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Additive Maps Preserving the Inner Local Spectral Radius

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Abstract. Let X be a complex Banach space and let $\mathcal{L}(X)$ be the algebra of all bounded linear operators on X. We characterize additive continuous maps from $\mathcal{L}(X)$ onto itself which preserve the inner local spectral radius at a nonzero fixed vector.

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1. Introduction

Throughout this paper, X and Y will denote infinite-dimensional complex Banach spaces and $\mathcal{L}(X)$ and $\mathcal{L}(Y)$ will denote the algebras of all bounded linear operators on X and Y with unit I, respectively. For $T \in \mathcal{L}(X)$ we will denote by $\sigma(T)$, $\sigma_{ap}(T)$, and $\sigma_{su}(T) := \{\lambda \in \mathbb{C} : \lambda - T \text{ is not surjective}\}$ the spectrum, the approximate point spectrum, and the surjectivity spectrum of T; respectively. The local resolvent set of $T \in \mathcal{L}(X)$ at a point $x \in X$, $\rho_T(x)$, is the set of all $\lambda \in \mathbb{C}$ for which there exists an open neighborhood U_λ of λ in $\mathbb{C}$ and an X-valued analytic function on U_λ such that $(\mu - T)f(\mu) = x$ for all $\mu \in U_\lambda$. Its complement denoted by $\sigma_T(x)$ is called the local spectrum of T at x. We denote as usual the spectral radius of T by $r(T) := \max\{|\lambda| : \lambda \in \sigma(T)\}$ which coincides, by Gelfand's formula for the spectral radius, with the limit of the convergent sequence $(\|T^n\|^{\frac{1}{n}})_n$. The lower-boundedness spectral radius $\ell(T)$ and the surjectivity spectral radius $\omega(T)$ of T are given by

$$\ell(T) = \sup\{\varepsilon \geq 0 : \lambda - T \text{ is bounded below for } |\lambda| < \varepsilon\},$$
$$\omega(T) = \sup\{\varepsilon \geq 0 : \lambda - T \text{ is surjective for } |\lambda| < \varepsilon\}.$$

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These quantities are quite useful for the localization of the approximate point (surjectivity) spectrum and the spectrum; see for instance [1] and [9]. In [10], E. Makai and J. Zemánek proved, in fact, that $\ell(T)$ (resp. $\omega(T)$) is nothing but the minimum modulus of $\sigma_{ap}(T)$ (resp. $\sigma_{su}(T)$) that coincides with the limit $\lim_{n\to\infty} m(T^n)^{\frac{1}{n}}$ (resp. $\lim_{n\to\infty} q(T^n)^{\frac{1}{n}}$). Here $m(T) := \inf\{\|Tx\| : x \in X, \|x\| \leq 1\}$ (resp. $q(T) := \sup\{\varepsilon \geq 0; \varepsilon B(0,1) \subseteq T(B(0,1))\}$) is the so-called minimum (resp. surjectivity) modulus of T; where $B(0,1)$ denotes the closed unit ball of X. In the same paper a counter-example was given showing that $\ell(T)$ and $\omega(T)$ are not determined by the spectrum of T. The inner local spectral radius of T at a point $x \in X$, $\iota_T(x)$, is defined by

$$\iota_T(x) := \sup\{\varepsilon \geq 0 : x \in \mathcal{X}_T(\mathbb{C} \setminus D(0,\varepsilon))\},$$

where $D(0,\varepsilon)$ denotes the open disc of radius ε centered at 0 and $\mathcal{X}_T(\mathbb{C} \setminus D(0,\varepsilon))$ is the so-called local spectral subspace of T associated with $\mathbb{C} \setminus D(0,\varepsilon)$, that is, the set of all $x \in X$ for which there is an X-valued analytic function f on $D(0,\varepsilon)$ such that $(\lambda - T)f(\lambda) = x$ for all $\lambda \in D(0,\varepsilon)$. The local spectral radius of T at x is given by

$$r_T(x) := \limsup_{n\to+\infty} \|T^n x\|^{\frac{1}{n}}.$$

The inner local (resp. local) spectral radius of T at x coincides with the minimum (resp. maximum) modulus of $\sigma_T(x)$ provided that T has the single-valued extension property; see [9] and [11]. Recall that T is said to have the single-valued extension property if for every open set U of $\mathbb{C}$, the equation $(T-\lambda)\phi(\lambda) = 0,\ (\lambda \in U)$, has no nontrivial analytic solution on U. For more details and basic facts concerning the spectral quantities $\ell(T)$, $\omega(T)$, and $\iota_T(x)$ we refer the reader to [1, 9, 10], and [11].

We will say that an additive map $\phi : \mathcal{L}(X) \to \mathcal{L}(X)$ compresses the local spectrum at a fixed nonzero vector $e \in X$ if $\sigma_{\phi(T)}(e) \subseteq \sigma_T(e)$ holds for all $T \in \mathcal{L}(X)$ and preserves the local spectrum (resp. local spectral radius) at e if the reverse set-inclusion holds too (resp. $r_{\phi(T)}(e) = r_T(e)$ for all $T \in \mathcal{L}(X)$).

In [7], Bračič and Müller characterized continuous surjective linear maps from $\mathcal{L}(X)$ into itself that preserve the local spectrum and the local spectral radius at a nonzero fixed vector in X. In [4], the authors treated the problem of characterizing locally spectrally bounded linear maps on the algebra $\mathcal{L}(\mathcal{H})$ of all bounded linear operators on a complex Hilbert space $\mathcal{H}$, and they described continuous linear maps from $\mathcal{L}(\mathcal{H})$ onto itself that compress the local spectrum at a fixed nonzero vector in $\mathcal{H}$. The surjective continuous additive mappings ϕ on $\mathcal{L}(X)$ which are local spectrum compressing or local spectral radius preserving at a nonzero vector were characterized in [5].

In this paper, we first collect in the next section some results concerning additive maps from $\mathcal{L}(X)$ onto $\mathcal{L}(Y)$ that preserve the lower-boundedness (surjectivity) of operators in both directions and the ones that preserve the lower-boundedness (surjectivity) spectral radius of operators. This allows us to characterize in the last section additive maps from $\mathcal{L}(X)$ onto $\mathcal{L}(Y)$ that preserve the inner local spectral radius at a fixed nonzero vector. It should be pointed out that our proofs use some arguments which are influenced by ideas from Bračič and Müller [7].

2. Preliminaries

We first fix some notation and terminology. The duality between the Banach spaces X and its dual, X^*, will be denoted by $\langle ., . \rangle$. For $x \in X$ and $f \in X^*$, as usual we denote by $x \otimes f$ the rank at most one operator on X given by $z \mapsto \langle z, f \rangle x$. For $T \in \mathcal{L}(X)$ we will denote by $\ker(T)$, $\mathrm{range}\,(T)$, and T^* the null space, the range, and the adjoint of T; respectively. The operator T is said to be semi-Fredholm if $\mathrm{range}\,(T)$ is closed and $\dim(\ker(T))$ or $\dim(X/\mathrm{range}\,(T))$ is finite, and is said to be semi-invertible if it is left or right invertible. An additive mapping $A : X \to Y$ is called semilinear if $A(\lambda x) = \tau(\lambda)A(x)$ holds for all scalars $\lambda \in \mathbb{C}$ and vectors $x \in X$, where τ is a ring automorphism of $\mathbb{C}$. It is called conjugate linear if $A(\lambda x) = \overline{\lambda}A(x)$ holds for all scalars $\lambda \in \mathbb{C}$ and vectors $x \in X$.

Recall that an additive map $\phi : \mathcal{L}(X) \to \mathcal{L}(Y)$ is called unital if $\phi(I) = I$, and is said to preserve the lower-boundedness of operators in both directions provided that $\phi(T)$ is bounded below if and only if T is. The additive maps preserving the surjectivity in both directions are defined in a similar way.

The following elementary lemmas, inspired by [3], are on the straightforward side. We include them for the sake of completeness.

Lemma 2.1. *Let $\phi : \mathcal{L}(X) \to \mathcal{L}(Y)$ be a surjective additive map. If ϕ either preserves lower-boundedness or surjectivity of operators in both directions, then either*

(i) *there exist invertible bounded both linear or both conjugate linear operators $A : X \to Y$ and $B : Y \to X$ such that $\phi(T) = ATB$ for all $T \in \mathcal{L}(X)$, or*

(ii) *there exist invertible bounded both linear or both conjugate linear operators $A : X^* \to Y$ and $B : Y \to X^*$ such that $\phi(T) = AT^*B$ for all $T \in \mathcal{L}(X)$.*

The last case occurs only if X and Y are reflexive.

Proof. Assume that ϕ preserves the lower-boundedness of operators in both directions. It is easy to check that T is lower bounded if and only if T is not left topological divisor of zero; i.e., there is no sequence $(S_n)_{n\geq 1} \subseteq \mathcal{L}(X)$ satisfying $\|S_n\| = 1$ and $TS_n \to 0$ as $n \to \infty$. So, by using the same approach as in [8, Theorem 3.1] one can see that ϕ is injective and either

(a) there exist semilinear bijective maps $C : X \to Y$ and $D : X^* \to Y^*$ such that $\phi(x \otimes f) = Cx \otimes Df$ for all $x \in X$ and all $f \in X^*$, or

(b) there exist semilinear bijective maps $C : X^* \to Y$ and $D : X \to Y^*$ such that $\phi(x \otimes f) = Cf \otimes Dx$ for all $x \in X$ and all $f \in X^*$.

Now, let us show that $\phi(I)$ is invertible. Note that $\phi(I)$ is injective with closed range, and let us show by way of contradiction that $\phi(I)$ is surjective. So, assume that there exists a nonzero element $y_0 \in Y \setminus \mathrm{range}\,(\phi(I))$. We claim that the operator $\phi(I) - y_0 \otimes g$ is injective with closed range for all $g \in Y^*$. Indeed, the operator $\phi(I)$ is semi-Fredholm since it is bounded below. Thus the operator $\phi(I) - y_0 \otimes g$ is semi-Fredholm for every $g \in Y^*$. On the other hand, $\phi(I) - y_0 \otimes g$ is injective because $\phi(I)$ is injective and $y_0 \notin \mathrm{range}\,\phi(I)$. This yields the claim. So, if the case (a) occurs we can find an element $x_0 \in X$ and a linear functional

$f_0 \in X^*$ such that $Ax_0 = y_0$ and $\langle x_0, f_0 \rangle = 1$. Thus, we have $I - x_0 \otimes f_0$ as well as $\phi(I - x_0 \otimes f_0) = \phi(I) - Ax_0 \otimes Cf_0$ is bounded below; which contradicts $\sigma(x_0 \otimes f_0) = \{0, 1\}$. By similarity, in the case when (b) occurs we get a contradiction, too. Hence $\phi(I)$ is invertible. Set

$$\chi(T) = \phi(I)^{-1}\phi(T), \ \ (T \in \mathcal{L}(X)).$$

The map χ is a unital surjective additive map preserving lower-boundedness of operators in both directions, and so by applying [8, Corollary 3.5] the map ϕ takes one of the desired forms.

The case when ϕ preserves the surjectivity of operators in both directions is treated in [3]; and the proof is therefore complete. □

Let us recall the following useful facts that will be often used in the sequel. For $T \in \mathcal{L}(X)$ it is straightforward that $\ell(T) > 0$ (resp. $\omega(T) > 0$) if and only if T is bounded below (resp. surjective), that is equivalent in the Hilbert space setting that T is left (resp. right) invertible. Notice that $\sigma_{ap}(T) = \sigma_{su}(T^*)$ and $\sigma_{su}(T) = \sigma_{ap}(T^*)$, and so $\ell(T) = \omega(T^*)$ and $\omega(T) = \ell(T^*)$; see [9] and [10].

We will say that an additive map $\phi : \mathcal{L}(X) \to \mathcal{L}(Y)$ preserves the lower-boundedness spectral radius if $\ell(\phi(T)) = \ell(T)$ for all $T \in \mathcal{L}(X)$. The additive maps preserving the surjectivity spectral radius are defined analogously.

Lemma 2.2. *Let $\varphi : \mathcal{L}(X) \to \mathcal{L}(Y)$ be a surjective additive map. If φ either preserves the lower-boundedness radius or surjectivity radius, then there exists a scalar $c \in \mathbb{C}$ of modulus one and either*

(i) *there exists an invertible bounded linear or conjugate linear operator $A : X \to Y$ such that $\varphi(T) = cATA^{-1}$ for all $T \in \mathcal{L}(X)$, or*
(ii) *there exists an invertible bounded linear or conjugate linear operator $A : X^* \to Y$ such that $\varphi(T) = cAT^*A^{-1}$ for all $T \in \mathcal{L}(X)$.*

The last case occurs only if X and Y are reflexive.

Proof. Note that, if ϕ preserves the spectral radius $\ell(.)$ (resp. $\omega(.)$) then ϕ preserves the lower-boundedness (resp. surjectivity) of operators in both directions; and thus by Lemma 2.1 either

(a) there exist invertible bounded both linear or both conjugate linear operators $A : X \to Y$ and $B : Y \to X$ such that $\phi(T) = ATB$ for all $T \in \mathcal{L}(X)$, or
(b) there exist invertible bounded both linear or both conjugate linear operators $A : X^* \to Y$ and $B : Y \to X^*$ such that $\phi(T) = AT^*B$ for all $T \in \mathcal{L}(X)$.

To complete the proof it suffices to show that AB is a multiple of the unit by a unimodular scalar.

Assume that ϕ preserves the lower-boundedness radius. First, we claim that

$$\ell(RQ) = \ell(Q) \tag{2.1}$$

for all $Q \in \mathcal{L}(Y)$, where $R := B^{-1}A^{-1}$. Indeed, if the case (a) occurs we have

$$\ell(R\phi(T)) = \ell(T) = \ell(\phi(T))$$

for all $T \in \mathcal{L}(X)$, and the surjectivity of ϕ yields the claim. If the case (b) occurs we have

$$\ell(R\phi(T)) = \ell(T^*) = \omega(T)$$

for all $T \in \mathcal{L}(X)$. Particulary we have

$$\begin{aligned} T \text{ is surjective } &\Leftrightarrow \phi(T) \text{ bounded below} \\ &\Leftrightarrow T \text{ is bounded below} \end{aligned}$$

for all $T \in \mathcal{L}(X)$. From this we infer that $\sigma_{ap}(T) = \sigma_{su}(T)$, and so $\ell(R\phi(T)) = \ell(T) = \ell(\phi(T))$ for all $T \in \mathcal{L}(X)$. Again the surjectivity of ϕ yields the claim in this case, too. Next, assume by way of contradiction that R and I are linearly independent. So, we can find a nonzero element $y_0 \in Y$ such that y_0 and Ry_0 are linearly independent, and let W be a topological complement of the linear subspace spanned by $\{y_0, Ry_0\}$ in Y. Fix a nonzero complex number α for which $|\alpha| < 1$, and define linearly the operator $Q_0 \in \mathcal{L}(Y)$ by

$$Q_0 y := \begin{cases} \alpha^{-1} R y_0 & \text{if } y = y_0 \\ \alpha y_0 & \text{if } y = Ry_0 \\ y & \text{if } y \in W \end{cases}$$

It easy to check that $\ell(Q_0) = 1$, and that $RQ_0(Ry_0) = \alpha Ry_0$. These show that

$$\ell(RQ_0) \leq |\alpha| < 1 = \ell(Q_0),$$

and lead to a contradiction; see (2.1). Thus AB as well as R is a multiple of the unit with a scalar $c \in \mathbb{C}$, and $| c | = \ell(AB) = \ell(I) = 1$.

By similarity, if ϕ preserves the surjectivity radius we have $\omega(QR) = \omega(Q)$ for all $Q \in \mathcal{L}(Y)$; and so $\ell(R^*Q^*) = \ell(Q^*)$ for all $Q \in \mathcal{L}(Y)$. Thus by what has been shown above, we have R as well as R^* is a multiple of the unit by a scalar of modulus one. The proof is therefore complete. □

In the finite-dimensional case, from the fact that a matrix T in the algebra $M_n(\mathbb{C})$ of all complex $n \times n$ matrices is invertible if and only if it is semi-invertible, one can see that

$$\ell(T) = \omega(T),$$

for all $T \in M_n(\mathbb{C})$.

The following characterizes additive maps from $M_n(\mathbb{C})$ onto itself that preserve the lower-boundedness or surjectivity spectral radius of matrices.

Proposition 2.3. *Let $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a surjective additive map. The following are equivalent:*

(i) $\ell(\phi(T)) = \ell(T)$ *for every* $T \in M_n(\mathbb{C})$.
(ii) $\omega(\phi(T)) = \omega(T)$ *for every* $T \in M_n(\mathbb{C})$.
(iii) *There exist a scalar $c \in \mathbb{C}$ of modulus one and an invertible matrix A in $M_n(\mathbb{C})$ such that either $\phi(T) = cATA^{-1}$, $\phi(T) = cAT^{tr}A^{-1}$, $\phi(T) = cAT^*A^{-1}$, or $\phi(T) = cA(T^{tr})^*A^{-1}$; for every $T \in M_n(\mathbb{C})$. Here T^{tr} denotes the transpose of the matrix T.*

Proof. As the sufficiency condition is obvious, we only need to prove the necessity. So assume that ϕ preserves either the lower boundedness or surjectivity spectral radius of matrices, and note that, in this case, ϕ is a bijective map preserving invertibility in both directions. So, using the same approach as in [2, Theorem 4.1] one can see that ϕ takes one of the desired forms; and the necessity condition is established. □

3. Main result and proof

We will say that an additive map $\phi : \mathcal{L}(X) \to \mathcal{L}(X)$ preserves the inner local spectral radius at a fixed nonzero vector $x \in X$ if $\iota_{\phi(T)}(x) = \iota_T(x)$ for all $T \in \mathcal{L}(X)$.

The following is the main result of this paper. It characterizes additive maps from $\mathcal{L}(X)$ onto itself that preserve the inner local spectral radius at a fixed nonzero vector and extends [6, Theorem 2.1] from linear case to additive case. Its proof use some arguments which are influenced by ideas from Bračič and Müller [7].

Theorem 3.1. *Let e be a fixed nonzero vector in X. An additive continuous map ϕ from $\mathcal{L}(X)$ onto itself preserves the inner local spectral radius at e if and only if there exist a scalar c of modulus one and a linear or conjugate linear bijective bounded operator $A : X \to X$ such that $Ae = e$, and $\phi(T) = cATA^{-1}$ for all $T \in \mathcal{L}(X)$.*

The proof of this theorem uses some auxiliary lemmas. The first is quoted from Bračič and Müller [7, Lemma 2.2].

Lemma 3.2. *Let e be a fixed nonzero vector in X, and let $T \in \mathcal{L}(X)$. If $\lambda \in \sigma_{su}(T)$, then for every $\varepsilon > 0$, there exists $T' \in \mathcal{L}(X)$ such that $\|T - T'\| < \varepsilon$ and $\lambda \in \sigma_{T'}(e)$.*

Proof. See [7, Lemma 2.2]. □

Lemma 3.3. *Let e be a fixed nonzero vector in X. For a linear or conjugate linear bijective bounded operator $A : X \to X$, the map $\phi : T \in \mathcal{L}(X) \mapsto ATA^{-1} \in \mathcal{L}(X)$ preserves the inner local spectrum at e if and only if $Ae = \lambda e$ for some $\lambda \in \mathbb{C}$.*

Proof. We shall only deal with the case when A is conjugate linear, because the linear case follows analogously. First, we claim that for every $T \in \mathcal{L}(X)$ and $\varepsilon > 0$ we have $Ae \in \mathcal{X}_{ATA^{-1}}(\mathbb{C} \setminus D(0, \varepsilon))$ whenever $e \in \mathcal{X}_T(\mathbb{C} \setminus D(0, \varepsilon))$. Indeed, assume that $e \in \mathcal{X}_T(\mathbb{C} \setminus D(0, \varepsilon))$ and let f be a X-valued analytic function on $D(0, \varepsilon)$ such that $(\mu - T)f(\mu) = x$ for all $\mu \in D(0, \varepsilon)$. We have

$$(\mu^\eta - ATA^{-1})Af(\mu) = Ae$$

for all $\mu \in D(0, \varepsilon)$; where $\eta : \mathbb{C} \to \mathbb{C}$ is the complex conjugation. Set

$$\widetilde{f}(\mu^\eta) := Af(\mu), \quad (\mu \in D(0, \varepsilon)),$$

and note that the map $\widetilde{f}$ is an analytic function on $D(0, \varepsilon)^\eta = D(0, \varepsilon)$ since

$$\lim_{h \to 0} \frac{\widetilde{f}(\mu^\eta + h) - \widetilde{f}(\mu^\eta)}{h} = \lim_{h \to 0} A\left(\frac{f(\mu + h^\eta) - f(\mu)}{h^\eta}\right) = Af'(\mu)$$

for all $\mu \in D(0,\varepsilon)$, where $f'(\mu)$ is the derivative of f at μ. This shows that $Ae \in \mathcal{X}_T(\mathbb{C} \setminus D(0,\varepsilon))$ and yields the claim. When Ae and e are linearly dependent, the reverse implication can be obtained by similarity, and thus, we in fact have $e \in \mathcal{X}_T(\mathbb{C}\setminus D(0,\varepsilon))$ if and only if $Ae \in \mathcal{X}_{ATA^{-1}}(\mathbb{C}\setminus D(0,\varepsilon))$ for all $\varepsilon > 0$; which show that $\iota_{ATA^{-1}}(e) = \iota_T(e)$ and ϕ preserves the inner local spectrum at e in this case.

Conversely, assume that ϕ preserves the inner local spectrum at e, but Ae and e are linearly independent. Let $f \in X^*$ be a linear functional such that $\langle e, f\rangle = 1$ and $\langle A^{-1}e, f\rangle = 0$. Set $T =: e \otimes f$ and note that $\iota_{ATA^{-1}}(e) = 0$ and $\iota_T(e) = 1$; which leads to a contradiction and completes the proof. □

We have now collected all the necessary ingredients and are therefore in a position to prove our main result.

Proof of Theorem 3.1. As the sufficiency condition is a consequence of the above Lemma, we only need to prove the necessity. So, assume that ϕ preserves the inner local spectral radius at e. We claim that ϕ preserves the spectral radius function $\omega(.)$. For this, let $T \in \mathcal{L}(X)$ and let $\lambda \in \sigma_{su}(\phi(T))$ satisfy $|\lambda| = \omega(\phi(T))$. The Lemma 3.2 ensures that for each integer $n \geq 1$ there exists an operator T'_n in $\mathcal{L}(X)$ such that $\|T'_n - \phi(T)\| < n^{-1}$ and $\lambda \in \sigma_{T'_n}(e)$. Since ϕ is continuous and surjective, by the Banach open mapping theorem there exists $\eta > 0$ such that $\eta B(0,1) \subseteq \phi(B(0,1))$, where $B(0,1)$ denotes the open unit ball of $\mathcal{L}(X)$. Therefore, for each n there exists $T_n \in \mathcal{L}(X)$ such that $\phi(T_n) = T'_n$ and $\|T_n - T\| \leq \eta^{-1}\|T'_n - \phi(T))\| \leq \eta^{-1}n^{-1}$. Thus $T_n \to T$ and $\lambda \in \sigma_{\phi(T_n)}(e)$ for all $n \geq 1$. On the other hand, again by the Banach open mapping theorem and by applying [12, Propositions 6.9 and 9.9] to the set of all surjective operators on X one can see that the surjectivity spectrum is an upper semi-continuity function. Thus, the spectral function $\omega(.)$ is upper semi-continuous and so

$$\omega(T) \leq \liminf_{n\to\infty} \omega(T_n) \leq \liminf_{n\to\infty} \iota_{T_n}(e) = \liminf_{n\to\infty} \iota_{\phi(T_n)}(e) \leq |\lambda| = \omega(\phi(T)).$$

To establish the reverse inequality, pick an arbitrary $T \in \mathcal{L}(X)$ and $\lambda \in \sigma_{su}(T)$ such that $|\lambda| = \omega(T)$. By Lemma 3.2 there exists a sequence of operators (T_n) in $\mathcal{L}(X)$ converging to T such that $\lambda \in \sigma_{T_n}(e)$ for all n, and consequently we have

$$\omega(\phi(T)) \leq \liminf_{n\to\infty} \omega(\phi(T_n)) \leq \liminf_{n\to\infty} \iota_{\phi(T_n)}(e) = \liminf_{n\to\infty} \iota_{T_n}(e) \leq |\lambda| = \omega(T).$$

From this, we infer that $\omega(\phi(T)) = \omega(T)$ for all $T \in \mathcal{L}(X)$, and so by Theorem 2.2, there exists a scalar c of modulus one and either there exists a linear or conjugate linear invertible bounded operator $A : X \to X$ such that $\phi(T) = cATA^{-1}$ for all $T \in \mathcal{L}(X)$, or there exists a linear or conjugate linear invertible bounded operator $A : X^* \to X$ such that $\phi(T) = cAT^*A^{-1}$ for all $T \in \mathcal{L}(X)$. By the same argument given in the end of the proof of Lemma 3.3, one can see that when A is defined from X^* into X we can find an operator $T \in \mathcal{L}(X)$ such that $\iota_T(e) = 1$ and $\iota_{AT^*A^{-1}}(e) = 0$; which shows that the second form is excluded, and consequently ϕ takes only the first one with $Ae = \lambda e$ for some nonzero $\lambda \in \mathbb{C}$. Dividing A by λ or its complex conjugate $\overline{\lambda}$ if necessary, we may assume that $Ae = e$, and thus the necessity condition is established. □

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