The Reproducing Kernel Property and Its Space: More or Less Standard Examples of Applications

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Contents

Abstract

This is a follow-up of the chapter ▸ The Reproducing Kernel Property and Its Space: The Basics, which is the first part of the two-chapter project by the present

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author appearing in this handbook under the common title "The Reproducing Kernel Property and Its Space."

Introduction

The applications in question are:

spaces of holomorphic functions;
dilation theory.

The latter has been sketched very much due to the limited capacity of this chapter. For more information, we refer to the introductory part of the chapter ▸ The Reproducing Kernel Property and its Space: The Basics appearing in this handbook.

Spaces of Holomorphic Functions

General Constructions

The following observation is basic.

Proposition 1. *Let $\Omega \subset \mathbb{C}^d$ be an open set. Furthermore, let K be a positive-definite kernel on Ω and $\mathcal{H}$ its Hilbert space Ω. Then the following conditions are equivalent:*

- *For any $z \in \Omega$, the function K_z is holomorphic.*
- *Any function $f \in \mathcal{H}$ is holomorphic.*
- *If $(e_\alpha)_\alpha$ is an arbitrary basis in $\mathcal{H}$, then every e_α is holomorphic.*

If K satisfies any of the equivalent conditions of Proposition 1, it is called a *holomorphic kernel* on Ω though it is holomorphic in the first variable and anti-holomorphic in the second.

As long as $d > 1$, the multi-index is used; it is easy to be recognized in any context, like that which follows. Let Ω be a subject to the following condition:

$$z \in \Omega \text{ and } w \in \mathbb{C}^d, \text{ and } |w_i| \leq |z_i|, 1 \leq i \leq d, \text{ imply } w \in \Omega. \qquad (\bullet)$$

In other words, Ω is the union of all polycylinders at 0 contained in it

Both *polycylinders* and the *ball* $B(a; R)$ (at center a and radius $R \leqslant +\infty$) satisfy ($\bullet$), and this is enough for this survey (*polydisc* is preferred for polycylinders if they are bounded).

Under ($\bullet$) for any function f in $\mathcal{O}(\Omega)$ ($\mathcal{O}(\Omega)$ stands for the totality of all holomorphic functions on $\Omega \subset \mathbb{C}^d$), there is a unique power series

$$\sum_{\alpha \in \mathrm{A}^d} c_\alpha z^\alpha, \tag{2.1}$$

convergent to $f(z)$ in every $z \in \Omega$; in other words, this is a global expansion in the whole of Ω.

One of the most frequent ways of generating reproducing kernel couples based on holomorphic functions is to follow the procedure (A), cf. [24]. The fact which is isolated here concerns that and is included in the following:

Proposition 2. *Let Ω be an open set in $\mathbb{C}^d$ and μ nonnegative measure on $\overline{\Omega}$. The inner product space*

$$\mathcal{H}^2(\mu) \stackrel{def}{=} \mathcal{L}^2(\mu) \cap \mathcal{O}(\Omega),$$

with the norm $\| \cdot \|_{L^2(\mu)}$, is a Hilbert space if and only if for every $z \in \Omega$ there is $c_z > 0$ such that for any $f \in \mathcal{H}^2(\mu)$

$$|f(z)|^2 \le c_z \int_\Omega |f(w)|^2 \, \mu(dw), \tag{2.2}$$

where c_z is bounded on Ωs which are compact.

Corollary 1. *If the Radon–Nikodym derivative h of the part μ_{a} of μ which is absolutely continuous with respect to the $2d$-dimensional Lebesgue measure m_{2d} satisfies the condition*

for every polydisc $D(a,r)$, $\overline{D(a,r)} \subset \Omega$, there is
$C > 0$ such that $h \ge C$ on $D(a,r)$ m_{2d}-almost everywhere,

then $\mathcal{H}^2(\mu)$ is a Hilbert space with reproducing kernel.

Proposition 2 simplifies a lot if one starts from a closed subspace.

Proposition 3. *Let Ω be an open set in $\mathbb{C}^d$ and μ nonnegative measure on $\overline{\Omega}$. Suppose*

$$\mathcal{H} \subset \mathcal{L}^2(\mu) \cap \mathcal{O}(\Omega)$$

is a closed subspace of $\mathcal{L}^2(\mu)$. The Hilbert space $\mathcal{H}$ is a RKHS if and only if for every $z \in \Omega$ there is $c_z > 0$ such that for every function $f \in \mathcal{H}$ the estimation (2.2) holds.

A Scheme

Let A stand for a set of indices. A^s is the "s-time product" $\mathrm{A} \times \cdots \times \mathrm{A}$. Given a family $\boldsymbol{k} = (k_\alpha)_{\alpha \in \mathrm{A}^s}$ of nonnegative numbers and a family $(f_\alpha)_{\alpha \in \mathrm{A}^s}$ of functions in $\mathcal{O}(\Omega \times \Omega^*)$ ($\Omega^* \overset{\text{def}}{=} \{\bar{z}\colon z \in \Omega\}$). Define the kernel

$$K(z, w) \overset{\text{def}}{=} \sum_{\alpha \in \mathrm{A}^s} k_\alpha f_\alpha(z, \bar{w}), \quad z, w \in \Omega. \tag{2.3}$$

The kernel K is positive definite if, in particular, for each $\alpha \in \mathrm{A}^s$

$$\sum_{i,j} \lambda_i \bar{\lambda}_j f_\alpha(z_i, \bar{z}_j) \geq 0, \quad (\lambda_i)_i \subset \mathbb{C}, \ (z_i)_i \subset \Omega \tag{2.4}$$

and then

$$\sum_{\alpha \in \mathrm{A}^s} k_\alpha f_\alpha(z, \bar{z}) < +\infty, \quad z \in \Omega$$

is sufficient for (2.3) to be finite. Consequently, (2.4) leads to a reproducing kernel couple $(K, \mathcal{H})$ such that, due to Proposition 1, members of $\mathcal{H}$ are in $\mathcal{O}(\Omega)$. This procedure will be specified later on.

Notice that (2.4) is certainly satisfied if for each α

$$f_\alpha(z, \bar{w}) = g_\alpha(z)\overline{g_\alpha(w)}, \quad g_\alpha \in \mathcal{O}(\Omega), \quad z, w \in \Omega.$$

Kernels on Polycylinders

Think of the polycylinder

$$D \overset{\text{def}}{=} D(0, r), r = (r_1, \ldots, r_d), \text{for every } i \ r_i = 1 \text{ or } r_i = +\infty. \tag{2.5}$$

Now $\boldsymbol{k} = (k_\alpha)_{\alpha \in \mathrm{A}^d}$ is a family of nonnegative numbers such that

$$\sum_{\alpha \in \mathrm{A}^d} k_\alpha z^\alpha \bar{z}^\alpha < +\infty, \quad z \in D.$$

The kernel K defined by (2.4) taking now the form

$$K(z, w) \overset{\text{def}}{=} \sum_{\alpha \in \mathrm{A}^d} k_\alpha z^\alpha \bar{w}^\alpha, \quad z, w \in D, \tag{2.6}$$

is positive definite, and the monomials $\left(k_\alpha{}^{1/2} Z^\alpha\right)_{\alpha\in \mathrm{A}^d}$ form an orthonormal basis (notice that Proposition 9 in [24] guaranties the polynomials $\mathbb{C}[Z]$ are in $\mathcal{H}$) of $\mathcal{H}$, the Hilbert space corresponding to K. According to Proposition 1, $\mathcal{H}$ is composed of functions holomorphic on D. Even more, the Parseval identity yields

$$f \in \mathcal{H} \quad \Longleftrightarrow \quad \|f\|^2 = \sum_{\alpha\in \mathrm{A}^d} |a_\alpha|^2 k_\alpha^{-1} < +\infty, \quad f = \sum_{\alpha\in \mathrm{A}^d} a_\alpha Z^\alpha. \tag{2.7}$$

The following is immediate.

Proposition 4. *Let K be defined by (2.6) with k_α's being nonzero, and let $\mathcal{H}$ be the corresponding Hilbert space. Moreover, let μ be a nonnegative measure on $\mathbb{C}^d$. Then the mapping $V\colon\mathcal{H} \to L^2(\mu)$ defined as*

$$V p_{\mathcal{H}} = p_{L^2(\mu)}$$

(subscript indicates the space to which the polynomial p belongs to) is an isometry if and only if

$$\int_{\mathbb{C}^d} z^\alpha \bar{z}^\beta \mu(dz) = k_\alpha^{-1} \delta_{\alpha,\beta}, \quad \alpha, \beta \in \mathrm{A}^d.$$

Proposition 4 does not treat explicitly the question whether there is a relation between D and $\operatorname{supp}\mu$. What becomes interesting in the sequel is to describe the ingredient appearing in Proposition 4. For this, go back to Proposition 2.

Let D be as in (2.5). Set

$$P \stackrel{\text{def}}{=} \sigma_1 \times \cdots \times \sigma_d, \quad \text{where} \quad \sigma_i = \begin{cases} [0,1] & \text{if } r_i = 1 \\ [0,+\infty) & \text{if } r_i = +\infty \end{cases}$$

and let ν be a probability measure on P with finite moments, that is,

$$\nu_\alpha \stackrel{\text{def}}{=} \int_P r^\alpha \,\nu(\mathrm{d}r) < +\infty \text{ for all } \alpha \in \mathrm{A}^d.$$

Because $\operatorname{supp}\nu \subset P$,

$$\lim_{n\to+\infty} \nu_{\alpha[n,i]}^{1/n} \le r_i,$$

where $\alpha[n,i] \stackrel{\text{def}}{=} (0,\dots,n,\dots,0) \in \mathrm{A}^d$ with n located at the i-th position (the limit always exists as the sequence is logarithmically convex).

The moments of the measure μ given as (χ_ρ stands for the characteristic (indicator) function of the set ρ)

$$\mu(\sigma) \overset{\text{def}}{=} \frac{1}{(2\pi)^d} \int_P \int_{[0,2\pi]^d} \chi_\sigma(r_1 \mathrm{e}^{\mathrm{i}\,t_1}, \dots, r_d \mathrm{e}^{\mathrm{i}\,t_d}) \,\mathrm{d}\, t \,\nu(\mathrm{d}\, r),$$
$$t = (t_1, \dots, t_d),\ r = (r_1, \dots, r_d) \tag{2.8}$$

are precisely

$$\mu_{\alpha,\beta} \overset{\text{def}}{=} \int_{\mathbb{C}^d} z^\alpha \bar{z}^\beta \mu(\mathrm{d}\, z) = \nu_{\alpha+\beta}\, \delta_{\alpha,\beta}, \quad \alpha, \beta \in \mathrm{A}^d.$$

Now the kernel (2.6) is at hand if $k_\alpha \overset{\text{def}}{=} \nu_{2\alpha}^{-1}$, namely,

$$K(z,w) \overset{\text{def}}{=} \sum_{\alpha \in \mathrm{A}^d} \nu_{2\alpha}^{-1} z^\alpha \bar{w}^\alpha, \quad z, w \in D. \tag{2.9}$$

Let $\mathcal{H}$ be the corresponding Hilbert space. The aim is to identify $\mathcal{H}$ with a subspace of $\mathcal{L}^2(\mu)$.

Lemma 1. *Suppose the measure* ν *is such that*

$$\overline{D(R, \dots, R)} \subset D \textit{ implies } \nu([R, +\infty)^d \cap P) > 0. \tag{2.10}$$

If μ *is defined by* (2.8)*, then for an arbitrary function* $f \in \mathcal{O}(D)$ *and an arbitrary subset* X *of* D*, there is* $C > 0$ *such that*

$$|f(z)| \le C \left(\int_D |f(z)|^2 \mu(dz) \right)^{1/2}, \quad z \in X, \tag{2.11}$$

provided the integral involved is finite.

Remark 1. The integral in (2.11) is finite at least in two cases which are the interest here:

1° The measure ν satisfies

$$\nu(P) = \nu([0, r_1) \times \cdots \times [0, r_d)), \tag{2.12}$$

for any $f \in \mathcal{L}^2(\mu|_D) = \mathcal{L}^2(\mu)$;

2° For $f \in \mathcal{O}(\Omega)$ with open Ω containing the closure of D.

Theorem 1. *Let* ν *be such that* (2.10) *and* (2.12) *hold. If* μ *is as in* (2.8)*, then*

$$\mathcal{H} = \mathcal{H}^2(\mu) \overset{\text{def}}{=} \mathcal{L}^2(\mu) \cap \mathcal{O}(D).$$

If ν does not satisfy (2.12), another way to go around has to be chosen; point 2° of Remark 1 is going to help. Take $0 < t < 1$ and let f_t the composition of f with $z \mapsto tz$, that is, $f_t(z) \stackrel{\text{def}}{=} f(tz)$, $z \in D$. If f is in $\mathcal{O}(D)$, then f_t is in $\mathcal{O}(t^{-1}D)$. Therefore, if $f = \sum_{\alpha \in \mathrm{A}^d} a_\alpha Z^\alpha$ is in $\mathcal{H}$, then f_t is in $\mathcal{H}$ as well. Because $f_t = \sum_{\alpha \in \mathrm{A}^d} a_\alpha t^{|\alpha|} Z^\alpha$ and, due to (2.7) and (2.9),

$$\|f_t\|_{\mathcal{H}} = \sum_{\alpha \in \mathrm{A}^d} |a_\alpha|^2 t^{2|\alpha|} \nu_{2\alpha}. \tag{2.13}$$

This implies

$$\lim_{t \to 1-} \|f_t\|_{\mathcal{H}} = \sup_{t \to 1-} \|f_t\|_{\mathcal{H}} = \|f\|_{\mathcal{H}}. \tag{2.14}$$

Consequently, for $f \in \mathcal{O}(D)$

$$\|f_t\|_{\mathcal{L}^2(\mu)} = \sum_{\alpha \in \mathrm{A}^d} |a_\alpha|^2 t^{2|\alpha|} \nu_{2\alpha}, \tag{2.15}$$

which compared with (2.13), yields $f_t \in \mathcal{H}$ and

$$\|f_t\|_{\mathcal{H}} = \|f_t\|_{\mathcal{L}^2(\mu)}. \tag{2.16}$$

Taking into account that $f_t \in \mathcal{H}$ and also that the right-hand side of (2.15) is equal to $\|f_t\|_{\mathcal{H}}$, one gets from (2.14) the equality

$$\lim_{t \to 1-} \|f_t\|_{\mathcal{L}^2(\mu)} = \|f\|_{\mathcal{H}}. \tag{2.17}$$

On the other hand, if $f \in \mathcal{H}$, then, because $\overline{D} \subset t^{-1}D$, $f_t \in \mathcal{O}(t^{-1}D)$. Applying point 2° of Remark 1 (notice that it follows from (2.17) that the involved integral is finite), one has

$$|f_t(z)| \le C \left(\int_D |f_t(z)|^2 \mu(\mathrm{d}\, z) \right)^{1/2}, \quad z \in X$$

for every compact X contained in D. This entails that $\mathcal{H}_t \stackrel{\text{def}}{=} \{f_t \colon f \in \mathcal{H}\}$ is a closed subspace of $\mathcal{L}^2(\mu)$ composed of holomorphic functions, which, by the way, is a RKHS.

Some definitions:

$$l(f) \stackrel{\text{def}}{=} \sup_{0<t<1} \int_D |f(tz)|^2 \mu(\mathrm{d}\, z), \quad f \in \mathcal{O}(D),$$

$$\mathcal{L}^2(D, \mu) \stackrel{\text{def}}{=} \{f \in \mathcal{L}(D) \colon l(f) < +\infty\},$$

$$\|f\|_{\mathcal{L}^2(D,\mu)} \stackrel{\text{def}}{=} l(f)^{1/2}, \quad f \in \mathcal{L}^2(D, \mu).$$

If $f \in \mathcal{L}^2(D,\mu)$, then from (2.16) and (2.14), one can derive $f \in \mathcal{H}$ and

$$\lim_{t\to 1-}\int_D |f(tz)|^2\mu(\mathrm{d}\,z) = \sup_{0<t<1}\int_D |f(tz)|^2\mu(\mathrm{d}\,z) = \|f\|^2_{\mathcal{H}}.$$

Remark 2. One has to distinguish $\mathcal{H}^2(\mu)$ from just-defined $\mathcal{H}^2(D,\mu)$. The latter could be different than the previous, because (2.12) may not hold; the measure μ may live out of D (however, if (2.12) is satisfied, then both notations $H^2(\mu)$ and $H^2(D,\mu)$ are exchangeable; from (2.17), one gets finiteness of the integral). Details will appear later on.

Theorem 2. *Let ν be such that (2.10) holds. If μ is defined as in (2.8), then the space $\mathcal{H}$ corresponding to the kernel K defined by (2.9) coincides with $\mathcal{H}^2(D,\mu)$.*

Remark 3. Notice a kind of dichotomy depending on whether (2.12) is satisfied or not. The choice is between Theorem 1 and Theorem 2.

Remark 4. The identification of the Hilbert space $\mathcal{H}$ with the kernel K defined by (2.9) that is offered by Theorems 1 and 2 has the property that it replaces the coefficient in the power series expansion by those appearing in the Fourier expansion in $\mathcal{H}$. More precisely,

$$f = \sum_{\alpha\in \mathrm{A}^d} a_\alpha z^\alpha, \quad z\in D,$$

for $f \in \mathcal{H}^2(D,\mu)$ (or, equivalently, $f \in \mathcal{H}^2(D)$) if and only if $(\sqrt{\nu_{2\alpha}}\,a_\alpha)_{\alpha\in\mathrm{A}^d} \in \ell^2(\mathrm{A}^d)$, that is if

$$\sum_{\alpha\in\mathrm{A}^d} |a|^2_\alpha \nu_{2\alpha} < +\infty.$$

If this happens

$$\sum_{\alpha\in\mathrm{A}^d} |a|^2_\alpha \nu_{2\alpha} = \|f\|^2_{\mathcal{H}^2(D,\mu)} \text{ and } \sum_{\alpha\in\mathrm{A}^d} |a|^2_\alpha \nu_{2\alpha} = \|f\|^2_{\mathcal{H}^2(D)}.$$

Kernels on the Ball

The temporary convention is either B stands for the ball $B(0,1)$ or for the whole space $\mathbb{C}^d$ which can be considered as a ball of infinite radius. The latter does not differ from a polycylinder, but the kernel proposed here is unlike.

Remind that for two vectors $z=(z_1,\dots,z_d)$ and $w=(w_1,\dots,w_d)$ in $\mathbb{C}^d$ their inner product is as follows: $\langle z,w\rangle_d \overset{\mathrm{def}}{=} z_1\bar{w}_1+\cdots+z_d\bar{w}_d$. Going back to the formula (2.3), consider $s = 1$ and take a sequence $(k_n)_{n=0}^{\infty}$ of nonnegative numbers such that

$$\sum_{n=0}^{\infty} k_n z^n < +\infty \text{ either for } |z| < 1 \text{ or for } z \in \mathbb{C}.$$

Then the kernel K given by

$$K(z, w) \overset{\text{def}}{=} \sum_n k_n (\langle z, w \rangle_d)^n, \quad z, w \in B \tag{2.18}$$

is invariant under the group of unitary mappings of $\mathbb{C}^d$, that is,

$$K(Uz, Uw) = K(z, w) \text{ for an arbitrary unitar operator } U \text{ on } \mathbb{C}^d.$$

For $d = 1$, the kernels (2.6) and (2.18) coincide.

The most interesting kernels of the form are (2.18) The above kernel can be obtained from a formulae like (2.6), because the monomials are orthonormal in the respective spaces; normalization determines the spaces accurately.

K	B	Kernel
$(1 - \langle z, w \rangle_d)^{-(d+1)}$	$B(0, 1)$	Bergman
$(1 - \langle z, w \rangle_d)^{-d}$	$B(0, 1)$	Cauchy, $d = 1$ Szegő
$(1 - \langle z, w \rangle_d)^{-1}$	$B(0, 1)$	$d = 1$ Szegő's, otherwise Drury-Arveson
$e^{\langle z, w \rangle_d}$	$\mathbb{C}^d$	Segal–Bargmann

The first two kernels are presented in [15]; for the Segal–Bargmann space special attention will be paid to later. The Bergman space is originated with $d = 1$ in [7] while Segal-Bargmann's (any d) is in [18] and [6]. Szgő's (again $d = 1$) is in [25].

Bergman Space

This is the space $\mathcal{O}(\mathbb{D}) \cap \mathcal{L}^2(B, \pi^{-1} m_2)$ (remind the standard notation: $\mathbb{D} \overset{\text{def}}{=} \{z \in \mathbb{C}: |z| < 1\}$ – the unit disc and $\mathbb{T} \overset{\text{def}}{=} \{z \in \mathbb{C}: |z| = 1\}$ – the circle). Its kernel

$$K(z, w) = (1 - z\bar{w})^{-2}, \quad z, w \in \mathbb{D}$$

can be defined in several ways, for instance, from the normalization of the monomials as $(\sqrt{n+1}\, Z^n)_{n=0}^{\infty}$. However, for $f \in H^2(\mathbb{D})$

$$\|f\|^2 = \sum_{n=0}^{\infty} \frac{|a_n|^2}{n+1}, \quad f = \sum_n a_n Z^n.$$

In trying to monitor what happens in several variables, consider the polydisc first. Let $\nu = 2^d r\, m_d(\mathrm{d}r)$, where m_d is d-dimensional Lebesgue measure on $[0,1]^d$. Its moments are $\nu_\alpha = 2^d\big((\alpha_1+2)\cdots(\alpha_d+2)\big)^{-1}$, $\alpha \in \mathrm{A}^d$. Defining μ as in (2.8), one can deduce that it is the restriction of the measure $\pi^{-d}m_{2d}$ to $\mathbb{D}^d$. According to (2.9), the kernel K takes the form

$$K(z,w) \overset{\mathrm{def}}{=} \sum_{\alpha\in\mathrm{A}^d} \frac{1}{(\alpha_1+1)\cdots(\alpha_d+1)} z^\alpha \bar{w}^\alpha = \prod_{i=1}^{d}(1-z_i\bar{w}_i)^{-2}, \quad z,w\in\mathbb{D}^d.$$

Because ν satisfies (2.10) and (2.12), Theorem 1 is applicable, ensuring the Hilbert space corresponding to K is nothing but $H^2(\pi^{-d}m_{2d})$. Notice the formula

$$\|f\|^2 = \sum_{\alpha\in\mathrm{A}^d} \frac{|a_\alpha|^2}{(\alpha_1+1)\cdots(\alpha_d+1)}, \quad f = \sum_{\alpha\in\mathrm{A}^d} a_\alpha Z^\alpha.$$

On the other hand, the orthonormal basis in $B(0,1)\subset\mathbb{C}^d$ is given as

$$\sqrt{\frac{d!\,\alpha!}{(d+|\alpha|)!}}\, Z^\alpha, \quad \alpha\in\mathrm{A}^d,$$

and therefore, the kernel is precisely $(z,\bar{w}) \mapsto (1 - z\cdot\bar{w})^{-(d+1)}$, cf. [15].

Hardy Space

It is right time to consider another classical space. Here, the situation is more delicate than in the Bergman case. Now it is more convenient to employ Theorem 2 instead of Theorem 1 as done before. The reason is (2.12) is not satisfied.

Referring to Theorem 2, take as ν the point mass at $x=1$, that is, $\nu=\delta_1$. Now μ is nothing else than the Lebesgue measure on $\mathbb{T}$, which is going to be denoted as $m_{\mathbb{T}}$.

According to Remark 4 to every f in $H^2(\mathbb{D}, m_{\mathbb{T}})$, there corresponds $(a_n^{(f)})_n \in \ell^2$, such that $f \mapsto (a^{(f)_n})_n$ is a unitary operator. On the other hand, given $\boldsymbol{a} = (a_n^{(f)})_n$ from ℓ^2, there is a function $f^{(\boldsymbol{a})}$ in $L^2(\mathbb{T})$, for which the sequence $(a_n^{(f)})_n$ constitutes the sequence of Fourier coefficients (more precisely, $(Z^n)_{n+-\infty}^{+\infty}$ is an orthonormal basis in $L^2(\mathbb{T})$ with respect to which negative Fourier coefficients of f vanish). In other words,

$$f^{(\boldsymbol{a})} \overset{\mathrm{def}}{=} \sum_{n=0}^{\infty} a_n Z^n \text{ in the norm of } L^2(\mathbb{T}).$$

Denoting by $H^2(\mathbb{T})$ the image, that is, all the functions $L^2(\mathbb{T})$ obtained in this way, another <u>unitary</u> operator appears. The carried-out diagram

$$\begin{array}{ccc} & \ell^2 & \\ \swarrow & & \nwarrow \\ H^2(\mathbb{T}) & & H^2(\mathbb{D}, m_{\mathbb{T}}) \end{array}$$

has to be completed at the bottom by another unitary, this time from $H^2(\mathbb{D}, m_{\mathbb{T}})$ to $H^2(\mathbb{T})$. The steps in achieving this are as follows, cf. [12] for the tools.

The classical result quoted below makes an isometry $H^2(\mathbb{T}) \to H^2(\mathbb{D}, m_{\mathbb{T}})$.

Theorem 3. *If f is in $H^2(\mathbb{T})$, then the function $\tilde{f}$ defined as*

$$\tilde{f}(r\,e^{it}) \stackrel{\text{def}}{=} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{is})P_r(t-s)\,ds, \quad 0 \le r < 1,\ t \in \mathbb{R}, \tag{2.19}$$

where

$$P_r(s) \stackrel{\text{def}}{=} \frac{1-r^2}{1-2r\cos s + r^2}, \quad P_r \text{ is the Poisson kernel},$$

is in $H^2(\mathbb{D}, m_{\mathbb{T}})$.

Surjectivity or in other words, the return from $H^2(\mathbb{D}, m_{\mathbb{T}})$ to $H^2(\mathbb{T})$ results from Fatou's theorem.

Theorem 4. *If F is in $H^2(\mathbb{D}, m_{\mathbb{T}})$, then for almost every z in $\mathbb{T}$*

$$f(z) = \lim_{r \to 1-} F(rz)$$

exists and the function f belongs to $H^2(\mathbb{T})$. Moreover, the relation between f and $\tilde{f}$ appearing in (2.19) holds.

Remark 5. The mapping

$$V\colon H^2(\mathbb{D}, m_{\mathbb{T}}) \ni f \mapsto \tilde{f} \in H^2(\mathbb{T})$$

materializes the unitary operator Proposition 4 mentions.

The diagram considered before now looks like

$$\begin{array}{ccc} & \ell^2 & \\ \swarrow & & \nwarrow \\ H^2(\mathbb{T}) & \longleftrightarrow & H^2(\mathbb{D}, m_{\mathbb{T}}) \end{array}$$

The kernel of the Hardy space is

$$(1 - z\bar{w})^{-1}, \quad z, w \in \mathbb{D},$$

and is, surprisingly, called after Szegő; this duality in names reflects complexity of the above construction.

What about more variables? The kernels are:

$$\frac{1}{(1 - z_1\bar{w}_1)\cdots(1 - z_d\bar{w}_d)}, \quad z, w \in \mathbb{D}^d \text{ (polydisc)},$$
$$\frac{1}{(1 - z \cdot \bar{w})^d}, \quad z, w \in B(0,1) \text{ (the ball)}.$$

Segal–Bargmann Space

This space is deeply settled in quantum optics. It can be introduced by means of Proposition 1.

The starting point is the space $\mathcal{L}^2(\pi^{-d}\,\mathrm{e}^{-z\cdot\bar{z}}\,\mathrm{d}\,z) = \mathcal{L}^2(\mathbb{C}^d, \pi^{-d}\,\mathrm{e}^{-z\cdot\bar{z}}\,\mathrm{d}\,z)$ of functions on $\mathbb{C}^d$ square integrable with respect to the Gaussian measure $\pi^{-d}\,\mathrm{e}^{-z\cdot\bar{z}}$. Take the holomorphic part of this space, that is,

$$\mathcal{H}^2(\mathbb{C}^d, \pi^{-d}\,\mathrm{e}^{-z\cdot\bar{z}}\,\mathrm{d}\,z) \stackrel{\text{def}}{=} \mathcal{O}(\mathbb{C}^d) \cap \mathcal{L}^2(\pi^{-d}\,\mathrm{e}^{-z\cdot\bar{z}}\,\mathrm{d}\,z).$$

Because the Gaussian measure fits in Proposition 1, the resulting space $\mathcal{L}^2(\mathbb{C}^d, \pi^{-d}\,\mathrm{e}^{-z\cdot\bar{z}}\,\mathrm{d}\,z)$ is a RKHS. The functions

$$e_\alpha \colon z \to \frac{z^\alpha}{\sqrt{\alpha!}}, \quad \alpha \in \mathrm{A}^d \tag{2.20}$$

constitute an orthonormal set in $\mathcal{H}^2(\mathbb{C}^d, \pi^{-d}\,\mathrm{e}^{-z\cdot\bar{z}}\,\mathrm{d}\,z)$. Let f be a function from $H^2(\mathbb{C}^d, \pi^{-d}\,\mathrm{e}^{-z\cdot\bar{z}}\,\mathrm{d}\,z)$ orthogonal to (2.20). Developing it as in (2.1), one gets

$$0 = \pi^{-d}\int_{\mathbb{C}^d}\sum_{\alpha\in\mathrm{A}^d} c_\alpha z^\alpha \overline{e_\beta(z)}\,\mathrm{e}^{-\langle z,z\rangle}\,\mathrm{d}\,z = \pi^{-d}\sum_{\alpha\in\mathrm{A}^d} c_\alpha\sqrt{\alpha!}\int_{\mathbb{C}^d} e_\alpha(z)\overline{e_\beta(z)}\,\mathrm{e}^{-\langle z,z\rangle}\,\mathrm{d}\,z,$$

which makes all the c_αs zero. Thus, $(e_\alpha)_\alpha$ is complete, hence a basis in $\mathcal{H}^2(\mathbb{C}^d, \pi^{-d}\,\mathrm{e}^{-z\cdot\bar{z}}\,\mathrm{d}\,z)$. The Zaremba formula defines the kernel as

$$K(z,w) = \sum_{\alpha\in\mathrm{A}^d}\frac{z^\alpha}{\sqrt{\alpha!}}\frac{\bar{w}^\alpha}{\sqrt{\alpha!}} = \sum_{\alpha\in\mathrm{A}^d}\prod_{i=1}^{d}\frac{(z_i\bar{w}_i)^{\alpha_i}}{\alpha_i!} = \mathrm{e}^{\langle z,w\rangle}, \quad z, w \in \mathbb{C}^d.$$

The Segal–Bargmann space $\mathcal{H}^2(\mathbb{C}^d, \pi^{-d}\, \mathrm{e}^{-z\cdot\bar{z}}\, \mathrm{d}\, z)$ is unitary equivalent to $\mathcal{L}^2(\mathbb{R}^d)$ by means of an integral transform. It is more convenient to define the inverse to the transformation in question, that is, a mapping from $\mathcal{L}^2(\mathbb{R}^d)$ to $\mathcal{H}^2(\mathbb{C}^d, \pi^{-d}\, \mathrm{e}^{-z\cdot\bar{z}}\, \mathrm{d}\, z)$ by

$$\pi^{-d/4} \int_{\mathbb{R}^d} \mathrm{e}^{-\frac{1}{2}(z\cdot\bar{z}+x\cdot x)+\sqrt{2}z\cdot x}\, \phi(x)\, \mathrm{d}\, x, \quad z \in \mathbb{C}^d, \quad \phi \in \mathcal{L}^2(\mathbb{R}^d). \tag{2.21}$$

Formula (2.21) defines a unitary transformation of $\mathcal{L}^2(\mathbb{R}^d)$ onto $\mathcal{H}^2(\mathbb{C}^d, \pi^{-d}\, \mathrm{e}^{-z\cdot\bar{z}}\, \mathrm{d}\, z)$, which transforms the bases as

$$h_\alpha \mapsto e_\alpha, \quad \alpha \in \mathrm{A}^d,$$

where h_α are Hermite functions

$$h_\alpha(x) \stackrel{\text{def}}{=} \sqrt{2^\alpha \alpha! \sqrt{\pi}}\, \mathrm{e}^{-\frac{x\cdot x}{2}}\, H_\alpha(x), \quad x \in \mathbb{R}$$

where H_α are d-dimensional (product) Hermite polynomials. The image of

$$x \mapsto \pi^{-d/4}\, \mathrm{e}^{-\frac{1}{2}(w\cdot\bar{w}+x\cdot x)+\sqrt{2}\,\bar{w}\cdot x}$$

from $\mathcal{L}^2(\mathbb{R})$ via the transformation (2.21) is in $\mathcal{H}^2(\mathbb{C}^d, \pi^{-d}\, \mathrm{e}^{-z\cdot\bar{z}}\, \mathrm{d}\, z)$ the kernel function $z \mapsto \mathrm{e}^{z\bar{w}}$.

The inverse transformation is of integral form as well with the kernel being the conjugate of (2.21).

Integrability of Positive-Definite Kernels

Let Ω be an open set in $\mathbb{C}^d$ satisfying (•), K be a holomorphic kernel on Ω (cf. Proposition 1), and $\mathcal{H}$ the corresponding RKHS. Because (•) guarantees the Taylor expansion, and by Proposition (9) from [24] does the polynomials are in $\mathcal{H}$. A couple $(\mathcal{H}, K)$ (and each of its members) is called *integrable*, if there is a nonnegative measure μ on $\mathbb{C}^d$ isometry $V\colon \mathcal{H} \to \mathcal{L}^2(\mu)$ such that

$$V p_\Omega = p_{\operatorname{supp}\mu}, \quad p \in \mathbb{C}[Z_1, \dots, Z_d]. \tag{2.22}$$

Most of the spaces of holomorphic function is integrable; some (even of analytic in nature) are not (warning!), cf. Dirichlet spaces.

Proposition 5. *A couple $(\mathcal{H}, K)$ is integrable if and only if the (bi)sequence $(c_{\alpha,\beta})_{\alpha,\beta\in\mathrm{A}^d}$*

$$c_{\alpha,\beta} \stackrel{\text{def}}{=} \langle Z^\alpha, Z^\beta\rangle_{\mathcal{H}}, \quad \alpha, \beta \in \mathrm{A}^d \tag{2.23}$$

is a complex moment sequence (for complex moment problem, refer to [11]), that is, μ *on* $\mathbb{C}^d$ *such that*

$$c_{\alpha,\beta} = \int_{\mathbb{C}^d} z^\alpha \bar{z}^\beta \mu(dz), \quad \alpha, \beta \in \mathrm{A}^d.$$

Because $\mathcal{L}^2(\mu)$ is closed with respect to complex conjugation and $\mathcal{H}$ is not, the following observation is not surprising.

Proposition 6. *If a couple* $(\mathcal{H}, K)$ *is integrable, then*

$$\sum_{\alpha,\beta,\gamma,\delta \in \mathrm{A}^d} \langle Z^{\alpha+\delta}, Z^{\beta+\gamma} \rangle_{\mathcal{H}} \lambda_{\alpha,\beta} \bar{\lambda}_{\gamma,\delta} \geq 0, \quad (\lambda_{\alpha,\beta})_{\alpha,\beta \in \mathrm{A}^d} \subset \mathbb{C}. \tag{2.24}$$

The problem is in the fact that (2.24) is nothing but a necessary condition. Though there are essential reasons behind like Hilbert's example concerning positive polynomials in two variables or Nelson's type examples concerning commutativity of essential self-adjoint operators, a direct construction is rather elaborate.

Example 1. This is based on [10] where further details can be found. Define a mapping $m\colon \{0,1,2\ldots\}^2 \to \{1,2\ldots\}$ as follows:

$$m(i,j) = \tfrac{1}{2}(i+j)(i+j+1) + j + 1, \quad \text{for } i+j \geq 4$$

$$m(0,0) = 1, \quad m(1,2) = 2, \quad m(2,1) = 3, \quad m(1,1) = 4, \quad m(1,0) = 5$$

$$m(0,1) = 6, \quad m(0,2) = 7, \quad m(3,0) = 8, \quad m(0,3) = 9.$$

m is a bijection, and therefore,

$$b_{2i+1,2j+1} = b_{2i+1,2j} = b_{2i,2j+1} = 0, \quad b_{2i,2j} = g_{m(i,j)}, \quad i,j = 0,1,\ldots$$

where $g_1 = g_2 = g_3 = 1$, $g_4 = 4$, $g_n = n!^{(n+1)!}$ for $n \geq 5$.

One can check repeating the arguments of [10] that the sequence

$$c_{m,n} = \sum_{i=0}^{m} \sum_{j=0}^{n} \binom{m}{i} \binom{n}{j} \mathrm{i}^{m-i}\, \mathrm{i}^{n-j}\, b_{i+j,m+n-i-j}, \quad m,n = 0,1,\ldots,$$

is positive definite in a sense which is close to (2.24), though it is not a complex moment sequence.

Another example of this kind is in [8]; Dirichlet space, which follows, is not integrable, but in this case, the necessary condition (2.24) fails to hold.

Now something productive. Consider $d = 1$, $D = \mathbb{D}$ or $D = \mathbb{C}$ and start with the following:

Proposition 7. *Let $\mathcal{H}$ be a space of holomorphic functions on $D \subset \mathbb{C}$ and its kernel K defined by* (2.6)*, that is,*

$$K(z,w) \stackrel{def}{=} \sum_{n=0}^{\infty} k_n z^n \bar{w}^n, \quad z,w \in D. \tag{2.25}$$

The following conditions are equivalent:

(a) *The couple $(\mathcal{H}, K)$ is integrable.*
(b) *Equation* (2.24) *holds.*
(c) *The sequence $(c_{m,n})_{m,n=0}^{\infty}$ defined by* (2.23) *is of the form*

$$c_{m,n} = k_n^{-1}\delta_{m,n}, \quad m,n = 0,1,\ldots \tag{2.26}$$

and moreover,

$$\sum_{m,n=0}^{N} k_{m+n}^{-1}\xi_m\bar{\xi}_n \geq 0, \quad \sum_{m,n=0}^{N} k_{m+n+1}^{-1}\xi_m\bar{\xi}_n \geq 0, \quad (\xi_n)_{n=0}^{N} \subset \mathbb{C}.$$

(d) *The sequence $(c_{m,n})_{m,n=0}^{\infty}$ defined by* (2.23) *is as in* (2.26) *and there is a nonnegative measure ν on $[0,+\infty)$ such that*

$$k_n = \int_0^{+\infty} x^n \nu(dx), \quad n = 0,1,\ldots$$

The second case refers rather to Proposition 5 than to Proposition 7.

Theorem 5. *$(c_{m,n})_{mn,=0}^{\infty}$ is a complex moment sequence if there exists a function $\omega\colon \{(m,n)\colon m+n \geqslant 0\} \to \mathbb{C}$ such that*

$$\sum_{\substack{m+n\geq 0\\ p+q\geq 0}} \omega(m+q,n+p)\lambda_{m,n}\bar{\lambda}_{p,q} \geq 0, \quad (\lambda_{m,n})_{m+n\geq 0} \subset \mathbb{C}$$

and

$$c_{m,n} = \omega(m,n), \quad m,n = 0,1,\ldots$$

The above theorem is in [19] where backward references can be found as well. For integrable RKHSs, a kind of converse to a fact included in Proposition 21 of [24] holds.

Proposition 8. *Let $(\mathcal{H}, K)$ be an integrable couple and the isometry V determined by (2.22) be a multiplicative operator such that*

$$\sup_{z \in \operatorname{supp} \mu} |V\varphi(z)| = \sup_{z \in \Omega} |\varphi(z)|, \text{ if both suprema are finite} \tag{2.27}$$

If φ is a multiplier of $(\mathcal{H}, K)$, then $\|M_\varphi\|_{\mathcal{H}} \leq \sup_{z \in \operatorname{supp} \mu} |\varphi(z)|$.

Together with Proposition 21 of [24], one has the following:

Corollary 2. *If a couple $(\mathcal{H}, K)$ is as in Proposition 8 and $K(z, z) > 0$ for $z \in \Omega$, then φ is a multiplier if and only if $\sup_{z \in \Omega} |\varphi(z)| < +\infty$. Then*

$$\|M_\varphi\|_{\mathcal{H}} = \sup_{z \in \Omega} |\varphi(z)|. \tag{2.28}$$

In the Hardy space case, though Ω and $\operatorname{supp} \mu$ are disjoint, the isometry V from Remark 5 satisfies condition (2.27). This is in fact a variant of Fatou's theorem (cf. [12]), which identifies bounded functions from $\mathcal{H}(\mathbb{D}, m_{\mathbb{T}})$ by its radial limits (as in Theorem 4) with members of $\mathcal{L}^\infty(\mathbb{T})$, allowing the formally looking definition

$$\mathcal{H}^\infty(\mathbb{T}) \stackrel{\text{def}}{=} \mathcal{H}(\mathbb{D}) \cap \mathcal{L}^\infty(\mathbb{T})$$

to make sense. Consequently, (2.28) looks like

$$\|M_\varphi\|_{\mathcal{H}} = \sup_{z \in \Omega} |\varphi(z)| = \|\tilde{\varphi}\|_{L^\infty(\mathbb{T})}.$$

Integrability of RKHSs allows to characterize densely defined multipliers as in the known theorem of Newman–Shapiro, cf. [13] and also [20].

In addition to the above integrability is a natural environment for considering the so-called Bergman projections.

Dirichlet Spaces

The common meaning of Dirichlet space $\mathbb{D}$ is for holomorphic functions on $\mathbb{D}$ with the following inner product

$$\langle f, g \rangle_{\mathcal{D}^2(\mathbb{D})} \stackrel{\text{def}}{=} \langle \partial f, \partial g \rangle_{\mathcal{H}^2(\pi^{-1} m_2)} + \langle f, g \rangle_{\mathcal{H}^2(\mathbb{T}, m_{\mathbb{T}})},$$
$$f, g \in \mathcal{H}(\mathbb{D}),\ \partial f, \partial g \in \mathcal{H}^2(\pi^{-1} m_2).$$

The space $\mathcal{D}^2(\mathbb{D})$ is complete. Like the spaces considered so far, it satisfies the basic condition (A) from [24]. Its kernel is just

$$-\frac{1}{z\bar{w}}\log(1-z\bar{w}), \quad z, w \in \mathbb{D},$$

with an orthonormal basis $(\sqrt{n+1}\,Z^n)_n$. Moreover,

$$\|f\|^2_{D^2(\mathbb{D})} = \sum_{n=0}^{\infty}(n+1)|a_n|^2, \quad f(z) = \sum_{n=0}^{\infty} a_n z^n,\ z \in \mathbb{D}.$$

The Dirichlet space is not integrable because (2.24) is not satisfied though the kernel is of the form (2.25). This means that (2.28) does not hold.

Looking at all those spaces defined on $\mathbb{D}$ considered so far, one may conclude that they belong to a one-parameter family of RKHS with norms defined as

$$\|f\|^2_{D^2_s(\mathbb{D})} = \sum_{n=0}^{\infty}(n+1)^{-s}|a_n|^2, \quad f(z) = \sum_{n=0}^{\infty} a_n z^n,\ z \in \mathbb{D},$$

where s is a real parameter.

The close relative of $D^2(\mathbb{D})$ is a Sobolev space $\mathcal{W}^2_1[01]$, which is also an RKHS.

de Branges–Rovnyak Spaces

This is the whole family of spaces serving for building models of contraction. For ϕ in $\mathcal{H}(\mathbb{D})$, consider the kernel

$$K_\phi(z,w) \stackrel{\text{def}}{=} \frac{1-\phi(z)\overline{\phi(w)}}{1-z\bar{w}}, \quad z, w \in \mathbb{D}.$$

Proposition 9. *The kernel K_ϕ is positive definite if and only if $|\phi(z)| \le 1$, $z \in \mathbb{D}$, and this happens if and only if ϕ is a multiplier of $\mathcal{H}^2(\mathbb{D}, m_\mathbb{T})$ and $\|M_\phi\| \le 1$.*

If K stands now for the Szegő kernel, the obvious equality $K - K_\phi = K\phi(\cdot)\phi(-)$, by Schur's lemma, yields

$$K_\phi \ll K,$$

which makes it possible to apply any of the conditions in (e) of [24]. In particular, one gets a contractive imbedding of the de Branges–Rovnyak space into the Hardy one, the topic analyzed in [17].

q–Spaces

Some definitions first. For x a positive integer and $q \in \mathbb{C}$, set

$$[x]_q \stackrel{\text{def}}{=} 1 + q \cdots + q^{x-1}.$$

Furthermore,

$$(a;q)_0 \stackrel{\text{def}}{=} 1, \quad (a;q)_k \stackrel{\text{def}}{=} (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{k-1}), \quad k = 1,2,3,\ldots$$

With $[n]_q! \stackrel{\text{def}}{=} [1]_q \cdots [n]_q$ for $n > 0$ and $[0]_q! \stackrel{\text{def}}{=} 1$, one has $[n]_q! = (q,q)_n(1-q)^{-n}$. Moreover,

$$(a;q)_\infty \stackrel{\text{def}}{=} \prod_{k=0}^{\infty}(1-aq^k), \quad |q| < 1.$$

The q-exponential can be defined in two ways (both tend to the standard exponential if $q \to 1$ and satisfy the Cauchy functional equation for q-commuting variables).

$$e_q(z) \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} \frac{1}{(q;q)_k} z^k, \quad z \in \omega_q,$$

$$E_q(z) \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} \frac{q^{\binom{k}{2}}}{(q;q)_k} z^k, \quad z \in \omega_{q^{-1}}, \quad q \neq 0,$$

where

$$\omega_q \stackrel{\text{def}}{=} \begin{cases} \{z\colon\ |z| < 1\} & \text{gdy } |q| < 1, \\ \mathbb{C} & \text{otherwise.} \end{cases}$$

Some details can be found in [4].

For $|q| < 1$, one gets $e_q(z) = ((z,q)_\infty))^{-1}$ and $E_q(z) = (-z,q)_\infty$. On the other hand, from

$$(q,q)_k = (-1)^k \frac{(q^{-1})^{\binom{k}{2}}}{(q^{-1};q^{-1})_k}, \quad k = 0,1,\ldots, \quad q \neq 0$$

the functions are related as

$$e_q(z) = E_{q^{-1}}(-z), \quad z \in \omega_q, \quad q \neq 0. \tag{2.29}$$

This is a kind of duality which allows to make a replacement

$$q \longleftrightarrow q^{-1}$$

in some formulae.

The *q-derivative* D_q defined as

$$D_q f(z) \overset{\text{def}}{=} \begin{cases} \frac{f(z)-f(qz)}{(1-q)z} & \text{if } z \neq 0 \\ f'(z) & \text{if } z = 0 \end{cases}$$

acts on $\mathbb{C}[Z]$ as $D_q Z^n = [n]_q Z^{n-1}$ and

$$D_q e_q(z) = e_q(z), \quad D_q E_q(z) = E_q(z).$$

Look how replacing the standard exponential by its q version impacts the Segal–Bargmann space.

The only kernel of the form (2.6) or, what is equivalent to, (2.18), and which mimics the Segal–Bargmann space, is

$$K(z, w) \overset{\text{def}}{=} \sum_{k=0}^{\infty} \frac{1}{[k]_q!} z^k \bar{w}^k = e_q((1-q)z\bar{w}) \quad \text{for } z, w \in |1-q|^{-1/2}\omega_q. \tag{2.30}$$

Now there are two possibilities, completely different: $0 < q < 1$ and $q > 1$, cf. [22].

ⓐ Consider the kernel (2.30) for $0 < q < 1$. Then the operator M_Z is bounded in the corresponding RKHS and $\|M_Z\| \leq (1-q)^{-1/2}$. The couple $(K, \mathcal{H})$ is integrable (cf. [3]) and the only measure is $\mu = G\, m_2$,

$$G(z) \overset{\text{def}}{=} \begin{cases} \frac{1}{2\pi}[q^{-k}(1-q)]^{1/2} E_q(-q(1-q)|z|^2) & \text{if } |z| = [q^k(1-q)^{-1}]^{1/2}, \\ & k = 0, \pm 1, \pm 2, \ldots \\ 0 & \text{otherwise} \end{cases}.$$

These spaces realize a kind of interpolation between the Hardy space ($q \to 0+$) and the Segal–Bargmann one ($q \to 1-$).

ⓑ The case $q > 1$ can be derived from that $0 < q < 1$, employing (2.29)

$$K(z, w) \overset{\text{def}}{=} E_q((1-q)z\bar{w}), \quad z, w \in |1-q|^{-1/2}\omega_q. \tag{2.31}$$

However, one can proceed in another way. Take $0 < q < 1$, and after setting

$$k_n \overset{\text{def}}{=} \frac{q^{n+\binom{n}{2}}(1-q)^n}{(q;q)_n},$$

define the kernel K as

$$K(z,w) \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} k_n z^n \bar{w}^n, \quad z, w \in \mathbb{C}.$$

Then

$$K(z,w) = E_q(q(1-q)z\bar{w})$$

is just precisely equal to the kernel defined by (2.31) with q replaced by $1-q+q^2$. With $A_q \stackrel{\text{def}}{=} (\int_0^{\infty} e_q(-(1-q)t)\,\mathrm{d}t)^{-1}$ i $B_q \stackrel{\text{def}}{=} (\sum_{k=-\infty}^{+\infty} e_q(-(1-q)q^k))^{-1}$, making use of calculations included in [5] (see also [22]), one gets two measures $\mu_i = G_i\, m_2$, $i = 1, 2$, where

$$G_1(z) = \frac{1}{\pi} A_q e_q(-(1-q)|z|^2)$$

and

$$G_2(z) \stackrel{\text{def}}{=} \begin{cases} \frac{1}{2\pi} B_q q^{\frac{1}{2}k} e_q(-(1-q)|z|^2) & \text{if } |z| = q^{\frac{1}{2}k}, k = 0, \pm 1, \pm 2, \ldots \\ 0 & \text{otherwise} \end{cases}$$

determining two different in nature $\mathcal{L}^2$ spaces in which the same RKHS $\mathcal{H}$ is contained. One of them, μ_1 is absolutely continuous with respect to the 2-dimensional Lebesgue measure, the other, μ_2, sits on a countable number of circles tending to the origin and from the other side to infinity; needless to say that each convex combination of these two (as well as weak limits) generates still another $\mathcal{L}^2$ "superspaces". I)

Pick–Nevanlinna Interpolation Problem

A beautiful example of usefulness of the developed theory so far is in the interpolation problem of Pick-Nevanlinna type, which goes back to the beginning of the twentieth century. It can be stated as follows:

Given $(z_\alpha)_{\alpha\in \mathrm{A}} \subset \mathbb{D}$ and $(w_\alpha)_{\alpha\in \mathrm{A}} \subset \mathbb{C}$, does there exist
A function $\varphi \in \mathcal{H}(\mathbb{D})$ such that $|\varphi(z)| \le 1$ for $z \in \mathbb{D}$
And $\varphi(z_\alpha) = w_\alpha$ for every $\alpha \in \mathrm{A}$?

The solution comes from Pick (A finite) and Nevanlinna (A arbitrary). The answer: this is possible if and only if

$$\sum_{i,j} \frac{1 - w_{\alpha_i}\bar{w}_{\alpha_j}}{1 - z_{\alpha_i}\bar{z}_{\alpha_j}} \lambda_i \bar{\lambda}_j \ge 0, \quad (\alpha_i)_i \subset \mathrm{A},\ (\lambda_i)_i \subset \mathbb{C}. \tag{2.32}$$

For more about this, look at [1, 2]. Besides analytic solution, so to speak, there are operator theoretic ones. Two of them are going to be outlined here.

Lifting Commutant Method

Notice that in (2.32), the appearing kernel is Szegő's; denote it by K. In the Hardy space $\mathcal{H}^2(\mathbb{D}, m_{\mathbb{T}})$, introduce the subspace $\mathcal{H}_{\mathrm{A}}$ equal to clolin$\{K_{z_\alpha}\colon\ \alpha \in \mathrm{A}\}$ and consider the operator R acting on (some) kernel functions as

$$R\colon \mathcal{H}_{\mathrm{A}} \ni K_{z_\alpha} \mapsto \bar{w}_\alpha K_{z_\alpha} \in \mathcal{H}_\alpha.$$

Condition (2.32) says that R is a contraction in $\mathcal{H}_{\mathrm{A}}$. If T is an operator $PM_Z|_{\mathcal{H}_{\mathrm{A}}}$, where M_Z is the multiplication by the independent variable in $\mathcal{H}^2(\mathbb{D}, m_{\mathbb{T}})$ and P is the orthogonal projection on $\mathcal{H}_{\mathrm{A}}$, then R commutes with T^*. Apply now the well-known theorem on lifting commutant to the couple T^* (contraction) and R (operator from the commutant of T^*). Because M_Z^* is a coisometry, one gets an extension S of R to the whole space $\mathcal{H}^2(\mathbb{D}, m_{\mathbb{T}})$ which preserves the norm as well as commutes with M_Z^*. Thus, $\|S\| \le 1$, because $\|R\| \leqslant 1$. Now the culminating moment: because S^* commutes with M_Z, it does so with every polynomial in M_Z and consequently with every M_{K_z} (K_z is a multiplier because it is bounded). Finally from Corollary 26 of [24] has $S^* = M_\varphi$, with $\varphi = S^*1$. Therefore, $\varphi \in \mathcal{M}(\mathcal{H}^2(\mathbb{D}, m_{\mathbb{T}}))$ and $\sup_{z\in\mathbb{D}} |\varphi(z)| \le 1$. A subtle calculation

$$\begin{aligned}\varphi(z_\alpha) &= \langle \varphi, K_{z_\alpha}\rangle_{\mathcal{H}^2(\mathbb{D},m_{\mathbb{T}})} = \langle M_\varphi 1, K_{z_\alpha}\rangle_{\mathcal{H}^2(\mathbb{D},m_{\mathbb{T}})}\\ &= \langle S^*1, K_{z_\alpha}\rangle_{\mathcal{H}^2(\mathbb{D},m_{\mathbb{T}})} = \langle 1, SK_{z_\alpha}\rangle_{\mathcal{H}^2(\mathbb{D},m_{\mathbb{T}})}\\ &= \langle 1, RK_{z_\alpha}\rangle_{\mathcal{H}_{\mathrm{A}}} = w_\alpha.\end{aligned}$$

completes the argument. Basic references are in [16] and [9].

Korány–Sz.-Nagy Method

Another operator method, less known, is this proposed in [26]. The tool for that consists in properties of resolvents of self-adjoint operators. In this way, the Pick–Nevanlinna interpolation involving Hardy space on a halfplane (see [14] for more details) comes out.

Elements of Dilation Theory

The previous section deals with the branch (A) encoded in [24] the present one shows some possibilities which opens the subdivision (B). Much more details are in Chapter 2 of [21] and extensions to $\mathcal{C}^*$–Hilbert modules as well as most of the references can be found in [23].

Operator Kernels and Their Generalizations

There is plenty of situations in which one considers kernels whose values are operators (bounded and unbounded) rather than scalars. The most common example is the kernel

$$\boldsymbol{K}: X \times X \to \mathbf{B}(\mathcal{H}).$$

Positive definiteness of $\boldsymbol{K}$ means now

$$\sum_{i,j=1}^{N} \langle \boldsymbol{K}(x_i, x_j) f_i, \bar{f_j} \rangle \geq 0, \quad x_1, \ldots, x_N \in X, \;\; f_1, \ldots f_N \in \mathcal{H}.$$

An easy trick reduces this situation to the scalar case. Instead of the set X, consider $X \times \mathcal{H}$; then $(x, f, y, g) \mapsto \langle g, \boldsymbol{K}(x, y) f \rangle_{\mathcal{H}}$ becomes a scalar kernel; positive definiteness of the new kernel will be sorted out immediately.

Declare, once and for all in this section, the following situation happens: given a set X, a linear space $\mathcal{E}$ and a kernel K on $X \times \mathcal{E}$; reorder variables of the kernel K having it defined as $K: X \times X \times \mathcal{E} \times \mathcal{E} \to \mathbb{C}$. Moreover, assume that always

$$g \mapsto K(x, y, f, g) \text{ is a linear function with fixed } x, y \in X \text{ and } f \in \mathcal{E}.$$

Having in mind that the first variable from the previous section is now the "first-third variable" group and the second variable is the "second-fourth variable" group, positive definiteness of a scalar valued kernel reads as

$$\sum_{i,j=1}^{N} K(x_i, x_j, f_i, f_j)\lambda_i \bar{\lambda}_j \geq 0, \; (x_1, f_1), \ldots, (x_N, f_N) \in X \times \mathcal{E}, \; \lambda_1, \ldots \lambda_N \in \mathbb{C}. \tag{2.33}$$

Symmetry of the kernel which satisfies the above condition of positive definiteness means now

$$K(x, y, f, g) = \overline{K(y, x, g, f)}, \quad x, y \in X, \;\; f, g \in \mathcal{E},$$

which in turn implies that with fixed $x, y \in X$ and $g \in \mathcal{E}$, the map $f \mapsto K(x, y, f, g)$ is a linear function. As a result, positive definiteness in the sense of (2.33) is equivalent to

$$\sum_{i,j=1}^{N} K(x_i, x_j, f_i, f_j) \geq 0, \quad x_1, \ldots, x_N \in X, \;\; f_1, \ldots f_N \in \mathcal{E}, \tag{2.34}$$

which becomes the certified definition of positive definiteness of kernels considered here.

Thus, $\mathcal{H}$ is a Hilbert space of functions on $X \times \mathcal{E}$ with reproducing kernel $K\colon X \times X \times \mathcal{E} \times \mathcal{E} \to \mathbb{C}$ on $X \times \mathcal{E}$, kernel functions $K_{x,f} \stackrel{\text{def}}{=} K(\,\cdot\,, -, x, f)$, $(x, f) \in X \times \mathcal{E}$ belonging to $\mathcal{H}$ and the reproduction property

$$F(x, f) = \langle F, K_{x,f} \rangle, \quad f \in \mathcal{H},\ x \in X,\ f \in \mathcal{E}. \tag{2.35}$$

From the general theory presented in [24], repeat the Corollary 6 of [24] giving it a status of:

Theorem 6. *If K is a positive-definite kernel on $X \times \mathcal{E}$, then there exists a Hilbert space $\mathcal{H}$ and a map $X \times \mathcal{E} \ni (x, f) \mapsto K_{x,f} \in \mathcal{H}$ such that*

$$\mathcal{H} = \operatorname{clolin}\{K_{x,f}\colon (x, f) \in X \times \mathcal{E}\},$$
$$K(x, f, y, g) = \langle K_{y,g}, K_{x,f} \rangle, \quad (x, f), (y, g) \in X \times \mathcal{E}.$$

For every $x \in X$ map $\mathcal{E} \ni f \mapsto K_{x,f} \in \mathcal{H}$ is linear. Space $\mathcal{H}$ is a Hilbert space with reproducing kernel K on $X \times \mathcal{E}$ whose elements are functions F on $X \times \mathcal{E}$ which are antilinear in the second variable.

Dilations on Semigroups

Assume additional structure on the set X.

$$\boxed{X = \mathfrak{S} \text{ is a (multiplicative) semigroup.}}$$

Since $\mathfrak{S}$ is not assumed to be commutative, it is written, as usually, the multiplicative notation for the semigroup operation. One can also consider X to be an arbitrary set, and $\mathfrak{S}$ a semigroup of the *action* on X, that is, if $\mathfrak{s} \in \mathfrak{S}$, then $\mathfrak{s}\colon X \to X$, no difficulty in carrying out alternative versions of the present investigations.

Assume that $\mathfrak{S}$ is a unital semigroup, that is, there exists an element $1 \in \mathfrak{S}$ such that $\mathfrak{s}1 = 1\mathfrak{s} = \mathfrak{s}$ for every $\mathfrak{s} \in \mathfrak{S}$. Kernel K will be called *non-degenerate*, if $K(1, f, 1, f) = 0$ implies $f = 0$. In case where $\mathcal{E}$ is a normed space, say the kernel K is *isometric*, if $K(1, f, 1, f) = \|f\|^2$. Putting

$$V\colon \mathcal{E} \ni f \to K_{1,f} \in \mathcal{H}, \tag{2.36}$$

one obtains a linear map, which in case of non-degenerate kernel is an injection and in case of isometric kernel is an isometry. Take $\mathfrak{u} \in \mathfrak{S}$, and for $F \in \mathcal{H}$, define a map $F_{\mathfrak{u}}$ on $X \times \mathcal{E}$ by

$$F_{\mathfrak{u}}(\mathfrak{s}, \mathfrak{t}) \stackrel{\text{def}}{=} F(\mathfrak{u}\mathfrak{s}, f), \quad (\mathfrak{s}, \mathfrak{t}) \in X \times \mathcal{E}, \tag{2.37}$$

and a linear space

$$\mathcal{D}(\mathfrak{u}) \stackrel{\text{def}}{=} \{F \in \mathcal{H}\colon\ F_{\mathfrak{u}} \in \mathcal{H}\}.$$

Define now two linear operators in $\mathcal{H}$ related to $\mathfrak{u} \in \mathfrak{S}$. First of them, $\Phi(\mathfrak{u})$, by

$$\mathcal{D}(\Phi(\mathfrak{u})) \stackrel{\text{def}}{=} \mathcal{D}(\mathfrak{u}), \quad \Phi(\mathfrak{u})F \stackrel{\text{def}}{=} F_{\mathfrak{u}}, \quad F \in \mathcal{D}(\mathfrak{u}).$$

The second, $\Phi_{\mathfrak{u}}$, will be given by

$$\mathcal{D}(\Phi_{\mathfrak{u}}) \stackrel{\text{def}}{=} \mathcal{D}_K, \quad \Phi_{\mathfrak{u}} \sum\nolimits_i \xi_i K_{s_i, f_i} = \sum\nolimits_i \xi_i K_{\mathfrak{u}s_i, f_i}, \quad (s_i)_i \subset \mathfrak{S},\ (f_i)_i \subset \mathcal{E}$$

is well defined if and only if

$$\sum\nolimits_i \xi_i K_{s_i, f_i} = 0 \implies \sum\nolimits_i \xi_i K_{\mathfrak{u}s_i, f_i} = 0. \tag{2.38}$$

Their basic properties are collected in the following:

Proposition 10. *If* (2.38) *holds, then*

$$\langle \Phi(\mathfrak{u})F, K_{\mathfrak{s},f}\rangle = \langle F, \Phi_{\mathfrak{u}} K_{\mathfrak{s},f}\rangle, \quad F \in \mathcal{D}(\mathfrak{u}),\ (\mathfrak{s}, f) \in X \times \mathfrak{S} \tag{2.39}$$

and the operator $\Phi(\mathfrak{u})$ is closed. Moreover, $\Phi_{\mathfrak{u}}^* = \Phi(\mathfrak{u})$; $\Phi_{\mathfrak{u}}$ is closable if and only if $\Phi(\mathfrak{u})$ is densely defined, and then $\bar{\Phi}_{\mathfrak{u}} = \Phi(\mathfrak{u})^*$.

Remark 6. The proof above implies that $\Phi(\mathfrak{u})$ is always closed.

Boundedness of the operators $\Phi(\mathfrak{u})$ is determined by:

Proposition 11. *$\Phi_{\mathfrak{u}}$ is a well-defined operator, which is a bounded operator if and only if there exists $c(u) \geq 0$ such that*

$$\sum_{i,j=1}^{N} K(\mathfrak{u}\mathfrak{s}_i, \mathfrak{u}\mathfrak{s}_j, f_i, f_j) \leq c(\mathfrak{u}) \sum_{i,j=1}^{N} K(\mathfrak{s}_i, \mathfrak{s}_j, f_i, f_j),$$
$$\mathfrak{s}_1, \ldots, \mathfrak{s}_N \in \mathfrak{S},\ f_1, \ldots f_N \in \mathcal{E}. \tag{2.40}$$

In such cases, $\Phi(\mathfrak{u})$ is a densely defined bounded operator, and $\|\Phi_{\mathfrak{u}}\| = \|\Phi(\mathfrak{u})\| \leq c(\mathfrak{u})$ and a posteriori (2.38) *holds.*

Have a look at algebraic properties of maps $\mathfrak{u} \mapsto \Phi(\mathfrak{u})$ and $\mathfrak{u} \mapsto \Phi_{\mathfrak{u}}$.

Proposition 12. *Suppose that for each* $\mathfrak{u} \in \mathfrak{S}$ (2.38) *holds. Then* $\mathcal{D}(\Phi_{\mathfrak{u}})$ *in invariant on* $\Phi_{\mathfrak{u}}$*, that is,* $\Phi_{\mathfrak{u}}\mathcal{D}(\Phi_{\mathfrak{u}}) \subset \mathcal{D}(\Phi_{\mathfrak{u}})$*, and the map* $u \mapsto \Phi_{\mathfrak{u}}$ *is multiplicative, that is,*

$$\Phi_{\mathfrak{u}\mathfrak{v}}F = \Phi_{\mathfrak{u}}\Phi_{\mathfrak{v}}F, \quad \mathfrak{u}, \mathfrak{v} \in \mathfrak{S}, \quad F \in \mathcal{D}_K.$$

The map $u \mapsto \Phi(\mathfrak{u})$ *is anti-multiplicative, which here means that*

$$\Phi(\mathfrak{u})\Phi(\mathfrak{v})F = \Phi(\mathfrak{v}\mathfrak{u})F, \quad \mathfrak{u}, \mathfrak{v} \in \mathfrak{S}, \quad F \in \mathcal{D}(\Phi(\mathfrak{u})\Phi(\mathfrak{v})).$$

Since all the ingredients are ready, it is time for the first and most general dilation theorem.

Theorem 7. *Let* $\mathfrak{S}$ *be a unital semigroup,* $\mathcal{E}$ *a normed space, and* K *a positive-definite kernel on* $X \times \mathfrak{S}$ *which is isometric. Then, in the Hilbert space* $\mathcal{H}$ *with the kernel* K*, one has formulae*

$$\mathcal{H} = \operatorname{clolin}\{\Phi_{\mathfrak{u}}f \colon \mathfrak{u} \in \mathfrak{S},\ f \in \mathcal{E}\},$$
$$K(\mathfrak{s}, \mathfrak{t}, f, g) = \langle \Phi_{\mathfrak{t}}Vg, \Phi_{\mathfrak{s}}Vf \rangle, \quad (\mathfrak{s}, f), (\mathfrak{t}, g) \in \mathfrak{S} \times \mathcal{E},$$

where all the objects mentioned in the conclusion have been already defined.

Dilations on Semigroups with Involution

Enrich the structure of the semigroup $\mathfrak{S}$, assuming that it is a semigroup with an *involution* or, alternatively, $*$-*semigroup*, that is, there exists a map $\mathfrak{S} \ni s \mapsto s^* \in \mathfrak{S}$ such that $\mathfrak{s}^{**} = \mathfrak{s}$, $(\mathfrak{s}\mathfrak{t})^* = \mathfrak{t}^*\mathfrak{s}^*$, and $1^* = 1$, the latter if $\mathfrak{S}$ is unital. An accompanying assumption is the kernel K to be invariant with respect to involution in $\mathfrak{S}$ in the sense that

$$K(\mathfrak{u}s, t, f, g) = K(s, \mathfrak{u}^*t, f, g), \quad \mathfrak{u}, s, t \in \mathfrak{S},\ f, g \in \mathcal{E}. \tag{2.41}$$

If $\mathfrak{S}$ is unital, then

$$\omega(\mathfrak{s}, f, g) \overset{\text{def}}{=} K(\mathfrak{s}, 1, f, g)$$

restores K and positive definiteness as in (2.34) takes the following form

$$\sum_{i,j=1}^{N} \omega(s_j^* s_i, f_i, f_j) \geq 0, \quad s_1, \dots, s_N \in \mathfrak{S},\ f_1, \dots f_N \in \mathcal{E}. \tag{2.42}$$

Everything done in the preceding subsection applies here. However, because the structure of $\mathfrak{S}$ is now richer, some additional facts have to be pointed out.

Proposition 13. *Implication* (2.38) *is true, which means that for each* $\mathfrak{u} \in \mathfrak{S}$*, an operator* $\Phi_{\mathfrak{u}}$ *is well defined. As a consequence,*

$$\text{for each } \mathfrak{u} \in \mathfrak{S},\ \mathcal{D}_\omega \subset \mathcal{D}(\Phi(\mathfrak{u})) \text{ and } \Phi_{\mathfrak{u}} = \Phi(\mathfrak{u}^*)|_{\mathcal{D}_\omega}, \tag{2.43}$$

an operator $\Phi_{\mathfrak{u}}$ *is closable and* $\bar{\Phi}_{\mathfrak{u}} = \Phi(\mathfrak{u})^* = \Phi(\mathfrak{u}^*)$.

Note also that for semigroups with involution, much more can be said about the boundedness condition (2.40).

Lemma 2. *Let* $\mathfrak{S}$ *be a unital* $*$*-semigroup. If the form* ω *on* $(\mathfrak{S}, X)$ *is positive definite, then the following conditions are equivalent:*

- *For every* $\mathfrak{u} \in \mathfrak{S}$*, there exists* $c(\mathfrak{u}) \geqslant 0$ *such that the inequality* (2.40) *holds, that is,*

$$\sum\nolimits_{i,j=1}^{N} \omega(\mathfrak{s}_i^* \mathfrak{u}^* \mathfrak{u} \mathfrak{s}_j, f_i, f_j) \leq c(\mathfrak{u}) \sum_{i,j=1}^{N} \omega(\mathfrak{s}_i^* \mathfrak{s}_j, f_i, f_j),$$
$$\mathfrak{s}_1, \dots, \mathfrak{s}_N \in \mathfrak{S},\ f_1, \dots f_N \in \mathcal{E}; \tag{2.44}$$

- *For every* $\mathfrak{u} \in \mathfrak{S}$ *exists* $d(\mathfrak{u}) \geq 0$ *such that*

$$\omega(\mathfrak{s}^* \mathfrak{u}^* \mathfrak{u} \mathfrak{s}, f, f) \leq d(\mathfrak{u}) \omega(\mathfrak{s}^* \mathfrak{s}, f, f), \quad \mathfrak{s} \in \mathfrak{S},\ f \in \mathcal{E}; \tag{2.45}$$

- *There exists a function* $\alpha \colon \mathfrak{S} \to \mathbb{R}_+$ *such that* $\alpha(\mathfrak{s}\mathfrak{t}) \leq \alpha(\mathfrak{s})\alpha(\mathfrak{t})$ *for* $\mathfrak{s}, \mathfrak{t} \in \mathfrak{S}$*, satisfying the following condition: for any* $f \in \mathcal{E}$*, there exists a constant* $C = C(f)$ *which allows for an evaluation*

$$|\omega(\mathfrak{u}, f, f)| \leq C\alpha(\mathfrak{u}), \quad \mathfrak{u} \in \mathfrak{S}; \tag{2.46}$$

- *There exists a function* $\alpha \colon \mathfrak{S} \to \mathbb{R}_+$ *such that* $\alpha(\mathfrak{s}\mathfrak{t}) \leq \alpha(\mathfrak{s})\alpha(\mathfrak{t})$ *for* $\mathfrak{s}$ *and* $\mathfrak{t}$ *such that* $\mathfrak{s} = \mathfrak{s}^*$ *oraz* $\mathfrak{t} = \mathfrak{t}^*$ *satisfying the following condition: for every* $\mathfrak{s}, \mathfrak{t} \in \mathfrak{S}$ *i* $f, g \in \mathcal{E}$ *there exists a constant* $C = C(\mathfrak{s}, \mathfrak{t}, f, g)$ *which allows for an evaluation*

$$|\omega(\mathfrak{s}\mathfrak{u}\mathfrak{t}, f, g)| \leq C\alpha(\mathfrak{u}), \quad \mathfrak{u} = \mathfrak{u}^*;$$

- *For every* $\mathfrak{s}, \mathfrak{t} \in \mathfrak{S}$ *and* $f, g \in \mathcal{E}$

$$\limsup_{k \to \infty} |\omega(\mathfrak{s}\mathfrak{u}^{2^k}\mathfrak{t}, f, g)|^{2^{-k}} < +\infty, \quad \mathfrak{u} = \mathfrak{u}^*;$$

- *For every* $\mathfrak{s}_1, \dots, \mathfrak{s}_N \in \mathfrak{S}$ *and* $f_1, \dots f_N \in \mathcal{E}$

$$\liminf_{k\to\infty} \sum_{i,j=1}^{N} \omega(\mathfrak{s}_i \mathfrak{u}^{2^k} \mathfrak{s}_j, f_i, f_j)^{2^{-k}} < +\infty, \quad \mathfrak{u} = \mathfrak{u}^*. \tag{2.47}$$

Corollary 3. *If $\mathcal{E}$ is a Hilbert space and ω is positive definite, that is,* (2.42) *is satisfied as well as it satisfies any of equivalent conditions of* Lemma 2*, then the are operators $\Phi_{\mathfrak{u}}$ and $\Phi(\mathfrak{u})$, $\mathfrak{u} \in \mathfrak{S}$, and an isometry V such that*

$$\boldsymbol{\omega}(\mathfrak{s}) = V^* \Phi_{\mathfrak{s}} V, \quad \mathfrak{s} \in \mathfrak{S}. \tag{2.48}$$

Remark 7. Formula (2.48) can be given in another, more suitable for a traditional meaning of the word *dilation*, form:

$$\boldsymbol{\omega}(\mathfrak{s}) = P\Phi_{\mathfrak{s}}|_{\mathcal{H}_{\mathcal{E}}}, \quad \mathfrak{s} \in \mathfrak{S},$$

where P is an orthogonal projection $\mathcal{H}$ onto $\mathcal{H}_{\mathcal{E}}$.

Subsequent Instances

This general scheme of dilating kernels contains among others the following topics: Stinespring and Powers theorems, GNS construction, dilations on groups (in particular, the Sz.-Nagy dilation theorem), dilations of positive operator valued measures (Naĭmark's dilation), normal extensions (including those of unbounded operators, closely related to integrability of RKHSs already discussed in this chapter), and more.

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