Quaternionic Analysis: Application to Boundary Value Problems

47

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Contents

Abstract

Generalizing the complex one-dimensional function theory the class of quaternion-valued functions, defined in domains of $\mathbb{R}^4$, will be considered. The null solutions of a generalized Cauchy–Riemann operator are defined as the $\mathbb{H}$-holomorphic functions. They show a lot of analogies to the properties

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of classical holomorphic functions. An operator calculus is studied that leads to integral theorems and integral representations such as the Cauchy integral representation, the Borel–Pompeiu representation, and formulas of Plemelj–Sokhotski type. Also a Bergman–Hodge decomposition in the space of square integrable functions can be obtained. Finally, it is demonstrated how these tools can be applied to the solution of non-linear boundary value problems.

Introduction

Quaternionic analysis stands for the function theory of a generalized Cauchy–Riemann operator or a Dirac operator, respectively. In this way it is a generalization or extension of complex analysis for holomorphic functions depending on one complex variable. Starting with the problem of extending the field of complex numbers it can be seen that real quaternions share a lot of algebraic and geometric properties with complex numbers. At the same time quaternions can be identified with vectors in the Euclidean space $\mathbb{R}^4$ analogously to the identification of $\mathbb{C}$ with $\mathbb{R}^2$. In this way it is tried to unify the advantages of numbers with the theory of Euclidean vector spaces. It is then a natural idea to consider quaternion-valued functions defined in $\mathbb{R}^4$ and to look for a class of functions that unifies results and strategies analogously to holomorphic functions with results known from vector analysis.

Looking at the notation "quaternionic analysis" it must be mentioned that there are different interpretations, depending on the historical development, the way of defining $\mathbb{H}$-holomorphic functions and on the desired field of applications. Fueter, for instance, worked in his pioneering works [7,8] with quaternion-valued functions, depending on a quaternion-valued variable and defined the regular functions as null solutions of a generalized Cauchy–Riemann operator, nowadays often called Cauchy–Fueter operator. Later on, in [2], monogenic functions are defined again not only as null solutions of a generalized Cauchy–Riemann operator but also as null solutions of a Dirac operator. Restricting the contents of [2] to there here considered case of quaternion-valued functions it became also possible to consider functions, defined in $\mathbb{R}^3$ with values in the full quaternions. This can be done by a reduced generalized Cauchy–Riemann operator or by the Moisil–Teodorescu operator, see [24]. For a complete survey it is recommended to read the paper [1]. The reduced Cauchy–Riemann operator can also be studied for functions, defined in $\mathbb{R}^3$ with values in $\mathbb{R}^3$, identified with the reduced quaternions. This leads to a special case of the Riesz system and was recently studied in [25] and earlier in [22]. All these approaches lead to very similar results concerning Cauchy integral formulas, Taylor series expansions, and power series representations. Therefore, in this chapter only one of these cases will be discussed.

Another line goes back to K. Habetha who pointed out (see, e.g., [17]) that a generalized function theory must have a Cauchy integral formula. This relates monogenic functions with integral operators and in the following the theory of right invertible operators [28] began to play a role in quaternionic analysis. With the identity operator I, the differential operator D (a generalized Cauchy–Riemann operator or a Dirac operator), and a right inverse T (Teodorescu operator) the initial value operator F is defined by $F := I - TD$ and one can show that this initial value operator from the theory of right invertible operators is the generalized Cauchy–Riemann operator. This operator relates the boundary values of an $\mathbb{H}$-holomorphic (monogenic, regular) function with its values in the whole domain and this property is what is needed for the solution of boundary value problems by integral representations. A first complete description of this approach can be found in [12] and [14]. In this chapter it will be shown how quaternionic analysis in the latter sense can be applied for the solution of boundary value problems and therefore the discussion of $\mathbb{H}$-holomorphic functions will be restricted to this goal.

The class of $\mathbb{H}$-holomorphic functions will be defined in this chapter as the set of functions belonging to the kernel of a generalized Cauchy–Riemann operator. Elements of a function theory for these functions will be described. The main goal is to obtain integral theorems and integral representations for the $\mathbb{H}$-holomorphic functions and to study their jumps on the boundary of a smoothly bounded domain. The consideration will be focused to this approach because it allows a direct and straight approach to the solution of boundary value problems and the representation of their solutions. Other equivalent approaches to such a function theory also in more general situations are contained in subsequent chapters. Basic idea in the presented approach is to derive all representations by using an operator calculus of the generalized Cauchy–Riemann operator, a right inverse that is defined by the convolution of the fundamental solution of the Cauchy–Riemann operator with the functions and the identity operator. Studying these operators, due to the factorization of the Laplace operator by the Cauchy–Riemann operator and its adjoint, also harmonic analysis is touched and it becomes visible that the considered function theory includes a refinement of harmonic analysis.

The most important result to connect the integral formulas and the $\mathbb{H}$-holomorphic functions with the solutions of general boundary value problems is the Bergman–Hodge decomposition. This orthogonal decomposition describes the complementary subspace of the subspace of square integrable $\mathbb{H}$-holomorphic functions in L_2. The Bergman projection together with the operators of the operator calculus and the Plemelj projections are the building blocks for constructing representation formulas for the solutions of boundary value problems. As an example a boundary value problem for the stationary Navier–Stokes equations will be studied.

Quaternions

Historical Notes

In 1833 Sir William Rowan Hamilton, one of the most fascinating scientists of the nineteenth century, proved that complex numbers form a so-called division algebra, i.e. there exist a unit element $1 \neq 0$ and all nonzero elements have a multiplicative inverse. All arithmetic operations (addition, subtraction, multiplication and division (for nonzero elements)) are defined and satisfy the usual rules. Hamilton recognized that there are two units: 1 and i with the algebraical rules

$$1^2 = 1 \quad \text{and} \quad i^2 = -1.$$

Every element can be written as $x + iy$ with $x, y \in \mathbb{R}$. In the following ten years he tried to extend this result to *triples*, i.e., the real unit 1 and two further imaginary units i and j. Such triples he called *vectors*. Over years he did not succeed in finding a division rule for vectors. In 1843 only after introducing a further imaginary unit and dropping the commutativity Hamilton was able to divide vectors. On the discovery the following story is told:

Hamilton had to chair a meeting at the Royal Irish Academy. His wife walked with him along the Royal Canals in Dublin. Suddenly he had an ingenious idea, took his pocket knife, and carved the fundamental relations of the skew-field of quaternions in a stone of the Brougham Bridge:

Here as he walked by on the 16th of October in 1843 Sir William Rowan Hamilton in a flash of genius discovered the fundamental formula for quaternion multiplication:

$$i^2 = j^2 = k^2 = ijk = -1$$

cut it on a stone of this bridge.

On November 14 in 1843 a first paper on quaternions appears in the Council Books of the Royal Academy.

W. Blaschke, celebrating the occasion of the 250th birth anniversary of Leonard Euler in his conference talk entitled: *Euler und die Geometrie* (Berlin 23.03.1957) stated that Euler had been the first one to define quaternions. This was done in a letter to Christian Goldbach on May, 4th in 1748 in his researches on parametric representations of movements in Euclidean space. Obviously, Euler defined (vectorial) quaternions without explicitly naming them.

Carl Friedrich Gauss was working with transformations of spaces in 1819 and he had to compose real quadruples. Given (a, b, c, d) and $(\alpha, \beta, \gamma, \delta)$ in $\mathbb{R}^4$ he obtained for the composite quadruple

$$(\alpha a - \beta b - \gamma c - \delta d, a\beta + b\alpha - c\delta + d\gamma, a\gamma + b\delta + c\alpha - d\beta, a\delta - b\gamma + c\beta + d\alpha).$$

After transposition of the second and third coordinates one gets the today usual notation of quaternionic multiplication (see also in [9, 20]).

Quaternion means in Latin *set of four*, the Greek translation of this word is "tetractys." Hamilton who knew the Greek language in depth and idolized the Pythagorean school has built a bridge between his structure and the Pythagorean tetractys, which was considered as the source of all things (P.G. Tait).

Calculation Rules and Representations

Let $\mathbb{H}$ be the skew-field of real quaternions and $a \in \mathbb{H}$, then $a = \alpha_0 e_0 + \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3$ with $\{\alpha_k\} \subset \mathbb{R}$. The multiplication is defined by

$$e_i e_j + e_j e_i = -2\delta_{ij} e_0 \quad \text{and} \quad e_0 e_j = e_j e_0;\ e_0^2 = e_0;\ e_1 e_2 = e_3 \quad i, j = 1, 2, 3.$$

The quaternionic conjugation is defined by

$$\overline{e}_0 = e_0, \overline{e}_k = -e_k \quad (k = 1, 2, 3), \quad \overline{x} = x_0 - \sum \alpha_k e_k = x_0 - \mathbf{x}.$$

$\mathbf{x} =: \mathrm{Vec}(x)$ is called after Hamilton *vector (part)* and $x_0 =: \mathrm{Sc}(x)$ *scalar (part)*. Further, we have

$$\overline{x} x = |x|^2_{\mathbb{R}^4} =: |x|^2_{\mathbb{H}}, \quad x^{-1} := \frac{1}{|x|^2}\overline{x} \quad \text{for } x \neq 0, \quad \overline{xy} = \overline{y}\,\overline{x}.$$

Let $\varphi = \arccos\frac{x_0}{|x|}$ and $\boldsymbol{\omega}(\mathbf{x}) = \frac{\mathbf{x}}{|\mathbf{x}|} \in S^2$. Then we have for any quaternion x with $x = |x|(\cos\varphi + \boldsymbol{\omega}(\mathbf{x})\sin\varphi)$ *de Moivre's formula*

$$(\cos\varphi + \boldsymbol{\omega}(\mathbf{x})\sin\varphi)^n = \cos n\varphi + \boldsymbol{\omega}(\mathbf{x})\sin n\varphi.$$

For vectors we obtain:

$$\mathbf{x}\,\mathbf{y} = -\mathbf{x} \cdot \mathbf{y} + \mathbf{x} \times \mathbf{y}.$$

The *inner product (scalar product)* was introduced by H.G. Grassmann [11] and the *cross product* by J.W. Gibbs (1881) [5]. The following algebraical properties can be proved immediately:

$$(i) \quad \mathbf{x} \cdot \mathbf{y} = -\frac{1}{2}(\mathbf{xy} + \mathbf{yx}), \quad \mathbf{x} \times \mathbf{y} = \frac{1}{2}(\mathbf{xy} - \mathbf{yx}),$$

(ii) $x \in \mathbb{R}$ if and only if $xy = yx$ for any $y \in \mathbb{H}$,

(iii) From $x^2 = y^2$ it does not follow that $x = \pm y$.

There are simple relations between quaternions and vectors:

Proposition 1. *Let $x \in \mathbb{H}$ then there exists a vector $\mathbf{y} \neq 0$ such that $x\mathbf{y}$ is a vector. Any quaternion x is the product of two vectors. The inverse of a vector is again a vector.*

Corollary 1. *In $\mathbb{H}$ there exist scalars and vectors.*

The advantage of the skew-field of real quaternions compared with the Euclidean vector spaces $\mathbb{R}^3$ or $\mathbb{R}^4$, respectively, becomes visible when one looks for a convenient descriptions of rotations.

Let $\rho_y(x) := yxy^{-1}$ for $y \in \mathbb{H}$ and $x \in \mathbb{R}^3$. This mapping is an automorphism of $\mathbb{R}^3$. One has:

$$\rho_y(\mathbf{x}) \times \rho_y(\mathbf{x}') = \rho_y(\mathbf{x} \times \mathbf{x}').$$

Euler–Rodrigues formula describes a rotation completely. We obtain

$$\mathbf{x}' := \rho_y(\mathbf{x}) = \mathbf{x}\cos 2\varphi + (\boldsymbol{\omega} \times \mathbf{x})\sin 2\varphi + (1 - \cos 2\varphi)(\boldsymbol{\omega} \cdot \mathbf{x})\boldsymbol{\omega}$$

with $y_0 = \cos\varphi$, $\boldsymbol{\omega} = \frac{\mathbf{y}}{|\mathbf{y}|}$ and $|\mathbf{y}| = \sin\varphi$. Moreover, *Cayley's theorem* can be proved.

Theorem 1. *The rotations in $\mathbb{H}$ are exactly the mappings $x \to x' = axb$ with $|a| = |b| = 1$ and $a, b \in \mathbb{H}$.*

For unimodular quaternions a nice representation formula exists. Let $u := x_0 + ix_1$, $v := x_2 + ix_3$ with $|u|^2 + |v|^2 = 1$ and the quaternion x can be written as $x = u + jv$. Then x can be represented by the matrix

$$X = \begin{pmatrix} u & v \\ -\overline{v} & \overline{u} \end{pmatrix}.$$

With $\operatorname{Re} u = \cos\frac{t}{2}$ the Nth power is given by [27]

$$X^N = \frac{\sin N\frac{t}{2}}{\sin\frac{t}{2}} X - \frac{\sin(N-1)\frac{t}{2}}{\sin\frac{t}{2}} I.$$

One denotes by $\mathbb{H}(\mathbb{C})$ the set of quaternions with complex coefficients $\alpha_k = \alpha_k^1 + i\alpha_k^2$ ($\alpha_k^i \in \mathbb{R}$), i.e.,

$$a = \alpha_0 e_0 + \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 = a^1 + ia^2, \quad (a^j \in \mathbb{H}).$$

Because of $ie_k = e_k i$, the denotation $\mathbb{CH} = \mathbb{H}(\mathbb{C})$ can be used. In the case of complex quaternions there are three possible conjugations:

(i) $\overline{a}^{\mathbb{C}} := a^1 - ia^2$,
(ii) $\overline{a}^{\mathbb{H}} := \overline{a}^1 + i\overline{a}^2$,
(iii) $\overline{a}^{\mathbb{CH}} := \overline{a}^1 - i\overline{a}^2$.

Formally, one gets a complex valued norm:

$$|a|_{\mathbb{C}}^2 = a\overline{a} = |a^1|^2 + |a^2|^2 + 2i\,[\alpha_0^1\alpha_0^2 + \mathbf{a}^1 \cdot \mathbf{a}^2].$$

A corresponding norm is then given by $\|a\|^4 := ||a|_{\mathbb{C}}^2|^2$. In $\mathbb{H}(\mathbb{C})$ we have to distinguish between scalars, vectors, bivectors, and pseudoscalars.

As is known, all real or complex $(n \times n)$ matrices form a ring, called the *full matrix ring* $\mathbb{R}^{n\times n}$ and $\mathbb{C}^{n\times n}$. It is often useful to look how to represent algebraic structures as isomorphic images in the matrix ring by automorphisms of $\mathbb{R}^4$ or also of $\mathbb{C}^2$. Let $x = x_0 + x_1e_1 + x_2e_2 + x_3e_3$ and $y = y_0 + y_1e_1 + y_2e_2 + +y_3e_3$ be two arbitrary quaternions. By multiplication xy, one obtains

$$\begin{aligned} xy = \quad & (x_0y_0 - x_1y_1 - x_2y_2 - x_3y_3) \\ & +(x_0y_1 + x_1y_0 + x_2y_3 - x_3y_2)e_1 \\ & +(x_0y_2 - x_1y_3 + x_2y_0 + x_3y_1)e_2 \\ & +(x_0y_3 + x_1y_2 - x_2y_1 + x_3y_0)e_3. \end{aligned}$$

By means of the usual isomorphism this quaternion will be associated with the $\mathbb{R}^4$-vector

$$\begin{pmatrix} x_0y_0 - x_1y_1 - x_2y_2 - x_3y_3 \\ x_1y_0 + x_0y_1 - x_3y_2 + x_2y_3 \\ x_2y_0 + x_3y_1 + x_0y_2 - x_1y_3 \\ x_3y_0 - x_2y_1 + x_1y_2 + x_0y_3 \end{pmatrix}.$$

Identifying the quaternion y with the $\mathbb{R}^4$-vector $y = (y_0, y_1, y_2, y_3)\top$ and introducing the matrix

$$L_x := \begin{pmatrix} x_0 & -x_1 & -x_2 & -x_3 \\ x_1 & x_0 & -x_3 & x_2 \\ x_2 & x_3 & x_0 & -x_1 \\ x_3 & -x_2 & x_1 & x_0 \end{pmatrix}$$

it is easy to check that

$$\begin{pmatrix} x_0y_0 - x_1y_1 - x_2y_2 - x_3y_3 \\ x_1y_0 + x_0y_1 - x_3y_2 + x_2y_3 \\ x_2y_0 + x_3y_1 + x_0y_2 - x_1y_3 \\ x_3y_0 - x_2y_1 + x_1y_2 + x_0y_3 \end{pmatrix} = L_xy.$$

Thus, the quaternion x is associated with the matrix L_x in a natural way

$$x \to L_x \quad \text{with} \quad xy = L_x y$$

for all $y \in \mathbb{H}$, such that L_x is called a *left representation* of the quaternion x in $\mathbb{R}^{4\times 4}$. It is not difficult to prove the following properties:

$$\begin{aligned} &\text{(i)} \;\; L_1 = E, \\ &\text{(ii)} \;\; L_{\overline{x}} = L_x^\top, \\ &\text{(iii)} \;\; L_{x\tilde{x}} = L_x L_{\tilde{x}}, \\ &\text{(iv)} \;\; \det L_x = |x|^4. \end{aligned}$$

Finally, one has the decomposition

$$L_x = x_0 E + X$$

with $X^\top = -X$.

Analogously one gets a *right representation*

$$R_x = \begin{pmatrix} x_0 & -x_1 & -x_2 & -x_3 \\ x_1 & x_0 & x_3 & -x_2 \\ x_2 & -x_3 & x_0 & x_1 \\ x_3 & x_2 & -x_1 & x_0 \end{pmatrix}$$

of the quaternion x obtained in $\mathbb{R}^{4\times 4}$, where $yx = R_x y$. The properties (i), (ii), and (iv) keep being valid, while (iii) is replaced by

$$R_{x\tilde{x}} = R_{\tilde{x}} R_x.$$

Other matrix representations in $\mathbb{R}^{4\times 4}$ are possible, but will not be considered here. One can show:

$$L_{e_1} L_{e_2} L_{e_3} = -E, \quad R_{e_1} R_{e_2} R_{e_3} = E,$$

where E is the identity matrix in the $\mathbb{R}^{4\times 4}$.

Quaternions can be represented as matrices in $\mathbb{C}^{2\times 2}$. Physicists prefer the following assignment

$$x_0 e_0 + e_1 x_1 + e_2 x_2 + e_3 x_3 \to \begin{pmatrix} x_0 - i x_3 & i x_1 - x_2 \\ -i x_1 + x_2 & x_0 + i x_3 \end{pmatrix}.$$

This assignment appears naturally to the orthogonal unit vectors e_0, e_1, e_2, e_3 using the so-called *Pauli matrices*, which are given by

$$\sigma_0 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; \sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \; \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

One has to identify subsequently e_0, e_1, e_2, e_3 with $\sigma_0, -i\sigma_1, -i\sigma_2, -i\sigma_3 = \sigma_1\sigma_2$. One gets a subalgebra of $\mathbb{C}^{2\times 2}$.

More information can be found in the books [13, 16].

Quaternion-Valued Elementary Functions

Only as an example the exponential function of a quaternion variable and its inverse will be defined. Starting from this one is able to deduce all other interesting elementary functions. A good source for such questions is the book [26].

The function e^x, x is a quaternion variable, defined by

$$e^x = e^{x_0}(\cos|\mathbf{x}| + \operatorname{sgn}(\mathbf{x})\sin|\mathbf{x}|) \tag{47.1}$$

is called *quaternion natural exponential function*. In the case that x is a real number, the definition of e^x is naturally extended to comply with the usual exponential function of real numbers. The previous representation is explained in the following calculation. For this purpose it will be necessary to define $e^x := \sum_{k=0}^{\infty} \frac{x^k}{k!}$. The reader should notice that this series converges normally for all x in analogy to the complex case, since we have $|x^k| \leq |x|^k$ for any quaternion x. Because $e^{|x|}$ converges, the comparison test yields that e^x converges for all x. Clearly, the series expansions $\sum_{k=0}^{\infty} \frac{{x_0}^k}{k!}$ and $\sum_{k=0}^{\infty} \frac{\mathbf{x}^k}{k!}$ converge normally. Therefore the Cauchy product of e^{x_0} and $e^{\mathbf{x}}$ leads to

$$\begin{aligned}\sum_{k=0}^{\infty}\left\{\sum_{j=0}^{k}\frac{x_0)^j}{j!}\frac{\mathbf{x}^{k-j}}{(k-j)!}\right\} &= \sum_{k=0}^{\infty}\frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}(x_0)^j\mathbf{x}^{k-j} \\ &= \sum_{k=0}^{\infty}\frac{(x_0+\mathbf{x})^k}{k!}.\end{aligned}$$

Consequently, $e^x = e^{x_0+\mathbf{x}} = e^{x_0}e^{\mathbf{x}}$. For the remaining term $e^{\mathbf{x}}$, it holds:

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{\mathbf{x}^k}{k!} &= \sum_{j=0}^{\infty}\frac{\mathbf{x}^{2j}}{(2j)!} + \sum_{j=0}^{\infty}\frac{\mathbf{x}^{2j+1}}{(2j+1)!} \\ &= \sum_{j=0}^{\infty}(-1)^j\frac{|\mathbf{x}|^{2j}}{(2j)!} + \frac{\mathbf{x}}{|\mathbf{x}|}\sum_{j=0}^{\infty}(-1)^{j+1}\frac{|\mathbf{x}|^{2j+1}}{(2j+1)!} \\ &= \cos|\mathbf{x}| + \operatorname{sgn}(\mathbf{x})\sin|\mathbf{x}|.\end{aligned}$$

From the practical point of view, the quaternion exponential function is an example of one that is defined by specifying its scalar and vector parts. More precisely, the scalar and vector parts of e^x are, respectively, $Sc(e^x) = e^{x_0} \cos|\mathbf{x}|$ and $Vec(e^x) = e^{x_0}\operatorname{sgn}(\mathbf{x}) \sin|\mathbf{x}|$. Thus, values of this quaternion function are found by expressing the point x as $x = x_0 + x_1 i + x_2 j + x_3 k$, and then substituting the values of x_0, x_1, x_2 and x_3 in the given expression. The following properties of the quaternion exponential function can be proved:

(i) $e^x \neq 0$, for all $x \in \mathbb{H}$,
(ii) $e^{-x}e^x = 1$, $e^{\mathbf{x}\pi} = -1$,
(iii) $(e^x)^n = e^{nx}$ for $n = 0, \pm 1, \pm 2, \ldots$ (*de Moivre's formula*),
(iv) $e^{(x^1)}e^{(x^2)} \neq e^{(x^1+x^2)}$ in general, unless x^1 and x^2 commute.

In particular, $e^x e^{-x} = e^{0_{\mathbb{H}}} = 1$. Then, by induction $(e^x)^n = e^{nx}$, where n is any positive or negative integer. For the latter, take, for example, $e^{\pi i}e^{\pi j} = (-1)(-1) = 1$, and $e^{\pi i+\pi j} = \cos(\pi\sqrt{2}) + \frac{i+j}{\sqrt{2}}\sin(\pi\sqrt{2}) \neq 1$.

This example shows that it depends very much on the property that has been generalized what the result will be. It is easy to prove that this elementary function is not an $\mathbb{H}$-holomorphic function. An alternative version of a quaternion exponential function is introduced in [13]. A. Hommel and K. Gürlebeck constructed in 2005 the following $\mathbb{H}$-holomorphic exponential function

$$\mathcal{E}(x) = e^{x_0}\Bigg[\Bigg(\cos\frac{x_1+x_2+x_3}{\sqrt{3}} + \sin\frac{x_1}{\sqrt{3}}\sin\frac{x_2}{\sqrt{3}}\sin\frac{x_3}{\sqrt{3}}\Bigg) + \frac{1}{\sqrt{3}}\Bigg((e_1+e_2+e_3)\sin\frac{x_1+x_2+x_3}{\sqrt{3}} - e_1\cos\frac{x_1}{\sqrt{3}}\sin\frac{x_2}{\sqrt{3}}\sin\frac{x_3}{\sqrt{3}} - e_2\sin\frac{x_1}{\sqrt{3}}\cos\frac{x_2}{\sqrt{3}}\sin\frac{x_3}{\sqrt{3}} - e_3\sin\frac{x_1}{\sqrt{3}}\sin\frac{x_2}{\sqrt{3}}\cos\frac{x_3}{\sqrt{3}}\Bigg)\Bigg].$$

Both mentioned exponential functions are $\mathbb{H}$-holomorphic extensions of the real-valued exponential function $\exp(x_0)$. Here, the property of the exponential function to coincide with its derivative is the main point in the generalization.

Remark 1. The quaternion natural logarithm [26] function $\ln(x)$ is defined by

$$\ln(x) = \log_e|x| + \operatorname{sgn}(\mathbf{x})\arg(x). \tag{47.2}$$

where $\operatorname{sgn}(\mathbf{x}) = \mathbf{x}/|\mathbf{x}|$. Here $\log_e|x|$ is the usual real natural logarithm of the positive number $|x|$ (and hence it is defined unambiguously). This quaternion function is another example of one that is defined by specifying its scalar and vector parts. More precisely, the scalar and vector parts of $\ln(x)$ are, respectively, $Sc(\ln(x)) = \log_e|x|$ and $Vec(\ln(x)) = \operatorname{sgn}(\mathbf{x})\arg(x)$. Because there are intrinsically infinitely many arguments of x, it is clear that the representation gives infinitely many solutions w

to the equation $e^w = x$ whether x is a nonzero quaternion number. By switching to polar form, one obtains the following alternative description of the quaternion logarithm:

$$\ln(x) = \begin{cases} \log_e |x| + \mathrm{sgn}(\mathbf{x}) \left(\arccos \frac{x_0}{|x|} + 2\pi n \right), & |\mathbf{x}| \neq 0, \\ \log_e |x_0| &, \quad |\mathbf{x}| = 0, \end{cases}$$

$$= \begin{cases} \log_e |x| + \mathrm{sgn}(\mathbf{x}) \left(\arctan \frac{|\mathbf{x}|}{x_0} + 2\pi n \right), & x_0 > 0, \\ \log_e |\mathbf{x}| + \mathrm{sgn}(\mathbf{x}) \left(\frac{\pi}{2} + 2\pi n \right) &, \quad x_0 = 0, \end{cases}$$

where $n = 0, \pm 1, \pm 2, \ldots$. Observe that the different values of $\ln(x)$ all have the same scalar part and that their vector parts differ by $2\pi n$. Each value of n determines what is known as a branch (or sheet), a single-valued component of the multiple-valued logarithmic quaternion function. When $n = 0$, one has a special situation.

$\mathbb{H}$-Holomorphic Functions and Quaternion Operator Calculus

$\mathbb{H}$-Holomorphic Functions

The main idea in the studies of quaternion-valued functions is to provide structures of vector fields with an algebraic structure to refine the well-known harmonic analysis and to generalize the complex analysis to higher dimensions. For this purpose one has to define the class of functions that should replace the holomorphic functions. Having in mind the corresponding complex approaches one will look for differentiability, directional derivatives, a generalized Cauchy–Riemann system and power (or polynomial) series expansions. The best and desired situation is that all these approaches can be generalized and keep their equivalence from the complex case. To speak about a function theory one should have at least a class of functions where the approaches found by Riemann, Cauchy, and Weierstrass, respectively, are equivalent. A detailed study of these approaches can be found in [13] and [21].

Here it will be used only the most popular way to define the desired class of functions as null solutions of a generalized Cauchy–Riemann system. The starting point is the definition of the differential operator

$$\overline{\partial} := \frac{\partial}{\partial x_0} + \frac{\partial}{\partial x_1} e_1 + \frac{\partial}{\partial x_2} e_2 + \frac{\partial}{\partial x_3} e_3.$$

One sees easily that $\frac{1}{2}\overline{\partial}$ is a formal generalization of the complex differential operator

$$\partial_{\overline{z}} := \frac{1}{2}(\partial_x + i\,\partial_y).$$

One can further see that the differential operator $\frac{1}{2}\partial$ with

$$\partial := \frac{\partial}{\partial x_0} - \frac{\partial}{\partial x_1}e_1 - \frac{\partial}{\partial x_2}e_2 - \frac{\partial}{\partial x_3}e_3$$

corresponds to

$$\partial_z := \frac{1}{2}(\partial_x - i\,\partial_y).$$

Therefore, the operators $\frac{1}{2}\overline{\partial}$ and $\frac{1}{2}\partial$ are called generalized Cauchy–Riemann operator and adjoint Cauchy–Riemann operator, respectively. An easy calculation shows that $\overline{\partial}$ and ∂ factorize the Laplacian, i.e.,

$$\overline{\partial}\partial = \partial\overline{\partial} = \Delta.$$

his property is analogous to the complex one-dimensional case and shows that one can get in this way a refinement of the harmonic analysis.

Definition 1. A function $f \in C^1(G)$ in a domain $G \subset \mathbb{H}$ and with values in $\mathbb{H}$ is there right- resp. left-$\mathbb{H}$-holomorphic if and only if

$$f\overline{\partial} = 0 \;\text{ resp. }\; \overline{\partial} f = 0.$$

These differential equations are also called Cauchy–Riemann differential equations (in $\mathbb{H}$). It should be mentioned that instead of "$\mathbb{H}$-holomorphic" synonymously the words "monogenic" or "regular" are in use.

In the special case that functions from $\mathbb{R}^3$ into $\mathbb{H}$ are considered there are two mainly discussed possibilities to embed the three-dimensional Euclidean space in $\mathbb{H}$. By identifying $x = (x_0, x_1, x_2)$ with the reduced quaternion $x = x_0e_0 + x_1e_1 + x_2e_2$ the *(reduced) Cauchy–Riemann operator* will be $\overline{\partial} = e_0\partial_0 + e_1\partial_1 + e_2\partial_2$. The resulting Cauchy–Riemann system is then the well-known *Riesz system*. A special property of this system is that its null solutions are left- and right-monogenic at the same time. If $x = (x_1, x_2, x_3)$, identified with $x = x_1e_1 + x_2e_2 + x_3e_3$ is used, then the corresponding differential operator is $D = e_1\partial_1 + e_2\partial_2 + e_3\partial_3$. This operator is called *Dirac operator* and the system of partial differential equations leads to the so-called *Dirac equation*. In this special case the operator is also known as the Moisil–Teodorescu operator. Identifying a quaternion-valued function $f = f_0e_0 + f_1e_1 + f_2e_2 + f_3e_3$ with the vector valued function $f = f_0 + \mathbf{f}$ the action of the Dirac operator D can be written in terms of the classical operators from vector analysis as $Df = -\operatorname{div}\mathbf{f} + \operatorname{grad} f_0 + \operatorname{curl}\mathbf{f}$. This opens the possibility to switch between both theories if it is necessary.

For several purposes it is useful to have the fundamental solution of the generalized Cauchy–Riemann operator available.

Definition 2. The function

$$E_3(x) := \frac{1}{\sigma_3} \frac{\overline{x}}{|x|^4} \quad (x \neq 0),$$

defined in $\mathbb{R}^4 \setminus \{0\}$, is called *Cauchy kernel*. Here σ_3 is the surface area of the unit sphere S^3 in $\mathbb{R}^4$. Using the Gamma function one has (see [13])

$$\sigma_3 = 2\,\Gamma^4\left(\frac{1}{2}\right) = 2\pi^2.$$

In $\mathbb{C}$ the Cauchy kernel simplifies to

$$E_1(x) = \frac{1}{2\pi} \frac{1}{x}.$$

It can be easily seen

Proposition 2. *The Cauchy kernel is left- and right-$\mathbb{H}$-holomorphic, i.e.* $\overline{\partial} u = 0$ *and* $(u\overline{\partial}) = 0$.

It can be shown that the Cauchy kernel is just the fundamental solution of the Cauchy–Riemann operator. Therefore, the convolution with the Cauchy kernel over the domain G defines a right inverse to the Cauchy–Riemann operator. From the theory of right invertible operators is known (see e.g. [28] and [30]) that the commutator of the Cauchy–Riemann operator and the convolution with the fundamental solution over the domain G defines the so-called initial operator. This operator is acting on functions defined on the boundary of G and it will be recognized as a generalized Cauchy integral operator. This fundamental property connects boundary values with solutions of partial differential equations in a convenient way (not only for the case of the Cauchy–Riemann operator). This operator calculus has to be introduced in the following. The main tool for obtaining all these results are the famous Gauss' and Stokes' theorems together with Green's formulae. The fundamental theorem is a formula of Borel–Pompeiu type as it is in any complex and hypercomplex analysis.

Integral Theorems for Quaternion-Valued Holomorphic Functions

Theorem 2 (Formula of Borel–Pompeiu). *Let* $G \subset \mathbb{R}^4$ *and* $\partial G = \Gamma$ *be a bounded domain with sufficiently smooth boundary and an outward pointing normal vector. Then one has for any* $u \in C^1(\overline{G})$

$$\int_{\partial G} E_3(y-x)u(y)d\Gamma_y - \int_G E_3(y-x)(\overline{\partial} u)(y)dG_y = \begin{cases} u(x), \ x \in G, \\ 0\ , \ x \in \mathbb{R}^4 \setminus \overline{G}. \end{cases}$$

The operators T_G and F_Γ, defined by

$$(T_G u)(x) := -\int_G E(y-x)u(y)dG_y; \quad (F_\Gamma u)(x) := \int_\Gamma E(y-x)\nu(y)u(y)d\Gamma_y$$

are called *Teodorescu transform* and *Cauchy–Fueter operator*, respectively, where $d\Gamma$ stands for the Lebesgue surface measure and dG denotes the volume measure. With these notations the relations $\partial\, T_G u = u$ in G and $F_\Gamma u + T_G\, \bar{\partial} u = u$ in G have been proved with the theorem above.

The most important consequence of Borel–Pompeiu's formula is Cauchy's integral formula, which appears as a simple corollary applying Borel–Pompeiu's formula to a left-holomorphic function u:

Theorem 3 (Cauchy's Integral Formula). *Let $G \in \mathbb{R}^4$ be a bounded domain with sufficiently smooth boundary and outward-pointing normal unit vector n. For a left-holomorphic function $u \in C^1(\overline{G})$ one has:*

$$\int_{\partial G} E_3(y-x)n(y)u(y)d\Gamma = \begin{cases} u(x)\ , & x \in G, \\ 0\ , & x \in \mathbb{R}^4 \setminus \overline{G}. \end{cases}$$

For a right-$\mathbb{H}$-holomorphic function u and $E_3(y-x)$ one has to interchange their positions.

Some consequences of Cauchy's formula will be mentioned, firstly an integral formula for the exterior domain. For this purpose it is assumed a *Jordan surface* Γ, which is a piecewise smoothly bounded manifold whose complement relatively to $\mathbb{R}^4$ consists of only two domains. Thus, $\mathbb{R}^4$ is split by Γ into two domains, one of them having the point ∞ as boundary point: this is called the *exterior domain* G^- of Γ. Correspondingly $G^+ = \mathbb{R}^4 \setminus (\Gamma \cup G^-)$ is called *interior domain* of Γ. Then it holds

Theorem 4 (Cauchy's Integral Formula for the Exterior Domain). *Let Γ be a Jordan surface with the exterior domain G^- and the interior domain G^+. The orientation of Γ is to be chosen so that the normal points into G^-. The function u is assumed to be left-holomorphic in G^- and continuously differentiable in $G^- \cup \Gamma$ and to have a limit value $u(\infty)$ at $x = \infty$. We then have:*

$$\int_\Gamma E_3(y-x)n(y)u(y)d\Gamma_y) = \begin{cases} -u(x) + u(\infty), & x \in G^-, \\ u(\infty), & x \in G^+. \end{cases}$$

There are other important theorems for $\mathbb{H}$-holomorphic functions. The first one is the mean value theorem, which is nothing more than the application of Cauchy's formula to a ball.

Corollary 2 (Mean Value Property). *An $\mathbb{H}$-holomorphic function u possesses the mean value property, i.e., for all $x_0 \in G$ and for all balls (disks) $\{x : |x - x_0| \le \rho\} \subset G$ we have:*

$$u(x) = \frac{1}{\sigma_3} \int\limits_{|y|=1} u(x_0 + \rho y) dS_y^3, \quad (S^3 = \partial B_1(0)).$$

That means that the value of u in the center of the ball is equal to the normalized integral mean of u over the boundary of the ball. By integrating over ρ one gets a mean value theorem over the whole ball of radius ρ.

The next theorem concerns a maximum principle.

Theorem 5 (Maximum Principle). *Let u be $\mathbb{H}$-holomorphic and bounded in a domain $G \subset \mathbb{R}^4$, i.e., $\sup_{x \in G} |u(x)| = M < \infty$. If $|u|$ attains the value M at a point of G, then u is constant in G with $|u(x)| = M$.*

Schwarz Formula for Quaternion-Valued Functions

It is necessary to introduce the notion of a Hardy space

Definition 3. Let $p \in (0, \infty)$, then an $\mathbb{H}$-holomorphic function f in $B_1(0) \subset \mathbb{H}$ belongs to the *Hardy space* (or $\mathbb{H}$-holomorphic Hardy space) $H^p(B_1(0))$ if the condition

$$\|f\|_{H^p} := \left(\sup_{0<r<1} \int\limits_{|x|=1} |f(rx)|^p |dS^3| \right)^{1/p} < \infty \tag{47.3}$$

is satisfied.

Given a function u, which is defined on $S^3 = \partial B_1(0) = \partial B$. It is assumed that $u \in L^2(S^3)$. Without loss of generality it can be further assumed that u is real-valued (otherwise u will be considered componentwise). Looking now for a function $U \in H^2(B_1(0))$ that satisfies

$$\lim_{r \to 1^-} \mathrm{Sc}(U(r\xi)) = u(\xi), \quad \text{a.e. on } S^3,$$

then, following the idea proposed in [2] and [29], such a function U can be constructed explicitly (not necessarily uniquely) by

$$U(x) = (Tu)(x) = \int_{|\omega|=1} S(x, \omega) u(\omega) do(\omega), \quad |x| < 1,$$

where $S(x,\omega) := P(x,\omega) + Q(x,\omega)$ is called *quaternionic Schwarz kernel*, with the *Poisson kernel*

$$P(x,\omega) = \frac{1}{2\pi^2}\frac{1-|x|^2}{|x-\omega|^4}$$

as well as

$$Q(x,\omega) = \mathrm{Vec}\Big(\int_0^1 t^2(\overline{\partial}P)(tx,\omega)x\,dt\Big) = \Big(\frac{1}{2\pi^2}\int_0^1 \frac{4t^2(1-t^2|x|^2)}{|tx-\omega|^6}dt\Big)\mathrm{Vec}(\overline{\omega}x)$$

$$= \frac{1}{2\pi^2}\left[\frac{(3+|x|^2)(3-\mathrm{Sc}(\overline{\omega}x))-8}{|x-\omega|^4} - \frac{\arctan\dfrac{\sqrt{|x|^2-(\mathrm{Sc}(\overline{\omega}x))^2}}{1-\mathrm{Sc}(\overline{\omega}x)}}{\sqrt{|x|^2-(\mathrm{Sc}(\overline{\omega}x))^2}}\right]$$

$$\times\frac{\mathrm{Vec}(\overline{\omega}x)}{|x|^2-(\mathrm{Sc}(\overline{\omega}x))^2}.$$

The kernel $Q(x,\omega)$ can be seen as a Cauchy-type harmonic conjugate of the Poisson kernel on the unit sphere. Similar to [29], one can prove that T is a bounded operator from $L^2(S)$ to $\mathcal{H}^2(B)$. This approach can be used to solve the problem of harmonic conjugates to given harmonic functions.

Formulae of Plemelj–Sokhotski Type

Analogously to the principle value integral of Cauchy type in the complex plane

$$(S_\Gamma u)(z) := \frac{1}{\pi i}\int_\Gamma \frac{u(t)}{t-z}dt$$

where Γ is a Liapunov curve, u a Hölder continuous $\mathbb{C}$-valued function, one introduces a corresponding integral over Liapunov surfaces in $\mathbb{H}$

$$(S_\Gamma u)(x) := \int_\Gamma E_3(y-x)\nu(y)u(y)d\Gamma_y \quad (x\in\Gamma)$$

where $\Gamma \subset \mathbb{H}$ is the piecewise Liapunov boundary of a domain in $\mathbb{H}$ and ν denotes the outward-pointing normal unit vector. It also exists as principle value integral. The operator S_Γ is called *operator of Cauchy–Bitzadse type*. It should be noted that analogous to the plane case the algebraic identity $S_\Gamma^2 = I$ holds.

Remark 2. In 1953 A.W. Bitzadse generalized the complex Cauchy integral to a principle value integral

$$(S_\Gamma u)(x) := \frac{1}{4\pi} \int_\Gamma \frac{x-y}{|x-y|^3} \nu(y) u(y) d\Gamma_y \quad (x \in \Gamma)$$

over vector functions on a Liapunov surface $\Gamma \subset \mathbb{R}^3$. Using a matrix calculus L.G. Magnaradse proved in 1957 the algebraic property $S_\Gamma^2 = I$. Already in 1955 T.G. Gegelia [10] proved the continuity of the operator S_Γ in $L_p(\Gamma)$, $p > 1$, i.e.

$$\|S_\Gamma u\|_p \leq C \|u\|_p.$$

It was A. McIntosh, who generalized in 1995 this result to Lipschitz surfaces.

What happens when the free variable in Cauchy–Fueter's integral tends to the boundary?

Let $u \in C^{0,\alpha}(\Gamma), 0 < \alpha \leq 1,\ \Gamma$ a piecewise Liapunov surface. Then for any regular point $x_0 \in \Gamma$

$$\lim_{\substack{x \to x_0 \\ x \in G^\pm}} (F_\Gamma u)(x) = \frac{1}{2}(\pm u(x_0) + (S_\Gamma u)(x_0)) \quad (\textit{Plemelj–Sokhotskiformulae})$$

with $G^+ := G$ and $G^- := \mathbb{R}^3 \setminus \overline{G^+}$. The name of this formula goes back to J.W. Sokhotski and J. Plemelj who got corresponding results for smooth curves in the plane. The operators

$$P_\Gamma := \frac{1}{2}(I + S_\Gamma) \quad \text{and} \quad Q_\Gamma := \frac{1}{2}(I - S_\Gamma)$$

are called *Plemelj projections*.

Theorem 6. *Plemelj projections have the following properties:*

(i)

$$P_\Gamma^2 = P_\Gamma,\ Q_\Gamma^2 = Q_\Gamma\ , P_\Gamma Q_\Gamma = Q_\Gamma P_\Gamma = 0,$$

(ii) im P_Γ consists of all functions, which can be $\mathbb{H}$-holomorphically extended into G^+,

(iii) im Q_Γ consists of all functions, which can be $\mathbb{H}$-holomorphically extended into G^- and vanish at infinity.

Remark 3. The operators P_Γ, Q_Γ have continuous extensions onto $L_p(\Gamma)$.

Bergman–Hodge Type Decompositions

To find the orthogonal complement of the square integrable $\mathbb{H}$-holomorphic functions in the quaternionic Hilbert space $L_2(G)$ turns out to be very important for the treatment of boundary value problems. The following statement holds:

Theorem 7. *The sets* $\ker \partial(G) \cap C^{m,\lambda}(\overline{G})$ *and* $\ker \partial(G) \cap W_p^k(G)$ *are closed subspaces in* $C^{m,\lambda}(G)$, $W_p^k(G)$, *respectively, and are called Bergman type spaces.*

In some applications it is useful to have the following generalization of the Bergman–Hodge decomposition.

Corollary 3. *Let A be an $\mathbb{R}$–linear L_2–homeomorphism with $B = A^{-1}$. Introducing in $L_2(G)$ the $\mathbb{R}$-linear scalar product*

$$(u, v)_A := \int_G \overline{Bu} Bv dG \in \mathbb{H},$$

then the decomposition

$$L_2(G) = (A \ker \bar{\partial} \cap L_2)(G) \oplus_A \partial \overset{\circ}{W}{}_2^1 (G)$$

holds.

This theorem is applied to stationary boundary value problems for pseudoparabolic equations, Navier–Stokes equations (NSE) with variable viscosity (also with heat conduction and magnetic dependence (Benard problem)), electro-magnetic equations with variable material coefficients, equations of linear elasticity with variable Young-modulus. Time-dependent boundary value problems of partial differential equations require an operator calculus for $(\partial + i\alpha)$ with $(\alpha \in \mathbb{C})$. A collection of applications can be found in [4, 14–16, 18, 19, 23] and a series of journal papers.

Bergman–Hodge decompositions also exist on lattices in the framework of discrete quaternionic analysis. In this example $\mathbb{R}^3$ has to be considered as isomorphically embedded in $\mathbb{H}$. A corresponding result for the discrete Dirac operator reads as follows. Let $G_h = G \cap \mathbb{R}_h^3$ with $\mathbb{R}_h^3 = \{(ih, jh, kh)^T : h > 0, i, j, k$ integer numbers$\}$. Further, let

$$\langle u, v\rangle_h = \sum_{s \in G_h} \overline{u(s)} v(s) h^3$$

be the corresponding scalar product in $L_{2,h}(G_h)$ and let the discrete Dirac operator $D_h^{\pm}$ be given by

$$(D_h^{\pm} u)(x) = \sum_{i=1}^{3} e_i (D_{i,h} u)(x), \quad (D_{i,h}^{\pm} u)(x) = \frac{\pm(u(V_{i,h}^{\pm} x) - u(x))}{h}, \text{ where } V_{i,h}^{\pm} x = x \pm h e_i.$$

Then

$$L_{2,h}(G_h) = \ker D_h^{\pm}(\mathrm{int} G_h) \oplus D_h^{\mp} \overset{\circ}{W}{}_2^{1,-}(G_h).$$

Notice that $D_h^{\mp} \overset{\circ}{W}{}^{1,-}(G_h)_2 = D_h^{\mp}(u : tr_\Gamma u = 0\}$. On the boundary we have the following decomposition

$$L_{2,h}(\Gamma_h) = \mathrm{im} tr_\Gamma F_{h,\Gamma}^{\pm} \oplus \mathrm{im} tr_\Gamma T_h^{\pm}$$

$F_{h,\Gamma}^{\pm}$, $T_h^{\pm}$ are the *discrete Cauchy–Fueter transform* and *discrete Teodorescu transform*, respectively. This approach was introduced in [12] and applied also in [16]. Further approaches to discrete function theories the reader can find in [6] and [3].

Remark 4. Similarly, a decomposition of for more general hypoelliptic operators [31] can be obtained. Let

$$P(\partial_1, \ldots, \partial_3) = \sum_{j_0,\ldots,j_3=0}^{m_1,\ldots,m_3} a_{j_0\ldots j_3} \partial_1^{j_0} \ldots \partial_3^{j_3}$$

where $a_{j_0\ldots j_3}$ are real constants. The adjoint operator is defined similarly by

$$\tilde{P}(\partial_0, \ldots, \partial_3) = \sum_{j_0,\ldots,j_3=0}^{m_0,\ldots,m_3} (-1)^{j_0+\cdots+j_3} a_{j_0\ldots j_3} \partial_0^{j_0} \ldots \partial_3^{j_3}.$$

Assume that P is *hypoelliptic* and maps $Pu \in C^\infty(G) \to u \in C^\infty(G)$. Then one has the decomposition

$$L_2(G) = \ker P \oplus \tilde{P} \overset{\circ}{W}{}_2^{m_0+\cdots+m_3}(G).$$

Quaternionic Analysis in Fluid Mechanics

Assume for the moment that the domain G is bounded by a piecewise smooth Liapunov surface. During the last years these assumptions could be considerably weakened. For this part T stands for the Teodorescu transform and $\mathbf{Q}$ is the Bergman projection onto the orthogonal complement of the subspace of square integrable $\mathbb{H}$-holomorphic functions. For the problems considered below one can work with the identity mapping instead of the more general mapping A from section "Bergman–Hodge Type Decompositions".

Linear Equations of Stokes' Type

$$-\Delta \mathbf{u} + \frac{1}{\eta}\nabla p = \frac{\rho}{\eta}\mathbf{f} \quad \text{in} \quad G$$

$$\operatorname{div} \mathbf{u} = f_0 \quad \text{in} \quad G$$

$$\mathbf{u} = \mathbf{g} \quad \text{on} \quad \Gamma.$$

Here η is the viscosity and ρ the density of the fluid. We have to look for the velocity u and the hydrostatic pressure p. Between f_0 and $\mathbf{g}$ we need the compatibility condition:

$$\int_G f_0 dx = \int_\Gamma \boldsymbol{\nu} \cdot \mathbf{g} d\Gamma.$$

For $\mathbf{g} = 0$ then the measure of compressibility f_0 has to satisfy the identity

$$\int_G f_0 dx = 0.$$

For all such real functions f_0 the unique solution can be represented (p is unique up to a real additive constant) as follows:

Theorem 8 ([16]). *Let* $f := f_0 + \mathbf{f} \in W_p^k(G)$ $(k \geq 0, 1 < p < \infty)$. *Then we have*

$$\mathbf{u} = \frac{\rho}{\eta} T \text{ Vec } T\mathbf{f} - \frac{\rho}{\eta} T \text{ Vec } F_\Gamma (tr_\Gamma T \text{Vec } F_\Gamma)^{-1} tr_\Gamma T \text{Vec } T\mathbf{f} - T f_0,$$

$$p = \rho \text{Sc} T\mathbf{f} - \rho \text{Sc } F_\Gamma (tr_\Gamma T \text{Vec} F_\Gamma)^{-1} \text{tr}_\Gamma T \text{Vec } T\mathbf{f} + \eta f_0.$$

In that way one separates velocity and pressure.

Problems of Navier–Stokes Type

In the stationary case Navier–Stokes equations are described in the following way:

$$-\Delta \mathbf{u} + \frac{\rho}{\eta}(\mathbf{u} \cdot \nabla)\mathbf{u} + \frac{1}{\eta}\nabla p = \frac{\rho}{\eta}\mathbf{f} \quad \text{in} \quad G, \tag{47.4}$$

$$\operatorname{div} \mathbf{u} = 0 \quad \text{in} \quad G,$$

$$\mathbf{u} = 0 \quad \text{on} \quad \Gamma.$$

We will abbreviate $M(u) := M^*(u) - \mathbf{f}$, where $M^*(u) := (\mathbf{u} \cdot \text{grad})\mathbf{u}$. The main result is now the following:

Theorem 9 ([12, 16]).

1. *Let* $\mathbf{f} \in L_2(G)$, $p \in W_2^1(G)$. *Every solution of (47.4) permits the operator integral representation*

$$\mathbf{u} = -\frac{\rho}{\eta} TQTM(\mathbf{u}) - \frac{1}{\eta} T\mathbf{Q}p \tag{47.5}$$
$$\frac{\rho}{\eta} Sc.TM(\mathbf{u}) - \frac{1}{\eta} Sc\mathbf{Q}p = 0.$$

2. *The system (47.5) has a unique solution* $\{\mathbf{u}, p\} \in \overset{\circ}{W}{}_2^1(G) \cap \ker(\mathrm{div}) \times L_2(G)$, *where* p *is unique up to a real constant, if*

$$(i) \quad \|\mathbf{f}\|_p \leq (18K^2C_1)^{-1}$$
$$\textit{with} \quad K := \frac{\rho}{\eta}\|T\|_{[L_2\cap im\mathbf{Q}, W_2^1]}\|T\|_{[L_p, L_2]}$$
$$(ii) \quad \mathbf{u}_0 \in \overset{\circ}{W}{}_2^1(G) \cap \ker(\mathrm{div})$$
$$\textit{with} \quad \|\mathbf{u}_0\|_{2,1} \leq min\left(V, \frac{1}{4KC_1} + W\right)$$

holds, with $V := 2KC_1)^{-1}$, $W := [(4KC_1)^{-2} - \frac{\rho\|f\|_p}{\eta C_1}]^{\frac{1}{2}}$ *and* $C_1 = 9^{\frac{1}{p}}C$, *where* C *is the embedding constant from* W_2^1 *in* L_2. *The iteration procedure (starting with* $\mathbf{u}_0$*)*

$$\mathbf{u}_n = \frac{\rho}{\eta} TQTM(\mathbf{u}_{n-1}) - \frac{1}{\eta} TQTDp_n$$
$$\frac{\rho}{\eta} Sc\mathbf{Q}TM(\mathbf{u}_{n-1}) = -\frac{1}{\eta} Sc\mathbf{Q}p_n$$
$$(\mathbf{u}_0 \in \overset{\circ}{W}{}_2^1(G) \cap \mathrm{kerdiv})$$

converges in $W_2^1(G) \times L_2(G)$ *to the solution* $(\mathbf{u}, p)$.

Conclusions

Quaternionic analysis as the function theory of a generalized Cauchy–Riemann operator shows a lot of analogies to complex analysis in case of functions depending on one complex variable. As demonstrated here one can obtain all important integral theorems, integral representation formulas, including the Cauchy integral formula and the Borel–Pompeiu formula as well as jump formulas of Plemelj–

Sokhotski type. This is based on an operator calculus built by the identity operator I, the Cauchy–Riemann operator $\bar{\partial}$, and a right inverse T to $\bar{\partial}$. Through the integral formulas it becomes also visible that quaternionic analysis can be seen as a refinement of harmonic analysis due to the factorization $\Delta = \bar{\partial}\partial = \partial\bar{\partial}$. An orthogonal Bergman–Hodge decomposition is a direct consequence and it can be seen that already with these few tools one has a powerful tool for the solution of boundary value problems available, including non-linear problems. Discrete versions of the operator calculus allow to establish efficient numerical methods by using finite difference methods. The presented method gives also possibilities to study qualitative properties of the solutions to boundary value problems as, for instance, existence and uniqueness, stability and well-posedness as well as a convergence analysis for the derived numerical methods. The presented concept can cover more general boundary value problems like parabolic or hyperbolic equations. This requires the generalization to complex quaternions and the corresponding function theory of the Cauchy–Riemann operator.

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