

Let

$$\rho_j \equiv \rho[d(\tau_j)], \quad j = \overline{1,s}.$$

Under condition (28) naturally we have that

$$\rho_j < 1, \quad j = \overline{1,s}.$$

Then, for each matrix $d(\tau_j) \in \mathbb{C}^{n\times n}$ satisfying the condition (28), there exists a non-singular matrix $B_j \in \mathbb{C}^{n\times n}$ such that (see, for instance, p. 316 in [4])

$$\left\| B_j d(\tau_j) B_j^{-1} \right\|_2 < 1, \quad j = \overline{1,s}.$$

Now let B be the non-singular polynomial matrix, without zeros on the closure of $\mathbb{T}_+$, defined by (see, for instance, Sections 0.9.11 in [4] and 6.1 in [5])

$$B(t) = B_1 L_1(t) + B_2 L_2(t) + \cdots + B_s L_s(t), \tag{31}$$

where

$$L_j(t) = \frac{\prod\limits_{\substack{i=1\\ i\neq j}}^{s} (t-\tau_i)}{\prod\limits_{\substack{i=1\\ i\neq j}}^{s} (\tau_j-\tau_i)}, \quad j = \overline{1,s},$$

are the Lagrange interpolating polynomials.

Then we define the continuous matrix function

$$b(t) = B(t)d(t)B^{-1}[\alpha(t)]. \tag{32}$$

[4]Proposition 3.1 substitutes Proposition 2.1 in [12].

We represent the function $b(t)$ in the form

$$b(t) = u(t)v(t), \tag{33}$$

where

$$u(t) \in C^{n\times n}(\mathbb{T}), \quad \max_{t\in\mathbb{T}} \|u(t)\|_2 = \gamma < 1, \tag{34}$$

and $v(t)$ is a continuous real-valued function on $\mathbb{T}$ such that

$$v(t) \geq \delta > 0, \quad t \in \mathbb{T}, \tag{35}$$

$$v(\tau_j) < 1, \quad j = \overline{1, s}. \tag{36}$$

Proceeding analogously as in the proof of Lemma 2.1 in [1], we construct a continuous real positive function $f(t)$ by the following way

$$f(t) = v[\alpha_{-1}(t)] + v[\alpha_{-1}(t)]v[\alpha_{-2}(t)] + \cdots + \prod_{k=1}^{m} v[\alpha_{-k}(t)],$$

where $\alpha_{-1}[\alpha(t)] \equiv t$, $\alpha_{-k}(t) \equiv \alpha_{-1}[\alpha_{-k+1}(t)]$.

Taking into account (35) we have

$$\inf_{t\in\mathbb{T}}\{f(t)\} \geq \delta > 0. \tag{37}$$

Moreover, due to conditions (35) and (36) and known properties of a shift function $\alpha(t)$ (see, for instance, Lemma 2, p. 28, in [7]; see also Lemma 2.2, p. 23, in [15]), the following inequality is valid for a sufficiently large m:

$$\prod_{k=1}^{m} v[\alpha_{-k}(t)] < 1. \tag{38}$$

With (38) we can show that

$$f[\alpha(t)] \geq f(t)v(t). \tag{39}$$

Now we introduce the function (see, for instance, Chapter IV in [6]; see also Chapter 4 in [20])

$$g(z) = \exp\left\{\frac{1}{2\pi}\int_{\mathbb{T}} \frac{t+z}{t-z} \ln f(t)|dt|\right\}.$$

The function $g(z)$ is continuous on $\mathbb{T}$, analytic on $\mathbb{T}_+$, and satisfies the following properties

a) $|g(t)| = f(t)$, $\forall t \in \mathbb{T}$;
b) $g(z) \neq 0$, $\forall z \in \mathbb{T} \cup \mathbb{T}_+$;
c) $g(t)$ can be uniformly approximated on the closure of $\mathbb{T}_+$ by a polynomial of a finite degree with any prescribed exactness ε (see, for instance, Volume 3, Section 12, pp. 97–100, in [18]; see also Section IV.E.1, p. 78, in [6]); due to property b) all the zeros of this polynomial belong to $\mathbb{T}_-$. Let $s_0(t)$ be such a polynomial and $\varepsilon < \delta$; therefore

$$|g(t) - s_0(t)| < \varepsilon < \delta. \tag{40}$$

Then we consider

$$s(t) = E_n s_0(t).$$

Taking into account (33), (34), (37), (39), (40) and the property a) above, we can estimate the norm of the function $s(t)b(t)s^{-1}[\alpha(t)]$. We have

$$\max_{t\in\mathbb{T}} \left\| s(t)b(t)s^{-1}[\alpha(t)] \right\|_2 \le \gamma \left\| s_0(t)v(t)s_0^{-1}[\alpha(t)] \right\|_{C(\mathbb{T})}.$$

Compare with (2.9), p. 6, in [1]; from here, doing exactly as in [1], pp. 6–7, we obtain

$$\max_{t\in\mathbb{T}} \left\| s(t)b(t)s^{-1}[\alpha(t)] \right\|_2 < 1.$$

Finally, recalling (32) and setting $r(t) = s(t)B(t)$, we obtain the inequality (29). Moreover, the polynomial matrix $r(t) = s(t)B(t) = E_n s_0(t)B(t)$ satisfies the condition (30) (recall (31) and the property c) above). □

Theorem 3.1. *Let $T_\alpha = I - cU_\alpha P_+$ be the operator defined by* (27)*, where $c \in C^{n\times n}(\mathbb{T})$ satisfies the condition* (28)*. Then the following estimate holds*

$$\dim\ker T_\alpha \le l(c),$$

where $l(c)$ is the number defined by (13) *for the matrix c.*

Proof. According to Proposition 3.1, there exists a polynomial matrix r such that the conditions (29) and (30) are verified for the matrix c. Taking into account Theorem 2.2, the result follows. □

3.2. Case 2

Now we consider a linear fractional non-Carleman shift preserving the orientation on $\mathbb{T}$

$$\beta(t) = \frac{at+b}{\overline{b}t+\overline{a}}, \quad t\in\mathbb{T}, \tag{41}$$

where $a, b \in \mathbb{C}$ are such that $|a|^2 - |b|^2 = 1$. This shift has two fixed points, τ_1 and τ_2 [5], given by the formula

$$\tau_{1,2} = \frac{a-\overline{a} \pm \sqrt{(a+\overline{a})^2 - 4}}{2\overline{b}}.$$

Obviously $\tau_1 \ne \tau_2$ if $|\mathrm{Re}\, a| \ne 1$.

[5]We note that in the case of a linear fractional shift with two fixed points, τ_1 and τ_2, the matrix defined by (31) in the proof of Proposition 3.1 has the simple form

$$B(t) = B_1 \frac{t-\tau_2}{\tau_1-\tau_2} + B_2 \frac{t-\tau_1}{\tau_2-\tau_1}.$$

In the case of a linear fractional shift having one fixed point ($\tau_1 = \tau_2$ if $|\mathrm{Re}\, a| = 1$, as we see below) the matrix defined by (31) is simply

$$B(t) = B_1 \in \mathbb{C}^{n\times n}.$$

To this case corresponds the shift on the real line $\beta_r(t) = t+\mu$, $t \in \overset{\circ}{\mathbb{R}} = \mathbb{R}\cup\{\infty\}$, μ is a fixed real number; the shift $\beta_r(t)$ has the only fixed point at infinity.

The shift $\beta(t)$ admits the factorization

$$\beta(t) = \beta_+(t) t \beta_-(t),$$

where

$$\beta_+(t) = \frac{1}{\overline{b}t + \overline{a}}, \quad \beta_-(t) = \frac{at+b}{t}.$$

We see that the functions $\beta_\pm, \beta_\pm^{-1}$ are analytic in $\mathbb{T}_\pm$ and continuous in the closure of $\mathbb{T}_\pm$, respectively.

For the linear fractional shift $\beta(t)$, it is convenient to consider the isometric shift operator

$$(U_\beta \varphi)(t) = \beta_+(t)\varphi[\beta(t)], \tag{42}$$

because U_β satisfies the additional property

$$U_\beta S = S U_\beta.$$

Then we consider the operator (1) (with $\omega = \beta$)

$$T_\beta = I - cU_\beta P_+ : L_2^n(\mathbb{T}) \to L_2^n(\mathbb{T}), \tag{43}$$

where we suppose now that $c \in C^{n\times n}(\mathbb{T})$ has the properties

$$\begin{aligned} &\sigma[c(\tau_j)] \subset \mathbb{T}_-, \quad && j = 1, 2, \\ &\det c(t) \neq 0, && \forall t \in \mathbb{T}. \end{aligned} \tag{44}$$

The non-singular continuous matrix function c admits the factorization (22) in $L_2^{n\times n}(\mathbb{T})$ and (23) is assumed. Then we apply Theorem 2.3 to the operator (43); this implies the estimate

$$\dim\ker T_\beta \le \dim\ker(I - \widetilde{c}U_\beta^{-1}P_+) + \sum_{\varkappa_j<0} |\varkappa_j|, \tag{45}$$

where $\widetilde{c} = c_+ c^{-1} c_+^{-1}(\beta_{-1})$.

Now we analyze the operator $I - \widetilde{c}U_\beta^{-1}P_+$.

We note that the matrices $\widetilde{c}(t)$ and $c^{-1}(t)$ are similar at the fixed points of the shift; indeed at τ_j, $j = 1, 2$,

$$\widetilde{c} = c_+ c^{-1} c_+^{-1}.$$

We have that $\sigma[c(\tau_j)] \subset \mathbb{T}_-$; then

$$\sigma[c^{-1}(\tau_j)] = \sigma[\widetilde{c}(\tau_j)] \subset \mathbb{T}_+, \quad j = 1, 2.$$

Therefore the operator $I - \widetilde{c}U_\beta^{-1}P_+$ satisfies all the conditions of Theorem 3.1; thus

$$\dim\ker(I - \widetilde{c}U_\beta^{-1}P_+) \le l(\widetilde{c}).$$

Finally, with (45) we get the following estimate.

Theorem 3.2. *Let* $T_\beta = I - cU_\beta P_+$ *be the operator defined by* (43), *where* $c \in C^{n\times n}(\mathbb{T})$ *satisfies the conditions* (44), (21), (22) *and* (23). *Then the following estimate holds*

$$\dim\ker T_\beta \le l(\widetilde{c}) + \sum_{\varkappa_j<0} |\varkappa_j|\,,$$

where $l(\widetilde{c})$ *is the number defined by* (13) *for the matrix* $\widetilde{c} = c_+ c^{-1} c_+^{-1}(\beta_{-1})$ *and* $\varkappa_j \in \mathbb{Z}$, $j = \overline{1,n}$ *are the partial indices of the matrix* c.

3.3. The case of an operator with polynomial coefficient relative to the shift operator

Let $\operatorname{ind} f$ denote the Cauchy index of a continuous function $f \in C(\mathbb{T})$, i.e.,

$$\operatorname{ind} f = \frac{1}{2\pi}\{\arg f(t)\}_{t\in\mathbb{T}};$$

as usual, $\operatorname{Ind} K$ denotes the index of a Fredholm operator K, i.e.,

$$\operatorname{Ind} K = \dim\ker K - \dim\operatorname{coker} K.$$

Now we consider the SIO with shift of the form

$$K_\beta = A_\beta P_+ + P_- : L_2(\mathbb{T}) \to L_2(\mathbb{T}), \tag{46}$$

where

$$A_\beta = I + \sum_{i=1}^{n} a_i U_\beta^i, \tag{47}$$

$a_i \in C(\mathbb{T})$, $i = \overline{1,n}$, and U_β is the shift operator defined by (42).

We define the nth degree polynomials

$$A_j(\xi) = 1 + \sum_{i=1}^{n} a_i(\tau_j)\xi^i, \quad \xi \in \mathbb{T}, \quad j = 1,2, \tag{48}$$

where τ_1 and τ_2 are the fixed points of the linear fractional shift β.

The invertibility of the operator (47) implies (see Section 3.4.12, pp. 142–145, in [7])

$$A_j(\xi) \ne 0, \quad \xi \in \mathbb{T}, \quad j = 1,2, \tag{49}$$

and the equality between the Cauchy indices of the polynomials A_1 and A_2

$$\operatorname{ind} A_1 = \operatorname{ind} A_2 =: \operatorname{ind} A, \tag{50}$$

where "=:" means "we denote". Therefore

$$0 \le \operatorname{ind} A \le n.$$

Consider also the matrix operator

$$\widetilde{T}_\beta = \widetilde{A}_\beta P_+ + P_- : L_2^n(\mathbb{T}) \to L_2^n(\mathbb{T}), \tag{51}$$

with

$$\widetilde{A}_\beta = I + aU_\beta, \tag{52}$$

where

$$a = \begin{pmatrix} a_1 & a_2 & \cdots & a_{n-1} & a_n \\ & & -E_{n-1} & & O_{(n-1)\times 1} \end{pmatrix}. \tag{53}$$

Now we define the polynomials

$$\widetilde{A}_j(\xi) = \det[E_n + a(\tau_j)\xi], \quad \xi \in \mathbb{T}, \quad j = 1, 2.$$

The invertibility of the operator (52) implies (see Section 3.4.1, pp. 118–120, in [7])

$$\widetilde{A}_j(\xi) \neq 0, \quad \xi \in \mathbb{T}, \quad j = 1, 2,$$

and the equality

$$\operatorname{ind} \widetilde{A}_1 = \operatorname{ind} \widetilde{A}_2 =: \operatorname{ind} \widetilde{A}.$$

Therefore

$$0 \leq \operatorname{ind} \widetilde{A} \leq n.$$

We note that

$$\widetilde{A}_j(\xi) = A_j(\xi), \quad \xi \in \mathbb{T}, \quad j = 1, 2. \tag{54}$$

Then

$$\operatorname{ind} \widetilde{A} = \operatorname{ind} A =: \operatorname{ind} A.$$

Furthermore, denoting by $\lambda_i(a, \tau_j)$, the eigenvalues of the matrices $a(\tau_j)$, $j = 1, 2$, and by $\xi_i(\tau_j)$ the roots of the polynomials $A_j(\xi)$, $j = 1, 2$, we have that

$$\lambda_i^{-1}(a, \tau_j) = -\xi_i(\tau_j), \quad i = \overline{1, n}, \quad j = 1, 2,$$

taking into account equality (54). Therefore we note that $\operatorname{ind} A$ coincides with the number of the roots of the polynomials $A_j(\xi)$, $j = 1, 2$, that are situated inside the unit disk, or, equivalently, with the eigenvalues of the matrices $a(\tau_j)$, $j = 1, 2$, that are situated outside the unit disk.

Analogously to the matrix case, the necessary and sufficient conditions of invertibility for the operator (47), and so the Fredholmness conditions for the operator (46), can not be expressed in an explicit form. In the two extreme cases, these conditions can be written in a simple form (see Section 3.5.1, pp. 148–151, in [7]):

Proposition 3.2. [7] *Let K_β be the operator defined by* (46), *$A_j(\xi)$, be the polynomials defined by* (48) *and let* (49) *and* (50) *be fulfilled. The following assertions hold:*

i) *If* $\operatorname{ind} A = 0$, *then the operator K_β is Fredholm and*

$$\operatorname{Ind} K_\beta = 0.$$

ii) *If* $\operatorname{ind} A = n$ *and* $a_n(t) \neq 0$ *for all* $t \in \mathbb{T}$, *then the operator K_β is Fredholm and*

$$\operatorname{Ind} K_\beta = \operatorname{ind} a_n^{-1}.$$

Remark. If $n = 1$, then the conditions i) and ii) of Proposition 3.2 are not only sufficient but also necessary for the Fredholmness of the operator K_β. This particular case is treated in Corollaries 4.3 and 4.4 below.

Then we can state the following results.

Theorem 3.3. *Let K_β be the operator defined by* (46)*, the conditions of Proposition* 3.2 *be satisfied and a be the matrix function defined by* (53)*. If* $\operatorname{ind} A = 0$*, then the following estimate holds*

$$\dim\ker K_\beta \leq l(a),$$

where $l(a)$ is the number defined by (13) *for the matrix a.*

Proof. Since $\operatorname{ind} A = 0$, i.e., $\sigma[a(\tau_j)] \subset \mathbb{T}_+$, $j = 1, 2$, we can apply Theorem 3.1 to the matrix operator defined by (51). Taking into account Proposition 2.4, the result follows. □

Theorem 3.4. *Let K_β be the operator defined by* (46)*, the conditions of Proposition* 3.2 *be satisfied and a be the matrix function defined by* (53)*. If* $\operatorname{ind} A = n$ *and $a_n(t) \neq 0$ for all $t \in \mathbb{T}$, then the following estimate holds*

$$\dim\ker K_\beta \leq l(\widetilde{a}) + \sum_{\varkappa_j < 0} |\varkappa_j|\,,$$

where $l(\widetilde{a})$ is the number defined by (13) *for the matrix $\widetilde{a} = a_+ a^{-1} a_+^{-1}(\beta_{-1})$, $a_\pm$ and $\varkappa_j \in \mathbb{Z}$, $j = \overline{1,n}$ are the external factors and the partial indices, respectively, of the factorization* (22) *of the matrix a, and* (23) *is fulfilled.*

Proof. As $a_n(t) \neq 0$, we have that $\det a(t) \neq 0$ for all $t \in \mathbb{T}$; then the continuous matrix function a admits the factorization (22).

Since $\operatorname{ind} A = n$, i.e., $\sigma[a(\tau_j)] \subset \mathbb{T}_-$, $j = 1, 2$, we can apply Theorem 3.2 to the matrix operator defined by (51). With Proposition 2.4 we are done. □

Corollary 3.1. *Let the conditions of Theorem* 3.4 *be satisfied; if $n = 2$, then the following estimate holds*

$$\dim\ker K_\beta \leq l(\widetilde{a}) + \max(0, -\operatorname{ind} a_2).$$

Proof. If $n = 2$ the matrix a has the form

$$a = \begin{pmatrix} a_1 & a_2 \\ -1 & 0 \end{pmatrix}.$$

We compute

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} a^T = \begin{pmatrix} a_2 & 0 \\ a_1 & -1 \end{pmatrix}.$$

The partial indices $\varkappa_{1,2}$ of the factorization

$$a^T = b_+ \Lambda b_-,$$

are equal to $\operatorname{ind} a_2$ and 0 if $\operatorname{ind} a_2 \le 1$ (see pp. 147–148 in [17]). Obviously the partial indices of the factorization

$$a = a_- \Lambda a_+,$$

are the same. Therefore negative partial indices are possible only if $\operatorname{ind} a_2 < 0$ and it follows the result. □

4. On the scalar case

4.1. The case of a general shift

Let us formulate the obtained results for the operator (3) in the scalar case:

$$T_\zeta = I - cU_\zeta P_+ : L_2(\mathbb{T}) \to L_2(\mathbb{T}). \tag{55}$$

Corollary 4.1. *Let T_ζ be the operator defined by* (55)*; if there exists a polynomial r of degree m, with zeros in $\mathbb{T}_-$,*

$$r(t) = \prod_{k=1}^{m} (t - \lambda_k), \quad |\lambda_k| > 1, \quad k = \overline{1, m},$$

such that

$$\left| r(t) c(t) r^{-1}[\zeta(t)] \right| < 1, \quad \forall t \in \mathbb{T}, \tag{56}$$

then

$$\dim \ker T_\zeta \le m.$$

Proof. Follows from Theorem 2.2 with $n = 1$. □

Now we consider the operator (20) in the scalar case:

$$T_\eta = I - cU_\eta P_+ : L_2(\mathbb{T}) \to L_2(\mathbb{T}), \tag{57}$$

where $c \in C(\mathbb{T})$ has the property

$$c(t) \ne 0, \quad \forall t \in \mathbb{T}. \tag{58}$$

The continuous function c admits the factorization in $L_2(\mathbb{T})$

$$c = c_- t^{\varkappa} c_+, \tag{59}$$

where

$$c_-^{\pm 1} \in L_2^-(\mathbb{T}), \quad c_+^{\pm 1} \in L_2^+(\mathbb{T}), \quad \varkappa = \operatorname{ind} c.$$

It is assumed that

$$c_\pm^{\pm 1} \in C(\mathbb{T}). \tag{60}$$

Suppose that a polynomial matrix r, satisfying the condition (56) for the function c and the shift η, does not exist, but there exists such one that (56) holds for the function $\widetilde{c} = c_+ c^{-1} c_+^{-1}(\eta_{-1})$. In this case we can state the following result.

Corollary 4.2. *Let T_η be the operator defined by* (57)*. Then the following estimate holds*

$$\dim\ker T_\eta \le m + \max(0, -\operatorname{ind} c),$$

where m is the degree of the polynomial r defined in Corollary 4.1 *for the function $\widetilde{c} = c_+ c^{-1} c_+^{-1}(\eta_{-1})$ and* $\operatorname{ind} c$ *is the Cauchy index of the function c.*

Proof. Follows from Theorem 2.4 with $n = 1$. □

4.2. The case of a non-Carleman shift

Consider the operator (27) on $L_2(\mathbb{T})$, with $c \in C(\mathbb{T})$,

$$T_\alpha = I - cU_\alpha P_+. \tag{61}$$

Corollary 4.3. *For every continuous function $c \in C(\mathbb{T})$ such that*

$$|c(\tau_j)| < 1, \quad j = \overline{1, s},$$

there exists a polynomial r of degree m, with zeros in $\mathbb{T}_-$,

$$r(t) = \prod_{k=1}^{m} (t - \lambda_k), \quad |\lambda_k| > 1, \quad k = \overline{1, m},$$

such that

$$\left| r(t)c(t)r^{-1}[\alpha(t)] \right| < 1, \quad \forall t \in \mathbb{T}. \tag{62}$$

Moreover

$$\dim\ker T_\alpha \le m,$$

where T_α is the operator defined by (61)*.*

Proof. [1]; follows from Theorem 3.1 with $n = 1$. □

Now consider the operator (43) on $L_2(\mathbb{T})$,

$$T_\beta = I - cU_\beta P_+, \tag{63}$$

where $c \in C(\mathbb{T})$ satisfies the properties (58), (59), (60) and

$$|c(\tau_j)| > 1, \quad j = 1, 2.$$

Corollary 4.4. *Let T_β be the operator defined by* (63)*. Then the following estimate holds*

$$\dim\ker T_\beta \le m + \max(0, -\operatorname{ind} c),$$

where m is the degree of the polynomial r defined in Corollary 4.3 *for the function $\widetilde{c} = c_+ c^{-1} c_+^{-1}(\beta_{-1})$ and* $\operatorname{ind} c$ *is the Cauchy index of the function c.*

Proof. Follows from Theorem 3.2 with $n = 1$. □

4.3. Two examples

Example 1. Let us consider the following shift, a rotation on the unit circle,

$$\gamma(t) = e^{i\theta}t, \quad \theta \in [0, 2\pi[, \quad t \in \mathbb{T}.$$

Note that, if $\frac{\theta}{2\pi} \in \mathbb{Q}$, then γ is a Carleman shift, and if $\frac{\theta}{2\pi} \in \mathbb{R} \setminus \mathbb{Q}$, then γ is a non-Carleman shift with an empty set of periodic points (a so-called ergodic shift). The corresponding isometric shift operator is defined as follows

$$(U_\gamma \varphi)(t) = \varphi(e^{i\theta}t).$$

Then we consider the SIO with shift on $L_2(\mathbb{T})$, with $c \in C(\mathbb{T})$,

$$T_\gamma = I - cU_\gamma P_+. \tag{64}$$

We recall the inequality (56)

$$\left| r(t)c(t)r^{-1}[\gamma(t)] \right| < 1,$$

or

$$|c(t)| < \left| r^{-1}(t) r[\gamma(t)] \right| \Leftrightarrow |c(t)|^2 < \left| r^{-1}(t) r[\gamma(t)] \right|^2 \Leftrightarrow$$
$$|c(t)|^2 < \left| \prod_{k=1}^{m} \frac{\gamma(t) - \lambda_k}{t - \lambda_k} \right|^2 \Leftrightarrow |c(t)|^2 < \prod_{k=1}^{m} \left| \frac{\gamma(t) - \lambda_k}{t - \lambda_k} \right|^2 .$$

Let

$$t = e^{ix} = \cos x + i \sin x, \quad x \in]-\pi, \pi],$$

and

$$\lambda_k = \rho_k e^{i\varphi_k} = \rho_k \cos\varphi_k + i\rho_k \sin\varphi_k, \quad \rho_k > 1, \quad \varphi_k \in]-\pi, \pi], \quad k = \overline{1, m}.$$

Then we have

$$\left| c(e^{ix}) \right|^2 < \prod_{k=1}^{m} y(x, \rho_k, \varphi_k), \tag{65}$$

where we denote

$$y(x, \rho_k, \varphi_k) = \frac{1 + \rho_k^2 - 2\rho_k \cos(x + \theta - \varphi_k)}{1 + \rho_k^2 - 2\rho_k \cos(x - \varphi_k)}.$$

From Corollary 4.1 we can state

Corollary 4.5. *Let T_γ be the operator defined by* (64) *and the conditions above be satisfied. If there exist numbers*

$$\rho_k, \varphi_k : \quad \rho_k > 1, \quad \varphi_k \in]-\pi, \pi], \quad k = \overline{1, m},$$

such that the inequality (65) *is fulfilled for all $x \in]-\pi, \pi]$, then*

$$\dim \ker T_\gamma \le m.$$

Suppose now that the function $c \in C(\mathbb{T})$ satisfies the properties (58), (59) and (60) and that the inequality (65) does not hold for the function c, but it is fulfilled for the function $\widetilde{c} = c_+ c^{-1} c_+^{-1}(\gamma_{-1})$, i.e.,

$$\left|\widetilde{c}(e^{ix})\right|^2 < \prod_{k=1}^{m} y(x, \rho_k, \varphi_k). \tag{66}$$

With Corollary 4.2, we can write

Corollary 4.6. *Let T_γ be the operator defined by* (64)*, where $c \in C(\mathbb{T})$ satisfies the properties* (58)*,* (59) *and* (60)*. If there exist numbers*

$$\rho_k, \varphi_k : \quad \rho_k > 1, \quad \varphi_k \in]-\pi, \pi], \quad k = \overline{1, m},$$

such that the inequality (66) *is fulfilled for all $x \in]-\pi, \pi]$, then*

$$\dim \ker T_\gamma \leq m + \max(0, -\operatorname{ind} c),$$

where $\operatorname{ind} c$ *is the Cauchy index of the function c.*

Remark. As an illustrative example, let us consider r a first degree polynomial and analyze the function $\left|r^{-1}(t) r[\gamma(t)]\right|$. By $\overline{\Gamma}_{pq}$, $p, q = 0, 1, 2$, we will denote the length of the arc Γ_{pq}; we have that

$$\left|r^{-1}(t) r[\gamma(t)]\right| = \begin{cases} < 1, & t \in \Gamma_{10} \\ \geq 1, & t \in \Gamma_{20} \end{cases},$$

where

$$\Gamma_{10} \cup \Gamma_{20} = \mathbb{T}, \quad \overline{\Gamma}_{10} = \overline{\Gamma}_{20} = \pi.$$

For instance, consider $\theta = 1$ and the function $y_1(x) = y(x; 10; 0)$, $x \in]-\pi, \pi]$; we have that $y_1(-0,5) = y_1(-0,5 + \pi) = 1$, the minimum of y_1 is $0,824$ (for $x = -1,821$) and the maximum is $1,213$ (for $x = 1,321$).

So, given a function $c \in C(\mathbb{T})$, coefficient of the operator defined by (64), with

$$|c(t)| = \begin{cases} < 1, & t \in \Gamma_{01} \\ \geq 1, & t \in \Gamma_{02} \end{cases}, \quad \Gamma_{01} \cup \Gamma_{02} = \mathbb{T},$$

if $\overline{\Gamma}_{01} > \overline{\Gamma}_{02}$, we should try to apply Corollary 4.5, and, if $\overline{\Gamma}_{01} < \overline{\Gamma}_{02}$, the Corollary 4.6. Note that if $\Gamma_{01} \equiv \mathbb{T}$, then $\dim \ker T_\gamma = 0$ and the operator T_γ is invertible.

Example 2. Let us recall the linear fractional non-Carleman shift defined by (41) (see Section 3.2)

$$\beta(t) = \frac{at + b}{\overline{b}t + \overline{a}}, \quad t \in \mathbb{T},$$

where $a, b \in \mathbb{C}$: $|a|^2 - |b|^2 = 1$, $a = |a|\, e^{i\mu}$, $b = |b|\, e^{i\nu}$, $\mu, \nu \in]-\pi, \pi]$; this shift has two fixed points, τ_1 and τ_2. The associated shift operator is defined by (42)

$$(U_\beta \varphi)(t) = \frac{1}{\overline{b}t + \overline{a}} \varphi[\beta(t)].$$

Consider the SIO with shift on $L_2(\mathbb{T})$,

$$T_\beta = I - cU_\beta P_+, \tag{67}$$

where $c \in C(\mathbb{T})$ is such that $|c(\tau_j)| < 1$, $j = 1, 2$.

Proceeding analogously to that we did in the previous example, from the inequality (62)

$$\left|r(t)c(t)r^{-1}[\beta(t)]\right| < 1,$$

we have

$$|c(t)|^2 < \left|r^{-1}(t)r[\beta(t)]\right|^2 \Leftrightarrow |c(t)|^2 < \prod\nolimits_{k=1}^{m} \left|\frac{\beta(t) - \lambda_k}{t - \lambda_k}\right|^2.$$

We set

$$t = e^{ix} = \cos x + i \sin x, \quad x \in]-\pi, \pi],$$

and

$$\lambda_k = \rho_k e^{i\varphi_k} = \rho_k \cos\varphi_k + i\rho_k \sin\varphi_k, \quad \rho_k > 1, \quad \varphi_k \in]-\pi, \pi], \quad k = \overline{1, m}.$$

Then

$$\left|c(e^{ix})\right|^2 < \prod\nolimits_{k=1}^{m} z(x, \rho_k, \varphi_k), \tag{68}$$

where we denote

$$z(x, \rho_k, \varphi_k) = \frac{A}{B},$$

$$\begin{aligned} A &= (1 + \rho_k^2)[|a|^2 + |b|^2 + 2|a||b|\cos(x + \mu - \nu)] - 2\rho_k \\ &\quad \times [2|a||b|\cos(\varphi_k - \mu - \nu) + |a|^2\cos(x - \varphi_k + 2\mu) + |b|^2\cos(x + \varphi_k - 2\nu)], \\ B &= [1 + \rho_k^2 - 2\rho_k\cos(x - \varphi_k)][|a|^2 + |b|^2 + 2|a||b|\cos(x + \mu - \nu)]. \end{aligned}$$

From Corollary 4.3 we can state

Corollary 4.7. *For every continuous function $c \in C(\mathbb{T})$ such that*

$$|c(\tau_j)| < 1, \quad j = 1, 2,$$

there exist numbers

$$\rho_k, \varphi_k : \quad \rho_k > 1, \quad \varphi_k \in]-\pi, \pi], \quad k = \overline{1, m},$$

such that the inequality (68) *is fulfilled for all $x \in]-\pi, \pi]$. Moreover*

$$\dim\ker T_\alpha \leq m, \tag{69}$$

where T_β is the operator defined by (67).

Suppose now that the function $c \in C(\mathbb{T})$ satisfies the properties (58), (59), (60) and

$$|c(\tau_j)| > 1, \quad j = 1, 2. \tag{70}$$

Corollary 4.4 and the considerations above yield the inequality

$$\left|\widetilde{c}(e^{ix})\right|^2 < \prod\nolimits_{k=1}^{m} z(x, \rho_k, \varphi_k), \tag{71}$$

where $\widetilde{c} = c_+ c^{-1} c_+^{-1}(\beta_{-1})$.

Therefore we can write

Corollary 4.8. *Let* T_β *be the operator defined by* (67)*, where* $c \in C(\mathbb{T})$ *satisfies the properties* (58)*,* (59)*,* (60) *and* (70)*. Then there exist numbers*

$$\rho_k, \varphi_k: \quad \rho_k > 1, \quad \varphi_k \in]-\pi, \pi], \quad k = \overline{1, m},$$

such that the inequality (71) *is fulfilled for all* $x \in]-\pi, \pi]$*. Moreover*

$$\dim \ker T_\beta \leq m + \max(0, -\operatorname{ind} c),$$

where $\operatorname{ind} c$ *is the Cauchy index of the function* c*.*

Remark. Again, as an illustrative example, consider the particular linear fractional non-Carleman shift

$$\delta(t) = \frac{\sqrt{2}t + 1}{t + \sqrt{2}}, \quad t \in \mathbb{T},$$

which has two fixed points $\tau_1 = 1$ and $\tau_2 = -1$. The associated shift operator is given by

$$(U_\delta \varphi)(t) = \frac{1}{t + \sqrt{2}} \varphi\left(\frac{\sqrt{2}t + 1}{t + \sqrt{2}}\right),$$

and consider the SIO with shift on $L_2(\mathbb{T})$,

$$T_\delta = I - cU_\delta P_+,$$

where $c \in C(\mathbb{T})$ is such that $|c(\tau_j)| < 1$, $j = 1, 2$.

The function z in the inequality (68) is given by

$$z(x, \rho_k, \varphi_k) = \frac{(1 + \rho_k^2)(3 + 2\sqrt{2}\cos x) - 2\rho_k[2\sqrt{2}\cos\varphi_k + 2\cos(x - \varphi_k) + \cos(x + \varphi_k)]}{[1 + \rho_k^2 - 2\rho_k \cos(x - \varphi_k)](3 + 2\sqrt{2}\cos x)}.$$

For instance, consider the function $z_1(x) = z(x; 1, 1; -\pi)$, $x \in]-\pi, \pi]$; note that the minimum of z_1 is 1 (at the fixed points of the shift: $\tau_{1,2} = \pm 1 \leftrightarrow x = 0, \pm\pi$). Then, given a function $c \in C(\mathbb{T})$, coefficient of the operator defined by (67), satisfying the conditions of Corollary 4.7, we can simply take a power m of the function z_1, such that the inequality (68) holds. Evidently to get the best (optimal) estimate (69) we have to carefully choose the functions $z(x, \rho_k, \varphi_k)$, i.e., the roots $\lambda_k = \rho_k e^{i\varphi_k}$ of the polynomial r. In [19] we constructed some examples which illustrate and show that the estimate (69), in a certain sense, is sharp.

References

[1] A.A. Baturev, V.G. Kravchenko, and G.S. Litvinchuk, *Approximate methods for singular integral equations with a non-Carleman shift.* J. Integral Equations Appl. **8** (1996), no. 1, 1–17.

[2] K. Clancey and I. Gohberg, *Factorization of Matrix Functions and Singular Integral Operators.* Operator Theory: Advances and Applications, vol. 3. Birkhäuser Verlag, Basel, 1981.

[3] T. Ehrhardt, *Invertibility theory for Toeplitz plus Hankel operators and singular integral operators with flip.* J. Funct. Anal. **208** (2004), n. 1, 64–106.

[4] R.A. Horn and C.R. Johnson, *Matrix Analysis.* Cambridge University Press, Cambridge, 1990.

[5] R.A. Horn and C.R. Johnson, *Topics in Matrix Analysis.* Cambridge University Press, Cambridge, 1991.

[6] P. Koosis, *Introduction to Hp spaces.* Cambridge University Press, Cambridge, 1998.

[7] V.G. Kravchenko and G.S. Litvinchuk, *Introduction to the Theory of Singular Integral Operators with Shift.* Mathematics and its Applications, vol. 289. Kluwer Academic Publishers, Dordrecht, 1994.

[8] V.G. Kravchenko, A.B. Lebre, and G.S. Litvinchuk, *Spectrum problems for singular integral operators with Carleman shift.* Math. Nachr. **226** (2001), 129–151.

[9] V.G. Kravchenko, A.B. Lebre, and J.S. Rodriguez, *Factorization of singular integral operators with a Carleman shift and spectral problems.* J. Integral Equations Appl. **13** (2001), no. 4, 339–383.

[10] V.G. Kravchenko, A.B. Lebre, and J.S. Rodriguez, *Factorization of singular integral operators with a Carleman shift via factorization of matrix functions.* Singular Integral Operators, Factorization and Applications, Operator Theory: Advances and Applications, vol. 142, 189–211. Birkhäuser Verlag, Basel, 2003.

[11] V.G. Kravchenko, A.B. Lebre, and J.S. Rodriguez, *The kernel of singular integral operators with a finite group of linear-fractional shifts.* Operator Theory 20, 143–154. Theta series in Advanced Mathematics, Bucharest, 2006.

[12] V.G. Kravchenko and R.C. Marreiros, *An estimate for the dimension of the kernel of a singular operator with a non-Carleman shift.* Factorization, Singular Operators and Related Problems, 197–204. Kluwer Academic Publishers, Dordrecht, 2003.

[13] V.G. Kravchenko and R.C. Marreiros, *On the kernel of some one-dimensional singular integral operators with shift. The Extended Field of Operator Theory*, Operator Theory: Advances and Applications, vol. 171, 245–257. Birkhäuser Verlag, Basel, 2007.

[14] N. Krupnik, *Banach Algebras with Symbol and Singular Integral Operators.* Operator Theory: Advances and Applications, vol. 26. Birkhäuser Verlag, Basel, 1987.

[15] G.S. Litvinchuk, *Boundary Value Problems and Singular Integral Equations with Shift.* Nauka, Moscow, 1977 (in Russian).

[16] G.S. Litvinchuk, *Solvability Theory of Boundary Value Problems and Singular Integral Equations with Shift.* Mathematics and its Applications, vol. 523. Kluwer Academic Publishers, Dordrecht, 2000.

[17] L.S. Litvinchuk and I.M. Spitkovskii, *Factorization of Measurable Matrix Functions*. Operator Theory: Advances and Applications, vol. 25. Birkhäuser Verlag, Basel, 1987.

[18] A.I. Markushevich, *Theory of Functions of a Complex Variable*. Chelsea Publishing Company, New York, 1985.

[19] R.C. Marreiros, *On the study of the dimension of the kernel of singular integral operators with non-Carleman shift using Mathematica software*. Proceedings of the 1st National Conference on Symbolic Computation in Education and Research, Instituto Superior Técnico, Lisboa, April 2–3, 2012, ID-08, Congress Centre, Lisboa, 2012.

[20] M. Rosenblum and J. Rovnyak, *Topics in Hardy Classes and Univalent Functions*. Birkhäuser Verlag, Basel, 1994.

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Inequalities Against Equations?

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Dedicated to Professor António Ferreira dos Santos

Abstract. The close connection between equations and inequalities will be illustrated on the example of the Hardy inequality.

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Keywords. Friedrichs' inequality, Hardy's inequality, Sturm–Liouville problem, spectral properties, variational eigenvalues.

1. No doubt, that there is a close, friendly and fruitful connection between (differential) equations and (integral) inequalities. Here, we want to show, on some simple examples, the symbiosis of these two phenomena and the way

(A) from inequalities to equations

and, vice versa,

(B) from equations to inequalities.

2. Everyone, who is dealing with differential equations, will need, after some time, an appropriate inequality. Let us quote G.H. Hardy, who at his election for President of the London Mathematical Society in 1926 told:

> "*All analysts spend half their time hunting through the literature for inequalities which they want to use and cannot prove.*"

Example 1. The Friedrichs inequality

$$\int_\Omega |f(x)|^2 dx \leq C \int_\Omega |\nabla f(x)|^2 dx \tag{1}$$

with Ω a domain in $\mathbb{R}^N$ holds for all functions $f \in C_0^\infty(\Omega)$ and is closely connected with the spectral problem for the Laplace operator Δ:

$$-\Delta f = \lambda f \text{ on } \Omega \tag{2}$$

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with homogeneous Dirichlet boundary condition: The (best) constant C in (1) depends on Ω and gives a lower bound for the spectrum,

$$\lambda \geq \frac{1}{C}.$$

This example illustrates the way (A): inequality (1) allows to describe the properties of the solution of equation (2).

3. As far as concerns the opposite way (B), many inequalities can be formulated as extrema of some functionals, e.g., of the type

$$J(y) = \int_a^b F(x, y, y')dx, \tag{3}$$

which leads to the solution of a differential equation, namely the Euler–Lagrange equation

$$\frac{\partial F}{\partial y} = \frac{d}{dx}\Big(\frac{\partial F}{\partial y'}\Big).$$

Even in the famous book "Inequalities" [1], there is a chapter entitled "Some Applications of the Calculus of Variations". The following example is taken from this book.

Example 2. The inequality

$$\int_0^1 \frac{y^2(x)}{x^2}dx \leq C \int_0^1 y'^2(x)dx \tag{4}$$

can be investigated via the infimum of the functional

$$J(y) = \int_0^1 \Big[Cy'^2(x) - \frac{y^2(x)}{x^2}\Big]dx,$$

i.e., of the functional J from (3) with $F(x, y, y') = Cy'^2 - \frac{y^2}{x^2}$, $C \geq 4$, for functions y such that $y' \in L^2(0,1)$, $y(0) = 0$, $y(1) = 1$. The corresponding Euler–Lagrange equation,

$$x^2y'' + \lambda y = 0, \;\; \lambda = \frac{1}{C},$$

has only one solution (up to a multiple constant), satisfying the conditions, namely

$$Y(x) = x^{\frac{1}{2}+a}, \;\; a = \frac{1}{2}\sqrt{1 - \frac{4}{C}}$$

(since the second solution, $Z(x) = x^{\frac{1}{2}-a}$, does not satisfy $Z' \in L^2(0,1)$). We can easily show that $J(Y) = \frac{2}{1-2a}$, and

$$J(y) = \int_0^1 \Big(Cy'^2 - \frac{y^2}{x^2}\Big)dx \geq J(Y) > 0$$

implies (4).

4. Here, we want to illustrate the connection between equations and inequalities on the N-dimensional Hardy inequality

$$\left(\int_\Omega |f(x)|^q u(x)dx\right)^{1/q} \le C\left(\int_\Omega |\nabla f(x)|^p v(x)dx\right)^{1/p} \tag{5}$$

with $\Omega \subset \mathbb{R}^N$ for functions $f \in C_0^\infty(\Omega)$; here the parameters p, q satisfy $1 < p, q < \infty$, and $u = u(x)$, $v = v(x)$ are appropriate weight functions, positive a.e. in Ω.

Let us consider the spectral problem

$$-\operatorname{div}(|\nabla f|^{p-2}\nabla f \cdot v) = \lambda |f|^{q-2} f u \quad \text{on } \Omega \tag{6}$$

with homogeneous Dirichlet boundary conditions on the boundary $\partial\Omega$ of Ω. Assume that there exists a *weak solution* of this problem, i.e., a function f on Ω such that $\nabla f \in L^p(\Omega, v)$ and that the integral identity

$$\int_\Omega |\nabla f|^{p-2}\nabla f v \nabla g dx = \lambda \int_\Omega |f|^{q-2} f u g dx$$

holds for every $g \in C_0^\infty(\Omega)$. If we choose $g = f$ we obtain that

$$\int_\Omega |\nabla f|^p v dx = \lambda \int_\Omega |f|^q u dx,$$

and the Hardy inequality implies

$$\int_\Omega |\nabla f|^p v dx \le \lambda C^q \left(\int_\Omega |\nabla f|^p v dx\right)^{q/p}.$$

If we take $q = p$ (or consider "normalized" functions f, i.e., such that $\int_\Omega |\nabla f|^p v dx = 1$), we obtain finally that

$$\lambda \ge \frac{1}{C^q}.$$

This allows to formulate the following proposition: *If the Hardy inequality* (5) *for* $p = q$ *holds, then the spectrum of the boundary value problem* (6) *is bounded from below.*

Remark 1. This proposition is in accordance with the result from Example 1, where the case $p = q = 2$, $u(x) = v(x) = 1$ was considered.

5. Much more information we can obtain, if we consider the one-dimensional case. The analogue of the Hardy inequality (5) for the case $N = 1$, $\Omega = (a, b)$ reads

$$\left(\int_a^b |f(x)|^q u(x)dx\right)^{1/q} \le C\left(\int_a^b |f'(x)|^p v(x)dx\right)^{1/p}. \tag{7}$$

We will consider this inequality for functions $f \in AC(a, b)$, for which it is $\|f'\|_{p,v}^p := \int_a^b |f'(x)|^p v(x)dx < \infty$ and which satisfy $f(a) = 0$. This set of functions, normed by $\|f'\|_{p,v}$, will be denoted $W_L^{1,p}(a, b; v)$ and called a *weighted Sobolev space.*

Let us recall (for details see, e.g., [2]) that the Hardy inequality (7) holds for all $f \in W_L^{1,p}(a,b;v)$ and for $1 < p \le q < \infty$ if and only if

$$\sup_{x\in(a,b)} A_M(x) < \infty \tag{8}$$

where

$$A_M(x) := \Big(\int_x^b u(t)dt\Big)^{1/q}\Big(\int_a^x v^{1-p'}(t)dt\Big)^{1/p'} \tag{9}$$

with $p' = \frac{p}{p-1}$.

Inequality (7) tells that the imbedding of the weighted Sobolev space $W_L^{1,p}(a,b;v)$ into the weighted Lebesgue space $L^q(a,b;u)$ is *continuous*,

$$W_L^{1,p}(a,b;v) \hookrightarrow L^q(a,b;u), \tag{10}$$

if and only if, for $1 < p \le q < \infty$, condition (8) is satisfied. Moreover, the imbedding (10) is *compact* if and only if, in addition, the conditions

$$\lim_{x\to a+} A_M(x) = \lim_{x\to b-} A_M(x) = 0 \tag{11}$$

are satisfied.

6. The connection between the Hardy inequality (7) and some differential equation has been investigated by P.R. Beesack [3] (for $1 < p \le q < \infty$) and G.A. Tomaselli [4] (for $p = q$). The corresponding result, whose proof can be found in [5, Theorem 4.1], reads:

Theorem 2. *Let be $1 < p \le q < \infty$, u and v weights, $v \in AC(a,b)$, $\int_a^x v^{1-p'}(t)dt < \infty$ for $x \in (a,b)$. Then the Hardy inequality (7) holds for every $f \in W_L^{1,p}(a,b;v)$ if and only if there is a number $\lambda > 0$ such that the differential equation*

$$\frac{d}{dx}\Big[v^{q/p}(x)\Big(\frac{dy}{dx}\Big)^{q/p'}\Big] + \lambda u(x)y^{q/p'}(x) = 0 \tag{12}$$

has a solution y satisfying

$$y' \in AC(a,b),\ y(x) > 0,\ y'(x) > 0 \textit{ for } x \in (a,b).$$

Now, let us denote, for $s \in \mathbb{R}$ and $r > 1$,

$$\phi_r(s) := |s|^{r-2}s = |s|^{r-1}\operatorname{sgn} s.$$

We will consider the equation

$$-(v(x)\Phi_p(y'(x)))' = \lambda u(x)\Phi_q(y(x)) \quad \text{on } (a,b) \tag{13}$$

with homogeneous Dirichlet boundary conditions.

Definition 3. Equation (13) is for $p, q \ne 2$ nonlinear. For $p = q$ ($\ne 2$) it is called *halflinear*, since it is homogeneous (with y, also cy is a solution) but not additive. – We say that equation (13) (or its eigenvalue λ) has the **BD-property**, if the spectrum is *bounded* from below and *discrete*.

Remark 4. Similarly as in the N-dimensional case (see Section 4 above) we can show that the validity of the Hardy inequality (7) implies (in the case $p = q$) that $\lambda \geq \frac{1}{C^p}$, i.e., the boundedness of the spectrum from below. But in some cases, the validity of the Hardy inequality is not only sufficient, but also necessary. Let us give some examples, which show, what is **necessary and sufficient** for the BD-property.

Example 3. (i) In the paper of I.S. Kac and M.G. Krein [6] it was shown, that the spectrum of the initial value problem

$$-y'' = \lambda\rho(x)y,\ y(0) = 0,\ y'(0) = 1,\ x \in (0, \infty) \tag{14}$$

is *bounded from below* if there exists a constant C_0 such that

$$A(x) := x \int_x^\infty \rho(t)dt < C_0 \quad \text{for every } x \in (0, \infty) \tag{15}$$

(with lower bound $1/(4C_0)$), and it is *discrete* if and only if

$$\lim_{x\to\infty} A(x) = 0. \tag{16}$$

In this case, we have the equation (13) with $(a, b) = (0, \infty)$, $p = q = 2$, $v(x) \equiv 1$ and $u(x) = \rho(x)$, and the function $A(x)$ from (15) is the square of the function $A_M(x)$ from (9). Hence, we see that in this case the spectrum of (14) has the BD-property if and only if the corresponding imbedding, realized by the Hardy inequality, is *compact*. (Note that (16) is one of the conditions (11); the second is satisfied automatically, if we suppose $\rho \in L^1(0, \infty)$.)

(ii) If we consider the Hardy inequality (7) for functions f which satisfy $f(b) = 0$, we have to replace the function $A_M(x)$ from (9) by

$$\widetilde{A}_M(x) := \left(\int_a^x u(t)dt\right)^{1/q} \left(\int_x^b v^{1-p'}(t)dt\right)^{1/p'} \tag{17}$$

and replace A_M by $\widetilde{A}_M$ also in the criteria (8) and (11) of the continuity and compactness, respectively, of the corresponding imbedding.

For the spectral problem

$$-(v(x)y')' = \lambda y(x),\ x \in (0, \infty),\ y(\infty) = 0, \tag{18}$$

we have again equation (13) with $(a, b) = (0, \infty)$, $p = q = 2$, $u(x) \equiv 1$, and

$$\widetilde{A}^2_M(x) = x \int_x^\infty \frac{1}{v(t)} dt.$$

Several authors have shown that the condition

$$\lim_{x\to\infty} x \int_x^\infty \frac{1}{v(t)} dt = 0 \tag{19}$$

is necessary and sufficient for the spectrum of (18) to have the BD-property.

If we formulate the **conjecture**, that *for the case* $p = q$, *the spectrum of the halflinear equation* (13) *has the* BD-*property if and only if the corresponding imbedding, described by the Hardy inequality, is compact*, then the foregoing examples confirm that this conjecture is true for $p = 2$.

7. The foregoing conjecture was confirmed for $p > 1$ in the following theorem, which is in fact an extension of the Sturm–Liouville theory from the linear to the halflinear case.

Theorem 5 (P. Drábek, A. Kufner [7]). *Let us consider equation* (13) *on* $(0, \infty)$ *for* $p = q > 1$ *with boundary conditions* $y'(0) = 0$, $y(\infty) = 0$ *and with* $u(x)$ *and* $v(x)$ *positive and continuous on* $[0, \infty)$. *Suppose that the function* $\widetilde{A}_M(x)$ *from* (17) *satisfies*

$$\lim_{x\to\infty} \widetilde{A}_M(x) = 0. \tag{20}$$

Then the set of all eigenvalues forms an increasing sequence $\{\lambda_n\}_{n=1}^{\infty}$ *such that* $\lambda_1 > 0$ *and* $\lim_{n\to\infty} \lambda_n = \infty$. *Every eigenvalue* λ_n *is simple and the eigenfunction* y_n *corresponding to* λ_n *has exactly* $n - 1$ *zeros in* $(0, \infty)$. *In particular,* y_1 *does not change sign in* $(0, \infty)$ *and for* $n \geq 3$, *between two consecutive zeros of* y_{n-1} *there is exactly one zero of* y_n.

Remark 6. Note that condition (20) guarantees the compactness of the corresponding imbedding. If condition (20) is violated, then examples show that there is either *no eigenvalue* at all, or (if $\widetilde{A}_M(x)$ is bounded) we have a *continuum* of eigenvalues, and the spectrum is bounded from below but *not discrete.*

Remark 7 (the case $p \neq q$). By our knowledge, there is no analogous result concerning the BD-property in connection with the Hardy inequality for *nonlinear* equations.

Remark 8 (the higher-order case). There are also results about the BD-property for higher-order differential equations. For example, in R.T. Lewis [8] it is shown that the condition

$$\lim_{x\to\infty} x^{2n-1} \int_x^\infty \frac{1}{v(t)} dt = 0 \tag{21}$$

is necessary and sufficient for the equation

$$(-1)^n (v(x) y^{(n)}(x))^{(n)} = \lambda y(x) \quad \text{on } (0, \infty)$$

to have the BD-property. [Notice, that the special case $n = 1$ was considered in Example 3 (ii) – see (18) and (19).]

This result is connected with the higher-order Hardy inequality

$$\left(\int_a^b |f(x)|^q u(x) dx \right)^{1/q} \leq C \left(\int_a^b |f^{(n)}(x)|^p v(x) dx \right)^{1/p}.$$

Condition (21) guarantees (in the case $(a, b) = (0, \infty)$, $p = q = 2$ and $u(x) \equiv 1$) the compactness of the corresponding imbedding.

Remark 9 (the *N*-dimensional case). For the case $N \geq 2$, there are not simple general criteria of the validity of the Hardy inequality (5). But we have a result analogous to Theorem 1, even for the more general extended (nonisotropic) Hardy inequality

$$\left(\int_\Omega |f(x)|^q u(x) dx \right)^{1/q} \leq C \left(\sum_{i=1}^N \int_\Omega \left| \frac{\partial f}{\partial x_i}(x) \right|^p v_i(x) dx \right)^{1/p}. \tag{22}$$

Inequality (22) *holds for* $f \in C_0^\infty(\Omega)$ *and for* $p = q$, *if there exists a solution* y *of the partial differential equation*

$$-\sum_{i=1}^N \frac{\partial}{\partial x_i} \left(v_i(x) \left| \frac{\partial y}{\partial x_i} \right|^{p-1} \operatorname{sgn} \frac{\partial y}{\partial x_i} \right) = u(x) |y(x)|^{p-1} \operatorname{sgn} y(x) \tag{23}$$

such that $y \neq 0$, $\frac{\partial y}{\partial x_i} \neq 0$ *a.e. in* Ω.

Equation (23) can be considered as the Euler–Lagrange equation for the functional

$$J(y) = \int_\Omega \left(\sum_{i=1}^N \left| \frac{\partial y}{\partial x_i} \right|^p v_i - |y|^p u \right) dx.$$

(For details, see [5, Section 14.1].)

References

[1] G.H. Hardy, J.E. Littlewood, G. Pólya: *Inequalities.* Cambridge Univ. Press, 1934.

[2] A. Kufner, L.E. Persson: *Weighted Inequalities of Hardy Type.* World Scientific, Singapore 2003.

[3] P.R. Beesack: *Hardy's inequality and its extensions.* Pacific J. Math. **11** (1961), 39–61.

[4] G.A. Tomaselli: *A class of inequalities.* Boll. Un. Mat. Ital. (2) **22** (1969), 622–631.

[5] B. Opic, A. Kufner: *Hardy-Type Inequalities.* Longman, Harlow 1990.

[6] I.S. Kac, M.G. Krein: *Criteria for discreteness of the spectra of a singular string.* Izv. Vyss. Ucheb. Zaved. Mat. no 2 (1958), 136–153. (Russian)

[7] P. Drábek, A. Kufner: *Hardy inequality and properties of the quasilinear Sturm–Liouville problem.* Atti. Accad. Naz. Lincei, Math. Appl. **18**, no. 2 (2007), 125–138.

[8] R.T. Lewis: *The discreteness of the spectrum of self-adjoint, even order, one term, differential operators.* Proc. Amer. Math. Soc. **42** (1974), 480–482.

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C^*-Algebra Generated by Mapping Which Has Finite Orbits

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Abstract. We consider the C^*-subalgebra of the algebra of all bounded operators on Hilbert space of square-summable functions on some countable set. The algebra being investigated is generated by a family of partial isometries. These isometries satisfy the relations defined by a preassigned mapping on the set. It is assumed that the mapping has elements with finite orbits. Under this assumption the algebra we consider contains the subalgebra of compact operators and its quotient algebra is $\mathbb{Z}$-graded. We consider the covariant system associated with the quotient algebra and construct the conditional expectation onto the fixed point subalgebra. We prove that the quotient algebra is nuclear and so is the algebra generated by mapping.

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Introduction

In this paper we continue to investigate the algebra $C^*_\varphi(X)$ which was introduced in [7, 8]. The algebra is generated by finite or countable family of partial isometries $\{U_k\}$. These are defined by the given mapping $\varphi : X \longrightarrow X$. In a simplest case when φ is injection (not bijection) the $C^*_\varphi(X)$ is isomorphic to the Toeplitz algebra.

Cuntz in [1] first considered the algebra $\mathfrak{O}_n$ generated by a finite family of noncommuting isometries $U_1, U_2, \dots, U_n$ whose range projections are summed to the identity. Later in some papers (for instance, see [2]–[6]) authors regarded the C^*-algebras generated by different families of partial isometries whose range and initial projections satisfy a specified set of conditions. The classical examples of such algebras are given by the Cuntz–Krieger algebra [3] and Cuntz–Pimsner algebra [4, 5].

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In papers [7, 8] it was proposed the construction of C^*-algebra $C^*_\varphi(X)$ generated by a finite family of partial isometries $\{U_k\}_{k=1}^m$ (or by generator T_φ which is a linear combination of these isometries). Their range and initial projections are summed to the noncommuting projections defined by a preassigned mapping φ on a countable set X.

Algebra $C^*_\varphi(X)$ generated by a countable family of partial isometries $\{U_k\}_{k=1}^\infty$ was considered in [11].

In [12] it was proposed the construction of C^*-algebra $\mathfrak{M}_\varphi$. The last is generated by a family of partial isometries $\{U_k\}_{k=1}^\infty$ satisfying the same relations and the multiplier algebra on $l^2(X)$.

In these papers the requirement on φ was that there is no element in X with finite orbit. All analyzed C^*-algebras are nuclear and $\mathbb{Z}$-graded with AF-subalgebra responding to zero.

In this paper the mapping φ need not satisfy the above-mentioned assumption. We show that in general case $C^*_\varphi(X)$ is still nuclear and has nontrivial AF-subalgebra but it is not $\mathbb{Z}$-graded.

The paper is organized as follows. Section 1 is intended to give the main notations and terminology. We briefly sketch the properties of the operator T_φ and define $C^*_\varphi(X)$. For the proofs we refer the reader to [8], [10], [11].

In Section 2 it is shown that $C^*_\varphi(X)$ contains the subalgebra of compact operators. For the quotient algebra $\mathfrak{C}^*_\varphi$ we introduce the notion of monomial and its index.

Section 3 is devoted to the study of structure of $\mathfrak{C}^*_\varphi$. The last is $\mathbb{Z}$-graded with AF-subalgebra responding to zero. Also we construct the conditional expectation to AF-subalgebra.

In Section 4 we consider the circle action on $\mathfrak{C}^*_\varphi$. We prove that $\mathfrak{C}^*_\varphi$ is nuclear and so is $C^*_\varphi(X)$.

1. The necessary information

Let $\varphi : X \longrightarrow X$ be a mapping of a countable set X into itself. We will denote by E_y the preimage of the element y ($\{x \in X : \varphi(x) = y\}$). Let $\gamma(y) = \operatorname{card} E_y$. We call an element $y \in X$ *φ-initial* (initial for short) if $\gamma(y) = 0$. We call an element $y \in X$ *φ-cyclic* (cyclic for short) if there is an $n \in \mathbb{N}$ such that $\varphi^n(y) = y$.

The family of functions $\{e_x\}_{x\in X}$ on X such that $e_x(y) = \delta_{x,y}$ ($\delta_{x,y}$ is the Kronecker delta), forms an orthonormal basis in Hilbert space $l^2(X) = \{f : X \to \mathbb{C} : \sum_{x\in X} |f(x)|^2 < \infty\}$.

Mapping φ induces a mapping

$$T_\varphi : \{e_x\}_{x\in X} \to \{e_x\}_{x\in X}$$

given by $T_\varphi e_x = e_{\varphi(x)}$. This mapping can be extended to a bounded linear operator on $l^2(X)$ (we denote it by T_φ again) if and only if the cardinalities of preimages are bounded in common under the action of mapping φ, $(T_\varphi f)(y) = \sum\limits_{x\in E_y} f(x)$. If this is not the case and the preimage's cardinality is only bounded then T_φ can be extended to a closed operator on $l^2(X)$. The algebras associated with mapping in the latter case were considered in [11].

Represent X as the union of the disjoint sets $X_k = \{y \in X : \gamma(y) = k\}$, we suppose $\sup\limits_{y\in X} \gamma(y) = m < \infty$. Correspondingly, the Hilbert space $l^2(X)$ can be represented as the direct sum of pairwise orthogonal Hilbert spaces $l^2(X_k)$, $0 \le k \le m$. If $X_k = \emptyset$ we suppose $l^2(X_k) = \{0\}$. Note that $T_\varphi T_\varphi^* f = kf$ for every $f \in l^2(X_k)$. Therefore, $T_\varphi T_\varphi^* = \bigoplus\limits_{k=1}^{m} kP_k$, where P_k is the projection from $l^2(X)$ onto $l^2(X_k)$.

Similarly, $l^2(X)$ can be represented as the direct sum of l_k^2, where $l_k^2 = \{f \in l^2(X) : T_\varphi^* T_\varphi f = kf\}$ if $1 \le k \le m$. In case $k = 0$ the subspace l_0^2 is defined as the orthogonal complement to all other l_k^2. It follows that $T_\varphi^* T_\varphi = \bigoplus\limits_{p=1}^{m} kQ_k$, where Q_k is the projection from $l^2(X)$ onto l_k^2.

The family of functions $\{e_x\}_{x\in X_k}$ forms an orthogonal basis in the subspace $l^2(X_k)$ and the family of $\{g_y\}_{y\in X_k}$, $g_y = \frac{1}{\sqrt{k}} \sum\limits_{x\in E_y} e_x$, does so for l_k^2.

Introduce the operator of partial isometry U_k, $k \neq 0$, by setting $U_k = \frac{1}{\sqrt{k}} T_\varphi P_k$. It can be easily checked that

$$U_k g_y = \begin{cases} e_y, & \text{if} \quad y \in X_k; \\ 0, & \text{if} \quad y \notin X_k. \end{cases} \qquad \text{Hence} \quad U_k^* e_y = \begin{cases} g_y, & \text{if} \quad y \in X_k; \\ 0, & \text{if} \quad y \notin X_k; \end{cases}$$

and $T_\varphi = U_1 + \sqrt{2}U_2 + \cdots + \sqrt{m}U_m$. Note that the family of partially isometric operators $\{U_k\}$ satisfies the relations:

$$\begin{cases} U_1^*U_1 + U_2^*U_2 + \cdots + U_m^*U_m = Q_1 + Q_2 + \cdots + Q_m = Q; \\ U_1U_1^* + U_2U_2^* + \cdots + U_mU_m^* = P_1 + P_2 + \cdots + P_m = P. \end{cases}$$

Let $C_\varphi^*(X)$ be the uniformly closed subalgebra of the algebra of all bounded operators on $l^2(X)$ generated by the operator T_φ. Since the spectra of the operators $T_\varphi T_\varphi^*$ and $T_\varphi^* T_\varphi$ are finite sets, we conclude that operators P_k and Q_k belong to $C_\varphi^*(X)$ as well as the operator U_k. Therefore the C^*-algebra $C_\varphi^*(X)$ is generated by the family of partial isometries $\{U_k\}_{k=1}^m$.

We call an element from $\{U_k\}_{k=1}^m \cup \{U_k^*\}_{k=1}^m$ the *primary monomial*. We define -1 (or 1) to be the *index* of primary monomial which is an element of the set $\{U_k\}_{k=1}^m$ (or $\{U_k^*\}_{k=1}^m$). We call V the *monomial* if it is any finite product of primary monomials not identically zero. Clearly, the same monomial can have different representations. We denote by $\mathrm{d}\,(V)$ the *length* of V – the least quantity of monomials in its representation.

Let $\{V\}$ be the family of all monomials of $C_\varphi^*(X)$. We can assume that $\{V\}$ contains zero monomial, then $\{V\}$ is an involutive semigroup with zero element.

We call the elements x_1, $x_2 \in X$ φ-*equivalent* (equivalent for short) *in the kth-order* if $\varphi^k(x_1) = \varphi^k(x_2) = y$. Let $E_y^k = \{x \in X : \varphi^k(x) = y\}$, where $k = 0, 1, 2, \ldots$. We assume $E_y^0 = \{y\}$ and $E_y^1 = E_y$. Since $\operatorname{card} E_y \le m$, $\operatorname{card} E_y^k \le m^k$ and if $y_1 \ne y_2$, then $E_{y_1}^k \cap E_{y_2}^k = \emptyset$. Therefore, the Hilbert space $l^2(X)$ can be represented as a direct sum of $l^2(E_y^k)$ for each fixed k,

$$l^2(X) = \bigoplus_{y \in X} l^2(E_y^k).$$

The lemma below was proved in [10]. It is still true for mapping with cyclic elements (or elements with finite orbits) and will be useful later on. For this purpose we adduce it in this paper.

Lemma 1.1. *Assume that the monomial V can be represented as a product of s factors from the set $\{U_p\}_{p=1}^m$ and l factors from the set $\{U_p^*\}_{p=1}^m$. Then V maps the subspace $l^2(E_y^k)$ into $l^2(E_y^{k-s+l})$ for each $k \ge s$ and Ve_y is contained in the Hilbert space $l^2(E_{\varphi^s(y)}^{k+l})$.*

Proof. Let $U_p e_x \ne 0$ for some e_x from $l^2(E_y^k)$. Hence $U_p e_x = \frac{1}{\sqrt{p}} e_{\varphi(x)}$. Since $x \in E_y^k$, we conclude that $\varphi(x) \in E_y^{k-1}$ and U_p maps $l^2(E_y^k)$ into $l^2(E_y^{k-1})$ for all $p = \overline{1, m}$. Similarly, $U_p^* e_x = g_x \in l^2(E_y^{k+1})$ if $U_p^* e_x$ does not vanish, hence U_p^* maps $l^2(E_y^k)$ into $l^2(E_y^{k+1})$ for all $p = \overline{1, m}$.

Thus, if the monomial V consists of s factors from $\{U_p\}_{p=1}^m$ and l factors from $\{U_p^*\}_{p=1}^m$, then it maps the subspace $l^2(E_y^k)$ into $l^2(E_y^{k-s+l})$. To complete the proof, note that from the definition of the set E_y^k it follows that the space $l^2(E_y^{k-s+l})$ is the subspace of $l^2(E_{\varphi^s(y)}^{k+l})$. Hence Ve_y belongs to $l^2(E_{\varphi^s(y)}^{k+l})$. □

Let the monomial V can be represented as a product of s_1 factors from the set $\{U_p\}_{p=1}^m$ and l_1 factors from the set $\{U_p^*\}_{p=1}^m$ as well as s_2 factors from $\{U_p\}_{p=1}^m$ and l_2 factors $\{U_p^*\}_{p=1}^m$. By Lemma 1.1, V moves the subspace $l^2(E_y^k)$ into the subspace $l^2(E_y^{k-s_1+l_1})$ as well as into $l^2(E_y^{k-s_2+l_2})$. If there is no element with finite orbit it immediately follows that $-s_1 + l_1 = -s_2 + l_2$.

The *index* of monomial V, denoted by $\operatorname{ind} V$, is defined to be the sum of indices of primary monomials participating in it's representation. In a case of absence of cyclic elements $C_\varphi^*(X)$ can be graded over the index of monomial. The subalgebra responding to zero (generated by monomials of zero index) is an AF-algebra. One can imbed the unit circle into the group of automorphisms of $C_\varphi^*(X)$, thus there is a covariant system $(C_\varphi^*(X), S^1, \kappa)$, in addition the fixed point subalgebra is an AF-algebra. The reader can find the detailed presentation of these results in [9, 10].

The purpose of the present work is to show that $C_\varphi^*(X)$ is a nuclear C^*-algebra with nontrivial AF-subalgebra in case of mapping with cyclic elements, and there is a covariant system associated with the last-mentioned mapping.

2. Monomials in $\mathfrak{C}^*_\varphi$

Let $\varphi : X \longrightarrow X$ be a mapping with cyclic element x_0, i. e. $\varphi^n(x_0) = x_0$ for some $n \geq 1$. We will denote by Γ_{x_0} the orbit of the element x_0. Let us associate the oriented graph (X, φ) with vertices in the points of X and edges $(x, \varphi(x))$ with a fixed mapping on the set. Without loss of generality we can assume (X, φ) to be connected (if it is not the case all arguments below are true for its connected components). Therefore any two cyclic elements are in Γ_{x_0}.

The problem is that there may be a monomial V, which has two representations: $V = U'_{j_1} U'_{j_2} \dots U'_{j_p}$; $V = U''_{i_1} U''_{i_2} \dots U''_{i_l}$, where U'_{j_s} and U''_{i_s} are from $\{U^*_k\}^m_{k=1} \cup \{U_k\}^m_{k=1}$ and $\sum\limits_{s=1}^{p} \mathrm{ind} U'_{j_s} \neq \sum\limits_{s=1}^{l} \mathrm{ind} U''_{i_s}$. Hence in general case $C^*_\varphi(X)$ is not $\mathbb{Z}$-graded.

Lemma 2.1. *Let V_0 be a monomial such that two representations exist: $V_0 = U'_{j_1} U'_{j_2} \dots U'_{j_p}$; $V_0 = U''_{i_1} U''_{i_2} \dots U''_{i_l}$, and $\sum\limits_{s=1}^{p} \mathrm{ind} U'_{j_s} \neq \sum\limits_{s=1}^{l} \mathrm{ind} U''_{i_s}$. Then V_0 is an operator of finite rank.*

Proof. Let us denote by n' (n'') the number of elements from the set $\{U_k\}^m_{k=1}$ participating in representation V_0 and by m' (m'') the ones from $\{U^*_k\}^m_{k=1}$ correspondingly. Note that $m' - n' \neq m'' - n''$. Let $F = \{x \in X : \varphi^r(x) \in \Gamma_{x_0}$ where $r = 1 + l + p + \mathrm{card}\Gamma_{x_0}\}$. It is obvious that F is a finite set. Consider the action of V_0 on e_y for an arbitrary y from $X \setminus F$. By Lemma 1.1, $V_0 e_y$ belongs to the Hilbert space $l^2(E^{m'}_{\varphi^{n'}(y)})$ as well as to $l^2(E^{m''}_{\varphi^{n''}(y)})$. But $E^{m'}_{\varphi^{n'}(y)} \cap E^{m''}_{\varphi^{n''}(y)} = \emptyset$ since $V_0 e_y = 0$ for every y from $X \setminus F$. Therefore V_0 is an operator of finite rank. $\square$

From Lemma 2.1 it follows that if the cyclic elements exist, then semigroup $\{V\}$ contains nontrivial subsemigroup of compact monomials $\{V_0\}$. Let $\{V_{\mathrm{inf}}\} = \{V\} \setminus \{V_0\}$.

Corollary 2.2. *Let monomial V from $\{V_{\mathrm{inf}}\}$ can be represented as*

$$V = U'_{j_1} U'_{j_2} \dots U'_{j_k} \quad \textit{as well as} \quad V = U''_{i_1} U''_{i_2} \dots U''_{i_l}.$$

Then $\sum\limits_{s=1}^{k} \mathrm{ind} U'_{j_s} = \sum\limits_{s=1}^{l} \mathrm{ind} U''_{i_s}$.

Thus we can define the index of monomials from $\{V_{\mathrm{inf}}\}$. Similarly to section 1 the *index* of $V \in \{V_{\mathrm{inf}}\}$, denoted by $\mathrm{ind}\, V$, is defined to be the sum of indices of primary monomials participating in it's representation.

Let F_N denote the set $\{x \in X : \varphi^N(x) \in \Gamma_{x_0}\}$. It is obvious that F_N is finite and $F_N \subset F_{N+1}$. There is no cyclic element in the set $X_N = X \setminus F_N$ for every N, and for every $V \in \{V_{\mathrm{inf}}\}$ and N there exists an x in X_N with $V e_x \neq 0$ (otherwise V is a compact operator).

Let $\mathcal{P}_N$ be a projection from $l^2(X)$ onto $l^2(F_N)$. Let us show that $\mathcal{P}_N$ belongs to $C^*_\varphi(X)$.

Let $V_0 \in \{V_0\}$. Then the self-adjoint operator $V_0 V_0^*$ is diagonable hence $C_\varphi^*(X)$ contains the finite-dimensional projections. Suppose P_f is one of these projections. Consider the action of the operator $P_f {T_\varphi^*}^k$ on $l^2(\Gamma_{x_0})$. By the properties of T_φ^* it is obvious that one can select such a k that

$$P_f {T_\varphi^*}^k e_x = 0, \quad \text{if} \quad e_x \notin \Gamma_{x_0}, \quad \text{and} \quad P_f {T_\varphi^*}^k e_x \neq 0, \quad \text{if} \quad e_x \in \Gamma_{x_0}.$$

One can check at once that the self-adjoint operator ${T_\varphi}^k P_f P_f {T_\varphi^*}^k = {T_\varphi}^k P_f {T_\varphi^*}^k$ moves the space $l^2(\Gamma_{x_0}) \equiv l^2(F_0)$ onto itself. Therefore there is projection $\mathcal{P}_0 \in C_\varphi^*(X)$.

Now consider the operator $T_\varphi^* \mathcal{P}_0$. By construction

$$T_\varphi^* \mathcal{P}_0 : l^2(\Gamma_{x_0}) \longrightarrow l^2(F_1).$$

Similarly we consider the operator $T_\varphi^* \mathcal{P}_0 T_\varphi$ and obtain projection $\mathcal{P}_1$. Continuing in the same manner we deduce that projection $\mathcal{P}_N$ belongs to $C_\varphi^*(X)$ (and so does $\mathcal{P}_N^\perp$).

Let V_1 and V_2 be monomials from $\{V_{\inf}\}$ of different indices with length s and n correspondingly. If we assume $y \in X$ to be an element with the property $\varphi^{s+n}(y) \in X_{N_{x_0}}$, then, by Lemma 1.1,

$$(V_1 e_y, V_2 e_y) = 0.$$

The following assertion will be useful later on. Let B be a non compact bounded operator on Hilbert space H and H_n be a finite-dimensional subspace of the last generated by the vectors $e_1, e_2, \ldots, e_n$ of orthonormal basis $\{e_i\}_{i=1}^\infty$ in H. Then $H = H_n \oplus H_n^\perp$ and

$$\lim_{n\to\infty} \|\mathcal{P}_n^\perp B \mathcal{P}_n^\perp\| > 0,$$

the projection $\mathcal{P}_n^\perp : H \longrightarrow H_n^\perp$ here being orthogonal to $\mathcal{P}_n : H \longrightarrow H_n$ ($\mathcal{P}_n^\perp + \mathcal{P}_n = I$).

One can represent the operator B in a form

$$B = \mathcal{P}_n B \mathcal{P}_n + \mathcal{P}_n B \mathcal{P}_n^\perp + \mathcal{P}_n^\perp B \mathcal{P}_n + \mathcal{P}_n^\perp B \mathcal{P}_n^\perp.$$

The first three are compact operators in contrast to the forth, therefore we deduce

$$\lim_{n\to\infty} \|B - \mathcal{P}_n B \mathcal{P}_n + \mathcal{P}_n B \mathcal{P}_n^\perp + \mathcal{P}_n^\perp B \mathcal{P}_n\| = \delta > 0.$$

Corollary 2.3. *Let V_1 and V_2 be monomials from $\{V_{\inf}\}$ of different indices. Then $V_1 - V_2$ is not a compact operator.*

By Lemma 2.1 it follows that $C_\varphi^*(X)$ contains the compact operators. Let J_φ be an ideal of compact operators. If $C_\varphi^*(X)$ is irreducible then J_φ coincides with $K(l^2(X))$ – the set of all compact operators on $l^2(X)$. In general case $J_\varphi = K(l^2(X)) \bigcap C_\varphi^*(X)$. Throughout the paper, K denotes a compact operator in $C_\varphi^*(X)$.

We denote by $\mathfrak{C}^*_\varphi$ the quotient algebra of $C^*_\varphi(X)$ over J_φ. The image of partial isometry U_k under the canonical mapping

$$C^*_\varphi(X) \longrightarrow \mathfrak{C}^*_\varphi$$

will be denoted by W_k. It is obvious that the images of some partial isometries can be zero. Let $\{W_i\}_{i\in\mathbb{I}}$ ($\mathbb{I}$ is a subset of integers from 1 to m depending on mapping φ) be non zero images of operators $\{U_i\}_{i=1}^m$. Since $W_iW_i^*$ is an image of projection P_i not equal to zero and under the canonical mapping the projection is moved either into zero or into projection, then $W_iW_i^*$ is a projection in $\mathfrak{C}^*_\varphi$, hence the family $\{W_i\}_{i\in\mathbb{I}}$ is the family of partial isometries which generates the C^*-algebra $\mathfrak{C}^*_\varphi$ (because the family $\{U_i\}_{i=1}^m$ generates $C^*_\varphi(X)$).

As before, we will call the finite product of elements from $\{W_i\}_{i\in\mathbb{I}} \cup \{W_i^*\}_{i\in\mathbb{I}}$ the *monomial.* The family $\{W\}$ of all monomials in $\mathfrak{C}^*_\varphi$ together with zero monomial forms the involutive semigroup $\{W\}$ with zero element.

The *index* of monomial $W = W'_{j_1}W'_{j_2}\dots W'_{j_k}$ is defined to be the number

$$\operatorname{ind} W = \sum_{s=1}^{k} \operatorname{ind} W'_{j_s}, \qquad \text{where} \quad \operatorname{ind} W'_{j_s} = \pm 1,$$

in accordance with W'_{j_s} being from $\{W_i^*\}_{i\in\mathbb{I}}$ or $\{W_i\}_{i\in\mathbb{I}}$ correspondingly.

Lemma 2.4. *The index of monomial W does not depend on its representation as a product.*

Proof. Assume this is not the case. We consider the monomial W to have the representation

$$W = W'_{j_1}W'_{j_2}\dots W'_{j_k} \qquad \text{as well as} \quad W = W''_{i_1}W''_{i_2}\dots W''_{i_l},$$

and

$$\sum_{s=1}^{k} \operatorname{ind} W'_{j_s} \neq \sum_{s=1}^{l} \operatorname{ind} W''_{i_s}.$$

Then in the preimage of W there exists an operator V from $C^*_\varphi(X)$ such that from the one hand

$$V = U'_{j_1}U'_{j_2}\dots U'_{j_k}$$

and from the other

$$V = U''_{i_1}U''_{i_2}\dots U''_{i_l} + K.$$

Here U'_{j_s} and U''_{i_s} are preimages of W'_{j_s} and W''_{i_s}. Since $W \neq 0$, V is not compact operator in $C^*_\varphi(X)$. We have

$$U'_{j_1}U'_{j_2}\dots U'_{j_k} - (U''_{i_1}U''_{i_2}\dots U''_{i_l} + K) = V - V = 0.$$

But from Corollary 2.3 the difference $U'_{j_1}U'_{j_2}\dots U'_{j_k} - U''_{i_1}U''_{i_2}\dots U''_{i_l}$ is not compact operator. The proof is complete. □

3. Structure of the algebra $\mathfrak{C}^*_\varphi$

Let $\mathfrak{C}^*_{\varphi,n}$ denote the closed linear subspace in $\mathfrak{C}^*_\varphi$ generated by monomials of index n.

Lemma 3.1. *The following conditions hold:*

1) $\mathfrak{C}^*_{\varphi,n}\mathfrak{C}^*_{\varphi,s} = \mathfrak{C}^*_{\varphi,n+s}$;
2) $\mathfrak{C}^*_{\varphi,n} \cap \mathfrak{C}^*_{\varphi,s} = \{0\}, \quad \text{if} \quad n \neq s.$

Proof. The proof of the first statement immediately follows from the definition of the index of monomial. Let us prove the second by contradiction. Let $C^{*'}_{\varphi,n}$ be the preimage of $\mathfrak{C}^*_{\varphi,n}$ under the canonical mapping of $C^*_\varphi(X)$ into $\mathfrak{C}^*_\varphi$. There is no loss of generality in assuming

$$\mathfrak{C}^*_{\varphi,0} \cap \mathfrak{C}^*_{\varphi,n} \neq \{0\}.$$

Then there exists a nonzero element

$$A \in C^{*'}_{\varphi,0} \cap C^{*'}_{\varphi,n},$$

which one of the preimages of some nonzero element of $\mathfrak{C}^*_{\varphi,0} \cap \mathfrak{C}^*_{\varphi,n}$. Since A is not compact,

$$\lim_{N\to\infty} \|\mathcal{P}^\perp_N A \mathcal{P}^\perp_N\| = \epsilon > 0.$$

The preimages of monomial W from $\mathfrak{C}^*_\varphi$ contain the monomial $V \in \{V_{\text{inf}}\}$ from $C^*_\varphi(X)$. The primary monomials U_{j_i} and $U^*_{j_l}$, $i,j = \overline{1,m}$, participating in representation of V, are the preimages of the elements W_{j_i} and $W^*_{j_l}$, $i,j = \overline{1, m' \leq m}$ in representation of monomial W. By Corollary 2.2,

$$\operatorname{ind} W = \operatorname{ind} V.$$

Therefore there exist elements

$$A' = \sum_{i=1}^{l} \alpha'_i V'_i + K' \in C^{*'}_{\varphi,0} \quad \text{and} \quad A'' = \sum_{j=1}^{k} \alpha''_j V''_j + K'' \in C^{*'}_{\varphi,n}$$

such that

$$\|A' - A\| < \frac{\epsilon}{8}, \quad \|A'' - A\| < \frac{\epsilon}{8},$$

in addition $\operatorname{ind} V'_i = 0 = s' - n'$ for all $i = \overline{1,l}$ and $\operatorname{ind} V''_j = n = s'' - n''$ for all $j = \overline{1,k}$. Then

$$\lim_{N\to\infty} \|\mathcal{P}^\perp_N A' \mathcal{P}^\perp_N\| \geq \lim_{N\to\infty} \|\mathcal{P}^\perp_N A \mathcal{P}^\perp_N\| - \lim_{N\to\infty} \|\mathcal{P}^\perp_N (A - A') \mathcal{P}^\perp_N\| \geq \epsilon - \frac{\epsilon}{8} = \frac{7\epsilon}{8}.$$

Similarly,

$$\lim_{N\to\infty} \|\mathcal{P}^\perp_N A'' \mathcal{P}^\perp_N\| > \frac{7\epsilon}{8}.$$

K' and K'' are compact, which gives

$$\lim_{N\to\infty} \|\mathcal{P}^\perp_N A' \mathcal{P}^\perp_N\| = \lim_{N\to\infty} \left\| \mathcal{P}^\perp_N \left(\sum_{i=1}^{l} \alpha'_i V'_i \right) \mathcal{P}^\perp_N \right\|$$

and

$$\lim_{N\to\infty} \|\mathcal{P}_N^{\perp} A'' \mathcal{P}_N^{\perp}\| = \lim_{N\to\infty} \left\| \mathcal{P}_N^{\perp} \left(\sum_{j=1}^{k} \alpha_j'' V_j'' \right) \mathcal{P}_N^{\perp} \right\|.$$

Let $N > 2\sum_{i=1}^{l} n_i' + \sum_{j=1}^{k} (s_j'' + n_j'')$. Consider the Hilbert space

$$H_N = \bigoplus_{y \in X \setminus F_N} l^2(E_y^N).$$

From the definition of E_y^N, it follows that if $\mathcal{P}_N^{\perp}$ is projection onto the space $l^2(X \setminus F_N)$, then $\mathcal{P}_{2N}^{\perp}$ is projection onto H_N. N is chosen to be such that for all y from $X \setminus F_N$ the space $l^2(E_y^N)$ is invariant in $l^2(X)$ for linear combination of monomials of index zero and the linear combination $\sum_{j=1}^{k} \alpha_j'' V_j''$ moves $l^2(E_y^N)$ onto $l^2(E_y^{N+s})$ (follows from Lemma 1.1). Hence

$$\left\| \sum_{i=1}^{l} \alpha_i' V_i' \mid_{H_N} \right\| = \sup_{y \in X \setminus F_N} \left\| \sum_{i=1}^{l} \alpha_i' V_i' \mid_{l^2(E_y^N)} \right\|.$$

Using the relations above and taking into account that operators A' and A'' are not compact, $\mathcal{P}_{2N}^{\perp}$ is projection onto H_N which is invariant for $\sum_{i=1}^{l} \alpha_i' V_i'$ as well as the every $l^2(E_y^N)$, and $\sum_{j=1}^{k} \alpha_j'' V_j''$ moves the latter into the subspace $l^2(E_y^{N+s})$ (the subspaces being orthogonal to each other), we obtain

$$\begin{aligned}
\frac{\epsilon}{4} &> \|A' - A''\| \geq \lim_{N\to\infty} \|\mathcal{P}_N^{\perp} A' \mathcal{P}_N^{\perp} - \mathcal{P}_N^{\perp} A'' \mathcal{P}_N^{\perp}\| \\
&= \lim_{N\to\infty} \left\| \mathcal{P}_{2N}^{\perp} \left(\sum_{i=1}^{l} \alpha_i' V_i' \right) \mathcal{P}_{2N}^{\perp} - \mathcal{P}_{2N}^{\perp} \left(\sum_{j=1}^{k} \alpha_j'' V_j'' \right) \mathcal{P}_{2N}^{\perp} \right\| \\
&\geq \lim_{N\to\infty} \sup_{y \in X \setminus F_N} \left\| \left(\mathcal{P}_{2N}^{\perp} \left(\sum_{i=1}^{l} \alpha_i' V_i' \right) \mathcal{P}_{2N}^{\perp} - \mathcal{P}_{2N}^{\perp} \left(\sum_{j=1}^{k} \alpha_j'' V_j'' \right) \mathcal{P}_{2N}^{\perp} \right) \mid_{l^2(E_y^N)} \right\| \\
&\geq \lim_{N\to\infty} \sup_{y \in X \setminus F_N} \left\| \mathcal{P}_{2N}^{\perp} \left(\sum_{i=1}^{l} \alpha_i' V_i' \right) \mathcal{P}_{2N}^{\perp} \mid_{l^2(E_y^N)} \right\| \\
&= \lim_{N\to\infty} \left\| \mathcal{P}_{2N}^{\perp} \left(\sum_{i=1}^{l} \alpha_i' V_i' \right) \mathcal{P}_{2N}^{\perp} \right\| > \frac{3\epsilon}{4},
\end{aligned}$$

a contradiction. The lemma is proved. □

An important consequence is:

Corollary 3.2. *Let B from $\mathfrak{C}^*_\varphi$ have the form*

$$B = \sum_{k=-n}^{n} B_k,$$

where B_k from $\mathfrak{C}^*_{\varphi,k}$. Then for all k

$$\|B_k\| \leq \|B\|.$$

Proof. Let us consider an element $B \in \mathfrak{C}^*_\varphi$ and an arbitrary element $A \in C^*_\varphi(X)$ from the coset $[B]$. We can write the latter in a form

$$A = \mathcal{P}_N A \mathcal{P}_N + \mathcal{P}_N A \mathcal{P}_N^\perp + \mathcal{P}_N^\perp A \mathcal{P}_N + \mathcal{P}_N^\perp A \mathcal{P}_N^\perp.$$

it is obvious that $\mathcal{P}_N^\perp A \mathcal{P}_N^\perp \in [B]$.

For every $A \in [B]$, $\|B\| \leq \|A\|$. From the other hand, for all $\epsilon > 0$ there is an operator A_0 such that $\|A_0\| < \|B\| + \epsilon$, which gives

$$\begin{aligned} \|B\| &\leq \lim_{N\to\infty} \|\mathcal{P}_N^\perp A \mathcal{P}_N^\perp\| = \lim_{N\to\infty} \|\mathcal{P}_N^\perp (A_0 + K) \mathcal{P}_N^\perp\| \\ &\leq \lim_{N\to\infty} \|\mathcal{P}_N^\perp A_0 \mathcal{P}_N^\perp\| + \lim_{N\to\infty} \|\mathcal{P}_N^\perp K \mathcal{P}_N^\perp\| \\ &= \lim_{N\to\infty} \|\mathcal{P}_N^\perp A_0 \mathcal{P}_N^\perp\| \leq \|A_0\| < \|B\| + \epsilon, \end{aligned}$$

and finally that

$$\|B\| = \lim_{N\to\infty} \|\mathcal{P}_N^\perp A \mathcal{P}_N^\perp\|.$$

Assertion being proved follows from the concluding inequalities of Lemma's 3.1 proof. □

In [10] it was shown that under the assumption of absence of cyclic elements the monomials of index zero generate AF-algebra $C^*_{\varphi,0}$. We now show that there exists a nontrivial AF-subalgebra of $C^*_\varphi(X)$ even in case we refuse this assumption. Actually, if there are cyclic elements for φ then, by Lemma 2.1, $C^*_\varphi(X)$ contains the algebra of compact operators as well as the algebra generated by all monomials of zero index from $\{V_{\inf}\}$, we denote the latter by $C^*_{\varphi,0}$. Then $C^*_{\varphi,0} + J_\varphi$ is closed subalgebra of $C^*_\varphi(X)$ and, clearly, AF-algebra. Hence the quotient algebra $\big(C^*_{\varphi,0} + J_\varphi\big)/J_\varphi = \mathfrak{C}^*_{\varphi,0}$ is AF-algebra.

From Lemma 3.1 and arguments mentioned above it immediately follows the theorem:

Theorem 3.3. *C^*-algebra $\mathfrak{C}^*_\varphi$ is $\mathbb{Z}$-graded with AF-subalgebra responding to zero.*

Remind that the *conditional expectation* from C^*-algebra $\mathfrak{A}$ to C^*-subalgebra $\mathfrak{B}$ is, by definition, the completely positive contracting mapping $\theta : \mathfrak{A} \longrightarrow \mathfrak{B}$ such that $\theta(B) = B$ and $\theta(B_1 A B_2) = B_1 \theta(A) B_2$ for all $B, B_1, B_2 \in \mathfrak{B}$ and $A \in \mathfrak{A}$ ([13], [14]). In other words the conditional expectation is a projection of norm 1. Tomiyama proved in [15] that the inverse statement is true ([13], [16]).

Lemma 3.4. *In $\mathfrak{C}^*_\varphi$ there is a conditional expectation onto the subalgebra $\mathfrak{C}^*_{\varphi,0}$.*

Proof. Define a mapping

$$\mathcal{P} : \mathfrak{C}^*_\varphi \longrightarrow \mathfrak{C}^*_{\varphi,0},$$

on a set being dense everywhere by $\mathcal{P}\Big(\sum\limits_{k=-n}^{n} A_k\Big) = A_0$, where $A_k \in \mathfrak{C}^*_{\varphi,k}$ and $A_0 \in \mathfrak{C}^*_{\varphi,0}$, and extend it by continuity.

In [11, Theorem 3] it was proved that $C^*_\varphi(X)$ generated by a finite family of partial isometries is unital. It is clear that $C^*_{\varphi,0}$ contains the unit, hence, by Corollary 3.2, $\|\mathcal{P}\| = 1$. □

4. Covariant system associated with $C^*_\varphi(X)$

In this section, in much the same way as in [10], we construct the covariant system associated with $C^*_\varphi(X)$ in general case.

Define $\mathfrak{C}^*_\varphi$-valued function $\widetilde{W}$ on the unit circle for every monomial W from semigroup $\{W\}$:

$$\widetilde{W}(e^{i\theta}) = e^{i\operatorname{ind} W\theta} W.$$

Thus we obtain the family of functions on S^1, we denote it by $\{\widetilde{W}\}$. It is clear that $\{\widetilde{W}\}$ is a semigroup isomorphic to $\{W\}$. Let $\widetilde{\mathfrak{C}}^*_\varphi$ be a C^*-subalgebra of $C(S^1, \mathfrak{C}^*_\varphi)$ generated by the semigroup $\{\widetilde{W}\}$. Since

$$\widetilde{W}(e^{i\theta_0} e^{i\theta}) = e^{i\operatorname{ind} W\theta_0} \widetilde{W}(e^{i\theta})$$

for all $\widetilde{W}$ from $\{\widetilde{W}\}$, $\widetilde{\mathfrak{C}}^*_\varphi$ is invariant with respect to the shifts by elements of S^1. Therefore we can define the covariant system

$$(\widetilde{\mathfrak{C}}^*_\varphi, S^1, \kappa),$$

where $\kappa : S^1 \longrightarrow \operatorname{Aut}\widetilde{\mathfrak{C}}^*_\varphi$ is a group homomorphism generated by shift operators,

$$(\kappa(e^{i\theta_0})\widetilde{B})(e^{i\theta}) = \widetilde{B}(e^{i(\theta+\theta_0)}).$$

Let

$$\widetilde{\mathfrak{C}}^*_{\varphi,n} = \{\widetilde{B} \in \widetilde{\mathfrak{C}}^*_\varphi : \kappa(e^{i\theta})(\widetilde{B}) = e^{in\theta}\widetilde{B}\}.$$

Theorem 4.1. *C^*-algebras $\widetilde{\mathfrak{C}}^*_\varphi$ and $\mathfrak{C}^*_\varphi$ are isomorphic.*

Proof. The mapping

$$\widetilde{B} \longrightarrow \widetilde{B}(e^{i0})$$

generates the continuous surjective *-homomorphism

$$\Phi : \widetilde{\mathfrak{C}}^*_\varphi \longrightarrow \mathfrak{C}^*_\varphi.$$

Let $\widetilde{B}$ be a function from $\ker\Phi$ with Fourier series $\sum\limits_{n=-\infty}^{\infty} \widetilde{B}_n$. Then $\sum\limits_{n=-\infty}^{\infty} \widetilde{B}_n(e^{i0})$ is Fourier series of the image $\widetilde{B}$ from $\mathfrak{C}^*_\varphi$. By Corollary 3.2, $\widetilde{B}_n(e^{i0}) = 0$. Since

$$\widetilde{B}_n(e^{i\theta}) = e^{in\theta} B_n,$$

B_n from $\mathfrak{C}^*_{\varphi,n}$, we obtain $\|\widetilde{B}_n\| = 0$. Hence $\widetilde{B} = 0$ and Φ is isomorphism. $\square$

It follows that there is a covariant system $(\mathfrak{C}^*_\varphi, S^1, \kappa)$ isomorphic to $(\widetilde{\mathfrak{C}}^*_\varphi, S^1, \kappa)$.

The action of a circle on C^*-algebras was studied in details in [17, 18] It is obvious that $\mathfrak{C}^*_{\varphi,n}$ is the n^{th} spectral subspace for κ. Summarizing all mentioned above, we have the following theorem:

Theorem 4.2. *If there are cyclic elements for φ then the last generates the covariant system $(\mathfrak{C}^*_\varphi, S^1, \kappa)$ with a semi-saturated action κ and the fixed point subalgebra $\mathfrak{C}^*_{\varphi,0}$ is AF-algebra.*

Note that the conditional expectation constructed in previous section is the one onto the fixed point subalgebra with respect to the action of the circle as a group of automorphisms of $\mathfrak{C}^*_\varphi$ ([13, II.6.10.4]).

Corollary 4.3. *Algebra $C^*_\varphi(X)$ is nuclear.*

Proof. We first show that $\mathfrak{C}^*_\varphi$ is nuclear. $\mathfrak{C}^*_{\varphi,0}$ is a fixed point subalgebra with respect to continuous action of compact group S^1 and AF-algebra by Theorem 3.3. Hence, by [[19], 4.5.2], $\mathfrak{C}^*_\varphi$ is nuclear. As far as J_φ is an ideal of compact operators (AF-algebra), $C^*_\varphi(X)$ is nuclear C^*-algebra ([13, IV.3.1.3]). $\square$

References

[1] J. Cuntz, *On the simple C^*-algebras generated by isometries.* Comn. Math. Phys. **57** (1977), 173–185.

[2] I. Cho and P. Jorgensen, *C^*-algebras generated by partial isometries.* J. Appl. Math. Comput. **26** (2008), 1–48.

[3] J. Cuntz and W. Krieger, *A class of C^*-algebras and topological Markov chains.* Invent. Math. **56**(3) (1980), 251–268.

[4] A. Kumjian, *On certain Cuntz–Pimsner algebras*, arXiv:math.OA/0108194 v1 (2001).

[5] V. Deaconu, A. Kumjian, and P. Muhly, *Cohomology of topological graphs and Cuntz–Pimsner algebras*, arXiv:math/9901094v1[math.OA](1999).

[6] R. Exel, M. Laca, and J. Quigg, *Partial dynamical systems and C^*-algebras generated by partial isometries*, arXiv:funct-an/9712007.

[7] S. Grigoryan and A. Kuznetsova, *C^*-algebras generated by mappings.* Lobachevskii J. of Math. **29**(1) (2008), 5–8.

[8] S.A. Grigoryan and A.Yu. Kuznetsova, *C^*-algebras Generated by Mappings.* Mat. Zametki 87(5) (2010), 694–703, [Russian]. English transl. Math. Notes **87**(5) (2010), 663–671.

[9] S.A. Grigoryan and A.Yu. Kuznetsova, *AF-subalgebras of a C^*-algebra generated by a mapping*. Izv. Vyssh. Zaved., Matem. 54(3) (2010), 82–87, [Russian]. English transl. Russian Mathematics (Iz. VUZ) **54**(3) (2010), 72–76.

[10] S. Grigoryan and A. Kuznetsova, *On a class of nuclear C^*-algebras*. An Operator Theory Summer, Proceedings of the 23rd international conference on operator theory (Timisoara, Romania, 2010), 39–50.

[11] A.Yu. Kuznetsova, *On a class of C^*-algebras generated by a countable family of partial isometries* . Izv. NAN Armen. Matem. 45(6) (2010), 51–62, [Russian]. English transl. Journal of Contemporary Mathematical Analysis **45**(6) (2010), 320–328.

[12] A.Yu. Kuznetsova and E.V. Patrin, *One class of a C^*-algebras generated by a family of partial isometries and multiplicators*. Izv. Vyssh. Zaved., Matem. 56(6) (2012), 44–55, [Russian]. English transl. Russian Mathematics (Iz. VUZ) **56**(6) (2012), 37–47.

[13] B. Blackadar, *Operator algebras*. Springer, 2006.

[14] U. Umegaki, *Conditional expectations in an operator algebra* I. Tôhoku Math. J. **6**(1) (1954), 177–181.

[15] J. Tomiyama, *On the projection of norm one in W^*-algebras*. Proc. Japan Acad. **33**(10) (1957), 608–612.

[16] S. Strătilă, *Modular theory in operator algebras*. Editura Academiei Republicii Socialistic România, Bucharest, 1981 [translation from the Romanian by the author].

[17] W.L. Pashke, *K-Theory for actions of the circle group on C^*-algebras*. J. Oper. Theory **6** (1981), 125–133.

[18] Ruy Exel, *Circle actions on C^*-algebras, partial automorphisms and generalized Pimsner-Voiculescu exact sequence*, arXiv:funct-an/9211001v1 22Nov (1992).

[19] Nathanial P. Brown and Narutaka Ozawa, *C^*-algebras and Finite-Dimensional Approximations*. Graduate Studies in Mathematics, vol. 88, American Mathematical Society, Providence, Rhode Island, 2008.

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On Spectral Subspaces and Inner Endomorphisms of Some Semigroup Crossed Products

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Abstract. This note provides a look into some of the abstract properties of the semigroup crossed product of a unital C^*-algebra by an action consisting of endomorphisms for which there is a left inverse. The objective is to describe in general terms some of the relations between spectral subspaces for the canonical coaction on the crossed product and certain eigenspaces of a time-evolution on the crossed product. The present analysis is inspired by certain constructions due to Cornelissen and Marcolli [2].

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Introduction

Semigroup crossed products as models for C^*-algebras arising in number theory were first considered by Laca and Raeburn in [8], with motivation provided by the work in [1]. In employing techniques of the theory of semigroup crossed products they were able to offer simplifications of the presentation of the reduced Hecke C^*-algebra in [1] and show that this algebra possessed a universal property. Similar classes of semigroup crossed products were studied later by many authors, and recently they appeared in work of Cornelissen and Marcolli [2].

The starting point in [2], from the point of view of semigroup crossed products, is common to many of the previous studies, and consists of a dynamical system (A, S, α) where A is a unital C^*-algebra, S is a semigroup with nice properties (typically part of a lattice ordered group (G, S)), and α is an action of S by injective endomorphisms of A for which there is a left inverse. The interest in [2] is in the crossed product $\mathcal{A} = A \rtimes_\alpha S$ as part of a quantum statistical mechanical system $(\mathcal{A}, \sigma)$ arising from a time-evolution σ_t on $\mathcal{A}$ for $t \in \mathbb{R}$. Cornelissen and Marcolli's study of isomorphism of two quantum statistical mechanical systems $(\mathcal{A}, \sigma)$ and $(\mathcal{A}', \sigma')$ associated to two number fields is heavily dependent on delicate number theoretic facts and arguments. However some of the ingredients introduced along the way pertain to the semigroup crossed product and may be looked at in general terms. This is the objective of the present note. We study here relations between the spectral subspaces of $A \rtimes_\alpha S$ associated to the canonical coaction of G and the eigenspaces of a time evolution σ on the crossed product. As application we identify inner endomorphisms (in the terminology of Cornelissen–Marcolli) of $(\mathcal{A}, \sigma)$ that are dagger inner endomorphisms of $A \rtimes_\alpha S$, where the dagger subalgebra is another new ingredient introduced in [2]. A crucial aspect in [2] is preservation of the dagger subalgebra under inner endomorphisms, and one of their results relies on characterizing dagger inner endomorphisms as those inner endomorphisms preserving the dagger subalgebra. It is a natural question whether it is possible to identify in greater generality elements among the inner endomorphisms that are dagger inner isomorphisms. We shall provide an affirmative answer to this question under some suitable conditions.

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1. Semigroup crossed products

Suppose that S is a subsemigroup of a (discrete) group G containing the identity element e and $\alpha : S \to \mathrm{End}(A)$ is an action of S by endomorphisms of a unital C^*-algebra A. The semigroup crossed product $A \rtimes_\alpha S$ is the unital C^*-algebra generated by (the image of) a pair (i_A, i_S) in which i_A a $*$-homomorphism of A and i_S is a semigroup homomorphism of S satisfying the *covariance condition*

$$i_S(s)i_A(a)i_S(s)^* = i_A(\alpha_s(a)) \tag{1.1}$$

for all $s \in S$ and $a \in A$; the pair (i_A, i_S) is universal in the sense that any pair (π, V) (into $B(H)$ for some Hilbert space H) satisfying the analogue of (1.1) factors through (i_A, i_S). If the system admits a pair (π, V) satisfying the covariance condition with π injective, then i_A is injective. We shall assume henceforth that i_A is injective.

We identify g in G with its image as a generating unitary inside $C^*(G)$. It is known that there is a homomorphism i_G from $C^*(G)$ into $C^*(G) \otimes C^*(G)$, with $\otimes$ denoting the minimal tensor product, such that $i_G : g \mapsto g \otimes g$. Recall that a coaction δ of G on a unital C^*-algebra $\mathcal{A}$ is an injective homomorphism $\delta : \mathcal{A} \to \mathcal{A} \rtimes C^*(G)$ which satisfies $(\delta \circ \mathrm{id}_{C^*(G)}) \circ \delta = (\mathrm{id}_{\mathcal{A}} \otimes i_G) \circ \delta$ and $\mathrm{span}(\delta(\mathcal{A})(1 \otimes C^*(G)))$ is dense in $\mathcal{A} \otimes C^*(G)$. We refer to [10] for basic properties of coactions of discrete groups on C^*-algebras.

With the notation of the first paragraph of this section, [7, Proposition 6.1] (see also their Remark 6.2 which says that one can work with the minimal tensor product), shows that there is a coaction δ of G on $A \rtimes_\alpha S$ such that $\delta(i_A(a)) = i_A(a) \otimes e$ and $\delta(i_S(s)) = i_S(s) \otimes s$ for $a \in A$ and $s \in S$.

Proposition 1.1. *The cosystem $(A \rtimes_\alpha S, G, \delta)$ is a maximal coaction in the sense of* [4].

Proof. Let $\mathcal{A}$ be the Fell bundle associated to $(A \rtimes_\alpha S, G, \delta)$, and let $C^*(\mathcal{A})$ be its full sectional C^*-algebra. Then the universal properties of both algebras imply that $A \rtimes_\alpha S \cong C^*(\mathcal{A})$, and the result follows because $C^*(\mathcal{A})$ with its canonical coaction of G is a maximal coaction system. □

Let S be a cancellative semigroup with identity e. We say that S is right-reversible (or S is an Ore semigroup) provided that $Ss \cap Ss' \neq \emptyset$ for any pair $s, s' \in S$. It follows that S embeds in its group of left quotients $G = S^{-1}S$. If $S \cap S^{-1} = \{e\}$, then $x \leq_r y \iff yx^{-1} \in S$ is a right-invariant order on G. Note that the right-reversibility condition ensures that any two elements $s, s' \in S$ have a common upper bound in S. In the next section we shall assume that there exists a least upper bound for any two elements in S (and in G).

It is known that $A \rtimes_\alpha S$ is the closed span of monomials $i_S(s)^* i_A(a) i_S(s')$ for $a \in A$ and $s, s' \in S$, see [6, 9]. The algebraic crossed product is

$$A \rtimes_\alpha^{\mathrm{alg}} S = \operatorname{span}\{i_S(s)^* i_A(a) i_S(s') \mid s, s' \in S, a \in A\}.$$

By extending the terminology from [2] for a certain class of semigroup crossed products, the dagger subalgebra $(A \rtimes_\alpha S)^\dagger$ of $A \rtimes_\alpha S$ is the algebra generated by $i_A(A)$ and the isometries $i_S(s)$ for $s \in S$. In this case, $(A \rtimes_\alpha S)^\dagger = \operatorname{span}\{i_A(a) i_S(s) \mid a \in A, s \in S\}$.

The next result was claimed in [9] without proof. We include some details of the proof here because they will be useful later. Recall that the spectral subspace at $g \in G$ for the coaction δ on $A \rtimes_\alpha S$ is the space $(A \rtimes_\alpha S)_g$ consisting of $x \in A \rtimes_\alpha S$ such that $\delta(x) = x \otimes g$.

Lemma 1.2. *The spectral subspaces corresponding to δ are given by*

$$(A \rtimes_\alpha S)_g^\delta = \overline{\operatorname{span}}\, \{i_S(s)^* i_A(a) i_S(s') : g = s^{-1}s', s, s' \in S, a \in A\}, \tag{1.2}$$

for $g \in G$. In particular, the fixed point algebra $(A \rtimes_\alpha S)_e^\delta$ for δ is

$$(A \rtimes_\alpha S)_e^\delta = \overline{\operatorname{span}}\, \{i_S(s)^* i_A(a) i_S(s) : s \in S, a \in A\}. \tag{1.3}$$

Proof. We prove first (1.3). Clearly $\overline{\operatorname{span}}\, \{i_S(s)^* i_A(a) i_S(s) : s \in S, a \in A\}$ is contained in $(A \rtimes_\alpha S)_e$. For the other inclusion, let $c \in (A \rtimes_\alpha S)_e$. We may assume $c = \sum_{j=1}^n i_S(s_j)^* i_A(a_j) i_S(s_j')$ for $s_j, s_j' \in S$ and $a_j \in A$. Let λ_G be the regular representation of G (extended to $C^*(G)$), and let i_G^r denote the embedding of G as unitaries in $C_r^*(G)$. The canonical trace τ on $C_r^*(G)$ carries $i_G^r(s_j^{-1}s_j')$ to 1 if $s_j^{-1}s_j' = e$ and to zero otherwise. Since $\delta(c) - c \otimes e = 0$ in $(A \rtimes_\alpha S) \otimes C^*(G)$, by applying $(\mathrm{id} \otimes \tau) \circ (\mathrm{id} \otimes \lambda_G)$ to the difference we get that the sum of terms $i_S(s_j)^* i_A(a_j) i_S(s_j')$ in c in which $s_j^{-1}s_j' \neq e$ is zero. This proves (1.3).

To prove the general case note that the right to left inclusion in (1.2) follows from the definition of δ. For the other inclusion let $c \in (A \rtimes_\alpha S)_g$ for some $g \in G$. Then $c^*c \in (A \rtimes_\alpha S)_g^*(A \rtimes_\alpha S)_g$, which is included in $(A \rtimes_\alpha S)_e$ by [10, Lemma

1.3]. Assume $c = \sum_{j=1}^n i_S(s_j)^* i_A(a_j) i_S(s'_j)$ for $s_j, s'_j \in S$ and $a_j \in A$. To simplify notation, let $M_j = i_S(s_j)^* i_A(a_j) i_S(s'_j)$. Then

$$\begin{aligned} c^*c &= \Big(\sum_j M_j\Big)^* \Big(\sum_k M_k\Big) \\ &= \sum_j M_j^* M_j + \sum_{j\neq k} M_j^* M_k \\ &= \sum_j i_S(s'_j)^* i_A(a_j^* \alpha_{s_j}(1) a_j) i_S(s'_j) + \sum_{j\neq k} M_j^* M_k; \end{aligned}$$

we compute the second sum separately. For each pair (j,k) with $j,k = 1,\dots,n$ and $j \neq k$ we choose $u_{(j,k)}, v_{(j,k)} \in S$ such that $s_j s_k^{-1} = u_{(j,k)}^{-1} v_{(j,k)}$. Then $w_{(j,k)} := u_{(j,k)} s_j = v_{(j,k)} s_k$. Letting

$$b_{(j,k)} := \alpha_{u_{(j,k)}}(a_j^*) \alpha_{w_{(j,k)}}(1) \alpha_{v_{(j,k)}}(a_k)$$

for $j \neq k$ we obtain

$$\begin{aligned} c^*c &= \sum_j i_S(s'_j)^* i_A(a_j^* \alpha_{s_j}(1) a_j) i_S(s'_j) + \sum_{j\neq k} M_j^* M_k \\ &= \sum_j i_S(s'_j)^* i_A(a_j^* \alpha_{s_j}(1) a_j) i_S(s'_j) + \sum_{j\neq k} i_S(u_{(j,k)} s'_j)^* i_A(b_{j,k}) i_S(v_{(j,k)} s'_k). \end{aligned}$$

Since $c^*c \in (A \rtimes_\alpha S)_e$, we obtain from two applications of (1.3) that

$$(u_{(j,k)} s'_j)^{-1} v_{(j,k)} s'_k = e$$

for all j,k with $j \neq k$. Thus we must have $(s'_j)^{-1} s_j s_k^{-1} s'_k = e$ or, equivalently, $s_j^{-1} s'_j = s_k^{-1} s'_k$ for all j,k with $j \neq k$. This means that

$$g' := s_j^{-1} s'_j \text{ for all } j = 1,\dots,n.$$

Then $\delta(c) = c \otimes g = c \otimes g'$, so $g = g' = s_j^{-1} s'_j$ for all j. □

2. Semigroup dynamical systems with left inverses

Semigroup dynamical systems of the form (A, S, α) with S a right-reversible Ore semigroup and α an action by injective endomorphisms of a unital C^*-algebra A are useful models of some C^*-algebras arising in number theory. In such examples, the action α has a left inverse. To formalise this, a *left inverse for* α is an action $\alpha' : S \to \mathrm{End}(A)$ such that $\alpha'_s \circ \alpha_s = \mathrm{id}$ and $\alpha_s \circ \alpha'_s$ is multiplication by the projection $\alpha_s(1)$ for all $s \in S$. We shall assume that $\alpha_e(1) = 1$ and $\alpha_s(1) \neq 1$ for $s \in S \setminus \{e\}$. If (A, S, α, α') is such a system, then i_A induces an isomorphism $i_A : A \xrightarrow{\cong} (A \rtimes_\alpha S)_e$, see, e.g., [9, Proposition 3.1(1)] (note that the properties (iii)–(iv) are not needed for assertion (1) in that proposition). Note also that $\alpha'_s(a) = i_S(s)^* i_A(a) i_S(s)$ for all $s \in S$ and $a \in A$. We next note that such systems have a gauge-invariant uniqueness property.

Proposition 2.1 (The gauge-invariant uniqueness property). *Let S be an Ore semigroup with enveloping group $G = S^{-1}S$ and $\alpha : S \to \operatorname{End}(A)$ an action of S on a unital C^*-algebra A for which there is a left inverse $\alpha' : S \to \operatorname{End}(A)$. A surjective homomorphism $\varphi : A \rtimes_\alpha S \to C$ is injective if and only if $\varphi|_{i_A(A)}$ is injective and there is a maximal coaction η of G on C such that φ is δ–η-equivariant.*

Proof. Since δ is maximal, this follows from the gauge-invariant uniqueness theorem for maximal coactions from [5, Corollary A.2]. □

Given a right-reversible semigroup S such that $S \cap S^{-1} = \{e\}$, assume moreover that for any two elements $s, s' \in S$ there is $\tilde{s} \in S$ satisfying $Ss \cap Ss' = S\tilde{s}$. Then the element $\tilde{s}$ is the least upper bound of s, s' with respect to the order $\leq_r$. We denote $\tilde{s} = s \vee_r s'$, and refer to S as being right-lattice ordered whenever any two elements admit a $\vee_r$.

Suppose that S is right-lattice ordered. Let $G = S^{-1}S$ be its group of left quotients. If $x = s^{-1}t \in G$, then $x \leq_r t$, so every element in G admits a right upper bound in S. It is not true in general that any $x \in G$ admits a least upper bound in S. Assume that every element in G admits a least upper bound in S with respect to $\leq_r$, in which case we refer to (G, S) as being right-lattice ordered. Then similar to the argument of [3, Lemma 7], which deals with the case of left-invariant orders, it follows that (G, S) is right-lattice ordered precisely when for each $g \in G = S^{-1}S$, there is a unique pair of elements $s, s' \in S$ such that:

1. $s \wedge_l s' = e$ (where $\wedge_l$ denotes greatest lower bound in G with respect to the left-invariant order $x \leq_l y \iff x^{-1}y \in S$),
2. $g = s^{-1}s'$, and
3. for any decomposition $g = z^{-1}z'$ with $z, z' \in S$ we have $s \leq_r z$ and $s' \leq_r z'$.

We denote $g_- := s$ and $g_+ := s'$ and refer to (g_-, g_+) as the minimal pair in $S \times S$ associated to g.

Under these assumptions, the spectral subspaces admit a particularly nice description.

Corollary 2.2. *Assume that (A, S, α) is a dynamical system where A is unital, (G, S) is right-lattice ordered, and α is an action by endomorphisms for which there is a left-inverse α'. Then for each $g \in G$ with associated minimal pair $(g_-, g_+) \in S \times S$ we have*

$$(A \rtimes_\alpha S)_g = \overline{\operatorname{span}}\,\{i_S(g_-)^* i_A(a) i_S(g_+) : a \in A\}.$$

Proof. By (1.2), it suffices to prove the left to right inclusion. Let $z, z' \in S$ such that $g = z^{-1}z'$. Then $g_- \leq_r z$ and $g_+ \leq_r z'$ with $z(g_-)^{-1} = z'(g_+)^{-1} \in S$, and the claim follows from the calculations

$$\begin{aligned} i_S(z)^* i_A(a) i_S(z') &= i_S(g_-)^* i_S(z(g_-)^{-1})^* i_A(a) i_S(z'(g_+)^{-1}) i_S(g_+) \\ &= i_S(g_-)^* i_A(\alpha'_{z(g_-)^{-1}}(a)) i_S(g_+). \end{aligned}$$ □

Let (G, S) be a right-lattice ordered group and (A, S, α) a semigroup dynamical system with injective endomorphisms of the unital C^*-algebra A. Let δ

be the canonical coaction of G on $A \rtimes_\alpha S$. Suppose that $N_G : G \to (0,\infty)$ is a homomorphism, where $(0,\infty)$ has its multiplicative structure. The universal property of $A \rtimes_\alpha S$ implies that there is a one-parameter group of automorphisms $\sigma : \mathbb{R} \to \mathrm{Aut}(A \rtimes_\alpha S)$ such that

$$\sigma_t(i_S(s)^* i_A(a) i_S(s')) = N_G(s^{-1}s')^{it} i_S(s)^* i_A(a) i_S(s'). \tag{2.1}$$

We need to recall some terminology from [2]. A quantum statistical mechanical system $(\mathcal{A}, \sigma)$ consists of a C^*-algebra $\mathcal{A}$ with a one-parameter group of automorphisms (a time evolution) σ. An element $c \in \mathcal{A}$ is an eigenvector of σ if there is $\lambda \in (0,\infty)$ such that $\sigma_t(c) = \lambda^{it} c$ for all $t \in \mathbb{R}$. An endomorphism of $(\mathcal{A}, \sigma)$ is a $*$-homomorphism $\varphi : \mathcal{A} \to \mathcal{A}$ such that $\varphi \circ \sigma_t = \sigma_t \circ \varphi$ for all $t \in \mathbb{R}$.

Definition 2.3 ([2, Definition 1.8]). Suppose that $(\mathcal{A}, \sigma)$ is a quantum statistical mechanical system. An *inner endomorphism* of $(\mathcal{A}, \sigma)$ is an endomorphism φ of $(\mathcal{A}, \sigma)$ such that there exists u an isometry in $\mathcal{A}$ and an eigenvector of σ for which $\varphi(x) = uxu^*$ for all $x \in \mathcal{A}$.

To simplify notation, we let $\mathcal{A}^\sigma_\lambda$ denote the space of eigenvectors c of σ such that $\sigma_t(c) = \lambda^{it} c$ for all $t \in \mathbb{R}$. By (2.1), $(A \rtimes_\alpha S)_g \subseteq (A \rtimes_\alpha S)^\sigma_{N_G(g)}$ for all $g \in G$. The next couple of results present partial converses to this inclusion.

Remark 2.4. Suppose (G, S) is a right-lattice ordered group, (A, S, α) is a semigroup dynamical system with a left inverse α', and σ is a time-evolution on $A \rtimes_\alpha S$ given as in (2.1). Let $(A \rtimes_\alpha S)^\dagger$ be the dagger subalgebra of $A \rtimes_\alpha S$. Suppose that S is abelian. We claim that every inner endomorphism of $(A \rtimes_\alpha S, \sigma)$ preserves the closure of the dagger subalgebra. To see this, assume first that $u = i_S(s)^* i_A(b) i_S(s')$ and $x = i_A(a) i_S(p) \in (A \rtimes_\alpha S)^\dagger$. Then

$$uxu^* = i_A\big(\alpha'_s(b\alpha_{s'}(a)\alpha_p(\alpha_{s'}(1)b^*))\big) i_S(p) \in (A \rtimes_\alpha S)^\dagger.$$

By continuity, we have $u\overline{(A \rtimes_\alpha S)^\dagger}u^* \subseteq \overline{(A \rtimes_\alpha S)^\dagger}$ for any isometry $u \in A \rtimes_\alpha S$, showing the claim.

There is no reason to expect in general that an endomorphism of $(A \rtimes_\alpha S, \sigma)$ that preserves the dagger subalgebra necessarily must be an inner endomorphism. In [2, Definition 1.8], a *dagger inner endomorphism* of $(\mathcal{A}, \sigma)$ is an inner endomorphism of the form $\varphi(x) = uxu^*$ for all $x \in \mathcal{A}$, where $u \in (A \rtimes_\alpha S)^\dagger$ is an isometry and an eigenvector of σ. We make the following slight modification of this notion.

Definition 2.5. A *dagger inner endomorphism* of $(\mathcal{A}, \sigma)$ is an inner endomorphism φ such that for some $u \in \overline{(A \rtimes_\alpha S)^\dagger}$ which is an eigenvector of σ we have $\varphi(x) = uxu^*$ for all $x \in \mathcal{A}$.

In the proof of [2, Proposition 10.1], it is observed that for the particular systems under consideration, inner endomorphisms that preserve the dagger subalgebra coincide with the dagger inner endomorphisms. Thus a natural question is whether it is possible to identify in greater generality elements among the inner endomorphisms that are dagger inner isomorphisms. We shall provide an affirmative answer to this question under some suitable conditions.

Assume that the homomorphism N_G which induces the time evolution in (2.1) satisfies the following two conditions:

Hom1. $N_G(S) \subseteq \mathbb{N} \setminus \{0\}$, and
Hom2. $N_G(s)$ and $N_G(s')$ are co-prime whenever $s, s' \in S \setminus \{e\}$ and $s \wedge_l s' = e$.

Lemma 2.6. *Suppose (G, S) is a right-lattice ordered group, (A, S, α) is a semigroup dynamical system with a left inverse α', and σ is the time-evolution on $A \rtimes_\alpha S$ associated to a homomorphism N_G that satisfies* (Hom1) *and* (Hom2).

Let $c \in (A \rtimes_\alpha S) \cap (A \rtimes_\alpha S)^\sigma_{m/n}$, where m, n are positive integers such that m, n are coprime. If $c^*c \in (A \rtimes_\alpha S)_e$, then there is $g \in G$ with $N_G(g_+) = m$ and $N_G(g_-) = n$ such that $c \in (A \rtimes_\alpha S)_g$.

Proof. From the hypothesis we have $\sigma_t(c) = (\frac{m}{n})^{it} c$. Suppose first that $c = \sum_{j=1}^J i_S(s_j)^* i_A(a_j) i_S(s'_j)$ for $s_j, s'_j \in S$ and $a_j \in A$. As in the proof of (1.2), the assumption that $c^*c \in (A \rtimes_\alpha S)_e$ implies that there is $g \in G$ such that $g = s_j^{-1} s'_j$ for all $j = 1, \ldots, J$. By Corollary 2.2, we may assume $c \in i_S(g_-)^* i_A(A) i_S(g_+)$, where $g = (g_-)^{-1} g_+$ is the minimal decomposition of g such that $g_- \wedge_l g_+ = e$. Then $\sigma_t(c) = (N_G((g_-)^{-1} g_+))^{it} c$, and thus $N_G(g_-)^{-1} N_G(g_+) = m/n$. By (Hom2), we must have $N_G(g_-) = n$ and $N_G(g_+) = m$, and the lemma follows. □

As a consequence of Lemma 2.6, it follows that the set of fixed-points of σ_t is exactly $i_A(A)$.

Corollary 2.7. *We have $i_A(A) = (A \rtimes_\alpha S)^\sigma_1$.*

Proof. Clearly the set of fixed-points of σ_t is a C^*-subalgebra of $A \rtimes_\alpha S$ that contains $i_A(A)$ by the definition of σ. Let $u \in A \rtimes_\alpha S$ be a unitary such that $\sigma_t(u) = u$. Lemma 2.6 implies that there are $s, s' \in S$ with $N_G(s) = N_G(s')$ and $s \wedge_l s' = e$ such that $u \in i_S(s)^* i_A(A) i_S(s')$. This in connection with (Hom2) forces $s = s' = e$ and the corollary follows. □

In examples, a system of the form (A, S, α, α') often has an additional feature. Denoting $v_s := i_S(s)$ for $s \in S$, assume that

$$v^*_{s'} v_s = v_s v^*_{s'} \text{ when } s \wedge_l s' = e \text{ in } S. \tag{2.2}$$

Assuming (2.2), it follows that for every $a \in A$, we have

$$\begin{aligned} i_A(\alpha'_{s'}(\alpha_s(a))) &= v^*_{s'} i_A(\alpha_s(a)) v_{s'} = v^*_{s'} v_s i_A(a) v^*_s v_{s'} \\ &= v_s v^*_{s'} i_A(a) v_{s'} v^*_s = i_A(\alpha_s(\alpha'_{s'}(a))). \end{aligned}$$

Thus (2.2) says that the actions α and α' satisfy the identity

$$\alpha'_{s'} \circ \alpha_s = \alpha_s \circ \alpha'_{s'} \text{ whenever } s, s' \in S \text{ with } s \wedge_l s' = e. \tag{2.3}$$

Note that the converse is valid, too: applying (2.3) to $a = 1$ and writing out the identity using the isometries v gives (2.2).

Lemma 2.8. *Assume that the actions α and α' satisfy* (2.3) *and $\alpha_s(1)$ are central projections in A for all $s \in S$. Let $u \in A \rtimes_\alpha S$ be an isometry with $u \in (A \rtimes_\alpha S)^\sigma_{m/n}$ for positive integers m, n such that m, n are coprime. Then $n = 1$ and there exists $s' \in S$ such that $N_G(s') = m$ and $u \in (A \rtimes_\alpha S)_{s'}$. In particular, u belongs to the closure of $(A \rtimes_\alpha S)^\dagger$.*

Proof. By Lemma 2.6, we may assume that $u = i_S(s)^* i_A(a) i_S(s')$ for $s, s' \in S$ with $N_G(s) = n, N_G(s') = m$ and $s \wedge_l s' = e$. Then

$$\begin{aligned} u^*u &= i_A\big(\alpha'_{s'}(a^*\alpha_s(1)a)\big) \\ &= i_A\big(\alpha'_{s'}(\alpha_s(1)a^*a)\big) \text{ since } \alpha_s(1) \text{ is central} \\ &= i_A(\alpha'_{s'} \circ \alpha_s \circ \alpha'_s(a^*a)) \\ &= i_A(\alpha_s \circ \alpha'_{s'} \circ \alpha'_s(a^*a))) \text{ by (2.3).} \end{aligned}$$

The assumption $u^*u = 1$ implies that $\alpha_s(a') = 1$ for $a' = \alpha'_{s'} \circ \alpha'_s(a^*a)$, from which we infer that $a' = \alpha'_s(\alpha_s(a')) = \alpha'_s(1) = 1$. Thus $\alpha_s(1) = 1$, which necessarily implies that $s = e$, and in particular that $n = 1$. The lemma follows. □

Theorem 2.9. *Suppose (G, S) is a right-lattice ordered group, (A, S, α) is a semigroup dynamical system with a left inverse α', and σ is the time-evolution on $A \rtimes_\alpha S$ associated to a homomorphism N_G that satisfies* (Hom1) *and* (Hom2). *Assume further that α and α' satisfy* (2.3) *and $\alpha_s(1)$ are central projections in A for all $s \in S$.*

Then every inner endomorphism of $(A \rtimes_\alpha S, \sigma)$ corresponding to a positive rational eigenvalue of the time evolution is a dagger inner endomorphism.

Proof. Assume that φ is an inner endomorphism of the form $\varphi(x) = uxu^*$ for $x \in A \rtimes_\alpha S$, where $u \in (A \rtimes_\alpha S)^\sigma_q$ for some rational q. Writing $q = m/n$ for $m, n \in \mathbb{N}$, $n \neq 0$, m, n co-prime, it follows from Lemma 2.8 that u is in the closure of $(A \rtimes_\alpha S)^\dagger$, as claimed. □

Given systems (A, S, α) and (B, R, β) where (G, S) and (K, R) are right-lattice ordered, assume α' is a left-inverse for α such that $\alpha'_s(1) = 1$, $\alpha_s(1)$ is central in A for all $s \in S$ and α, α' satisfy (2.3), and similarly β' is a left-inverse for β such that $\beta'_r(1) = 1$, $\beta_r(1)$ are central in B for all $r \in R$ and β, β' satisfy (2.3). Let $N_G : G \to (0, \infty)$ and $N_K : K \to (0, \infty)$ be homomorphisms satisfying (Hom1)–(Hom2), and let σ and τ, respectively, be the time evolutions on $A \rtimes_\alpha S$ and $B \rtimes_\beta R$ given as in (2.1). Suppose that $\phi : A \rtimes_\alpha S \to B \rtimes_\beta R$ is an isomorphism such that $\phi \circ \sigma_t = \tau_t \circ \phi$ for all $t \in \mathbb{R}$. The first observation is that $\phi|_{i_A(A)} : i_A(A) \to i_B(B)$ is an isomorphism. Indeed, since $\phi(i_A(a))$ is a fixed point of τ_t, it follows that $\phi(i_A(a)) \in i_B(B)$ by Corollary 2.7. Hence $\phi(i_A(A)) \subseteq i_B(B)$, and the opposite direction is similar using the inverse ϕ^{-1}.

Let $y \in (A \rtimes_\alpha S)^\delta_s$ where $N_G(s) = m$ for some positive integer $m \geq 1$. The equivariance of ϕ with respect to σ and τ shows that $\phi(y) \in (B \rtimes^{\mathrm{alg}}_\beta R) \cap (B \rtimes_\beta R)^\tau_m$. Thus by Lemma 2.6 there is $r \in R$ with $N_K(r) = m$ such that $\phi(y) \in (B \rtimes_\beta R)^\varepsilon_r$. It is not clear in general that an isomorphism ϕ as above will preserve the dagger subalgebras. In the situation of [2], it is part of the assumptions that an isomorphism between $(\mathcal{A}, \sigma)$ and $(\mathcal{B}, \tau)$ preserves the dagger subalgebras. In the present setup, the following partial result holds true.

Corollary 2.10. *If ϕ is an isomorphism of quantum statistical mechanical systems $(A \rtimes_\alpha S, \sigma)$ and $(B \rtimes_\beta R, \tau)$ as above, then for every $m \geq 1$, ϕ is an isomorphism*

$$\phi : \overline{(A \rtimes_\alpha S)^\dagger} \cap (A \rtimes_\alpha S)^\sigma_m \to \overline{(R \rtimes_\beta R)^\dagger} \cap (B \rtimes_\beta R)^\varepsilon_m.$$

References

[1] J.B.-Bost and A. Connes, *Hecke algebras, type III factors and phase transitions with spontaneous symmetry breaking in number theory.* Selecta Math. (N.S.) **1** (1995), no. 3, 411–457.

[2] G. Cornelissen and M. Marcolli, *Quantum statistical mechanics, L-series and anabelian geometry*, arXiv:1009.0736v5[math.NT].

[3] J. Crisp and M. Laca, *On the Toeplitz algebras of right-angled and finite-type Artin groups.* J. Austral. Math. Soc. **72** (2002), 223–245.

[4] S. Echterhoff, S. Kaliszewski and J. Quigg, *Maximal coactions.* Internat. J. Math. **15** (2004), 47–61.

[5] S. Kaliszewski, N.S. Larsen and J. Quigg, *Inner coactions, Fell bundles, and abstract uniqueness theorems.* Münster J. Math. **5** (2012), 209–232.

[6] M. Laca, *From endomorphisms to automorphisms and back: dilations and full corners.* J. London Math. Soc. **61** (2000), 893–904.

[7] M. Laca and I. Raeburn, *Semigroup crossed products and the Toeplitz algebras of nonabelian groups.* J. Funct. Anal. **139** (1996), 415–440.

[8] M. Laca and I. Raeburn, *A semigroup crossed product arising in number theory.* J. London Math. Soc **59** (1999), no. 1, 330–344.

[9] N.S. Larsen, *Crossed products by semigroups of endomorphisms and groups of partial automorphisms.* Canad. Math. Bull. **46** (2003), 98–112.

[10] J. Quigg, *Discrete coactions and C^*-algebraic bundles.* J. Austral. Math. Soc. (Series A) **60** (1996), 204–221.

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Ut-Majorization on $\mathbb{R}^n$ and its Linear Preservers

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Abstract. Let $\mathbf{M}_n$ be the set of all $n \times n$ real matrices. A matrix $R \in \mathbf{M}_n$ with nonnegative entries is called row stochastic if $Re = e$, where $e = (1, \dots, 1)^t \in \mathbb{R}^n$. For $x, y \in \mathbb{R}^n$, we say that x is upper triangular majorized by y (written as $x \prec_{ut} y$) if there exists an upper triangular row stochastic matrix R such that $x = Ry$. In the present paper, some properties of ut-majorization are investigated. Also, the structure of all linear functions $T : \mathbb{R}^n \to \mathbb{R}^n$ preserving (resp. strongly preserving) ut-majorization with additional condition $Te_1 \neq 0$ (resp. with no condition) will be characterized.

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1. Introduction

Majorization is a topic of much interest in various areas of mathematics and statistics. In recent years, this concept has been attended specially. A matrix R with nonnegative entries is called row stochastic if the sum of every row of R is 1.
The following notation will be fixed throughout the paper: $\mathbb{R}^n$ for the set of all $n \times 1$ (*column*) real vectors; $\mathcal{R}_n^{ut}$ for the collection of all $n \times n$ row stochastic upper triangular matrices; $\{e_1, \dots, e_n\}$ for the standard basis of $\mathbb{R}^n$; $A[\alpha_1, \dots, \alpha_k]$ for the submatrix of A with $a_{\alpha_i \alpha_j}$ as (i, j) entry, where $A = (a_{ij}) \in \mathbf{M}_n$ and $\{\alpha_1, \dots, \alpha_k\} \subseteq \{1, 2, \dots, n\}$; $\mathbb{N}_k$ for the set $\{1, \dots, k\} \subset \mathbb{N}$; $\mathcal{P}(n)$ for the set of all $n \times n$ permutation matrices; A^t for the transpose of a given matrix A; $[T]$ for the matrix representation of a linear function $T : \mathbb{R}^n \to \mathbb{R}^n$ with respect to the standard basis.

Let $\sim$ be a relation on $\mathbb{R}^n$. A linear function $T : \mathbb{R}^n \to \mathbb{R}^n$ is said to be a linear preserver (or strong linear preserver) of $\sim$, if $Tx \sim Ty$ whenever $x \sim y$ (or $Tx \sim Ty$ if and only if $x \sim y$).

Let $x, y \in \mathbb{R}^n$. Then x is said to be left matrix majorized by y, denoted $x \prec_l y$, if $x = Ry$ for some row stochastic matrix R. In [4] and [5], the authors described the structure of all linear functions $T : \mathbb{R}^n \to \mathbb{R}^n$ preserving left matrix majorization as follows.

Proposition 1.1. *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear function. Then T preserves $\prec_l$ if and only if $Tx = (aI + bQ)x$ for all $x \in \mathbb{R}^n$, where $Q \in \mathcal{P}(n)$, $Q \neq I$, and a and b are real numbers such that $ab \leq 0$, and, if $n \neq 2$, $ab = 0$. In case $n \neq 2$, $aI + bQ = cP$, for some $c \in \mathbb{R}$ and some $P \in \mathcal{P}(n)$ and, hence, $Tx = dPx$ for some $d \in \mathbb{R}$.*

Proposition 1.2. *A linear function $T : \mathbb{R}^n \to \mathbb{R}^n$ strongly preserves $\prec_l$ if and only if there exist an $n \times n$ permutation matrix P and a nonzero real number a such that $Tx = aPx$ for all $x \in \mathbb{R}^n$.*

In this paper, we define the relation $\prec_{ut}$ on $\mathbb{R}^n$ and characterize all linear functions preserving ut-majorization with additional condition $Te_1 \neq 0$ and linear functions strongly preserving ut-majorization on $\mathbb{R}^n$. Some types of majorization and their linear preservers are presented in [1], [2], [3] and [6].

2. Ut-majorization on $\mathbb{R}^n$

In this section, first we state an equivalent condition for $\prec_{ut}$ on $\mathbb{R}^n$ and afterwards obtain some facts for finding the structure of all (strong) linear preserver of this relation on $\mathbb{R}^n$.

Definition 2.1. For $x, y \in \mathbb{R}^n$, it is said that x is ut-majorized by y, denoted $x \prec_{ut} y$, if there exists $R \in \mathcal{R}_n^{ut}$ such that $x = Ry$.

For a subset $A \subset \mathbb{R}^n$ the convex hull of A is the following set:

$$Co(A) := \left\{ \sum_{i=1}^{m} \lambda_i a_i \mid m \in \mathbb{N},\ \lambda_i \geq 0,\ \sum_{i=1}^{m} \lambda_i = 1,\ a_i \in A,\ \ i \in \mathbb{N}_m \right\}.$$

Note that if $A \subseteq \mathbb{R}$ is compact, then $Co(A) = [\min A, \max A]$. The following remark gives an equivalent condition for ut-majorization on $\mathbb{R}^n$.

Remark 2.2. Let $x = (x_1, \ldots, x_n)^t$, $y = (y_1, \ldots, y_n)^t \in \mathbb{R}^n$. Then $x \prec_{ut} y$ if and only if

$$x_i \in Co\{y_i, \ldots, y_n\},\ \forall i \in \mathbb{N}_n.$$

The following set-theoretic representation of $\prec_l$ and $\prec_{ut}$ on $\mathbb{R}^2$ may be of interest. Suppose that $a \leq b$, then

$$\begin{aligned} \{(x,y)^t : (x,y)^t \prec_\ell (a,b)^t\} &= [a,b] \times [a,b], \\ \{(x,y)^t : (x,y)^t \prec_{ut} (a,b)^t\} &= [a,b] \times \{b\}. \end{aligned}$$

We use the following lemmas to prove the main results.

Lemma 2.3. *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear preserver of $\prec_{ut}$. Assume $S : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ is a linear function such that $[S] = [T][2, 3, \dots, n]$. Then S preserves $\prec_{ut}$ on $\mathbb{R}^{n-1}$.*

Proof. Let $x' = (x_2, \dots, x_n)^t, y' = (y_2, \dots, y_n)^t \in \mathbb{R}^{n-1}$ and let $x' \prec_{ut} y'$. Then, by Remark 2.2, $x := (0, x_2, \dots, x_n)^t \prec_{ut} y := (0, y_2, \dots, y_n)^t$ and hence $Tx \prec_{ut} Ty$. This implies that $Sx' \prec_{ut} Sy'$. Therefore, S preserves $\prec_{ut}$ on $\mathbb{R}^{n-1}$. □

Lemma 2.4. *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear preserver of $\prec_{ut}$. Then $[T]$ is an upper triangular matrix.*

Proof. Let $[T] = [a_{ij}]$. Use induction on n. For $n = 1$, there is nothing to prove. If $n = 2$, then we only to prove that $a_{21} = 0$. Set $x = e_1$ and $y = 2e_1$. Since $x \prec_{ut} y$, it implies that $Tx \prec_{ut} Ty$. So $a_{21} = 2a_{21}$ and hence $a_{21} = 0$. For $n > 2$, assume that the matrix representation of every linear preservers of $\prec_{ut}$ on $\mathbb{R}^{n-1}$ is an upper triangular matrix. Let $S : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ be the linear function with $[S] = [T][2, 3, \dots, n]$. By Lemma 2.3, the linear function S preserves $\prec_{ut}$ on $\mathbb{R}^{n-1}$. The induction hypothesis insures that $[S]$ is an $n-1 \times n-1$ upper triangular matrix. So it is enough to show that $a_{21} = a_{31} = \dots = a_{n1} = 0$. Put $x = e_1$ and $y = e_2$. Then $x \prec_{ut} y$ and hence $Tx = (a_{11}, a_{21}, \dots, a_{n-11}, a_{n1})^t \prec_{ut} (a_{12}, a_{22}, 0, \dots, 0)^t = Ty$. By Remark 2.2, it implies that $a_{31} = a_{41} = \dots = a_{n1} = 0$. So it is enough to show that $a_{21} = 0$. Assume that $a_{21} \neq 0$. Set $x = e_1$ and $y = (\frac{-a_{22}}{a_{21}}, 1, 0, \dots, 0)^t$. So $x \prec_{ut} y$ and hence $Tx \prec_{ut} Ty$. It follows that $a_{21} \in Co\{0\}$ and so $a_{21} = 0$, which is a contradiction. Thus $a_{21} = 0$ and hence the induction argument is completed. Therefore, $[T]$ is an upper triangular matrix. □

The following theorem characterizes the linear functions $T : \mathbb{R}^n \to \mathbb{R}^n$ preserving ut-majorization with additional condition $Te_1 \neq 0$.

Theorem 2.5. *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear function. Assume $[T] = [a_{ij}]$, and $Te_1 \neq 0$. Then T preserves $\prec_{ut}$ if and only if*

$$[T] = \begin{pmatrix} a_{11} & 0 & 0 & \dots & 0 & 0 & a_{1n} \\ 0 & a_{22} & 0 & \dots & 0 & 0 & a_{2n} \\ 0 & 0 & a_{33} & \dots & 0 & 0 & a_{3n} \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & a_{n-1n-1} & a_{n-1n} \\ 0 & 0 & 0 & \dots & 0 & 0 & a_{nn} \end{pmatrix},$$

such that $a_{11} + a_{1n} = a_{22} + a_{2n} = \dots = a_{n-1n-1} + a_{n-1n} = a_{nn}$, and the finite sequence $(0, a_{11}, a_{22}, \dots, a_{n-1n-1})$ is monotonically increasing or decreasing.

Proof. It is clear that T preserves $\prec_{ut}$ if and only if αT preserves $\prec_{ut}$, for all $\alpha \in \mathbb{R} \setminus \{0\}$. So we may and shall assume without loss of generality that, $a_{11} = 1$. First, suppose T preserves $\prec_{ut}$. The result is trivial for $n = 1$. Suppose that $n = 2$. By Lemma 2.4, it is enough to show that $a_{11} + a_{12} = a_{22}$. Let $y = ((a_{22} - a_{12})/a_{11}, 1)^t$ and $x = (1, 1)^t$. Thus $x \prec_{ut} y$ and hence $Tx = (a_{11} + a_{12}, a_{22})^t \prec_{ut} (a_{22}, a_{22})^t = Ty$. Therefore, $a_{11} + a_{12} = a_{22}$. Now assume that $n > 2$. By Lemma 2.4, $[T]$ is an upper triangular matrix. We show that $a_{1j} = 0$ whenever $2 \leq j \leq n-1$. Assume that there is some j $(2 \leq j \leq n-1)$ such that $a_{1j} \neq 0$. Let $\epsilon = \frac{a_{1j}}{|a_{1j}|}$. Choose $x_1, y_1 \in \mathbb{R}$ such that $x_1 - |a_{1j}| < y_1 < x_1 < \min\{0, \epsilon, \epsilon a_{2j}, \epsilon a_{3j}, \ldots, \epsilon a_{jj}\}$. Set $x = x_1 e_1$ and $y = y_1 e_1 + \epsilon e_j$. Since $y_1 < x_1 < \epsilon$, it follows that $x \prec_{ut} y$ and, hence $Tx \prec_{ut} Ty$. This implies that

$$x_1 \in Co\{y_1 + |a_{1j}|, \epsilon a_{2j}, \epsilon a_{3j}, \ldots, \epsilon a_{jj}, 0\};$$

a contradiction. Therefore, $a_{1j} = 0$, whenever $2 \leq j \leq n-1$.
Now, we prove $a_{22} \geq 1$. Since $e_1 \prec_{ut} e_2$, $Te_1 \prec_{ut} Te_2$ and so $1 \in Co\{0, a_{22}\}$. Thus $a_{22} \geq 1$.

Let $S : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ be the linear function with $[S] = [T][2, 3, \ldots, n]$. By Lemma 2.3, S preserves $\prec_{ut}$ on $\mathbb{R}^{n-1}$. Since $a_{22} > 0$, it follows from the induction hypothesis that,

$$[S] = \begin{pmatrix} a_{22} & 0 & 0 & \ldots & 0 & 0 & a_{2n} \\ 0 & a_{33} & 0 & \ldots & 0 & 0 & a_{3n} \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & a_{n-1n-1} & a_{n-1n} \\ 0 & 0 & 0 & \ldots & 0 & 0 & a_{nn} \end{pmatrix},$$

$a_{22}+a_{2n} = a_{33}+a_{3n} = \cdots = a_{n-1n-1}+a_{n-1n} = a_{nn}$, and $(0, a_{22}, a_{33}, \ldots, a_{n-1n-1})$ is monotone. Thus, by the previous discussion,

$$[T] = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 & 0 & a_{1n} \\ 0 & a_{22} & 0 & \ldots & 0 & 0 & a_{2n} \\ 0 & 0 & a_{33} & \ldots & 0 & 0 & a_{3n} \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & a_{n-1n-1} & a_{n-1n} \\ 0 & 0 & 0 & \ldots & 0 & 0 & a_{nn} \end{pmatrix}.$$

Since $(0, a_{22}, \ldots, a_{n-1n-1})$ is increasing and $a_{22} \geq 1$, it follows that the finite sequence $(0, 1, a_{22}, \ldots, a_{n-1n-1})$ is increasing. It remains to prove $1 + a_{1n} = a_{nn}$. If $1 + a_{1n} \neq a_{nn}$, then choose $\epsilon \in \mathbb{R} \setminus \{0\}$ such that $1 + a_{1n} \notin Co\{1 + a_{1n} + \epsilon, a_{nn}\}$. Set $x = (1, \ldots, 1)^t$ and $y = (1 + \epsilon, 1, \ldots, 1)^t$. Thus $x \prec_{ut} y$ and hence $Tx \prec_{ut} Ty$. It follows that $1 + a_{1n} \in Co\{1 + \epsilon + a_{1n}, a_{nn}\}$, which is a contradiction. Therefore, $1 + a_{1n} = a_{nn}$.

To prove the sufficiency, we proceed by induction on n. Clearly, the assertion holds for $n = 1$. Assume it holds in any space of dimension at most $n-1$. Let $x = (x_1, \ldots, x_n)^t, y = (y_1, \ldots, y_n)^t \in \mathbb{R}^n$ and let $x \prec_{ut} y$. Since $x_n = y_n$, it is easy to see that

$$Tx = (x_1 + a_{1n}y_n, a_{22}x_2 + a_{2n}y_n, \ldots, a_{n-1n-1}x_{n-1} + a_{n-1n}y_n, a_{nn}y_n)^t$$

and

$$Ty = (y_1 + a_{1n}y_n, a_{22}y_2 + a_{2n}y_n, \ldots, a_{n-1n-1}y_{n-1} + a_{n-1n}y_n, a_{nn}y_n)^t.$$

Let $S : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ be the linear function with $[S] = [T][2, 3, \dots, n]$. Put $x' = (x_2, \dots, x_n)^t$ and $y' = (y_2, \dots, y_n)^t$. Then $x' \prec_{ut} y'$ and hence

$$Sx' = (a_{22}x_2 + a_{2n}y_n, \dots, a_{n-1n-1}x_{n-1} + a_{n-1n}y_n, a_{nn}y_n)^t$$

and

$$Sy' = (a_{22}y_2 + a_{2n}y_n, \dots, a_{n-1n-1}y_{n-1} + a_{n-1n}y_n, a_{nn}y_n)^t.$$

By induction hypothesis, $Sx' \prec_{ut} Sy'$. To prove $Tx \prec_{ut} Ty$ it is enough to show that

$$x_1 + a_{1n}y_n \in Co\{y_1 + a_{1n}y_n, a_{22}y_2 + a_{2n}y_n, \dots, a_{n-1,n-1}y_{n-1} + a_{n-1,n}y_n, a_{nn}y_n\}.$$

Write $x_1 = s_1y_1 + \cdots + s_ny_n$ with $\sum_i s_i = 1$ and $s_i \geq 0$ for $i = 1, 2, \dots, n$. Note that $0 \leq t_i := \frac{s_i}{a_{ii}} \leq 1$ for $i = 1, 2, \dots, n-1$ and $\sum_{i=1}^{n-1} t_i \leq \sum_{i=1}^{n-1} s_i \leq 1$. Then $0 \leq t_n := 1 - t_1 - t_2 - \cdots - t_{n-1} \leq 1$ and $x_1 + a_{1n}y_n = \sum_{i=1}^{n-1} t_i(a_{ii}y_i + a_{in}y_n) + t_na_{nn}y_n$. ☐

Corollary 2.6. *If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear preserver of $\prec_{ut}$ such that $Te_1 \neq 0$, then $rank[T] \geq n - 1$.*

As a consequence of Proposition 1.1, one can show that for $n \geq 3$ every linear preserver of left matrix majorization on $\mathbb{R}^n$ has rank 0 or n. But the following example demonstrates that this is not true for ut-majorization on $\mathbb{R}^n$. In fact it shows that for every $k \in \mathbb{N}_n$ there exists a linear preserver of $\prec_{ut}$ of rank k.

Example. Let $k \in \mathbb{N}_n$. Put $A = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \in \mathbf{M}_n$, where $I \in \mathbf{M}_k$ and define $T : \mathbb{R}^n \to \mathbb{R}^n$ by $Tx = Ax$.

We need the following lemma to prove the last result of the paper.

Lemma 2.7. *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear function that strongly preserves $\prec_{ut}$. Then T is invertible.*

Proof. Let $x \in \mathbb{R}^n$ and let $Tx = 0$. Since $Tx = T0$ and T strongly preserves $\prec_{ut}$, $x \prec_{ut} 0$. So $x = 0$. Therefore, T is invertible. ☐

The following theorem characterizes all the linear functions $T : \mathbb{R}^n \to \mathbb{R}^n$ which strongly preserve ut-majorization.

Theorem 2.8. *A linear function $T : \mathbb{R}^n \to \mathbb{R}^n$ strongly preserves $\prec_{ut}$ if and only if there exist $a, b \in \mathbb{R}$ such that $a, a + b \neq 0$, and*

$$[T] = \begin{pmatrix} a & 0 & 0 & \dots & 0 & 0 & b \\ 0 & a & 0 & \dots & 0 & 0 & b \\ 0 & 0 & a & \dots & 0 & 0 & b \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & a & b \\ 0 & 0 & 0 & \dots & 0 & 0 & a+b \end{pmatrix}. \tag{2.1}$$

Proof. First, assume T strongly preserves $\prec_{ut}$. By Lemma 2.7, T is invertible and hence $Te_1 \neq 0$. Now, by applying Theorem 2.5,

$$[T] = \begin{pmatrix} a_{11} & 0 & 0 & \dots & 0 & 0 & a_{1n} \\ 0 & a_{22} & 0 & \dots & 0 & 0 & a_{2n} \\ 0 & 0 & a_{33} & \dots & 0 & 0 & a_{3n} \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & a_{n-1n-1} & a_{n-1n} \\ 0 & 0 & 0 & \dots & 0 & 0 & a_{nn} \end{pmatrix},$$

where $a_{11} + a_{1n} = a_{22} + a_{2n} = \cdots = a_{n-1n-1} + a_{n-1n} = a_{nn} \neq 0$, and the finite sequence $(0, a_{11}, a_{22}, \dots, a_{n-1n-1})^t$ is monotone. By a simple calculation, one can show that

$$[T^{-1}] = \begin{pmatrix} \frac{1}{a_{11}} & 0 & 0 & \dots & 0 & 0 & \frac{-a_{1n}}{a_{11}a_{nn}} \\ 0 & \frac{1}{a_{22}} & 0 & \dots & 0 & 0 & \frac{-a_{2n}}{a_{22}a_{nn}} \\ 0 & 0 & \frac{1}{a_{33}} & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & \frac{1}{a_{n-1n-1}} & \frac{-a_{n-1n}}{a_{n-1n-1}a_{nn}} \\ 0 & 0 & 0 & \dots & 0 & 0 & \frac{1}{a_{nn}} \end{pmatrix}.$$

Since T strongly preserves $\prec_{ut}$, the operator T^{-1} is a linear preserver of $\prec_{ut}$ and hence $(0, \frac{1}{a_{11}}, \frac{1}{a_{22}}, \dots, \frac{1}{a_{n-1n-1}})^t$ is monotone, by Theorem 2.5. On the other hand $(0, a_{11}, a_{22}, \dots, a_{n-1n-1})^t$ is monotone and thus $a_{11} = a_{22} = \cdots = a_{n-1n-1}$, as desired.

Conversely, assume that there exist $a, b \in \mathbb{R}$ such that $a, a+b \neq 0$ and (2.1) holds. It easy to see that

$$[T^{-1}] = \begin{pmatrix} \frac{1}{a} & 0 & 0 & \dots & 0 & 0 & \frac{-b}{a(a+b)} \\ 0 & \frac{1}{a} & 0 & \dots & 0 & 0 & \frac{-b}{a(a+b)} \\ 0 & 0 & \frac{1}{a} & \dots & 0 & 0 & \frac{-b}{a(a+b)} \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & \frac{1}{a} & \frac{-b}{a(a+b)} \\ 0 & 0 & 0 & \dots & 0 & 0 & \frac{1}{a+b} \end{pmatrix}.$$

Then both of T and T^{-1} preserve $\prec_{ut}$, by Theorem 2.5, and therefore, T strongly preserves $\prec_{ut}$. $\square$

In view of Proposition 1.1, for $n \geq 3$ it is easy to see that every invertible linear preserver of $\prec_l$ is strong. But the following example shows that this is not true for $\prec_{ut}$.

Example. Let $A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$. Define $T : \mathbb{R}^3 \to \mathbb{R}^3$ by $Tx = Ax$. It is clear that T is invertible and, by Theorem 2.5, T preserves $\prec_{ut}$. But T is not strong, by Theorem 2.8.

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References

[1] T. Ando, *Majorization, Doubly stochastic matrices, and comparision of eigenvalues*. Linear Algebra Appl. **118** (1989), 163–248.

[2] A. Armandnejad and H. Heydari, *Linear functions preserving gd-majorization from* $\mathbf{M}_{n,m}$ *to* $\mathbf{M}_{n,k}$. Bull. Iranian Math. Soc. **37** No. 1 (2011), 215–224.

[3] A. Armandnejad and A. Ilkhanizadeh Manesh, *gut-majorizations and its linear preservers*. Electron. J. Linear Algebra **23** (2012), 646–654.

[4] A.M. Hasani and M. Radjabalipour, *On linear preservers of (right) matrix majorization*. Linear Algebra Appl. **423** (2007), 255–261.

[5] A.M. Hasani and M. Radjabalipour, *The structure of linear operators strongly preserving majorizations of matrices*. Electron. J. Linear Algebra **15** (2006), 260–268.

[6] F. Khalooei and A. Salemi, *The Structure of linear preservers of left matrix majorization on* $\mathbb{R}^p$. Electron. J. Linear Algebra **18** (2009), 88–97.

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Arithmetic Aspect of $C_r^*SL(2)$

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Abstract. Let $G = SL(2, F)$ where F is a local field of characteristic zero. We use R-groups to study the reducibility of the unitary principal series of G. We show how the arithmetic of F has implications on the topology of the tempered dual of G and on the structure of the underlying reduced C^*-algebra.

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1. Introduction

Let F be a local field with characteristic zero. Artin reciprocity studies the finite extensions E/F such that the Galois group $\mathrm{Gal}(E/F)$ is commutative. Many aspects of the arithmetic of the field F are encoded in reciprocity laws [17].

The unitary principal series representations of the special linear group $SL(n, F)$ consists of the representations unitarily induced from a continuous unitary character of the upper triangular group. Unlike the case of the general linear group $GL(n, F)$ where these representations are always irreducible, for $SL(n, F)$ reducibility may occur, see [7, 8].

In this article we consider the case of $SL(2)$ over a local field. To study reducibility, the direct analytic approach of [8] (see also [21] for a modern survey) consists of computing explicitly intertwining operators. And those operators depend on the residue characteristic. For instance, the case of $\mathbb{Q}_2$ is absent from [8, 21] (see [21, p. 46] for a comment on the difficulty of computing intertwining operators for $SL(2, \mathbb{Q}_2)$). The case of the archimedean local fields $\mathbb{R}$ and $\mathbb{C}$, are also analytically dealt differently. Although an intricate theory, R-groups [9, 12] have the advantage of being a tool that works for any local field of characteristic zero, including $\mathbb{Q}_2$ and the archimedean fields. Since, as far as we know, there is no notion of R-groups in positive characteristic, we exclude the case of local function fields. See [16] for an account of $SL(2)$ over a local function field.

Intense research has been made to study the topology of the dual spaces of C^*-algebras, particularly reduced group C^*-algebras, see Connes' book [3] and [5], [10], [11], [13], [14], [16], [18], [19], [20], [22]. It is well known that the arithmetic of the ground field is related with the topology of the dual space. In [15], Plymen and the author uncovered a relationship between the topological K-theory of the tempered dual of $GL(n, F)$ and some arithmetic invariants of F, where F is a finite Galois extension of $\mathbb{Q}_p$.

The unitary principal series representations for $SL(2, \mathbb{Q}_p)$ are (unitarily) parabolic induced from characters of $\mathbb{Q}_p^\times$, see §3. When the character is quadratic, the induced representation is reducible. The principal series representations decompose as a direct sum of two irreducible components [8, 21]. We prove this result (Theorem 3.10) by using R-groups.

The tempered dual $\mathrm{Irr}^t(G)$ of $G = SL(2, \mathbb{Q}_p)$ has a topology, called the hull-kernel topology. Let $G_P^\wedge$ denote the union of the irreducible unitary principal series and the irreducible components of the reducible unitary principal series. Then, $G_P^\wedge \subset \mathrm{Irr}^t(G)$. The parameter space $\mathcal{Q}$ has also a natural topology [19, 20]. The hull-kernel and the natural topology coincide [19]. The tempered dual fails to be Hausdorff as a consequence of G admitting reducible representations. We illustrate this with the unitary principal series representations of $G = SL(2, F)$ for $F = \mathbb{Q}_p, \mathbb{R}$ and $\mathbb{C}$. If $F = \mathbb{C}$, the unitary principal series representations are always irreducible. If $F = \mathbb{R}$ or $\mathbb{Q}_p$ reducibility may occur. The link with the topology of the dual space and the arithmetic of the ground field is the following: quadratic characters are attached to quadratic extensions E/F via class field theory. The Artin symbol $(., E/F) : F^\times \to \mathrm{Gal}(E/F)$ can be used to manufacture a quadratic character, and all the quadratic characters of $F^\times$ are of this form. In §4 we show how the arithmetic of F influences the topology on the tempered dual of $SL(2, F)$ (Theorem 3.10 and §4.2).

The reducibility of the principal series of $SL(2, F)$ has also implications in the structure of the reduced C^*-algebra $\mathfrak{A} = C_r^* SL(2, F)$. In fact, the arithmetic of F determine the existence of sub-C^*-algebras of $\mathfrak{A}$ which are the crossed product of a commutative algebra by a finite group (the R-group to be precise) [10, 13]. As a result, $\mathfrak{A}$ has noncommutative summands. We explain this relationship in the main result of §5, Theorem 5.2.

Our intention in this work is to show that even for a simple example as $SL(2)$, the arithmetic of the ground field already influence the structure of the underlying C^*-algebra. By working with both archimedean and nonarchimedean fields we intend to illustrate with the example of $SL(2)$ the Lefschetz principle as formulated by Harish-Chandra, which says whatever is true for real reductive groups is also true for p-adic groups.

I would like to thank Roger Plymen for a valuable exchange of emails and the referee for several constructive comments. The title is, of course, borrowed from [13].

2. Local fields

Let F be a local field with zero characteristic. Then F is one of the archimedean fields $\mathbb{R}$ and $\mathbb{C}$, or it is nonarchimedean, i.e., a finite extension of the field $\mathbb{Q}_p$ of p-adic numbers, where p is a prime number. We will not consider local fields with positive characteristic, i.e., finite extensions of the field of Laurent series $\mathbb{F}_p((x))$. If F is nonarchimedean we denote by $|.|_F$ the absolute value on F. The ring of integers is denoted by $\mathfrak{o}$ and $\mathfrak{p} \subset \mathfrak{o}$ denotes its maximal ideal. An element $\varpi \in \mathfrak{p}$ such that $\mathfrak{p} = \varpi\mathfrak{o}$ is called a uniformizer. ν denotes a normalized valuation on F so that $\nu(\varpi) = 1$ and $\nu(F) = \mathbb{Z} \cup \{\infty\}$. The group of units is denoted by $\mathfrak{o}^\times$. The residue field is $k_F = \mathfrak{o}/\mathfrak{p}$, and $\sharp k_F = p^f$. The valuation and the absolute value are related by $|x|_F = q^{-\nu(x)}$, for every $x \in F$.

When $F = \mathbb{Q}_p$ we use the standard notation. $\mathbb{Z}_p$ is the ring of integers, $p\mathbb{Z}_p$ its maximal ideal, p is a uniformizer and $\mathbb{Z}_p^\times$ is the group of units. The residue field of $\mathbb{Q}_p$ is $\mathbb{F}_p = \mathbb{Z}_p/p\mathbb{Z}_p$. When $F = \mathbb{Q}_p$ we write simply $|\ | := |\ |_F$ instead of the common notation $|\ |_p$.

The group $F^\times$ is a locally compact, Hausdorff topological abelian group. There is a canonical isomorphism of topological groups

$$F^\times \simeq \langle \varpi \rangle \times \mathfrak{o}^\times \simeq \mathbb{Z} \times \mathfrak{o}^\times. \tag{2.1}$$

The group $\mathfrak{o}^\times$ is compact. In particular,

$$\mathbb{Q}_p^\times \simeq \langle p \rangle \times \mathbb{Z}_p^\times \simeq \mathbb{Z} \times \mathbb{Z}_p^\times. \tag{2.2}$$

There is also a canonical isomorphism for archimedean fields

$$\mathbb{R}^\times \simeq \mathbb{R}_{>0} \times \mathbb{Z}/2\mathbb{Z} \tag{2.3}$$

and

$$\mathbb{C}^\times \simeq \mathbb{R}_{>0} \times \mathbb{T} \tag{2.4}$$

where $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ is the unit circle.

Definition 2.1. A quasi-character of $\mathbb{Q}_p^\times$ is a continuous group homomorphism $\chi : \mathbb{Q}_p^\times \to \mathbb{C}^\times$. A character is a unitary quasi-character.

The set of characters of $\mathbb{Q}_p^\times$ will be denoted $\mathbb{Q}_p^{\times\wedge}$ and is called the Pontryagin dual of $\mathbb{Q}_p^\times$. Note that $\mathbb{Q}_p^{\times\wedge}$ is an abelian group under pointwise multiplication. From 2.2, a character of $\mathbb{Q}_p^\times$ may be written as

$$\chi(p^{\nu(x)}u) = z^{\nu(x)}\chi^*(u), \tag{2.5}$$

where $x = p^{\nu(x)}u$, $z \in \mathbb{T}$, $u \in \mathbb{Z}_p^\times$ and $\chi^* \in \mathbb{Z}_p^{\times\wedge}$.

Definition 2.2. A character $\chi \in \mathbb{Q}_p^{\times\wedge}$ is called unramified if it is trivial when restricted to $\mathbb{Z}_p^\times$, i.e., $\chi^* \equiv 1$.

Definition 2.3. A character $\chi \in \mathbb{Q}_p^{\times\wedge}$ is called quadratic if $\chi^2 = 1$ and $\chi \neq 1$.

Note that $\chi \in \mathbb{Q}_p^\times$ is quadratic if and only if $\chi \in (\mathbb{Q}_p^\times/\mathbb{Q}_p^{\times 2})^\wedge$ and $\chi \neq 1$.

The central result of local class field theory is Artin's reciprocity law, see [17]. Let E be a finite Galois extension of $\mathbb{Q}_p$. The local reciprocity law is an isomorphism of topological groups

$$Art_{E/\mathbb{Q}_p} : \mathbb{Q}_p^\times/N_{E/\mathbb{Q}_p}(E^\times) \to \mathrm{Gal}(E/\mathbb{Q}_p)^{ab}, \tag{2.6}$$

where $N_{E/\mathbb{Q}_p}$ is the norm map and $\mathrm{Gal}(E/\mathbb{Q}_p)^{ab}$ is the abelianization of $\mathrm{Gal}(E/\mathbb{Q}_p)$. Composing with the canonical morphism we obtain a map

$$\mathbb{Q}_p^\times \to \mathbb{Q}_p^\times/N_{E/\mathbb{Q}_p}(E^\times) \to \mathrm{Gal}(E/\mathbb{Q}_p)^{ab}.$$

We denote the composition map by

$$x \mapsto (x, E/\mathbb{Q}_p),$$

and call $(x, E/\mathbb{Q}_p)$ the **Artin symbol** associated with $E/\mathbb{Q}_p$ and x.

Remark 2.4. (Archimedean reciprocity) There is also an archimedean reciprocity law. The field $\mathbb{C}$ is algebraically closed and therefore does not have nontrivial finite extensions. The only nontrivial finite extension of $\mathbb{R}$ is $\mathbb{C}/\mathbb{R}$, and we have an Artin symbol

$$\mathbb{R}^\times \to \mathrm{Gal}(\mathbb{C}/\mathbb{R}), x \mapsto (x, \mathbb{C}/\mathbb{R})$$

with kernel

$$N_{\mathbb{C}/\mathbb{R}}(\mathbb{C}^\times) = \{z.\overline{z} : z \in \mathbb{C}\} = \{|z|^2 : z \in \mathbb{C}\} \simeq \mathbb{R}_{>0}.$$

Hence

$$\mathbb{R}^\times/N_{\mathbb{C}/\mathbb{R}}(\mathbb{C}^\times) \simeq \mathbb{R}^\times/\mathbb{R}_{>0} \simeq \mathbb{Z}/2\mathbb{Z} = \mathrm{Gal}(\mathbb{C}/\mathbb{R}).$$

As a consequence of Artin's reciprocity, if $E/\mathbb{Q}_p$ is a finite extension then the index $(\mathbb{Q}_p^\times : N_{E/\mathbb{Q}_p}(E^\times))$ is finite. In general, it divides $[E : \mathbb{Q}_p]$, being equal if and only if the extension is abelian. Moreover, if we fix an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$, there is a one-one correspondence between finite abelian Galois extensions $E/\mathbb{Q}_p$, $\mathbb{Q}_p \subset E \subset \overline{\mathbb{Q}}_p$, and open subgroups of $\mathbb{Q}_p^\times$ with finite index

$$E \mapsto N_{E/\mathbb{Q}_p}(E^\times).$$

See [17] for more details.

Quadratic extensions of $\mathbb{Q}_p$. When p is odd, we have

$$\mathbb{Q}_p^\times/\mathbb{Q}_p^{\times 2} \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

Therefore, there are exactly three quadratic extensions of $\mathbb{Q}_p$, up to isomorphism, one for each nontrivial class in $\mathbb{Q}_p^\times/\mathbb{Q}_p^{\times 2}$. These quadratic extensions are given by $\mathbb{Q}_p(\sqrt{a})$, for $a \in \{\epsilon, p, \epsilon p\}$, where $\epsilon \in \mathbb{Z}_p^\times \backslash \mathbb{Z}_p^{\times 2}$ is the smallest nonquadratic residue of p, i.e., $(\frac{\epsilon}{p}) = -1$, where $(\frac{\cdot}{p})$ is the Legendre symbol. The extension $\mathbb{Q}_p(\sqrt{\epsilon})$ is unramified. The remaining extensions, $\mathbb{Q}_p(\sqrt{p})$ and $\mathbb{Q}_p(\sqrt{\epsilon p})$, are totally ramified.

Quadratic extensions of $\mathbb{Q}_2$. In this case, we have

$$\mathbb{Q}_2^\times/\mathbb{Q}_2^{\times 2} \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

As a consequence, there are exactly seven quadratic extensions of $\mathbb{Q}_2$, up to isomorphism, one for each nontrivial class in $\mathbb{Q}_2^\times/\mathbb{Q}_2^{\times 2}$. We may choose the following representatives of the nontrivial equivalent classes:

$$\{-1, \pm 2, \pm 5, \pm 10\}.$$

$\mathbb{Q}_2(\sqrt{5})$ is the (unique) quadratic unramified extension of $\mathbb{Q}_2$, up to isomorphism. The remaining extensions

$$\mathbb{Q}_2(\sqrt{-1}), \mathbb{Q}_2(\sqrt{-5}), \mathbb{Q}_2(\sqrt{2}), \mathbb{Q}_2(\sqrt{10}), \mathbb{Q}_2(\sqrt{-2}), \mathbb{Q}_2(\sqrt{-10})$$

are totally ramified.

Proposition 2.5. *Let $\mathbb{Q}_p(\sqrt{a})$ be a quadratic extension of $\mathbb{Q}_p$, where $\overline{a}$ is a nontrivial class in $\mathbb{Q}_p^\times/\mathbb{Q}_p^{\times 2}$. The quadratic character assigned by local class field theory to $\mathbb{Q}_p(\sqrt{a})$ is precisely the Artin symbol $(., \mathbb{Q}_p(\sqrt{a})/\mathbb{Q}_p)$ associated with $\mathbb{Q}_p(\sqrt{a})$, with kernel*

$$N_{\mathbb{Q}_p(\sqrt{a})/\mathbb{Q}_p}(\mathbb{Q}_p(\sqrt{a})^\times).$$

Proof. Let $\mathbb{Q}_p(\sqrt{a})/\mathbb{Q}_p$ be a quadratic extension. The reciprocity law for local fields yields an isomorphism

$$\mathbb{Q}_p^\times/N_{\mathbb{Q}_p(\sqrt{a})/\mathbb{Q}_p}(\mathbb{Q}_p(\sqrt{a})^\times) \simeq \mathrm{Gal}(\mathbb{Q}_p(\sqrt{a})/\mathbb{Q}_p) \simeq \mathbb{Z}/2\mathbb{Z}.$$

Let $\mu_2(\mathbb{C}) = \{\pm 1\}$ denote the group of roots of unity. Then, composing the Artin symbol with the isomorphism $\mathbb{Z}/2\mathbb{Z} \simeq \mu_2(\mathbb{C})$, we obtain a quadratic character χ_a which we still denote by $\chi_a = (., \mathbb{Q}_p(\sqrt{a})/Q_p)$ □

Example. (Quadratic characters of $\mathbb{Q}_p^\times$ for odd p)
Let $x = p^n u \in \mathbb{Q}_p^\times$, with $n \in \mathbb{Z}$ and $u \in \mathbb{Z}_p^\times$. The quadratic character associated with the unramified quadratic extension $\mathbb{Q}_p(\sqrt{\epsilon})/\mathbb{Q}_p$ is given by

$$(p^n u, \mathbb{Q}_p(\sqrt{\epsilon})/\mathbb{Q}_p) = (-1)^n.$$

Note that $(n \mapsto (-1)^n)$ is the unique quadratic character of the group $\langle p \rangle \simeq \mathbb{Z}$. The quadratic extension $\mathbb{Q}_p(\sqrt{p})/\mathbb{Q}_p$ has an associated quadratic character given by

$$(p^n u, \mathbb{Q}_p(\sqrt{p})/\mathbb{Q}_p) = \left(\frac{\overline{u}}{p}\right),$$

where $(\frac{\div}{p})$ is the Legendre symbol and $\overline{u}$ denotes the class of $u \in \mathbb{Z}_p^\times$ in $\mathbb{F}_p^\times$. For $\mathbb{Q}_p(\sqrt{\epsilon p})/\mathbb{Q}_p$, the associated quadratic character is

$$(p^n u, \mathbb{Q}_p(\sqrt{\epsilon p})/\mathbb{Q}_p) = (-1)^n \left(\frac{\overline{u}}{p}\right).$$

3. Unitary principal series

Let $G = SL(2, \mathbb{Q}_p)$. We define the following subgroups of G.

The Borel subgroup, given by

$$B = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : a \in \mathbb{Q}_p^\times, b \in \mathbb{Q}_p \right\},$$

The maximal torus

$$T = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a \in \mathbb{Q}_p^\times \right\} \simeq \mathbb{Q}_p^\times,$$

and the subgroups

$$N = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{Q}_p \right\} \simeq \mathbb{Q}_p \quad \text{and} \quad \overline{N} = \left\{ \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} : c \in \mathbb{Q}_p \right\} \simeq \mathbb{Q}_p.$$

Note that N and $\overline{N}$ are normal in B. The product $B = NT$ is called the Levi decomposition of B. Now, every character of T (which is, in fact, a character of $\mathbb{Q}_p^\times$ since $T \simeq \mathbb{Q}_p^\times$) may be extended trivially to B, also denoted by χ, by setting

$$\chi(an) = \chi(a),$$

for any $a \in T$ and $n \in N$. Note that χ is well defined since N is a normal subgroup.

Definition 3.1. The set of all representations which are unitary induced to G from representations of B that arise in this way is called the unitary principal series representations of G and is denoted

$$\pi(\chi) = \operatorname{Ind}_B^G \chi.$$

The representation $\pi(\chi)$, corresponding to $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, is defined in the separable Hilbert space $L^2(\overline{N})$ (identified with $L^2(\mathbb{Q}_p)$) as follows:

$$\pi(\chi)(g)f(x) = f\left(\frac{ax + c}{bx + d}\right)\chi(bx + d)|bx + d|^{-1},$$

with $\chi \in \mathbb{Q}_p^{\times\wedge}$ and $f \in L^2(\mathbb{Q}_p)$.

Remark 3.2. Since χ is unitary it can be shown that the induced representation $\operatorname{Ind}_B^G \chi$ is also unitary.

Remark 3.3. The definition of the unitary principal series of $SL(2)$ over $\mathbb{R}$ and $\mathbb{C}$ is analogous. See [14] for the real case. For $F = \mathbb{C}$, $|bx + d|^{-1}$ should be replaced by $|bx + d|^{-2}$, see [6].

Let $\chi \in \mathbb{Q}_p^\times$. We will use the theory of R-groups to study the reducibility of $\pi(\chi)$. As we will see, the representation $\pi(\chi)$ is reducible if and only if χ is quadratic. The reducibility of the unitary principal series of $SL(2, \mathbb{R})$ was handled in [14], and for $SL(2, \mathbb{Q}_p)$, odd p, in [8]. The case of $SL(2, \mathbb{Q}_2)$ could be handled in a similar analytic way (see [8, p. 198]), but presents more difficulties, see [21, p. 46]. The use of R-groups is independent of the residue characteristic and applies to archimedean fields also.

Let $W = W(T) = N_G(T)/T$ denote the Weyl group of T, where $N_G(T)$ is the normalizer of T in G. Then, $W = \mathbb{Z}/2\mathbb{Z} = \langle\omega\rangle$.

Given $\chi, \eta \in \mathbb{Q}_p^{\times\wedge}$, the representations $\pi(\chi)$ and $\pi(\eta)$ are unitarily equivalent if and only if $\chi = \eta$ or $\chi = \eta^{-1}$, see [8, p. 163] and [19, p. 411].

The Weyl group acts on T simply by permuting the elements:

$$\omega.\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix}, a \in \mathbb{Q}_p^{\times}. \tag{3.1}$$

Hence, it acts on $\mathbb{Q}_p^{\times}$ by inverting the elements, and on its unitary dual $\mathbb{Q}_p^{\times\wedge}$. Since $\mathbb{Q}_p^{\times\wedge} \simeq \mathbb{T} \times \mathbb{Z}_p^{\times\wedge}$, a character $\chi \in \mathbb{Q}_p^{\times\wedge}$ will split as a pair $\chi = (z, \chi^*) \in \mathbb{T} \times \mathbb{Z}_p^{\times\wedge}$. The action of W on $\mathbb{Q}_p^{\times\wedge}$ is as follows

$$\omega.\chi = (z^{-1}, \chi^{*-1}). \tag{3.2}$$

We have [12, p. 351]

$$\mathrm{Ind}_B^G \chi \simeq \mathrm{Ind}_B^G \omega\chi.$$

Denote by $W_\chi = \{w \in W : w\chi = \chi\}$ the isotropy subgroup of χ in W. It is known (by Bruhat theory) that the length of the composition series of $\mathrm{Ind}_B^G \chi$ is bounded by $|W_\chi|$ [12, p. 352].

Remark 3.4. It follows from the above discussion that $\mathrm{Ind}_B^G \chi$ is irreducible if $W_\chi = \{1\}$. Moreover, since $W = \mathbb{Z}/2\mathbb{Z}$, if $\mathrm{Ind}_B^G \chi$ is reducible it has at most two irreducible components.

Gelbart and Knapp studied in detail the problem of reducibility of the unitary principal series of $SL(n)$ over a nonarchimedean field with characteristic zero, see [7]. Their results were extended further by Goldberg [9]. We will apply these results to the case of $G = SL(2)$. In order to understand the reducibility of $\pi(\chi) = \mathrm{Ind}_B^G \chi$, one should look into the commuting algebra $\mathrm{End}_G(\mathrm{Ind}_B^G \chi)$. An important tool to understand this commuting algebra is a finite group $R(\chi)$, the so-called R-group, introduced by Knapp–Stein, see [12, p. 363] and [9, p. 81]. In fact, it is known that the commuting algebra $\mathrm{End}_G(\mathrm{Ind}_B^G \chi)$ has a basis of operators parametrized by $R(\chi)$, see [7, p. 315], [12, p. 365].

We will use the framework and notation as in [9]. Let $\mathcal{E}_2(G)$ denote the equivalence classes of irreducible square integrable representations of G. These representations are called discrete series. The discrete series are tempered hence we have $\mathcal{E}_2(G) \subset \mathrm{Irr}^t(G)$, where $\mathrm{Irr}^t(G)$ denotes the tempered dual of G.

Now, let $G = SL(2, \mathbb{Q}_p)$ and let $\widetilde{G} = GL(2, \mathbb{Q}_p)$. Let $T \subset SL(2, \mathbb{Q}_p)$ denote the maximal torus. Then the elements of $\mathcal{E}_2(T)$ are the characters of $T \simeq \mathbb{Q}_p^{\times}$. The maximal split torus $\widetilde{T}$ of $\widetilde{G}$ is

$$\widetilde{T} = \left\{ \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} : a_1, a_2 \in \mathbb{Q}_p^\times \right\} \simeq \mathbb{Q}_p^\times \times \mathbb{Q}_p^\times,$$

and an element of $\mathcal{E}_2(\widetilde{T})$ is a character $\widetilde{\chi}$ of $\widetilde{T}$, which in fact is a product $\widetilde{\chi} = \chi_1\chi_2$ of two characters χ_1 and χ_2 of $\mathbb{Q}_p^\times$. If we denote an element $g = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \in \widetilde{T}$ simply by $g = (a_1, a_2)$, then $\widetilde{\chi}(a_1, a_2) = \chi_1(a_1)\chi_2(a_2)$.

The Weyl group $W = \mathbb{Z}/2\mathbb{Z}$ acts also on $\mathcal{E}_2(\widetilde{T})$, the set of characters of $\widetilde{T}$:

$$\omega.\widetilde{\chi}(a_1, a_2) = \widetilde{\chi}(a_{\omega(1)}, a_{\omega(2)}).$$

Let $\chi \in \mathcal{E}_2(T)$. We will now manufacture a particular character of $\widetilde{T}$ associated with χ. By [9, p. 82] Theorem 1.6(c), there is an element

$$\pi_\chi \in \mathcal{E}_2(\widetilde{T}), \tag{3.3}$$

such that $\pi_\chi|_T \supset \chi$.

Definition 3.5. The character $\pi_\chi \in \mathcal{E}_2(\widetilde{T})$ takes the form $\pi_\chi = \chi_1\chi_2$, where $\chi_1 = \chi$ and $\chi_2 \equiv 1$ is the trivial character.

The representation $\mathrm{Ind}_B^G \chi$ is equivalent to the restriction of the representation $\mathrm{Ind}_{\widetilde{B}}^{\widetilde{G}} \pi_\chi$.

Remark 3.6. We are using two different notations here. On one hand, $\pi(\chi) = \mathrm{Ind}_B^G \chi$ denotes the unitary principal series induced by a character χ of $\mathbb{Q}_p^\times$. On the other hand, $\pi_\chi \in \mathcal{E}_2(\widetilde{T})$ denotes the character of $\widetilde{T}$ defined above.

As in [9, p. 85], we denote $\pi_\chi \otimes \eta \circ \det$ by $\pi_\chi \otimes \eta$, where η is a character of $\mathbb{Q}_p^\times$ and det is the determinant. We will use a realization of the R-group due to Goldberg. Define the following subgroups of $\mathbb{Q}_p^{\times\wedge}$:

$$\overline{L}(\pi_\chi) = \{\eta \in \mathbb{Q}_p^{\times\wedge} : \pi_\chi \otimes \eta \simeq w\pi_\chi, \text{ for some } w \in W\},$$

$$X(\pi_\chi) = \{\eta \in \mathbb{Q}_p^{\times\wedge} : \pi_\chi \otimes \eta \simeq \pi_\chi\}.$$

Definition 3.7. The R-group of χ is given by

$$R(\chi) \simeq \overline{L}(\pi_\chi)/X(\pi_\chi).$$

(See [9, p. 87], Theorem 2.4.)

Remark 3.8. For unitary principal series of $SL(2, \mathbb{Q}_p)$, the formula is even simpler. In fact,

$$X(\pi_\chi) = \{\eta \in \mathbb{Q}_p^{\times\wedge} : \pi_\chi \otimes \eta \simeq \pi_\chi\} = \{\eta \in \mathbb{Q}_p^{\times\wedge} : \chi\eta \simeq \chi\} = \{1\}.$$

Therefore, for unitary principal series of $SL(2, \mathbb{Q}_p)$, $R(\chi) \simeq \overline{L}(\pi_\chi)$. (See Remark on [9, p. 87].)

The following lemma is a particular case of Keys' result [12, p. 365], adapted to $SL(2)$.

Lemma 3.9. *Let* $\chi \in \mathbb{Q}_p^{\times\wedge}$. *Then,*

(a) *The number of inequivalent irreducible components of* $\mathrm{Ind}_B^G \chi$ *is equal to the order of* $R(\chi)$*;*
(b) *The multiplicity of each irreducible component of* $\mathrm{Ind}_B^G \chi$ *is equal to* 1*;*
(c) $\mathrm{End}_G(\mathrm{Ind}_B^G \chi) \simeq \mathbb{C}[R(\chi)]$.

Proof. It follows from [12, p. 365], Corollary 1. (a) follows from the fact that $R(\chi)$ is abelian ($R(\chi) \subset W = \mathbb{Z}/2\mathbb{Z}$), and so the number of conjugacy classes equals the order of $R(\chi)$. □

We may now use the R-groups to study the reducibility of the unitary principal series of $SL(2, \mathbb{Q}_p)$. We emphasize that the result is independent of the prime p.

Theorem 3.10. *Let* χ *be a character of* $\mathbb{Q}_p^\times$. *Then,* $R(\chi) = \mathbb{Z}/2\mathbb{Z}$ *if and only if* χ *is quadratic, in which case* $\pi(\chi) = \mathrm{Ind}_B^G \chi$ *is reducible. Moreover, in case of reducibility, the representation* $\pi(\chi) = \mathrm{Ind}_B^G \chi$ *decomposes into a direct sum with precisely two irreducible components*

$$\pi(\chi) = \pi^-(\chi) \oplus \pi^+(\chi).$$

Proof. Let χ be a character of $\mathbb{Q}_p^\times$. The isotropy subgroup of χ in W is

$$W_\chi = \{w \in W : w\chi = \chi\}.$$

Since

$$\omega\chi = \chi \Leftrightarrow \chi^{-1} = \chi \Leftrightarrow \chi^2 = 1,$$

we conclude the following:

If χ is not quadratic then $W_\chi = \{1\}$. Since the length of the composition series of $\mathrm{Ind}_B^G \chi$ is bounded by $|W_\chi|$ it follows that $\mathrm{Ind}_B^G \chi$ is irreducible.

If χ is quadratic then $W_\chi = \mathbb{Z}/2\mathbb{Z}$. From Remark 3.8, we have only to compute $\overline{L}(\pi_\chi)$. Then, for $g = (a_1, a_2) \in \widetilde{T}$,

$$\begin{aligned}(\pi_\chi \otimes \eta)(a_1, a_2) = (\omega.\pi_\chi)(a_1, a_2) &\Leftrightarrow \chi(a_1)\eta \circ det(a_1, a_2) = \chi(a_2)\\ &\Leftrightarrow \eta(a_1a_2) = \chi^{-1}(a_1)\chi(a_2) \Leftrightarrow \eta(a_1a_2) = \chi(a_1a_2).\end{aligned}$$

Therefore, $R(\chi) = \langle\chi\rangle \simeq \mathbb{Z}/2\mathbb{Z}$ and $\mathrm{Ind}_B^G \chi$ is reducible by Lemma 3.9. From Remark 3.2, the representation is unitary and so it is semisimple. From Remark 3.4, we conclude that $\pi(\chi)$ decomposes into a direct sum of precisely two irreducible components,

$$\pi(\chi) = \pi^-(\chi) \oplus \pi^+(\chi).$$

Finally, suppose $\chi \equiv 1$ is trivial. Then, $W_\chi = \mathbb{Z}/2\mathbb{Z}$. We have, for $g = (a_1, a_2) \in \widetilde{T}$,

$$\begin{aligned}(\pi_1 \otimes \eta)(a_1, a_2) = (\omega.\pi_1)(a_1, a_2) &\Leftrightarrow 1(a_1)\eta \circ \det(a_1, a_2) = 1(a_2)\\ &\Leftrightarrow \eta(a_1a_2) = 1.\end{aligned}$$

And so

$$R(1) \simeq \overline{L}(\pi_1) = \{1\}.$$

We conclude from Lemma 3.9 that $\pi(1) = \mathrm{Ind}_B^G(1)$ is irreducible for $G = SL(2, \mathbb{Q}_p)$. □

Now, we use the theory of R-groups to discuss the reducibility of the unitary principal series of $SL(2)$ over archimedean fields.

The case of $\boldsymbol{SL(2,\mathbb{C})}$. The multiplicative group $\mathbb{C}^\times$ has no nontrivial characters of finite order. Hence, $R(\chi) = \{1\}$ and $\mathrm{Ind}_B^G \chi$ is irreducible for every character of $\mathbb{C}^\times$. In fact, this result holds not only for $SL(2, \mathbb{C})$ but for every Chevalley group over $\mathbb{C}$, see [12, p. 353].

The case of $\boldsymbol{SL(2,\mathbb{R})}$. The field $F = \mathbb{R}$ has only one nontrivial finite extension, $\mathbb{C}/\mathbb{R}$, which is quadratic. Therefore, $\mathbb{R}^\times$ has only one nontrivial quadratic character, given by

$$\mathrm{sgn}(x) = x/|x|,$$

or, in the notation of Section 2, $\mathrm{sgn}(x) = (x, \mathbb{C}/\mathbb{R})$, where $(., \mathbb{C}/\mathbb{R})$ is the archimedean Artin symbol. It follows $R(\mathrm{sgn}) = \mathbb{Z}/2\mathbb{Z}$ and

$$\pi(\mathrm{sgn}) = \mathrm{Ind}_B^G(\mathrm{sgn}) = \pi^-(\mathrm{sgn}) \oplus \pi^+(\mathrm{sgn}).$$

For any other character χ of $\mathbb{R}^\times$, $R(\chi) = \{1\}$ and $\pi(\chi) = \mathrm{Ind}_B^G(\chi)$ is irreducible. Note that this result also holds other Chevalley groups over $\mathbb{R}$, see [12, p. 353].

4. The tempered dual

The tempered dual $\mathrm{Irr}^t(G)$ of $G = SL(2, F)$ comprises the discrete series and the irreducible components in the unitary principal series. This is a consequence of the Plancherel theorem of Harish-Chandra, which is valid for any local field F.

Let F be nonarchimedean. There is a Bernstein decomposition [1, 2]

$$\mathrm{Irr}^t(G) = \bigsqcup_{\mathfrak{s} \in \mathfrak{B}(G)} \mathrm{Irr}^t(G)^{\mathfrak{s}} \tag{4.1}$$

where the index set $\mathfrak{B}(G)$ is called the Bernstein spectrum of G. For $G = SL(2, F)$, a point $\mathfrak{s}$ of the Bernstein spectrum corresponds to either supercuspidal representations or parabolically induced representations of G. For the latter, the only ones we will consider, a point $\mathfrak{s}$ in $\mathfrak{B}(G)$ is of the form $\mathfrak{s} = [T, \chi]_G$, where $T \cong F^\times$ is the standard maximal torus of F and χ is a character of $F^\times$.

In [20], Plymen described the topology on the tempered dual $\mathrm{Irr}^t(G)$ of a reductive p-adic group. It has a canonical topology, called the hull-kernel topology, see [5]. Let $\mathfrak{A} = C_r^*(G)$ denote the reduced C^*-algebra of G (see next section for the definition of $C_r^*(G)$). The C^*-dual $\widehat{\mathfrak{A}}$ may be identified with the tempered dual of G [4]:

$$\widehat{\mathfrak{A}} \simeq \mathrm{Irr}^t(G) \tag{4.2}$$

Let $G_P^\wedge$ be the union of the irreducible unitary principal series and the irreducible components of the reducible unitary principal series. Hence $G_P^\wedge \subset \mathrm{Irr}^t(G)$.

According to [20], $\mathfrak{A} = C_r^*(SL(2,\mathbb{Q}_p))$ is a C^*-direct sum of certain fixed C^*-algebras. Let $\mathfrak{A}_P \subset \mathfrak{A}$ denote the sub-C^*-algebra which correspond to unitary principal series. Then, $\widehat{\mathfrak{A}}_P \simeq G_P^\wedge$. We conclude that $\mathfrak{A}_P$ is a C^*-direct sum indexed by characters of $\mathbb{Q}_p^\times$.

The space $\mathbb{Q}_p^{\times\wedge}$ is given the Pontryagin dual topology, so that $\mathbb{Q}_p^{\times\wedge}$ has countably many components. Each component is a circle in its Euclidean topology. Recall that the Weyl group $W = \mathbb{Z}/2\mathbb{Z}$ acts on $\mathbb{Q}_p^{\times\wedge}$.

Definition 4.1. The parameter space of the unitary principal series is the space

$$\mathcal{Q} = \mathbb{Q}_p^{\times\wedge}/W = (\mathbb{T} \times \mathbb{Z}_p^{\times\wedge})/W$$

with the quotient topology.

4.1. Extended quotient

In order to understand the parameter space we introduce the concept of extended quotient, which is central in a recent geometric conjecture due to Aubert, Baum and Plymen, see [1, 2].

Let Γ be a finite abelian group and let X be a compact Hausdorff topological space. Suppose Γ is acting on X as continuous automorphisms of X. Define

$$\widetilde{X} = \{(\gamma, x) \in \Gamma \times X : \gamma x = x\}.$$

Then Γ acts on $\widetilde{X}$:

$$\alpha(\gamma, x) = (\alpha\gamma\alpha^{-1}, \alpha x), \text{ with } (\gamma, x) \in \widetilde{X}, \alpha \in \Gamma.$$

Definition 4.2. The extended quotient, denoted $X//\Gamma$, is defined to be

$$X//\Gamma := \widetilde{X}/\Gamma.$$

Thus the extended quotient is the ordinary quotient for the action of Γ on $\widetilde{X}$. The extended quotient always contains the ordinary quotient. The projection $\Gamma \times X \to X$, $(\gamma, x) \mapsto x$ induces a map

$$\theta : X//\Gamma \to X \ , \ \Gamma(\gamma, x) \mapsto \Gamma x$$

called the projection of the extended quotient on to the ordinary quotient.

In [19], Plymen proved that the topology on $G_P^\wedge$ and the quotient topology on the parameter space $\mathcal{Q}$ coincide. The connected components of $\mathcal{Q}$ have been identified with extended quotients $\mathbb{T}//W$, see [1, 2]. For $G = SL(2,\mathbb{Q}_p)$ it has been proved that, for any $\mathfrak{s} = [T, \chi]_G \in \mathfrak{B}(SL(2,\mathbb{Q}_p))$, there is a natural bijection

$$\mathrm{Irr}^t(G)^{\mathfrak{s}} \cong \mathbb{T}//W_{\mathfrak{s}}. \tag{4.3}$$

$W_{\mathfrak{s}}$ is trivial if and only if the character χ associated with $\mathfrak{s} = [T, \chi]_G$ is nonquadratic and ramified. Otherwise, $W_{\mathfrak{s}} = \mathbb{Z}/2\mathbb{Z}$ is the Weyl group of $SL(2)$.

Therefore, the extended quotient $\mathbb{T}//W_{\mathfrak{s}}$ is either $\mathbb{T}$ or a disjoint union of the ordinary quotient with two isolated points

$$\mathbb{T}//W_{\mathfrak{s}} = \mathbb{T}/W \sqcup \{-1\} \sqcup \{1\}.$$

We now describe bijection (4.3).

Case 1: $\mathfrak{s} = [T, \chi]$, χ ramified and $\chi^2 \neq 1$. Then $W_{\mathfrak{s}} = \{1\}$. Let ξ be an unramified character of T. Then, there is a natural bijection

$$\mathrm{Irr}^t(G)^{\mathfrak{s}} \cong \mathbb{T}, \quad \pi(\xi\chi) \mapsto \xi(\varpi)$$

Case 2: $\mathfrak{s} = [T, \chi]$, χ ramified and $\chi^2 = 1$. Then $W_{\mathfrak{s}} = \mathbb{Z}/2\mathbb{Z}$. Let ϵ be the unique quadratic unramified character of T. Let $(\pi^+(\chi), \pi^-(\chi))$ denote the pair of irreducible constituents of $\pi(\chi)$ and $(\pi^+(\epsilon\chi), \pi^-(\epsilon\chi))$ denote the pair of irreducible constituents of $\pi(\epsilon\chi)$. There is a bijective map

$$\mathrm{Irr}^t(G)^{\mathfrak{s}} \cong \mathbb{T}//W_{\mathfrak{s}},$$

given by

(i) $\xi^2 \neq 1$, $\pi(\xi\chi) \mapsto \xi(\varpi) \in \mathbb{T}/W_{\mathfrak{s}}$
(ii) $\xi = 1$, $(\pi^+(\chi), \pi^-(\chi)) \mapsto 1 \in \mathbb{T}//W_{\mathfrak{s}}$
(iii) $\xi = \epsilon$, $(\pi^+(\epsilon\chi), \pi^-(\epsilon\chi)) \mapsto -1 \in \mathbb{T}//W_{\mathfrak{s}}$

Case 3: $\mathfrak{s} = [T, \chi]$, χ unramified. Then $W_{\mathfrak{s}} = \mathbb{Z}/2\mathbb{Z}$. Let triv_G, St_G denote, respectively, the trivial representation and the Steinberg representation of G. There is a bijective continuous map which is *not* a homeomorphism

$$\mathrm{Irr}^t(G)^{\mathfrak{s}} \cong \mathbb{T}//W_{\mathfrak{s}},$$

defined by

(i) $\xi^2 \neq 1$, $\pi(\xi\chi) \mapsto \xi(\varpi) \in \mathbb{T}/W_{\mathfrak{s}}$
(ii) $\xi = 1$, $(\mathrm{triv}_G, St_G) \mapsto 1 \in \mathbb{T}//W_{\mathfrak{s}}$
(iii) $\xi = \epsilon$, $(\pi^+(\epsilon), \pi^-(\epsilon)) \mapsto -1 \in \mathbb{T}//W_{\mathfrak{s}}$

We conclude the following result on the tempered dual corresponding to induced elements.

Theorem 4.3. *There is a Bernstein decomposition of the parameter space as a disjoint union*

$$\mathcal{Q} = \bigsqcup_{\mathfrak{s}=[T,\chi]} \mathbb{T}//W_{\mathfrak{s}}$$

where each connected component is an extended quotient. The extended quotient is as follows:

(a) *a circle, if χ is a non quadratic and ramified character of $\mathbb{Q}_p^\times$;*
(b) *a closed semi-circle with two double-points, if χ is a ramified quadratic character $\mathbb{Q}_p^\times$;*
(c) *a closed semi-circle with one double-point, if χ is an unramified character of $\mathbb{Q}_p^\times$.*

Along the lines of [19], we will now give a portrayal of the extended quotients, i.e., of the topology in the tempered dual associated with induced elements.

The quadratic characters $(.,\mathbb{Q}_p(\sqrt{p})/\mathbb{Q}_p)$ and $(.,\mathbb{Q}_p(\sqrt{\epsilon p})/\mathbb{Q}_p)$ pair together in the same semi-circle, since

$$(.,\mathbb{Q}_p(\sqrt{p})/\mathbb{Q}_p)|_{\mathbb{Z}_p^\times} = (.,\mathbb{Q}_p(\sqrt{\epsilon p})/\mathbb{Q}_p)|_{\mathbb{Z}_p^\times} = (\frac{\cdot}{p}),$$

where $(\frac{\cdot}{p})$ is the Legendre symbol.

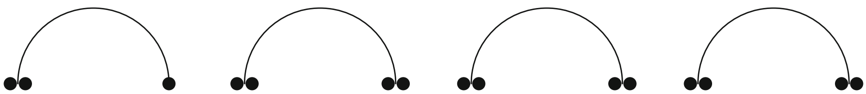
The unramified quadratic character $(.,\mathbb{Q}_p(\sqrt{\epsilon})/\mathbb{Q}_p)$ and the trivial character $\chi \equiv 1$ belong to the same semi-circle, since

$$(.,\mathbb{Q}_p(\sqrt{\epsilon})/\mathbb{Q}_p)|_{\mathbb{Z}_p^\times} = 1.$$

Let $\pi(\chi_a)$ denote the unitary principal series associated with the quadratic character $\chi_a = (.,\mathbb{Q}_p(\sqrt{a})/\mathbb{Q}_p)$. When p is odd, we have

The double points in the above picture correspond to the irreducible constituents $(\pi^+(\chi_a),\pi^-(\chi_a))$ of $\pi(\chi_a)$, i.e., to quadratic characters. There are countably many nonquadratic characters and their contribution to $\mathcal{Q}$ is a disjoint union of countably many unit circles.

When $p=2$, if we choose representatives $\{-1,\pm 2,\pm 5,\pm 10\}$ to represent the quadratic extensions of $\mathbb{Q}_2$, we have:



There are countably many nonquadratic characters and their contribution to $\mathcal{Q}$ is with a disjoint union of countably many unit circles. The existence of nontrivial quadratic characters is responsible for the failure of the tempered dual to be Hausdorff.

4.2. The archimedean case

As far as we know, the geometric conjecture [1, 2] has not been even stated for reductive real or complex groups.

For $G = SL(2,\mathbb{C})$ we have the following result: the space $G_P^\wedge$ coincides with the all tempered dual

$$G_P^\wedge = \mathrm{Irr}^t(SL(2,\mathbb{C})).$$

The unitary principal series representations of $SL(2,\mathbb{C})$ are parametrized by characters

$$(\nu,\mu) \in \mathbb{C}^{\times\wedge} \simeq \mathbb{Z}\times\mathbb{R},$$

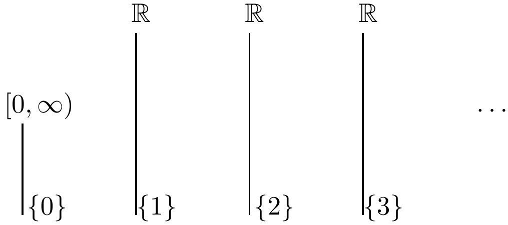
modulo the action of the Weyl group $W = \mathbb{Z}/2\mathbb{Z}$. Each character $(\nu, \mu) \in \mathbb{Z} \times \mathbb{R}$ determines an irreducible unitary principal series representation $\pi(\nu, \mu)$. The only equivalence relation is the following

$$\pi(\nu, \mu) \simeq \pi(-\nu, -\mu).$$

Therefore,

$$\mathcal{Q} = \mathbb{C}^{\times\wedge}/(\mathbb{Z}/2\mathbb{Z}) \simeq \{0\} \times [0, \infty) \sqcup \{1\} \times \mathbb{R} \sqcup \{2\} \times \mathbb{R} \sqcup \{3\} \times \mathbb{R} \sqcup \cdots .$$

The space $\mathcal{Q}$ is a disjoint union of an half-line and countably many real lines



There are no nontrivial quadratic characters of $\mathbb{C}^\times$. The tempered dual of $SL(2, \mathbb{C})$ is Hausdorff.

The unitary principal series representations of $SL(2, \mathbb{R})$ are parametrized by characters

$$(\nu, \mu) \in \mathbb{R}^{\times\wedge} \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{R},$$

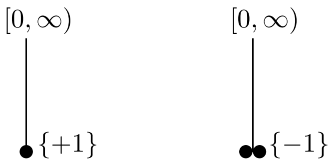
modulo the action of the Weyl group $W = \mathbb{Z}/2\mathbb{Z}$. Here, unlike the complex case, the representation $\pi(-1, 0)$ is reducible and decomposes as a direct sum

$$\pi(-1, 0) = \pi^-(-1) \oplus \pi^+(-1).$$

The irreducible representations $\{\pi^-(-1), \pi^+(-1)\}$ are called "limits of discrete series". Therefore, we have a disjoint union

$$\mathcal{Q} = \mathbb{R}^{\times\wedge}/(\mathbb{Z}/2\mathbb{Z}) \simeq \{+1\} \times [0, \infty) \sqcup \{-1\} \times [0, \infty).$$

Here, $+1$ and -1 denote the trivial and the nontrivial character of $\mathbb{Z}/2\mathbb{Z}$, respectively. The space $\mathcal{Q}$ is a disjoint union of two half-lines, one of them with a double point.



The group $\mathbb{R}^\times$ has a unique quadratic character $\operatorname{sgn}(.) = (., \mathbb{C}/\mathbb{R})$, responsible for the double point in $\mathcal{Q}$. As a consequence, the tempered dual $\operatorname{Irr}^t(SL(2, \mathbb{R}))$ fails to be Hausdorff.

5. The C^*-algebra $C_r^*SL(2)$

Let F be either $\mathbb{Q}_p$, $\mathbb{R}$ or $\mathbb{C}$. The reducibility of the unitary principal series of $SL(2,F)$ has also implications in the structure of the reduced C^*-algebra $\mathfrak{A} = C_r^*SL(2,F)$. The arithmetic link is the following: depending on F having quadratic extensions, the C^*-algebra $\mathfrak{A}$ has summands which are not commutative. Quite explicit, these summands are crossed products of a commutative algebra by a finite group. The finite group is precisely the R-group, see [13] and [10]. Note that this fits into the framework of noncommutative geometry [3], where a well-known dictionary relates the topological property of a space being Hausdorff with the algebraic property of the algebra of continuous functions on that space being commutative.

We start by recalling the definition of a reduced group C^*-algebra. The definition works for any locally compact group. We choose a left invariant Haar measure on G and form the Hilbert space $L^2(G)$. The left regular representation λ of $L^1(G)$ on $L^2(G)$ is given by

$$(\lambda(f))(h) = f * h,$$

where $f \in L^1(G), h \in L^2(G)$ and $*$ denotes the convolution. The C^*-algebra generated by the image of λ is the reduced C^*-algebra $C_r^*(G)$.

The C^*-algebra $\mathfrak{A} = C_r^*SL(2,F)$ was studied in full detail independently by Plymen [19] and Tadić [22] for F nonarchimedean, by Miličić [14] for $F = \mathbb{R}$, and by Fell [6] for $F = \mathbb{C}$. In [13], Plymen and Leung have determined the structure of those fixed-algebras whose duals contains a particular class of representations of $SL(\ell)$, with ℓ prime. In [10], Plymen and Jawdat extended those results for $SL(N)$ with N not necessarily prime.

Let $\mathcal{K}$ denote the C^*-algebra of compact operators on some separable Hilbert space $\mathcal{H}$. We now define the notion of stably isomorphic C^*-algebras (See [3, p. 152] for more details).

Definition 5.1. A C^*-algebra $\mathfrak{A}$ is said to be stable if $\mathfrak{A} \otimes \mathcal{K} \simeq \mathfrak{A}$. We call $\mathfrak{A} \otimes \mathcal{K}$ the stabilization of $\mathfrak{A}$. Two C^*-algebra $\mathfrak{A}$ and $\mathfrak{B}$ are called stably isomorphic (or stably equivalent) when $\mathfrak{A} \otimes \mathcal{K} \simeq \mathfrak{B} \otimes \mathcal{K}$.

We now quote a result of Plymen and Leung on the structure of the reduced C^*-algebra of $SL(\ell)$. The standard maximal torus T of $SL(\ell)$ is a minimal Levi subgroup and contributes to the reduced C^*-algebra, see [13, p. 256], Theorem 2.5. Part of that contribution is given by a noncommutative component. Quite explicit, Leung and Plymen proved in [13, Theorem 5.4] that the reduced C^*-algebra of the p-adic group $SL(\ell)$ admits ℓ direct summands stably isomorphic to

$$C(\mathbb{T}^\ell/\mathbb{T}) \rtimes \mathbb{Z}/\ell\mathbb{Z}.$$

Now, consider the case of $SL(2,\mathbb{Q}_p)$. The above quotient becomes $\mathbb{T}^2/\mathbb{T}$. Let $(z_1 : z_2) \in \mathbb{T}^2/\mathbb{T}$ be a homogeneous coordinate, with $|z_1| = |z_2| = 1$. Since

$(z_1 : z_2) \sim (z_1/z_2 : 1)$, there is a homeomorphism

$$\mathbb{T}^2/\mathbb{T} \cong \mathbb{T}, \quad (z:1) \mapsto z.$$

Therefore, by [13, Theorem 5.4], the reduced C^*-algebra of $SL(2,\mathbb{Q}_p)$ admits summands stably isomorphic to

$$C(\mathbb{T}^2/\mathbb{T}) \rtimes \mathbb{Z}/2\mathbb{Z} \cong C(\mathbb{T}) \rtimes \mathbb{Z}/2\mathbb{Z}.$$

The number of such summands is determined by pairs of Artin symbols, which correspond to tamely ramified quadratic extensions of the ground field. For odd p there is only one pair

$$(., \mathbb{Q}_p(\sqrt{p})/\mathbb{Q}_p) \ , \ (., \mathbb{Q}_p(\sqrt{\epsilon p})/\mathbb{Q}_p).$$

Although Leung and Plymen considered only the case of odd residue characteristic, the result can be extended to $\mathbb{Q}_2$. There are three pairs of Artin symbols

$$(., \mathbb{Q}_2(\sqrt{-1})/\mathbb{Q}_2) \ , \ (., \mathbb{Q}_2(\sqrt{-5})/\mathbb{Q}_2).$$
$$(., \mathbb{Q}_2(\sqrt{2})/\mathbb{Q}_2) \ , \ (., \mathbb{Q}_2(\sqrt{10})/\mathbb{Q}_2).$$
$$(., \mathbb{Q}_2(\sqrt{-2})/\mathbb{Q}_2) \ , \ (., \mathbb{Q}_2(\sqrt{-10})/\mathbb{Q}_2).$$

Note that $\mathfrak{A}_P$ contains also countably many commutative summands of the form $C(\mathbb{T})$, which correspond to irreducible representations $\pi(\chi)$ induced by nonquadratic and nontrivial characters χ of $\mathbb{Q}_p^\times$, and also the contribution of the unitary unramified principal series representations (see Remark 5.3 bellow).

We now concentrate on the case when the ground field F is archimedean. Let X denote a locally compact Hausdorff topological space and denote by $C_0(X)$ the C^*-algebra of complex-valued continuous functions on X, vanishing at infinity, i.e., for each $\varepsilon > 0$ there is a compact subset $K \subset X$ such that $|f(x)| \leq \varepsilon, \forall x \in X \backslash K$.

For $SL(2,\mathbb{R})$, the C^*-algebra $\mathfrak{A}_P$ is noncommutative and is stably isomorphic to the following C^*-direct sum (cf. §4.2)

$$\mathfrak{A}_P \sim_s C_0(\mathbb{R}/(\mathbb{Z}/2\mathbb{Z})) \oplus C_0(\mathbb{R}) \rtimes \mathbb{Z}/2\mathbb{Z},$$

where $\sim_s$ means stably isomorphic.

The noncommutative summand $C_0(\mathbb{R}) \rtimes \mathbb{Z}/2\mathbb{Z}$ corresponds to the unique nontrivial quadratic extension of $\mathbb{R}$ and so is induced by the archimedean Artin symbol $(., \mathbb{R}/\mathbb{C})$.

The remaining case is the complex group $SL(2,\mathbb{C})$, where all the contribution for the tempered dual comes from the unitary principal series representations. In other words, $\mathfrak{A}_P = \mathfrak{A}$.

The C^*-algebra $\mathfrak{A}$ is stably isomorphic to the following commutative C^*-direct sum (cf. §4.2)

$$\mathfrak{A} \sim_s C_0([0,\infty)) \oplus \left(\bigoplus_{n \geq 1} C_0(\mathbb{R}) \right).$$

In this case, there is no noncommutative summand since there is no quadratic extension of $\mathbb{C}$.

Putting together sections 3, 4 and the present section, we have the following rather general result for $C_r^*SL(2)$ over any local field F of zero characteristic.

Theorem 5.2. *Let F be a local field of characteristic zero. Let $\mathfrak{A}_P$ denote the sub-C^*-algebra whose tempered dual corresponds to the unitary principal series representations. Then, the Artin symbol determine the existence of noncommutative summands in $\mathfrak{A}_P$, depending on whether or not F admits quadratic extensions.*

Remark 5.3. Denote the sub-C^*-algebra of $C_r^*SL(2,\mathbb{Q}_p)$ corresponding to the unitary unramified principal series representations by $\mathfrak{A}_s$. This is called the spherical C^*-algebra of $SL(2,\mathbb{Q}_p)$. According to [11, p. 113], for $SL(2,\mathbb{Q}_p)$, $\mathfrak{A}_s$ can be be identified with the fixed-point algebra

$$\mathfrak{A}_s \cong C(\mathbb{T},\mathcal{K})^W.$$

Surprisingly, for more general p-adic groups, $\mathfrak{A}_s$ has a very subtle structure. Even without realizing the spherical C^*-algebra as a crossed product (modulo Morita equivalence), Kamran and Plymen where able to compute its K-theory for any split, simply connected, almost simple p-adic group, see [11, Theorem 3.1].

References

[1] A.-M. Aubert, P. Baum, and R.J. Plymen, *The Hecke algebra of a reductive p-adic group: a geometric conjecture.* Aspects Math. **37** (2006), 1–34.

[2] A.-M. Aubert, P. Baum, and R.J. Plymen, *Geometric structure in the representation theory of p-adic groups II.* Contemporary Math. **543** (2011), 71–90.

[3] A. Connes, *Noncommutative geometry.* Academic Press, New York, 1994.

[4] J. Dixmier, *C^*-algebras.* North-Holland, 1977.

[5] J.M.G. Fell, *The dual spaces of C^*-algebras.* Trans. Amer. Math. Soc. **94** (1960), 365–403.

[6] J.M.G. Fell, *The structure of algebras of operator fields.* Acta Math. **106** (1961), 233–280.

[7] S.S. Gelbart and A.W. Knapp, *Irreducible constituents of principal series of $SL_n(k)$.* Duke Math. J. **48** (2) (1981), 313–326.

[8] I.M. Gelfand, M.I. Graev, and I.I. Pyatetskii-Shapiro, *Representation theory and Automorphic Functions.* Saunders, Philadelphia, 1969.

[9] D. Goldberg, *R-groups and elliptic representations for SL_n.* Pacific J. Math. **165** (1994), 77–92.

[10] J. Jawdat and R.J. Plymen, *R-groups and geometric structure in the representation theory of $SL(N)$.* J. Noncommut. Geom. **4** (2010), 265–279.

[11] T. Kamran and R.J. Plymen, *K-theory and the connection index.* Bull. London Math. Soc. **45** (2013), 111–119.

[12] D. Keys, *On the decomposition of reducible principal series representations of p-adic Chevalley groups.* Pacific J. Math. **101** (1982), 351–388.

[13] C.W. Leung and R.J. Plymen, *Arithmetic aspect of operator algebras*. Compositio Math. **77** (1991), 293–311.

[14] D. Miličić, *Topological representations of the group C^*-algebra of $SL(2,\mathbb{R})$*. Glas. Mat. **6** (1971), 231–246.

[15] S. Mendes and R.J. Plymen, *Base change and K-theory for $GL(n)$*. J. Noncommut. Geom. **1** (2007), 311–331.

[16] S. Mendes and R.J. Plymen, *L-packets and formal degrees for $SL_2(K)$ with K a local function field of characteristic* 2, arXiv:1302.6038 [math.RT].

[17] J. Neukirch, *Algebraic Number Theory*. Springer-Verlag, Berlin, 1999.

[18] R.J. Plymen, *The reduced C^*-algebra of the p-adic group $GL(n)$*. J. Funct. Anal. **72** (1987), 1–12.

[19] R.J. Plymen, *K-theory of the reduced C^*-algebra of $SL_2(\mathbb{Q}_p)$*. In "Lecture Notes in Mathematics, Volume **1132**", Springer-Verlag, New York (1985), 409–420.

[20] R.J. Plymen, *Reduced C^*-Algebra for Reductive p-adic Groups*. J. Funct. Anal. **88** (1990), 251–266.

[21] P.J. Sally, *An Introduction to p-adic Fields, Harmonic Analysis and the Representation theory of SL_2*. Lett. Math. Phys. **46** (1998), 1–47.

[22] M. Tadić, *The C^*-algebra of $SL(2,k)$*. Glas. Mat. **7** (1982), 249–263.

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Inequalities and Convexity

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Dedicated to Professor António Ferreira dos Santos

Abstract. It is a close connection between various kinds of inequalities and the concept of convexity. The main aim of this paper is to illustrate this fact in a unified way as an introduction of this area. In particular, a number of variants of classical inequalities, but also some new ones, are derived and discussed in this general frame.

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1. Introduction

Different kinds of inequalities are very important in various areas of mathematics and its applications. Today the knowledge about inequalities has been developed to be an independent area with many papers, Journals, conferences and books (see, e.g., [6], [13], [18], [20], [21], and the references given there). Moreover, there are also some books fully or partly devoted to convexity techniques (see, e.g., [14], [22], [28], and the references given there). These areas are of independent interest but there are also a huge numbers of examples how these subjects have supported each other in the further developments of these areas but also of other areas within mathematical sciences and even in other more applied areas.

Already G.H. Hardy, J.E. Littlewood and G. Polya in their classical book [13] clearly understood the crucial role of convexity to develop the theory of inequalities. Our intention with this paper is to complement and on some points further develop the content of this book. The main idea is to further explain and use the crucial role of convexity (Jensen's inequality), to further develop and explain the rich area of inequalities in an elementary and unified way.

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As an example of the importance of inequalities we mention the following classical sentence by G.H. Hardy in his Presidental address at the meeting of the London Mathematical Society in November 8, 1928: "All analysts spend half their time hunting through the literature for inequalities which they want to use and cannot prove". Our hope and main aim is that this paper can help these researchers to find what they are looking for, e.g., by directly finding the inequality and if not to give powerful ideas and hints to be able to derive even inequalities not explicitly stated in our paper.

It is maybe a matter of fate that Hardy himself never discovered that also his famous inequality (6.1) (after a simple substitution) in fact follows directly from Jensen's inequality. Moreover, also his first powerweighted versions of his inequality is a consequence of the same simple technique. And maybe even more remarkable is that "all" powerweighted Hardy inequalities (for $p = q$) are, in fact, equivalent to the simple inequality

$$\int_0^\infty \left(\frac{1}{x} \int_0^x f(x) \right)^p \frac{dx}{x} \leq \int_0^\infty f^p(x) \frac{dx}{x}, p \geq 1, \tag{1.1}$$

which easily follows from Jensen's inequality. Moreover, also a number of other classical inequalities (including those by Carleman, Pólya–Knopp etc.) follows directly from this fact.

We pronounce that all inequalities derived in this paper are sharp.

The content is organized as follows: In Section 2 we point out a number of elementary inequalities which follow more or less directly from convexity (Young's inequality. Clarcson's discrete inequality, two fundamental inequalities [6], etc.). In Section 3 we reformulate Jensen's inequality as an equivalence theorem connected to the concept of convexity. In the next sections the following inequalities are derived and analyzed by using convexity arguments from Sections 2–3.

Section 4: Classical Hölder's inequality and various variants of this inequality (including a version for infinite many Lebesgue spaces).

Section 5: Classical Minkowski's inequality and various variants of this inequality (including an integral version of Fubini type).

Section 6: Some classical inequalities (by Hardy, Carleman and Pólya–Knopp).

It is also pointed out that all these inequalities (via substitutions and limit arguments) can be derived from the same basic inequality (1.1), which, in turn, follows from Jensen's inequality. In particular, these calculations show that "all" powerweighted Hardy inequalities are in fact equivalent because they are equivalent to this basic inequality.

Section 7: Some more Hardy type inequalities including variants with finite intervals involved (a precise equivalence result is proved, and thus improving and making the statements in Section 6 more clear).

Section 8 is reserved for some further results and final remarks. It is shortly mentioned how also interpolation theory is closely related to the concept of convexity. This fact is further explained and developed in [25]. As an example there we just derive Young's integral inequality (including a limit case) via interpolation and convexity. We also mention a fairly new idea that Jensen's inequality can be "refined" if convex functions are replaced by superquadratic functions. We point out that in particular our technique and this fact implies a new refined Hardy type inequality with "breaking point" $p = 2$ (for $p = 2$ we even get a new integral identity). This is in contrast to usual Hardy type inequalities where the "breaking point" is $p = 1$.

2. Convexity – some elementary inequalities

Let I denote a finite or infinite interval on $\mathbb{R}_+$. We say that a function f is convex on I if, for $0 < \lambda < 1$, and all $x, y \in I$,

$$f((1-\lambda)x + \lambda y) \leq (1-\lambda)f(x) + \lambda f(y).$$

If the inequality holds in the reversed direction, then we say that the function f is concave.

Moreover, we say that the function f is midpoint convex on I if, for all $x, y \in I$,

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2}.$$

There are many well-known facts concerning convex functions, see, e.g., the book [22] by C.P. Niculescu and L.E. Persson. Here we just mention a few introductory but useful facts:

* It follows directly from the definition of convexity that if f is convex on $I = [a, b]$, then, for all $x \in [a, b]$,

$$f(x) \leq f(a) + \frac{f(b) - f(a)}{b - a}(x - a). \tag{2.1}$$

* Assume that $f(x)$ is continuous on I. Then
  a) f is convex if and only if it is midpoint convex,
  b) f is convex if and only if

$$f(x+h) + f(x-h) - 2f(x) \geq 0.$$

Example 1. The function $f(x) = x^p, p \geq 1, x \geq 1$, is an elementary example of a convex function and as we will see later on this simple fact implies, e.g., the Hardy inequality (see (6.1)). And since this function is convex also when $p < 0$ this inequality holds also for $p < 0$, a fact which was also not noted by Hardy himself. Another elementary example of convex function is $f(x) = e^x$ and by using this function in a similar way we obtain a "trivial" proof of the Pólya–Knopp inequality (6.4), see Examples 23 and 25, and Remark 6.4.

But there are also many non-trivial examples of convex functions which have been important for the development. We will here only present one example which we will use later, namely the following one by M. Riesz which was crucial when he proved his convexity theorem, which was very important when interpolation theory was initiated via the famous Riesz–Thorin interpolation theorem, see, e.g., the book [7] by J. Bergh and J. Löfström.

Example 2. Let a and b be complex numbers. Then the function

$$f(\alpha,\beta) = \log\max\frac{\left(|a+b|^{1/\alpha}+|a-b|^{1/\alpha}\right)^{\alpha}}{\left(|a|^{1/\beta}+|b|^{1/\beta}\right)^{\beta}}$$

is convex on the triangle $T : 0 \le \alpha \le \beta \le 1$.

We shall now present some useful elementary inequalities, which follow directly from convexity, sometimes combined by some other argument from calculus.

Example 3. Let $a, b > 0$. Then

$$a^p + b^p \le (a+b)^p \le 2^{p-1}(a^p+b^p),\ \ p \ge 1,$$
$$2^{p-1}(a^p+b^p) \le (a+b)^p \le a^p+b^p, 0 < p \le 1.$$

Proof. The function $f(u) = u^p$ is convex when $p \ge 1$ and concave when $0 < p < 1$. Hence,

$$\text{if } p \ge 1, \text{then } \left(\frac{a+b}{2}\right)^p \le \frac{a^p+b^p}{2},\ \text{ i.e., } (a+b)^p \le 2^{p-1}(a^p+b^p),$$

and

$$\text{if } 0 < p < 1, \text{then } \left(\frac{a+b}{2}\right)^p \ge \frac{a^p+b^p}{2},\ \text{ i.e., } (a+b)^p \ge 2^{p-1}(a^p+b^p).$$

We may without loss of generality assume that $b \le a$. Consider the function

$$f(t) = (1+t)^p - t^p, [t = b/a]$$

and note that

$$f'(t) = p(1+t)^{p-1} - pt^{p-1}.$$

Hence $f'(t) \ge 0\ for\ p \ge 1$ and $f'(t) \le 0$ for $0 < p < 1$. Moreover, $f(0) = 1$ and we conclude that

$$f(t) \ge 0 \text{ for } p \ge 1 \Leftrightarrow \left(1+\frac{b}{a}\right)^p - 1 - \left(\frac{b}{a}\right)^p \ge 0 \text{ for } p \ge 1$$
$$\Leftrightarrow a^p + b^p \le (a+b)^p\ for\ p \ge 1,$$

and

$$f(t) \le 0 \text{ for } 0 < p \le 1 \Leftrightarrow \left(1+\frac{b}{a}\right)^p - 1 - \left(\frac{b}{a}\right)^p \le 0\ for\ 0 < p < 1$$
$$\Leftrightarrow (a+b)^p \le a^p + b^p\ for\ 0 < p < 1.$$

□

Remark 2.1. The inequalities in Example 3 can be unified as follows

$$c_1(a^p+b^p) \le (a+b)^p \le c_2(a^p+b^p), p > 0, \tag{2.2}$$

where $c_1 = \min\{2^{p-1}, 1\}$ and $c_2 = \max\{2^{p-1}, 1\}$. When (2.2) holds for some positive numbers c_1 and c_2, we also write $(a+b)^p \approx (a^p+b^p)$. This equivalence notion can be generalized to more general situations in a natural way.

By using induction we can generalize Example 3 as follows:

Example 4. Let $a_1, a_2, \ldots, a_n$ be positive numbers. Then

$$\text{(a)} \qquad \sum_{i=1}^{n} a_i^p \leq \left(\sum_{i=1}^{n} a_i\right)^p \leq n^{p-1} \sum_{i=1}^{n} a_i^p, \quad p \geq 1,$$

$$\text{(b)} \qquad n^{p-1} \sum_{i=1}^{n} a_i^p \leq \left(\sum_{i=1}^{n} a_i\right)^p \leq \sum_{i=1}^{n} a_i^p, \quad 0 < p \leq 1.$$

Example 5. (Two fundamental inequalities). If $x > 0$ and $\alpha \in \mathbb{R}$, then

$$\begin{cases} x^\alpha - \alpha x + \alpha - 1 \geq 0 & \text{for} \quad \alpha > 1 \text{ and } \alpha < 0 \\ x^\alpha - \alpha x + \alpha - 1 \leq 0 & \text{for} \quad 0 < \alpha < 1. \end{cases} \tag{2.3}$$

Remark 2.2. In the book [6] E.F. Beckenbach and R. Bellman called (2.3) "A fundamental relationship" (see page 12). In particular, they showed later in the book that several well-known inequalities follow directly from (2.3), e.g., the AG-inequality, Hölder's inequality, Minkowski's inequality, etc. In [6] it was given two different proofs of (2.3) but in view of the main argument in this paper we mention another "proof" namely that (2.3) follows directly from the fact that the function $f(x) = x^\alpha$ is convex for $\alpha > 1$ and $\alpha < 0$ and concave for $0 < \alpha < 1$. In fact, if $f(x) = x^\alpha$, then the equation for the tangent at $x = 1$ is equal to $y = \alpha(x-1)+1$ and (2.3) follows directly. Moreover, this proof shows that we have equality in both inequalities in (2.3) if and only if $x = 1$ for all α.

Example 6 (Discrete Young inequality). For any $a, b > 0, p, q \in \mathbb{R} \setminus \{0\}, \frac{1}{p} + \frac{1}{q} = 1$, it yields that

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \text{ if } p > 1 \tag{2.4}$$

and

$$ab \geq \frac{a^p}{p} + \frac{b^q}{q}, \text{ if } p < 1, p \neq \{0\}. \tag{2.5}$$

Proof. In fact, (2.4) follows directly from (2.3) applied with $x = \frac{a}{b}$ and $\alpha = \frac{1}{p}$ (the case $0 < \alpha < 1$) and (2.5) follows from (2.3) in the same way by instead applying (2.3) in the cases $\alpha > 1$ and $\alpha < 0$. □

Remark 2.3. Another proof of (2.4) is obtained by directly using the fact that $f(x) = e^x$ is convex:

$$ab = e^{\ln ab} = e^{\frac{1}{p}\ln a^p + \frac{1}{q}\ln b^q} \leq \frac{1}{p} e^{\ln a^p} + \frac{1}{q} e^{\ln b^q} = \frac{1}{p} a^p + \frac{1}{q} b^q.$$

By using the same argument and induction (cf. also Proposition 3.1) we obtain the following generalization of Young's inequality (2.4): Let $a_i > 0, p_i > 1, n =$

$1, 2, \ldots, n, n \in \mathbb{Z}_+, n \geq 2, \sum_{i=1}^{n} \frac{1}{p_i} = 1$. Then

$$\prod_{i=1}^{n} a_i \leq \sum_{i=1}^{n} \frac{1}{p_i} a_i^{p_i}.$$

It seems not to be possible to derive a similar generalization of (2.5).

Example 7 (A generalization of (2.3)). Our simple proof of (2.3) gives directly the following more general result: Let $\Phi(x)$ be a convex function on $\mathbb{R}_+$, which is differentiable at $x = 1$. Then

$$\Phi(x) - \Phi'(1)x + \Phi'(1) - \Phi(1) \geq 0. \tag{2.6}$$

If instead $\Phi(x)$ is concave, then (2.6) holds in the reverse direction.

Another way to understand and generalize (2.4) is as follows:

Example 8 (Generalized discrete Young inequality). Let $\Phi(x)$ be a continuous and strictly increasing function for $x \geq 0$ and $\Phi(0) = 0$. The inverse of Φ is denoted Ψ (draw the figure of the situation). By examining the areas in this figure we see that

$$ab \leq \int_0^a \Phi(x)dx + \int_0^b \Psi(y)dy. \tag{2.7}$$

Inequality (2.4) is obtained by applying (2.7) with $\Phi(x) = x^{p-1}, p > 1$. This argument also shows that we have equality in (2.7) exactly when $b = \Phi(a)$, in particular we have equality in (2.4) exactly when $b = a^{p-1}$. It seems not to be possible to have some similar generalization of inequality (2.5).

We finish this section by showing that also another useful inequality follows from convexity via Example 2.

Example 9 (Discrete Clarkson inequality). Let $a, b \in \mathbb{R}, 1 < p \leq 2$ and $q = \frac{p}{p-1}$. Then

$$\left(|a+b|^q + |a-b|^q\right)^{1/q} \leq 2^{1/q} \left(|a|^p + |b|^p\right)^{1/p}. \tag{2.8}$$

The inequality is sharp, i.e., $2^{1/q}$ can not be replaced by any smaller number.

Proof. Consider the convex function $f(\alpha, \beta)$ defined in Example 2. By using the parallelogram law

$$|a+b|^2 + |a-b|^2 = 2(|a|^2 + |b|^2) \tag{2.9}$$

we see that $f(1/2, 1/2) = \frac{1}{2}\log 2$. Moreover, we easily find that $f(0+, 0+) = \log 2$ and $f(0+, 1-) = 0$.

The linear function $g(\alpha, \beta)$ that coincides with $f(\alpha, \beta)$ at the points $(1/2, 1/2)$, $(0, 0)$ and $(0, 1)$ is $g(\alpha, \beta) = (1 - \beta)\log 2$. Moreover, the convexity implies that $f(\alpha, \beta) \leq g(\alpha, \beta) = (1 - \beta)\log 2$, and by choosing $\beta = 1/p, 1 \leq p \leq 2$, and $\alpha = 1 - \beta = 1/q$ we obtain that

$$\log \frac{\left(|a+b|^q + |a-b|^q\right)^{1/q}}{\left(|a|^p + |b|^p\right)^{1/p}} \leq \frac{1}{q} \log 2 = \log 2^{1/q}$$

and (2.8) is proved. Finally, we note that by putting $a = b$ in (2.8) we have equality in (2.8). The proof is complete. □

Remark 2.4. We see that in the case $p = q = 2$, we have even equality in (2.8) via the parallelogram law (2.9) and in the other extreme case when $p \to 1$ we have

$$\max(|a+b|, |a-b|) \leq (|a| + |b|), \tag{2.10}$$

which is just the triangle inequality. Hence, (2.8) is just some "interpolated" inequality between these two extreme cases. This argument can be formalized to a formal proof by considering the operator $T : (a, b) \to (a + b, a - b)$ and note that it maps $\ell_2^2 \to \ell_2^2$ with norm $\sqrt{2}$ (see (2.9)) and $\ell_1^2 \to \ell_\infty^2$ with norm 1 (see (2.10)) and the usual Riesz–Thorin interpolation theorem (see [7]) gives the result.

Remark 2.5. Inequality (2.8) is fundamental for proving Clarkson type inequalities and its applications to uniformly convex spaces. Moreover, by combining (2.8) with other convexity inequalities we obtain more general inequalities, which are also important for applications.

3. Convexity = Jensen's inequality

Proposition 3.1. (*Discrete Jensen inequality*). *Let $n \in \{2, 3, \dots\}$ and let $a = \{a_k\}_1^n$ be a sequence of positive real numbers. If $\Phi(x)$ is convex on an interval including a, then, for $\lambda_k > 0, \sum_{k=1}^n \lambda_k = 1$, it yields that*

$$\Phi\left(\sum_{k=1}^n \lambda_k a_k\right) \leq \sum_{k=1}^n \lambda_k \Phi(a_k). \tag{3.1}$$

Proof. In fact, for $n = 2$ it is just the definition of convexity and for $n = 3$ it follows by using the definition two times:

$$\begin{aligned}\Phi(\lambda_1 a_1 + \lambda_2 a_2 + \lambda_1 a_3) &= \Phi\left(\lambda_1 a_1 + (\lambda_1 + \lambda_2)\left[\frac{\lambda_2}{\lambda_2 + \lambda_3} a_2 + \frac{\lambda_3}{\lambda_2 + \lambda_3} a_3\right]\right) \\ &\leq \lambda_1 \Phi(a_1) + (\lambda_2 + \lambda_3)\Phi\left(\frac{\lambda_2}{\lambda_2 + \lambda_3} a_2 + \frac{\lambda_3}{\lambda_2 + \lambda_3} a_3\right) \\ &\leq \lambda_1 \Phi(a_1) + \lambda_2 \Phi(a_2) + \lambda_3 \Phi(a_3).\end{aligned}$$

The proof follows by repeating this argument and formalize it via induction. □

Of course the above argument shows that in fact the discrete Jensen inequality is equivalent to the definition of convexity. We shall now continue by reformulating the classical Jensen inequality

$$\Phi\left(\int_\Omega f d\mu\right) \leq \int_\Omega \Phi(f) d\mu, \tag{3.2}$$

where $\mu(\Omega) = 1$, as a more general form of such an equivalence statement.

Here and in the sequel we let μ denote a positive measure on a σ-algebra in a set Ω.

Theorem 3.2. *Let f be a real μ-measurable function on Ω such that $-\infty \leq a < f(x) < b \leq +\infty$ for all $x \in \Omega$ and Φ be a function on $I = (a, b)$. Then the following conditions are equivalent:*

(i) Φ *is convex,*

(ii) *the inequality*

$$\Phi\left(\frac{1}{\mu(\Omega)}\int_\Omega f d\mu\right) \leq \frac{1}{\mu(\Omega)}\int_\Omega \Phi(f) d\mu \tag{3.3}$$

holds for all measures such that $0 < \mu(\Omega) < \infty$.

Proof. (ii)⇒(i): Apply (3.3) with the measure μ defined as the point mass $1 - \lambda, (0 < \lambda < 1)$ at x and λ at y for $x, y \in I$ and we find by (3.3) that

$$\Phi((1-\lambda)x + \lambda y) \leq (1-\lambda)\Phi(x) + \lambda\Phi(y),$$

i.e., Φ is convex.

(i)⇒(ii): It is obviously sufficient to prove (3.3) with the restriction $\mu(\Omega) = 1$, i.e., that (3.2) holds. First we note that since Φ is convex (cf. (2.1)) it yields that

$$\frac{\Phi(t) - \Phi(s)}{t - s} \leq \frac{\Phi(u) - \Phi(t)}{u - t} \tag{3.4}$$

whenever $a < s < t < u < b$.

Let $t = \int_\Omega f d\mu$. Then $a < t < b$.

Put $\beta =$ supremum of all quotients to the left in (3.4) for fixed $u \in (t, b)$.

Hence $\frac{\Phi(t)-\Phi(s)}{t-s} \leq \beta$ so that $\Phi(s) \geq \Phi(t) + \beta(s - t)$.

Thus, for all $x \in \Omega$ (with $s = f(x)$) it yields that

$$\Phi(f(x)) - \Phi(t) - \beta(f(x) - t) \geq 0.$$

We integrate and get that

$$\begin{aligned}
&\int_\Omega \Phi(f(x))d\mu - \int_\Omega \Phi(t)d\mu - \beta\int_\Omega (f(x) - t)d\mu \\
&\quad = \int_\Omega \Phi(f(x))d\mu - \Phi(t) - \beta\int_\Omega f(x)d\mu + \beta\int_\Omega t d\mu \\
&\quad = \int_\Omega \Phi(f(x))d\mu - \Phi\left(\int_\Omega f d\mu\right) - \beta\int_\Omega f(x)d\mu + \beta\int_\Omega f(x)d\mu \\
&\quad = \int_\Omega \Phi(f(x))d\mu - \Phi\left(\int_\Omega f d\mu\right) \geq 0.
\end{aligned}$$

The proof is complete. □

Remark 3.3. The arguments in the proof of (i)⇒(ii) are the same as those in most Functional Analysis books but the formulation of Theorem 3.2 as an equivalence theorem is important and done for our further purposes.

Our proof of Theorem 3.2 shows that we also have the following characterization of concave functions:

Theorem 3.4. *Let f and Φ be defined as in Theorem 3.2. Then the following conditions are equivalent:*

(iii) *Φ is concave,*
(iv) *the inequality (3.3) holds in the reversed direction for all measures μ such that $0 < \mu(\Omega) < \infty$.*

Remark 3.5. If $\Omega = \mathbb{R}_+, n = 2, 3, \dots,\ \mu = \sum_{k=1}^n \lambda_k \delta_k$ (δ_k is the unity mass at $t = k$), $\lambda_k > 0$ and $\sum_{k=1}^n \lambda_k = 1$, then Jensen's inequality (3.2) coincides with the discrete Jensen inequality (3.1), with $f(k) = a_k$. Moreover, if Φ is concave, then Theorem 3.4 shows that (3.1) holds in the reversed direction.

The original forms of Jensen's inequality traces back to his original papers [15] and [16] from 1905–06.

4. Various variants of Hölder's inequality via convexity

As usual, the space $L_p = L_p(\mu), 0 < p \le \infty$, consists of all functions f on Ω such that

$$\|f\|_{L_p} := \left(\int_\Omega |f|^p d\mu\right)^{1/p} < \infty, \text{ if } 0 < p < \infty,$$

and

$$\|f\|_{L_\infty} := \operatorname*{ess\,sup}_{x\in\Omega} |f(x)| \quad < \infty, \text{ if } p = \infty.$$

Example 10 (Hölder's inequality). Let $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$\|fg\|_{L_1} \le \|f\|_{L_p}\|g\|_{L_q},$$

i.e.,

$$\int_\Omega |fg| d\mu \le \left(\int_\Omega |f|^p d\mu\right)^{1/p} \left(\int_\Omega |g|^q d\mu\right)^{1/q}. \tag{4.1}$$

Proof 1. First we let $\|f_0\|_{L_p} = \|g_0\|_{L_q} = 1$ and use convexity of the exponential function via Young's inequality (2.4) and find that

$$\int_\Omega |f_0 g_0| d\mu \le \frac{1}{p}\int_\Omega |f_0|^p d\mu + \frac{1}{q}\int_\Omega |g_0|^q d\mu = \frac{1}{p}\cdot 1 + \frac{1}{q}\cdot 1 = 1.$$

Apply this inequality with $f_0 = \frac{f}{\|f\|_{L_p}}$ and $g_0 = \frac{g}{\|g\|_{L_q}}$ and (4.1) is proved. □

Another even more direct convexity proof is the following one:

Proof 2. We may without loss of generality assume that $0 < \int_\Omega |g| d\mu < \infty$ and apply Jensen's inequality (3.2) with the convex function $\Phi(u) = u^p$ to obtain that

$$\left(\frac{1}{\int_\Omega |g| d\mu}\int_\Omega |fg| d\mu\right)^p \le \left(\int_\Omega |g| d\mu\right)^{-1}\int_\Omega |f|^p |g| d\mu,$$

i.e., that

$$\int_\Omega |fg|d\mu \le \left(\int_\Omega |g|d\mu\right)^{1-1/p} \left(\int_\Omega |f|^p|g|d\mu\right)^{1/p}.$$

Put $|f||g|^{1/p} = |f_1|$ and $|g|^{1/q} = |g_1|$ and we find that

$$\int_\Omega |f_1g_1|d\mu \le \left(\int_\Omega |f_1|^p d\mu\right)^{1/p} \left(\int_\Omega |g_1|^q d\mu\right)^{1/q}.$$

We just change notation and (4.1) is proved. □

Remark 4.1. It is easy to use the first proof to find (all) cases of equality in Hölder's inequality namely when $g(x) = (f(x))^{p-1}$ (see Example 8). In particular, this means that the following important relation

$$\left(\int_\Omega |f|^p d\mu\right)^{1/p} = \sup \int_\Omega |f|\varphi d\mu, \tag{4.2}$$

yields for each $p > 1$, where supremum is taken over all $\varphi \ge 0$ such that $\int_\Omega \varphi^q d\mu = 1$. This technique is example of a technique called quasi-linearization.

Example 11 (Hölder's inequality – the reversed form). Let $\frac{1}{p} + \frac{1}{q} = 1, 0 < p < 1$. Then

$$\int_\Omega |fg|d\mu \ge \left(\int_\Omega |f|^p d\mu\right)^{1/p} \left(\int_\Omega |g|^q d\mu\right)^{1/q}. \tag{4.3}$$

Proof. Note that the function $\Phi(u) = u^p$ is convex also for $p < 0$. Therefore as in the second proof of Hölder's inequality we find that (with the same notation)

$$\left(\int_\Omega |f_1g_1|d\mu\right)^p \le \int_\Omega |f_1|^p d\mu \left(\int_\Omega |g_1|^q d\mu\right)^{p-1}.$$

Hence

$$\int_\Omega |f_1g_1|d\mu \ge \left(\int_\Omega |f_1|^p d\mu\right)^{1/p} \left(\int_\Omega |g_1|^q d\mu\right)^{1/q}$$

for $p < 0$ (and $0 < q < 1$) and, hence, by interchanging the completely symmetric roles of p and q and change notation we obtain (4.3) and the proof is complete. □

We shall now formulate a more general result, which includes Examples 10 and 11 as special cases.

Example 12 (Hölder's inequality – completely symmetric form). Let p, q and r be real numbers $\neq 0$ such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. Then

$$\text{(a)} \qquad \left(\int_\Omega |fg|^r d\mu\right)^{\frac{1}{r}} \le \left(\int_\Omega |f|^p d\mu\right)^{1/p} \left(\int_\Omega |g|^q d\mu\right)^{1/q}$$

if $p > 0, q > 0, r > 0$ or $p < 0, q > 0, r < 0$ or $p > 0, q < 0, r < 0$, and

$$\text{(b)} \qquad \left(\int_\Omega |fg|^r d\mu\right)^{\frac{1}{r}} \ge \left(\int_\Omega |f|^p d\mu\right)^{1/p} \left(\int_\Omega |g|^q d\mu\right)^{1/q}$$

if $p > 0, q < 0, r > 0$ or $p < 0, q > 0, r > 0$ or $p < 0, q < 0, r < 0$. (Whenever some parameter is negative we assume that the involved function is strictly positive.)

Proof. The case $\underline{p > 0, q > 0, r > 0.}$ First we note that the convexity of the function $f(u) = e^u$ implies that

$$\begin{aligned}|f(t)g(t)|^r &= \exp(r(\ln|f(t)| + \ln|g(t)|))\\ &= \exp\left(\frac{r}{p}\ln|f(t)|^p + \frac{r}{q}\ln|g(t)|^q\right) \le \frac{r}{p}|f(t)|^p + \frac{r}{q}|g(t)|^q.\end{aligned}$$

By now integrating and discussing as in the proof of the special case $r = 1$ (see Proof 1 of Example 10) we obtain (a).

The case $\underline{p > 0, q < 0, r > 0.}$ By using the estimate we just have proved we find that

$$\begin{aligned}\left(\int_\Omega |f(t)|^p d\mu\right)^{1/p} &= \left(\int_\Omega |f(t)g(t)|^p \frac{1}{|g(t)|^p} d\mu\right)\\ &\le \left(\int_\Omega |f(t)g(t)|^r d\mu\right)^{1/r} \left(\int_\Omega \left|\frac{1}{g(t)}\right|^{-q} d\mu\right)^{1/-q},\end{aligned}$$

i.e.,

$$\left(\int_\Omega |f(t)|^p d\mu\right)^{1/p} \left(\int_\Omega |g(t)|^q d\mu\right)^{1/q} \le \left(\int_\Omega |f(t)g(t)|^r d\mu\right)^{1/r},$$

which means that (b) holds.

By symmetry we see that (b) holds also for the case $\underline{p < 0,\ q > 0,\ r > 0.}$

For the cases $\underline{p < 0, q > 0, r < 0.}$ and $\underline{p > 0, q < 0, r < 0,}$ we use the obtained results with r, p, q replaced by $-r, -p, -q$, respectively, and obtain that

$$\begin{aligned}\left(\int_\Omega |f(t)g(t)|^r d\mu\right)^{1/r} &= \left(\int_\Omega \left|\frac{1}{f(t)g(t)}\right|^{-r} d\mu\right)^{1/-r}\\ &\le \left(\int_\Omega \left|\frac{1}{f(t)}\right|^{-p} d\mu\right)^{1/-p} \Big/ \left(\int_\Omega \left|\frac{1}{g(t)}\right|^{-q} d\mu\right)^{1/-q},\end{aligned}$$

which means that (a) holds.

The proof of the case $\underline{p < 0, q < 0, r < 0}$ is similar. □

Another well-known generalization of Example 10 is the following:

Example 13 (Hölder's inequality for $n{-}L^p$ spaces). Let $p_1, p_2, \ldots, p_n, n = 2, 3, \ldots,$ be positive numbers such that $\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_n} = 1$. Then

$$\int_\Omega |f_1 f_2 \cdots f_n| d\mu \le \left(\int_\Omega |f_1|^{p_1} d\mu\right)^{1/p_1} \cdots \left(\int_\Omega |f_n|^{p_n} d\mu\right)^{1/p_n}. \tag{4.4}$$

The proof follows by just using (4.1) and induction or by using directly the discrete Jensen inequality (3.1) and discussing as in Proof 1 of (4.1).

Remark 4.2. Note that if we put $1/p = \theta, 0 < \theta < 1$, and replace $|f|$ by $|f|^\theta$ and $|g|$ by $|g|^{1-\theta}$, then Hölder's inequality (4.1) can be written

$$\int_\Omega |f|^\theta |g|^{1-\theta} d\mu \leq \left(\int_\Omega |f| d\mu\right)^\theta \left(\int_\Omega |g| d\mu\right)^{1-\theta}, \tag{4.5}$$

where $0 < \theta < 1$.

Remark 4.3. If a and b are positive numbers, then the number $a^{1-\theta}b^\theta, 0 < \theta < 1$, is a geometric type mean of the numbers a and b (for $\theta = 1/2$ we have the usual geometric mean). Moreover, the geometric mean of a positive function f over an interval $[0, b]$ is defined as follows

$$G_f := \exp\left(\frac{1}{b}\int_0^x \ln f(t) dt\right).$$

Accordingly to the Remarks 4.2 and 4.3 it is tempting to think that Example 14 can be generalized to the case with infinite many L^p spaces (cf. (4.7) below) and in fact this is also true. The reader shall here think of that the functions $f_t(x), t \in (0, b)$, belongs to the space $L_{p(t)}$, where $p(t)$ is sufficiently "smooth" so the involved integrals make sense.

Example 14 (A Hölder inequality for infinite many functions involved). Let $p(t)$ be positive on $[0, b]$ and let p be defined by

$$\frac{1}{p} = \frac{1}{b}\int_0^b \frac{1}{p(t)} dt. \tag{4.6}$$

Then

$$\begin{aligned} &\left(\int_\Omega \left(\exp \frac{1}{b}\int_0^b \log |f_t(x)| dt\right)^p d\mu\right)^{1/p} \\ &\leq \exp \frac{1}{b}\int_0^b \log \left(\int_\Omega |f_t(x)|^{p(t)} d\mu\right)^{1/p(t)} dt. \end{aligned} \tag{4.7}$$

Remark 4.4. If we put $b = 1, 0 = a_0 < a_1 < a_2 < \cdots < a_n = 1, \alpha_i = a_{i+1} - a_i, i = 1, 2, \ldots, n, f_t(x) = f_i(x)$ for $a_i < t \leq a_{i+1}, i = 1, 2, \ldots, n$, then $p(t) = \frac{1}{a_i}, i = 1, 2, \ldots, n$, so (4.7) reads

$$\int_\Omega \left(\prod_{i=1}^n |f_i(x)|^{\alpha_i}\right) d\mu \leq \prod_{i=1}^n \left(\int_\Omega |f_i(x)| d\mu\right)^{\alpha_i},$$

where $\sum_{i=1}^n \alpha_i = 1$, which is a generalization of (4.5) and equivalent to (4.4).

Remark 4.5. Inequality (4.7) was stated and proved in a little different form in the paper [23] by L. Nikolova and L.E. Persson, where they used the theory of interpolation between infinite many Banach spaces. However, here we shall finish this section by presenting another proof, which shows that also (4.7) follows essentially from Jensen's inequality (convexity).

Proof. To prove (4.7) first we note that it is sufficient to prove that

$$I_0 := \int_\Omega \left(\exp \frac{1}{b} \int_0^b \log |g_t(x)| dt \right)^p d\mu \le 1, \tag{4.8}$$

where

$$g_t = g_t(x) = \frac{f_t(x)}{\left(\int_\Omega |f_t(x)|^{p(t)} d\mu\right)^{1/p(t)}}.$$

Since

$$\left(\exp \frac{1}{b} \int_0^b \log |g_t(x)| dt \right)^p = \exp \int_0^b \log |g_t(x)|^{p(t)} \frac{p}{p(t)} \frac{1}{b} dx$$

the function $\Phi(u) = e^u$ is convex and, by (4.6), $\int_0^b \frac{p}{p(t)} \frac{1}{b} dt = 1$, we can use Jensen's inequality to obtain that

$$\left(\exp \frac{1}{b} \int_0^b \log |g_t(x)| dt \right)^p \le \frac{1}{b} \int_0^b |g_t(x)|^{p(t)} \frac{p}{p(t)} dt.$$

Hence, by integrating and using Fubini's theorem and (4.6), we find that

$$\begin{aligned} I_0 \le \int_\Omega \left(\frac{p}{b} \int_0^b |g_t(x)|^p(t) \frac{1}{p(t)} dt \right) d\mu &= \int_0^b \frac{p}{b} \frac{1}{p(t)} \int_\Omega \left(\frac{|f_t(x)|^p(t)}{\int_\Omega |f_t(x)|^{p(t)} d\mu} \right) d\mu \\ &= \int_0^b \frac{p}{b} \frac{1}{p(t)} dt = 1, \end{aligned}$$

and (4.8) is proved. The proof is complete. □

5. Various variants of Minkowski's inequality via convexity

The standard variant of Minkowski's inequality reads:

Example 15 (Minkowski's inequality). If $p \ge 1$, then

$$\left(\int_\Omega |f+g|^p d\mu \right)^{1/p} \le \left(\int_\Omega |f|^p d\mu \right)^{1/p} + \left(\int_\Omega |g|^p d\mu \right)^{1/p}. \tag{5.1}$$

Remark 5.1. The inequality (5.1) can be written

$$\|f+g\|_{L_p(\Omega)} \le \|f\|_{L_p(\Omega)} + \|g\|_{L_p(\Omega)},$$

which is the triangle inequality in $L_p(\Omega)$-spaces. This is the crucial property that the spaces $L_p(\Omega)$ are normed spaces, even Banach spaces, for $p \ge 1$.

Proof 1. By the triangle inequality and Hölder's inequality we have that

$$\begin{aligned} \int_\Omega |f+g|^p d\mu = \int_\Omega |f+g|^{p-1} |f+g| d\mu &\le \int_\Omega |f+g|^{p-1} (|f| + |g|) d\mu \\ &= \int_\Omega |f+g|^{p-1} |f| d\mu + \int_\Omega |f+g|^{p-1} |g| d\mu \end{aligned}$$

$$\leq \left(\int_\Omega |f|^p d\mu\right)^{1/p} \left(\int_\Omega |f+g|^p d\mu\right)^{1/q} + \left(\int_\Omega |g|^p d\mu\right)^{1/p} \left(\int_\Omega |f+g|^p d\mu\right)^{1/q}$$
$$= \left(\int_\Omega |f+g|^p d\mu\right)^{1/q} \left[\left(\int_\Omega |f|^p d\mu\right)^{1/p} + \left(\int_\Omega |g|^p d\mu\right)^{1/p}\right].$$

Hence

$$\left(\int_\Omega |f+g|^p d\mu\right)^{1-1/q} \leq \left(\int_\Omega |f|^p d\mu\right)^{1/p} + \left(\int_\Omega |g|^p d\mu\right)^{1/p}$$

and since $1 - 1/q = 1/p$ we obtain (5.1). □

Proof 1 is the most common proof in Functional Analysis books but we present here also another proof of (5.1), which is easier to generalize and which is a special case of a general technique called quasi-linearization. In our case we do this linearization by using (4.2).

Proof 2. The fact that we have equality in Hölder's inequality means that

$$\left(\int_\Omega |f|^p d\mu\right)^{1/p} = \sup_{\varphi \geq 0} \int_\Omega |f|\varphi d\mu,$$

(see (4.2)), where supremum is taken over all φ such that for $q = p/(p-1)$

$$\left(\int_\Omega \varphi^q d\mu\right)^{1/q} \equiv 1.$$

Hence, by the usual triangle inequality for numbers and an obvious estimate, we have that

$$\left(\int_\Omega |f+g|^p d\mu\right)^{1/p} = \sup_{\varphi \geq 0} \int_\Omega |f+g|\varphi d\mu \leq \sup_{\varphi \geq 0} \int_\Omega (|f|\varphi + |g|\varphi) d\mu$$
$$\leq \sup_{\varphi \geq 0} \int_\Omega |f|\varphi d\mu + \sup_{\varphi \geq 0} \int_\Omega |g|\varphi d\mu = \left(\int_\Omega |f|^p d\mu\right)^{1/p} + \left(\int_\Omega |g|^p d\mu\right)^{1/p}. \quad \square$$

A generalization of (5.1) reads:

Example 16 (Minkowski's inequality for n functions $f_1, f_2, \dots, f_n$). If $p \geq 1$, $n = 2, 3, \dots$, then

$$\left(\int_\Omega |f_1 + f_2 + \cdots + f_n|^p d\mu\right)^{1/p} \leq \left(\int_\Omega |f_1|^p d\mu\right)^{1/p} + \cdots + \left(\int_\Omega |f_n|^p d\mu\right)^{1/p}.$$

The proof of this inequality follows by generalizing Proof 2 above of (5.1) in an obvious way or simply by using induction and (5.1).

Next we shall present Minkowski's inequality for infinite many functions $K_y(x) = K(x, y)$, which usually is called Minkowski's integral inequality.

Example 17 (Minkowski's integral inequality). Let $-\infty \leq a \leq b \leq \infty, -\infty \leq c \leq d \leq \infty$, and let $K(x,y)$ be measurable on $[a,b] \times [c,d]$.
If $p \geq 1$, then

$$\left(\int_a^b \left(\int_c^d K(x,y)dy\right)^p dx\right)^{1/p} \leq \int_c^d \left(\int_a^b K^p(x,y)dx\right)^{1/p} dy. \tag{5.2}$$

Proof. Let $p > 1$. We use again the quasi-linearization idea from (4.2) and obtain that

$$I_0 := \left(\int_a^b \left(\int_c^d K(x,y)dy\right)^p dx\right)^{1/p} = \sup_{\varphi \geq 0} \int_a^b \varphi(x) \int_c^d K(x,y)dydx,$$

where the supremum is taken over all measurable φ such that $\int_a^b \varphi^q(x)dx = 1, q = p/(p-1)$. Hence, by using the Fubini theorem and an obvious estimate, we have that

$$\begin{aligned} I_0 = \sup_{\varphi \geq 0} \int_c^d \int_a^b K(x,y)\varphi(x)dx\ dy &\leq \int_c^d \left(\sup_{\varphi \geq 0} \int_a^b K(x,y)\varphi(x)dx\right) dy \\ &= \int_c^d \left(\int_a^b K^p(x,y)dx\right)^{1/p} dy. \end{aligned}$$

For $p = 1$ we have even equality in (5.2) because of the Fubini theorem, so the proof is complete. □

Next we shall consider a special case of Example 17, which is useful, e.g., when working with mixed-norm L_p spaces and we need some estimate replacing the Fubini theorem. More exactly, we put

$$K(x,y) = \begin{cases} k(x,y)\Psi(y)\Psi_0^{1/p}(x), & a \leq y \leq x, \\ 0 & , \ x < y \leq b, \end{cases}$$

where $k(x,y), \Psi(y)$ and $\Psi_0(x)$ are measurable so that Minkowski's integral inequality (5.2) can be used. Under this assumption we have the following:

Example 18 (Minkowski's integral inequality of Fubini type). If $p \geq 1, -\infty \leq a < b \leq \infty$, then

$$\begin{aligned} &\left(\int_a^b \left(\int_a^x k(x,y)\Psi(y)dy\right)^p \Psi_0(x)dx\right)^{1/p} \\ &\quad \leq \int_a^b \left(\int_y^b k^p(x,y)\Psi_0(x)dx\right)^{1/p} \Psi(y)dy. \end{aligned} \tag{5.3}$$

Remark 5.2. With the same proof as above we can also formulate more general forms of the estimates (5.2) and (5.3) by replacing the measures dx and dy by general measures $d\mu(x)$ and $d\mu(y)$, respectively, and thus, e.g., also cover cases with double sums instead of double integrals.

In particular, we have the following discrete variant of (5.3):

Example 19. Let $p \geq 1$ and let $\{a_{k\ell}\}_{k,\ell=1}^{\infty,\infty}, \{b_k\}_{k=1}^{\infty}$ and $\{c_k\}_{k=1}^{\infty}$ be positive sequences. Then

$$\left(\sum_{k=1}^{\infty}\left(\sum_{\ell=1}^{k} a_{k\ell}b_{\ell}\right)^{p} c_k\right)^{1/p} \leq \sum_{\ell=1}^{\infty}\left(\sum_{k=\ell}^{\infty} a_{k\ell}^{p}c_k\right)^{1/p} b_{\ell}. \tag{5.4}$$

Remark 5.3. In the same way we can prove the following associate variants of (5.3) and (5.4):

$$\left(\int_a^b\left(\int_x^b k(x,y)\Psi(y)dy\right)^p \Psi_0(x)dx\right)^{1/p}$$
$$\leq \int_a^b\left(\int_a^y k^p(x,y)\Psi_0(x)dx\right)^{1/p} \Psi(y)dy,$$

respectively,

$$\left(\sum_{k=1}^{\infty}\left(\sum_{\ell=k}^{\infty} a_{k\ell}b_{\ell}\right)^{p} c_k\right)^{1/p} \leq \sum_{\ell=1}^{\infty}\left(\sum_{1}^{\ell} a_{k\ell}^{p}c_k\right)^{1/p} b_{\ell}.$$

6. Some classical inequalities (by Hardy, Carleman and Pólya–Knopp) via convexity

The main information in this and the next section is mainly taken from the recent paper [27] (cf. also [26]) by L.E. Persson and N. Samko. But the formulation of some crucial results are different and put to this more general frame.

Example 20 (Hardy's inequality (continuous form)). If f is non-negative and p-integrable over $(0,\infty)$, then

$$\int_0^{\infty}\left(\frac{1}{x}\int_0^x f(y)dy\right)^p dx \leq \left(\frac{p}{p-1}\right)^p \int_0^{\infty} f^p(x)dx, \quad p>1. \tag{6.1}$$

Example 21 (Hardy's inequality (discrete form)). If $\{a_n\}_1^{\infty}$ is a sequence of non-negative numbers, then

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{n=1}^{n} a_i\right)^p \leq \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{\infty} a_n^p, \quad p>1. \tag{6.2}$$

Remark 6.1. The dramatic more than 10 years period of research until Hardy stated in 1920 (see [10]) and proved in 1925 (see [11]) his inequality (6.1) was recently described in [19]. It is historically clear that Hardy's original motivation when he discovered his inequalities was to find a simple proof of Hilbert's double series inequality, so first he even only considered the case $p = 2$.

Remark 6.2. It is clear that (6.1) ⇒ (6.2), which can be seen by applying (6.1) with step functions. This was pointed out to Hardy in a private letter from F. Landau already in 1921 and here Landau even included a proof of (6.2).

Example 22 (Carleman's inequality). If $\{a_n\}_1^\infty$ is a sequence of positive numbers, then

$$\sum_{n=1}^{\infty} \sqrt[n]{a_1 \cdots a_n} \quad \leq \quad e \sum_{n=1}^{\infty} a_n. \tag{6.3}$$

Remark 6.3. This inequality was proved by T. Carleman in 1922 (see [8]) in connection to this important work on quasianalytical functions. Carleman's idea of proof was to find maximum of $\sum_{i=1}^{n} (a_1 \cdots a_i)^{1/i}$ under the constraint $\sum_{i=1}^{n} a_i = 1, n \in Z_+$. However, (6.3) is in fact a limit inequality (as $p \to \infty$) of the inequalities (6.2) according to the following:

Replace a_i with $a_i^{1/p}$ in the Hardy discrete inequality (6.2) and we obtain that

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{n=1}^{n} a_i^{1/p} \right)^p \leq \left(\frac{p}{p-1} \right)^p \sum_{n=1}^{\infty} a_n, \; p > 1.$$

Moreover, when $p \to \infty$ we have that

$$\left(\frac{1}{n} \sum_{i=1}^{n} a_i^{1/p} \right)^p \to \left(\prod_{i=1}^{n} a_i \right)^{1/n} \qquad \text{and} \qquad \left(\frac{p}{p-1} \right)^p \to e.$$

In view of the fact that Carleman and Hardy had a direct cooperation at that time (see, e.g., [9]) it is maybe a surprise that Carleman did not mention this fact and simpler proof in his paper.

Example 23 (The Pólya–Knopp inequality). If f is a positive and integrable function on $(0, \infty)$, then

$$\int_0^{\infty} \exp \left(\frac{1}{x} \int_0^x \ln f(y) dy \right) dx \leq e \int_0^{\infty} f(x) dx. \tag{6.4}$$

Remark 6.4. Sometimes (6.4) is referred to as the Knopp inequality with reference to his 1928 paper [17]. But it is clear that it was known before and in his 1925 paper [11] Hardy informed that G. Pólya had pointed out the fact that (6.4) is in fact a limit inequality (as $p \to \infty$) of the inequality (6.1) and the proof is literally the same as that above that (6.2) implies (6.3), see Remark 6.3. Accordingly, nowadays (6.4) is many times referred to as the Pólya–Knopp inequality and we have adopted this terminology.

All inequalities above are sharp, i.e., the constants in the inequalities can not be replaced by any smaller constants.

In particular, the discussion above shows that indeed all the inequalities (6.1)–(6.4) are proved as soon as (6.1) is proved. Our next aim is to present a really simple ("miracle") proof of this inequality via convexity, but first we need the following:

Basic observation 6.5. We note that for $p > 1$ it yields that

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(y)dy\right)^p dx \le \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)dx, \tag{6.5}$$

$$\Leftrightarrow$$

$$\int_0^\infty \left(\frac{1}{x}\int_0^x g(y)dy\right)^p \frac{dx}{x} \le 1 \cdot \int_0^\infty g^p(x)\frac{dx}{x}, \tag{6.6}$$

where $f(x) = g(x^{1-1/p})x^{-1/p}$. In fact, consider (6.5) (= (6.1)) and we find that:

$$\begin{aligned}\left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)dx &= \left(\frac{p}{p-1}\right)^p \int_0^\infty g^p(x^{1-1/p})\frac{dx}{x}\\ &= \left(\frac{p}{p-1}\right)^{p+1} \int_0^\infty g^p(y)\frac{dy}{y},\end{aligned}$$

and

$$\begin{aligned}\int_0^\infty \left(\frac{1}{x}\int_0^x g(t^{1-1/p})t^{-1/p}dt\right)^p dx &= \left(\frac{p}{p-1}\right)^p \int_0^\infty \left(\frac{1}{x}\int_0^{x^{1-1/p}} g(s)ds\right)^p dx\\ &= \left(\frac{p}{p-1}\right)^{p+1} \int_0^\infty \left(\frac{1}{y}\int_0^y g(s)ds\right)^p \frac{dy}{y},\end{aligned}$$

which proves this statement.

According to the Basic observation 6.5 we have proved (6.1) (and thus also (6.2)–(6.4)) as soon as (6.6) is proved and here is the ("miracle") proof of (6.1): By Jensen's inequality and Fubini's theorem we have that

$$\begin{aligned}\int_0^\infty \left(\frac{1}{x}\int_0^x g(y)dy\right)^p \frac{dx}{x} &\le \int_0^\infty \left(\frac{1}{x}\int_0^x g^p(y)dy\right)\frac{dx}{x}\\ &= \int_0^\infty g^p(y)\int_y^\infty \frac{dx}{x^2}dy = \int_0^\infty g^p(y)\frac{dy}{y}.\end{aligned} \tag{6.7}$$

Remark 6.6. Since the function $\Phi(u) = u^p$ is convex also when $p < 0$ this simple proof shows that (6.1) in fact also holds for $p < 0$, a fact which was not noted by Hardy himself.

Remark 6.7. In 1927 G.H. Hardy himself (see [12]) proved the first weighted version of his inequality (6.1) namely the following: The inequality

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(y)dy\right)^p x^a dx \le \left(\frac{p}{p-1-a}\right)^p \int_0^\infty f^p(x)x^a dx \tag{6.8}$$

holds for all measurable and non-negative functions f on $(0,\infty)$ whenever $a < p-1, p \ge 1$.

Hardy obviously believed that this was a generalization of (6.1) but, in fact, by making the substitution

$$f(t) = g(t^{\frac{p-1-\alpha}{p}})t^{-\frac{1+\alpha}{p}}$$

and calculations like in the Basic observation 6.5 we see that also (6.8) for any considered α is equivalent to (6.6).

Remark 6.8. There exists also an associate variant of (6.8), namely the following:

$$\int_0^\infty \left(\frac{1}{x}\int_x^\infty f(y)dy\right)^p x^{\alpha_0}dx \le \left(\frac{p}{\alpha_0+1-p}\right)^p \int_0^\infty f^p(x)x^{\alpha_0}dx, \tag{6.9}$$

which holds for all measurable and non-negative functions on $(0,\infty)$ whenever $\alpha_0 > p-1, p \ge 1$. In fact, also this inequality is equivalent to the basic inequality (6.6) so, in particular, (6.8) and (6.9) are equivalent (with the relation $\alpha_0 = -\alpha - 2 + 2p$ as we will see later on). Moreover, since the function $\Phi(u) = u^p$ is convex also for $p < 0$ it yields that

a) (6.8) holds also in the case $p < 0, \alpha > p-1$,
b) (6.9) holds also in the case $p < 0, \alpha_0 < p-1$,

and these inequalities are equivalent also then.

Finally, we note that since the function $\Phi(u) = u^p$ is concave for $0 < p < 1$ it yields that (6.8) and (6.9) hold in the reversed direction for $0 < p < 1$ with the same restrictions on α and α_0.

This important remark is a special case of a more general statement (Proposition 7.3) proved and discussed in detail in our next section.

7. More Hardy type inequalities via convexity

The same convexity argument as that in the proof (see (6.7)) of the basic inequality (6.6) shows that we have the following more general statement:

Example 24. Let f be a measurable function on $\mathbb{R}_+$ and let Φ be a convex function on $D_f = \{f(x)\}$. Then

$$\int_0^\infty \Phi\left(\frac{1}{x}\int_0^x f(y)dy\right)\frac{dx}{x} \le \int_0^\infty \Phi\big(f(x)\big)\frac{dx}{x}. \tag{7.1}$$

If Φ instead is positive and concave, then the reversed inequality holds.

In fact, by Jensen's inequality and Fubini's theorems we have that

$$\begin{aligned}\int_0^\infty \Phi\left(\frac{1}{x}\int_0^x f(y)dy\right)\frac{dx}{x} &\le \int_0^\infty\int_0^x \Phi(f(y))dy\frac{dx}{x^2}\\ &= \int_0^\infty \Phi(f(y))\int_y^\infty \frac{1}{x^2}dx\; dy = \int_0^\infty \Phi(f(y))\frac{dy}{y}.\end{aligned}$$

If Φ is concave, then the only inequality holds in the reverse direction.

Example 25. Consider the convex function $\Phi(u) = e^u$ and replace $f(y)$ with $\ln\ f(y)$. Then (7.1) reads

$$\int_0^\infty \exp\left(\frac{1}{x}\int_0^x \ln f(y)dy\right)\frac{dx}{x} \le \int_0^\infty f(x)\frac{dx}{x}. \tag{7.2}$$

By now making the substitution $f(x) = xg(x)$ we transform (7.2) to the inequality

$$\int_0^\infty \exp\left(\frac{1}{x}\int_0^x \ln g(y)dy\right)dx \le e\int_0^\infty g(x)dx,$$

i.e., we obtain another proof of the Pólya–Knopp inequality (6.4) without going via the limit argument mentioned in the Remark 6.4.

It is also known that the Hardy inequality (6.1) holds for finite intervals, e.g., that

$$\int_0^\ell \left(\frac{1}{x}\int_0^x f(y)dy\right)^p dx \le \left(\frac{p}{p-1}\right)^p \int_0^\ell f^p(x)dx, \quad p > 1, \tag{7.3}$$

holds for any $\ell, 0 < \ell \le \infty$, and the constant $\left(\frac{p}{p-1}\right)^p$ is still sharp also for $\ell < \infty$.

But the inequality (7.3) can be improved to the following:

$$\int_0^\ell \left(\frac{1}{x}\int_0^x f(y)dy\right)^p dx \le \left(\frac{p}{p-1}\right)^p \int_0^\ell f^p(x)\left[1-\left(\frac{x}{\ell}\right)^{\frac{p-1}{p}}\right]dx, \tag{7.4}$$

where $p > 1$ or $p < 0$.

This fact is a special case of the more general result (Proposition 7.3) we next aim to prove and discuss. As a preparation of independent interest we first state the following generalization of our previous basic inequality (6.6):

Lemma 7.1. *Let g be a non-negative and measurable function on* $(0,\ell), 0 < \ell \le \infty$.

a) *If $p < 0$ or $p \ge 1$, then*

$$\int_0^\ell \left(\frac{1}{x}\int_0^x g(y)dy\right)^p \frac{dx}{x} \le 1 \cdot \int_0^\ell g^p(x)\left(1-\frac{x}{\ell}\right)\frac{dx}{x}. \tag{7.5}$$

(*In the case $p < 0$ we assume that $g(x) > 0, 0 < x \le \ell$.*)

b) *If $0 < p \le 1$, then* (7.5) *holds in the reversed direction.*

c) *The constant $C = 1$ is sharp in both* a) *and* b).

Proof. By using Jensen's inequality with the convex function $\Phi(u) = u^p, p \ge 1$ or $p < 0$, and reversing the order of integration, we find that

$$\begin{aligned}\int_0^\ell \left(\frac{1}{x}\int_0^x g(y)dy\right)^p \frac{dx}{x} &\le \int_0^\ell \frac{1}{x}\int_0^x g^p(y)dy\frac{dx}{x} = \int_0^\ell g^p(y)\left(\int_y^\ell \frac{1}{x^2}dx\right)dy \\ &= \int_0^\ell g^p(y)\left(\frac{1}{y}-\frac{1}{\ell}\right)dy = \int_0^\ell g^p(y)\left(1-\frac{y}{\ell}\right)\frac{dy}{y}.\end{aligned}$$

The only inequality in this proof holds in the reversed direction when $0 < p \le 1$ so the proof of b) follows in the same way.

Concerning the sharpness of the inequality (7.5) we first let $\ell < \infty$ and assume that

$$\int_0^\ell \left(\frac{1}{x}\int_0^x g(y)dy\right)^p \frac{dx}{x} \le C \cdot \int_0^\ell g^p(x)\left(1-\frac{x}{\ell}\right)\frac{dx}{x} \tag{7.6}$$

for all non-negative and measurable functions g on $(0,\ell)$ with some constant $C, 0 < C < 1$. Let $p \geq 1$ and $\varepsilon > 0$ and consider $g_\varepsilon(x) = x^\varepsilon$ (for the case $p < 0$ we assume that $-1 < \varepsilon < 0$). By inserting this function into (7.6) we obtain that

$$C \geq (\varepsilon p + 1)(\varepsilon + 1)^{-p},$$

so that, by letting $\varepsilon \to 0_+$ we have that $C \geq 1$. This contradiction shows that the best constant in (7.5) is $C = 1$. In the same way we can prove that the constant $C = 1$ is sharp also in the case b). For the case $\ell = \infty$ the sharpness follows by just making a limit procedure with the result above in mind. The proof is complete. □

Remark 7.2. For the case $\ell = \infty$ (7.5) coincides with the basic inequality (6.6) and, thus, the constant $C = 1$ is sharp, which, in its turn, implies the well-known fact that the constant $C = \left(\frac{p}{p-1}\right)^p$ in Hardy's inequality (6.1) is sharp for $p > 1$ and as we see above this holds also for $p < 0$.

Moreover, since also the weighted variants (6.8) and (6.9) are equivalent to the basic inequality (6.6) via substitutions we conclude that also these constants are sharp in all considered cases.

We are now ready to formulate our main result in this section.

Proposition 7.3. *Let* $0 < \ell \leq \infty$, $\ell_0 = 1/\ell$, $a, a_0 \in \mathbb{R}$ *and let* $p \in \mathbb{R} \setminus \{0\}$ *be a fixed number and* f *be a non-negative function.*

a) *Let* f *be a measurable function on* $(0, \ell]$. *Then*

$$\begin{aligned} &\int_0^\ell \left(\frac{1}{x}\int_0^x f(y)dy\right)^p x^a dx \\ &\leq \left(\frac{p}{p-1-a}\right)^p \int_0^\ell f^p(x)x^a \left[1 - \left(\frac{x}{\ell}\right)^{\frac{p-a-1}{p}}\right] dx \end{aligned} \tag{7.7}$$

holds for the following cases:

a₁) $p \geq 1$, $a < p - 1$,
a₂) $p < 0$, $a > p - 1$.

b) *For the case* $0 < p < 1, a < p - 1$, *inequality* (7.7) *holds in the reversed direction in both cases* a₁) *and* a₂).

c) *Let* f *be a measurable function on* $[\ell, \infty)$. *Then*

$$\begin{aligned} &\int_{\ell_0}^\infty \left(\frac{1}{x}\int_x^\infty f(y)dy\right)^p x^{a_0} dx \\ &\leq \left(\frac{p}{a_0+1-p}\right)^p \int_{\ell_0}^\infty f^p(x)x^{a_0} \left[1 - \left(\frac{\ell_0}{x}\right)^{\frac{a_0+1-p}{p}}\right] dx \end{aligned} \tag{7.8}$$

holds for the following cases:

c₁) $p \geq 1$, $a_0 > p - 1$,
c₂) $p < 0$, $a_0 < p - 1$.

d) *For the case* $0 < p \leq 1, a_0 > p - 1$ *inequality* (7.8) *holds in the reversed direction in both cases* c_1) *and* c_2).

e) *All inequalities above are sharp.*

f) *Let* $p \geq 1$ *or* $p < 0$. *Then, the inequalities* (7.7) *and* (7.8) *are equivalent for all permitted* a *and* a_0 *because they are in all cases equivalent to* (7.5) *via substitutions.*

g) *Let* $0 < p < 1$. *Then, the reversed inequalities* (7.7) *and* (7.8) *are equivalent for all permitted* a *and* a_0.

Remark 7.4. The formal relation between the parameters a and a_0 is that $a = 2p - a_0 - 2$, but this is not important according to f) and g).

Proof. First we prove that (7.7) in the case (a_1) in fact is equivalent to (7.5) via the relation

$$f(x) = g\left(x^{\frac{p-a-1}{p}}\right) x^{-\frac{a+1}{p}}.$$

In fact, with $f(x) = g\left(x^{\frac{p-a-1}{p}}\right) x^{-\frac{a+1}{p}}$ and $\ell_0 = \ell^{\frac{p}{p-a-1}}$, in (7.7) we get that

$$\begin{aligned}
&\left(\frac{p}{p-1-a}\right)^p \int_0^{\ell_0} g^p\left(x^{\frac{p-a-1}{p}}\right)\left[1-\left(\frac{x}{\ell_0}\right)^{\frac{p-1-a}{p}}\right]\frac{dx}{x}\\
&= \left(\frac{p}{p-1-a}\right)^{p+1} \int_0^{\ell^{\frac{p-a-1}{p}}} g^p(y)\left[1-\frac{y}{\ell_0^{\frac{p-1-a}{p}}}\right]\frac{dy}{y}\\
&= \left(\frac{p}{p-1-a}\right)^{p+1} \int_0^{\ell} g^p(y)\left[1-\frac{y}{\ell}\right]\frac{dy}{y},
\end{aligned}$$

where $y = x^{\frac{p-a-1}{p}}$, $dy = x^{-\frac{a+1}{p}}\left(\frac{p-1-a}{p}\right)dx$, and

$$\begin{aligned}
&\int_0^{\ell_0}\left(\frac{1}{x}\int_0^x g\left(y^{\frac{p-a-1}{p}}\right) y^{-\frac{a+1}{p}}dy\right)^p x^a dx\\
&= \left(\frac{p}{p-1-a}\right)^p \int_0^{\ell_0}\left(\frac{1}{x^{\frac{p-a-1}{p}}}\int_0^{x^{\frac{p-a-1}{p}}} g(s)ds\right)^p\frac{dx}{x}\\
&= \left(\frac{p}{p-1-a}\right)^{p+1} \int_0^{\ell}\left(\frac{1}{y}\int_0^y g(s)ds\right)^p\frac{dy}{y}.
\end{aligned}$$

Since we have only equalities in the calculations above we conclude that (7.5) and (7.7) are equivalent and, thus, by Lemma 7.1, a) is proved for the case (a_1).

For the case (a_2) all calculations above are still valid and, according to Lemma 7.1, (7.5) holds also in this case and a) is proved also for the case (a_2).

For the case $0 < p \leq 1, a < p - 1$, all calculations above are still true and both (7.5) and (7.7) hold in the reversed direction according to Lemma 7.1. Hence also b) is proved.

For the proof of c) we consider (7.7) with $f(x)$ replaced by $f(1/x)$, with a replaced by a_0 and with ℓ replaced by $\ell_0 = 1/\ell$:

$$\begin{aligned}\int_0^{\ell_0}\left(\frac{1}{x}\int_0^x f(1/y)dy\right)^p x^{a_0}dx \\ \leq \left(\frac{p}{p-1-a_0}\right)^p \int_0^{\ell_0} f^p(1/x)x^{a_0}\left[1-\left(\frac{x}{\ell_0}\right)^{\frac{p-a_0-1}{p}}\right]dx.\end{aligned} \tag{7.9}$$

Moreover, by making first the variable substitution $x = 1/s$ and after that $y = 1/x$, we find that left-hand side of (7.9) is equal to

$$\begin{aligned}\int_0^{\ell_0}\left(\frac{1}{x}\int_{1/x}^{\infty}\frac{f(s)}{s^2}ds\right)^p x^{a_0}dx &= \int_\ell^\infty\left(y\int_y^\infty \frac{f(s)}{s^2}ds\right)^p y^{-a_0-2}dy \\ &= \int_\ell^\infty\left(\frac{1}{y}\int_y^\infty \frac{f(s)}{s^2}ds\right)^p y^{-a_0-2+2p}dy\end{aligned}$$

[Put $\frac{f(s)}{s^2} = g(s)$]

$$= \int_\ell^\infty\left(\frac{1}{y}\int_y^\infty g^p(y)\right)^p y^{2p-a_0-2}dy,$$

and, by using the substitution $y = 1/x$, we obtain that right-hand side of (7.9) is equal to

$$\begin{aligned}&\left(\frac{p}{p-1-a_0}\right)^p\int_\ell^\infty f^p(y)y^{-a_0}\left[1-\left(\frac{\ell}{y}\right)^{\frac{p-a_0-1}{p}}\right]y^{-2}dy \\ &= \left(\frac{p}{p-1-a_0}\right)^p\int_\ell^\infty g^p(y)y^{2p-a_0-2}\left[1-\left(\frac{\ell}{y}\right)^{\frac{p-a_0-1}{p}}\right]dy.\end{aligned}$$

Now replace $2p-a_0-2$ by a and g by f and we have that $a_0 = 2p-a-2, p-1-a_0 = a+1-p$. Hence, it yields that

$$\int_\ell^\infty\left(\frac{1}{x}\int_x^\infty f(s)ds\right)^p x^a dx \leq \left(\frac{p}{a+1-p}\right)^p\int_\ell^\infty f^p(x)x^a\left[1-\left(\frac{\ell}{x}\right)^{\frac{a+1-p}{p}}\right]dx$$

and, moreover,

$$a_0 < p-1 \Leftrightarrow 2p-a-2 < p-1 \Leftrightarrow a > p-1.$$

By changing notation and using the symmetry between the parameters, we find that c) with the conditions (c_1) and (c_2) are in fact equivalent to a) with the conditions (a_1) and (a_2), respectively, and also c) is proved. (The formal relation between the parameters is $a = 2p - a_0 - 2$ and $\ell_0 = 1/\ell$.)

The calculations above hold also in the case d) and the only inequality holds in the reversed direction in this case so also d) is proved.

Finally, we note that the proof above only consists of suitable substitutions and equalities to reduce all inequalities to the sharp inequality (7.5), or the reversed inequality (7.5), and we obtain a proof also of the statements e) and f) according to Lemma 7.1. The proof is complete. □

8. Some further results and final remarks

Remark 8.1. Our presented simple convexity technique to prove powerweighted Hardy inequalities can be useful for several generalizations. We only present here the following generalization of our fundamental inequality (7.5) for the case of piecewise-constant $p(x)$:

Example 26. Let

$$p(x) = \begin{cases} p_0, & 0 \le x \le 1, \\ p_1, & x > 1, \end{cases} \tag{8.1}$$

where $p_0, p_1 \in \mathbb{R} \setminus \{0\}$. Let $0 < \ell \le \infty$, and let $p(x)$ be defined by (8.1). Then, for every non-negative and measurable function f,

$$\int_0^\ell \left(\frac{1}{x} \int_0^x f(t)dt \right)^{p(x)} \frac{dx}{x} \le 1 \cdot \int_0^\ell (f(x))^{p(x)} \left(1 - \frac{x}{\ell}\right) \frac{dx}{x} \tag{8.2}$$
$$+ \max\left\{0, 1 - \frac{1}{\ell}\right\} \int_0^1 \left[(f(x))^{p_1} - (f(x))^{p_0} \right] dx,$$

whenever $p(x) \ge 1$ or $p(x) < 0$ (for the case $p(x) < 0$ we also assume that $f(x) > 0$.)

For the case $0 < p(x) < 1$ inequality (8.2) holds in the reversed direction.

The constant $C = 1$ in front of the first integral is sharp.

Remark 8.2. The proof of this and more general statements of this type can be found in [27]. Note that for the case $p_0 = p_1 = p$ (8.2) coincides with (7.5). Hence, our inequality (8.2) is a genuine generalization not only of (7.5) but also of all Hardy type inequalities we have derived from (7.5) in this paper (see, e.g., Proposition 7.3).

Remark 8.3. We have already mentioned that convexity was very important when modern interpolation theory was initiated (see Example 2 and [7]). Hence, it is not surprising that interpolation theory is also very important tool when proving inequalities. The main aim of [25] is to illustrate and develop this close connection between Convexity, Interpolation and Inequalities. Here we just mention the following example:

Example 27 (Young's integral inequality). Consider the convolution operator T defined by

$$Tf(x) = \int_{-\infty}^{+\infty} k(x-y) f(y) dy = k * f(x).$$

If $k \in L_r = L_r(-\infty, +\infty)$ and $f \in L_p = L_p(-\infty, +\infty)$, where $1 < p < r' = r/(r-1)$, then $k * f \in L_q$, where $1/q + 1/p - 1/r'$ and

$$\|k * f\|_{L_q} \le \|k\|_{L_r} \|f\|_{L_p}. \tag{8.3}$$

Proof. By our variant of Minkowski's inequality we have that

$$\|Tf\|_{L_r} \leq \|k\|_{L_r}\|f\|_{L_1}$$

and, by Hölder's inequality,

$$\|Tf\|_{L_\infty} \leq \|k\|_{L_r}\|f\|_{L_{r'}}.$$

This means that $T : L_1 \to L_r$ and $T : L_{r'} \to L_\infty$ with norm $\|k\|_{L_r}$ in both cases. By interpolating between these two situations with the usual relation for the parameters in intermediate spaces we obtain (8.3) and the proof is complete (cf. [7], p. 6). □

Remark 8.4. Note that this argument of proof does not work for the case $r = 1$ (so that $p = q$) but this limit case holds also, which can be seen by just using a direct convexity argument.

Example 28. If $f \in L_p, 1 \leq p \leq \infty$, and $g \in L_1$, then $f * g \in L^p$ and, moreover,

$$\|f * g\|_{L_p} \leq \|g\|_{L_1}\|f\|_{L_p}.$$

Proof. We shall prove that

$$\left(\int_{-\infty}^{+\infty}\left|\int_{-\infty}^{+\infty} f(y)g(x-y)dy\right|^p dx\right)^{1/p} \leq \int_{-\infty}^{+\infty}|g(x)|dx\left(\int_{-\infty}^{+\infty}|f(x)|^p dx\right)^{1/p}.$$

The cases $p = 1$ and $p = \infty$ are trivial so we assume that $1 < p < \infty$. First we note that, by Hölder's inequality, for each $x \in \mathbb{R}$ we have that

$$\begin{aligned}|f * g(x)| &= \left|\int_{-\infty}^{+\infty} f(y)g(x-y)dy\right| \leq \int_{-\infty}^{+\infty}|f(y)||g(x-y)|^{1/p}|g(x-y)|^{1/p'}dy \\ &\leq \left(\int_{-\infty}^{+\infty}|f(y)|^p|g(x-y)|dy\right)^{1/p}\left(\int_{-\infty}^{+\infty}|g(x-y)|dy\right)^{1/p'}.\end{aligned}$$

We now take the L_p norm of both sides and use Fubini's theorem to obtain that

$$\begin{aligned}\|f * g\|_{L_p} &\leq (\|g\|_{L_1})^{1/p'}\left(\int_{-\infty}^{+\infty}|f(y)|^p\left(\int_{-\infty}^{+\infty}|g(x-y)|dx\right)dy\right)^{1/p} \\ &= (\|g\|_{L_1})^{1/p'}(\|g\|_{L_1})^{1/p}\left(\int_{-\infty}^{+\infty}|f(y)|^p dy\right)^{1/p} = \|g\|_{L_1}\|f\|_{L_p}.\end{aligned}$$ □

Remark 8.5. We claim that also a number of other classical and new inequalities can be derived and understood in this uniform way via convexity and interpolation. For further information concerning this we refer to [25].

We shall finish this section by shortly discussing the possibility to change the concept of convexity a little and thus be able to prove some refined versions of classical inequalities.

In this connection we mention that the following concept of super-quadratic (sub-quadratic) function was introduced in 2004 by S.Abramovich et al. in [2]:

Definition 8.6. [2, Definition 2.1]. A function $\varphi : [0, \infty) \to \mathbb{R}$ is superquadratic provided that for all $x \geq 0$ there exists a constant $C_x \in \mathbb{R}$ such that

$$\varphi(y) - \varphi(x) - \varphi(|y-x|) \geq C_x (y-x)$$

for all $y \geq 0$.

We say that f is subquadratic if $-f$ is superquadratic.

Remark 8.7. It is easy to see that the function $f(u) = u^p$ is super-quadratic for $p \geq 2$ and sub-quadratic for $1 < p \leq 2$.

In the paper [2] the authors proved the following remarkable refinement of Jensen's inequality for super-quadratic functions:

Theorem 8.8. *Let (Ω, μ) be a measure space with $\mu(\Omega) = 1$. The inequality*

$$\varphi\left(\int_\Omega f(s)d\mu(s)\right) \leq \int_\Omega \varphi(f(s))d\mu(s) - \int_\Omega \varphi\left(\left|f(s) - \int_\Omega f(s)d\mu(s)\right|\right) d\mu(s) \quad (8.4)$$

holds for all probability measures μ and all nonnegative μ-integrable functions f if and only if φ is super-quadratic. Moreover, (8.4) *holds in the reversed direction if and only if φ is sub-quadratic.*

In view of Remark 8.7 we have the following important special case of Theorem 8.8:

Example 29. Let (Ω, μ) be a measure space with $\mu(\Omega) = 1$. If $p \geq 2$, then

$$\left(\int_\Omega f(s)d\mu(s)\right)^p \leq \int_\Omega (f(s))^p d\mu(s) - \int_\Omega \left|f(s) - \int_\Omega f(s)d\mu(s)\right|^p d\mu(s) \quad (8.5)$$

holds and the reversed inequality holds when $1 < p \leq 2$ (see also [1, Example 1, p. 1448]).

By now using the same technique as in our previous sections but with this refined Jensen inequality (see Example 29) we can obtain for example the following refined Hardy type inequalities (the details in the calculations can be found in the paper [24] by J. Oguntuase and L.E. Persson).

Example 30. Let $p > 1$, $k > 1$, $0 < b \leq \infty$, and let the function f be locally integrable on $(0, b)$ such that $0 < \int\limits_0^b x^{p-k} f^p(x)dx < \infty$.

(i) If $p \geq 2$, then

$$\begin{aligned}
&\int_0^b x^{-k}\left(\int_0^x f(t)dt\right)^p dx + \frac{k-1}{p}\int_0^b \int_t^b \left|\frac{p}{k-1}\left(\frac{t}{x}\right)^{1-\frac{k-1}{p}} f(t) \right.\\
&\qquad \left. - \frac{1}{x}\int_0^x f(t)dt\right|^p \cdot x^{p-k-\frac{k-1}{p}} dx t^{\frac{k-1}{p}-1} dt \qquad (8.6)\\
&\quad \leq \left(\frac{p}{k-1}\right)^p \int_0^b \left[1 - \left(\frac{x}{b}\right)^{\frac{k-1}{p}}\right] x^{p-k} f^p(x)dx.
\end{aligned}$$

(ii) If $1 < p \leq 2$, then inequality (8.6) holds in the reversed direction.

Remark 8.9. Note that (8.6) with $b = \infty$ means that if $p \geq 2$, then the classical Hardy inequality for $k > 1$ can be refined by adding a second term on the left-hand side. In fact, this factor is so big that the inequality holds in the reversed direction for $1 < p \leq 2$ so that, in particular, for $p = 2$ we have the following identity:

$$\int_0^\infty x^{-k}\left(\int_0^x f(t)dt\right)^2 dx + \frac{k-1}{2}\int_0^\infty \int_t^\infty \left(\frac{2}{k-1}\left(\frac{t}{x}\right)^{1-\frac{k-1}{2}} f(t) - \frac{1}{x}\int_0^x f(t)dt\right)^2 \cdot x^{2-k-\frac{k-1}{2}} dx t^{\frac{k-1}{2}-1} dt = \left(\frac{2}{k-1}\right)^2 \int_0^\infty x^{2-k} f^2(x)dx.$$

Remark 8.10. As we have seen the "normal" behaviour in Hardy type inequalities is that the natural "breaking point" (the point where it reverses) is $p = 1$ but in the refined Hardy inequality (8.6) the "breaking point" is $p = 2$. Further research in this direction can be found in recent papers by S. Abramovich and the present authors (see [3], [4] and [5]).

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References

[1] S. Abramovich, S. Banić, and M. Matić, *Superquadratic functions in several variables.* J. Math. Anal. Appl. **327**(2) (2007), 1444–1460.

[2] S. Abramovich, G. Jameson, and G Sinnamon, *Refining of Jensen's inequality.* Bull. Math. Soc. Sci. Math. Roumanie (N.S) 47(95) (2004), 3–14.

[3] S. Abramovich and L.E. Persson, *Some new scales of refined Hardy type inequalities via functions related to superquadraticity.* Math. Inequal Appl., to appear.

[4] S. Abramovich and L.E. Persson, *Hardy type inequalities with breaking points $p = 2$ and $p = 3$.* Volume 236 of *Operator Theory: Advances and Applications*, Birkhäuser Verlag, Basel, 2013, to appear.

[5] S. Abramovich, L.E. Persson, and N. Samko, *On some new developments of Hardy-type inequalities.* AIP (American Institute of Physics), 1493 (2012), 739–746.

[6] E.F. Beckenbach and R. Bellman, *Inequalities.* Volume 30 of *Ergebnisse der Mathematik und ihrer Grenzgebiete.* Springer-Verlag, Berlin-Heidelberg-New York-Tokyo, 1983, forth ed.

[7] J. Bergh and J. Löfström, *Interpolation spaces: An Introduction.* Grundlehren der Mathematischen Wissenschaften, N 223, Springer-Verlag, Berlin-New York, 1976.

[8] T. Carleman, *Sur les functions quasi-analytiques.* In Comptes Rendus du V Congrès des Mathématiciens Scandinaves, Helsinki, pages 181–196, 1922.

[9] T. Carleman and G.H. Hardy, *Fourier series and analytic functions.* Proc. R. Soc. Lond. Ser. A **101** (1922), 124–133.

[10] G.H. Hardy, *Note on a theorem of Hilbert.* Math. Z. **6** (1920), 314–317.

[11] G.H. Hardy, *Notes on some points in the integral calculus, LX. An inequality between integrals.* Messenger of Math. **54** (1925), 150–156.

[12] G.H. Hardy. *Notes on some points in the integral calculus, LXIV. Further inequalities between integrals.* Messenger of Math. **57** (1927), 12–16.

[13] G.H. Hardy, J.E. Littlewood, and G. Pólya, *Inequalities.* Cambridge University Press, Cambridge, 1934.

[14] L. Hörmander, *Notions of Convexity.* Birkhäuser Verlag Boston, Inc., Boston, 1994.

[15] J.L.W.V. Jensen, *Om konvexe funktioner og uligheter mellom middelvaerdier* (in Norwegian). Nyt. Tidsskr. Math. **16**B (1905), 49–69.

[16] J.L.W.V. Jensen, *Sur les fonctions convexes et les inégalités entre les valeurs moyennes.* Acta Math. **30** (1906), 175–193.

[17] K. Knopp, *Über Reihen mit positiven Gliedern.* J. London Math. Soc. **3** (1928), 205–212.

[18] V. Kokilashvili, A. Meskhi, and L.E. Persson, *Weighted norm inequalities for integral transforms with product kernels.* Mathematics Research Developments Series, Nova Science Publishers, New York, 2010.

[19] A. Kufner, L. Maligranda, and L.E. Persson, *The prehistory of the Hardy inequality.* Amer. Math. Monthly **113**(8) (2006), 715–732.

[20] A. Kufner, L. Maligranda, and L.E. Persson, *The Hardy Inequality: About its History and Some Related Results.* Vydavatelsky Servis Publishing House, Pilsen, 2007.

[21] A. Kufner and L.E. Persson, *Weighted inequalities of Hardy type.* World Scientific Publishing Co. Inc., River Edge, NJ, 2003.

[22] C.P. Niculescu and L.E. Persson, *Convex Functions and their Applications – A Contemporary Approach.* CMS Books in Mathematics, Springer, New York, 2006.

[23] L.N. Nikolova and L.E. Persson, *Some properties of X^p-spaces.* Function Spaces (Poznan, 1989) Teubner-Texte Math., 120 (1991), 174–185.

[24] J.A. Oguntuase and L.E. Persson, *Refinement of Hardy's inequalities via superquadratic and subquadratic functions.* J. Math. Anal. Appl. **339** (2008), 1305–1312.

[25] L.E. Persson and N. Samko, *Inequalities via Convexity and Interpolation.* Book manuscript in preparation.

[26] L.E. Persson and N. Samko, *Some remarks and new developments concerning Hardy-type inequalities.* Rend. Circ. Mat. Palermo, serie II, 82 (2010), 1–29.

[27] L.E. Persson and N. Samko, *What should have happened if Hardy had discovered this?* J. Inequal. Appl., SpringerOpen, 2012:29, 2012.

[28] R.T. Rockafellar, *Convex Analysis.* Princeton University Press, Princeton, N.J., 1970.

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On a Question by M. Seidel and the Answer by D. Dragičević et al.

Steffen Roch

Dedicated to Prof. António Ferreira dos Santos

Abstract. According to [1], Markus Seidel asked whether certain homomorphisms which identify local algebras can be also viewed as lifting homomorphisms. The authors of [1] give an affirmative answer in the context of concrete Banach algebras. The purpose of this short note is to show that this question *always* has an affirmative answer, with the meaning of "always" explained below.

Mathematics Subject Classification (2010). Primary 65R20; Secondary 46L99, 47N40.

Keywords. Lifting theorem, central localization, strong limit homomorphism.

Introduction. To keep the paper simple and short, we consider the Hilbert space setting only. In principle, all notions can be adapted to the setting of separable reflexive Banach spaces as well, and the basic arguments remain valid in this context.

Given a sequence of separable Hilbert spaces H_n with identity operators I_n, write $\mathcal{F}$ for the set of all bounded sequences $(A_n)_{n\in\mathbb{N}}$ of operators $A_n \in L(H_n)$. Provided with pointwise defined operations and the supremum norm, $\mathcal{F}$ becomes a C^*-algebra, and the set $\mathcal{G}$ of all sequences $(G_n) \in \mathcal{F}$ with $\|G_n\| \to 0$ is a closed ideal of $\mathcal{F}$. A basic task of numerical analysis is, for a given unital C^*-subalgebra $\mathcal{A}$ of $\mathcal{F}$ which contains $\mathcal{G}$, to examine the stability of sequences (A_n) in $\mathcal{A}$ or, equivalently, the invertibility of the coset $(A_n) + \mathcal{G}$ in $\mathcal{F}/\mathcal{G}$.

This task is usually performed in two steps: a lifting step, which provides us with a C^*-subalgebra $\mathcal{F}^{\mathcal{J}}$ of $\mathcal{F}$ which contains $\mathcal{A}$, and with a closed ideal $\mathcal{J}$ of $\mathcal{F}^{\mathcal{J}}$ which contains $\mathcal{G}$, such that the invertibility of a coset $(A_n) + \mathcal{J}$ can be lifted by a family of lifting homomorphisms and such that the quotient $\mathcal{F}^{\mathcal{J}}/\mathcal{J}$ possesses a sufficiently large C^*-algebra $\mathcal{C}$ in its center, and a second, localization, step where localization over $\mathcal{C}$ is employed in order to study invertibility in $\mathcal{F}^{\mathcal{J}}/\mathcal{J}$. The basic tools of these steps, the lifting theorem and the local principle by Allan–Douglas, are briefly described in the following paragraphs.

A basic technical ingredient employed in both steps are *strong limit homomorphisms*. These are defined in terms of a sequence $E := (E_n)$ of isometries E_n on H_n with values in an Hilbert space H^E, i.e., it is $E_n^* E_n = I_n$, and we assume that the projections $E_n E_n^*$ converge strongly to the identity operator I^E on H^E. Note that the latter requirement ensures the separability of H^E. Let $\mathcal{F}^E$ be the set of all sequences $\mathbf{A} = (A_n)$ in $\mathcal{F}$ for which the limits

$$W^E(\mathbf{A}) := \text{s-lim}\, E_n A_n E_n^* \quad \text{and} \quad \text{s-lim}\, E_n A_n^* E_n^*$$

exist in the strong operator topology (in which case we call $(E_n A_n E_n^*)$ a **-strongly convergent* sequence). Then $\mathcal{F}^E$ is a closed unital subalgebra of $\mathcal{F}$, W^E is a unital *-homomorphism from $\mathcal{F}^E$ to $L(H^E)$, and the set

$$\mathcal{J}^E := \{(E_n^* K E_n + G_n) : K \in L(H^E) \text{ compact, } (G_n) \in \mathcal{G}\}$$

is a closed ideal of $\mathcal{F}^E$.

Lifting. Let now $\{E^t\}_{t\in T}$ be a family of sequences $E^t = (E_n^t)$ of isometries as above and with the additional property that

$$E_n^s (E_n^t)^* \to 0 \text{ weakly as } n \to \infty \text{ whenever } s \neq t. \tag{1}$$

For $t \in T$, we put $W_t := W^{E^t}$ and $\mathcal{J}_t := \mathcal{J}^{E^t}$, We further set $\mathcal{F}_T := \cap_{t\in T} \mathcal{F}^{E^t}$ and write $\mathcal{J}_T$ for the smallest closed ideal of $\mathcal{F}_T$ which contains all ideals $\mathcal{J}_t$ with $t \in T$. The condition (1) implies that $\mathcal{J}_s \cap \mathcal{J}_t = \mathcal{G}$ whenever $s \neq t$ (use Theorem 5.51 in [2]). We therefore refer to (1) as the *ideal separation condition.*

Theorem 1 (Lifting theorem). (*Theorems* 5.37 *and* 5.51 *in* [2]). *Let the family $\{E^t\}_{t\in T}$ satisfy the ideal separation condition* (1)*. Then a sequence $\mathbf{A} \in \mathcal{F}_T$ is stable if and only if the operators $W_t(\mathbf{A})$ are invertible for every $t \in T$ and if the coset $\mathbf{A} + \mathcal{J}_T$ is invertible in $\mathcal{F}_T/\mathcal{J}_T$.*

Localization. Let $\mathcal{A}$ be a closed unital C^*-subalgebra of $\mathcal{F}$. In many circumstances, one is able to find a family $\{E^t\}_{t\in T}$ of sequences of isometries which satisfy the ideal separation condition (1) and for which

- $\mathcal{A} \subseteq \mathcal{F}_T$ and
- the quotient algebra $(\mathcal{A} + \mathcal{J}_T)/\mathcal{J}_T$ has a non-trivial center $\mathcal{C}$.

In this setting, it is an evident idea to use central localization in order to study the invertibility of a coset $\mathbf{A} + \mathcal{J}_T$ in $(\mathcal{A} + \mathcal{J}_T)/\mathcal{J}_T$ (equivalently, in $\mathcal{F}_T/\mathcal{J}_T$). The context of the general central localization theorem by Allan and Douglas is as follows. We are given a unital C^*-algebra $\mathcal{B}$ and a C^*-subalgebra $\mathcal{C}$ of the center of $\mathcal{B}$ which contains the unit element. For every maximal ideal s of the commutative C^*-algebra $\mathcal{C}$, let I_s denote the smallest closed ideal of $\mathcal{B}$ which contains s, and write Φ_s for the canonical homomorphism from $\mathcal{B}$ to $\mathcal{B}/I_s$. In this context the following holds.

Theorem 2 (Allan–Douglas). *An element $b \in \mathcal{B}$ is invertible in $\mathcal{B}$ if and only if $\Phi_s(b)$ is invertible in $\mathcal{B}/I_s$ for every maximal ideal s of $\mathcal{C}$.*

The question. Applying the local principle in order to study invertibility in $(\mathcal{A} + \mathcal{J}_T)/\mathcal{J}_T$ requires to study the local algebras $((\mathcal{A} + \mathcal{J}_T)/\mathcal{J}_T)/I_s$, where $s \in S$, the maximal ideal space of $\mathcal{C}$. We suppose that this can be done again by a family of strong limit homomorphisms $\{F^s\}_{s\in S}$, i.e., for every $s \in S$, there is a sequence $F^s = (F_n^s)$ of isometries such that

- $\mathcal{A} + \mathcal{J}_T \subseteq \mathcal{F}^{F^s}$, $\mathcal{J}_T \subseteq \ker W^{F^s}$, and
- the quotient mapping $W^{F^s}/\mathcal{J}_T : \mathbf{A} + \mathcal{J}_T \mapsto W^{F^s}(\mathbf{A})$ has the ideal I_s in its kernel, and the quotient mapping $(W^{F^s}/\mathcal{J}_T)/I_s$ is injective on $((\mathcal{A} + \mathcal{J}_T)/\mathcal{J}_T)/I_s$ (thus, a coset $(\mathbf{A} + \mathcal{J}_T) + I_s$ in $((\mathcal{A} + \mathcal{J}_T)/\mathcal{J}_T)/I_s$ is invertible if and only if the operator $W^{F^s}(\mathbf{A})$ in invertible on H^{F^s}).

We thus have two families of strong limit homomorphisms: one which provides us with the lifting mechanism, and one which identifies local algebras. Markus Seidel's question was if one can include both families into a large family which still satisfies the conditions of the lifting theorem, and how this would affect the structure of the local algebras.

The answer. We formulate the first part of Seidel's question in a slightly more general way. Suppose we are given two families of strong limit homomorphisms: one defined by a family $\{E^t\}_{t\in T}$ of sequences of isometries which satisfies the ideal separation condition (1), and one by a family $\{F^s\}_{s\in S}$ of sequences of isometries with $\mathcal{J}_T \subseteq \ker W^{F^s}$ for all s which satisfies the following *point separation condition*

- for every pair of distinct points s_1, $s_2 \in S$, there is a sequence $\mathbf{A} \in \mathcal{A}$ such that $W^{F^{s1}}(\mathbf{A}) = I$ and $W^{F^{s2}}(\mathbf{A}) = 0$.

It is clear that a family which allows identification of the local algebras satisfies this condition (the algebra of the Gelfand transforms of a commutative C^*-algebra separates the points of the maximal ideal space).

The only thing we have to check is if the family $\{E^t\}_{t\in T} \cup \{F^s\}_{s\in S}$ satisfies the ideal separation condition (1). We employ the following elementary observation for sequences (A_n), (B_n) of bounded linear operators:

1. *If $A_n \to A$ strongly and $B_n \to B$ weakly, then $B_nA_n \to BA$ weakly.*
2. *If $A_n^* \to A^*$ strongly and $B_n \to B$ weakly, then $A_nB_n \to AB$ weakly.*
3. *If $A_n \to 0$ strongly and (B_n) is bounded, then $B_nA_n \to 0$ strongly.*

Let $E = (E_n)$ and $F = (F_n)$ be sequences in $\{E^t\}_{t\in T} \cup \{F^s\}_{s\in S}$.

We distinguish between three cases.

Case 1: $\boldsymbol{E}, \boldsymbol{F} \in \{\boldsymbol{E^t}\}_{\boldsymbol{t\in T}}$. Then (1) holds by assumption.

Case 2: $\boldsymbol{E} \in \{\boldsymbol{E^t}\}_{\boldsymbol{t\in T}}$ and $\boldsymbol{F} \in \{\boldsymbol{F^s}\}_{\boldsymbol{s\in S}}$. Then the ideal $\mathcal{J}^E$ lies in $\ker W^F$ by assumption. Hence, $F_nE_n^*KE_nF_n^* \to 0$ strongly for every compact operator K. This implies the weak convergence of $E_nF_n^*$ to zero as in the proof of Theorem 5.51 in [2]. Taking adjoints we get the weak convergence of $F_nE_n^*$ to zero as well.

Case 3: $\boldsymbol{E}, \boldsymbol{F} \in \{\boldsymbol{F_s}\}_{\boldsymbol{s\in S}}$. By the point separation condition, there is a sequence $\mathbf{A} = (A_n)$ such that $E_nA_nE_n^* \to I$ and $F_nA_nF_n^* \to 0$ *-strongly. In particular, with $Y_n := F_nE_n^*$,

$$E_nA_nE_n^* = E_n(F_n^*F_n)A_n(F_n^*F_n)E_n^* = Y_n^*F_nA_nF_n^*Y_n \to I \tag{2}$$

*-strongly. Since $Y_nY_n^* = F_nE_n^*E_nF_n^* = F_nF_n^* \to I$ and $F_nA_nF_n^* \to 0$ *-strongly, we conclude that $Y_nY_n^*F_nA_nF_n^* \to 0$ *-strongly.

Suppose that $Y_n \to Y$ weakly for some operator Y. From Observation 2 and (2) we then conclude that $Y_nY_n^*F_nA_nF_n^*Y_n \to 0$ weakly. On the other hand, we can use Observation 3 to conclude from (2) that $Y_nY_n^*F_nA_nF_n^*Y_n - Y_n \to 0$ strongly. Hence, $Y_n \to 0$ weakly. The same argument shows that whenever a subsequence of (Y_n) converges weakly, then it converges weakly to zero.

It remains to show that the sequence (Y_n) indeed converges weakly to 0. Suppose it does not. Then there are vectors $x \in H^E$, $y \in H^F$ and an $\varepsilon > 0$ such that $|\langle Y_{n_k}x, y\rangle| \ge \varepsilon$ for all elements in an (infinite) subsequence (Y_{n_k}) of (Y_n). A standard diagonal argument yields that this subsequence has a weakly convergent subsequence[1], which then converges weakly to zero as shown before. Contradiction.

In particular, we have seen that the point separation property implies the ideal separation property. Thus, whenever the combination of lifting theorem and local principle makes sense, the first part of Seidel's question has an affirmative answer (the general answer to the second part is already in [1]).

References

[1] D. Dragičević, P.A. Santos, and M. Szamotulski, *On a Question by Markus Seidel.* This volume, pp. 159–172.

[2] R. Hagen, S. Roch, and B. Silbermann, *C^*-Algebras and Numerical Analysis.* Marcel Dekker, Inc., New York, Basel 2001.

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[1]Choose dense countable subsets $H_E \subseteq H^E$ and $H_F \subseteq H^F$ and let $n \mapsto (x(n), y(n))$ be a bijection from $\mathbb{N}$ onto $H_E \times H_F$. Set $Y_k^{(0)} := Y_{n_k}$. For every $n \ge 1$, choose a subsequence $(Y_k^{(n)})_{k\ge1}$ of $(Y_k^{(n-1)})_{k\ge1}$ such that $\langle Y_k^{(n)}x(n), y(n)\rangle$ converges as $k \to \infty$. Set $Z_k := Y_k^{(k)}$. Then (Z_k) is the desired subsequence.

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A Tour to Compact Type Operators and Sequences Related to the Finite Sections Projection

Steffen Roch and Pedro A. Santos

Dedicated to Prof. António Ferreira dos Santos

Abstract. We prove that all compact operators acting on $L^p(\mathbb{R})$ belong to the algebra generated by the operator of multiplication by the characteristic function of the positive half-axis and by the convolution operators with continuous generating function. This result, together with the similar classical result on the algebra generated by the operators of multiplication and the singular integral operator, is then used to prove that certain ideals of compact-like operator sequences in infinite products of Banach algebras are included in the algebra generated by convolution and multiplication operators and the finite section projection sequence.

Mathematics Subject Classification (2010). Primary 65R20; Secondary 45E10, 47B35, 47L80, 65J10.

Keywords. Finite sections method, compact operator, sequence algebra, sequence of compact type.

1. Introduction

In [10], we studied the finite sections method for operators which are composed by operators of multiplication by a piecewise continuous function, operators of (Fourier) convolution by a piecewise continuous Fourier multiplier, and by a certain flip operator. This class of operators is extremely large; some prominent members

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of this class are Toeplitz plus Hankel operators on Hardy spaces H^p, and Wiener–Hopf plus Hankel operators on Lebesgue spaces L^p. The techniques we used to tackle the stability problem for these operators were of algebraic nature; for example the stability of a sequence is equivalent to the invertibility of an associated element in a suitably constructed Banach algebra (some details will be given below).

At some point in [10] we needed that certain sequences with very special properties (the sequences of compact type, mentioned in the title of the paper) belong to our algebra and form a closed ideal there. Roughly speaking, the only reason why we needed these sequences was to be able to factor them out. That's why we decided not to spend much time with them; we just made our algebra a little bit larger by including all desired and needed sequences by hand. Although this practice was successful, we were not satisfied with it. The question remained if the enlargement of the algebra was really necessary, or if the needed sequences were already contained in the smaller original algebra.

Questions of this type occur frequently in operator theory and numerical analysis. For a concrete example, suppose we are interested in the Fredholm theory of singular integral operators $aI + bS$. Here I is the identity operator, a and b are operators of multiplication by (say, continuous) functions, and S is the singular integral operator

$$(S_\Gamma u)(x) := \frac{1}{\pi \mathbf{i}} \int_\Gamma \frac{u(y)}{y - x}\, dy, \quad x \in \Gamma, \tag{1.1}$$

with the integral understood in the sense of the Cauchy principal value. It is well known that this operator is bounded on $L^p(\Gamma)$ if $1 < p < \infty$ and if Γ is the unit circle $\mathbb{T}$ in the complex plane $\mathbb{C}$ or the real line $\mathbb{R}$, for instance.

Since the Fredholm property of a bounded operator A on $L^p(\Gamma)$ is equivalent to its invertibility modulo the ideal of the compact operators on $L^p(\Gamma)$, and since invertibility problems are typically studied in algebras which should not be too large, this leads naturally to the question: *Is the ideal of the compact operators contained in the smallest closed algebra which contains all operators* $aI + bS$ *we are interested in?* In this setting, the answer is well known and turns out to be YES, and the following is a (well-known) prototype of the results we will meet in this paper.

Theorem 1.1. *The ideal of the compact operators on* $L^p(\Gamma)$ *is contained in the smallest closed algebra which contains all singular integral operators* $aI + bS$ *with* a, b *continuous on* Γ *if* $\Gamma = \mathbb{T}$ *and continuous on the one point compactification of* Γ *if* $\Gamma = \mathbb{R}$.

So we decided to tackle the above-mentioned problem again, and after some efforts we were indeed able to show that the original algebra was already large enough to include all needed sequences. On the way to this result we will encounter a lot of results in the same spirit, both in the context of operator theory and of numerical analysis.

Throughout this paper, we let $1 < p < \infty$. Moreover, for a Banach space X, we denote the Banach algebra of all bounded linear operators on X by $\mathcal{B}(X)$

and the set of the compact operators on X by $\mathcal{K}(X)$. If $\mathcal{A}$ is a non-empty subset of $\mathcal{B}(X)$ then $\mathrm{alg}\mathcal{A}$ and $\mathrm{clos\,alg}\mathcal{A}$ stand for the smallest subalgebra and for the smallest closed subalgebra of $\mathcal{B}(X)$ which contain all operators in $\mathcal{A}$, respectively.

We are grateful to Peter Junghanns for stimulating discussions and for bringing the reference [6] to our attention.

2. On the unit circle $\mathbb{T}$

We start our tour on the unit circle $\mathbb{T}$ in the complex plane $\mathbb{C}$. Let $\mathcal{P}_\mathbb{T}$ stand for the algebra of all trigonometric polynomials on $\mathbb{T}$. We write the elements of $\mathcal{P}_\mathbb{T}$ as

$$\sum_{r=-\infty}^{\infty} f_r t^r, \quad f_r \in \mathbb{C},$$

where only a finite number of the f_r do not vanish. Throughout what follows we suppose that $\alpha \in \mathbb{R}$ is such that

$$0 < \frac{1}{p} + \alpha < 1. \tag{2.1}$$

Let $L^p(\mathbb{T}, \alpha)$ denote the space of all Lebesgue-integrable functions f on $\mathbb{T}$ with

$$\|f\|_{L^p(\mathbb{T},\alpha)} := \left(\int_\mathbb{T} |f(t)|^p |1-t|^{\alpha p} \, |dt| \right)^{1/p} < \infty.$$

Lemma 2.1. *The following statements hold:*

(i) *$\mathcal{P}_\mathbb{T}$ is dense in $L^p(\mathbb{T}, \alpha)$.*

(ii) *The operator*

$$P_\mathbb{T} : \mathcal{P}_\mathbb{T} \to \mathcal{P}_\mathbb{T}, \quad \sum_{r=-\infty}^{\infty} f_r t^r \mapsto \sum_{r=0}^{\infty} f_r t^r$$

extends to a bounded linear operator on $L^p(\mathbb{T}, \alpha)$.

(iii) *Let $m \in \mathbb{Z}$. The operator*

$$M_m^\mathbb{T} : \mathcal{P}_\mathbb{T} \to \mathcal{P}_\mathbb{T}, \quad \sum_{r=-\infty}^{\infty} f_r t^r \mapsto \sum_{r=-\infty}^{\infty} f_r t^{r+m}$$

extends to a bounded linear operator on $L^p(\mathbb{T}, \alpha)$, the operator of multiplication by t^m.

Assertions (i) and (ii) are taken from [2, 1.44 and 5.9], whereas (iii) is evident since $|t^m| = 1$. We denote the extensions of the operators in (ii) and (iii) by $P_\mathbb{T}$ and $M_m^\mathbb{T}$ again and remark that $\|M_m^\mathbb{T}\|_{L(L^p(\mathbb{T},\alpha))} = 1$ for $m \in \mathbb{Z}$.

For $u, v \in \mathcal{P}_\mathbb{T}$, consider the operator

$$K_{u,v} : \mathcal{P}_\mathbb{T} \to \mathcal{P}_\mathbb{T}, \quad f \mapsto \langle f, u \rangle \, v$$

where $\langle f, u \rangle := \int_\mathbb{T} f(t)\overline{u(t)} \, |dt|$.

Lemma 2.2. $K_{u,v} \in \operatorname{alg}\left\{P_{\mathbb{T}}, M^{\mathbb{T}}_{\pm 1}\right\}$ *for* $u, v \in \mathcal{P}_{\mathbb{T}}$.

Proof. It is sufficient to prove the assertion for $u(t) = t^k$ and $v(t) = t^l$, with $k, l \in \mathbb{Z}$. For these u,v and for $f \in \mathcal{P}_{\mathbb{T}}$, we have

$$K_{u,v} f = \langle f, u\rangle\, v = \left\langle f, M^{\mathbb{T}}_k \mathbf{1}\right\rangle M^{\mathbb{T}}_l \mathbf{1} = \left\langle M^{\mathbb{T}}_{-k} f, \mathbf{1}\right\rangle M^{\mathbb{T}}_l \mathbf{1}$$

which implies that

$$K_{u,v} = M^{\mathbb{T}}_l K_{\mathbf{1},\mathbf{1}} M^{\mathbb{T}}_{-k} = (M^{\mathbb{T}}_1)^l K_{\mathbf{1},\mathbf{1}} (M^{\mathbb{T}}_{-1})^k, \tag{2.2}$$

where $\mathbf{1}$ refers to the constant function $t \mapsto 1$ on $\mathbb{T}$. Further,

$$K_{\mathbf{1},\mathbf{1}} = P_{\mathbb{T}} - M^{\mathbb{T}}_1 P_{\mathbb{T}} M^{\mathbb{T}}_{-1}. \tag{2.3}$$

The identities (2.2) and (2.3) imply that $K_{u,v} \in \operatorname{alg}\left\{P_{\mathbb{T}}, M^{\mathbb{T}}_{\pm 1}\right\}$ for all $k, l \in \mathbb{Z}$, whence the assertion follows. □

Since $\mathcal{P}_{\mathbb{T}}$ is dense in $L^p(\mathbb{T}, \alpha)$ and in $(L^p(\mathbb{T}, \alpha))^* = L^q(\mathbb{T}, -\alpha)$, with $1/p + 1/q = 1$, by Lemma 2.1, the operators $K_{u,v}$, with $u, v \in \mathcal{P}_{\mathbb{T}}$, span a dense subset of $K(L^p(\mathbb{T}, \alpha))$. So we conclude the following result from Lemma 2.2 (see also [1, Lemma 8.23], where this result is proved in a more general setting).

Theorem 2.3. $\mathcal{K}(L^p(\mathbb{T}, \alpha)) \subseteq \operatorname{clos}\operatorname{alg}\left\{P_{\mathbb{T}}, M^{\mathbb{T}}_{\pm 1}\right\} = \operatorname{clos}\operatorname{alg}\left\{P_{\mathbb{T}}, C(\mathbb{T})I\right\}$.

3. From $\mathbb{T}$ to $\mathbb{R}$

Given $p \in (1, \infty)$, we now specify $\alpha := 1 - 2/p$. Note that then

$$1 < p < \infty \Leftrightarrow 0 < 1/p < 1 \Leftrightarrow 0 < 1 - 1/p < 1 \Leftrightarrow 0 < 1/p + \alpha < 1$$

for this special value of α. Hence, the pair (p, α) satisfies (2.1). The basic observation to pass from $\mathbb{T}$ to $\mathbb{R}$ is given by the following lemma, whose proof can be found in [5, Chapter 1, Theorem 5.1] and [7, page 56]. Similar operators (which leave the natural orientations of $\mathbb{R}$ and $\mathbb{T}$ invariant, in contrast to the $B^{\pm 1}$ below) are also used in [2, Section 9.1].

Lemma 3.1. *The operator*

$$B : L^p(\mathbb{R}) \to L^p(\mathbb{T}, \alpha), \quad (B\varphi)(t) := \frac{1}{t-1}\varphi\left(\mathbf{i}\frac{t+1}{t-1}\right) \quad (t \in \mathbb{T})$$

is bounded and invertible. Its inverse is given by

$$B^{-1} : L^p(\mathbb{T}, \alpha) \to L^p(\mathbb{R}), \quad (B\psi)(s) := \frac{2\mathbf{i}}{s-\mathbf{i}}\psi\left(\frac{s+\mathbf{i}}{s-\mathbf{i}}\right) \quad (s \in \mathbb{R}).$$

Assertion (i) of the following lemma is evident; assertion (ii) is proved in [7, page 56], [5, Chapter 1, Theorem 5.2] and [1, pages 370–371], with the difference that the authors of the first mentioned reference arrive at $B^{-1} S_{\mathbb{T}} B = +S_{\mathbb{R}}$ (with a plus sign). For that reason, we sketch the proof here.

Lemma 3.2.

(i) $B^{-1}M_m^{\mathbb{T}}B =: M_m^{\mathbb{R}}$ *is the operator of multiplication by the function* $s \mapsto (\frac{s+\mathbf{i}}{s-\mathbf{i}})^m$ *for every* $m \in \mathbb{Z}$*;*

(ii) $B^{-1}S_{\mathbb{T}}B = -S_{\mathbb{R}}$.

Proof. As already mentioned, we only prove the second assertion. First note that

$$(B^{-1}S_{\mathbb{T}}B\varphi)(s) = \frac{2\mathbf{i}}{\pi\mathbf{i}(s-\mathbf{i})}\int_{\mathbb{T}} \frac{\varphi(\mathbf{i}\frac{x+1}{x-1})}{(x-1)(x-\frac{s+\mathbf{i}}{s-\mathbf{i}})}\,dx. \tag{3.1}$$

We substitute $\mathbf{i}\frac{x+1}{x-1} = t$, respective $x = \frac{t+\mathbf{i}}{t-\mathbf{i}}$, and

$$\frac{dx}{dt} = \frac{(t-\mathbf{i})-(t+\mathbf{i})}{(t-\mathbf{i})^2} = \frac{-2\mathbf{i}}{(t-\mathbf{i})^2}.$$

Note that if t moves on $\mathbb{R}$ from 0 to $+\infty$, then x moves on $\mathbb{T}$ in the clockwise direction. Since the standard orientation on $\mathbb{T}$ is the counter-clockwise one, this gives a minus sign. Thus, (3.1) becomes

$$\begin{aligned}(B^{-1}S_{\mathbb{T}}B\varphi)(s) &= \frac{-2}{\pi(s-\mathbf{i})}\int_{\mathbb{R}} \frac{\varphi(t)}{(\frac{t+\mathbf{i}}{t-\mathbf{i}}-1)(\frac{t+\mathbf{i}}{t-\mathbf{i}}-\frac{s+\mathbf{i}}{s-\mathbf{i}})}\frac{-2\mathbf{i}}{(t-\mathbf{i})^2}\,dt\\
&= \frac{4\mathbf{i}}{\pi(s-\mathbf{i})}\int_{\mathbb{R}} \frac{\varphi(t)}{(t+\mathbf{i}-(t-\mathbf{i}))\big(t+\mathbf{i}-\frac{(s+\mathbf{i})(t-\mathbf{i})}{s-\mathbf{i}}\big)}\,dt\\
&= \frac{4\mathbf{i}}{\pi}\int_{\mathbb{R}} \frac{\varphi(t)}{(t+\mathbf{i}-(t-\mathbf{i}))((t+\mathbf{i})(s-\mathbf{i})-(s+\mathbf{i})(t-\mathbf{i}))}\,dt\\
&= \frac{4\mathbf{i}}{\pi}\int_{\mathbb{R}} \frac{\varphi(t)}{2\mathbf{i}(\mathbf{i}s-\mathbf{i}t-\mathbf{i}t+\mathbf{i}s)}\,dt\\
&= -\frac{1}{\mathbf{i}\pi}\int_{\mathbb{R}} \frac{\varphi(t)}{t-s}\,dt \;=\; -(S_{\mathbb{R}}\varphi)(s).\end{aligned}$$

□

Corollary 3.3. *With* $P_\Gamma := (I+S_\Gamma)/2$ *and* $Q_\Gamma := (I-S_\Gamma)/2$*, one obtains*

$$B^{-1}P_{\mathbb{T}}B = Q_{\mathbb{R}}, \quad B^{-1}Q_{\mathbb{T}}B = P_{\mathbb{R}}.$$

The following is just a translation of the corresponding results on $\mathbb{T}$ stated in Lemmas 2.1 and 2.2 and in Theorem 2.3.

Lemma 3.4.

(i) *The set* $\mathcal{P}_{\mathbb{R}} := B^{-1}\mathcal{P}_{\mathbb{T}}$ *is dense in* $L^p(\mathbb{R})$*.*

(ii) *For* $u, v \in \mathcal{P}_{\mathbb{R}}$ *the operator* $K_{u,v} : \mathcal{P}_{\mathbb{R}} \to \mathcal{P}_{\mathbb{R}}, f \mapsto \langle f, u\rangle_{\mathbb{R}}\, v$ *belongs to* $\operatorname{alg}\{Q_{\mathbb{R}}, M_{\pm 1}^{\mathbb{R}}\}$.

(iii) $K(L^p(\mathbb{R})) \subseteq \operatorname{clos}\operatorname{alg}\{Q_{\mathbb{R}}, M_{\pm 1}^{\mathbb{R}}\}$.

Let $J : L^p(\mathbb{R}) \to L^p(\mathbb{R})$ denote the flip operator $(Jf)(t) := f(-t)$. It is well known that

$$JP_{\mathbb{R}}J = Q_{\mathbb{R}}, \quad JQ_{\mathbb{R}}J = P_{\mathbb{R}} \tag{3.2}$$

and easy to check that

$$(JM_m^{\mathbb{R}}Jf)(s) = \left(\frac{-s+\mathbf{i}}{-s-\mathbf{i}}\right)^m f(s) = \left(\frac{s+\mathbf{i}}{s-\mathbf{i}}\right)^{-m} f(s),$$

whence

$$JM_m^{\mathbb{R}}J = M_{-m}^{\mathbb{R}} \quad \text{for } m \in \mathbb{Z}. \tag{3.3}$$

Summarizing Lemma 3.4 (iii) and (3.2)–(3.3) we arrive at the next stop of our tour.

Theorem 3.5. $\mathcal{K}(L^p(\mathbb{R})) \subseteq \operatorname{clos\,alg}\left\{P_{\mathbb{R}}, M_{\pm 1}^{\mathbb{R}}\right\} = \operatorname{clos\,alg}\left\{P_{\mathbb{R}}, C(\dot{\mathbb{R}})\right\}.$

Here, $\dot{\mathbb{R}}$ stands for the compactification of the real line by one point ∞.

4. From $\mathbb{R}$ to $\mathbb{R}$ by Fourier transform

The next step will lead us to a statement which can be viewed as the Fourier-symmetric version of Theorem 3.5. We define the Fourier transform for functions in the Schwartz space by

$$(Fu)(y) = \int_{-\infty}^{\infty} e^{-2\pi \mathbf{i} yx} u(x)\, dx, \quad y \in \mathbb{R}. \tag{4.1}$$

Then its inverse is given by

$$(F^{-1}v)(x) = \int_{-\infty}^{\infty} e^{2\pi \mathbf{i} xy} v(y)\, dy, \qquad x \in \mathbb{R}. \tag{4.2}$$

It is well known that F and F^{-1} extend continuously to bounded and unitary operators on the Hilbert space $L^2(\mathbb{R})$, which we denote by F and F^{-1} again. Thus, if A is a bounded operator on $L^2(\mathbb{R})$, then the composition $F^{-1}AF$ is well defined, and it is bounded on $L^2(\mathbb{R})$ again.

We call an operator $A \in \mathcal{B}(L^2(\mathbb{R}))$ a *p-Fourier multiplier* if $F^{-1}AFu \in L^p(\mathbb{R})$ whenever $u \in L^2(\mathbb{R}) \cap L^p(\mathbb{R})$ and there is a constant c_p such that $\|F^{-1}AFu\|_p \leq c_p \|u\|_p$ for all $u \in L^2(\mathbb{R}) \cap L^p(\mathbb{R})$. If A owns this property, then the composition $F^{-1}AF$ extends continuously to a bounded operator on $L^p(\mathbb{R})$. We denote this extension by A^F and call it the *Fourier image* of A. For some general facts on these operators, see [10].

It is well known that $P_{\mathbb{R}}$ and $M_{\pm 1}^{\mathbb{R}}$ are p-Fourier multipliers for every $p \in (1, \infty)$ (note that the functions $s \mapsto (\frac{s+i}{s-i})^m$ have bounded total variation on $\mathbb{R}$; so they are Fourier multipliers by Stechkin'n inequality, see [2, 9.3 (e)]) and that $P_{\mathbb{R}}^F$ is the operator of multiplication by the characteristic function of $[0, \infty)$. It makes thus sense to consider

$$\operatorname{alg}\left\{P_{\mathbb{R}}^F,\, (M_{\pm 1}^{\mathbb{R}})^F\right\} = \operatorname{alg}\left\{\chi_+ I,\, (M_{\pm 1}^{\mathbb{R}})^F\right\}.$$

By Lemma 3.4(ii), this algebra contains all operators $K_{u,v}^F$ with $u,\, v \in \mathcal{P}_{\mathbb{R}}$ (here we only use the algebraic properties of the mapping $A \mapsto A^F$). Since

$$K_{u,v}^F \varphi = \langle F\varphi,\, u\rangle\, F^{-1}v = \left\langle \varphi,\, F^{-1}u\right\rangle F^{-1}v = K_{F^{-1}u,\, F^{-1}v}\varphi,$$

it follows that $K^F_{u,v} = K_{F^{-1}u,F^{-1}v}$, and we conclude that

$$K_{F^{-1}u,F^{-1}v} \in \operatorname{alg}\left\{\chi_+ I,\, (M^{\mathbb{R}}_{\pm 1})^F\right\} \quad \text{for } u,v \in \mathcal{P}_{\mathbb{R}}. \tag{4.3}$$

It would follow from this line that $\operatorname{clos}\operatorname{alg}\left\{\chi_+ I, (M^{\mathbb{R}}_{\pm 1})^F\right\}$ contains *all* compact operators if we would know that the linear span of $\left\{K_{F^{-1}u,F^{-1}v} : u,v \in \mathcal{P}_{\mathbb{R}}\right\}$ is dense in $\mathcal{K}(L^p(\mathbb{R}))$. This, on its hand, would be clear if we would know that $F^{-1}\mathcal{P}_{\mathbb{R}}$ is dense in $L^p(\mathbb{R})$, for every $p \in (1,\infty)$. We are going to show this now.

Recall that $\mathcal{P}_{\mathbb{R}} = B^{-1}\mathcal{P}_{\mathbb{T}}$ is generated by the functions

$$s \mapsto \frac{1}{s-\mathbf{i}}\left(\frac{s+\mathbf{i}}{s-\mathbf{i}}\right)^m, \quad m \in \mathbb{Z}.$$

The inverse Fourier transforms of these functions can be calculated using residue calculus (see the theorem in [9, Section 14.2.1]). What results is the known fact that $F^{-1}\mathcal{P}_{\mathbb{R}}$ consists of all functions of the form

$$r(t) = \begin{cases} e^{2\pi t} p_1(t) & \text{if } \; t < 0 \\ e^{-2\pi t} p_2(t) & \text{if } \; t \geq 0 \end{cases} \tag{4.4}$$

where p_1 and p_2 are (algebraic) polynomials. The functions in (4.4) are dense in $L^1(\mathbb{R})$ (see [4, Section I.8]). We need the same property for $L^p(\mathbb{R})$ with $p > 1$. It is clearly sufficient to prove this for the semi-axes considered separately.

Lemma 4.1. *$\{e^{-at} f(t) : f$ a polynomial$\}$ is dense in $L^p(\mathbb{R}^+)$ for $p > 1$ and $a > 0$.*

Proof. The result is essentially stated in [6]. The argument runs as follows. Rescaling we can assume that $a = 1$. Because C_0^∞ is dense in $L^p(\mathbb{R}^+)$, it suffices to show that every function in C_0^∞ can be approximated in the L^p norm by functions of the form $e^{-t} f(t)$ with f a polynomial. So let $u \in C_0^\infty$. Then $e^t u$ is still in C_0^∞. If now Π_n denotes the set of all polynomials of degree less than or equal to n then, by [6, 2.5.32],

$$\inf_{p \in \Pi_n} \left\|e^t u - f\right\|_{L^p(\mathbb{R}^+, e^{-t})} \leq C w(e^t u, 1/\sqrt{n}), \tag{4.5}$$

where w is a (certain) module of continuity introduced in [6]. Since

$$\left\|e^t u - f\right\|_{L^p(\mathbb{R}^+, e^{-t})} = \left\|(e^t u - f)e^{-t}\right\|_{L^p(\mathbb{R}^+)} = \left\|u - e^{-t} p\right\|_{L^p(\mathbb{R}^+)}$$

and $w(e^t u, 1/\sqrt{n}) \to 0$ as $n \to \infty$, the estimate (4.5) indeed implies the desired density result. □

Corollary 4.2. *The following holds for every $p \in (1,\infty)$:*

(i) *$F^{-1}\mathcal{P}_{\mathbb{R}}$ is dense in $L^p(\mathbb{R})$;*
(ii) $\operatorname{span}\left\{K_{F^{-1}u,F^{-1}v} : u,v \in \mathcal{P}_{\mathbb{R}}\right\}$ *is dense in $\mathcal{K}(L^p(\mathbb{R}))$.*

We already mentioned that every operator of multiplication by a continuous function a with bounded total variation on $\mathbb{R}$ is a Fourier multiplier. We denote the closure in the norm of $\mathcal{B}(L^p(\mathbb{R}))$ of the set of all operators $(aI)^F$ with a of this form by $W^0(C_p)$, in accordance with the notation in [10]. Thus, $W^0(C_p)$ is a closed subalgebra of $\mathcal{B}(L^p(\mathbb{R}))$. The following is then an immediate consequence of the preceding corollary.

Theorem 4.3. $\mathcal{K}(L^p(\mathbb{R})) \subseteq \operatorname{clos\,alg}\{\chi_+ I,\, (M^{\mathbb{R}}_{\pm 1})^F\} = \operatorname{clos\,alg}\{\chi_+ I,\, W^0(C_p)\}$.

This is the end point on the operator theory side of our tour. We would not like to stop without mentioning that there is a lot of results of the same spirit in the literature; see, e.g., [2, 9.9] and [8, Proposition 3.3.1].

5. On the side of numerical analysis

Now we turn to the side of numerical analysis. First we introduce an algebra the role of which is comparable with that of the algebra $\mathcal{B}(L^p(\mathbb{R}))$ in operator theory. Let $\mathcal{E}$ denote the set of all bounded functions $\mathbf{A} : (0,\infty) \to \mathcal{B}(L^p(\mathbb{R}))$, and write A_τ for the value of $\mathbf{A} \in \mathcal{E}$ at $\tau \in (0,\infty)$. Sometimes we will also use the notation $(A_\tau)_{\tau>0}$ in place of $\mathbf{A}$. Provided with pointwise defined operations and the norm

$$\|\mathbf{A}\|_{\mathcal{E}} := \sup_{\tau\in(0,\infty)} \|A_\tau\|_{\mathcal{B}(L^p(\Gamma))},$$

$\mathcal{E}$ becomes a Banach algebra, and the set $\mathcal{G}$ of all functions $\mathbf{G} \in \mathcal{E}$ for which $\lim_{\tau\to\infty} \|G_\tau\| = 0$ forms a closed two-sided ideal of $\mathcal{E}$. Every operator $A \in \mathcal{B}(L^p(\mathbb{R}))$ gives rise to a constant function $\tau \mapsto A$ in $\mathcal{E}$ which we denote by A again. The importance of the quotient algebra $\mathcal{E}/\mathcal{G}$ stems from the following elementary, but basic, observation: a function $\mathbf{A} = (A_\tau) \in \mathcal{E}$ is stable in the sense of numerical analysis if and only if the coset $\mathbf{A} + \mathcal{G}$ is invertible in $\mathcal{E}/\mathcal{G}$ (see, for instance, [12, Section 6.2]).

To state our results we need some more notation. For $s \in \mathbb{R}$, let $(V_s u)(x) := u(x-s)$ be the operator of shift by s on $L^p(\mathbb{R})$, and let U_s be the operator of multiplication by the function $x \mapsto e^{-2\pi \mathbf{i} xs}$. For $\tau > 0$ let P_τ denote the operator of multiplication by the characteristic function of the interval $[-\tau,\tau]$, set $Q_\tau := I - P_\tau$, and define R_τ, S_τ and $S_{-\tau}$ by

$$(R_\tau u)(x) = \begin{cases} u(\tau - x) & \text{if } \ 0 < x < \tau \\ u(-\tau - x) & \text{if } \ -\tau < x < 0 \\ 0 & \text{if } \ |x| > \tau \end{cases}, \tag{5.1}$$

$$(S_\tau u)(x) = \begin{cases} 0 & \text{if } \ |x| < \tau \\ u(x-\tau) & \text{if } \ x > \tau \\ u(x+\tau) & \text{if } \ x < -\tau \end{cases}, \tag{5.2}$$

$$(S_{-\tau} u)(x) = \begin{cases} u(x+\tau) & \text{if } \ x > 0 \\ u(x-\tau) & \text{if } \ x < 0 \end{cases}. \tag{5.3}$$

These operators are bounded and have norm 1 on every $L^p(\mathbb{R})$. If $\chi_\pm$ denotes the characteristic function of the positive (negative) semi-axis of $\mathbb{R}$, then

$$\chi_\pm P_\tau = \chi_\pm V_{\pm\tau}\chi_\mp V_{\mp\tau}\chi_\pm = \chi_\pm V_{\pm\tau}\chi_\mp V_{\mp\tau} = V_{\pm\tau}\chi_\mp V_{\mp\tau}\chi_\pm, \tag{5.4}$$

$$\chi_\pm R_\tau = J\chi_\mp V_{\mp\tau}\chi_\pm I = \chi_\pm V_{\pm\tau}\chi_\mp J, \tag{5.5}$$

$$\chi_\pm S_\tau = V_{\pm\tau}\chi_\pm I, \tag{5.6}$$

$$\chi_\pm S_{-\tau} = \chi_\pm V_{\mp\tau}. \tag{5.7}$$

Further we adopt our earlier notation and let now $\operatorname{clos\,alg} M$ stand for the smallest closed subalgebra of $\mathcal{E}$ which contains all sequences in the subset M of $\mathcal{E}$. (There will be no confusion because if M consists of constant sequences only, then $\operatorname{clos\,alg} M$ also consists of constant sequences and can, hence, be identified with a subalgebra of $\mathcal{B}(L^p(\mathbb{R}))$.)

The sequences in Theorems 5.1, 5.2 and 5.3 below are the "compact type sequences" addressed to in the title of the paper.

Theorem 5.1. *Let K_1, K_2, $K_3 \in \mathcal{K}(L^p(\mathbb{R}))$. Then the sequence $(K_1 + V_{-\tau}K_2V_\tau + V_\tau K_3 V_{-\tau})_{\tau>0}$ belongs to the algebra* $\operatorname{clos\,alg}\left\{\chi_+ I, W^0(C_p), (P_\tau)_{\tau>0}\right\}$.

Proof. Let $K \in \mathcal{K}(L^p(\mathbb{R}))$. Then $K \in \operatorname{clos\,alg}\left\{\chi_+ I, W^0(C_p)\right\}$ by Theorem 4.3. Hence, and because the operators in $W^0(C_p)$ are shift invariant,

$$\begin{aligned}(V_{-\tau}KV_\tau)_{\tau>0} &\in \operatorname{clos\,alg}\left\{(V_{-\tau}\chi_+V_\tau)_{\tau>0},\, W^0(C_p)\right\}\\ &= \operatorname{clos\,alg}\left\{(\chi_{[-\tau,\infty)})_{\tau>0},\, W^0(C_p)\right\}\\ &= \operatorname{clos\,alg}\left\{(P_\tau + \chi_+Q_\tau)_{\tau>0},\, W^0(C_p)\right\}.\end{aligned}$$

Similarly,

$$\begin{aligned}(V_\tau KV_{-\tau})_{\tau>0} &\in \operatorname{clos\,alg}\left\{(V_\tau\chi_+V_{-\tau})_{\tau>0},\, W^0(C_p)\right\}\\ &= \operatorname{clos\,alg}\left\{(\chi_{[\tau,\infty)})_{\tau>0},\, W^0(C_p)\right\}\\ &= \operatorname{clos\,alg}\left\{(\chi_+Q_\tau)_{\tau>0},\, W^0(C_p)\right\},\end{aligned}$$

which implies the assertion. □

Theorem 5.2. *Let K_1, K_2, $K_3, K_4 \in \mathcal{K}(L^p(\mathbb{R}))$. Then the sequence*

$$(R_\tau K_1 R_\tau + R_\tau K_2 S_{-\tau} + S_\tau K_3 R_\tau + S_\tau K_4 S_{-\tau})_{\tau>0}$$

belongs to the algebra $\operatorname{clos\,alg}\left\{J,\, \chi_+ I,\, W^0(C_p),\, (P_\tau)_{\tau>0}\right\}$.

Proof. First consider $(R_\tau K R_\tau)_{\tau>0}$ with K compact. Write this sequence as

$$(R_\tau\chi_+K\chi_+R_\tau) + (R_\tau\chi_+K\chi_-R_\tau) + (R_\tau\chi_-K\chi_+R_\tau) + (R_\tau\chi_-K\chi_-R_\tau).$$

By (5.5)–(5.7), the latter is equal to

$$\begin{aligned}&(\chi_+V_\tau\chi_-JKJ\chi_-V_{-\tau}\chi_+I) + (\chi_+V_\tau\chi_-JKJ\chi_+V_\tau\chi_-I)\\ &\qquad + (\chi_-V_{-\tau}\chi_+JKJ\chi_-V_{-\tau}\chi_+I) + (\chi_-V_{-\tau}\chi_+JKJ\chi_+V_\tau\chi_-I)\end{aligned} \tag{5.8}$$

The first and the last sequence in (5.8) are of the form

$$(\chi_+V_\tau K_1 V_{-\tau}\chi_+I) \quad \text{and} \quad (\chi_-V_{-\tau}K_2V_\tau\chi_-I), \tag{5.9}$$

with $K_1 := \chi_-JKJ\chi_-$ and $K_2 := \chi_+JKJ\chi_+$ compact. These sequences are in

$$\operatorname{clos\,alg}\left\{\chi_+I, W^0(C_p), (P_\tau)_{\tau>0}\right\}.$$

by Theorem 5.1. The second sequence in (5.8) can be written as

$$(J\chi_-V_{-\tau}\chi_+KJ\chi_+V_\tau\chi_-I) = (J\chi_-V_{-\tau}K_3V_\tau\chi_-I)$$

with $K_3 := \chi_+ K J \chi_+ I$ compact. Again by Theorem 5.1, this sequence is in

$$\operatorname{clos\,alg}\left\{J, \chi_+ I, W^0(C_p), (P_\tau)_{\tau>0}\right\}.$$

Similarly, the third sequence in (5.8) is in this algebra. Thus, the assertion is proved for the sequences $(R_\tau K R_\tau)$. The other sequences can be treated similarly. □

Theorem 5.3. *Let* $K_1, K_2, K_3, K_4 \in \mathcal{K}(L^p(\mathbb{R}))$. *Then the sequence*

$$(R_\tau^F K_1 R_\tau^F + R_\tau^F K_2 S_{-\tau}^F + S_\tau^F K_3 R_\tau^F + S_\tau^F K_4 S_{-\tau}^F)_{\tau>0}$$

belongs to the algebra $\operatorname{clos\,alg}\left\{J,\, P_{\mathbb{R}},\, C(\dot{\mathbb{R}}),\, (P_\tau^F)\right\}$.

Proof. Let K be a compact operator. Again starting from Theorem 3.5, we get $K \in \operatorname{clos\,alg}\left\{P_{\mathbb{R}},\, C(\dot{\mathbb{R}})\right\}$ and, since the operators in $C(\dot{\mathbb{R}})$ commute with the U_s,

$$(U_{-s} K U_s)_{s>0} \in \operatorname{clos\,alg}\left\{(U_{-s} P_{\mathbb{R}} U_s)_{s>0},\, C(\dot{\mathbb{R}})\right\}.$$

Now, from

$$\begin{aligned} U_{-s} P_{\mathbb{R}} U_s &= U_{-s} W^0(\chi_+) U_s \;=\; F^{-1} V_s \chi_+ V_{-s} F \\ &= W^0(\chi_{[s,\infty)}) \\ &= W^0(\chi_+) W^0(\chi_{(-\infty,-s]} + \chi_{[s,\infty)}) \\ &= P_{\mathbb{R}} Q_s^F = P_{\mathbb{R}}(1 - P_s^F), \end{aligned}$$

we conclude that $(U_{-s} K U_s)_{s>0} \in \operatorname{clos\,alg}\left\{P_{\mathbb{R}},\, C(\dot{\mathbb{R}}),\, (P_\tau^F)\right\}$. Similarly, the sequence $(U_s K U_{-s})_{s>0}$ belongs to this algebra. We now continue as in the proof of the previous theorem to get the assertion. □

6. Why we need these results

We will now briefly indicate where and why the results of Theorems 5.1, 5.2 and 5.3 are useful.

We say that a bounded function $\mathbf{A} : (0,\infty) \to \mathcal{B}(L^p(\mathbb{R}))$ converges **-strongly* if it converges strongly as $\tau \to \infty$ and if the adjoint function $\mathbf{A}^*$ (which takes the value A_τ^* at the point τ) converges strongly on the dual space as $\tau \to \infty$. The *-strong limit of $\mathbf{A}$ is denoted by s-lim* $\mathbf{A}$. Let now $\mathcal{E}$ be the set formed by all bounded functions $\mathbf{A}$. With pointwise-defined sum and product, and the supremum norm, $\mathcal{E}$ becomes a Banach algebra which contains the ideal $\mathcal{G}$ of all operator-valued functions converging (in the operator norm) to zero as $\tau \to \infty$. The quotient algebra $\mathcal{E}/\mathcal{G}$ is an example of a "suitable constructed Banach algebra" that was referred in the Introduction. In fact, a bounded sequence of operators is an element of $\mathcal{E}$ and its stability is equivalent to invertibility of the corresponding coset in $\mathcal{E}/\mathcal{G}$.

Unfortunately, it is not possible to characterize invertibility in the very large algebra $\mathcal{E}/\mathcal{G}$. So one tries to find smaller algebras where invertibility can be tackled.

One way is to consider additional families of strong limits, generated through (auto)morphisms over $\mathcal{E}$. These strong limits highlight properties of the operator function, and its existence defines a subalgebra with a rich ideal structure and whose elements are, in some sense, nicer. Of course, care must be taken so that the concrete operator functions (*i.e.*, sequences) we are interested belong to this smaller algebra.

Let $\{\mathsf{W}_{t,\bullet}\}_{t\in\mathsf{T}}$ be a family of algebra automorphisms with the following properties:

1. $0 \in \mathsf{T}$, and $\mathsf{W}_{0,\bullet}$ is the identity automorphism;
2. $\|\mathsf{W}_{t,\bullet}\| = 1$ for every $t \in \mathsf{T}$;
3. $\mathsf{W}_{t,\bullet}(\mathbf{A})^* = \mathsf{W}_{t,\bullet}(\mathbf{A}^*)$ for every $\mathbf{A} \in \mathcal{E}$ and $t \in \mathsf{T}$;
4. $\text{s-lim}^* \, \mathsf{W}_{t,\bullet}(\mathsf{W}_{s,\bullet}^{-1}(\mathbf{A})) = 0$ for every $\mathbf{A} \in \mathcal{E}$ and $t \neq s$.

Define now $\mathcal{F}$ as the set of all functions $\mathbf{A} \in \mathcal{E}$ with the property that, for every $t \in \mathsf{T}$, the function $\mathsf{W}_{t,\bullet}(\mathbf{A})$ converges *-strongly, and set

$$\mathsf{W}_t(\mathbf{A}) := \text{s-lim}^* \, \mathsf{W}_{t,\bullet}(\mathbf{A}).$$

The set $\mathcal{F}$ is a closed and inverse-closed subalgebra of $\mathcal{E}$ that includes the ideal $\mathcal{G}$, the mappings W_t act as bounded homomorphisms on $\mathcal{F}$, and the ideal $\mathcal{G}$ is in the kernel of each of these homomorphisms [11, Proposition 4.1]. Moreover, the sets

$$\mathcal{J}_t := \mathsf{W}_{t,\bullet}^{-1}(\mathcal{K}) + \mathcal{G}, \tag{6.1}$$

where $\mathcal{K}$ is the ideal of compact operators, are closed two-sided ideals of $\mathcal{F}$. The relation between the ideals $\mathcal{J}_t$ and the algebra $\mathcal{F}$ can be seen as the analogue of the relation between $\mathcal{K}$ and $\mathcal{B}(L^p(\Gamma))$.

Moreover, given a suitable choice of the strong limits family (equivalently, of the morphisms), defining the ideal $\mathcal{J}$ as the smallest ideal containing all ideals $\mathcal{J}_t$ and using a lifting theorem (see, for instance, [12, Section 6.3]) it is even possible to completely characterize invertibility for (interesting) elements of $\mathcal{F}$ with the invertibility of the images of associated strong limit homomorphisms. This happens when invertibility of cosets in $\mathcal{F}/\mathcal{J}$ already follows from the invertibility of those images. An example of the procedure, using the above notation is presented in [3]. When the chosen strong limits family does not completely characterize stability, further steps must be performed to characterize invertibility in $\mathcal{F}/\mathcal{J}$, using, for instance, the two projections theorem to identify local algebras that result after applying local principles to a suitable subalgebra of $\mathcal{F}/\mathcal{J}$ (as was done in [10] and [11]).

For a more concrete setting, given a family of operators in $\mathcal{B}(L^p(\mathbb{R}))$ and a sequence of projections P_τ with complementary projections $Q_\tau := I - P_\tau$ such that $\text{s-lim}^* \, P_\tau = I$, that generate an algebra $\mathcal{A} \in \mathcal{F}$, one tries to find a suitable family of compatible automorphisms $\{\mathsf{W}_{t,\bullet}\}_{t\in\mathsf{T}}$ so that it is possible to characterize invertibility in $\mathcal{F}/\mathcal{G}$, and thus the stability of the related operator sequences. If the family of operators belongs to the subalgebra of multiplication and convolution operators on $L^p(\mathbb{R})$ generated by piecewise continuous functions, and if $P_\tau =$

$\chi_{[-\tau,\tau]}I$, the operator of multiplication by the characteristic function of the interval $[-\tau,\tau]$, then the relevant automorphisms $\mathsf{W}_{t,\bullet}$ are

$$\mathsf{W}_{0,\bullet} : (A_\tau) \mapsto (A_\tau);$$
$$\mathsf{W}_{-1,\bullet} : (A_\tau) \mapsto (V_{-\tau} A_\tau V_\tau);$$
$$\mathsf{W}_{1,\bullet} : (A_\tau) \mapsto (V_\tau A_\tau V_{-\tau})$$

(see [11]). This simple picture changes if one also wants to consider Hankel operators. Then it is necessary to include the flip operator $(Ju)(x) := u(-x)$ into the family of operators. But because the (constant) sequence (J) is not included in the algebra $\mathcal{F}$ defined by the above family of automorphisms, a more complex construction is necessary.

Instead of considering only automorphisms in $\mathcal{E}$ we consider now also a homomorphism between the algebras $\mathcal{E}$ and $\mathcal{E}^{2\times 2}$ given by

$$\mathsf{W}_{1,\bullet} : (A_\tau) \mapsto \left(\begin{bmatrix} R_\tau \\ S_{-\tau} \end{bmatrix} A_\tau \begin{bmatrix} R_\tau & S_\tau \end{bmatrix} \right) \tag{6.2}$$

(see [10]). In this regard note that

$$\begin{bmatrix} R_\tau & S_\tau \end{bmatrix} \begin{bmatrix} R_\tau \\ S_{-\tau} \end{bmatrix} = R_\tau R_\tau + S_\tau S_{-\tau} = P_\tau + Q_\tau = I.$$

It is also possible to consider the projection $P_\tau^F := F^{-1} P_\tau F$ associated with the Fourier finite section method. In this case, the homomorphism is defined as [10]:

$$\mathsf{W}_{1,\bullet}^F : (A_\tau) \mapsto \left(\begin{bmatrix} R_\tau^F \\ S_{-\tau}^F \end{bmatrix} A_\tau \begin{bmatrix} R_\tau^F & S_\tau^F \end{bmatrix} \right). \tag{6.3}$$

Summarizing, the results in Section 5 show that the ideal $\mathcal{J}_1$ defined by (6.1) using the inverse of (6.2) applied to $\mathcal{K}^{2\times 2}$ belongs to the subalgebra of $\mathcal{F}$ generated by the constant sequences of the singular integral operator and the operators of multiplication by continuous functions, and the non-constant projection sequence (P_τ). The same holds in the Fourier-symmetric setting, that is, the ideal $\mathcal{J}_1^F$ related to (6.3) is generated by convolution operators with continuous symbol by the operator of multiplication by the characteristic function of the positive half-axis, and by the projection sequence (P_τ^F).

Note that we have not proved that the ideal $\mathcal{G}$ belongs to the algebra $\operatorname{clos\,alg}\left\{PC(\mathbb{R}), W^0(PC_p), J, (P_\tau), (P_\tau^F)\right\}$. Thus, Theorems 5.2 and 5.3 do *not* imply that the ideals $\mathcal{J}_1$ and $\mathcal{J}_1^F$ belong to that algebra. But the ideal $\mathcal{G}$ can be explicitly introduced and then be factored out, because one is usually interested in invertibility on $\mathcal{E}/\mathcal{G}$. In any case, we have

$$\mathcal{J}_0/\mathcal{G},\ \mathcal{J}_1/\mathcal{G},\ \mathcal{J}_1^F/\mathcal{G} \subseteq \operatorname{clos\,alg}\left\{PC, PC_p, \mathcal{J}, (P_\tau), (P_\tau^F), \mathcal{G}\right\}/\mathcal{G}.$$

References

[1] A. Böttcher and Yu.I. Karlovich, *Carleson Curves, Muckenhoupt Weights, and Toeplitz Operators*. Birkhäuser, Basel, 1997.

[2] A. Böttcher and B. Silbermann, *Analysis of Toeplitz Operators*. Springer-Verlag, Berlin, second edition, 2006.

[3] D. Dragičević, G. Preto, P.A. Santos, and M. Szamotulski. On a question by Markus Seidel. Volume 242 of *Oper. Theory Adv. Appl.* Birkhäuser, 2014 (this volume), 159–172.

[4] I. Gohberg and I.A. Feldman, *Convolution Equations and Projection Methods for their solution*. Amer. Math. Soc., Providence, R.I., 1974. First published in Russian, Nauka, Moscow, 1971.

[5] I. Gohberg and N. Krupnik, *One-dimensional Linear Singular Integral Equations* (*I*). Birkhäuser, Basel, 1992.

[6] G. Mastroianni and G.V. Milovanovic, *Interpolation Processes: Basic Theory and Application*. Springer-Verlag, Berlin, Heidelberg, 2008.

[7] S.G. Mikhlin and S. Prössdorf, *Singular Integral Operators*. Springer-Verlag, Berlin, 1986.

[8] V.S. Rabinovich, S. Roch, and B. Silbermann, *Limit Operators and their Applications in Operator Theory*. Birkhäuser, Basel, 2004.

[9] R. Remmert, *Theory of Complex Functions*. Volume 122 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1991.

[10] S. Roch and P.A. Santos, *Finite section method in an algebra of convolution, multiplication and flip operators on* L^p. *To appear*, 2013.

[11] S. Roch, P.A. Santos, and B. Silbermann, *A sequence algebra of finite sections, convolution and multiplication operators on* $L^p(\mathbb{R})$. *Numer. Funct. Anal. Optim.* **31**(1) (2010), 45–77.

[12] S. Roch, P.A. Santos, and B. Silbermann, *Non-commutative Gelfand Theories*. Springer-Verlag, London, 2011.

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A Chen-type Modification of Hadamard Fractional Integro-Differentiation

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Dedicated to Professor António Ferreira dos Santos

Abstract. The so-called Chen modification of the Liouville fractional integrals (LFI) allows to study LFI of functions which may have arbitrary behaviour at both $-\infty$ and $+\infty$. We develop a similar approach for dilation invariant Hadamard fractional integro-differentiation on $\mathbb{R}_+$. We introduce several types of truncation of the corresponding Marchaud form of fractional Chen–Hadamard fractional derivatives and show that these truncations applied to Chen–Hadamard fractional integral of a function f in $L^p_{\mathrm{loc}}(\mathbb{R}_+)$ or $L^p(\mathbb{R}_+)$ converge to this function in L^p-norm, locally or globally, respectively. In the local case, we admit functions f with an arbitrary growth both at the origin and infinity.

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1. Introduction

The fractional integro-differentiation introduced by J. Hadamard [5], is applied to functions defined on the half-axis $\mathbb{R}_+$ and is dilation invariant, i.e., it commutes with the operator

$$(\Pi_t f)(x) = f(tx), \quad t > 0.$$

Hadamard fractional integration of order $\alpha > 0$ is realized in the form

$$\Im^\alpha_+ \varphi = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{\varphi(t)}{\left(\ln \frac{x}{t}\right)^{1-\alpha}} \frac{dt}{t}, \qquad \Im^\alpha_- \varphi = \frac{1}{\Gamma(\alpha)} \int_x^\infty \frac{\varphi(t)}{\left(\ln \frac{t}{x}\right)^{1-\alpha}} \frac{dt}{t}. \tag{1.1}$$

For properties of the Hadamard fractional integro-differentiation we refer to Section 18 in the book [8]. After the book [8] appeared, further advances in the study of Hadamard constructions have been made in [1], [2], [3], [6], [7].

As can be seen from (1.1), the construction of the Hadamard fractional integration may be applied only to functions vanishing at the origin (in the case of the left-hand-sided integral) or at infinity (in the case of the right-hand-sided integral), similarly to the case where the Liouville fractional integrals are applied to functions vanishing at the corresponding infinity.

A modification of the Liouville fractional integrals which may be applied to functions with an arbitrary behaviour at both infinities, suggested by Y.W. Chen in [4], was developed in [9], [10], where the inversion of this modification with L^p-densities was obtained by means of the Chen–Marchaud type constructions. In this paper we develop a similar approach to the corresponding Chen–Hadamard type fractional integrals and Chen–Hadamard–Marchaud fractional derivatives.

In Section 2 we present necessary definitions and various auxiliary properties of the Hadamard–Chen fractional integro-differentiation, and in Section 3 we propose different methods of "truncation" of the Marchaud–Hadamard–Chen fractional derivative. In Section 4, which is central in a sense from the point of view of the technique of the proofs, we we prove integral representations for truncated fractional derivatives applied to fractional derivatives of functions in L^p, and in Section 5 we prove the main results: the correspondingly defined convergence of truncated Chen–Hadamard fractional derivatives generates inversion of Chen–Hadamard fractional integrals of functions in L^p.

In the sequel, by $L^p_{\mathrm{loc}}(\mathbb{R}_+)$ we denote the space of functions f on $\mathbb{R}_+$, whose restrictions on an interval (a,b) are p-integrable on (a,b) for all $0<a<b<\infty$, which will be equipped with the natural topology of convergence in L^p-norm on any such interval.

We also use the standard notation $x_+^a = \begin{cases} x^a, & x>0, \\ 0, & x<0. \end{cases}$

2. On Hadamard–Chen fractional integro-differentiation

We will also need the Hadamard fractional differentiation in the Marchaud form:

$$\mathbb{D}^\alpha_+ f(x) := \frac{\alpha}{\Gamma(1-\alpha)} \int\limits_0^x \frac{f(x)-f(t)}{\left(\ln \frac{x}{t}\right)^{1+\alpha}} \frac{dt}{t}, \tag{2.1}$$

and

$$\mathbb{D}^\alpha_- f(x) := \frac{\alpha}{\Gamma(1-\alpha)} \int\limits_x^\infty \frac{f(x)-f(t)}{\left(\ln \frac{t}{x}\right)^{1+\alpha}} \frac{dt}{t}, \tag{2.2}$$

see [8], p. 332.

2.1. Main constructions

Definition 2.1. Let $0 < c < \infty$ be any fixed point and $\alpha > 0$. We refer to

$$(\Im_c^\alpha \varphi)(x) := \begin{cases} \frac{1}{\Gamma(\alpha)} \int\limits_x^c \frac{\varphi(t)}{\left(\ln \frac{t}{x}\right)^{1-\alpha}} \frac{dt}{t}, & 0 < x < c, \\ \frac{1}{\Gamma(\alpha)} \int\limits_c^x \frac{\varphi(t)}{\left(\ln \frac{x}{t}\right)^{1-\alpha}} \frac{dt}{t}, & c < x < \infty, \end{cases} \tag{2.3}$$

as the Hadamard–Chen fractional integral of order α, or Chen-type modification of the Hadamard fractional integrals.

Note that the Chen-type modification of Liouville fractional integrals was studied in [9].

In the notation

$$(P_{c+}\varphi)(x) = \varphi_{c+}(x) = \begin{cases} \varphi(x), & x > c, \\ 0, & x < c, \end{cases}$$

$$(P_{c-}\varphi)(x) = \varphi_{c-}(x) = \begin{cases} 0, & x > c, \\ \varphi(x), & x < c \end{cases}$$

the fractional integration (2.3) has the form

$$(\Im_c^\alpha \varphi)(x) = (\mathfrak{I}_+^\alpha \varphi_{c+})(x) + (\Im_-^\alpha \varphi_{c-})(x),$$

or in the operator form

$$\Im_c^\alpha = \Im_+^\alpha P_{c+} + \Im_-^\alpha P_{c-} = P_{c+}\Im_+^\alpha P_{c+} + P_{c-}\Im_-^\alpha P_{c-},$$

so that the operators $\Im_c^\alpha$ possess the semi-group property

$$\Im_c^\alpha \Im_c^\beta \varphi = \Im_c^{\alpha+\beta}\varphi$$

for $\varphi \in L^p_{\text{loc}}(\mathbb{R}_+)$ and $\alpha > 0, \beta > 0$.

The Hadamard–Chen fractional differentiation is then correspondingly defined as

$$(\mathfrak{D}_c^\alpha f)(x) := (\Im_c^\alpha)^{-1} f(x) = \frac{1}{\Gamma(1-\alpha)} \begin{cases} x\frac{d}{dx} \int\limits_c^x \frac{f(t)}{\left(\ln \frac{x}{t}\right)^\alpha} \frac{dt}{t}, & c < x < \infty, \\ -x\frac{d}{dx} \int\limits_x^c \frac{f(t)}{\left(\ln \frac{t}{x}\right)^\alpha} \frac{dt}{t}, & 0 < x < c \end{cases} \tag{2.4}$$

when $0 < \alpha < 1$, and for $\alpha \geq 1$ we put

$$(D_c^\alpha f)(x) = (D_c^{[\alpha]} D_c^{\{\alpha\}} f)(x) = [\text{sign}(x-c)]^{[\alpha]} \left(x\frac{d}{dx}\right)^{[\alpha]} D_c^{\{\alpha\}} f,$$

where $\alpha = [\alpha] + \{\alpha\}$. Making use of the Marchaud form for Hadamard fractional differentiation (see [8], formula (18.58)), we arrive at the following Marchaud type form for the differentiation (2.4)

$$(\mathbb{D}_c^\alpha f)(x) = \frac{f(x)}{\Gamma(1-\alpha)\left|\ln \frac{x}{c}\right|^\alpha} + \frac{\alpha}{\Gamma(1-\alpha)} \int\limits_{\min(x,c)}^{\max(x,c)} \frac{f(x) - f(t)}{\left|\ln \frac{x}{t}\right|^{1+\alpha}} \frac{dt}{t}. \tag{2.5}$$

for $0<\alpha<1$, at the least for "nice" functions f, so that $(\mathbb{D}_c^\alpha f)(x)=(D_c^\alpha f)(x)$ for such functions. In terms of the left- and right-hand-sided Marchaud forms (2.1)–(2.2) of the Hadamard fractional differentiation in has the form

$$\mathbb{D}_c^\alpha f=\frac{\alpha}{\Gamma(1-\alpha)}\int\limits_0^1 \frac{f(x)-f_{c+}(xt)-f_{c-}(xt^{-1})}{|\ln t|^{1+\alpha}}\,\frac{dt}{t}=D_+^\alpha f_{c+}+D_-^\alpha f_{c-}. \tag{2.6}$$

As is known, when applied to "not so nice" functions, for instance, to fractional integrals of L^p-functions, the Marchaud construction should be interpreted as the limit of the corresponding truncations of the hypersingular integral in these constructions. The appearance of the point $c\in(0,\infty)$ in our constructions, which breaks the invariance with respect to dilations, leads to certain problems in these interpretation. We consider this question in the next section.

3. On different approaches to truncation of the fractional derivative (2.6)

Since the construction (2.6) contains the truncations $f_{c\pm}$ of the function f itself, related to the point c, we have to co-ordinate the truncation of the integral with those of the function. It may be done in different ways. We consider several ways, the first one made with a constant step in the truncation of the integral, and second one with this step depending on the point x (more precisely, on the distance $|\ln x-\ln c|$). The first way might be called "quasi-convolution dilation invariant" by the reason which will be clear below. One more modification of "quasi-difference" approach to truncation will be also considered.

3.1. Quasi-convolution truncation

Let $0<\rho<1$. For $0<\alpha<1$, we put

$$\begin{aligned}\mathbb{D}_{c,1-\rho}^\alpha f&=\frac{\alpha}{\Gamma(1-\alpha)}\int\limits_0^\rho \frac{f(x)-f_{c+}(xt)-f_{c-}(xt^{-1})}{|\ln t|^{1+\alpha}}\,\frac{dt}{t}\\&=\mathfrak{D}_{+,1-\rho}^\alpha f_{c+}+\mathfrak{D}_{-,1-\rho}^\alpha f_{c-},\end{aligned} \tag{3.1}$$

where $\mathfrak{D}_{\pm,1-\rho}^\alpha$ are the corresponding truncations of the Hadamard–Chen–Marchaud derivatives (2.1) and call the constructions (3.1) Hadamard–Chen–Marchaud truncated fractional derivative. For "not so nice" functions f we put by definition

$$\mathbb{D}_c^\alpha f=\lim_{\rho\to1-0}\mathbb{D}_{c,1-\rho}^\alpha f$$

The components $\mathfrak{D}_{\pm,1-\rho}^\alpha f_{c\pm}$ in (3.1), can be explicitly written:

$$\mathfrak{D}_{\pm,1-\rho}^\alpha f_{c\pm}=\frac{f(x)}{\Gamma(1-\alpha)\ \left|\ln\frac{x}{c}\right|^\alpha}+\frac{\alpha}{\Gamma(1-\alpha)}\psi_{\pm,1-\rho}(x), \tag{3.2}$$

where

$$\psi_{+,1-\rho}(x) = \begin{cases} \int\limits_c^{x\rho} \frac{f(x)-f(t)}{\left(\ln \frac{x}{t}\right)^{1+\alpha}} \frac{dt}{t}, & x > \frac{c}{\rho} \\ \frac{f(x)}{\alpha}\left[\frac{1}{\left(\ln \frac{1}{\rho}\right)^{\alpha}} - \frac{1}{\left(\ln \frac{x}{c}\right)^{\alpha}}\right], & c < x < \frac{c}{\rho}, \end{cases} \tag{3.3}$$

and

$$\psi_{-,1-\rho}(x) = \begin{cases} \int\limits_{\frac{x}{\rho}}^{c} \frac{f(x)-f(t)}{\left(\ln \frac{t}{x}\right)^{1+\alpha}} \frac{dt}{t}, & x < c\rho, \\ \frac{f(x)}{\alpha}\left[\frac{1}{\left(\ln \frac{1}{\rho}\right)^{\alpha}} - \frac{1}{\left(\ln \frac{c}{x}\right)^{\alpha}}\right], & c\rho < x < c. \end{cases} \tag{3.4}$$

The equalities (3.3)–(3.4) actually mean that $\mathfrak{D}^{\alpha}_{\pm,1-\rho} f_{c\pm}$ have been built as the corresponding truncations of Hadamard–Marchaud derivatives of the function $f(x)$ continued as identical zero from one of the portions $(0,c), (c,\infty)$ to another one.

We can also rewrite this truncation in the form

$$\mathbb{D}^{\alpha}_{c,1-\rho} f = \frac{f(x)}{\Gamma(1-\alpha)\left|\ln \frac{x}{c}\right|^{\alpha}} + \begin{cases} \frac{\alpha}{\Gamma(1-\alpha)} \int\limits_{\ln \frac{1}{\rho}}^{\left|\ln \frac{x}{c}\right|} \frac{f(x)-f\left(xe^{-t\ \mathrm{sign}(x-c)}\right)}{t^{1+\alpha}} dt, & x \notin \left(c\rho, \frac{c}{\rho}\right), \\ 0, & x \in \left(c\rho, \frac{c}{\rho}\right). \end{cases} \tag{3.5}$$

Note that so introduced truncation "quasi commutes" with the dilation operator Π_δ in the following sense:

$$\mathbb{D}^{\alpha}_{c,1-\rho} \Pi_\delta f = \Pi_\delta \mathbb{D}^{\alpha}_{c\delta,1-\rho} f. \tag{3.6}$$

3.2. Truncation in dependence on $\left|\ln \frac{x}{c}\right|$

We will see that for the Chen type constructions it is natural and convenient to introduce also a truncation depending on the distance $|\ln x - \ln c|$. Namely, we denote

$$\tilde{\rho} = \tilde{\varrho}(x) := \left|\ln \frac{x}{c}\right| \ln \frac{1}{\rho}, \qquad \text{where} \quad \frac{1}{e} < \rho < 1,$$

and introduce the following truncation:

$$D^{\alpha}_{c,1-\tilde{\rho}} f := \begin{cases} D^{\alpha}_{c+,1-\tilde{\rho}} f, & x > c \\ D^{\alpha}_{c-,1-\tilde{\rho}} f, & x < c \end{cases} \tag{3.7}$$

where

$$D^{\alpha}_{c\pm,1-\tilde{\rho}} f := \frac{f(x)}{\Gamma(1-\alpha)\left|\ln \frac{x}{c}\right|^{\alpha}} + \psi_{\pm,\tilde{\rho}}(x)$$

and

$$\psi_{+,\tilde{\rho}}(x) = \int\limits_c^{x\rho^{\ln \frac{x}{c}}} \frac{f(x)-f(t)}{\left(\ln \frac{x}{t}\right)^{1+\alpha}} \frac{dt}{t} = \int\limits_{\frac{c}{x}}^{\rho^{\ln \frac{x}{c}}} \frac{f(x)-f(xy)}{\left(\ln \frac{1}{y}\right)^{1+\alpha}} \frac{dy}{y}, \quad x > c, \tag{3.8}$$

$$\psi_{-,\tilde{\rho}}(x) = \int\limits_{\frac{x}{\rho^{\ln\frac{c}{x}}}}^{c} \frac{f(x)-f(t)}{\left(\ln\frac{t}{x}\right)^{1+\alpha}} \frac{dt}{t} = \int\limits_{\frac{x}{c}}^{\rho^{\ln\frac{c}{x}}} \frac{f(x)-f(xy^{-1})}{\left(\ln\frac{1}{y}\right)^{1+\alpha}} \frac{dy}{y}, \quad x<c. \tag{3.9}$$

Under this way (3.7)–(3.8) of truncation one does not need a separate definition of the functions $\psi_{\pm,\tilde{\rho}}(x)$ on an interval $c\rho < x < \frac{c}{\rho}$, small as $\rho \to 1$, i.e., we do not use the zero continuation of the function $f(x)$ but deal with proper values of the function $f(x)$ on the semi-axis.

This truncation does not have the quasi-commutation property (3.6). Note also that

$$\left\{ \begin{array}{ll} \psi_{+,\tilde{\rho}}, & x>c \\ \psi_{-,\tilde{\rho}}, & x<c \end{array} \right. = \frac{\alpha}{\Gamma(1-\alpha)} \int\limits_{\left|\ln\frac{x}{c}\right| \ln\frac{1}{\rho}}^{\left|\ln\frac{x}{c}\right|} \frac{f(x)-f\left(xe^{-t\,\mathrm{sign}\left(\ln\frac{x}{c}\right)}\right)}{t^{1+\alpha}} dt.$$

Then we define

$$\left(D_c^\alpha f\right)(x) = \lim_{\rho\to 1} (D^\alpha_{c,1-\tilde{\rho}} f)(x). \tag{3.10}$$

It is clear that definitions (2.1) and (3.10) coincide on sufficiently nice functions $f(x)$. In the sequel, we will study their coincidence in the case of "not so nice" functions.

3.3. Modification of the "quasi-convolutive" truncation

As mentioned above, in (3.3)–(3.4) we used the zero continuation of the function $f(x)$ when necessary. Such a method is natural in the study of left-sided or right-sided fractional differentiation, when the functions are given only unilaterally from the initial point (in our case point c). In our consideration the function $f(x)$ is given on the whole semi-axis, and it is more natural to have truncations with the use of its proper values instead of the zero continuation.

Such a truncation is provided by (3.7)–(3.9). On the other hand, it is not well suited to dilations on $\mathbb{R}_+$ since it has no property (3.6). By this reason, together with the above two constructions, we introduce also the following truncation

$$\dot{D}^\alpha_{c,1-\rho} f = \frac{f(x)}{\Gamma(1-\alpha)\left|\ln\frac{x}{c}\right|^\alpha} + \frac{\alpha}{\Gamma(1-\alpha)} \int\limits_{\ln\frac{1}{\rho}}^{\left|\ln\frac{x}{c}\right|} \frac{f(x)-f\left(xe^{-t\,\mathrm{sign}(x-c)}\right)}{t^{1+\alpha}} dt, \tag{3.11}$$

for all $x \in (0,\infty)$ where both the cases $\left|\ln\frac{x}{c}\right| > \ln\frac{1}{\rho}$ and $\left|\ln\frac{x}{c}\right| < \ln\frac{1}{\rho}$ are admitted.

This way of truncation achieves both the goals, i.e., it satisfies the quasi-commutation condition (3.6) and does not use the zero continuation, leads to more complicated integral representations for truncated derivatives of fractional integrals, this complication having place on the "small" interval $c\rho < x < \frac{c}{\rho}$.

4. Integral representations for truncated fractional derivatives

By

$$\Im_c^\alpha(X) = \{f : f(x) = (\Im_c^\alpha \varphi)(x), \varphi(x) \in X\},$$

we denote the range of the Hadamard–Chen fractional integration operator over a function space X. We will work with the space $X = L^p_{\text{loc}}(\mathbb{R}_+)$. In the next statements we provide integral representations of different truncated fractional derivatives of functions in such a range $\Im_c^\alpha(L^p_{\text{loc}}(\mathbb{R}_+))$.

Lemma 4.1. *The operator* $\Im_c^\alpha, \alpha > 0$ *is continuous in the space* $L^p_{\text{loc}}(\mathbb{R}_+)$, $1 \le p < \infty$ *and admits the estimate*

$$\|\Im_c^\alpha \varphi\|_{L_{p(a,b)}} \le C\,\|\varphi\|_{L_{p(a^*,b^*)}}, \qquad \textit{when} \qquad (a,b) \ni c, \tag{4.1}$$

where $0 < a < b < \infty, a^* = \min\left(a, \frac{c^2}{b}\right), b^* = \max\left(b, \frac{c^2}{a}\right)$ *and* $C = C(a,b,c;p,\alpha)$.

Proof. We consider only the case $a < c < b$, the case $c \notin (a,b)$ is easy. We have

$$\|\Im_c^\alpha \varphi\|_{L^p(a,b)} \le \left\{ \int_c^b \left| \frac{1}{\Gamma(\alpha)} \int_{\frac{c}{b}}^1 \frac{\varphi(xy)}{\left(\ln \frac{1}{y}\right)^{1-\alpha}} \frac{dy}{y} \right|^p dx \right\}^{1/p}$$

$$+ \left\{ \int_a^c \left| \frac{1}{\Gamma(\alpha)} \int_{\frac{a}{c}}^1 \frac{\varphi(xy^{-1})}{\left(\ln \frac{1}{y}\right)^{1-\alpha}} \frac{dy}{y} \right|^p dx \right\}^{1/p}.$$

Therefore,

$$\|\Im_c^\alpha \varphi\|_{L^p(a,b)} \le \frac{1}{\Gamma(\alpha)} \int_{\frac{c}{b}}^1 \left(\ln \frac{1}{y}\right)^{\alpha-1} \frac{dy}{y} \left\{ \int_c^b |\varphi(xy)|^p \, dx \right\}^{\frac{1}{p}}$$

$$+ \frac{1}{\Gamma(\alpha)} \int_{\frac{a}{c}}^1 \left(\ln \frac{1}{y}\right)^{\alpha-1} \frac{dy}{y} \left\{ \int_a^c |\varphi(xy^{-1})|^p \, dx \right\}^{\frac{1}{p}}.$$

After the substitutions $xy = \xi$ and $xy^{-1} = \eta$, we have

$$\|\Im_c^\alpha \varphi\|_{L^p(a,b)} \le \frac{1}{\Gamma(\alpha)} \int_{\frac{c}{b}}^1 \left(\ln \frac{1}{y}\right)^{\alpha-1} \frac{dy}{y^{1+\frac{1}{p}}} \left\{ \int_{cy}^{by} |\varphi(\xi)|^p \, d\xi \right\}^{\frac{1}{p}}$$

$$+ \frac{1}{\Gamma(\alpha)} \int_{\frac{a}{c}}^1 \left(\ln \frac{1}{y}\right)^{\alpha-1} \frac{dy}{y^{1-\frac{1}{p}}} \left\{ \int_{\frac{a}{y}}^{\frac{c}{y}} |\varphi(\eta)|^p \,, d\eta \right\}^{\frac{1}{p}}.$$

from which (4.1) follows. □

Lemma 4.2. *Let $1 \le p < \infty$ and $0 < \alpha < 1$. The truncated fractional derivative $D^\alpha_{c,1-\rho} f$ has the following representation for functions $f = \Im^\alpha_c \varphi, \quad \varphi \in L^p_{\mathrm{loc}}(\mathbb{R}_+)$:*

$$\left(D^\alpha_{c,1-\rho} f\right)(x) = \int\limits_0^\infty K^+_\alpha(y) \ \left[\varphi_{c+}(x\rho^y) + \varphi_{c-}(x\rho^{-y})\right] dy$$

$$= \int\limits_0^{r(x,\rho)} K^+_\alpha(y)\, \varphi \left(x\rho^{y \operatorname{sign}\left(\ln \frac{x}{c}\right)}\right) dy,$$

where $r(x,\rho) = \left|\ln \frac{x}{c}\right| / \ln \frac{1}{\rho}$,

$$K^+_\alpha(y) = \frac{\sin \alpha\pi}{\pi} \frac{y^\alpha_+ - (y-1)^\alpha_+}{y} \in L_1(\mathbb{R}_+) \quad \textit{and} \quad \int\limits_0^\infty K^+_\alpha(y) dy = 1. \tag{4.2}$$

Proof. For $0 < t < 1$ we have

$$f_{c+}(x) - f_{c+}(xt) + f_{c-}(x) - f_{c-}(xt^{-1})$$
$$= \left(\ln \frac{1}{t}\right)^\alpha \int\limits_0^\infty k_\alpha(y) \left[\varphi_{c+}(xt^y) + \varphi_{c-}(xt^{-y})\right] dy,$$

where

$$k_\alpha(y) = \frac{1}{\Gamma(\alpha)} \begin{cases} y^{\alpha-1}, & 0 < y < 1, \\ y^{\alpha-1} - (y-1)^{\alpha-1}, & y > 1. \end{cases}$$

It is known that $k_\alpha(y) \in L_1(\mathbb{R}_+)$ and $\int\limits_0^\infty k_\alpha(y) dy = 0$, see [8], p. 124. From (3.8) we obtain

$$D^\alpha_{c,1-\rho} f = \frac{\alpha}{\Gamma(1-\alpha)} \int\limits_0^\rho \frac{1}{\ln \frac{1}{t}} \frac{dt}{t} \int\limits_0^\infty k_\alpha(y) \left[\varphi_{c+}(xt^y) + \varphi_{c-}(xt^{-y})\right] dy$$

$$= \frac{\alpha}{(1-\alpha)} \int\limits_{\ln \frac{1}{\rho}}^\infty \frac{d\xi}{\xi^2} \int\limits_0^\infty k_\alpha\left(\frac{\tau}{\xi}\right) \left[\varphi_{c+}(xe^{-\tau}) + \varphi_{c-}(xe^{\tau})\right] d\tau$$

after the substitutions $\ln \frac{1}{t} = \xi$ and $y\xi = \tau$. Hence

$$D^\alpha_{c,1-\rho} f = \frac{\alpha}{\Gamma(1-\alpha)} \int\limits_0^\infty \left[\varphi_{c+}(xe^{-\tau}) + \varphi_{c-}(xe^{\tau})\right] \int\limits_{\ln \frac{1}{\rho}}^\infty k_\alpha\left(\frac{\tau}{\xi}\right) \frac{d\xi}{\xi^2}$$

$$= \frac{\alpha}{\Gamma(1-\alpha)} \int\limits_0^\infty \left[\varphi_{c+}(x\rho^t) + \varphi_{c-}(x\rho^{-t})\right] \left(\frac{1}{t} \int\limits_0^t k_\alpha(s) ds\right) dt,$$

which proves the lemma. Properties (4.2) of the function $k_\alpha(t)$ are known (see [8], p. 125). □

Corollary 4.3. *Let $1 \le p < \infty$ and $0 < \alpha < 1$. The truncated fractional derivative $D^{\alpha}_{c,1-\tilde{\rho}} f$ has the following representation for functions $f(x) = (\Im^{\alpha}_{c}\varphi)(x)$, $\varphi \in L^{p}_{\mathrm{loc}}(\mathbb{R}_{+})$:*

$$D^{\alpha}_{c,1-\tilde{\rho}} f = \int\limits_{0}^{\frac{1}{\ln \frac{1}{\rho}}} K^{+}_{\alpha}(y)\varphi\left(x\rho^{y \ln \frac{x}{c}}\right) dy.$$

Actually, it suffices to replace $\ln\frac{1}{\rho}$ by $\tilde{\rho} = \left|\ln\frac{x}{c}\right| \ln\frac{1}{\rho}$ in Lemma 4.2.

Lemma 4.4. *Let $1 \le p < \infty$ and $0 < \alpha < 1$. The truncated fractional derivative $\dot{D}^{\alpha}_{c,1-\rho} f, 0 < \rho < 1$, of fractional integrals $f = \Im^{\alpha}_{c}\varphi$ with $\varphi \in L^{p}_{\mathrm{loc}}(\mathbb{R}_{+})$, has the following representation*

$$\dot{D}^{\alpha}_{c,1-\rho} f = \int\limits_{0}^{1} K_{c}(x,\rho;t)\,\varphi\left(x\rho^{t\,\mathrm{sign}(x-c)}\right) dt, \tag{4.3}$$

for the case $c\rho < x < \frac{c}{\rho}$, where

$$K_{c}(x,\rho;t) = \frac{\sin\alpha\pi}{\pi}\begin{cases} t^{\alpha-1}, & 0 < t < \frac{\left|\ln\frac{x}{c}\right|}{\ln\frac{1}{\rho}}, \\ \frac{(1-t)^{\alpha}}{t}, & \frac{\left|\ln\frac{x}{c}\right|}{\ln\frac{1}{\rho}} < t < 1. \end{cases}$$

Proof. For $f(x) = (\Im^{\alpha}_{c+}\varphi)(x)$ we have

$$f(x) - f(xt) = \frac{1}{\Gamma(\alpha)}\int\limits_{c}^{x} \frac{\varphi(\tau)}{\left(\ln\frac{x}{\tau}\right)^{1-\alpha}}\frac{d\tau}{\tau} - \frac{1}{\Gamma(\alpha)}\int\limits_{x\,t}^{c} \frac{\varphi(\tau)}{\left(\ln\frac{\tau}{xt}\right)^{1-\alpha}}\frac{d\tau}{\tau}$$

when $c < x < \frac{c}{\rho}$, and

$$f(x) - f(xt^{-1}) = \frac{1}{\Gamma(\alpha)}\int\limits_{x}^{c} \frac{\varphi(\tau)}{\left(\ln\frac{\tau}{x}\right)^{1-\alpha}}\frac{d\tau}{\tau} - \frac{1}{\Gamma(\alpha)}\int\limits_{xt^{-1}}^{c} \frac{\varphi(\tau)}{\left(\ln\frac{xt^{-1}}{\tau}\right)^{1-\alpha}}\frac{d\tau}{\tau},$$

when $c\rho < x < c$. Hence, after the substitutions $\tau = xt^{y}, \tau = xt^{-y}$, we obtain

$$f(x) - f(xt) = \frac{\left(\ln\frac{1}{t}\right)^{\alpha}}{\Gamma(\alpha)}\left[\int\limits_{0}^{\frac{\ln\frac{x}{c}}{\ln\frac{1}{t}}} y^{\alpha-1}\varphi(xt^{y})dy - \int\limits_{\frac{\ln\frac{x}{c}}{\ln\frac{1}{t}}}^{1} \frac{\varphi(xt^{y})}{(1-y)^{1-\alpha}}dy\right]$$

and

$$f(x) - f(xt^{-1}) = \frac{\left(\ln\frac{1}{t}\right)^{\alpha}}{\Gamma(\alpha)}\left[\int\limits_{0}^{\frac{\ln\frac{c}{x}}{\ln\frac{1}{t}}} y^{\alpha-1}\varphi(xt^{-y})dy - \int\limits_{\frac{\ln\frac{c}{x}}{\ln\frac{1}{t}}}^{1} \frac{\varphi(xt^{-y})}{(1-y)^{1-\alpha}}dy\right].$$

We substitute this into the following expressions of the truncated derivatives

$$\varphi_{+,\rho}(x) := -\int_{\rho}^{\frac{c}{x}} \frac{f(x)-f(xt)}{\left(\ln\frac{1}{t}\right)^{1+\alpha}}\frac{dt}{t}, \quad x < \frac{c}{\rho},$$

$$\varphi_{-,\rho}(x) := -\int_{\rho}^{\frac{x}{c}} \frac{f(x)-f(xt^{-1})}{\left(\ln\frac{1}{t}\right)^{1+\alpha}}\frac{dt}{t}, \quad x > c\rho,$$

and get

$$\varphi_{+,\rho}(x) = -\frac{1}{\Gamma(\alpha)}\int_{\rho}^{\frac{c}{x}} \frac{1}{\ln^2\frac{1}{t}} \; \frac{dt}{t}\left(\int_{\frac{c}{x}}^{1} \frac{\varphi(x\tau)}{\left(\frac{\ln\frac{1}{\tau}}{\ln\frac{1}{t}}\right)^{1-\alpha}}\frac{d\tau}{\tau} - \int_{t}^{\frac{c}{x}} \frac{\varphi(x\tau)}{\left(1-\frac{\ln\frac{1}{\tau}}{\ln\frac{1}{t}}\right)^{1-\alpha}}\frac{d\tau}{\tau}\right),$$

$$\varphi_{-,\rho}(x) = -\frac{1}{\Gamma(\alpha)}\int_{\rho}^{\frac{x}{c}} \frac{1}{\ln^2\frac{1}{t}} \; \frac{dt}{t}\left(\int_{\frac{x}{c}}^{1} \frac{\varphi(x\tau^{-1})}{\left(\frac{\ln\frac{1}{\tau}}{\ln\frac{1}{t}}\right)^{1-\alpha}}\frac{d\tau}{\tau} - \int_{t}^{\frac{x}{c}} \frac{\varphi(x\tau^{-1})}{\left(1-\frac{\ln\frac{1}{\tau}}{\ln\frac{1}{t}}\right)^{1-\alpha}}\frac{d\tau}{\tau}\right),$$

with the substitution $t^y = \tau$ taken into account. Changing the order of integration on the right-hand side, we arrive at

$$\varphi_{+,\rho}(x) = -\frac{1}{\Gamma(\alpha)}\Bigg[\int_{\frac{c}{x}}^{1}\varphi(x\tau)\frac{d\tau}{\tau}\int_{\rho}^{\frac{c}{x}}\left(\frac{\ln\frac{1}{\tau}}{\ln\frac{1}{t}}\right)^{\alpha-1}\frac{dt}{t\ln^2\frac{1}{t}}$$
$$-\int_{\rho}^{\frac{c}{x}}\varphi(x\tau)\frac{d\tau}{\tau}\int_{\rho}^{\tau}\left(1-\frac{\ln\frac{1}{\tau}}{\ln\frac{1}{t}}\right)^{\alpha-1}\frac{dt}{t\ln^2\frac{1}{t}}\Bigg],$$

$$\varphi_{-,\rho}(x) = -\frac{1}{\Gamma(\alpha)}\Bigg[\int_{\frac{x}{c}}^{1}\varphi(x\tau^{-1})\frac{d\tau}{\tau}\int_{\rho}^{\frac{x}{c}}\left(\frac{\ln\frac{1}{\tau}}{\ln\frac{1}{t}}\right)^{\alpha-1}\frac{dt}{t\ln^2\frac{1}{t}}$$
$$-\int_{\rho}^{\frac{x}{c}}\varphi(x\tau^{-1})\frac{d\tau}{\tau}\int_{\rho}^{\tau}\left(1-\frac{\ln\frac{1}{\tau}}{\ln\frac{1}{t}}\right)^{\alpha-1}\frac{dt}{t\ln^2\frac{1}{t}}\Bigg].$$

Substitution $\frac{\ln\frac{1}{\tau}}{\ln\frac{1}{t}} = \xi$ gives the following representation

$$\varphi_{+,\rho}(x) = -\frac{1}{\Gamma(\alpha)}\left[\int_{\frac{c}{x}}^{1}\frac{\varphi(x\tau)d\tau}{\tau\ln\frac{1}{\tau}}\int_{\frac{\ln\frac{1}{\tau}}{\ln\frac{1}{\rho}}}^{\frac{\ln\frac{1}{\tau}}{\ln\frac{x}{c}}}\frac{d\xi}{\xi^{1-\alpha}} - \int_{\rho}^{\frac{c}{x}}\frac{\varphi(x\tau)d\tau}{\tau\ln\frac{1}{\tau}}\int_{\frac{\ln\frac{1}{\tau}}{\ln\frac{1}{\rho}}}^{1}\frac{d\xi}{(1-\xi)^{1-\alpha}}\right],$$

and

$$\varphi_{-,\rho}(x) = -\frac{1}{\Gamma(\alpha)}\left[\int\limits_{\frac{x}{c}}^{1} \frac{\varphi(x/\tau)\,d\tau}{\tau \ln\frac{1}{\tau}} \int\limits_{\frac{\ln\frac{1}{\tau}}{\ln\frac{1}{\rho}}}^{\frac{\ln\frac{1}{\tau}}{\ln\frac{c}{x}}} \frac{d\xi}{\xi^{1-\alpha}} - \int\limits_{\rho}^{\frac{x}{c}} \frac{\varphi(x/\tau)\,d\tau}{\tau \ln\frac{1}{\tau}} \int\limits_{\frac{\ln\frac{1}{\tau}}{\ln\frac{1}{\rho}}}^{1} \frac{d\xi}{(1-\xi)^{1-\alpha}}\right].$$

Further we obtain

$$\varphi_{+,\rho}(x) = -\frac{f(x)}{\alpha\left(\ln\frac{x}{c}\right)^{\alpha}} + \frac{1}{\alpha\Gamma(\alpha)}\left[\int\limits_{0}^{\frac{\ln\frac{x}{c}}{\ln\frac{1}{\rho}}} \frac{\varphi(x\rho^{y})}{y^{1-\alpha}}dy + \int\limits_{\frac{\ln\frac{x}{c}}{\ln\frac{1}{\rho}}}^{1} \frac{(1-y)^{\alpha}}{y}\varphi(x\rho^{y})dy\right],$$

$$\varphi_{-,\rho}(x) = -\frac{f(x)}{\alpha\left(\ln\frac{c}{x}\right)^{\alpha}} + \frac{1}{\alpha\Gamma(\alpha)}\left[\int\limits_{0}^{\frac{\ln\frac{c}{x}}{\ln\frac{1}{\rho}}} \frac{\varphi\left(\frac{x}{\rho^{y}}\right)}{y^{1-\alpha}}dy + \int\limits_{\frac{\ln\frac{c}{x}}{\ln\frac{1}{\rho}}}^{1} \frac{(1-y)^{\alpha}}{y}\varphi(\frac{x}{\rho^{y}})dy\right].$$

With the notation $r(x,\rho) = \frac{\left|\ln\frac{x}{c}\right|}{\ln\frac{1}{\rho}}$ we can also rewrite the last formulas in the form

$$\varphi_{\pm,\rho}(x) = -\frac{f(x)}{\alpha\left|\ln\frac{x}{c}\right|^{\alpha}} + \frac{1}{\alpha\Gamma(\alpha)}\left[\int\limits_{0}^{r(x,\rho)} y^{\alpha-1}\varphi\left(x\rho^{y\,\mathrm{sign}\left(\ln\frac{x}{c}\right)}\right)dy\right.$$
$$\left. + \int\limits_{r(x,\rho)}^{1}\left(\frac{(1-y)^{\alpha}}{y}\varphi(x\rho^{y\,\mathrm{sign}\left(\ln\frac{x}{c}\right)}\right)dy\right]$$

for $c\rho < x < \frac{c}{\rho}$. From (3.11) we obtain

$$\dot{D}^{\alpha}_{c,1-\rho}f = \frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)}\left[\int\limits_{0}^{r(x,\rho)} y^{\alpha-1}\varphi\left(x\rho^{y\,\mathrm{sign}(x-c)}\right)dy\right.$$

$$\left. + \int\limits_{r(x,\rho)}^{1}\left(\frac{(1-y)^{\alpha}}{y}\varphi(x\rho^{y\,\mathrm{sign}(x-c)}\right)dy\right]$$

for $c\rho < x < \frac{c}{\rho}$, which proves (4.3). □

5. The inversion theorem in terms of "quasi-convolution" truncated fractional derivatives

Theorem 5.1. *Let* $f(x) = (\Im_c^\alpha)(x)$*, where* $\varphi \in L^p(\mathbb{R}_+)$, $1 \le p < \infty$, $0 < \alpha < 1$. *Then*

$$\varphi(x) = (D_c^\alpha f)(x),$$

where $(D_c^\alpha f)(x)$ *is interpreted as*

$$D_c^\alpha f = \lim_{\substack{\rho\to 1-0 \\ (L^p(\mathbb{R}_+))}} \left(D_{c,1-\rho}^\alpha f\right)(x).$$

Proof. Theorem 5.1 is proved by means of the representation proved in Lemma 4.2. By that representation we have

$$D_{c,1-\rho}^\alpha f - \varphi = \int_0^\infty K_\alpha^+(t)\left[\varphi_{c+}(x\rho^t) + \varphi_{c-}(x\rho^{-t}) - \varphi(t)\right] dt.$$

Hence by the Minkowski inequality we obtain

$$\begin{aligned} \left\|D_{c,1-\rho}^\alpha f - \varphi\right\|_{L^p} &\le \int_0^\infty K_\alpha^+(t)\left\|\varphi_{c+}(x\rho^t) + \varphi_{c-}(x\rho^{-t})\varphi(x) - \varphi(t)\right\|_{L^p} dy \\ &\le \int_0^\infty K_\alpha^+(t)\left(\left\|\varphi(x\rho^t) - \varphi(x)\right\|_{L^p} + \left\|\varphi(x\rho^{-t}) - \varphi(x)\right\|_{L^p}\right) dt \to 0 \end{aligned}$$

as $\rho \to 1$, in view of mean continuity of functions $f \in L^p$ and the Lebesgue dominated convergence theorem. □

Theorem 5.2. *Let* $f(x) = (\Im_c^\alpha \varphi)(x)$*, where* $\varphi \in L^p_{\text{loc}}(\mathbb{R}_+)$, $1 \le p < \infty$, $0 < \alpha < 1$. *Then*

$$(D_c^\alpha f)(x) = \lim_{\rho\to 1-0} D_{c,1-\rho}^\alpha f = \varphi(x).$$

in the topology of the space $L^p_{\text{loc}}(\mathbb{R}_+)$.

Proof. Making use of Lemma 4.2, we get

$$\left(D_{c,1-\rho}^\alpha\right)(x) - \varphi(x) = \int_0^{\frac{\left|\ln\frac{x}{c}\right|}{\ln\frac{1}{\rho}}} K_\alpha^+(y)\left[\varphi\left(x\rho^{y\,\mathrm{sign}(x-c)}\right) - \varphi(x)\right] dy,$$

for $\left|\ln \frac{x}{c}\right| > \ln \frac{1}{\rho}$, while for $\left|\ln \frac{x}{c}\right| < \ln \frac{1}{\rho}$, according to (3.5), we have

$$\left(D^\alpha_{c,1-\rho} f\right)(x) - \varphi(x) = \frac{f(x)}{\Gamma(1-\alpha)\left(\ln \frac{1}{\rho}\right)^\alpha}$$

$$= \frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)\left(\ln \frac{1}{\rho}\right)^\alpha} \int\limits_{\min(x,c)}^{\max(x,c)} \frac{\varphi(cxy^{-1})}{\left|\ln \frac{y}{c}\right|^{1-\alpha}} \, \frac{dy}{y}.$$

Let (a,b) be any interval on $\mathbb{R}_+$ containing the point c. We have

$$\begin{aligned}
&\left\|D^\alpha_{c,1-\rho} f - \varphi\right\|_{L^p(a,b)} \\
&\quad \le \int_0^\infty \left|K^+_\alpha(y)\right| \left\{\int_{\frac{c}{\rho}}^b \left|\theta_+\left(\ln \frac{x}{c} - y \ln \frac{1}{\rho}\right)\varphi(x\rho^y) - \varphi(x)\right|^p dx\right\}^{\frac{1}{p}} dy \\
&\qquad + \int_0^\infty \left|K^+_\alpha(y)\right| \left\{\int_a^{c\rho} \left|\theta_-\left(\ln \frac{x}{c} + y \ln \frac{1}{\rho}\right)\varphi(x\rho^{-y}) - \varphi(x)\right|^p dx\right\}^{\frac{1}{p}} dy \\
&\qquad + \frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)\left(\ln \frac{1}{\rho}\right)^\alpha}\left[\int_{c\rho}^c \left(\ln \frac{c}{y}\right)^{\alpha-1} \frac{dy}{y} \left\{\int_{c\rho}^y \left|\varphi\left(x\frac{c}{y}\right)\right|^p dx\right\}^{\frac{1}{p}}\right. \\
&\qquad \left. + \int_c^{\frac{c}{\rho}} \left(\ln \frac{y}{c}\right)^{\alpha-1} \frac{dy}{y} \left\{\int_y^{\frac{c}{\rho}} \left|\varphi\left(x\frac{c}{y}\right)\right|^p dx\right\}^{\frac{1}{p}}\right].
\end{aligned}$$

Hence

$$\begin{aligned}
&\left\|D^\alpha_{c,1-\rho} f - \varphi\right\|_{L^p(a,b)} \\
&\quad \le \int_0^\infty \left|K^+_\alpha(y)\right| \left\|\theta_+\left(\ln \frac{x}{c} - y \ln \frac{1}{\rho}\right)\varphi(x\rho^y) - \varphi(x)\right\|_{L^p\left(\frac{c}{\rho},b\right)} dy \\
&\qquad + \int_0^\infty K^+_\alpha \left\|\theta_-\left(\ln \frac{x}{c} + y \ln \frac{1}{\rho}\right)\varphi(x\rho^{-y}) - \varphi(x)\right\|_{L^p(a,c\rho)} dy + d\,\|\varphi\|_{L^p\left(c\rho,\frac{c}{\rho}\right)},
\end{aligned}$$

where $d = \frac{\sin \alpha\pi}{\alpha\pi}\left(1 + (\frac{1}{\rho})^{\frac{1}{p}}\right)$. The passage to the limit in the above estimate is now justified by the Lebesgue dominated convergence theorem. □

Theorem 5.3. *Let $f(x) = (I^\alpha_c \varphi)(x)$, where $\varphi \in L^p(\mathbb{R}_+)$ or $L^p_{\mathrm{loc}}(\mathbb{R}_+)$, $1 < p < \infty$, $0 < \alpha < 1$, $c > 0$. Then*

$$\left(\dot{D}^\alpha_c f\right)(x) = \lim_{\rho \to 1-0} \left(\dot{D}^\alpha_{c,1-\rho} f\right)(x) = \varphi(x),$$

in the topology of the space $L^p(\mathbb{R}_+)$, or $L^p_{\mathrm{loc}}(\mathbb{R}_+)$, respectively.

Proof. We consider the case where $\varphi \in L^p(\mathbb{R}_+)$. The proof for the space $L^p_{\mathrm{loc}}(\mathbb{R}_+)$ follow the same lines.

The proof is prepared by Lemma 4.4. Since

$$\left(\dot{D}^{\alpha}_{c,1-\rho}f\right)(x) = \left(D^{\alpha}_{c,1-\rho}\right)(x) \quad \text{for} \quad x \notin \left[c\rho, \frac{c}{\rho}\right],$$

we have

$$\left\|\dot{D}^{\alpha}_{c,1-\rho}f - \varphi\right\|_{L^p(\mathbb{R}_+)} \le \left\|D^{\alpha}_{c,1-\rho}f - \varphi\right\|_{L^p(\mathbb{R}_+\setminus\left(c\rho,\frac{c}{\rho}\right))} + \left\|\dot{D}^{\alpha}_{c,1-\rho}f - \varphi\right\|_{L^p\left(c\rho,\frac{c}{\rho}\right)}$$
$$\le \left\|D^{\alpha}_{c,1-\rho}f - \varphi\right\|_{L^p(\mathbb{R}_+)} + \left\|\dot{D}^{\alpha}_{c,1-\rho}f - \varphi\right\|_{L^p\left(c\rho,\frac{c}{\rho}\right)} + \|\varphi\|_{L^p\left(c\rho,\frac{c}{\rho}\right)}.$$

Here the first term on the right-hand side tends to zero by Theorem 3.5, the same is obvious for the third one. To estimate the middle term, in view of the symmetry, we will consider this term with integration only along the integral $(c, c/\rho)$, estimation along the left interval $(c\rho, c)$ being similar. By (4.3) we have

$$\left\|\dot{D}^{\alpha}_{c,1-\rho}f\right\|_{L^p(c,c/\rho)} \le \left\|\int_0^1 K(x,\rho;t)\varphi(x\rho^t)dt\right\|_{L^p(c,c/\rho)}$$
$$\le d\left\|\int_0^{\ln\frac{x}{c}/\ln\frac{1}{\rho}} t^{\alpha-1}\varphi(x\rho^t)dt\right\|_{L_p((c,c/\rho))} + d\left\|\int_{\ln\frac{x}{c}/\ln\frac{1}{\rho}}^1 (1-t)^{\alpha}\frac{1}{t}\varphi(x\varphi^t)dt\right\|_{L_p((c,c/\rho)}$$
$$=: I_1 + I_2,$$

where $d = \frac{\sin\alpha\pi}{\alpha}$. For I_1 by the Minkowski inequality, we obtain

$$I_1 \le d\int_0^1 t^{\alpha-1}\left\{\int_c^{c/\rho}\left|\theta_+\left(\ln\frac{x}{c}/\ln\frac{1}{\rho} - t\right)\varphi(x\rho^t)\right|^p dx\right\}^{1/p} dt$$
$$\le C\int_0^1 t^{\alpha-1}\rho^{-\frac{1}{p}t}\left\{\int_c^{c\rho^{t-1}}|\varphi(y)|^p dy\right\}^{1/p} dt \le C_1\|\varphi\|_{L_p(c,c/\rho)}.$$

For I_2, with the changes $\rho^t = \left(\frac{x}{c}\right)^{-y}$ and $\ln\frac{x}{c} = \frac{1}{z}\ln\frac{1}{\rho}$ give:

$$I_2 \le C\left\|\int_{\ln\frac{x}{c}/\ln\frac{1}{\rho}}^1 (1-t)^{\alpha}\frac{1}{t}\varphi(x\rho^t)dt\right\|_{L^p(c,c/\rho)}$$
$$\le C\left\{\int_1^{\infty}\left|\int_1^z \left(1-\frac{y}{z}\right)^{\alpha}\varphi\left(c\left(\frac{1}{\rho}\right)^{\frac{1-y}{z}}\right)\frac{dy}{y}\right|^p c\left(\frac{1}{\rho}\right)^{\frac{1}{z}}\ln\frac{1}{\rho}\frac{dz}{z^2}\right\}^{1/p}.$$

Applying the Minkowski inequality, we get

$$I_2 \le C\left(c\ln\frac{1}{\rho}\right)^{\frac{1}{p}}\int_1^{\infty}\left\{\int_y^{\infty}\left|\varphi\Big(c\left(\frac{1}{\rho}\right)^{\frac{1-y}{z}}\Big)\right|^p\left(\frac{1}{\rho}\right)^{\frac{1}{z}}\frac{dz}{z^2}\right\}^{1/p}\frac{dy}{y}$$

and then the change $c\left(\frac{1}{\rho}\right)^{\frac{1-y}{z}} = \xi$ yields

$$I_2 \le C_2\int_1^{\infty}\left\{\int_{c(\frac{1}{p})^{\frac{1-y}{y}}}^{c}|\varphi(\xi)|^p\xi^{\frac{1}{1-y}-1}c^{-\frac{1}{1-y}}d\xi\right\}^{1/p}\frac{dy}{y(y-1)^{1/p}}$$

$$\leq C_2 \int_1^\infty \left\{ \int_{c\rho}^c |\varphi(\xi)|^p d\xi \right\}^{1/p} \frac{dy}{y(y-1)^{1/p}} = C_3 \|\varphi\|_{L_p(c\rho,c)} \to 0$$

as $\rho \to 1$, which completes the proof. □

References

[1] P.L. Butzer, A.A. Kilbas, and J.J. Trujillo, *Compositions of Hadamard-type fractional integration operators and the semigroup property.* J. Math. Anal. Appl. **269**(2) (2002), 387–400.

[2] P.L. Butzer, A.A. Kilbas, and J.J. Trujillo, *Fractional calculus in the Mellin setting and Hadamard-type fractional integrals.* J. Math. Anal. Appl. **269**(1) (2002), 1–27.

[3] P.L. Butzer, A.A. Kilbas, and J.J. Trujillo, *Mellin transform analysis and integration by parts for Hadamard-type fractional integrals.* J. Math. Anal. Appl. **270**(1) (2002), 1–15.

[4] Y.W. Chen, *Entire solutions of a class of differential equations of mixed type.* In Part. Diff. Equat. and Cont. Mech., pages 336–337. Madison, Wisconsin Press, 1961.

[5] J. Hadamard, *Essai sur l'étude des functions données par leur développment de Taylor.* J. Math. Pures Appl. **8**: Ser. 4 (1892), 101–186.

[6] A. Kilbas, *Hadamard-type fractional calculus.* J. Korean Math. Soc. **38**(6) (2001), 1191–1204.

[7] A.A. Kilbas, *Hadamard-type integral equations and fractional calculus operators.* In *Singular Integral Operators, Factorization and Applications*, Oper. Theory Adv. Appl. **142** (2003), 175–188, Birkhäuser, Basel.

[8] S.G. Samko, A.A. Kilbas, and O.I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications.* London-New York: Gordon & Breach. Sci. Publ., (Russian edition – *Fractional Integrals and Derivatives and some of their Applications*, Minsk: Nauka i Tekhnika, 1987), 1993, 1012 pages.

[9] S.G. Samko and M.U. Yakhshiboev, *A modification of Riemann–Liouville fractional integro-differentiation applied to functions on R^1 with arbitrary behaviour at infinity.* Izv. Vyssh. Uchebn. Zaved. Mat. **4** (1992), 96–99.

[10] S.G. Samko and M.U. Yakhshiboev, *On a class of identity approximation operators of a non-convolution type.* Frac. Calc. Appl. Anal. **4**(4) (2001), 523–530.

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Poisson Geometry of Difference Lax Operators and Difference Galois Theory

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Abstract. We discuss the lift of Poisson structures associated with auxiliary linear problems for the differential and difference Lax equations to the space of wave functions. Due to a peculiar symmetry breaking, the corresponding differential and difference Galois groups become Poisson Lie Groups.

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1. Virasoro algebra and its Galois extension

The study of Poisson structures on the spaces of differential and difference operators is motivated by applications to non-linear equations of Lax type which arise as compatibility conditions for linear differential and difference equations. Our first object to discuss is the space of second-order Schrödinger differential operators on the line. This space carries a natural Poisson structure (in fact, even several compatible ones, but we shall not dwell upon that), such that spectral invariants of a second-order differential operator form a maximal involutive family. This Poisson structure admits several descriptions; perhaps the most straightforward one is the identification of the space of Schrödinger operators on the line with a hyperplane in the dual space of the Virasoro algebra. Recall that the dual space of any Lie algebra carries a natural Poisson bracket, called the Lie–Poisson bracket, and its symplectic leaves are the coadjoint orbits of the associated Lie group. The Virasoro algebra is a central extension of the Lie algebra $\mathrm{Vect}\, S^1$ of vector fields on the circle $S^1 = \mathbb{R}/\mathbb{Z}$. The elements of $\mathrm{Vect}\, S^1$ are linear differential operators on the line with periodic coefficients of the form $\xi_f = f\,\partial_x,\ f \in C^\infty(S^1)$ with the Lie bracket

$$[\xi_f,\, \xi_g] = \xi_{w(f,g)}, \quad \text{where} \quad w(f,\, g) = f'g - g'f. \tag{1.1}$$

The dual space of smooth linear functionals on $\operatorname{Vect} S^1$ consists of quadratic differentials $F_u = u\, dx^2,\ u \in C^\infty(S^1)$; the action of a linear functional F_u on a vector field is given by the coupling

$$F_u(\xi_f) = \int \langle u\, dx^2, \xi_f \rangle = \int u\, f\, dx. \tag{1.2}$$

The Lie group which corresponds to $\operatorname{Vect} S^1$ is the group $\operatorname{Diff} S^1$ of diffeomorphisms of the circle; its adjoint and coadjoint representations correspond to the standard change of variables for a vector field and for a quadratic differential, respectively. We have

$$\operatorname{Ad}^* \phi \cdot F_u = (\phi^{-1})^* F_u = \phi'(x)^{-2} u(\phi^{-1}(x))\, dx^2. \tag{1.3}$$

According to a fundamental theorem of Gelfand and Fuchs, the second cohomology group of $\operatorname{Vect} S^1$ is one-dimensional; it is generated by the 2-cocycle

$$\Omega(\xi_f,\, \xi_g) = \int f''' g\, dx. \tag{1.4}$$

The central extension of $\operatorname{Vect} S^1$ associated with the cocycle (1.4) is called the *Virasoro algebra*. Commutation relations in the Virasoro algebra are frequently written with the help of a standard basis in the complexified Lie algebra of vector fields,

$$\xi_k = ie^{ikx}\partial_x,\ k \in \mathbb{Z};$$

Remark 1.1. Vector fields ξ_0, ξ_1, ξ_{-1} generate the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$; since this algebra is simple, the restriction of the Gelfand–Fuchs cocycle to this algebra is trivial. It is convenient to modify the cocycle (1.4) in such a way that this restriction is identically zero. The modified cocycle is given by

$$\widehat{\Omega}(\xi_f,\, \xi_g) = \int (f''' g - f' g)\, dx. \tag{1.5}$$

With these conventions the commutation relations in the Virasoro algebra take the form

$$[\xi_k,\, \xi_l] = (k - l)\xi_{k+l} + e\,(k^3 - k)\delta_{k+l,0}. \tag{1.6}$$

The coadjoint representation of the Virasoro algebra is in fact a representation of the quotient algebra $\operatorname{Vect}(S^1)$ and integrates to the group $\operatorname{Diff} S^1$.

Proposition 1.1. *The coadjoint representation of* $\operatorname{Diff} S^1$ *on*

$$\widehat{\operatorname{Vir}}^* \simeq \operatorname{Vect}(S^1)^* \dot{+} \mathbb{R}$$

is given by

$$\operatorname{Ad}^* \phi \cdot (F_u,\, e) = \left((\phi^{-1})^* F_u + e\, S(\phi^{-1}) dx^2,\, e\right), \tag{1.7}$$

where

$$S(\phi) = \frac{\phi'''}{\phi'} - \frac{3}{2} \frac{(\phi'')^2}{(\phi')^2} \tag{1.8}$$

is the Schwarzian derivative of ϕ.

The Schwarz derivative of a function ϕ is identically zero if and only if ϕ is a fractional linear transformation; thus the additional term in (1.7) vanishes precisely on the projective group $PSL(2,\mathbb{R}) \subset \operatorname{Diff} S^1$.

The complicated formula (1.7) is related to the *change of variables in an auxiliary linear problem.* Namely, consider the Schrödinger equation with periodic potential on the line,

$$H_u\psi = -\psi'' + u\psi = 0. \tag{1.9}$$

Let us denote by Ω_α the space of degree α densities on the line. We shall regard the Schrödinger operator as a mapping

$$H_u\colon \Omega_{-1/2} \longrightarrow \Omega_{3/2},$$

i.e., we assume that under the change of variables ϕ the wave function transforms according to the rule

$$\psi \longmapsto \phi^*\psi = (\phi')^{-\frac{1}{2}}\psi \circ \phi, \tag{1.10}$$

and $H_u\psi$ acquires an extra square of derivative. This transformation law is in fact the only one possible in order to preserve the form of the Schrödinger equation.

Lemma 1.2. *Under the change of variables ϕ the potential in the Schrödinger equation goes to $(\phi')^2 u(\phi(x)) - \frac{1}{2}S(\phi)$.*

We conclude that the space $\mathcal{H}$ of Schrödinger operators may be identified with a hyperplane in the dual space of the Virasoro algebra (with central charge $e = -1/2$); accordingly, this space carries a natural Poisson structure, the Lie–Poisson bracket of the Virasoro algebra. This bracket is completely characterized by the Poisson bracket relations for linear functions on $\mathcal{H}$ which, by definition, coincide with the commutation relations in the Virasoro algebra and are explicitly given by formula (1.6).

The space $\mathcal{H}$ is precisely the phase space for the celebrated KdV hierarchy. The Hamiltonians for the KdV equation and its higher analogs are spectral invariants of Schrödinger operator; explicitly they are expressed as integrals of local densities which are polynomial in u and its derivatives. The question which we shall address in this section is the existence of a natural Poisson structure on the space of *wave functions*, i.e., the solutions of the Schrödinger equation. This question is of practical interest in applications to a family of KdV-like equations. Its particular interest lies in the rather non-trivial character of the answer.

For a given u the space $V = V_u$ of solutions of the Schrödinger equation is 2-dimensional and for any two solutions ϕ, ψ their wronskian $W = \phi\psi' - \phi'\psi$ is constant. Any $w \in V$ may be regarded as a non-degenerate quasi-periodic plane curve (the non-degeneracy condition means that $w \wedge w'$ is nowhere zero). There exists a matrix $M \in SL(2,\mathbb{R})$ (the monodromy matrix) such that, writing elements of V as row vectors $w = (\phi, \psi)$,

$$w(x + 2\pi n) = w(x)M^n, \quad n \in \mathbb{Z}.$$

It is useful to pass to the corresponding projective curve with values in $\mathbb{R}P_1 \simeq S^1$.

Theorem 1.3 ([8]).

(i) *Any pair of linearly independent solutions of the Schrödinger equation defines a non-degenerate*[1] *quasi-periodic projective curve* $\gamma : \mathbb{R} \to \mathbb{R}P_1$ *such that* $\gamma(x+2\pi) = \gamma(x)M$.

(ii) *Conversely, any such curve uniquely defines a second-order differential on the line with periodic potential; any two projective curves associated with a given Schrödinger equation are related by a global projective transformation.*

Without restricting the generality we may fix an affine coordinate on $\mathbb{C}P_1$ in such a way that ∞ corresponds to the zeros of the second coordinate ψ of the point on the plane curve; with this choice γ is replaced with the affine curve $x \mapsto \eta(x) = \phi(x)/\psi(x)$. The potential u may be restored from η by the formula

$$u = \tfrac{1}{2} S(\eta), \tag{1.11}$$

where S is the Schwarzian derivative.

We may regard the space of wave functions as a kind of extension of the space of potentials. In fact, there is an interesting class of non-linear evolution equations of KdV type which can be defined on this space. These equations are related to each other by differential substitutions. At the top of this tower of "KdV-like equations" is the "Schwarz–KdV equation"

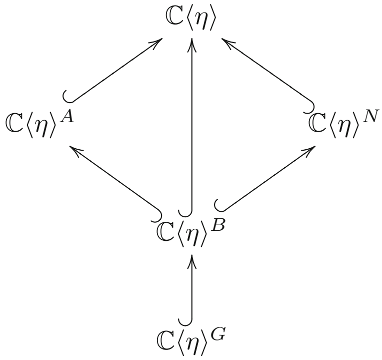
$$\eta_t = S(\eta)\eta_x,$$

This equation has got a peculiar invariance property: if η is its solution, so is $\frac{a\eta+c}{b\eta+d}$ for all $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2)$. The original KdV equation holds for $u = \frac{1}{2}S(\eta)$ which is $PSL(2)$-invariant. Other KdV-like equations are related to differential invariants of various subgroups of $PSL(2)$.

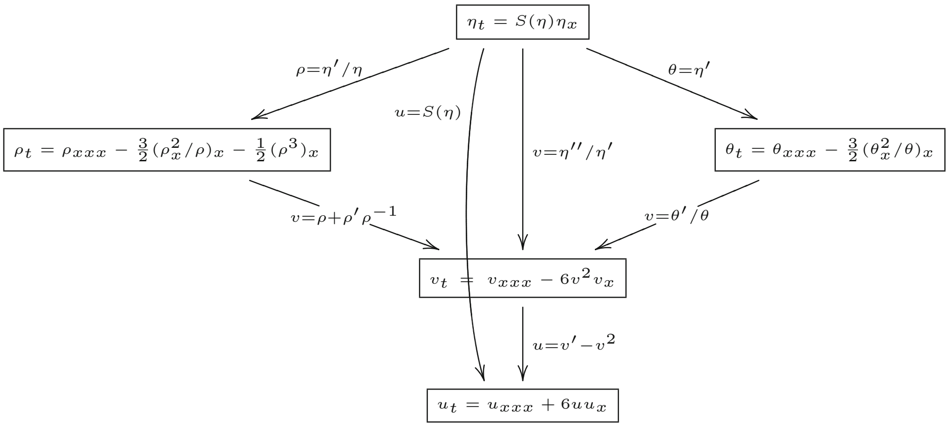
In the spirit of elementary variational calculus we may associate observables for various KdV-like equations with densities which are rational functions of ψ_1, ψ_2 and its derivatives. In the language of differential Galois theory (which was first applied to the study of KdV-like equations by G. Wilson [11]), we define the differential field $\mathcal{F} = \mathbb{C}\langle\psi_1, \psi_2\rangle$ as a free algebra of rational functions in an infinite set of variables $\psi_1, \psi_2, \psi_1', \psi_2', \psi_1'', \psi_2'', \ldots$ with a formal derivation ∂ such that $\partial\psi_i^{(n)} = \psi_i^{(n+1)}$. A *differential automorphism* is an automorphism of $\mathbb{C}\langle\psi_1, \psi_2\rangle$ (as an algebra) which commutes with ∂. All differential automorphisms are induced by linear transformations $(\psi_1, \psi_2) \mapsto (\psi_1, \psi_2) \cdot g,\ g \in GL(2, \mathbb{C})$. Automorphisms which preserve the wronskian $W = \psi_1\psi_2' - \psi_1'\psi_2$ belong to $G = SL(2)$. Observables for the KdV equations form its differential subfield generated by the potential $u = \frac{1}{2}S(\psi_1/\psi_2)$ which may be characterized as the subfield of invariants of the differential Galois group $G = SL(2)$ which acts on wave functions. Different subgroups of G give rise to intermediate subfields of $\mathcal{F}$. Let $Z = \{\pm 1\}$ be the center of G and N, A, $B = AN$ its standard subgroups (nilpotent, split Cartan & Borel).

[1] A parameterized curve $\gamma : R \to \mathbb{R}P_1$ is called nondegenerate if its velocity is nowhere zero.

The subfields of invariants form a tower of extensions



It is easy to see that each subfield of invariants is a free differential algebra generated by a single function (a differential invariant of the corresponding subgroup), $\mathbb{C}\langle\phi,\psi\rangle^Z = \mathbb{C}\langle\eta\rangle$, $\eta = \phi/\psi$, $\mathbb{C}\langle\eta\rangle^A = \mathbb{C}\langle\rho\rangle$, $\rho = \eta'/\eta$, $\mathbb{C}\langle\eta\rangle^N = \mathbb{C}\langle\theta\rangle$, $\theta = \eta'$, $\mathbb{C}\langle\eta\rangle^B = \mathbb{C}\langle v\rangle$, $v = \eta''/\eta'$, $\mathbb{C}\langle\eta\rangle^G = \mathbb{C}\langle u\rangle$, $u = S(\eta)$. The tower of compatible integrable KdV-like equations associated with subgroups of $PSL(2)$ is represented in the diagram of the tower of KdV-like equations below.



As already mentioned, the KdV equation is Hamiltonian with respect to the Poisson bracket associated with the Virasoro algebra. It is natural to expect that there is a Poisson structure on all levels of the extension tower such that all KdV-like equations are Hamiltonian and all differential substitutions are Poisson mappings. However, the extension of the Virasoro–Poisson bracket to upper levels of the tower proves to be non-trivial, since all arrows go in the 'wrong' direction: Poisson brackets cannot be pulled back.[2] The final result looks somewhat fancy, since it incorporates a peculiar symmetry breaking: it is possible to define the

[2]An alternative idea is to use the symplectic form [11], but this option also leads to unexpected obstructions, due to the monodromy of the wave functions.

appropriate Poisson brackets, but they are not G-invariant. More precisely, the differential Galois group $G = SL(2)$ becomes a Poisson Lie group and all relevant Poisson brackets are *Poisson covariant.* We shall recall the main definitions related to Poisson Lie groups in the next section; here we simply quote the final formulae for the Poisson brackets [7] which give the Poisson bracket relations for the *evaluation functionals* on our phase space which assign the value $u(x)$ to the function u (by an abuse of language, which is generally adopted in the physics literature, we do not distinguish these evaluation functionals and their values on u)[3]:

$$\{\eta(x), \eta(y)\} = \eta(x)^2 - \eta(y)^2 - \operatorname{sign}(x-y)\left(\eta(x) - \eta(y)\right)^2 . \tag{1.12}$$

For $\theta = \eta'$ we have

$$\{\theta(x), \theta(y)\} = 2 \operatorname{sign}(x-y)\theta(x)\theta(y).$$

For $v = \frac{1}{2}\eta''/\eta' = \frac{1}{2}\theta'/\theta$ we have

$$\{v(x), v(y)\} = \tfrac{1}{2}\delta'(x-y). \tag{1.13}$$

For $u = \frac{1}{2}v' - v^2 = S(\eta)$ we have

$$\{u(x), u(y)\} = \tfrac{1}{2}\delta'''(x-y) + \delta'(x-y)\left[u(x) + u(y)\right]. \tag{1.14}$$

In these formulas $\operatorname{sign}(x-y)$ is the distribution kernel of the operator $2\partial_x^{-1}$ and $\delta'(x-y)$ is the distribution kernel of ∂_x.

It is easy to see that (1.14) is precisely the Lie–Poisson bracket for the Virasoro algebra (formula (1.14) gives the Poisson bracket relations for the evaluation functionals, while (1.6) gives Poisson brackets for the functionals ξ_k which assign to u its Fourier coefficients).

The potential v, which satisfies the modified KdV equation, is associated with the factorized form of the Schrödinger operator

$$D^2 + u = (D+v)(D-v), \tag{1.15}$$

which implies the relation $u = v' - v^2$ called *Miura transform.*

2. First-order difference operators

The choice of the Poisson structure for the wave functions becomes more transparent if we pass to first-order matrix differential operators and then to difference operators, which are the main subject of the present paper. Let us recall that the space of first-order matrix differential operators on the line carries a natural Poisson structure (sometimes called the *Schwinger–Poisson structure*); indeed, it may be identified with the dual space of the canonical central extension of the associated loop algebra. The passage to the case of second order (or, more generally, nth-order) scalar differential operators is the subject of the so-called *Drinfeld–Sokolov theory* [3]. Its starting point is the well-known equivalence of a

[3]Formula (1.12) was introduced independently in the study of zero modes of the Liouville model by Faddeev and Takhtajan [5]

second-order differential equation to a matrix 2×2 first-order equation. Drinfeld and Sokolov proved that the Poisson–Virasoro algebra associated with second-order differential operators arises from that for matrix 2×2 1st-order differential operators by Hamiltonian reduction. The analogue of the Drinfeld–Sokolov theory for first-order difference operators is more complicated (and less widely known). The study of difference operators starts with the definition of suitable Poisson structures. Poisson structures on the space of first-order matrix difference operators are not quite easy to describe; they are also much less canonical than in the differential case, since the choice of such a structure depends on the choice of a classical r-matrix. For abstract difference operators this choice is largely arbitrary. However, in the special case of higher-order scalar difference operators this choice is in fact very rigid, so that there are no free parameters left. The next question which will be of primary interest for us in the present paper is the extension of the Poisson structure originally defined on the space of difference operators to the space of 'wave functions', i.e., of solutions of difference equations. We shall discuss the generalized Drinfeld–Sokolov theory in Section 5. In the present section we discuss abstract first-order difference equations. We start with the definition of abstract difference operators.

Let $\mathbb{G}$ be a Lie group equipped with an automorphism τ Let $G = \mathbb{G}^\tau$ be the group of "quasi-constants", $G = \{g \in \mathbb{G}; g^\tau = g\}$. We suppose, in addition, that the Lie algebra $\mathbf{g}$ of $\mathbb{G}$ is equipped with a non-degenerate invariant inner product and that the induced automorphism of $\mathbf{g}$ is orthogonal.

Three basic examples are constructed as follows:

1. $\mathbb{G} = G^\Gamma$ is the group of functions on the lattice $\Gamma = \mathbb{Z}$ with values in $G = GL(n)$ and with pointwise multiplication, and τ is the translation, $g_n^\tau = g_{n+1}$.
2. $\mathbb{G}$ is the group of functions on the line with values in $G = GL(n)$, and τ is the shift, $g^\tau(x) = g(x+1)$.
3. $\mathbb{G}$ is the group of functions with values in $G = GL(n)$ which are meromorphic in the punctured complex plane $\mathbb{C}^*$, and τ is the dilation automorphism, $g^\tau(z) = g(qz)$, with $q \in \mathbb{C}$, $|q| \leq 1$.

The "auxiliary linear problem" reads:

$$\psi^\tau \psi^{-1} = L. \tag{2.1}$$

The natural action of $\mathbb{G}$ on itself by left multiplication induces gauge transformations for L:

$$g \colon \psi \mapsto g \cdot \psi, \quad L \mapsto g^\tau L g^{-1}. \tag{2.2}$$

The quasi-constants act by right multiplications, $\psi \mapsto \psi h$ and leave L invariant. We shall call 'potentials' L in the auxiliary linear problem (2.1) *difference connections*. Solutions of the auxiliary linear problem will be called *wave functions*.

In the case of first-order differential equations the space of potentials carries a natural Poisson structure (the Lie–Poisson bracket of the central extension of the loop algebra) and gauge transformations are Hamiltonian. The case of first-order difference operators is more complicated: now the set of potentials is a group manifold and the gauge action is *not* Hamiltonian. Rather, the gauge group must be treated as a Poisson group and the gauge action becomes a Poisson action. The choice of the Poisson bracket on the gauge group represents an extra structure; its choice is further restricted (and becomes partially or totally rigid) when we pass from the first-order matrix difference operators to nth-order scalar difference operators. Let us recall that a Poisson group G is a Lie group equipped with a multiplicative Poisson structure. i.e., a Poisson bracket such that the multiplication

$$m : G \times G \to G$$

is a Poisson mapping. An action $G \times M \to \mathcal{M}$ of G on a smooth Poisson manifold $\mathcal{M}$ is called a Poisson action if the mapping

$$G \times \mathcal{M} \to \mathcal{M} : (g, x) \mapsto g \cdot x$$

is Poisson. In both definitions it is assumed that the Cartesian products $G \times G$, $G \times \mathcal{M}$ are equipped with the product Poisson structure. In typical cases, the Poisson structure on a Lie group G is given by the *Sklyanin bracket* associated with a classical r-matrix. We shall assume that the Lie algebra $\mathfrak{g}$ of G is equipped with a non-degenerate invariant inner product; in this case we may regard a classical r-matrix as a skew-symmetric linear operator $r \in \operatorname{End}\mathfrak{g}$; the Sklyanin bracket of smooth functions $\phi, \psi \in C^\infty(G)$ is given by

$$\{\phi, \psi\} = \tfrac{1}{2}\langle r(\nabla_\phi),\, \nabla_\psi\rangle - \tfrac{1}{2}\langle r(\nabla'_\phi),\, \nabla'_\psi\rangle, \tag{2.3}$$

where $\nabla_\phi, \nabla'_\phi$ are left and right gradients of ϕ defined by

$$\langle\nabla_\phi(x),\, \xi\rangle = \left(\tfrac{d}{dt}\right)_{t=0} \phi(e^{t\xi}x), \quad \langle\nabla'_\phi(x),\, \xi\rangle = \left(\tfrac{d}{dt}\right)_{t=0} \phi(xe^{t\xi}).$$

The Jacobi identity for the Sklyanin bracket imposes strong restrictions on the choice of r; a sufficient condition is the famous classical Yang–Baxter identity

$$[rX, rY] - r([rX, Y] + [X, rY]) = -[X, Y], \quad X, Y \in \mathfrak{g}. \tag{2.4}$$

Example. Let P_+, P_-, P_0 be the complementary projection operators associated with the decomposition of matrices into upper triangular, lower triangular and diagonal parts. Then $r = P_+ - P_- \in \operatorname{End} sl(n)$ satisfies (2.4) and is skew-symmetric with respect to the inner product $\langle X,\, Y\rangle = \operatorname{tr} XY$. This r-matrix defines the so-called standard Poisson structure on $G = SL(n)$. In the special case $n = 2$ one can show [7] that the projective action

$$\eta \mapsto \frac{a\eta + c}{b\eta + d}$$

of $SL(2)$ equipped with the standard Poisson structure on the space of wave functions $\eta = \psi_1/\psi_2$ with the bracket (1.12) is a Poisson group action.

Let us now fix an r-matrix R defining a multiplicative Poisson bracket on the gauge group G acting by gauge transformations (2.2) on the space of difference connections.

Theorem 2.1 ([9]). *There exists a unique Poisson structure on the space* $\mathcal{C}$ *of difference connections which makes the gauge action* (2.2) *a Poisson action. Explicitly, this structure is given by*

$$\begin{aligned}\{\varphi,\psi\}_\tau = \tfrac{1}{2}\left\langle R\left(\nabla\varphi\right),\nabla\psi\right\rangle + \tfrac{1}{2}\left\langle R\left(\nabla'\varphi\right),\nabla'\psi\right\rangle \\ - \left\langle \tau\circ R_+\left(\nabla'\varphi\right),Y\right\rangle - \left\langle R_-\circ\tau^{-1}\left(\nabla\varphi\right),\nabla'\psi\right\rangle,\end{aligned} \tag{2.5}$$

where $R_\pm = \frac{1}{2}(R\pm Id)$.

The key feature of the bracket (2.5) is the presence of two terms of 'mixed chirality' (left gradients coupled to right gradients) which also contain the automorphism τ and its inverse.

Our main goal will be now to extend the Poisson structure to the space of wave functions in a way which is consistent with the bracket (2.5) on the space of difference connections. An appropriate class of Poisson structures which are compatible with the expected covariance properties are the so-called *affine Poisson structures*; they are defined by the formula

$$\{f_1,f_2\} = \tfrac{1}{2}\langle l(\nabla_{f_1}),\nabla_{f_2}\rangle + \tfrac{1}{2}\langle r(\nabla'_{f_1}),\nabla'_{f_2}\rangle, \tag{2.6}$$

where l and r are two (a priori, different) r-matrices. We denote by $G^{(l,r)}$ the Lie group G equipped with this Poisson structure. Let G^l, G^r be two copies of G equipped with the Sklyanin bracket associated with l and r, respectively. It is easy to check that the action

$$G^l\times G^{(l,r)}\times G^r \to G^{(l,r)}\colon (g,\psi,h)\mapsto g\psi h$$

is Poisson; in other words, $G^{(l,r)}$ may be regarded as a principle homogeneous space for the left action of G^l and the right action of G^r. Remarkably, in the category of Poisson spaces principle homogeneous spaces for left and right actions are not unique: one can vary the choice of Poisson structure.

At first glance, the choice of left and right r-matrices in (2.6) is totally free. One condition which links them comes from the Yang–Baxter equation: to ensure the Jacobi identity for the bracket (2.6) both l and r must satisfy the Yang–Baxter identity (2.4) *with the same normalization of the r. h. s.*[4] We shall see now that the choice of l and r which is compatible with the Poisson structure (2.5) on the space of difference connections is very rigid; both l and r are fixed completely, although their rôles are very different.

Let us write

$$\{f_1,f_2\}^l = \tfrac{1}{2}\langle l(\nabla_{f_1}),\nabla_{f_2}\rangle,\quad \{f_1,f_2\}^r = \tfrac{1}{2}\langle r(\nabla'_{f_1}),\nabla'_{f_2}\rangle. \tag{2.7}$$

(Note that separately these brackets do not necessarily satisfy the Jacobi identity.) Let us study first the 'left' bracket. Let us choose $F_1,F_2\in C^\infty(G)$; let $f_1(\psi)=F_1(\psi^\tau\psi^{-1})$, $f_2(\psi)=F_2(\psi^\tau\psi^{-1})$. We denote by X_1,X_2,X'_1,X'_2 left and right gradients of F_1,F_2, respectively.

[4]One can replace $-[X,Y]$ in the r. h. s. of (2.4) by its multiple, e.g., by rescaling r, which does not affect the Jacobi identity for (2.3); for the bracket (2.6) such rescaling must be done simultaneously for l and r.

Proposition 2.1. *Left bracket on G admits reduction to the space $\mathcal{C}$ of difference connections. The quotient bracket is given by*

$$\{F_1, F_2\}^l(\psi^\tau \psi^{-1}) := \{f_1, f_2\}^l(\psi) \tag{2.8}$$
$$= \tfrac{1}{2}\langle l(X_1), X_2\rangle + \tfrac{1}{2}\langle l(X_1'), X_2'\rangle - \tfrac{1}{2}\langle l \circ \tau^{-1}(X_1), X_2'\rangle - \tfrac{1}{2}\langle \tau \circ l(X_1'), X_2\rangle. \tag{2.9}$$

Formula (2.8) resembles (2.5) (with $l = R$), but differs from it in some crucial terms: in fact, it contains l everywhere rather than $l_\pm$, as in (2.5). This difference is very important; as expected, when l satisfies (2.4), the bracket (2.8) does not satisfy the Jacobi identity. The remedy should come from the contribution of the right bracket, however, at first glance this seems impossible: indeed, since right gradients are *not* right-invariant, the right bracket $\{f_1, f_2\}^r$ depends in principle on ψ and not on the combination $\psi^\tau \psi^{-1}$. Explicitly, we have:

Proposition 2.2. *The right bracket of f_1, f_2 is given by*

$$\{f_1, f_2\}^r(\psi) = \tfrac{1}{2}\langle (1-\tau) r (1-\tau^{-1}) \mathrm{Ad}\psi^{-1} X_1', \mathrm{Ad}(\psi^\tau)^{-1} X_2\rangle. \tag{2.10}$$

The proof of (2.10) is based on the following simple formula for the right gradient of f.

Lemma 2.2. *The right gradient of $f : \psi \mapsto F(\psi^\tau \psi^{-1})$ is given by*

$$\nabla' f(\psi) = (\tau^{-1} - 1)\mathrm{Ad}\psi^{-1} X_F' = (\tau^{-1} - 1)\mathrm{Ad}(\psi^\tau)^{-1} X_F. \tag{2.11}$$

Theorem 2.3. *The mapping $\psi \mapsto \psi^\tau \psi^{-1}$ is Poisson if and only if*

$$(1-\tau) r (1-\tau^{-1}) = \tau - \tau^{-1}. \tag{2.12}$$

We shall assume that r commutes with τ; in that case (2.12) amounts to

$$2r - \tau \cdot r - r \cdot \tau^{-1} = \tau - \tau^{-1}. \tag{2.13}$$

The Lie algebra $\mathfrak{g}$ admits an orthogonal decomposition $\mathfrak{g} = \mathrm{Im}(1-\tau) \oplus \mathfrak{g}_0$, where $\mathfrak{g}_0 = \mathrm{Ker}\,(1-\tau)$ is the subalgebra of quasi-constants. It is clear that on $\mathfrak{g}_0$ condition (2.12) is void, so it is important to define r on the subspace $\mathfrak{g}' = \mathrm{Im}(1-\tau) \subset \mathfrak{g}$.

Proposition 2.3. *The restriction of r to $\mathfrak{g}'$ is the Cayley transform,*

$$r = (1+\tau)(1-\tau)^{-1}. \tag{2.14}$$

Proof. It is clear that if $r = (1+\tau)(1-\tau)^{-1}$, condition (2.12) is satisfied. Since $1-\tau$ is invertible on $\mathfrak{g}'$, the converse is also true.

We now come up to the check of condition (2.12). The mapping $\psi \mapsto \psi^\tau \psi^{-1}$ is Poisson if and only $\mathrm{Ad}\psi$, $\mathrm{Ad}(\psi^\tau)$ disappear in the r. h. s. of (2.10). Let us assume that (2.12) holds. Since $\mathrm{Ad}\psi^{-1} X_i' = \mathrm{Ad}(\psi^\tau)^{-1} X_i$, $i = 1, 2$, we get

$$\begin{aligned} 2\{f_1, f_2\}^r(\psi) &= \langle (\tau - \tau^{-1}) \cdot \mathrm{Ad}\psi^{-1} X_1',\, \mathrm{Ad}(\psi^\tau)^{-1} X_2\rangle \\ &= \langle \tau \cdot \mathrm{Ad}\ \psi^{-1} \cdot X_1', \mathrm{Ad}(\psi^\tau)^{-1} \cdot X_2\rangle - \langle \tau^{-1} \cdot \mathrm{Ad}(\psi^\tau)^{-1} X_1,\, \mathrm{Ad}\psi^{-1} X_2'\rangle \\ &= \langle \mathrm{Ad}(\psi^\tau)^{-1} \cdot \tau \cdot X_1',\, \mathrm{Ad}(\psi^\tau)^{-1} \cdot X_2'\rangle - \langle \mathrm{Ad}\psi^{-1} \tau^{-1} \cdot X_1,\, \mathrm{Ad}\psi^{-1}\rangle \\ &= \langle \tau \cdot X_1',\, X_2\rangle - \langle X_1, \tau \cdot X_2'\rangle; \end{aligned} \tag{2.15}$$

At the final stage, to get rid of the operators $\mathrm{Ad}(\psi^\tau)^{-1}, \mathrm{Ad}\psi^{-1}$ we used the invariance of the inner product. It is clear that this argument is reversible, and hence condition (2.12) is necessary and sufficient.

Let us note that (2.15) provides precisely the two terms which were missing in the left bracket (2.8) in order to convert it into the correct Poisson bracket (2.5) on the space of difference connections.

So far we did not discuss the Jacobi identity for the right bracket. Note that the condition imposed on r amounts to a linear equation in the vector space $\mathrm{End}\,\mathfrak{g}$. This condition is already quite rigid, so there is not too much space for maneuver.

Proposition 2.4. (i) *Operator* (2.14) *on* $\mathfrak{g}'$ *satisfies the classical Yang–Baxter identity* (2.4). (ii) *Put*

$$[X,Y]_r = \tfrac{1}{2}\left([rX,Y] + [x,rY]\right), \quad X,Y \in \mathfrak{g}'. \tag{2.16}$$

Then $[X,Y]_r \in \mathfrak{g}'$ *and formula* (2.16) *defines the structure of a Lie algebra on* $\mathfrak{g}'$.

Proof. Suppose that $X,Y \in \mathrm{Im}(1-\tau)$, $X=(1-\tau)\xi, Y=(1-\tau)\eta$. Then

$$\begin{aligned}2[X,Y]_r &= [(1+\tau)\xi,(1-\tau)\eta] + [(1-\tau)\xi,(1+\tau)\eta]\\ &= [\xi,\eta]+[\tau\xi,\eta]-[\xi,\tau\eta]-[\tau\xi,\tau\eta]+[\xi,\eta]-[\tau\xi,\eta]+[\xi,\tau\eta]-[\tau\xi,\tau\eta]\\ &= 2(1-\tau)[\xi,\eta].\end{aligned}$$

Hence $r\left([rX,Y]+[x,rY]\right) = 2(1+\tau)[\xi,\eta]$. On the other hand,

$$\begin{aligned}[rX,rY]+[X,Y] &= [(1+\tau)\xi,(1+\tau)\eta] + [(1-\tau)\xi,(1-\tau)\eta]\\ &= (1+\tau)[\xi,\eta]+[\xi,\tau\eta]+[\tau\xi,\eta]+(1+\tau)[\xi,\eta]-[\xi,\tau\eta]-[\tau\xi,\eta]\\ &= 2(1+\tau)[\xi,\eta].\end{aligned}$$ □

3. Exchange r-matrices and singular integrals

Let us start with the following simple example. In the lattice case (Example 1 on page 347) the functional equation (2.13) may be solved explicitly. We identify linear operators acting in the Lie algebra $\mathfrak{g}^\Gamma$ with their kernels, i.e., elements of $\mathfrak{g}^\Gamma \otimes \mathfrak{g}^\Gamma$ with the help of the invariant inner product in $\mathfrak{g}$. We denote $t \in \mathfrak{g}\otimes\mathfrak{g}$ the Casimir element (the kernel of the identity operator in $\mathfrak{g}$). Set

$$r(n,m) = \tfrac{1}{2}t\epsilon(n-m) + r_0, \tag{3.1}$$

where

$$\epsilon(n) = \mathrm{sign}\, n = \begin{cases} 1, & n>0,\\ 0, & n=0,\\ -1, & n<0.\end{cases}$$

and $r_0 \in \mathfrak{g}\otimes\mathfrak{g}$ is a constant r-matrix independent of n,m. The automorphism τ acts on $r(n,m)$ by translation of n, $r^\tau(n,m) = r(n+1-m)$; clearly, we have $\epsilon(n+1-m)-\epsilon(n-m) = \delta(n+1-m)$, $\epsilon(n-1-m)-\epsilon(n-m) = \delta(n-1-m)$, where $\delta(n-m)$ is the Kronecker delta. The constant r-matrix r_0 is chosen so as to satisfy the classical Yang–Baxter identity. Instead of r_0 one can introduce ${}^0r_\pm = \frac{1}{2}(r_0 \pm t)$. One has:

$$ {}^0r_+ + {}^0r_- = r_0, \quad {}^0r_+ - {}^0r_- = t, $$

and the r-matrix (3.1) may be rewritten as

$$ r(n,m) = {}^0r_+\theta(n-m) + {}^0r_-\theta(m-n), \tag{3.2} $$

where

$$ \theta(n) = \begin{cases} 1, & n \geq 0, \\ 0, & n < 0. \end{cases} $$

Poisson algebra with this r is usually called *lattice exchange algebra*.

This example is in fact quite similar to the continuous case. Let $\mathcal{C}(\mathfrak{g})$ be the space of connections on the line with values in a Lie algebra $\mathfrak{g}$ equipped with a non-degenerate invariant inner product. $\mathcal{C}(\mathfrak{g})$ may be identified with (a hyperplane in) the dual space of the central extension of the current algebra $C^\infty(\mathbb{R};\mathfrak{g})$ and hence carries a natural Poisson structure (the so-called *Schwinger–Poisson bracket*). The current group $C^\infty(\mathbb{R};G)$ acts on $\mathcal{C}(\mathfrak{g})$ by gauge transformations,

$$ g : L \mapsto \mathrm{Ad}g \cdot L + \partial_x g \cdot g^{-1}. \tag{3.3} $$

Let $W(\mathfrak{g})$ be the space of wave functions, i.e., of G-valued solutions of the differential equation

$$ \partial_x \psi = L\psi. $$

Gauge transformations act on $W(\mathfrak{g})$ by left translations, $g : \psi \mapsto g \cdot \psi$. We want to equip $W(\mathfrak{g})$ with a Poisson structure such that the natural mapping

$$ W(\mathfrak{g}) \to \mathcal{C}(\mathfrak{g}) : \psi \mapsto \partial_x\psi \cdot \psi^{-1} $$

is Poisson. Since the gauge action (3.3) is Hamiltonian with respect to the natural Poisson structure on $\mathcal{C}(\mathfrak{g})$, the Poisson structure on $W(\mathfrak{g})$ should be left-invariant. In complete analogy with (3.1) let choose this Poisson structure in the following form:

$$ \{\psi_1(x),\, \psi_2(y)\} = \psi_1(x)\psi_2(y)r_{12}(x-y), \tag{3.4} $$

where

$$ r(x-y) = r_+\theta(x-y) + r_-\theta(y-x) \tag{3.5} $$

and $r_+, r_- \in \mathfrak{g}\otimes\mathfrak{g}$ have the same meaning as above. (Here and in the sequel we freely use the tensor notation for the kernels of our Poisson operators.) With this choice we get the Poisson brackets for $L(x) = \psi'(x)\psi^{-1}(x)$:

$$ \begin{aligned} \{L_1(x), L_2(y)\} = {} & \frac{\partial^2}{\partial x \partial y}\left(\psi_1\psi_2 r_{12}\right)\psi_1^{-1}\psi_2^{-1} - L_2\frac{\partial}{\partial x}(\psi_1\psi_2 r_{12})\psi_1^{-1}\psi_2^{-1} \\ & - L_1\frac{\partial}{\partial y}(\psi_1\psi_2 r_{12})\psi_1^{-1}\psi_2^{-1} + L_1L_2\psi_1\psi_2 r_{12}\psi_1^{-1}\psi_2^{-1}. \end{aligned} \tag{3.6} $$

Taking into account that

$$ r'_x(x-y) = -r'_y(x-y) = t\delta(x-y),\; r''_{xy}(x-y) = -t\delta'(x-y), \tag{3.7} $$

we get, after several remarkable cancellations based on the identity

$$ {}^0r_+ - {}^0r_- = t $$

and on the standard properties of the Casimir element t:

$$ \{L_1(x), L_2(y)\} = [t,\, L_1(x) - L_2(y)]\delta(x-y) - t\delta'(x-y), \tag{3.8} $$

i.e., the correct Schwinger–Poisson structure on $\mathcal{C}(\mathfrak{g})$. The bracket (3.4) is called the (continuous) exchange bracket. The kernel (3.2) may be written equivalently as $r(x-y) = \frac{1}{2}t\epsilon(x-y) + r_0$; of course, $\epsilon(x-y)$ is the distribution kernel of ∂_x^{-1}. In other words, r is an extension of the partially defined operator ∂_x^{-1}.

It is instructive to rewrite the r-operator (3.2) in the Fourier representation. Setting as usual

$$ \hat{X}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(x)e^{-ikx}\,dx, $$

we get, using the standard Fourier transform of step function,

$$ rX(x) = r_0(\hat{X}(0)) + \text{v.p.}\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\hat{X}(k)}{ik}e^{ikx}\,dk, \tag{3.9} $$

or, equivalently,

$$ rX(x) = {}^0r_+\left(\frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{\hat{X}(k)}{k+i0}e^{ikx}\,dk\right) + {}^0r_-\left(\frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{\hat{X}(k)}{k-i0}e^{ikx}\,dk\right), \tag{3.10} $$

where we assume that the finite-dimensional r-matrices ${}^0r_\pm$ are acting pointwise on the values of the integral. In other words, the exchange r-matrix is a regularized singular integral operator; the finite-dimensional r-matrix is necessary to correctly regularize the zero Fourier modes.

When τ is a translation operator, $g^\tau(x) = g(x+1)$, the Cayley transform $r = (1+\tau)(1-\tau)^{-1}$ is again a singular integral operator which is formally given by

$$ (rf)(x) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\operatorname{cotan}(k/2)\hat{f}(k)e^{ikx}\,dk. \tag{3.11} $$

For $f \in C_0^\infty(\mathbb{R};\mathfrak{g})$ we set

$$ F(x) = \sum_{n=-\infty}^{\infty} f(x+n); $$

clearly, F is 1-periodic and hence lies in the kernel of $1-\tau$. In order to regularize (3.11) we use the standard decomposition of $\operatorname{cotan}(k/2)$ into simple fractions and the Poisson formula

$$ F(x) = \sum_{n=-\infty}^{\infty}\hat{f}(2\pi n)e^{2\pi inx}. $$

The proper way to regularize (3.11) is given by the formula

$$\begin{aligned}(rf)(x) = {}^0r_+\left(\frac{1}{2\pi i}\int_{-\infty}^{\infty}\hat{X}(k)\,\mathrm{cotan}(k+i0)e^{ikx}\,dk\right)\\ +{}^0r_-\left(\frac{1}{2\pi i}\int_{-\infty}^{\infty}\hat{X}(k)\,\mathrm{cotan}(k-i0)e^{ikx}\,dk\right),\end{aligned} \tag{3.12}$$

in complete analogy with (3.10). Using the Sokhotsky and Poisson formulas. we can rewrite (3.12) in equivalent form:

$$(rf)(x) = r_0(F(x)) + \text{v.p.}\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{cotan}(k/2)\hat{f}(k)e^{ikx}\,dk, \tag{3.13}$$

where r_0 is acting pointwise in the subspace of quasiconstants.

In the q-difference case, when $g^\tau(z) = g(qz)$, the Cayley transform $r = (1+\tau)(1-\tau)^{-1}$ is completely characterized by its distribution kernel given by the formal series

$$r(z,z') = r\delta(z/z'), \quad \text{where} \quad \delta(z/z') = \sum_{n=-\infty}^{\infty}(z/z')^n$$

is the Dirac delta function. Set $z/z' = t$; we get

$$\begin{aligned}r(z,z') &= \sum_{n=-\infty,n\neq 0}^{\infty}\frac{1+q^n}{1-q^n}t^n = \sum_{n=1}^{\infty}\left(\frac{1+q^n}{1-q^n}t^n + \frac{1+q^{-n}}{1-q^{-n}}t^{-n}\right)\\ &= \sum_{n=1}^{\infty}(t^n - t^{-n}) + 2\sum_{n=1}^{\infty}\frac{q^n}{1-q^n}(t^n - t^{-n})\\ &= \frac{t+1}{t-1} + 2\sum_{n=1}^{\infty}\frac{q^n}{1-q^n}(t^n - t^{-n}).\end{aligned}$$

Put $z = e^{ix}, z' = e^{ix'}$; we get a Fourier series expansion of $r(z,z')$ on the unit circle,

$$r(z,z') = \frac{1}{i}\,\mathrm{cotan}(x - x'/2) + 4i\sum_{n=1}^{\infty}\frac{q^n}{1-q^n}\sin n(x-x'). \tag{3.14}$$

It is easy to see that the r.h.s. is essentially the logarithmic derivative of Jacobi's theta function θ_1 (the difference is due to the fact that in the standard definition the (quasi)-periods of θ_1 are 2 and $q^2 = e^{2\pi i\tau}$, while in our case they are 2π and q). Note also that $r(z,z')$ is not quite 'elliptic', since the logarithmic derivative of θ_1 is invariant with respect to translations by $2\pi i\tau$ only up to an additive constant. If we regard (3.14) as the kernel of a singular integral operator on the unit circle, its regularization may be achieved in the same way as above (now the only pole of the kernel which lies on the unit circle is $x = 0$ (other poles, which coincide with the roots of θ_1 are associated with the points of the multiplicative lattice q^n).

Remark. As in rational and trigonometric case, the regularization introduces just a finite-dimensional r-matrix r_0. This simple prescription is in fact perfectly in line with the difference Galois theory: according to a theorem of P. Etingof [4], the difference Galois group associated with the difference equation $\psi(qz) = L(z)\psi(z)$ is a finite-dimensional; it consists of constant matrices (and hence is a subgroup of the infinite-dimensional group of quasi-constants) and is generated by the *values* of the associated Birkhoff transition matrix. This finite-dimensional group naturally inherits the Poisson structure induced by the finite-dimensional r-matrix r_0. Generically, this group coincides with the full special linear group $SL(n)$.

4. Poisson brackets for the monodromy matrix

Let us return to the lattice case and assume that the 'potential' in the auxiliary linear problem is N-periodic, $L_{n+N} = L_n$. We also assume that the gauge group acting by left translations to consists of N-periodic functions on the lattice. In that case the wave functions are of course only quasi-periodic, which leads to the standard notion of monodromy. There are two definitions of the monodromy matrix in this setting:

- $M = \psi_N \psi_0^{-1}$, or
- $\tilde{M} = \psi_0^{-1}\psi_N$.

Of course, M and $\tilde{M}$ are conjugate, but their transformation properties are very different: M is invariant with respect to the right action of the group of quasi-constants (generically, in the lattice case it coincides with the difference Galois group), while $\tilde{M}$ is gauge invariant. Hence $\tilde{M}$ is adapted to reduction over the subgroups of the gauge group (this reduction will be the main subject of sections 5, 6). On the other hand, we have obviously

$$M = \overset{\curvearrowleft}{\prod_n} L_n,$$

and hence the Poisson bracket relations for M are easily computable. Explicitly we have:

Theorem 4.1. *Let us assume that the space of difference connections on* $\mathfrak{g}^\Gamma$ *carries the Poisson bracket* (2.5) (*with* $R = l$). (i) *Equip* G *with the Poisson bracket*

$$\{f_1, f_2\}^l(M) = \langle l(X_1), X_2\rangle + \langle l(X_1'), X_2'\rangle - \langle l_+(X_1), X_2'\rangle - \langle l_-(X_1'), X_2\rangle,$$

where as usual X_1, X_2, X_1', X_2' *stand for left and right gradients of* f_1, f_2. *The mapping*

$$\mathbb{G} \to G : (L_n) \mapsto M = \overset{\curvearrowleft}{\prod} L_n$$

is Poisson. (ii) *Gauge action of the gauge group* (*equipped with the Sklyanin bracket associated with* l) *on the monodromy by conjugation is Poisson.*

By contrast, Poisson bracket relations for $\tilde{M}$ depend mainly on r. Using the explicit form of r described in (3.2) it is easy to prove the following assertion.

Theorem 4.2.

(i) *The mapping* $\tilde{M} : \mathbb{G} \to G : \psi \mapsto \psi_0^{-1}\psi_N$ *is Poisson with respect to the dual Poisson bracket on* G *associated with* r_0,

$$\begin{aligned}\{f_1, f_2\}^{r_0}(\tilde{M}) = \tfrac{1}{2}\langle r_0(X_1), X_2\rangle + \tfrac{1}{2}\langle r_0(X_1'), X_2'\rangle \\ - \langle r_+^0(X_1), X_2'\rangle - \langle r_-^0(X_1'), X_2\rangle.\end{aligned} \tag{4.1}$$

(ii) *We equip the group of quasi-constants* (*alias, the difference Galois group*) *with the Sklyanin bracket associated with* r_0. *The bracket* (4.1) *is Poisson covariant with respect to the action of the difference Galois group by conjugation.*

There is a simple way to reconcile the Poisson structures associated with two alternative definitions of the monodromy matrix: just choose $r_0 = l$. This simple choice eliminates the residual freedom in the definition of r (once l is chosen) and is almost satisfactory (we shall see that in applications to difference Drinfeld–Sokolov reduction it has to be slightly modified).

5. Difference Drinfeld–Sokolov theory

So far the choice of the 'left' r-matrix remains arbitrary. There exists an important special case in which this choice is totally rigid: this is the theory of higher-order difference equations. Before passing to the study of difference operators we shall briefly recall the treatment of the differential case. Recall that an nth-order differential equation

$$D^n\phi + u_{n-1}D^{n-1}\phi + u_{n-2}D^{n-2}\phi + \cdots + u_1 D\phi + u_0\phi = 0 \tag{5.1}$$

may be rewritten as a first-order matrix equation

$$D\psi + L\psi = 0, \tag{5.2}$$

where the column vector $\psi = (\phi, \phi', \ldots, \phi^{(n-1)})^t$ and L is the companion matrix,

$$L = \begin{pmatrix} 0 & -1 & 0 & \ldots & \\ 0 & 0 & -1 & 0 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \\ 0 & 0 & \ldots & 0 & -1 \\ u_0 & u_1 & \ldots & u_{n-2} & u_{n-1} \end{pmatrix}. \tag{5.3}$$

Without restriction of generality we may assume that $\operatorname{tr} L = u_{n-1} = 0$.

The assignment $\phi \mapsto (\phi, \phi', \ldots, \phi^{(n-1)})^t$ is in fact not canonical: we may replace $\phi^{(k)}$ by its linear combination with all derivatives of smaller order. This introduces the natural gauge group consisting of lower triangular (unipotent) matrices. The gauge group acts freely on the set of potentials of the form

$$L = \begin{pmatrix} * & -1 & 0 & \ldots & \\ * & * & -1 & 0 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \\ * & * & \ldots & * & -1 \\ * & * & \ldots & * & * \end{pmatrix}. \tag{5.4}$$

The set of companion matrices of the form (5.3) gives a (non-canonical) global cross-section of gauge action, hence the space of scalar nth-order differential operators may be identified with the quotient space over the action of gauge group; the set of companion matrices (5.3) provides a model for this quotient space. As noticed by Drinfeld and Sokolov, this description of higher-order differential operator matches perfectly well with the Hamiltonian reduction. Let $\mathfrak{g}$ be the Lie algebra $sl(n)$ equipped with the standard invariant inner product, $\langle X, Y\rangle = \operatorname{tr} XY$, and $\mathfrak{n}_\pm$ its nilpotent subalgebras of upper (lower) triangular matrices. Note that $\mathfrak{n}_+$ and $\mathfrak{n}_-$ are set in duality by this inner product, so that $\mathfrak{n}_+ \simeq \mathfrak{n}_-^*$. Recall that the space of first-order differential operators with potential $L \in C^\infty(\mathbb{R}, \mathfrak{g})$ carries a natural Poisson bracket, the Lie–Poisson bracket of the central extension of the loop algebra $C^\infty(\mathbb{R}, \mathfrak{g})$ of the Lie algebra $\mathfrak{g}$, alias the Schwinger–Poisson bracket which we have already briefly discussed in section 3 (formula (3.8)). The action of the unipotent gauge group $C^\infty(\mathbb{R}, N_-)$ is Hamiltonian and admits a natural moment map $\mu : C^\infty(\mathbb{R}, \mathfrak{g}) \to C^\infty(\mathbb{R}, \mathfrak{n}_-)^* \simeq C^\infty(\mathbb{R}, \mathfrak{n}_+)$. The set U_f of constrained potentials of the form (5.4) is the level set of the moment map μ, $U_f = \mu^{-1}(f)$, where

$$f = \begin{pmatrix} 0 & -1 & 0 & \dots & 0 \\ 0 & 0 & -1 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & -1 \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix} \tag{5.5}$$

is the standard nilpotent matrix (a principal nilpotent element of $\mathfrak{g}$) regarded as an element of the dual space $C^\infty(\mathbb{R}, \mathfrak{n}_-)^* \simeq C^\infty(\mathbb{R}, \mathfrak{n}_+)$. An important point is that $f \in \mathfrak{n}_+ \simeq \mathfrak{n}_-^*$ defines a character of the Lie algebra $\mathfrak{n}_-$; this implies the important technical property of the set of constraints which define the level surface (5.4): these constraints are *first class* according to Dirac, i.e., their Poisson brackets vanish on the level surface. As a result, the level surface U_f is invariant with respect to the action of the entire gauge group $C^\infty(\mathbb{R}, N_-)$ and its quotient is isomorphic to the Hamiltonian reduced space (over the point f). For $n = 2$ the quotient Poisson structure is precisely the Poisson–Virasoro algebra, for $n > 2$ this is the so-called classical W-algebra.

After this brief reminder, we pass to the study of the difference case. In complete analogy with (5.3), an nth-order difference equation

$$\tau^n \phi + u_{n-1}\tau^{n-1} \cdot \phi + u_{n-2}\tau^{n-2} \cdot \phi + \dots + u_1 \tau \cdot \phi + \phi = 0 \tag{5.6}$$

may be rewritten as a matrix first-order equation

$$\tau \cdot \psi + L\psi = 0,$$

where $\psi = (\phi, \phi^\tau, \dots, \phi^{\tau^{n-1}})^t$ and L is a companion matrix, as in (5.3). This time, however, L satisfies the normalization condition $\det L = 1$ (rather than $\operatorname{tr} L = 0$, as in the differential case); hence we have $u_0 = 1$. This marks an important difference with the differential case: the potential L admits a canonical factorization $L = Us$,

where

$$s = \begin{pmatrix} 0 & -1 & 0 & \dots & 0 \\ 0 & 0 & -1 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & -1 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix} \tag{5.7}$$

and U is a unipotent matrix,

$$U = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 1 & 0 \\ u_1 & u_2 & \dots & u_{n-1} & 1 \end{pmatrix}. \tag{5.8}$$

Note that unlike f in (5.5) s is an invertible semi-simple matrix; this is in fact a *Coxeter element*, i.e., conjugation by s induces an outer automorphism of the Cartan subgroup $H \subset SL(n)$ consisting of diagonal matrices which is the product of all simple reflections. The abelian unipotent group $N_0 \subset N_-$ of one-row matrices of the form (5.8) is generated by the root vectors associated with the negative roots which change sign under the Coxeter transformation.

The natural gauge group $\mathbf{N}_-$ associated with nth-order difference operators consists again of functions with values in the unipotent group $N_- \subset SL(n)$. The constrained set of potentials which are gauge equivalent to $L = Us$ are again given by (5.4). One can prove that the gauge group acts freely on this set and that matrices of the form $L = Us$, $U \in N_0$, provide a global cross-section of the gauge action.

The Poisson theory of nth-order difference equations starts with the choice of an appropriate classical r-matrix. The Poisson structure on the space of difference $SL(n)$-connections is given by (2.5) and is Poisson covariant with respect to the action of the 'big' gauge group equipped with the corresponding Sklyanin bracket. The counterpart of Hamiltonian reduction in our case is *Poisson reduction.* A subgroup of the gauge group is called *admissible* if its invariants in the algebra of functions form a Lie subalgebra with respect to the chosen Poisson bracket (i.e., the Poisson bracket of two invariant functions is again an invariant function). The r-matrix should be chosen in such a way that the 'little' gauge group consisting of functions with values in the unipotent group $N_- \subset SL(n)$ is admissible. Let

$$\mathfrak{g} = \mathfrak{n}_- \dot{+} \mathfrak{h} \dot{+} \mathfrak{n}_+ \tag{5.9}$$

be the standard triangular decomposition in the Lie algebra $\mathfrak{g} = sl(n)$. We denote by $\mathbf{h}, \mathbf{n}_-, \mathbf{n}_+$ the Lie algebras of functions with values in $\mathfrak{h}, \mathfrak{n}_-, \mathfrak{n}_+$ (our choice of the algebra of function corresponds to one of the three basic examples described on p. 347). We denote by P_+, P_-, P_0 the projection operators onto $\mathfrak{n}_+, \mathfrak{n}_-, \mathfrak{h}$ parallel to the complement in the decomposition (5.9); let $\mathbf{P}_+, \mathbf{P}_-, \mathbf{P}_0$ be the corresponding pointwise projection operators in $\mathbf{g}$.

Proposition 5.1. *The subgroup* $\mathbf{N}_-$ *of the gauge group is admissible if and only if the r-matrix* $R \in \operatorname{End}\mathbf{g}$ *has the form*

$$R = \mathbf{P}_+ - \mathbf{P}_- + \rho \cdot \mathbf{P}_0, \tag{5.10}$$

where $\rho \in \operatorname{End}\mathbf{h}$ *is a skew symmetric linear operator.*

The residual freedom in the definition of the classical r-matrix, which is represented by the possibility to choose $\rho \in \operatorname{End}\mathbf{h}$, is a well-known phenomenon which goes back to [2]. In the present case it plays a crucial role.

The key condition which has now to be taken into account is the Poisson properties of the constraint set (5.4). Unlike the differential case, in general these constraints are now *second class* (i.e., their Poisson brackets computed according to (2.5) do not vanish on the constraint surface). If this is the case, the description of the space of difference operators as a quotient over the gauge group will be inconsistent with our choice of the Poisson structure (the quotient will not be a Poisson submanifold in the reduced manifold $\mathcal{C}/\mathbf{N}_-$). Luckily, there is a smart way to avoid this difficulty by an appropriate choice of ρ. Let $T_s \in \operatorname{End}\mathfrak{h}$ be the linear operator induced by the Coxeter element of the Weyl group; we shall denote by the same letter its pointwise extension to $\mathbf{h}$, $T_s(f)(x) = T_s(f(x))$.

Theorem 5.1. *The constraints* (5.4) *are first class with respect to the Poisson bracket* (2.5) *on the space of difference connections if and only if*

$$\rho = (I + \tau \circ T_s^{-1})(I - \tau \circ T_s^{-1})^{-1}. \tag{5.11}$$

In the special case when $n = 2$ *we have* $T_s = -I$ *and* $\rho = (I - \tau)(I + \tau)^{-1}$.

Once the r-matrix is chosen in this specific way, it is easy to compute the Poisson bracket relations for the coefficients of the difference operator (5.6). The corresponding Poisson algebra is called the *deformed W-algebra* (or, for $n = 2$, the *deformed Poisson–Virasoro algebra*). We shall give explicit formulas only for the deformed Virasoro case; the general case is treated similarly, but leads to rather bulky expressions. It is technically convenient to start with an extended algebra generated by the coefficients of the factorized operator

$$\tau^2 + u\tau + 1 = (\tau + v)(\tau + v^{-1}) \tag{5.12}$$

Clearly, we have

$$u = v + (v^{-1})^\tau.$$

The mapping $v \mapsto v + (v^{-1})^\tau$ is the generalized Miura transform which replaces ordinary Miura transform described in Section 1 (formula (1.15)). The main simplification in the case when $n = 2$ is due to the fact that the Coxeter transform is scalar, $T_s = -1$. As a consequence, in the q-difference case the correction term in the definition of r-matrix amounts to a scalar function

$$\varphi(z/z') = \sum_{n=-\infty}^{\infty} \frac{1 - q^n}{1 + q^n} (z/z')^n. \tag{5.13}$$

Note that φ is elliptic and the series (5.13) essentially coincides with the Fourier expansion of the Jacobi elliptic function $\mathrm{dn}\,(q^{1/2}z/z')$. The Poisson bracket relations for v is best written in form of he generating series,

$$\{v(z), v(w)\} = \varphi\left(\frac{w}{z}\right) v(z)v(w). \tag{5.14}$$

Expressing $u(z)$ by means of the q-Miura transform we then get:

$$\{u(z), u(w)\} = \varphi\left(\frac{w}{z}\right) u(z)u(w) + \delta\left(\frac{wq}{z}\right) - \delta\left(\frac{w}{zq}\right), \tag{5.15}$$

where, as usual,

$$\delta\,(z) = \sum_{n=-\infty}^{\infty} z^n.$$

In the case of difference on the line the same calculations lead to the singular kernel

$$\varphi(x - x') = v.p.\,\frac{1}{2\pi i}\int_{-\infty}^{\infty} \tan\left(\frac{k}{2}\right) e^{ik(x-x')}\,dk \tag{5.16}$$

and to the Poisson bracket relations

$$\begin{aligned}&\{v(x), v(x')\} = \varphi(x - x')v(x)v(x'),\\ &\{u(x), u(x')\} = \varphi(x - x')u(x)u(x') + \delta(x - x' - 1) - \delta(x - x' + 1).\end{aligned} \tag{5.17}$$

6. Difference Drinfeld–Sokolov theory. II. The space of wave functions

We now come up to the extension of the deformed Poisson–Virasoro structure to the space of wave functions. Fundamental solutions of the first-order linear problem $\psi^\tau = L\psi$ are functions with values in G. By contrast, wave functions of a scalar nth-order difference equation form an n-tuple $(\phi_1, \ldots, \phi_n) \in \mathbb{C}^n \setminus \{0\}$. Up to a natural equivalence this set of wave functions defines a point in $\mathbb{C}P_{n-1}$. To match these descriptions note that the gauge group $\mathbb{N}_-$ acts on wave functions by left translations, $n \cdot \psi(x) = n(x)\psi(x)$. The quotient space $\mathbb{N}_-\backslash\mathbb{G}$ may be identified with the space of functions with values in the principal affine space $N_-\backslash G$. The Cartan subgroup $H \subset G$ normalizes N_- and hence there is a natural action $H \times N_-\backslash G \to N_-\backslash G$ and the associated pointwise action $\mathbb{H} \times \mathbb{N}_-\backslash\mathbb{G} \to \mathbb{N}_-\backslash\mathbb{G}$. When $G = SL(2,\mathbb{C})$, the quotient $HN_-\backslash G$ is isomorphic to the projective space $\mathbb{C}P_1$ and $\mathbb{H}\mathbb{N}_-\backslash\mathbb{G}$ is the space of projectivized wave functions of the second-order difference equation.

In the general case, when the potential of the first order matrix difference equation in canonical form $L = Us$ as in (5.7–5.8), its matrix wave function ψ has the simple 'Vandermonde' form,

$$\psi = \begin{pmatrix} \phi_1 & \phi_2 & \dots & \phi_n \\ \phi_1^\tau & \phi_2^\tau & \dots & \phi_n^\tau \\ \dots & \dots & \dots & \\ \phi_1^{\tau^{n-1}} & \phi_2^{\tau^{n-1}} & \dots & \phi_n^{\tau^{n-1}} \end{pmatrix}. \tag{6.1}$$

and is completely determined by its first row. Let $H_0 \subset H \subset SL(n)$ be the 1-dimensional subgroup consisting of degenerate diagonal matrices $h = \mathrm{diag}(t, \dots, t, s)$, $\det h = 1$. Let M_0' be the centaliser of H_0 in $G = SL(n)$ and M_0 its maximal semisimple subgroup; let $N_0 \subset N_-$ be the unipotent subgroup defined in (5.8). The subgroup $P_0 = M_0 H_0 N_0 \subset G$ is a maximal parabolic subgroup which contains the standard Borel subgroup $B = HN_-$. We regard $M_0 N_0 \backslash G$ as an affine algebraic variety; its affine ring $\mathbb{A}(M_0 N_0 \backslash G)$ is generated by the matrix coefficients of the first row of the matrix $g \in G$. Since $M_0 N_0 \supset N$, this affine ring is canonically embedded into the affine ring of $N_- \backslash G$. The multi-scaling action of the Cartan subgroup H on the affine ring $\mathbb{A}(N \backslash G)$ induced by the natural action $H \times N_- \backslash G \to N_- \backslash G$ may be restricted to $\mathbb{A}(M_0 N_0 \backslash G)$ and amounts to the scaling action of the rank 1 subgroup $H_0 \subset H$. The quotient $M_0 H_0 N_0 \backslash G$ is isomorphic to the projective space $\mathbb{C}P_{n-1}$; the associated space of functions with values in $M_0 H_0 N_0 \backslash G$ is precisely the space of projectivized wave functions of the n-th order difference equations (cf. [8], where the authors treat the case of n-th order differential equations).

If we take and the natural projection $\mathbb{G} \to \mathbb{M}_0 \mathbb{N}_0 \backslash \mathbb{G}$ assigns to this matrix its first row. The natural idea is to endow $\mathbb{G}$ (and hence also its quotient $\mathbb{N}_- \backslash \mathbb{G}$) with the Poisson structure such that the affine ring

$$\mathbb{A}(\mathbb{M}_0 \mathbb{N}_0 \backslash \mathbb{G}) \subset \mathbb{A}(\mathbb{N}_- \backslash \mathbb{G})$$

inherits a Poisson structure compatible with that of the deformed W-algebra described in the previous section.

Following the pattern described above in Sections 2 and 4, we chose the left r-matrix as in (5.10); note that this r-matrix is slightly more general than those considered, since it explicitly depends on τ. As a result, we must modify the choice of the right r-matrix r_0 acting in the space of quasiconstants: we simply take

$$r_0 = P_+ - P_- + \rho_0 P_0, \quad \text{where} \quad \rho_0 = (I + T_s^{-1})(I - T_s^{-1})^{-1} \tag{6.2}$$

(cf. formulae (5.10, 5.11) above). It is easy to see that with this choice of the right r-matrix the Poisson properties of the monodromy matrix described in Theorem 4.1 are maintained. Note that for $G = SL(2)$ we have $\rho_0 = 0$ and hence the right r-matrix coincides with the standard r-matrix on $sl(2)$.

We may regard the image of the Vandermonde map as a constrained set in $\mathbb{G}$; our condition on the choice of left r-matrix assures that the corresponding constraints are again first class; this allows to compute the quotient Poisson brackets for the wave functions simply by computing the Poisson brackets in $\mathbb{G}$ and then restricting them to the image of the Vandermonde map. This computation is particularly easy in the $sl(2)$ case and yields the exchange algebra Poisson brackets relations for the wave functions of the second-order difference operator as described in [7]. The details of this computation in the general case will be described in a separate publication.

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References

[1] Babelon, O. *Exchange formula and lattice deformation of the Virasoro algebra*, Phys. Lett. **B 238** (1990), no. 2-4, 234–238.

[2] Belavin, A.A., Drinfeld, V.G. Solutions of the classical Yang–Baxter equation for simple Lie algebras, *Funct. Anal. Appl.*, **16** (1982), 159–180.

[3] Drinfeld, V.G., Sokolov, V.V. *Lie algebras and equations of Korteweg–de Vries type*, Sov. Math. Dokl. **23** (1981) 457–462.

[4] Etingof, P. *Galois groups and connections matrices of q-difference equations*, Electronic Res. Announc. of the Amer. Math. Soc. **1**, 1 (1995).

[5] Faddeev, L.D., Takhtajan, L.A. *Liouville model on the lattice*, Lect. Notes in Phys. **246** (1986), 166–179.

[6] Frenkel, E., Reshetikhin, N., Semenov-Tian-Shansky, M.A. *Drinfeld–Sokolov reduction for difference operators and deformations of W-algebras. I: The case of Virasoro algebra*, Commun. Math. Phys. **192** (1998) 605–629.

[7] Marshall I., Semenov-Tian-Shansky, M.A. *Poisson groups and differential Galois theory of Schrödinger equation on the circle*, Commun. Math. Phys. 2008, **284**, 537–552.

[8] Ovsienko, V., Tabachnikov, S. *Projective differential geometry old and new: from Schwarzian derivative to cohomology of diffeomorphism group.* Cambridge Tracts in Mathematics, 165. Cambridge University Press, Cambridge, 2005.

[9] Semenov-Tian-Shansky, M.A. *Dressing action transformations and Poisson-Lie group actions*, Publ. RIMS. **21** (1985), 1237–1260.

[10] Semenov-Tian-Shansky, M.A., Sevostyanov, A.V. *Drinfeld-Sokolov reduction for difference operators and deformations of W-algebras. II: The general semisimple case*, Commun. Math. Phys. **192** (1998) 631–647.

[11] Wilson, G. *On the quasi-Hamiltonian formalism of the KdV equation*, Physics Letters A **132** (1988), 445–450.

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The Boundedness of the Riesz Potential Operator from Generalized Grand Lebesgue Spaces to Generalized Grand Morrey Spaces

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Dedicated to Professor A.F. dos Santos

Abstract. We introduce weighted generalized Grand Morrey spaces and prove that the boundedness of linear operators from the generalized Grand Lebesgue spaces to generalized Morrey spaces may be derived from their boundedness from classical weighted Lebesgue spaces into weighted Morrey spaces. As an application we prove a theorem on mapping properties of the Riesz potential operator from weighted generalized Grand Lebesgue spaces to weighted generalized Grand Morrey spaces with Muckenhoupt–Wheeden $A_{p,q}$-weights, under some natural assumptions on the way how we generalize grand spaces.

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1. Introduction

Morrey spaces introduced by Charles B. Morrey, jr. [20] in 1938 for the study of regularity properties of solutions to differential equations, nowadays are widely used not only in the theory of PDE, but also for the study of operators of harmonic analysis. The boundedness of classical operators of harmonic analysis, such as maximal, singular and potential type operators and others, in Morrey spaces were investigated, for instance, in [1, 2, 3, 7, 22, 27, 8], see also references therein.

In 1992 T. Iwaniec and C. Sbordone [10] introduces a new class of function spaces called *grand Lebesgue spaces*. Such spaces and their generalizations were afterwards studied in [4, 6, 9, 11, 12, 13, 15, 16, 17] on bounded sets.

In [26] there were introduced such Grand Lebesgue spaces on sets of infinite measure, where the "Grandization" of the space was made by introducing an additional parameter, corresponding to a weight introduced to control behaviour of functions at infinity.

The ideas of [26] were further developed in the recent paper [30], where there were introduce generalized Grand Lebesgue spaces, depending on a *functional parameter* and studied the boundedness of Hardy–Littlewood maximal and Calderón–Zygmund singular operators in such weighted Grand Lebesgue spaces.

Grand Morrey spaces were introduced in [18, 19] similarly to the approach for Grand Lebesgue spaces on bounded sets, i.e., with "grandization" with respect to p. Recently in [23] and [14] there were introduced more general Grand Morrey spaces with "grandization" with respect both to p and λ. and proved the boundedness of some classical operators of harmonic analysis.

We generalize the notion of Grand Morrey spaces via the "grandization" of weighed Morrey spaces with respect to p, λ and also the weight function w.

The paper is organized as follows. In Section 2 we give some preliminaries on generalized Grand Lebesgue spaces. In Section 3 we introduce *generalized Grand Morrey spaces* $\mathcal{L}^{p),\lambda}_{a,\nu}(\Omega, w)$. the main statements are contained in Sections 4 and 5. In Theorem 4.1 we prove, under some assumptions on the weight w, that linear operators are bounded from generalized Grand Lebesgue $L^{p)}_a(\Omega, w)$ space into generalized Grand Morrey space $\mathcal{L}^{p),\lambda}_{a,\nu}(\Omega, w)$, if, roughly speaking, they have such a property in the classical setting. In this "transference result" an essential role is played by an interpolation theorem from Lebesgue to Morrey spaces with change of weights, proved in [29]. By means of Theorem 4.1 in Section 5 we prove the boundedness of the Riesz potential operator from weighted generalized Grand Lebesgue space to a weighted generalized Grand Morrey space, under some assumptions on the weight.

Notation

- C, c are various absolute positive constants not depending on the involved variables. which may have different values even in the same line;
- Ω is an open set in $\mathbb{R}^n$;
- $|A|$ is the Lebesgue measure of a measurable set $A \subset \Omega$; Q is an arbitrary cube in $\mathbb{R}^n$ with sides parallel to coordinate planes;
- w is a weight on Ω, i.e., a non-negative locally integrable function with $|\{x \in \Omega : w(x) = 0\}| = 0$;
- $B(x, r) = \{y \in \mathbb{R}^n : \ |y - x| < r\}$;
- $\widetilde{B}(x, r) = B(x, r) \cap \Omega$;
- $\hookrightarrow$ stands for continuous embedding;

We use the following notation

$$L^p(\Omega, w) = \left\{ f : \ \|f\|_{L^p(\Omega)} := \left(\int_\Omega |f(x)|^p w(x) dx < \infty \right)^{\frac{1}{p}} < \infty \right\}$$

for weighted Lebesgue spaces.

2. Preliminaries

2.1. Generalized Grand Lebesgue spaces

The Grand Lebesgue space $L^{p)}(\Omega)$ on a bounded set Ω, introduced in [10] is defined by the norm

$$\|f\|_{L^{p)}(\Omega)} := \sup_{0<\varepsilon<p-1} \varepsilon^{\frac{1}{p-\varepsilon}} \|f(x)\|_{L^{p-\varepsilon}(\Omega)}, \ 1<p<\infty. \tag{2.1}$$

For unbounded sets we base ourselves on the following generalization of weighted Grand Lebesgue spaces introduced in [30].

Definition 2.1. Let $1<p<\infty$, $\Omega \subseteq \mathbb{R}^n$ and w be a weight on Ω. By the generalized Grand Lebesgue space $L_a^{p)}(\Omega, w)$ we call the space, defined by the norm

$$\|f\|_{L_a^{p)}(\Omega,w)} := \sup_{0<\varepsilon<p-1} \varepsilon^{\frac{1}{p-\varepsilon}} \|f\|_{L^{p-\varepsilon}(\Omega, wa^{\varepsilon})}, \tag{2.2}$$

where the weight a is suppose to satisfy the condition $a \in L^p(\Omega, w)$.

2.2. Auxiliary lemmas

Let $W(x)$ be the Lambert function, i.e., the solution of the equation $W(x) \times e^{W(x)} = x$. It has two branches with domains in $[-e^{-1}, +\infty)$ and $[-e^{-1}, 0)$ and value in $[-1,\infty)$ and $(-\infty,-1]$, respectively. In the sequel W stands for the first of these branches.

Lemma 2.2. *Let $p>1$, $s>0$ and $t \in [0, p-1)$. Then*

$$\sup_{t<x<p-1} (xs)^{\frac{1}{p-x}} = \begin{cases} (p-1)s, & s \le \frac{1}{p-1} e^{-\frac{1}{p-1}} \\ e^{W(-pse^{-1})/p}, & \frac{1}{p-1} e^{-\frac{1}{p-1}} < s < \frac{1}{t} e^{1-\frac{1}{t}} \\ (ts)^{1/(p-t)}, & s \ge \frac{1}{t} e^{1-\frac{1}{t}}. \end{cases}$$

Proof. We have $\left((xs)^{1/(p-x)}\right)'_x = 0 \Leftrightarrow \ln x + \ln s + \frac{p}{x} - 1 = 0 \Leftrightarrow W(-pse^{-1}) = -\frac{p}{x} < -1 \Rightarrow x_{\max} = -\frac{p}{W(-pse^{-1})}$. Treating separately the cases $0 < x_{\max} \le t$, $t < x_{\max} < p-1$ and $x_{\max} \ge p-1$, we arrive at the statement of the lemma. $\square$

By A_p, $1<p<\infty$, we denote the Muckenhoupt class of weights w satisfying the condition

$$\sup_{Q \subset \mathbb{R}^n} \left(\frac{1}{|Q|}\int_Q w(x)dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{-\frac{1}{p-1}}dx\right)^{p-1} < \infty,$$

and by $A(p,q)$, $1<p,q<\infty$, we denote the Muckenhoupt–Wheeden class od weights w satisfying the condition

$$\sup_{Q \subset \mathbb{R}^n} \left(\frac{1}{|Q|}\int_Q [w(x)]^{\frac{q}{p}}dx\right)^{\frac{p}{q}}\left(\frac{1}{|Q|}\int_Q w(x)^{-\frac{1}{p-1}}dx\right)^{p-1} < \infty.$$

As is known, $w \in A(p,q)$ if and only if $w^{\frac{q}{p}} \in A_{1+\frac{q}{p'}}$.

We will need the following lemma.

Lemma 2.3. *Let $1 < p < \infty$ and $w \in A(p,p^*)$, where $\frac{1}{p^*} = \frac{1}{p} - \gamma$, $0 < \gamma < \frac{1}{p}$. If $v^\delta \in A(p-\delta,(p-\delta)^*)$ for some $\delta \in (0, p-1)$, then there exists a $\varepsilon \in (0,\delta)$ such that $wv^\varepsilon \in A(p-\varepsilon,(p-\varepsilon)^*)$.*

Proof. We have to show that

$$\Delta_1 \cdot \Delta_2 := \left(\frac{1}{|Q|}\int_Q [w(x)v(x)^\varepsilon]^{\frac{(p-\varepsilon)^*}{p-\varepsilon}} dx\right)^{\frac{p-\varepsilon}{(p-\varepsilon)^*}} \times \left(\frac{1}{|Q|}\int_Q [w(x)v(x)^\varepsilon]^{-\frac{1}{p-\varepsilon-1}} dx\right)^{p-\varepsilon-1} < \infty.$$

From the assumption $w \in A(p,p^*)$ by properties of Muckenhoupt weights there exists a $\theta > 1$ such that $w^\theta \in A(p,p^*)$, i.e.,

$$\Delta := \sup_{Q\subset\mathbb{R}^n} \left(\frac{1}{|Q|}\int_Q [w(x)]^{\frac{\theta p^*}{p}} dx\right)^{\frac{p}{p^*}} \left(\frac{1}{|Q|}\int_Q w(x)^{-\frac{\theta}{p-1}} dx\right)^{p-1} < \infty. \tag{2.3}$$

To estimate the factor Δ_1, we use the Hölder inequality with the exponent $t = \frac{p^*(p-\varepsilon)}{p(p-\varepsilon)^*}\theta > 1$ and obtain

$$\Delta_1 \le \left(\frac{1}{|Q|}\int_Q [w(x)]^{\frac{\theta p^*}{p}} dx\right)^{\frac{p}{\theta p^*}} \left(\frac{1}{|Q|}\int_Q [v(x)^\varepsilon]^{\frac{t'(p-\varepsilon)^*}{p-\varepsilon}} dx\right)^{\frac{p-\varepsilon}{t'(p-\varepsilon)^*}}.$$

For Δ_2 we use the Hölder inequality with the exponent $\tau = \frac{p-\varepsilon-1}{p-1}\theta > 1$ (assuming that $\varepsilon < \frac{p-1}{\theta'}$). Then

$$\Delta_2 \le \left(\frac{1}{|Q|}\int_Q [w(x)]^{-\frac{\theta}{p-1}} dx\right)^{\frac{p-1}{\theta}} \left(\frac{1}{|Q|}\int_Q [v(x)^\varepsilon]^{-\frac{\tau'}{p-\varepsilon-1}} dx\right)^{\frac{p-\varepsilon-1}{\tau'}}.$$

Consequently

$$\Delta_1 \cdot \Delta_2 \le \Delta^{\frac{1}{\theta}} \Delta_3,$$

where

$$\Delta_3 = \left(\frac{1}{|Q|}\int_Q [v(x)^\varepsilon]^{\frac{t'(p-\varepsilon)^*}{p-\varepsilon}} dx\right)^{\frac{p-\varepsilon}{t'(p-\varepsilon)^*}} \left(\frac{1}{|Q|}\int_Q [v(x)^\varepsilon]^{-\frac{\tau'}{p-\varepsilon-1}} dx\right)^{\frac{p-\varepsilon-1}{\tau'}}.$$

After easy transformations we arrive at

$$\Delta_3^{\theta'} = \left(\frac{1}{|Q|}\int_Q [v(x)^{\varepsilon\theta'}]^{\frac{1}{r}} dx\right)^r \left(\frac{1}{|Q|}\int_Q [v(x)^{\varepsilon\theta'}]^{-\frac{1}{s}} dx\right)^s,$$

where

$$r = \left(\frac{p-\varepsilon}{(p-\varepsilon)^*} - \frac{p}{p^*}\right)\theta' + \frac{p}{p^*}, \quad s = p - \varepsilon\theta' - 1.$$

This completes the proof after replacing $\varepsilon\theta'$ by δ. $\square$

3. Generalized Grand Morrey spaces

3.1. Weighted Morrey spaces

Let $1 \le p < \infty$, $0 \le \lambda < n$ w a weight on Ω. The classical weighted Morrey space denoted by $\mathcal{L}^{p,\lambda}(\Omega, w)$ is defined by the norm

$$\|f\|_{\mathcal{L}^{p,\lambda}(\Omega,w)} := \sup_{x\in\Omega, r>0} r^{-\frac{\lambda}{p}} \|f\|_{L^p(\widetilde{B}(x,r),w)}. \tag{3.1}$$

In the case $\lambda = 0$ it recovers the weighted Lebesgue space $L^p(\Omega, w)$.

Examples with "limiting exponents" given below illustrate the inclusion of power functions into Morrey spaces, see, for instance, [24, 25]:

1) $|x|^{-\frac{n}{p}} \notin L^p(\mathbb{R}^n)$, $|x|^{\frac{\lambda-n}{p}} \in \mathcal{L}^{p,\lambda}(\mathbb{R}^n)$, $\lambda > 0$;
2) $|x|^{\frac{\lambda-n}{p}} \ln^{\nu} \frac{1}{|x|} \notin \mathcal{L}^{p,\lambda}(B(0,1))$, $\nu > 0$.

The following lemma on embedding of weighted Morrey spaces is proved by means of the corresponding Hölder inequality.

Lemma 3.1. *Let $1 \le p_1 < p_2 < \infty$, $0 \le \lambda_1, \lambda_2 < n$. If the weights w_1 and w_2 on Ω satisfy the condition*

$$K := \sup_{x\in\Omega, r>0} r^{\frac{\lambda_2}{p_2} - \frac{\lambda_1}{p_1}} \left\{ \int_{\widetilde{B}(x,r)} \left(\frac{w_1(y)^{p_2}}{w_2(y)^{p_1}} \right)^{\frac{1}{p_2-p_1}} dy \right\}^{\frac{1}{p_1} - \frac{1}{p_2}} < \infty, \tag{3.2}$$

then

$$\|f\|_{\mathcal{L}^{p_1,\lambda_1}(\Omega,w_1)} \le K \|f\|_{\mathcal{L}^{p_2,\lambda_2}(\Omega,w_2)}. \tag{3.3}$$

In the case $p_1 = p_2$, the condition (3.2) *should be replaced by*

$$K = \sup_{x\in\Omega, r>0} r^{\lambda_2-\lambda_1} \sup_{y\in\widetilde{B}(x,r)} \frac{w_1(y)}{w_2(y)} < \infty.$$

3.2. Riesz–Thorin–Stein–Weiss $L^p(\Omega, v) \to \mathcal{L}^{q,\lambda}(\Omega, w)$-interpolation theorem

In [28] there was proved an $L^p \to L^p$-interpolation theorem for linear operators with change of measure. We will use the following version of such a result, in terns of change of weights, for operators from Lebesgue to Morrey spaces, proved in [29].

Theorem 3.2. *Let $p_i, q_i \in [1, \infty)$ and v_i, w_i, $i = 1, 2$, be weights on Ω, and T a linear operator defined on $L^{p_1}(\Omega, v_1) \cup L^{p_2}(\Omega, v_2)$. If*

$$\|Tf\|_{\mathcal{L}^{q_i,\lambda_i}(\Omega,w_i)} \le K_i \|f\|_{L^{p_i}(\Omega,v_i)}$$

for all $f \in L^{p_i}(\Omega, v_i)$, $i = 1, 2$, then the operator T is bounded from $L^p(\Omega, v)$ to $\mathcal{L}^{q,\lambda}(\Omega, w)$, where p, q, λ and the weights v, w are defined by

$$\frac{1}{p} = \frac{1-t}{p_1} + \frac{t}{p_2}, \quad \frac{1}{q} = \frac{1-t}{q_1} + \frac{t}{q_2}, \quad \frac{\lambda}{q} = (1-t)\frac{\lambda_1}{q_1} + t\frac{\lambda_2}{q_2},$$
$$v = v_1^{(1-t)\frac{p}{p_1}} v_2^{t\frac{p}{p_2}}, \quad w = w_1^{(1-t)\frac{q}{q_1}} w_2^{t\frac{q}{q_2}}$$

and

$$\|Tf\|_{\mathcal{L}^{q,\lambda}(\Omega,w)} \le K_1^{1-t} K_2^t \|f\|_{L^p(\Omega,v)}, \quad 0 \le t \le 1. \tag{3.4}$$

3.3. Definition of generalized Grand Morrey spaces

We define generalized weighted Morrey spaces $\mathcal{L}^{p,\lambda}(\Omega, w)$, making these spaces grand with respect to p, λ (as in [23, 14]) and also the weight w. The change of the weight function $w(x)$ is taken in the form $w(x)[a(x)]^\varepsilon$ with a non-negative function $a(x)$, which is natural from the point of view of the usage of interpolation technique.

Definition 3.3. Let $1 < p < \infty$, $0 \le \lambda < n$, and w a weight on Ω. The generalized Grand Morrey space $\mathcal{L}^{p),\lambda}_{a,\nu}(\Omega, w)$ is defined as the set of functions $f : \Omega \to R$ having the finite norm

$$\|f\|_{\mathcal{L}^{p),\lambda}_{a,\nu}(\Omega,w)} := \sup_{0<\varepsilon<p-1} \varepsilon^{\frac{1}{p-\varepsilon}} \|f\|_{\mathcal{L}^{p-\varepsilon,\lambda-\nu\varepsilon}(\Omega,wa^\varepsilon)}, \tag{3.5}$$

where ν is a real number and a is a non-negative function on Ω.

In the following lemma we show that the weighted Morrey space is contained in such a weighted generalized Grand Morrey space under some natural assumptions on the choices of the function a and of the parameter ν.

Lemma 3.4. *Let $p > 1$. If $a \in \mathcal{L}^{p,\lambda-\nu p}(\Omega, w)$ and $-\frac{n-\lambda}{p} < \nu \le \frac{\lambda}{p}$, then the embedding*

$$\mathcal{L}^{p,\lambda}(\Omega, w) \hookrightarrow \mathcal{L}^{p),\lambda}_{a,\nu}(\Omega, w) \tag{3.6}$$

holds and

$$\|f\|_{\mathcal{L}^{p),\lambda}_{a,\nu}(\Omega,w)} \le C_{p,a,\nu} \|f\|_{\mathcal{L}^{p,\lambda}(\Omega,w)} \tag{3.7}$$

where

$$C_{p,a,\nu} = \begin{cases} (p-1)\|a\|_{\mathcal{L}}^{p-1}, & \|a\|_{\mathcal{L}}^p \le \frac{1}{p-1} e^{-\frac{1}{p-1}} \\ \|a\|_{\mathcal{L}}^{-1} e^{W(-p\|a\|_{\mathcal{L}}^p e^{-1})/p}, & \|a\|_{\mathcal{L}}^p > \frac{1}{p-1} e^{-\frac{1}{p-1}}, \end{cases}$$

$\|a\|_{\mathcal{L}} = \|a\|_{\mathcal{L}^{p,\lambda-\nu p}}$. *In the case $\nu = 0$ the norm of the operator of this embedding is $C_{p,a,0}$.*

Proof. We have

$$\|f\|_{\mathcal{L}^{p),\lambda}_{a,\nu}(\Omega,w)} = \sup_{0<\varepsilon<p-1} \varepsilon^{\frac{1}{p-\varepsilon}} \|f\|_{\mathcal{L}^{p-\varepsilon,\lambda-\nu\varepsilon}(\Omega,wa^\varepsilon)}.$$

Now we make use of Lemma 3.1 with

$$p_1 = p,\ p_2 = p - \varepsilon,\ w_1 = w,\ w_2 = wa^\varepsilon,\ \lambda_1 = \lambda,\ \lambda_2 = \lambda - \nu\varepsilon.$$

We obtain

$$\begin{aligned} K &= \sup_{x\in\Omega, r>0} r^{\frac{\lambda-\nu\varepsilon}{p-\varepsilon} - \frac{\lambda}{p}} \left\{ \int_{\widetilde{B}(x,r)} w(y)[a(y)]^p dy \right\}^{\frac{1}{p-\varepsilon} - \frac{1}{p}} \\ &= \|a\|^{\frac{\varepsilon}{p-\varepsilon}}_{\mathcal{L}^{p,\lambda-\nu p}(\Omega,w)} < \infty,\ 0 < \varepsilon < p - 1. \end{aligned}$$

Hence

$$\|f\|_{\mathcal{L}^{p),\lambda}_{a,\nu}(\Omega,w)} \le \|a\|^{-1}_{\mathcal{L}^{p,\lambda-\nu p}(\Omega,w)} \sup_{0<\varepsilon<p-1} \left(\varepsilon \|a\|^p_{\mathcal{L}^{p,\lambda-\nu p}(\Omega,w)} \right)^{\frac{1}{p-\varepsilon}} \|f\|_{\mathcal{L}^{p,\lambda}(\Omega,w)}.$$

It remains to use Lemma 2.2 with $t = 0, x = \varepsilon$ and $s = \|a\|^p_{\mathcal{L}^{p,\lambda-\nu p}(\Omega,w)}$.

In the case $\nu = 0$ we have

$$\begin{aligned}
\|a\|_{\mathcal{L}^{p),\lambda}_{a,0}(\Omega,w)} &= \sup_{0<\varepsilon<p-1} \varepsilon^{\frac{1}{p-\varepsilon}} \|a\|_{\mathcal{L}^{p-\varepsilon,\lambda}(\Omega,wa^\varepsilon)} \\
&= \sup_{0<\varepsilon<p-1} \varepsilon^{\frac{1}{p-\varepsilon}} \sup_{x,r} \left(r^{-\lambda} \int_{B(x,r)} a^{p-\varepsilon} a^\varepsilon w dy \right)^{\frac{1}{p-\varepsilon}} \\
&= \sup_{0<\varepsilon<p-1} \varepsilon^{\frac{1}{p-\varepsilon}} \|a\|^{\frac{p}{p-\varepsilon}}_{\mathcal{L}^{p,\lambda}(\Omega,w)} \\
&= \sup_{0<\varepsilon<p-1} \varepsilon^{\frac{1}{p-\varepsilon}} \|a\|^{\frac{\varepsilon}{p-\varepsilon}}_{\mathcal{L}^{p,\lambda}(\Omega,w)} \|a\|_{\mathcal{L}^{p,\lambda}(\Omega,w)} \\
&= C_{p,a,0} \|a\|_{\mathcal{L}^{p,\lambda}(\Omega,w)}.
\end{aligned}$$

□

The statement of the next lemma follows from the definition of the norm in the generalized Grand Morrey space.

Lemma 3.5. *If* $a \in \mathcal{L}^{p,\lambda-\nu p}(\Omega, w)$, $-\frac{n-\lambda}{p} < \nu \le \frac{\lambda}{p}$, *then for every* $0 < \varepsilon < p-1$ *the embedding*

$$\mathcal{L}^{p),\lambda}_{a,\nu}(\Omega, w) \hookrightarrow \mathcal{L}^{p-\varepsilon,\lambda-\nu\varepsilon}(\Omega, wa^\varepsilon)$$

is valid.

By standard means the following lemma is proved.

Lemma 3.6. $\mathcal{L}^{p),\lambda}_{a,\nu}(\Omega, w)$ *is a Banach space.*

Note that the generalized Grand Morrey space coincides with the generalized Grand Lebesgue space when $\lambda = \nu = 0$:

$$\mathcal{L}^{p),0}_{a,0}(\Omega, w) = L^{p)}_a(\Omega, w).$$

Below we give examples illustrating inclusion of power-logarithmic functions into generalized Morrey spaces $\mathcal{L}^{p),\lambda}_{a,\nu}(\Omega, w)$.

Example. Let $1 < p < \infty$ and $0 \le \lambda < n$. Then

a) $|x|^{\frac{\lambda-n}{p}} \in \mathcal{L}^{p,\lambda}(\mathbb{R}^n)$,

b) $|x|^{\frac{\lambda-n}{p}} \notin \mathcal{L}^{p-\varepsilon,\lambda-\nu\varepsilon}(\mathbb{R}^n)$, $\forall\varepsilon \in (0, p-1)$, $\forall\nu \in \left(-\frac{n-\lambda}{p}, \frac{\lambda}{p}\right]$,

c) $|x|^{\frac{\lambda-n}{p}} \notin \mathcal{L}^{p),\lambda}_{1,\nu}(\mathbb{R}^n)$,

d) $|x|^{\frac{\lambda-n}{p}} \in \mathcal{L}^{p),\lambda}_{1,\nu}(\Omega)$ in the case Ω is bounded,

e) $|x|^{\frac{\lambda-n}{p}} \in \mathcal{L}^{p),\lambda}_{a,\nu}(\mathbb{R}^n)$, where $a(x) = (1 + |x|)^{\frac{\lambda-n}{p}}$.

4. The boundedness of linear operators from generalized Grand Lebesgue spaces to generalized Grand Morrey spaces

4.1. The main statement

In [30] under some assumptions on the weight w there was studied the boundedness of linear operators in generalized Grand Lebesgue spaces $L_a^{p)}(\Omega, w)$. The next theorem represents a version of such a study for the case of linear operators from generalized Grand Lebesgue spaces $L_a^{p)}(\Omega, w)$ to generalized Grand Morrey spaces $\mathcal{L}_{a,\nu}^{p),\lambda}(\Omega, w)$.

Theorem 4.1. *Let $1 < p_0 < p < \infty$ and $\lambda, \lambda_0 \in [0, n)$. Suppose that a linear operator T*

i) *is bounded from the Lebesgue space $L^p(\Omega, u)$ into the Morrey space $\mathcal{L}^{p,\lambda}(\Omega, v)$ with the norm M_1,*

ii) *is bounded from the Lebesgue space $L^{p_0}(\Omega, ua^{p-p_0})$ into the Morrey space $\mathcal{L}^{p_0,\lambda_0}(\Omega, vb^{p-p_0})$ with the norm M_2, for some non-negative on Ω functions $a \in L^p(\Omega, u)$ and $b \in \mathcal{L}^{p,\lambda-\nu p}(\Omega, v)$, $\nu = \frac{\lambda-\lambda_0}{p-p_0}$.*

Then T is bounded from the generalized Grand Lebesgue space $L_a^{p)}(\Omega, u)$ into the generalized Grand Morrey space $\mathcal{L}_{b,\nu}^{p),\lambda}(\Omega, v)$ with the norm $M \leq \max\{M_1, M_2\}$.

Proof. We have

$$\|Tf\|_{\mathcal{L}_{b,\nu}^{p),\lambda}(\Omega,v)} = \sup_{0<\varepsilon<p-1} \varepsilon^{\frac{1}{p-\varepsilon}} \|Tf\|_{\mathcal{L}^{p-\varepsilon,\lambda-\nu\varepsilon}(\Omega,vb^\varepsilon)} = \max\{A, B\},$$

where

$$A = \sup_{0<\varepsilon\leq p-p_0} \varepsilon^{\frac{1}{p-\varepsilon}} \|Tf\|_{\mathcal{L}^{p-\varepsilon,\lambda-\nu\varepsilon}(\Omega,vb^\varepsilon)},$$

$$B = \sup_{p-p_0<\varepsilon<p-1} \varepsilon^{\frac{1}{p-\varepsilon}} \|Tf\|_{\mathcal{L}^{p-\varepsilon,\lambda-\nu\varepsilon}(\Omega,vb^\varepsilon)}.$$

Estimation of A. By the assumptions of the theorem $\|Tf\|_{\mathcal{L}^{p,\lambda}(\Omega,v)} \leq M_1\|f\|_{L^p(\Omega,u)}$ and $\|Tf\|_{\mathcal{L}^{p_0,\lambda_0}(\Omega,vb^{p-p_0})} \leq M_2\|f\|_{L^{p_0}(\Omega,ua^{p-p_0})}$. We make use of the interpolation theorem 3.2 with

$$p_1 = q_1 = p,\ p_2 = q_2 = p_0,\ v_1 = u,\ w_1 = v,\ v_2 = ua^{p-p_0},\ w_2 = vb^{p-p_0},$$
$$\lambda_1 = \lambda,\ \lambda_2 = \lambda_0$$

and notation $\left(\frac{1-t}{p} + \frac{t}{p_0}\right)^{-1} =: p - \varepsilon$. We obtain that the operator T is bounded from the space $L^{p-\varepsilon}(\Omega, ua^\varepsilon)$ into the space $\mathcal{L}^{p-\varepsilon,\lambda-\nu\varepsilon}(\Omega, vb^\varepsilon)$, $\nu = \frac{\lambda-\lambda_0}{p-p_0}$, i.e.,

$$\|Tf\|_{\mathcal{L}^{p-\varepsilon,\lambda-\nu\varepsilon}(\Omega,vb^\varepsilon)} \leq K\|f\|_{L^{p-\varepsilon}(\Omega,ua^\varepsilon)}, \tag{4.1}$$

and $K = M_1^{1-t}M_2^t$, where $t = \frac{\varepsilon p_0}{(p-\varepsilon)(p-p_0)}$.

Consequently,

$$A \le \sup_{0<\varepsilon\le p-p_0} M_1^{1-t} M_2^t \varepsilon^{\frac{1}{p-\varepsilon}} \|f\|_{L^{p-\varepsilon}(\Omega, ua^{\varepsilon})}$$
$$= \sup_{0<\varepsilon\le p-p_0} M_1^{1-t} M_2^t \|f\|_{L_a^{p)}(\Omega,u)} = \max\{M_1, M_2\} \|f\|_{L_a^{p)}(\Omega,u)}.$$

Estimation of B. We apply Lemma 3.1 and obtain

$$\|Tf\|_{\mathcal{L}^{p-\varepsilon,\lambda-\nu\varepsilon}(\Omega,vb^{\varepsilon})} \le \|b\|^{p\left(\frac{1}{p_0}-\frac{1}{p-\varepsilon}\right)}_{\mathcal{L}^{p,\lambda-\nu p}(\Omega,v)} \|Tf\|_{\mathcal{L}^{p_0,\lambda-\nu(p-p_0)}(\Omega,vb^{(p-p_0)})},$$

so that

$$B \le \sup_{p-p_0<\varepsilon<p-1} \varepsilon^{\frac{1}{p-\varepsilon}} \|b\|^{p\left(\frac{1}{p-\varepsilon}-\frac{1}{p_0}\right)}_{\mathcal{L}^{p,\lambda-\nu p}(\Omega,v)} \|Tf\|_{\mathcal{L}^{p_0,\lambda_0}(\Omega,vb^{p-p_0})}$$
$$= (p-p_0)^{-\frac{1}{p_0}} \|b\|^{-\frac{p}{p_0}}_{\mathcal{L}^{p,\lambda-\nu p}(\Omega,v)}$$
$$\times \sup_{p-p_0<\varepsilon<p-1} \varepsilon^{\frac{1}{p-\varepsilon}} \|b\|^{\frac{p}{p-\varepsilon}}_{\mathcal{L}^{p,\lambda-\nu p}(\Omega,v)} (p-p_0)^{\frac{1}{p_0}} \|Tf\|_{\mathcal{L}^{p_0,\lambda-\nu(p-p_0)}(\Omega,vb^{p-p_0})}$$
$$\le \inf_{0<p-p_0<p-1} \left([h(p-p_0)]^{-1} \sup_{p-p_0<\varepsilon<p-1} h(\varepsilon) \right) \cdot A = A,$$

where we denoted $h(\varepsilon) := \varepsilon^{\frac{1}{p-\varepsilon}} \|b\|^{\frac{p}{p-\varepsilon}}_{\mathcal{L}^{p,\lambda-\nu p}(\Omega,v)}$.

It remains to gather the estimates for A and B. □

5. Application to the Riesz potential operator

Let

$$I^\alpha f := \int_\Omega |x-y|^{\alpha-n} f(y)dy, \; 0<\alpha<n,$$

be the Riesz potential on $\Omega \subseteq \mathbb{R}^n$. The following result is well known ([21]).

Theorem 5.1. *Let $\Omega = \mathbb{R}^n$, $0<\alpha<n$, $1<p<n/\alpha$, $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$. The operator I^α is bounded from $L^p(\mathbb{R}^n, w)$ into $L^q(\mathbb{R}^n, w^{q/p})$, if and only if $w \in A(p,q)$.*

The following theorem can be derived from Theorem 5.1. It gives us an example of a bounded linear operator acting from a weighted Lebesgue space into a weighted Morrey space.

Theorem 5.2. *Let $0<\alpha<n$, $1<p<n/\alpha$, $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$, and w be a restriction to Ω of a weight in the class $A(p,q)$. Then*

$$\|I^\alpha f\|_{\mathcal{L}^{p,\lambda}(\Omega,w)} \le c\|f\|_{L^p(\Omega,w)}$$

for $\lambda \le p\alpha$, if Ω is bounded and $\lambda = p\alpha$, otherwise.

Proof. By the Hölder inequality with the exponent q/p, we get

$$\|I^\alpha f\|_{\mathcal{L}^{p,\lambda}(\Omega,w)} = \sup_{x\in\Omega,0<r\leq diam\Omega} r^{-\frac{\lambda}{p}} \left\{ \int_{B(x,r)} |I^\alpha f|^p w(y)dy \right\}^{\frac{1}{p}}$$
$$\leq \sup_{x\in\Omega,0<r\leq diam\Omega} r^{-\frac{\lambda}{p}+n\frac{q-p}{qp}} \|I^\alpha f\|_{L^q(\Omega,w^{q/p})}.$$

To complete the proof, it remains to apply Theorem 5.1. □

Theorem 5.3. *Let* $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. *Suppose that*

1) $w \in A(p,q)$,
2) $a \in L^p(\mathbb{R}^n, w)$ *and* $a^\delta \in A(p-\delta, q_\delta)$ *for some* $\delta \in (0, p-1)$, *where* $\frac{1}{q_\delta} = \frac{1}{p-\delta} - \frac{\alpha}{n}$.

Then the operator I^α *is bounded from the generalized Grand Lebesgue space* $L_a^{p)}(\mathbb{R}^n, w)$ *into the generalized Grand Morrey space* $\mathcal{L}_{a,\alpha}^{p),\alpha p}(\mathbb{R}^n, w)$.

Proof. It suffices to show that the operator $T = I^\alpha$ satisfies the conditions i)–ii) of Theorem 4.1. The former follows by the assumption 1) and Theorem 5.2. The latter follows by Lemma 2.3 and Theorem 5.2. □

References

[1] D.R. Adams, *A note on Riesz potentials.* Duke Math. J. **42**(4) (1975), 765–778.

[2] C. Capone and A. Fiorenza, *On small Lebesgue spaces.* J. Funct. Spaces Appl. **3** (2005), 73–89.

[3] G. Di Fazio and M.A. Ragusa, *Commutators and Morrey spaces.* Bollettino U.M.I. 7(5-A) (1991), 323–332.

[4] G. Di Fratta and A. Fiorenza, *A direct approach to the duality of grand and small Lebesgue spaces.* Nonlinear Anal. **70**(7) (2009), 2582–2592,

[5] Y. Ding and C.-C. Lin, *Two-weight norm inequalities for the rough fractional integrals.* Int. J. Math. Math. Sci. **25**(8) (2001), 517–524.

[6] A. Fiorenza, B. Gupta, and P. Jain, *The maximal theorem in weighted grand Lebesgue spaces.* Studia Math. **188**(2) (2008), 123–133.

[7] M. Giaquinta, *Multiple integrals in the calculus of variations and non-linear elliptic systems.* Princeton Univ. Press, 1983.

[8] D. Gilbarg and N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order.* 2nd Edition, Springer-Verlag, Berlin, 1983.

[9] L. Greco, T. Iwaniec, and C. Sbordone, *Inverting the p-harmonic operator.* Manuscripta Math. **92** (1997), 249–258.

[10] T. Iwaniec and C. Sbordone, *On the integrability of the Jacobian under minimal hypotheses.* Arch. Ration. Mech. Anal. **119** (1992), 129–143.

[11] V. Kokilashvili, *Boundedness criterion for the Cauchy singular integral operator in weighted grand Lebesgue spaces and application to the Riemann problem.* Proc. A. Razmadze Math. Inst. **151** (2009), 129–133.

[12] V. Kokilashvili, *The Riemann boundary value problem for analytic functions in the frame of grand $L^{p)}$ spaces*. Bull. Georgian Natl. Acad. Sci. **4**(1) (2010), 5–7.

[13] V. Kokilashvili and A. Meskhi, *A note on the boundedness of the Hilbert transform in weighted grand Lebesgue spaces*. Georgian Math. J. **16**(3) (2009), 547–551.

[14] V. Kokilashvili, A. Meskhi, and H. Rafeiro, *Riesz type potential operators in generalized grand Morrey spaces*. Georgian Math. J. **20**(1) (2013), 43–64.

[15] V. Kokilashvili and S. Samko, *Boundedness of weighted singular integral operators on a Carleson curves in Grand Lebesgue spaces*. In ICNAAM 2010: Intern. Conf. Numer. Anal. Appl. Math., AIP Confer. Proc., volume 1281 (2010), 490–493.

[16] V. Kokilashvili and S. Samko, *Boundedness of weighted singular integral operators in Grand Lebesgue spaces*. Georgian Math. J. **18**(2) (2011), 259–269.

[17] A. Meskhi, *Criteria for the boundedness of potential operators in grand Lebesgue spaces*, arXiv:1007.1185.

[18] A. Meskhi, *Maximal functions and singular integrals in Morrey spaces associated with grand Lebesgue spaces*. Proc. A. Razmadze Math. Inst. **151** (2009), 139–143.

[19] A. Meskhi, *Integral operators in maximal functions, potentials and singular integrals in grand Morrey spaces*. Complex Var. Elliptic Equ., Doi.10.1080/ 17476933.2010.534793, 2011, 1–19.

[20] C.B. Morrey. *On the solutions of quasi-linear elliptic partial differential equations*. Amer. Math. Soc. **43** (1938), 126–166.

[21] B. Muckenhoupt and R.L. Wheeden, *Weighted norm inequalities for fractional integrals*. Trans. Amer. Math. Soc. **192** (1974), 261–274.

[22] J. Peetre, *On the theory of $\mathcal{L}_{p,\lambda}$ spaces*. J. Funct. Anal. **4** (1969), 71–87.

[23] H. Rafeiro, *A note on boundedness of operators in grand grand Morrey spaces*. Operator Theory: Advances and Applications **229** (2013), 349-356.

[24] N.G. Samko, *Weighted Hardy and potential operators in Morrey spaces*. J. Funct. Spaces Appl. **2012** (2012), Article ID 678171, doi: 10.1155/2012/678171.

[25] N.G. Samko, *Weighted Hardy and singular operators in Morrey spaces*. J. Math. Anal. Appl. **350**(1) (2009), 56–72.

[26] S.G. Samko and S.M. Umarkhadzhiev, *On Iwaniec–Sbordone spaces on sets which may have infinite measure*. Azerb. J. Math. **1**(1) (2011), 67–84.

[27] Y. Sawano and H. Tanaka, *Morrey spaces for non-doubling measures*. Acta Math. Sin. (Engl. Ser.) **21**(6) (2005), 1535–1544.

[28] E.M. Stein and G. Weiss, *Interpolation of operators with change of measures*. Trans. Amer. Math. Soc. **87** (1958), 159–172.

[29] S.M. Umarkhadzhiev, *Riesz–Thorin–Stein–Weiss Interpolation Theorem in a Lebesgue–Morrey Setting*. Operator Theory: Advances and Applications **229** (2013), 387–392.

[30] S.M. Umarkhadzhiev, *A generalization of the notion of Grand Lebesgue space*. Russian Math. (Iz. VUZ), to appear.

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