Contractions and the Commutant Lifting Theorem in Kreĭn Spaces

11

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Contents

Introduction 220
The Commutant Lifting Problem on Hilbert Spaces 220
The Sz.-Nagy and Foias Version of the Problem 220
The Commutant Lifting Problem as Formulated by Sarason 223
The Operator Matrix Completion Problem 223
Commutant Lifting and Andô's Theorem 225
Further Generalizations and Some Applications 225
The Commutant Lifting Problem on Kreĭn Spaces 227
A Précis of Kreĭn Space Operator Theory and Notation Used 227
Dilations and Operator Matrix Completions in Kreĭn Spaces 230
A Solution to the Commutant Lifting Problem 232
An Approach to Commutant Lifting Via Diagram Chasing 234
Intertwining Operators with a Finite Number of Negative Squares and Some Open Questions 237
References 237

Abstract

A brief survey of the commutant lifting theorem is presented. This is initially done in the Hilbert space context in which the commutant lifting problem was initially considered, both in Sarason's original form and that of the later generalization due to Sz.-Nagy and Foias. A discussion then follows of the connection with contraction operator matrix completion problems, as well as

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with the Sz.-Nagy and Andô dilation theorems. Recent work in abstract dilation theory is outlined, and the application of this to various generalizations of the commutant lifting theorem are indicated. There is a short survey of the relevant Kreĭn space operator theory, focusing in particular on contraction operators and highlighting the fundamental differences between such operators on Kreĭn spaces and Hilbert spaces. The commutant lifting theorem is formulated in the Kreĭn space context, and two proofs are sketched, the first using a multistep extension procedure with a Kreĭn space version of the contraction operator matrix completion theorem, and the second diagrammatic approach which is a variation on a method due to Arocena. Finally, the problem of lifting intertwining operators which are not necessarily contractive is mentioned, as well as some open problems.

Introduction

The commutant lifting problem was originally formulated and solved in the Hilbert space setting by Donald Sarason in the 1960s [43]. Not long after, Sz.-Nagy and Foias presented and proved an abstract generalization of Sarason's theorem [44]. Since then many different proofs have been discovered and connections with other important theorems and applications have been noted, especially in complex function theory, and particularly as applied to interpolation problems [32]. Many of these purely mathematical ideas were driven by the needs of applied mathematicians and engineers working in such areas as signal processing and linear control.

It happens that the concepts needed to state the commutant lifting problem in Hilbert spaces have analogues in Kreĭn spaces, and these neatly revert to the original forms when the Kreĭn spaces under consideration happen to be Hilbert spaces. Despite this, there were major hurdles to be overcome in finding a proof of a commutant lifting theorem in the Kreĭn space setting, primarily because even though the condition for being a contraction has an equivalent algebraic formulation in both Hilbert and Kreĭn spaces, in the latter contractions can no longer be described metrically.

The Commutant Lifting Problem on Hilbert Spaces

The Sz.-Nagy and Foias Version of the Problem

Problem 1 (The Commutant Lifting Problem). Let $T_1 \in \mathscr{L}(\mathscr{H}_1)$, $T_2 \in \mathscr{L}(\mathscr{H}_2)$ be Hilbert space contractions, and let $V_1 \in \mathscr{L}(\mathscr{K}_1)$, $V_2 \in \mathscr{L}(\mathscr{K}_2)$ be minimal isometric dilations of T_1 and T_2. Suppose that $A \in \mathscr{L}(\mathscr{H}_1, \mathscr{H}_2)$ is a contraction and that $AT_1 = T_2 A$. Does there exist $\tilde{A} \in \mathscr{L}(\mathscr{K}_1, \mathscr{K}_2)$ lifting A with the property that $\tilde{A} V_1 = V_2 \tilde{A}$?

The *commutant lifting theorem* states that a positive solution exists to this problem. Further on, methods are discussed for proving the commutant lifting theorem when the problem is reformulated in the Kreĭn space setting.

Here is an explanation of some of the terminology. An operator T on a Hilbert space $\mathscr{H}$ is a *contraction* if it has norm less than or equal to one. Equivalently, $1 - T^*T \geq 0$ in the usual ordering for self-adjoint operators, where 1 is used to denote the identity operator. Since on a Hilbert space the norm of an operator and that of its adjoint are equal, it is automatic that $1 - TT^* \geq 0$. Write $\tilde{\mathscr{D}}$ and $\mathscr{D}$ for the closures of the ranges of $1 - T^*T$ and $1 - TT^*$. These are referred to as *defect spaces* for the operators T and T^*, while the operators $\tilde{D} := (1 - T^*T)^{1/2} : \tilde{\mathscr{D}} \to \mathscr{H}$ and $D := (1 - TT^*)^{1/2} : \mathscr{D} \to \mathscr{H}$ are the corresponding *defect operators*. It is clear that if the defect spaces and operators are altered up to isomorphism, then they still function in the same fashion, and so these will usually be chosen in a manner convenient to the context, referring to *a* rather than *the* defect operator and defect space. The operator $\begin{pmatrix} T \\ \tilde{D}^* \end{pmatrix}$ is an isometry and it can be verified that there is an operator $L : \tilde{\mathscr{D}} \to \mathscr{D}$ such that $\begin{pmatrix} D \\ -L^* \end{pmatrix}$ is its defect operator (see [31, Theorem 2.3]). As a consequence, there exists a unitary operator $\begin{pmatrix} T & D \\ \tilde{D}^* & -L^* \end{pmatrix}$: $\mathscr{H} \oplus \mathscr{D} \to \mathscr{H} \oplus \tilde{\mathscr{D}}$, called a *Julia operator*. Note that in this context L is just the restriction of T to $\tilde{\mathscr{D}}$ when the explicit choices of $\tilde{D}$ and D given above are made, since $T(1 - T^*T)^{1/2} = (1 - TT^*)^{1/2}T$.

The operator $V \in \mathscr{L}(\mathscr{K})$ is an *isometric dilation* of $T \in \mathscr{L}(\mathscr{H})$ if $\mathscr{H} \subseteq \mathscr{K}$ and T^* is the restriction of V^* to an invariant subspace (equivalently stated, T is the restriction of V to the co-invariant subspace $\mathscr{H}$). It will then automatically be the case that V is a so-called power dilation of T; that is, for $n = 0, 1, 2, \ldots$, $T^n P_{\mathscr{H}} = P_{\mathscr{H}} V^n$, where $P_{\mathscr{H}}$ is the orthogonal projection from $\mathscr{K}$ to $\mathscr{H}$. The isometric dilation is *minimal* if the only subspace $\mathscr{K}' \subseteq \mathscr{K}$ containing $\mathscr{H}$ such that $V|_{\mathscr{K}'}$ is an isometry is $\mathscr{K}$ itself. Equivalently, the closed linear span of the spaces $V^n\mathscr{H}$ is $\mathscr{K}$. The notion of a *lifting* is closely allied; $\tilde{A} \in \mathscr{L}(\mathscr{K}_1, \mathscr{K}_2)$ lifts the operator $A \in \mathscr{L}(\mathscr{H}_1, \mathscr{H}_2)$ if $A|_{\mathscr{H}_1} = P_{\mathscr{H}_2}\tilde{A}$ and $\|\tilde{A}\| \leq \|A\|$. Obviously, if A is assumed to be a contraction, then the lifting $\tilde{A}$ is also required to be a contraction. Because of the form of liftings and dilations, if $T_2 = T_1 = T$, $T, A \in \mathscr{L}(\mathscr{H})$ and $n, m = 0, 1, 2, \ldots$, then $A^m T^n P_{\mathscr{H}} = P_{\mathscr{H}} V^n \tilde{A}^m$, where V is an isometric dilation of T.

In addition to the isometric dilation of an operator $T \in \mathscr{L}(\mathscr{H})$, it is also possible to construct a *unitary dilation*. This is a unitary operator $U \in \mathscr{L}(\tilde{\mathscr{K}})$ with the property that $\mathscr{H} \subseteq \tilde{\mathscr{K}}$ and $\mathscr{H}$ is a *semi-invariant subspace* for U; that is, it is the intersection of two subspaces of $\tilde{\mathscr{K}}$, one of which is invariant for U while the other is *co-invariant* (invariant for U^*). There is a similar notion of minimality for unitary dilations. Semi-invariance ensures that U is in fact a power dilation, in that for $n = 0, 1, 2, \ldots$, $T^n = P_{\mathscr{H}} U^n|_{\mathscr{H}}$ and $T^{*n} = P_{\mathscr{H}} U^{-n}|_{\mathscr{H}}$, where restriction operator $|_{\mathscr{H}}$ is the adjoint of $P_{\mathscr{H}}$. Sz.-Nagy was the first to note the existence of a minimal unitary dilation U of a contraction T. His proof is constructive. Define the space $\tilde{\mathscr{K}} = \bigoplus_1^\infty \mathscr{D} \oplus \mathscr{H} \oplus \bigoplus_1^\infty \tilde{\mathscr{D}}$ and on $\tilde{\mathscr{K}}$ set

$$U = \begin{pmatrix} \ddots & & & & & & \\ & 1 & & & & & \\ & & 1 & & & & \\ & & & D & T & & \\ & & & -L^* & \tilde{D}^* & & \\ & & & & & 1 & \\ & & & & & & 1 \\ & & & & & & & \ddots \end{pmatrix}, \tag{11.1}$$

where unspecified entries are 0 and T maps $\mathscr{H}$ to itself in $\tilde{\mathscr{K}}$. In essence the unitary dilation of T is a Julia operator padded with a unilateral shift and the adjoint of a unilateral shift so as to make up for the fact that in general the Julia operator acts between different spaces.

Once the unitary dilation is available, the isometric dilation V comes for free by restricting U to the invariant subspace, $\mathscr{K} = \mathscr{H} \oplus \bigoplus_1^\infty \tilde{\mathscr{D}}$. Explicitly,

$$V = \begin{pmatrix} T & & & \\ \tilde{D}^* & & & \\ & 1 & & \\ & & 1 & \\ & & & \ddots \end{pmatrix}. \tag{11.2}$$

It is not hard to verify that V constructed in this way also enjoys the additional property of being minimal. Furthermore, all minimal unitary dilations of T are unitarily equivalent, as are all minimal isometric dilations, and so these constructions are canonical.

In case the commutant lifting theorem holds, it follows that it also holds when isometric dilations are replaced by unitary dilations. This can be seen by first using the theorem to lift with isometric dilations, taking adjoints, isometrically dilating V_1^* and V_2^* to obtain unitary operators, and then employing the commutant lifting theorem one more time.

Many times the commutant lifting problem is stated with the assumption that the intertwining operator A is simply bounded rather than being a contraction, though without loss of generality, A can always be scaled to be a contraction if it is not already one. There are various other restrictions which can be made to simplify solving the problem though still allowing for the obtaining of a solution of the general problem. For example, T_1 and T_2 can be replaced by a single operator $T = T_1 \oplus T_2$ and A by $A' = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix}$ to get $TA' = A'T$. The direct sum of minimal isometric dilations of T_1 and T_2 will be a minimal isometric dilation of T, and a dilation of A is easily constructed from one of A'.

Alternatively, T_1 can be replaced by its minimal isometric dilation V_1 and A by the direct sum of A with 0s on the defect spaces of T_1.

Finally, a problem which has also occupied a number of authors has been to parameterize the set of solutions of the commutant lifting problem [24,32,35]. More will be said about some of these matters below.

The Commutant Lifting Problem as Formulated by Sarason

In the Sarason version of the commutant lifting problem [43], it is assumed that the adjoints of T_1 and T_2 are the restrictions of adjoints of unilateral shift operators (with multiplicities) to invariant subspaces. The problem then reduces to asking whether a contractive intertwining operator A lifts to a contraction $\tilde{A}$ intertwining these shift operators. Following the terminology of Rosenblum and Rovnyak [42], an operator commuting with a unilateral shift is said to be *analytic*, since analytic functions are precisely those which commute with multiplication by z. It is this which makes the solution of the commutant lifting problem so useful in such applications as interpolation.

The Operator Matrix Completion Problem

As was noted, the Julia operator provides a way of embedding a Hilbert space contraction operator inside of a unitary operator. This motivates the following related problems: given a contraction $T \in \mathscr{L}(\mathscr{H})$, is there some useful way of describing the operators X and Y such that

$$C := \begin{pmatrix} T \\ X \end{pmatrix} \qquad \text{and} \qquad R := \begin{pmatrix} T & Y \end{pmatrix}$$

are contractions? Moreover, supposing that the operator T has fixed contractive column and row completions C and R, is there a useful description of those Z making

$$A := \begin{pmatrix} T & Y \\ X & Z \end{pmatrix}$$

a contraction? This latter question is known as the *Parrott completion problem*. It will be seen to be intimately bound up with the commutant lifting problem [38].

The column completion problem is addressed by first considering the apparently simpler problem of describing those operators X such that C is an isometry; that is, those X for which $1 - T^*T - X^*X = 0$. Equivalently, $X^*X = 1 - T^*T =$

$\tilde{D}\tilde{D}^*$, where as before, $\tilde{D} : \tilde{\mathscr{D}} \to \mathscr{H}$ is a defect operator for T. By Douglas' lemma, there is an isometry V such that $X = V\tilde{D}^*$. Now returning to the original problem where the column C is simply assumed to be contractive, observe that by appending a defect operator, an isometry $\begin{pmatrix} C \\ \tilde{D}_C^* \end{pmatrix}$ is obtained, and so $\begin{pmatrix} X \\ \tilde{D}_C^* \end{pmatrix} = V\tilde{D}^*$ for some isometry V. Consequently, there is a contraction $\tilde{G}$ such that $X = \tilde{G}^*\tilde{D}^*$. Obviously the converse is also true; namely, if X has this form, then C is a contraction. As a bonus, it is found that $\tilde{D}_C = \tilde{D}D_{\tilde{G}}$, where $D_{\tilde{G}}$ is a defect operator for $\tilde{G}^*$.

The row completion question can be addressed by taking adjoints in the column case, but since this will not necessarily work when it comes to considering contractions on Kreĭn spaces, an alternative approach is sketched, hinging on the simple observation that both $\begin{pmatrix} R & D_R \end{pmatrix}$ and $\begin{pmatrix} T & D \end{pmatrix}$ are co-isometries, where D_R is the defect operator for R^* and D is the defect operator for T^*. Then by Douglas' lemma, there is a co-isometry W such that $\begin{pmatrix} Y & D_R \end{pmatrix} = DW$. From it is read off that $Y = DG$, where G, being a restriction of W, is a contraction. Again it is clear that if Y has this form, then R is a contraction, and that it is possible to choose $D_R = DD_G$, where D_G is a defect operator for G^*.

The solution to the Parrott completion problem is an application of the row and column results. Based on the assumptions made, there are contractions $\tilde{G}$ and G so that $C = \begin{pmatrix} T \\ \tilde{G}^*\tilde{D}^* \end{pmatrix}$ and $R = \begin{pmatrix} T & DG \end{pmatrix}$. A straightforward calculation shows that a defect operator for C^* is

$$D_C = \begin{pmatrix} D & 0 \\ -\tilde{G}^*L^* & \tilde{D}_{\tilde{G}} \end{pmatrix},$$

where $\tilde{D}_{\tilde{G}}$ is a defect operator for $\tilde{G}$. By the solution to the row completion problem, $A = \begin{pmatrix} C & D_C E \end{pmatrix}$ for some contraction $E = \begin{pmatrix} E_1 \\ E_2 \end{pmatrix}$. Examining the first entry of $D_C E$, it is seen that $E_1 = G$. By the solution to the column completion problem, $E_2 = F\tilde{D}_G^*$ for some contraction F, and so the second entry of $D_C E$, which is just Z, has the form $-\tilde{G}^*L^*G + \tilde{D}_{\tilde{G}}F\tilde{D}_G^*$. A somewhat tedious calculation shows that whenever X, Y, and Z have these explicit forms, the operator A is a contraction. The work can be simplified by considering the product of unitary operators

$$\begin{pmatrix} 1 & & & \\ & \tilde{G}^* & \tilde{D}_{\tilde{G}} & \\ & D_{\tilde{G}}^* & -L_{\tilde{G}} & \\ & & & 1 \end{pmatrix} \begin{pmatrix} T & D & & \\ \tilde{D}^* & -L^* & & \\ & & F & D_F \\ & & \tilde{D}_F^* & -L_F^* \end{pmatrix} \begin{pmatrix} 1 & & & \\ & G & D_G & \\ & \tilde{D}_G^* & -L_G^* & \\ & & & 1 \end{pmatrix}$$

(unspecified entries 0), since A is then the compression to the upper left 2×2 block of this product [31]. It is also possible to identify defect operators and spaces for A, as well as a link operator in the Julia operator for A in this way.

Commutant Lifting and Andô's Theorem

There is a two variable analogue of the Sz.-Nagy dilation theorem, called Andô's theorem. It states that a pair of commuting contractions T_1 and T_2 on a Hilbert space $\mathscr{H}$ dilate to a pair of commuting unitary operators U_1 and U_2. As in the case of the Sz.-Nagy dilation theorem, these are the so-called power dilations; that is, for $n, m = 0, 1, 2, \ldots$, $T_1^n T_2^m = P_{\mathscr{H}} U_1^n U_2^m|_{\mathscr{H}}$. Of course Andô's theorem implies the commutant lifting theorem in Hilbert spaces [2, Theorem 10.29], since as was noted above, there is no loss of generality in assuming $T_1 = T_2$ in the statement of the commutant lifting problem.

The converse is also true. If it is assumed that the commutant lifting theorem is valid, then it is possible to prove Andô's theorem [38]. Though not Parrott's original argument, here is a rough sketch of a way of proving this following ideas from [28]. Without loss of generality, take $T_2 = T_1 = T \in \mathscr{L}(\mathscr{H})$ with isometric dilation V_1. (Throughout, all isometric dilations are taken to be of the canonical form as in (11.2).) By assumption the contraction A such that $AT = TA$ lifts to a contraction, denoted by W_1, such that $W_1 V_1 = V_1 W_1$. Let W_2 be the canonical minimal isometric dilation of W_1 and lift V_1 to a contraction V_2 which commutes with this. Since V_1 is an isometry and since it is assumed that W_2 is in canonical form, it is not difficult to see that V_2 is the direct sum of V_1 and $\bigoplus_{n=1}^{\infty} V_1'$, where V_1' is a contraction. Continue in this manner alternating dilating and lifting of the Vs and Ws. Taking a direct limit to obtain operators V_∞ and W_∞ on some Hilbert space $\mathscr{H}_\infty$. Define $\mathscr{K} = \bigvee_{m,n=0}^{\infty} V_\infty^m W_\infty^n \mathscr{H}$, an invariant subspace for both V_∞ and W_∞. Set $V = V_\infty|_{\mathscr{K}}$ and $W = W_\infty|_{\mathscr{K}}$ to get two commuting isometries dilating T and A, respectively. Taking adjoints of V and W and applying the same argument yields unitary operators with the properties stated in Andô's theorem.

Further Generalizations and Some Applications

The commutant lifting problem has given rise to a number of generalizations over the years. Just a few of these are now mentioned.

As observed in the last section, Andô's theorem gives that commuting contractions have commuting isometric dilations. So suppose now that there are three commuting contractions and two of these are dilated in this manner. Is there a lifting of the third which commutes with these dilations? There are a number of examples, starting with one due to Varopoulos [45], which show that in general it is impossible to construct a lifting. However under certain restrictions it is possible to lift. If, as in Sarason's version of the commutant lifting problem, the dilated operators are unilateral shifts and certain restrictions are placed on the intertwining operator, then a lifting will exist [5, 13, 37].

Another variation is to assume that a row contraction $T := \begin{pmatrix} T_1 & \cdots & T_d \end{pmatrix}$ is given. This means that $\sum_1^d \|T_k\|^2 \leq 1$) and A intertwines T entrywise. Note that there is no a priori assumption that the operators in T commute. Without the assumption of commutativity (which would then naturally require that entries of the dilation also commute), it is not necessary to alter by much the standard proofs of the classical commutant lifting theorem to find a proof in this context [33, 41]. A more challenging problem is to find such a theorem in the commutative case. This has recently been accomplished in [22] (see also [14]).

There is a somewhat more general framework which is worthwhile considering. For $T \in \mathscr{L}(\mathscr{H})$ the map $p \mapsto p(T)$ defines a unital representation of the algebra of polynomials $\mathscr{A}$ with norm $\|p\|$ the supremum norm over the unit disk. If the operator norm $\|p(T)\| \leq \|p\|$ for all $p \in \mathscr{A}$, the representation is said to be *contractive*. Since any contraction T has a unitary dilation by the Sz.-Nagy dilation theorem, it follows from the functional calculus for unitary operators that an operator T defines a contractive representation of $\mathscr{A}$ if and only if T is a contraction (this is essentially a restatement of the von Neumann inequality). The same argument shows that for any $n \in \mathbb{N}$ and $p \in M_n(\mathbb{C}) \otimes \mathscr{A}$, the algebra of $n \times n$ matrix-valued polynomials with norm the supremum of the operator norm of $p(z)$ as z ranges over $\mathbb{D}$, it is the case that $\|p(T)\| \leq \|p\|$. In other words, the representation is *completely contractive*. A straightforward argument using the Arveson extension theorem and the Stinespring dilation theorem (giving an alternate proof of the existence of a unitary dilation of a contraction) implies that $\pi : \mathscr{A} \to \mathscr{L}(\mathscr{H})$ is a completely contractive representation if and only if $\pi(z) = T$ for some contraction T (see, for example, [39]).

This is a special example of the following [28]. Let $\mathscr{A}$ be a unital operator algebra (this can either be thought of concretely, or else abstractly as an algebra with a matricial norm structure obeying the Blecher-Effros-Ruan axioms [39]). Consider the collection $\mathscr{R}$ of all completely contractive representations of $\mathscr{A}$. These can be partially ordered, in the sense that for $\pi_1, \pi_2 \in \mathscr{R}$ mapping into $\mathscr{L}(\mathscr{H}_1)$ and $\mathscr{L}(\mathscr{H}_2)$, respectively, $\pi_2 \geq \pi_1$ if $\mathscr{H}_1 \subseteq \mathscr{H}_2$, $\mathscr{H}_1$ is invariant for π_2, and $\pi_2|_{\mathscr{H}_1} = \pi_1$. A representation is called π_1 *extremal* if whenever $\pi_2 \geq \pi_1$, π_2 contains π_1 as a direct summand. It can be shown that all representations in $\mathscr{R}$ dilate to extremal representations [28], and they play the role of the adjoints of isometric dilations of contractions.

One can likewise partially order the elements of $\mathscr{R}$ by $\pi_2 \geq \pi_1$ if $\mathscr{H}_1 \subseteq \mathscr{H}_2$, $\mathscr{H}_1$ is semi-invariant for π_2 and $P_{\mathscr{H}_1}\pi_2|_{\mathscr{H}_1} = \pi_1$. Those irreducible representations π_1 which have the property that, $\pi_2 \geq \pi_1$ with respect to this partial ordering implies π_2 contains π_1 as a direct summand, are called *boundary representations*. The existence of such representations without the condition of irreducibility was first proved in [28], and a refined version of the arguments found there were given in [21] showing that all completely contractive representations extend to boundary representations (see also [10]). Boundary representations play the role of minimal unitary dilations of contractions.

While the abstract theory is satisfying in that it ensures the existence of representations playing the role of isometric and unitary dilations, for any fixed algebra there is often a difficult hurdle which needs to be overcome at the start; namely, concretely characterize the set $\mathscr{R}$ of completely contractive representations. There are a few instances in which this can be done. For example, by the Sz.-Nagy dilation theorem and Andô's theorem, completely contractive representations of the algebra of polynomials with supremum norm on the disk and bi-disk are obtained from contractions and pairs of commuting contractions, respectively, though for higher dimensional polydisks this fails (this is the interpretation of Varopoulos' example in the present context), and an example due to Parrott shows that contractive representations over higher dimensional polydisks need not be completely contractive [38]. There is a positive solution in the case of the annulus, since contractive representations of the algebra of rational functions with poles off of the domain are completely contractive [1], but over domains of higher connectivity, again this fails to be the case [3, 29, 40]. The question as to whether over a bounded domain in $\mathbb{C}^d$ the collection of contractive representations coincides with the collection of completely contractive representations for the algebra of rational functions with poles off of the domain is known as the *rational dilation problem*.

The commutant lifting problem in this context is as follows.

Problem 2 (Abstract Commutant Lifting Problem). Let $\mathscr{A}$ be an operator algebra with collection of completely contractive representations $\mathscr{R}$. Suppose that $\pi_1, \pi_2 \in \mathscr{R}$ and that there is a contraction a such that for all $p \in \mathscr{A}$, $a\pi_1(p) = \pi_2(p)a$, and let $\tilde{\pi}_1, \tilde{\pi}_2$ be extremal representations dilating π_1 and π_2, respectively. Does there exist a lifting $\tilde{a}$ of a such that for all $p \in \mathscr{A}$, $\tilde{a}\tilde{\pi}_1(p) = \tilde{\pi}_2(p)\tilde{a}$?

A lifting here is defined exactly as before, and a positive solution to the problem for a given algebra is called a *commutant lifting theorem*. As was seen in the example of the algebra of polynomials over the bi-disk, a commutant lifting theorem may not exist in general. There are some examples where there are positive solutions though. See, for example, [14, 20, 36].

The Commutant Lifting Problem on Kreĭn Spaces

A Précis of Kreĭn Space Operator Theory and Notation Used

A *Kreĭn space* $\mathscr{H}$ is the direct sum of two Hilbert spaces $\mathscr{H} = \mathscr{H}_+ \oplus \mathscr{H}_-$ with an indefinite inner product defined

$$\langle f_+ \oplus f_-, g_+ \oplus g_- \rangle_{\mathscr{H}} = \langle f_+, g_+ \rangle_{\mathscr{H}_+} - \langle f_-, g_- \rangle_{\mathscr{H}_-}.$$

It is assumed in all cases that the underlying field is that of the complex numbers, though some of what follows can also be done over the reals. The usual axioms for inner products hold, except that it no longer the case that $\langle f, f \rangle \geq 0$ or that $\langle f, f \rangle = 0$ implies that $f = 0$. The space $\mathscr{H}_-$ with inner product being the negative of the usual Hilbert space inner product is sometimes referred to as the *anti-space of a Hilbert space*, and it is then said that a Kreĭn space is the direct sum of a Hilbert space and the anti-space of a Hilbert space. If $\dim \mathscr{H}_- < \infty$, $\mathscr{H}$ is usually called a *Pontryagin space*.

Generally, Kreĭn spaces have lots of subspaces (closed linear manifolds in the Hilbert space topology), though most of these will not themselves be Kreĭn spaces. A subspace in which all vectors have non-positive self inner product is called a *negative subspace*. Positive subspaces are defined analogously, and neutral subspaces are those in which all self inner products (and hence all inner products) are 0. Those positive or negative subspaces which are themselves Hilbert spaces or anti-spaces of Hilbert spaces are said to be *uniformly positive* or *negative*. *Maximal* positive or negative subspaces are those which are not properly contained in any subspace of the same kind, while maximal uniformly definite (i.e., positive or negative) subspaces are maximal subspaces which are also uniformly definite. It can be shown that all definite subspaces are contained in maximal definite subspaces, and these can be chosen to be uniformly definite if the original space is [31].

For a Kreĭn space $\mathscr{H}$, the dimensions of $\mathscr{H}_+$ and $\mathscr{H}_-$ are called the *positive* and *negative indices* of the Kreĭn space $\mathscr{H}$, and are notated as $\mathrm{ind}_\pm \mathscr{H}$. Hilbert spaces are precisely those spaces for which $\mathrm{ind}_- \mathscr{H} = 0$, while Pontryagin spaces have $\mathrm{ind}_- \mathscr{H} < \infty$. There is no fixed convention for notation in this field. This article follows the camp which uses Hilbert space notation with Kreĭn spaces, in part because Hilbert spaces are special cases of Kreĭn spaces, and also to stress the similarity between many Kreĭn space results and their Hilbert space counterparts. Details for much of what follows in this section can be found in a number of sources [4, 11, 15, 30, 31]. Notationally, the present paper is closest to [31].

The Kreĭn space defined above has associated to it an operator $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ with respect to the given decomposition, known as a *fundamental symmetry*. Except in trivial cases, neither the decomposition nor the fundamental symmetry is unique. However each decomposition of a Kreĭn space $\mathscr{H}$ gives rise to an associated Hilbert space, and all of these Hilbert spaces have equivalent topologies. Consequently, the class of bounded linear operators $\mathscr{L}(\mathscr{H})$ on $\mathscr{H}$ is well defined.

Outside of being used in defining a Kreĭn space in the first place and a small number of proofs, the decomposition of a Kreĭn space and its fundamental symmetry play no role in what is done. What is more important, as well as being invariant under the choice of decomposition, are the positive and negative indices.

The notion of the *adjoint* T^* of an operator $T \in \mathscr{L}(\mathscr{H})$ is defined on a Kreĭn space as it is in a Hilbert space, and this can be related to the Hilbert space adjoint on an associated Hilbert space via multiplication with a fundamental symmetry to see that $T^* \in \mathscr{L}(\mathscr{H})$. Since Kreĭn spaces are almost exclusively used without referring to an underlying Hilbert space, here and elsewhere no bother is made to add the words "Kreĭn space" in front of "adjoint" or any of the other notions introduced having Hilbert space counterparts unless there is some chance of confusion. In particular, from here on all spaces will be Kreĭn spaces unless otherwise noted.

Once adjoints of operators on Kreĭn spaces are defined, by mimicking what is done in Hilbert spaces it is straightforward to give definitions for a host of classes of operators. Here is a list of some of the more useful ones. Let $T \in \mathscr{L}(\mathscr{H}, \mathscr{K})$. Then T is *self-adjoint* if $\mathscr{K} = \mathscr{H}$ and $T^* = T$, and it is *positive* (written $T \geq 0$) if it is self-adjoint and for $h \in \mathscr{H}$, $\langle Tf, f \rangle \geq 0$. It is a *projection* if it is self-adjoint and $T^2 = T$. While projections in Kreĭn spaces need not be positive operators, such an operator has the property that the range is also a Kreĭn space. An operator T is an *isometry* if it preserves inner products; that is, $1 - T^*T = 0$. A *co-isometry* is the adjoint of an isometry. It is a *partial isometry* if $T = TT^*T$.

For the purposes of this article, a particularly important class of operators is that of the contractions. On a Hilbert space, such an operator has several equivalent definitions. The one that works on Kreĭn spaces is that T is a *contraction* if $\langle f, f \rangle - \langle Tf, Tf \rangle \geq 0$ for all f, or equivalently, assuming that T is bounded, $1 - T^*T \geq 0$. Unlike the situation on Hilbert spaces, there is no metrical equivalent to this condition. Indeed, there exist unbounded operators which are contractions. Furthermore, even if an operator T is a contraction, this does not guarantee that its adjoint is a contraction, contrary to what happens on Hilbert spaces. A simple example illustrating this is to take $\mathscr{H}_\pm$ to be one dimensional, and $T \in \mathscr{L}(\mathscr{H})$ to be any operator such that $T\mathscr{H}_+ = \mathscr{H}_-$ and $T\mathscr{H}_- = \{0\}$. While isometries are obviously contractions, it is not difficult to come up with examples of co-isometries which are not. Likewise, partial isometries and projections need not be contractions.

Those operators T which have the property that both T and T^* are contractions are called *bicontractions*. Contractions and bicontractions have special properties when it comes to how they map certain positive and negative subspaces. For example, the kernel of a contraction is uniformly positive. Also, it is clear that contractions must map negative subspaces to negative subspaces, and the same goes for uniformly negative subspaces. As it happens, this gives a geometric characterization of bicontractions: these are precisely the contractions which map maximal uniformly negative subspaces to maximal uniformly negative subspaces. By a fixed point argument, it can be shown that there is a maximal uniformly negative subspace $\mathscr{H}_-$ such that $T\mathscr{H}_- = \mathscr{H}_-$ when T is a bicontraction (see, for example, [31]).

While not all contractions are bicontractions, there are circumstances in which this is the case. In particular, if $T \in \mathscr{L}(\mathscr{H})$ is a contraction on a Pontryagin space $\mathscr{H}$, then T is a bicontraction.

The proofs of some of these results use an interesting connection between bicontractions and Hilbert space contractions via the so-called *Potopov-Ginsburg transform*. If $T \in \mathscr{L}(\mathscr{H})$ is a bicontraction and $\mathscr{H}_-$ is a maximal uniformly negative subspace fixed by T, then there is a fundamental decomposition $\mathscr{H} = \mathscr{H}_+ \oplus \mathscr{H}_-$. Write $T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$ with respect to this decomposition. Changing the sign of the inner product on $\mathscr{H}_-$ so that $\mathscr{H}$ becomes a Hilbert space, the operator

$$S = \begin{pmatrix} T_{11} - T_{12}T_{22}^{-1}T_{21} & T_{12}T_{22}^{-1} \\ -T_{22}^{-1}T_{21} & T_{22}^{-1} \end{pmatrix}$$

is a Hilbert space contraction. There is an interpretation of S as the scattering matrix for a linear system, and the upper left corner is a Schur complement. Given a Hilbert space contraction S, it is also possible to recover a Kreĭn space bicontraction T via an inverse transform. See Section 1.3 of [30] for more details.

Dilations and Operator Matrix Completions in Kreĭn Spaces

As was seen, contractions on Hilbert spaces have associated to them unitary operators called Julia operators, and with these can be used to construct in a canonical way minimal isometric and unitary dilations. The situation is similar on Kreĭn spaces, though in this setting, a Julia operator can be found for *any* operator, and as a consequence, isometric and unitary dilations always exist, even when the operator is not a contraction [23].

In order to see why this is the case, the following lemma is needed.

Lemma 1 (Bognár–Krámli Factorization [16]). *On a Kreĭn space $\mathscr{H}$, for any self-adjoint operator $A \in \mathscr{L}(\mathscr{H})$, there is a Kreĭn space $\mathscr{D}$ and $D \in \mathscr{L}(\mathscr{D}, \mathscr{H})$ with* $\ker D = \{0\}$ *such that $A = DD^*$. Furthermore, $A \geq 0$ if and only if $\mathscr{D}$ is a Hilbert space.*

The proof of this is one of the few circumstances where it seems that fundamental symmetries and associated Hilbert spaces must be used. The argument essentially reduces to the polar decomposition of a Hilbert space self-adjoint operator. Details can be found in [31]. The notation $\mathrm{ind}_\pm A := \mathrm{ind}_\pm \mathscr{D}$ is used for the positive and negative *indices* of the self-adjoint operator A. Positive operators are those self-adjoint operators for which $\mathrm{ind}_- A = 0$. Even up to isomorphism of the intermediate space $\mathscr{D}$, the factorization will not in general be unique, though it is if either $\mathrm{ind}_+ A$ or $\mathrm{ind}_- A$ is finite [17, 31].

The Bognár-Krámli factorization lemma now allows for the definition of defect operators for any operator $T \in \mathscr{L}(\mathscr{H}, \mathscr{K})$. Factor $1 - T^*T = \tilde{D}\tilde{D}^*$, $\tilde{D} \in \mathscr{L}(\tilde{\mathscr{D}}, \mathscr{H})$ with $\ker \tilde{D} = \{0\}$ and $1 - TT^* = DD^*$, $D \in \mathscr{L}(\mathscr{D}, \mathscr{K})$ with $\ker D = \{0\}$. Call $\tilde{D}$ and $\tilde{\mathscr{D}}$ a *defect operator* and *defect space* for T, and D and $\mathscr{D}$ a defect operator and defect space for T^*. Following the method outlined in the Hilbert space case, construct a *Julia operator* for T, $\begin{pmatrix} T & D \\ \tilde{D}^* & -L^* \end{pmatrix}$, where $L \in \mathscr{L}(\tilde{\mathscr{D}}, \mathscr{D})$ is again referred to as the *link operator*. See also [9]. It is also possible to construct minimal isometric and unitary dilations, but this time for *any* bounded operator on a Kreĭn space. It should be noted though that unless the operator is a bicontraction, at least one of the defect spaces will be a Kreĭn space. For a contraction T, $1 - T^*T \geq 0$ and so the defect space $\tilde{\mathscr{D}}$ will be a Hilbert space, while if T is a bicontraction (so in particular, if it is a contraction on a Pontryagin space or Hilbert space), it is also the case that $1 - TT^* \geq 0$, which means that in addition the defect space $\mathscr{D}$ is a Hilbert space. It is not difficult to see that minimal isometric dilations of contractions are isomorphic, while minimal unitary dilations are isomorphic for bicontractions (see Section 3 of [30]).

One can try to characterize contractive matrix extensions of an operator $T \in \mathscr{L}(\mathscr{H}, \mathscr{K})$ as before. Unfortunately the sort of decomposition in terms of defect operators and contractions which was seen in section "The Operator Matrix Completion Problem" may fail. As a simple example, let T be an isometry, so that $\tilde{\mathscr{D}} = \{0\}$, and N be a non-zero operator such that $N^*N = 0$, meaning that the range of N is contained in a neutral subspace. Then $\begin{pmatrix} T \\ N \end{pmatrix}$ is an isometry (and so a contraction), yet it is not possible to write $N = \tilde{G}^*\tilde{D}^*$ as $\tilde{D} = 0$.

There are conditions which can be placed on T and the extension spaces which then give rise to the sorts of decompositions found for Hilbert space contractions in section "The Operator Matrix Completion Problem", details of which can be found in Section 3 of [31]. Only a special case is considered here, since this suffices for giving a proof of a commutant lifting theorem.

Assume that $T \in \mathscr{L}(\mathscr{H}, \mathscr{K})$ is a contraction with defect operators and spaces labeled as above, $\mathscr{E}$ is a Kreĭn space, and $\mathscr{F}$ a Hilbert space. The operator matrices

$$C := \begin{pmatrix} T \\ X \end{pmatrix} \in \mathscr{L}(\mathscr{H}, \mathscr{K} \oplus \mathscr{F}) \qquad \text{and} \qquad R := \begin{pmatrix} T & Y \end{pmatrix} \in \mathscr{L}(\mathscr{H} \oplus \mathscr{E}, \mathscr{K})$$

are contractions if and only if $X = \tilde{G}^*\tilde{D}^*$ and $Y = DG$, where $\tilde{G}$ and G are contractions. (Note in this case that $\tilde{G}$ will be a contraction between Hilbert spaces.) It can be shown that $\operatorname{ind}_-(1 - RR^*) \leq \operatorname{ind}_-(1 - TT^*)$ with equality if and only if $\mathscr{E}$ is a Hilbert space and that R is a bicontraction if and only if G is one. On the other hand operator $\operatorname{ind}_-(1 - CC^*) = \operatorname{ind}_-(1 - TT^*)$, so C will be a bicontraction precisely when T is. The proofs of these statements are virtually identical to that sketched for contractions on Hilbert spaces.

The Parrott extension problem also has an analogous statement and solution in this setting. It is assumed once again that $T \in \mathscr{L}(\mathscr{H}, \mathscr{K})$ is a contraction with defect operators and spaces labeled as above, $\mathscr{E}$ is a Kreĭn space, and $\mathscr{F}$ a Hilbert space. It is also assumed that fixed contractive column and row completions C and R of T as above are given. Then

$$A := \begin{pmatrix} T & Y \\ X & Z \end{pmatrix} \in \mathscr{L}(\mathscr{H} \oplus \mathscr{E}, \mathscr{K} \oplus \mathscr{F})$$

is a contraction if and only if $Z = -\tilde{G}^* L^* G + \tilde{D}_{\tilde{G}} F \tilde{D}_G^*$, where, since the defect spaces for G and $\tilde{G}$ are Hilbert spaces, the operator F is a Hilbert space contraction. Again, the proof outlined in the Hilbert space case works equally well here. The operator A will be a bicontraction if G is one. In particular, if T is a bicontraction, $\mathscr{E}$ will need to be a Hilbert space if G is going to be a contraction, and thus it will be automatic in this case that G is bicontractive, and so likewise for A ([30, Cor. 2.4.3] and [31, Section 3]).

A Solution to the Commutant Lifting Problem

Theorem 1 (The Commutant Lifting Theorem for Contractions). *Let $T_1 \in \mathscr{L}(\mathscr{H}_1)$, $T_2 \in \mathscr{L}(\mathscr{H}_2)$ be Kreĭn space contractions, and let $V_1 \in \mathscr{L}(\mathscr{K}_1)$, $V_2 \in \mathscr{L}(\mathscr{K}_2)$ be isometric dilations of T_1 and T_2. Suppose that $A \in \mathscr{L}(\mathscr{H}_1, \mathscr{H}_2)$ is a contraction and that $AT_1 = T_2 A$. Then there exists a contraction $\tilde{A} \in \mathscr{L}(\mathscr{K}_1, \mathscr{K}_2)$ lifting A with the property that $\tilde{A} V_1 = V_2 \tilde{A}$.*

There have been a number of different proofs of the commutant lifting theorem for contractions on Hilbert spaces [32], as well as several on Kreĭn spaces. Below is sketched one using the matrix completion ideas from the previous subsection. Full details can be found in [30, Theorem 3.2.1].

Let $T_1 \in \mathscr{L}(\mathscr{H}_1)$, $T_2 \in \mathscr{L}(\mathscr{H}_2)$ be two contractions on Kreĭn spaces $\mathscr{H}_1, \mathscr{H}_2$ with isometric dilations $V_1 \in \mathscr{L}(\mathscr{K}_1)$, $V_2 \in \mathscr{L}(\mathscr{K}_2)$. The spaces $\bigvee_{k=0}^{\infty} V_i^k \mathscr{H}_j$, are invariant for V_i, $i = 1, 2$, and when V_1 and V_2 are restricted to these subspaces, they give rise to minimal isometric dilations for T_1 and T_2. If a lifting intertwining the minimal isometric dilations can be found, then padding with zeros gives a lifting intertwining the original isometric dilations. Hence without loss of generality, it is assumed that V_1 and V_2 are minimal. Then since any two minimal isometric dilations of a contraction are isomorphic, there is no loss in generality in assuming these dilations have the canonical form

$$V_i = \begin{pmatrix} T_i & & & \\ \tilde{D}_i^* & & & \\ & 1 & & \\ & & 1 & \\ & & & \ddots \end{pmatrix} \in \mathscr{L}(\mathscr{H}_i \oplus \tilde{\mathscr{D}}_i \oplus \tilde{\mathscr{D}}_i \cdots) \qquad i = 1, 2.$$

For $i = 1, 2$, set $\mathscr{K}_{i,0} = \mathscr{H}_i$ and for $j = 1, 2, \ldots$, define $\mathscr{K}_{i,j} = \mathscr{H}_i \oplus \tilde{\mathscr{D}}_i \oplus \cdots \oplus \tilde{\mathscr{D}}_i$, where there are j copies of $\tilde{\mathscr{D}}_i$. Then set $V_{i,j}$ to be the compression of V_i to $\mathscr{K}_{i,j}$. These are contractions since $\tilde{\mathscr{D}}_1$ and $\tilde{\mathscr{D}}_2$ are Hilbert spaces.

For the commutant lifting problem, a contraction $A \in \mathscr{L}(\mathscr{H}_1, \mathscr{H}_2)$ such that $AT_1 = T_2 A$ is given, and it is desired to lift this to $\tilde{A} \in \mathscr{L}(\mathscr{K}_1, \mathscr{K}_2)$. The intertwining relation $AT_1 = T_2 A$ can be rewritten as $A_0 V_{1,0} = V_{2,0} A_0$, where

$A_0 = A$. The proof then proceeds by induction. Assume that for $j = 1, \ldots, n$, contractions A_j have been found such that

$$A_{j-1} P_{\mathscr{K}_{1,j-1}} = P_{\mathscr{K}_{2,j-1}} A_j, \tag{11.3}$$

$$A_j V_{1,j} = V_{2,j} A_j. \tag{11.4}$$

Notice that (11.3) implies that each A_j is a lower triangular operator matrix and that $AP_{\mathscr{H}_1} = P_{\mathscr{H}_2} A_n$ on $\mathscr{K}_{1,n}$.

Decompose $\mathscr{K}_{1,n+1} = V_{1,n+1}\mathscr{K}_{1,n+1} \oplus (\mathscr{K}_{1,n+1} \ominus V_{1,n+1}\mathscr{K}_{1,n+1})$ and $\mathscr{K}_{2,n+1} = \mathscr{K}_{2,n} \oplus (\mathscr{K}_{2,n+1} \ominus \mathscr{K}_{2,n})$, and write

$$A_{n+1} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$$

with respect to this decomposition. It is necessary to ensure that however A_{n+1} is chosen, it is a contraction satisfying (11.3) and (11.4) when $j = n + 1$. Because of the form of $V_{2,n+1}$, $V_{2,n+1} A_{n+1} = V_{2,n+1} P_{\mathscr{K}_{2,n}} A_n|_{\mathscr{K}_{2,n}}$, and since it is assumed that A_n is given, the equation $A_{n+1} V_{1,n+1} = V_{2,n+1} A_{n+1}$ specifies A_{n+1} on the range of $V_{1,n+1}$; that is, it fixes $C = \begin{pmatrix} C_{11} \\ C_{21} \end{pmatrix}$. Likewise, (11.3) fixes $R = \begin{pmatrix} C_{11} & C_{22} \end{pmatrix}$. These should give the same operator for C_{11}, which is the case since both are seen to be given by $P_{\mathscr{K}_{2,n}} A_{n+1}|_{V_{1,n+1}\mathscr{K}_{1,n+1}}$. By assumption A_n is a contraction, and as noted, for all j, $V_{1,j}$ and $V_{2,j}$ are contractions. Hence both C and R are contractions. Finally, observe that $\mathscr{K}_{2,n+1} \ominus \mathscr{K}_{2,n}$ is isomorphic to $\tilde{\mathscr{D}}_2$ which is a Hilbert space. Apply the Kreĭn space version of the Parrott extension found in the last subsection to obtain an operator C_{22} such that A_{n+1} is a contraction.

Next take a directed limit of the A_ns. This requires that they be uniformly bounded on an associated Hilbert spaces, which can be shown to be the case, thus yielding the lifting $\tilde{A}$. In the next section, a proof which uses a single application of the Parrott extension theorem is given, thus avoiding the need to take limits and find bounds.

There is a version of this theorem with unitary dilations rather than isometric dilations. However, to be able to use the solution of the Parrott extension problem then requires that the operators T_1 and T_2 are bicontractions in order to guarantee that the bottom row of the matrix extension continues to map into a Hilbert space. Other than this, the proof can be done in a more or less identical fashion to that sketched above for contractions and isometric dilations [26, 30].

Theorem 2 (The Commutant Lifting Theorem with Unitary Dilations). *Let $T_1 \in \mathscr{L}(\mathscr{H}_1)$, $T_2 \in \mathscr{L}(\mathscr{H}_2)$ be Kreĭn space bicontractions, and let $U_1 \in \mathscr{L}(\mathscr{K}_1)$, $U_2 \in \mathscr{L}(\mathscr{K}_2)$ be minimal unitary dilations of T_1 and T_2. Suppose that $A \in \mathscr{L}(\mathscr{H}_1, \mathscr{H}_2)$ is a bicontraction and that $AT_1 = T_2 A$. Then there exists a bicontraction $\tilde{A} \in \mathscr{L}(\mathscr{K}_1, \mathscr{K}_2)$ lifting A with the property that $\tilde{A}U_1 = U_2\tilde{A}$.*

There is also a version of Andô's theorem in Kreĭn spaces stating that two commuting contractions lift to commuting isometric dilations. If the operators are bicontractions, then taking adjoints and applying the isometric version of the theorem a second time gives a lifting to commuting unitary dilations. The proof follows the lines of the standard one on Hilbert spaces [12, Lemma 2.1].

An Approach to Commutant Lifting Via Diagram Chasing

There is an alternate way of approaching the commutant lifting theorem in terms of commutative diagrams. Details can be found in [27]. A different proof in the Hilbert space case which is closer to a diagrammatic version of the proof given in the last section can be found in [25].

As is standard, $\mathbb{C}[z]$ is used to denote the algebra of complex polynomials in one variable. Representations of algebras have already been briefly mentioned. This has an equivalent viewpoint in terms of modules, which is stated only for this particular algebra. Any unital representation π of $\mathbb{C}[z]$ is determined by $\pi(z)$, which in the present context will be some operator $T \in \mathscr{L}(\mathscr{H})$, where $\mathscr{H}$ is a Kreĭn space. Alternatively, it is possible to think of $\mathscr{H}$ as a (*left*) $\mathbb{C}[z]$-*module*, where the action of $\mathbb{C}[z]$ on $\mathscr{H}$ (i.e., the map taking $\mathbb{C}[z]\times\mathscr{H}$ to $\mathscr{H}$ obeying the various module rules, such as distributivity of addition in $\mathscr{H}$) is given by $p \cdot f = p(T)f$ for $p \in \mathbb{C}[z]$ and $f \in \mathscr{H}$. Since the module action is determined by where the generator is sent, $\mathscr{H}_T$ is often written for the module where $z \cdot f = Tf$.

A *module map* $\alpha : \mathscr{H}_{T_1} \to \mathscr{H}_{T_2}$ is a bounded linear map satisfying $\alpha(T_1 f) = T_2\alpha(f)$; in other words, α acts as an intertwining map. For example, in the setup for the commutant lifting problem, $\alpha(f) = Af$ and the assumed property that $AT_1 = T_2 A$ is what makes this into a module map. In the particular category that this construction is carried out, modules are assumed to be contractive, in that z is mapped to a contraction, and all intertwining maps are assumed to be contractive. Note that if A and T are contractions, then

$$1 - A^*T^*TA = 1 - A^*A + A^*(1 - T^*T)A \geq 0,$$

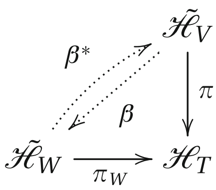
implies that composition of contractive maps is contractive and so the category is well defined.

There are other circumstances where module maps naturally occur. For example, if V is an isometric dilation of T, then the map π projecting from $\mathscr{H}_V$ onto $\mathscr{H}_T$ is a module map since $\mathscr{H}_T$ is an invariant subspace for V^*. In this example, the map π is a contractive co-isometry as well, and somewhat confusingly, in the language of categories, π is called a *cokernel*, while if it is isometric, it is a *kernel* (note that these are not the proper definitions of kernel and cokernel, but happens to be what they amount to in this context).

The notion of an *extension* of a module is also needed. This is given by a diagram of the form where $\mathscr{L}_X$ is a Hilbert space, κ is an isometry (i.e., a kernel), ν is a contractive co-isometry (i.e., a cokernel), and the range of κ equals the kernel of

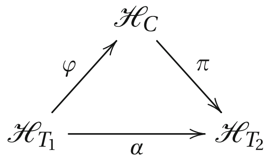
ν. That is, A is (isometrically isomorphic to) an operator of the form $\begin{pmatrix} T & 0 \\ Q & X \end{pmatrix}$: $\mathscr{H}_T \oplus \mathscr{L}_X \to \mathscr{H}_T \oplus \mathscr{L}_X$, where Q is such that the operator matrix is a contraction.

Let $\tilde{\mathscr{H}}_V$ and $\tilde{\mathscr{H}}_W$ be minimal isometric extensions of $\mathscr{H}_T$. As was noted, this means that there exist contractive co-isometric module maps π_V and π_W from $\tilde{\mathscr{H}}_V$ and $\tilde{\mathscr{H}}_W$ to $\mathscr{H}_T$. The fact that minimal isometric extensions are isomorphic is expressed diagrammatically by the existence of an isometric module map β : $\tilde{\mathscr{H}}_V \to \tilde{\mathscr{H}}_W$ such that the adjoint β^* : $\tilde{\mathscr{H}}_W \to \tilde{\mathscr{H}}_V$ is also an isometric module map and the following diagram commutes:



If on the other hand, V is not minimal, it is still true that β is a contractive co-isometric module map.

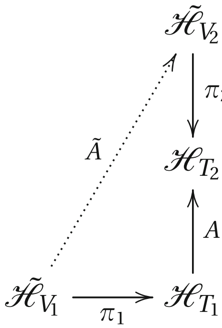
The idea behind the Parrott extension theorem can also b expressed diagrammatically. Let $\mathscr{H}_{T_1}$ and $\mathscr{H}_{T_2}$ be contractive Kreĭn modules, $\alpha : \mathscr{H}_{T_1} \to \mathscr{H}_{T_2}$ a contractive module map. The theorem then reads that there exists a contractive Kreĭn module $\mathscr{H}_C$, an isometric module map $\varphi : \mathscr{H}_{T_1} \to \mathscr{H}_C$, and a contractive co-isometric module map $\pi : \mathscr{H}_C \to \mathscr{H}_{T_2}$ such that the following diagram commutes:



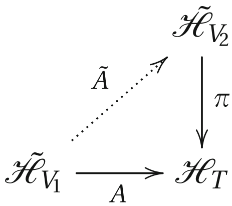
If, in addition, the module $\mathscr{H}_{T_1}$ is isometric (i.e., T_1 is an isometry), then C can be replaced by its minimal isometric dilation meaning that C will be an isometric dilation of T_2 and so $\mathscr{H}_C$ is isometric. While the last statement was never verified, it is easily seen from looking at the defect operator for C which can be deduced from its Julia operator.

The first statement gives the form of C on the range of φ (so in a sense fixes a column of C), while π specifies the form of C on a row. The fact that φ and π are module maps guarantees that the row and column agree where they overlap. Parrott's theorem then ensures that the map α exists.

The commutant lifting problem may be abstractly formulated as follows. Suppose $\mathscr{H}_{T_1}$, $\mathscr{H}_{T_2}$ are contractive Kreĭn modules with $\tilde{\mathscr{H}}_{V_1}$, and $\tilde{\mathscr{H}}_{V_2}$ the Kreĭn modules corresponding to their minimal isometric dilations. So π_1 and π_2 are cokernels (i.e., contractive co-isometric module maps), and it is further assumed that A is a contractive module map:



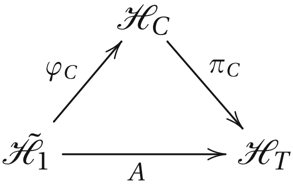
The desire is to find $\tilde{A}$ so that the diagram commutes. By composing π_1 and A and doing some relabeling this can be changed to the following equivalent problem. Find a contractive module map $\tilde{A}$ so that the following diagram commutes:



with π a cokernel, and A is a contractive module map.

With the given setup, a diagrammatic proof of the commutant lifting theorem is now obtained, roughly translating a Hilbert space proof due to Arocena [6] (see also [35]). Details can be found in [27].

Using the diagrammatic form of Parrott's lemma, factor the map A as



where φ_C and π_C are isometric and co-isometric module maps, and $\mathscr{H}_C$ is an isometric module. This module may not correspond to a minimal isometric dilation, but as noted above, there will in any case be a contractive co-isometric module map $\beta : \mathscr{H}_C \to \tilde{\mathscr{H}}_{V_2}$ so that the following diagram commutes:

$$\begin{array}{ccc} & & \tilde{\mathscr{H}}_{V_2} \\ & \overset{\beta}{\nearrow} & \Big\downarrow \pi_2 \\ \mathscr{H}_C & \xrightarrow[\pi_C]{} & \mathscr{H}_T \end{array}$$

Combining this diagram with the last one and setting $\tilde{A} = \beta \circ \varphi_C$, the required lifting results.

A slightly more detailed argument can be used to prove that not only does the contraction A lift to a contraction $\tilde{A}$, but that $\operatorname{ind}_- (1 - \tilde{A}\tilde{A}^*) = \operatorname{ind}_- (1 - AA^*)$, see [27]. Consequently, if A is a bicontraction, then $\tilde{A}$ is one as well.

Intertwining Operators with a Finite Number of Negative Squares and Some Open Questions

What happens if the condition that the intertwining operator A in the statement of the commutant lifting problem is a contraction is relaxed? On Hilbert spaces this would not be a problem since in this setting it is possible to scale A to be a contraction, but on Kreĭn spaces, the problem does not have a solution in general [18]. Nevertheless, there will be a lifting, at least upon restricting to a certain subspace of co-dimension equal to $\operatorname{ind}_- (1 - A^*A)$. See [7, 8, 18, 19].

As indicated earlier, the study of operator systems and operator spaces has revolutionized dilation theory and commutant lifting. However these all rely on tools that are essentially Hilbert space based. Notions such as complete positivity and complete contractivity may be defined for maps into the bounded operators on a Kreĭn space, but even the most basic results for such maps into the bounded operators on a Hilbert space do not appear to have obvious analogues. Can a similar theory be derived in the Kreĭn space context?

One could also ask about applications. As mentioned earlier, the Hilbert space commutant lifting theorem has been particularly useful in addressing interpolation problems. What are the analogous problems to which the Kreĭn space version of the commutant lifting theorem might be applied? Kreĭn spaces do appear naturally in certain interpolation problems [34], and the original question as to whether there is a Kreĭn space version of the commutant lifting theorem for contractions was motivated by de Branges' work on the Bieberbach conjecture.

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