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Non-negative Self-adjoint Extensions in Rigged Hilbert Space

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Abstract. We study non-negative self-adjoint extensions of a non densely defined non-negative symmetric operator $\dot{A}$ with the exit in the rigged Hilbert space constructed by means of the adjoint operator $\dot{A}^*$ (bi-extensions). Criteria of existence and descriptions of such extensions and associated closed forms are obtained. Moreover, we introduce the concept of an extremal non-negative bi-extension and provide its complete description. After that we state and prove the existence and uniqueness results for extremal non-negative bi-extensions in terms of the Kreĭn–von Neumann and Friedrichs extensions of a given non-negative symmetric operator. Further, the connections between positive boundary triplets and non-negative self-adjoint bi-extensions are presented.

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1. Introduction

In order to describe the main ideas and results of the current paper, we first recall the notion of the rigged Hilbert spaces. A triplet $\mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_-$ is a rigged Hilbert space constructed upon a symmetric operator $\dot{A}$ in a Hilbert space $\mathcal{H}$ if $\mathcal{H}_+ = \mathrm{Dom}(\dot{A}^*)$ with an inner product defined by

$$(f,g)_+ = (f,g) + (\dot{A}^* f, \dot{A}^* g), \quad f,g \in \mathrm{Dom}(A^*). \tag{1.1}$$

and $\mathcal{H}_-$ is the space of all anti-linear functional on $\mathcal{H}_+$ that are continuous w.r.t. $\|\cdot\|_+$. An extension theory of symmetric operators in rigged Hilbert spaces was thoroughly covered in [7]. One of the objects of this theory is a self-adjoint bi-extension $\mathbb{A}$ of a symmetric operator $\dot{A}$ whose definition is given below in Preliminaries section. Throughout this entire article, by a non-negative operator in a rigged Hilbert space we understand an operator $\mathbb{T}$ such that $(\mathbb{T}f, f) \geq 0$ for all $f \in \mathrm{Dom}(\mathbb{T})$. In this paper we put our main focus on non-negative bi-extensions of a non-negative symmetric operator. The theory of extensions of non-negative symmetric operators originates in the works of von Neumann, Friedrichs, and Kreĭn (see survey [12]). That is why most of the main results of the paper are given in terms of the Kreĭn–von Neumann and Friedrichs extensions of a given non-negative symmetric operator that are described in details in Section 3. The existence conditions for non-negative bi-extensions are presented in Section 4 and rely on the concepts if disjointness and transversality of self-adjoint extensions that were introduced in Preliminaries. Here we also give a descriptions of the non-negative self-adjoint bi-extensions and associated closed quadratic forms. Section 5 is solely dedicated to extremal self-adjoint bi-extensions and contains existence and uniqueness results. The connections between non-negative self-adjoint bi-extensions and boundary triplets is established in Section 6.

The results of the current paper complement and enhance the classical results of the theory of extensions of non-negative symmetric operators as well as some new developments of this theory in rigged Hilbert spaces discussed in [7], [8]. Applications of these results may be used in solving realization problems for Stieltjes and inverse Stieltjes functions in infinite-dimensional Hilbert spaces similarly to finite-dimensional cases treated in [13] and [14].

2. Preliminaries

For a pair of Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$ we denote by $[\mathcal{H}_1, \mathcal{H}_2]$ the set of all bounded linear operators from $\mathcal{H}_1$ to $\mathcal{H}_2$. Let $\dot{A}$ be a closed, densely defined, symmetric operator in a Hilbert space $\mathcal{H}$ with inner product (f, g), $f, g \in \mathcal{H}$.

Consider the rigged Hilbert space (see [15], [31]) $\mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_-$, where $\mathcal{H}_+ = \mathrm{Dom}(\dot{A}^*)$ and $(f, g)_+$ is defined by (1.1). Note that by the second representation theorem [20] we have

$$\mathrm{Dom}(I + \dot{A}\dot{A}^*)^{1/2} = \mathcal{H}_+, \ \mathrm{Ran}(I + \dot{A}\dot{A}^*)^{1/2} = \mathcal{H},$$

and

$$(f, g)_+ = ((I + \dot{A}\dot{A}^*)^{1/2} f, (I + \dot{A}\dot{A}^*)^{1/2} g), \quad f, g \in \mathcal{H}.$$

The Hilbert space $\mathcal{H}_+$ admits the following $(+)$-orthogonal decomposition

$$\mathcal{H}_+ = \mathrm{Dom}(\dot{A}) \oplus \mathfrak{N}_{-i} \oplus \mathfrak{N}_i,$$

where $\mathfrak{N}_\lambda := \ker(\dot{A}^* - \lambda I)$, $\mathrm{Im}\,\lambda \neq 0$ is the defect subspace of $\dot{A}$. Denote

$$\mathfrak{M} = \mathfrak{N}_{-i} \oplus \mathfrak{N}_i$$

and let

$$P^+_{\mathrm{Dom}(\dot{A})}, \ P^+_{\mathfrak{N}_{-i}}, \ P^+_{\mathfrak{N}_i}, \ P^+_{\mathfrak{M}}$$

be $(+)$-orthogonal projections in $\mathcal{H}_+$ onto $\mathrm{Dom}(\dot{A})$, $\mathfrak{N}_{-i}$, $\mathfrak{N}_i$, and $\mathfrak{M}$, respectively.

Recall that $\mathcal{H}_-$ can be identified with the space of all anti-linear functional on $\mathcal{H}_+$ and continuous w.r.t. $\|\cdot\|_+$. Let $\mathcal{R}$ be the *Riesz–Berezansky operator* (see [7]) which maps $\mathcal{H}_-$ onto $\mathcal{H}_+$ such that $(f,g) = (f,\mathcal{R}g)_+$ and $\|\mathcal{R}g\|_+ = \|g\|_-$ for all $f \in \mathcal{H}_+$, $g \in \mathcal{H}_-$. Clearly

$$\mathcal{R}\upharpoonright\mathcal{H} = (I + \dot{A}\dot{A}^*)^{-1}.$$

Definition 2.1. Let $\mathbb{A}$ be a linear operator with $\mathrm{Dom}(\mathbb{A})$ dense in $\mathcal{H}_+$ and with values in $\mathcal{H}_-$. Then the adjoint operator $\mathbb{A}^*$ is defined as follows:

$$\begin{aligned}\mathrm{Dom}(\mathbb{A}^*) &= \{u \in \mathcal{H}_+ : \exists\, \psi \in \mathcal{H}_- \ \mid\ (u,\mathbb{A}f) = (\psi, f) \text{ for all } f \in \mathrm{Dom}(\mathbb{A})\}\,,\\ \mathbb{A}^*u &= \psi.\end{aligned}$$

It is easy to see $\mathcal{R}\mathbb{A}^* : \mathcal{H}_+ \supseteq \mathrm{Dom}(\mathbb{A}^*) \to \mathcal{H}_+$ is the $(+)$-adjoint operator to $\mathcal{R}\mathbb{A}$ acting in $\mathcal{H}_+$.

Definition 2.2. An operator $\mathbb{A} : \mathcal{H}_+ \supset \mathrm{Dom}(\mathbb{A}) \to \mathcal{H}_-$ is called a *generalized self-adjoint* if $\mathrm{Dom}(\mathbb{A})$ is dense in $\mathcal{H}_+$ and $\mathbb{A}^* = \mathbb{A}$.

Definition 2.3. A generalized self-adjoint operator $\mathcal{H}_+ \supset \mathrm{Dom}(\mathbb{A}) \to \mathcal{H}_-$ is called *self-adjoint bi-extension* of a symmetric operator $\dot{A}$ if $\mathbb{A} \supset \dot{A}$.

The formula (see [9], [7])

$$\mathbb{A} = \dot{A}^* + \mathcal{R}^{-1}\left(\mathcal{S} - \frac{i}{2}P^+_{\mathfrak{N}_i} + \frac{i}{2}P^+_{\mathfrak{N}_{-i}}\right)P^+_{\mathfrak{M}} = \dot{A}^* + \mathcal{R}^{-1}\left(\mathcal{S} - \frac{1}{2}\dot{A}^*\right)P^+_{\mathfrak{M}} \tag{2.1}$$

establishes a one-to-one correspondence between the set of all self-adjoint bi-extensions of $\dot{A}$ and the set of all $(+)$-self-adjoint operators $\mathcal{S}$ in $\mathfrak{M}$.

Let $\mathbb{A}$ be a self-adjoint bi-extension of $\dot{A}$ and let the operator $\widehat{A}$ in $\mathcal{H}$ be defined as follows:

$$\mathrm{Dom}(\widehat{A}) = \{f \in \mathcal{H}_+ : \widehat{A}f \in \mathcal{H}\}, \quad \widehat{A} = \mathbb{A}\upharpoonright\mathrm{Dom}(\widehat{A}).$$

The operator $\widehat{A}$ is called a *quasi-kernel* of a self-adjoint bi-extension $\mathbb{A}$ (see [31]). We say that a self-adjoint bi-extension $\mathbb{A}$ of $\dot{A}$ is *twice-self-adjoint* or *t-self-adjoint* (see [7]) if its quasi-kernel $\widehat{A}$ is a self-adjoint operator in $\mathcal{H}$.

For the existence, description, and analog of von Neumann's formulas for bounded self-adjoint bi-extensions and $(*)$-extensions see [7] and references therein. In what follows we suppose that $\dot{A}$ has equal deficiency indices. Recall that two self-adjoint extensions A_1 and A_0 of $\dot{A}$ are called *disjoint* if

$$\mathrm{Dom}(A_1) \cap \mathrm{Dom}(A_0) = \mathrm{Dom}(\dot{A}) \tag{2.2}$$

and *transversal* if

$$\mathrm{Dom}(A_1) + \mathrm{Dom}(A_0) = \mathrm{Dom}(\dot{A}^*).$$

Note that it immediately follows from von Neumann formulas that two transversal self-adjoint extensions are automatically disjoint.

The following statements for two self-adjoint extensions A_1 and A_0 of $\dot A$ are evident:

$$A_1, A_0 \text{ are disjoint } \iff \overline{\mathrm{Ran}}\left((A_1-\lambda I)^{-1}-(A_0-\lambda I)^{-1}\right)=\mathfrak{N}_\lambda,$$
$$A_1, A_0 \text{ are transversal } \iff \mathrm{Ran}\left((A_1-\lambda I)^{-1}-(A_0-\lambda I)^{-1}\right)=\mathfrak{N}_\lambda$$

for at least one $\lambda\in\rho(A_1)\cap\rho(A_0)$.

Thus, if the deficiency numbers of $\dot A$ are finite (and equal), then two self-adjoint extensions of $\dot A$ are transversal if and only they are disjoint.

Let $\dot A$ be a closed densely defined symmetric operator and let A_1 be its self-adjoint extension. It has been shown in [2], [9] that any self-adjoint bi-extension $\mathbb{A}$ of $\dot A$ such that $\mathbb{A}\supset A_1$ is *generated* by a disjoint to A_1 self-adjoint extension A_0 of $\dot A$ via the formulas

$$\mathrm{Dom}(\mathbb{A})=\mathrm{Dom}(A_1)+\mathrm{Dom}(A_0),$$
$$\mathbb{A}f=\dot A^*f-\mathcal{R}^{-1}\dot A^*\mathcal{P}_Gf,\quad f\in\mathrm{Dom}(\mathbb{A}),$$

where $\mathcal{P}_G$ is a skew projection operator in $\mathrm{Dom}(\mathbb{A})$ onto G parallel to $\mathrm{Dom}(A_1)$ and G is defined from the $(+)$-orthogonal decomposition

$$\mathrm{Dom}(A_0)=\mathrm{Dom}(\dot A)\oplus G. \tag{2.3}$$

The operator $\mathcal{S}$ corresponding to $\mathbb{A}$ in (2.1) is of the form

$$\begin{aligned}\mathcal{S}f&=\frac{1}{2}\dot A^*f,\qquad f\in\mathrm{Dom}(A_1)\ominus\mathrm{Dom}(\dot A),\\ \mathcal{S}g&=-\frac{1}{2}\dot A^*g,\quad g\in\mathrm{Dom}(A_0)\ominus\mathrm{Dom}(\dot A).\end{aligned}\tag{2.4}$$

In particular,

$$\mathbb{A}g=(\dot A^*-\mathcal{R}^{-1}\dot A^*P^+_{\mathfrak{M}})g,\quad g\in\mathrm{Dom}(A_0).$$

The following formula immediately follows from (2.3)

$$(\mathbb{A}f,f)=(A_1f_1,f_1)+(A_0f_0,f_0)+2\mathrm{Re}\,(A_1f_1,f_0), \tag{2.5}$$

where $f=f_1+f_0$, $f_l\in\mathrm{Dom}(A_l)$, $(l=0,1)$.

Let $\mathbb{A}$ be a self-adjoint bi-extension of $\dot A$. We define a *dual* extension $\mathbb{A}'$ on $\mathrm{Dom}(\mathbb{A})$ by the formula

$$(\mathbb{A}'f,g)=(\dot A^*f,g)+(f,\dot A^*g)-(\mathbb{A}f,g),\quad f,g\in\mathrm{Dom}(\mathbb{A}). \tag{2.6}$$

We note that $\dot A^*\in[\mathcal{H}_+,\mathcal{H}]\subset[\mathcal{H}_+,\mathcal{H}_-]$ and the *generalized adjoint* of $\dot A^*$ takes the form [7]

$$\left(\dot A^*\right)^*=\dot A^*-\mathcal{R}^{-1}\dot A^*P^+_{\mathfrak{M}}. \tag{2.7}$$

It follows from (2.1) that if

$$\mathbb{A}=\dot A^*+\mathcal{R}^{-1}\left(\mathcal{S}-\frac{i}{2}P^+_{\mathfrak{N}_i}+\frac{i}{2}P^+_{\mathfrak{N}_{-i}}\right)P^+_{\mathfrak{M}}$$

is a self-adjoint bi-extension of $\dot{A}$, then $\mathbb{A}'$ is of the form

$$\mathbb{A}' = \dot{A}^* + \mathcal{R}^{-1}\left(-\mathcal{S} - \frac{i}{2}P^+_{\mathfrak{N}_i} + \frac{i}{2}P^+_{\mathfrak{N}_{-i}}\right)P^+_{\mathfrak{M}}.$$

So, if $\mathbb{A}$ is a self-adjoint bi-extension of $\dot{A}$, then $\mathbb{A}'$ is a self-adjoin bi-extension of $\dot{A}$ as well. It was also shown in [2] that if $\mathbb{A}$ is a t-self-adjoint of $\dot{A}$, then $\mathbb{A}'$ is also a t-self-adjoint bi-extension of $\dot{A}$. Moreover, if $\hat{A}$ is a quasi-kernel of $\mathbb{A}$ and $\mathbb{A}$ is generated by a disjoint to $\hat{A}$ self-adjoint extension A, then the quasi-kernel of $\mathbb{A}'$ coincides with A and $\mathbb{A}'$ is generated by $\hat{A}$. Clearly, $(\mathbb{A}')' = \mathbb{A}$.

Notice that from (2.6) and the inequality

$$2|(\dot{A}^* f, f)| \le 2||f||\,||\dot{A}^* f|| \le ||f||^2 + ||\dot{A}^* f||^2 = ||f||^2_+,$$

we get

$$-||f||^2_+ \le (\mathbb{A}f, f) + (\mathbb{A}'f, f) \le ||f||^2_+.$$

3. The Friedrichs and Kreĭn–von Neumann extensions

Let $\tau[\cdot,\cdot]$ be a sesquilinear form in a Hilbert space $\mathcal{H}$ defined on a linear manifold $\mathrm{Dom}(\tau)$. The form τ is called symmetric if $\tau[u,v] = \overline{\tau[v,u]}$ for all $u, v \in \mathrm{Dom}(\tau)$ and non-negative if $\tau[u] := \tau[u,u] \ge 0$ for all $u \in \mathrm{Dom}(\tau)$.

A sequence $\{u_n\}$ is called τ-*converging* to the vector $u \in \mathcal{H}$ [20] if

$$\lim_{n\to\infty} u_n = u \quad \text{and} \quad \lim_{n,m\to\infty} \tau[u_n - u_m] = 0.$$

The form τ is called *closed* if for every sequence $\{u_n\}$ τ- converging to a vector u it follows that $u \in \mathrm{Dom}(\tau)$ and $\lim\limits_{n\to\infty} \tau[u - u_n] = 0$. The form τ is *closable* [20], i.e., there exists a minimal closed extension (the closure) of τ. We recall that a symmetric operator $\dot{B}$ is called *non-negative* if

$$(\dot{B}f, f) \ge 0, \quad \forall f \in \mathrm{Dom}(\dot{B}).$$

If τ is a closed, densely defined non-negative form, then according to First Representation Theorem [23], [20] there exists a unique self-adjoint non-negative operator T in $\mathfrak{H}$, associated with τ, i.e.,

$$(Tu, v) = \tau[u,v] \quad \text{for all} \quad u \in \mathrm{Dom}(T) \quad \text{and} \quad \text{for all} \quad v \in \mathrm{Dom}(\tau).$$

According to the Second Representation Theorem [23], [20] the identities hold:

$$\mathrm{Dom}(\tau) = \mathrm{Dom}(T^{1/2}), \quad \tau[u,v] = (T^{1/2}u, T^{1/2}v).$$

Let $\dot{B}$ be a non-negative symmetric operator in a Hilbert space $\mathcal{H}$. It is known [20] that the non-negative sesquilinear form $\tau_{\dot{B}}[f,g] = (\dot{B}f, g)$, $\mathrm{Dom}(\tau) = \mathrm{Dom}(\dot{B})$, is closable. Following the M. Kreĭn notations we denote by $\dot{B}[\cdot,\cdot]$ the closure of $\tau_{\dot{B}}$ and by $\mathcal{D}[\dot{B}]$ its domain. By definition $\dot{B}[u] = \dot{B}[u,u]$ for all $u \in \mathcal{D}[\dot{B}]$. Because $\dot{B}[u,v]$ is closed, it possesses the property: if

$$\lim_{n\to\infty} u_n = u \quad \text{and} \quad \lim_{n,m\to\infty} \dot{B}[u_n - u_m] = 0,$$

then $\lim_{n\to\infty} \dot B[u-u_n]=0$. For a densely defined $\dot B$, the *Friedrichs extension* B_F of $\dot B$ is defined as a non-negative self-adjoint operator associated with the form $\dot B[\cdot,\cdot]$ by the First Representation Theorem. If $\dot B$ is densely defined then, clearly,

$$\mathrm{Dom}(B_F)=\mathcal{D}[\dot B]\cap \mathrm{Dom}(\dot B^*),\quad B_F=\dot B^*\upharpoonright \mathrm{Dom}(B_F).$$

The Friedrichs extension B_F is a unique non-negative self-adjoint extension having the domain in $\mathcal{D}[\dot B]$. Notice that by the Second Representation Theorem [20] one has

$$\mathcal{D}[\dot B]=\mathcal{D}[B_F]=\mathrm{Dom}(B_F^{1/2}),\quad \dot B[u,v]=(B_F^{1/2}u,B_F^{1/2}v),\quad u,v\in\mathcal{D}[\dot B].$$

If $\dot B$ is non-densely defined, then its Friedrichs extension B_F is a non-negative linear relation of the form (see [28])

$$B_F=\left\{\left\langle x,(\dot B_0)_F x\right\rangle,\ x\in \mathrm{Dom}((\dot B_0)_F)\right\}\oplus\langle 0,\mathfrak{B}\rangle,$$

where $(B_0)_F$ is the Friedrichs extension of the operator $\dot B_0:=P_{\overline{\mathrm{Dom}}(\dot B)}\dot B$ in the subspace $\overline{\mathrm{Dom}}(\dot B)$ and $\mathfrak{B}=\mathcal{H}\ominus\mathrm{Dom}(\dot B)$.

The Kreĭn–von Neumann extension is defined as follows [1], [16]:

$$\dot B_K=((\dot B^{-1})_F)^{-1},$$

where $\dot B^{-1}$ is the linear relation inverse to the graph of $\dot B$.

Theorem 3.1 ([1]). *The following relations describing $\mathcal{D}[B_K]$ and $B_K[u]$ hold:*

$$\begin{aligned}\mathcal{D}[B_K]&=\left\{u\in\mathcal{H}:\sup_{f\in\mathrm{Dom}(\dot B)}\frac{|(\dot Bf,u)|^2}{(\dot Bf,f)}<\infty\right\},\\ B_K[u]&=\sup_{f\in\mathrm{Dom}(\dot B)}\frac{|(\dot Bf,u)|^2}{(\dot Bf,f)},\quad u\in\mathcal{D}[B_K].\end{aligned}\tag{3.1}$$

We note the equalities for an arbitrary non-negative self-adjoint operator B in a Hilbert space $\mathcal{H}$:

$$\begin{aligned}\mathrm{Ran}(B^{1/2})&=\left\{g\in\mathcal{H}:\sup_{f\in\mathrm{Dom}(B)}\frac{|(f,g)|^2}{(Bf,f)}<\infty\right\},\\ \|B^{[-1/2]}g\|^2&=\sup_{f\in\mathrm{Dom}(B)}\frac{|(f,g)|^2}{(Bf,f)},\quad g\in\mathrm{Ran}(B^{1/2}),\end{aligned}$$

where $B^{[-1]}$ is the Moore–Penrose inverse. The Kreĭn–von Neumann extension of a non-densely defined non-negative operator $\dot B$ is an operator (not just a linear relation) if and only if the domain $\mathcal{D}[B_K]$ is dense in $\mathfrak{H}$. According to [1] a non-negative operator $\dot B$ is called *positively closable* if from $\lim_{n\to\infty}\dot B\varphi_n=g$ and $\lim_{n\to\infty}(\dot B\varphi_n,\varphi_n)=0$ follows $g=0$ ($\{\varphi_n\}\subset\mathrm{Dom}(\dot B)$). Notice that a densely defined $\dot B$ is positively closable. A theorem of Ando and Nishio [1] states that $\dot B$

admits non-negative self-adjoint extensions, which are operators, if and only if $\dot{B}$ is positively closable.

A non-negative self-adjoint extension $\widetilde{B}$ of $\dot{B}$ is called *extremal* [3], [5], [6] if the relation

$$\inf\left\{\left(\widetilde{B}(u-\varphi), u-\varphi\right) : \varphi \in \mathrm{Dom}(\dot{B})\right\} = 0$$

holds for every $u \in \mathrm{Dom}(\widetilde{B})$. A characterization of the Kreĭn–von Neumann extension B_K is obtained in [5] and [6]: *the Kreĭn–von Neumann extension B_K is the unique extremal non-negative self-adjoint extension of $\dot{B}$ having maximal domain of its closed associated sesquilinear form.*

Theorem 3.2. *Let $\widetilde{B}$ be a non-negative self-adjoint extension of $\dot{B}$. Then*

$$B_K \le \widetilde{B} \le B_F \tag{3.2}$$

in the sense of quadratic forms. More precisely

$$\begin{aligned} &\mathcal{D}[\dot{B}] \subseteq \mathcal{D}[\widetilde{B}] \subseteq \mathcal{D}[B_K], \\ &\widetilde{B}[u] \ge B_K[u] \qquad \textit{for all} \quad u \in \mathcal{D}[\widetilde{B}], \\ &\widetilde{B}[v] = \dot{B}[v] \qquad \textit{for all} \quad v \in \mathcal{D}[\dot{B}]. \end{aligned}$$

Besides,

$$\mathcal{D}[\widetilde{B}] = \mathcal{D}[\dot{B}]\dot{+}(\mathcal{D}[\widetilde{B}] \cap \mathcal{N}_z), \tag{3.3}$$

where $\mathcal{N}_z$ is the defect subspace of $\dot{B}$, $z \in \mathbb{C} \setminus [0, +\infty)$.

For a densely defined non-negative $\dot{B}$ inequalities (3.2) in the equivalent form

$$(B_F + I)^{-1} \le (\widetilde{B} + I)^{-1} \le (B_K + I)^{-1}$$

and equality (3.3) for $z < 0$ were established by M. Kreĭn [23]. For a sectorial operator $\dot{B}$ with vertex at zero and for sectorial linear relations all statements of Theorem 3.2 can be found in [5] and [6].

The next theorem gives a descriptions of all closed forms associated with non-negative self-adjoint extensions of $\dot{B}$.

Theorem 3.3 ([5]). *If $\widetilde{B}$ is a non-negative self-adjoint extension of a non-negative symmetric operator $\dot{B}$, then the form*

$$(\widetilde{B}u, v) - B_K[u, v], \quad u, v \in \mathrm{Dom}(\widetilde{B})$$

is non-negative and closable in the Hilbert space $\mathcal{D}[B_K]$. Moreover, the formulas

$$\begin{aligned} &\mathcal{D}[\widetilde{B}] = \mathcal{D}[\tau], \\ &\widetilde{B}[u, v] = B_K[u, v] + \tau[u, v], \ u, v \in \mathcal{D}[\widetilde{B}] \end{aligned}$$

give a one-to-one correspondence between all closed forms $\widetilde{S}[\cdot,\cdot]$ associated with non-negative self-adjoint extensions $\widetilde{B}$ of $\dot{B}$ and all non-negative forms $\tau[\cdot,\cdot]$ closed in the Hilbert space $\mathcal{D}[B_K]$ and such that $\tau[\varphi] = 0$ for all $\varphi \in \mathcal{D}[\dot{B}]$.

In addition, the closed forms associated with extremal extensions are closed restrictions of the form $B_K[\cdot,\cdot]$ on the linear manifolds $\mathcal{M}$ such that

$$\mathcal{D}[\dot{B}] \subseteq \mathcal{M} \subseteq \mathcal{D}[B_K].$$

The next theorem can be found in [29], [30], [6], [19].

Theorem 3.4. *Let $\dot{B}$ be a bounded non-densely defined non-negative symmetric operator in a Hilbert space $\mathcal{H}$, $\mathrm{Dom}(\dot{B}) = \mathcal{H}_0$. Let $\dot{B}^* \in [\mathcal{H}, \mathcal{H}_0]$ be the adjoint of $\dot{B}$. Put $\dot{B}_0 = P_{\mathcal{H}_0}\dot{B}$, $\mathcal{N} = \mathcal{H} \ominus \mathcal{H}_0$, where $P_{\mathcal{H}_0}$ is an orthogonal projection in $\mathcal{H}$ onto $\mathcal{H}_0$. Then the following statements are equivalent*

(i) *$\dot{B}$ admits bounded non-negative self-adjoint extensions in $\mathcal{H}$;*

(ii) $\sup\limits_{f\in\mathcal{H}_0} \dfrac{||\dot{B}f||^2}{(\dot{B}f,f)} < \infty$;

(iii) $\dot{B}^*\mathcal{N} \subseteq \mathrm{Ran}(\dot{B}_0^{1/2})$.

Let $\dot{B}$ be a non-negative closed symmetric operator. Consider the symmetric contractions

$$\dot{S} = (I - \dot{B})(I + \dot{B})^{-1},$$

defined on $\mathrm{Dom}(\dot{S}) = (I + \dot{B})\mathrm{Dom}(\dot{B})$. Notice that the orthogonal complement $\mathfrak{N} = \mathcal{H} \ominus \mathrm{Dom}(\dot{S})$ coincides with the defect subspace $\mathfrak{N}_{-1}$ of the operator $\dot{B}$. There is a one-to-one correspondence given by the Cayley transform

$$B = (I - S)(I + S)^{-1}, \quad S = (I - B)(I + B)^{-1},$$

between all non-negative self-adjoint extensions B (linear relations in general) of the operator $\dot{B}$ and all self-adjoint contractive (*sc*) extensions S of $\dot{S}$. As was established by M. Kreĭn in [23], [24] the set of all *sc*-extensions of $\dot{A}$ forms an operator interval $[S_\mu, S_M]$. Following M. Kreĭn's notations we call the endpoints S_μ and S_M by the *rigid* and the *soft* extensions, respectively. They possess the properties

$$\begin{aligned} &\inf_{\varphi\in\mathrm{Dom}(\dot{S})} ((I + S_\mu)(f - \varphi), (f - \varphi) = 0, \\ &\inf_{\varphi\in\mathrm{Dom}(\dot{S})} ((I - S_M)(f - \varphi), (f - \varphi) = 0, \end{aligned} \tag{3.4}$$

for all $f \in \mathcal{H}$. The operator interval $[S_\mu, S_M]$ can be parameterized as follows

$$S = \frac{1}{2}(S_M + S_\mu) + \frac{1}{2}(S_M - S_\mu)^{1/2} X (S_M - S_\mu)^{1/2}, \tag{3.5}$$

where X is a self-adjoint contraction in the subspace $\overline{\mathrm{Ran}(S_M - S_\mu)} (\subseteq \mathfrak{N})$.

Notice that for each $S \in [S_\mu, S_M]$ the equalities (3.4) imply

$$\begin{aligned} &\inf_{\varphi\in\mathrm{Dom}(\dot{S})} ((I + S)(f - \varphi), (f - \varphi) = ((S - S_\mu)f, f), \\ &\inf_{\varphi\in\mathrm{Dom}(\dot{S})} ((I - S)(f - \varphi), (f - \varphi) = ((S_M - S)f, f), \ f \in \mathcal{H}. \end{aligned} \tag{3.6}$$

Using the relation (see [23])

$$\inf_{\varphi\in\mathrm{Dom}(\dot S)}((I+S)(f-\varphi),(f-\varphi) = ||P_\Omega(I+S)^{1/2}f||^2,$$

where

$$\Omega=\{g\in\mathcal{H}:(I+S)^{1/2}g\in\mathfrak{N}\},$$

from (3.6) we get the equalities

$$\begin{aligned}(I+S)^{1/2}\Omega&=\mathrm{Ran}((S-S_\mu)^{1/2}),\\ ||(I+S)^{[-1/2]}f||&=||(S-S_\mu)^{[-1/2]}f||^2,\quad f\in\mathrm{Ran}((S-S_\mu)^{1/2}).\end{aligned}\tag{3.7}$$

Let L be a bounded non-negative self-adjoint operator in the Hilbert space $\mathcal{H}$ and let $\mathcal{M}$ be a subspace in $\mathcal{H}$. The Kreĭn shorted operator $L_{\mathcal{M}}$ [23], [1] is given by the following definition

$$L_{\mathcal{M}}=\max\{X\le L\mid \mathrm{Ran}(X)\subseteq\mathcal{M}\}.$$

It is shown in [23], that

$$L_{\mathcal{M}}=L^{1/2}QL^{1/2},\tag{3.8}$$

where Q is an orthoprojection operator onto the subspace $\mathrm{Ran}(Q)=(L^{1/2})^{-1}(\mathcal{M})$. Moreover, [23]

$$(L_{\mathcal{M}}f,f)=\inf_{\varphi\in\mathcal{H}\ominus\mathcal{M}}(L(f-\varphi),f-\varphi),\ f\in\mathcal{H}.\tag{3.9}$$

Thus, from (3.6) we have

$$(I+S)_{\mathfrak{N}}=S-S_\mu,\ (I-S)_{\mathfrak{N}}=S_M-S.$$

The next result describes the sesquilinear form $B[u,v]$ by the means of the fractional-linear transformation $S=(I-B)(I+B)^{-1}$. The following proposition can be found in [7].

Proposition 3.5.

(1) *Let B be a non-negative self-adjoint operator and let $S=(I-B)(I+B)^{-1}$ be its Cayley transform. Then*

$$\begin{aligned}\mathcal{D}[B]&=\mathrm{Ran}((I+S)^{1/2}),\\ B[u,v]&=-(u,v)+2\left((I+S)^{-1/2}u,(I+S)^{-1/2}v\right),\quad u,v\in\mathcal{D}[B].\end{aligned}$$

(2) *Let $\dot B$ be a closed densely defined non-negative symmetric operator and let B be its non-negative self-adjoint extension. If $\dot S=(I-\dot B)(I+\dot B)^{-1}$, $S=(I-B)(I+B)^{-1}$, then*

$$\mathcal{D}[B]=\mathrm{Ran}(I+S_\mu)^{1/2}\dotplus\mathrm{Ran}(S-S_\mu)^{1/2}.\tag{3.10}$$

We note that $\mathrm{Ran}(B^{1/2})=\mathrm{Ran}((I-S)^{1/2})$. Now let S_μ and S_M be the rigid and the soft extensions of $\dot S$. Then the Friedrichs and Kreĭn–von Neumann extensions of $\dot B$ are given by

$$B_F=(I-S_\mu)(I+S_\mu)^{-1},\ B_K=(I-S_M)(I+S_M)^{-1}.$$

4. Non-negative self-adjoint bi-extensions

4.1. Disjointness and tranversality of non-negative self-adjoint extensions

Proposition 4.1. *Let $\dot{A}$ be a non-negative closed densely defined operator. Then the following statements hold true for a non-negative self-adjoint extensions A of $\dot{A}$:*

$$A \text{ is disjoint with } A_F \iff \mathcal{D}[A]\cap\mathcal{H}_+ \text{ is dense in } \mathcal{H}_+,$$
$$A \text{ is transversal with } A_F \iff \mathcal{D}[A]\supset\mathcal{H}_+.$$

Proof. Using equality (3.3) in the form

$$\mathcal{D}[A]=\mathcal{D}[\dot{A}]\dot{+}\left(\mathfrak{N}_{-1}\cap\mathcal{D}[A]\right)$$

and the relation $\mathrm{Dom}(A_F)=\mathcal{D}[\dot{A}]\cap\mathrm{Dom}(\dot{A}^*)$, we get that

$$\mathcal{D}[A]\cap\mathcal{H}_+=\mathrm{Dom}(A_F)\dot{+}\left(\mathfrak{N}_{-1}\cap\mathcal{D}[A]\right), \tag{4.1}$$

where $\mathfrak{N}_\lambda$ is the defect subspace of $\dot{A}$. Taking into account the equality

$$\mathcal{H}_+=\mathrm{Dom}(A_F)\dot{+}\mathfrak{N}_{-1},$$

we get that $\mathcal{D}[A]\cap\mathcal{H}_+$ is dense in $\mathcal{H}_+$ if and only if $\mathfrak{N}_{-1}\cap\mathcal{D}[A]$ is dense in $\mathfrak{N}_{-1}$ and

$$\mathcal{D}[A]\cap\mathcal{H}_+=\mathcal{H}_+ \iff \mathfrak{N}_{-1}\subset\mathcal{D}[A].$$

Put

$$\dot{S}=(I-\dot{A})(I+\dot{A}),\quad S_\mu=(I-A_F)(I+A_F),\quad S=(I-A)(I+A).$$

Then

$$S-S_\mu=(A+I)^{-1}-(A_F+I)^{-1}. \tag{4.2}$$

Now the equality (see (3.10))

$$\mathcal{D}[A]=\mathcal{D}[\dot{A}]\dot{+}\mathrm{Ran}(S-S_\mu)^{1/2} \tag{4.3}$$

implies the validity of the statements in the proposition. □

From (4.2) and (4.3) we get the following equalities

$$\mathcal{D}[A]\cap\mathcal{H}_+=\mathrm{Dom}(A_F)\dot{+}\mathrm{Ran}(S-S_\mu)^{1/2}=\mathrm{Dom}(A)\dot{+}\mathrm{Ran}(S-S_\mu)^{1/2}.$$

Notice that the equivalence

$$A_F \text{ and } A_K \text{ are transversal} \iff \mathrm{Dom}(\dot{A}^*)\subseteq\mathcal{D}[A_K]$$

has been shown in [25] (see also [11]). The next statement provides one more criteria for A_F and A_K to be transversal.

Proposition 4.2.

$$A_F \text{ and } A_K \text{ are transversal} \iff \sup_{f\in\mathrm{Dom}(\dot{A})}\frac{\|(I+\dot{A}\dot{A}^*)^{-1/2}\dot{A}f\|^2}{(\dot{A}f,f)}<\infty. \tag{4.4}$$

Proof. Let A be a non-negative self-adjoint extension of $\dot{A}$. Since $A^{1/2}$ is closed in $\mathcal{H}$, the closed graph theorem yields that

$$\mathcal{H}_+ \subset \mathcal{D}[A] = \mathrm{Dom}(A^{1/2}) \iff A^{1/2}\upharpoonright \mathcal{H}_+ \in [\mathcal{H}_+, \mathcal{H}],$$

i.e., there exists a number $c > 0$ such that

$$||A^{1/2}u||^2 = A[u] \le c||u||_+^2 \quad \text{for all} \quad u \in \mathcal{H}_+.$$

Take $A = A_K$. Then for $u \in \mathcal{D}[A_K] \cap \mathcal{H}_+$

$$||A_K^{1/2}u||^2 = \sup_{f\in\mathrm{Dom}(\dot{A})} \frac{|(\dot{A}f, u)|^2}{(\dot{A}f, f)} = \sup_{f\in\mathrm{Dom}(\dot{A})} \frac{|(\mathcal{R}\dot{A}f, u)_+|^2}{(\dot{A}f, f)}.$$

Hence

$$|(\mathcal{R}\dot{A}f, u)_+|^2 \le ||A_K^{1/2}u||^2 \, (\dot{A}f, f).$$

Then

$$\begin{aligned}
\mathcal{H}_+ \subset \mathcal{D}[A_K] &\iff |(\mathcal{R}\dot{A}f, u)_+|^2 \le c||u||_+^2 \, (\dot{A}f, f), \ \forall u \in \mathcal{H}_+, \ \forall f \in \mathrm{Dom}(\dot{A}) \\
&\iff ||\mathcal{R}\dot{A}f||_+^2 = \sup_{u\in\mathcal{H}_+} \frac{|(\mathcal{R}\dot{A}f, u)_+|^2}{||u||_+^2} \le c\,(\dot{A}f, f), \ \forall f \in \mathrm{Dom}(\dot{A}) \\
&\iff \sup_{f\in\mathrm{Dom}(\dot{A})} \frac{||\mathcal{R}\dot{A}f||_+^2}{(\dot{A}f, f)} < \infty.
\end{aligned}$$

Since

$$||\mathcal{R}g||_+^2 = ||(I + \dot{A}\dot{A}^*)^{-1/2}g||^2, \ g \in \mathcal{H},$$

we arrive at (4.4). □

Notice that due to Theorem 3.4 condition

$$\sup_{f\in\mathrm{Dom}(\dot{A})} \frac{||(I + \dot{A}\dot{A}^*)^{-1/2}\dot{A}f||^2}{(\dot{A}f, f)} < \infty$$

means that the operator $\mathcal{R}\dot{A}$ admits (+)-bounded (+)-self-adjoint non-negative extensions. It is not difficult to show that

$$(I + \dot{A}\dot{A}^*)^{-1/2}\dot{A}f = \dot{A}(I + \dot{A}^*\dot{A})^{-1/2}f, \quad f \in \mathrm{Dom}(\dot{A}).$$

This relation implies that if $\dot{A}$ is positively definite, then A_F and A_K are transversal. Indeed,

$$||(I+\dot{A}\dot{A}^*)^{-1/2}\dot{A}f||^2 = ||\dot{A}(I+\dot{A}^*\dot{A})^{-1/2}f||^2 \le C||f||^2 \le m(\dot{A}f, f), \ f \in \mathrm{Dom}(\dot{A}).$$

Hence,

$$\sup_{f\in\mathrm{Dom}(\dot{A})} \frac{||(I + \dot{A}\dot{A}^*)^{-1/2}\dot{A}f||^2}{(\dot{A}f, f)} < \infty.$$

4.2. Non-negative self-adjoint bi-extensions

Existence. Let $\mathfrak{H}_+ \subset \mathfrak{H} \subset \mathfrak{H}_-$ be a rigged Hilbert space. If $\mathcal{T}$ is a non-negative, densely defined in $\mathfrak{H}_+$ and closed sesquilinear form in $\mathfrak{H}_+$, then there exists a non-negative generalized self-adjoint operator $\mathbb{T}$ acting from $\mathrm{Dom}(\mathbb{T})$ into $\mathcal{H}_-$ *associated* with the form $\mathcal{T}$ in the following sense

$$(\mathbb{T}u, v) = \mathcal{T}[u, v] \text{ for all } u \in \mathrm{Dom}(\mathbb{T}) \text{ and all } v \in \mathrm{Dom}(\mathcal{T}). \tag{4.5}$$

Actually, due to the First Representation Theorem, there is a (+)-non-negative self-adjoint operator $\mathfrak{T}$ associated with the form $\mathcal{T}$ in $\mathfrak{H}_+$, i.e.,

$$(\mathfrak{T}u, v)_+ = \mathcal{T}[u, v] \text{ for all } u \in \mathrm{Dom}(\mathfrak{T}) \text{ and all } v \in \mathrm{Dom}(\mathcal{T}).$$

If $\mathcal{J} \in [\mathfrak{H}_-, \mathfrak{H}_+]$ is the Riesz–Berezansky operator, then $\mathbb{T} = \mathcal{J}^{-1}\mathfrak{T}$ satisfies (4.5). If a non-negative form is defined on $\mathfrak{H}_+$ and is bounded in $\mathfrak{H}_+$, then, clearly, the associated non-negative self-adjoint operator belongs to $[\mathfrak{H}_+, \mathfrak{H}_-]$.

If $\mathfrak{T} : \mathfrak{H}_+ \supseteq \mathrm{Dom}(\mathfrak{T}) \to \mathfrak{H}_-$ is a non-negative generalized self-adjoint operator in the rigged Hilbert space $\mathfrak{H}_+ \subset \mathfrak{H} \subset \mathfrak{H}_-$, i.e., $(\mathfrak{T}f, f) \geq 0$ for all $f \in \mathrm{Dom}(\mathfrak{T})$ and $\mathfrak{T} = \mathfrak{T}^*$, then the sesquilinear form

$$\mathcal{T}_{\mathfrak{T}}[f, g] = (\mathfrak{T}f, g), \quad \mathrm{Dom}(\mathcal{T}_{\mathfrak{T}}) = \mathrm{Dom}(\mathfrak{T})$$

is closable in $\mathfrak{H}_+$. We will denote by $\mathfrak{T}[\cdot,\cdot]$ its closure and by $\mathcal{D}[\mathfrak{T}]$ its domain.

Now we consider a closed non-negative symmetric densely defined operator $\dot{A}$. Let $\mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_-$ be the rigged Hilbert space, where $\mathcal{H}_+ = \mathrm{Dom}(\dot{A}^*)$ and (+)-inner product is defined by (1.1). We are going to study non-negative self-adjoint bi-extensions of the operator $\dot{A}$. Clearly, the operator

$$\dot{B} = \mathcal{R}\dot{A}$$

is non-densely defined in $\mathcal{H}_+$, (+)-bounded and (+)-non-negative. Each non-negative (+)-self-adjoint extension B of $\dot{B}$ in $\mathcal{H}_+$, which is an operator, determines a non-negative self-adjoint bi-extension of $\dot{A}$ by the formula $\mathbb{A} = \mathcal{R}^{-1}B$. Since

$$||\dot{B}\varphi||_+ = ||\mathcal{R}\dot{A}\varphi||_+ = ||(I + \dot{A}\dot{A}^*)^{-1}\dot{A}\varphi||_+ = ||(I + \dot{A}\dot{A}^*)^{-1/2}\dot{A}\varphi||, \ \varphi \in \dot{A},$$

and $(\dot{B}\varphi, \varphi)_+ = (\dot{A}\varphi, \varphi)$, we can use the Ando and Nishio theorem (see [1]) about positively closable symmetric operator and get the following statement.

Proposition 4.3. *A non-negative densely defined closed symmetric operator $\dot{A}$ admits non-negative self-adjoint bi-extension if and only if from*

$$\lim_{n\to\infty} (I + \dot{A}\dot{A}^*)^{-1/2}\dot{A}\varphi_n = g \quad \textit{and} \quad \lim_{n\to\infty} (\dot{A}\varphi_n, \varphi_n) = 0$$

follows $g = 0$, where $\{\varphi_n\} \subset \mathrm{Dom}(\dot{A})$.

Theorem 4.4. *Let $\dot{A}$ be a non-negative closed densely defined operator. The following conditions are equivalent:*

(i) *$\dot{A}$ admits a non-negative self-adjoint bi-extension,*
(ii) *$\dot{A}$ admits t-self-adjoint bi-extension with quasi-kernel A_K,*
(iii) *the Friedrichs and Kreĭn–von Neumann extensions of $\dot{A}$ are disjoint.*

Proof. Clearly (ii)⇒(i). Let us show that (iii)⇒(ii). Suppose that the Friedrichs extension A_F and the Kreĭn–von Neumann extension A_K of the operator $\dot{A}$ are disjoint. Then $\mathrm{Dom}(A_F) + \mathrm{Dom}(A_K)$ is (+)-dense in $\mathcal{H}_+$ or coincides with $\mathcal{H}_+$ (when A_F and A_K are transversal). Then it follows that $\mathcal{D}[A_K]\cap\mathcal{H}_+$ is (+)-dense in $\mathcal{H}_+$ or coincides with $\mathcal{H}_+$. Clearly, the sesquilinear form

$$A_K[u,v], \quad u,v\in\mathcal{D}[A_K]\cap\mathcal{H}_+,$$

is closed in $\mathcal{H}_+$. Because it is at least (+)-densely defined in $\mathcal{H}_+$, there is an associated self-adjoint non-negative operator $\mathbb{A}_K\colon \mathcal{H}_+\supseteq\mathrm{Dom}(\mathbb{A}_K)\to\mathcal{H}_-$, i.e.,

$$(\mathbb{A}_K u,v) = A_K[u,v] \text{ for all } u\in\mathrm{Dom}(\mathbb{A}_K) \quad \text{and for all} \quad v\in\mathcal{D}[A_K]\cap\mathcal{H}_+.$$

Because $(A_K u,v) = A_K[u,v]$ for all $u\in\mathrm{Dom}(A_K)$ and all $v\in\mathcal{D}[A_K]$, we get that $\mathbb{A}_K\supset A_K$, i.e., the quasi-kernel of $\mathbb{A}_K$ is A_K and therefore, A_K is t-self-adjoint bi-extension of $\dot{A}$.

Let us prove (i)⇒(iii). Suppose that $\dot{A}$ admits non-negative self-adjoint bi-extensions. Then the Kreĭn–von Neumann extension B_K of the operator $\dot{B}=\mathcal{R}\dot{A}$ in $\mathcal{H}_+$ is an operator. Due to the formula (3.1) the domain $\mathcal{D}[B_K]$ is at least dense in $\mathcal{H}_+$. On the other hand since

$$\frac{|(\dot{B}f,u)_+|^2}{(\dot{B}f,f)_+} = \frac{|(\dot{A}f,u)|^2}{(\dot{A}f,f)},$$

from (3.1) we get

$$\mathcal{D}[B_K] = \mathcal{D}[A_K]\cap\mathcal{H}_+$$

and $B_K[u] = A_K[u]$ for all $u\in\mathcal{D}[A_K]\cap\mathcal{H}_+$. It follows from (4.1) that

$$\mathcal{D}[A_K]\cap\mathcal{H}_+ = \mathrm{Dom}(A_F)\dot{+}(\mathfrak{N}_{-1}\cap\mathcal{D}[A_K]).$$

Therefore, the density of $\mathcal{D}[A_K]\cap\mathcal{H}_+$ implies the density of $\mathfrak{N}_{-1}\cap\mathcal{D}[A_K]$ in $\mathfrak{N}_{-1}$. Equality (3.10) yields that

$$\overline{\mathrm{Ran}}\left((A_K+I)^{-1}-(A_F+I)^{-1}\right) = \mathfrak{N}_{-1},$$

i.e., A_F and A_K are at least disjoint. □

Theorem 4.5.

1) *Let A be a non-negative self-adjoint extension of $\dot{A}$. Then there exists a t-self-adjoint bi-extension $\mathbb{A}$ of $\dot{A}$ with quasi-kernel A if and only if A is disjoint with A_F.*
2) *If a non-negative self-adjoint extension A of $\dot{A}$ is disjoint with A_F, then t-self-adjoint bi-extension $\mathbb{A}$ with quasi-kernel A and generated by A_F is associated with the sesquilinear form $A[u,v]$, $u,v\in\mathcal{D}[A]\cap\mathcal{H}_+$.*

Proof. The form $A[u,v]$ defined on $\mathcal{D}[A]\cap\mathcal{H}_+$ is closed in $\mathcal{H}_+$. By Proposition 4.1 A is disjoint with A_F if and only if the linear manifold $\mathcal{D}[A]\cap\mathcal{H}_+$ is dense in $\mathcal{H}_+$ in which case the non-negative sesquilinear form $A[u,v]$, $u,v\in\mathcal{D}[A]\cap\mathcal{H}_+$ is closed in $\mathcal{H}_+$. The latter implies the existence of a non-negative self-adjoint operator $\mathbb{A}:\mathcal{H}_+\supseteq\mathrm{Dom}(\mathbb{A})\to\mathcal{H}_-$ associated with $A[u,v]$, $u,v\in\mathcal{D}[A]\cap\mathcal{H}_+$.

Since $(Au,v)=A[u,v]$ for all $u\in \mathrm{Dom}(A)$ and all $v\in\mathcal{D}[A]$, we get that $\mathbb{A}\supset A$, i.e., the quasi-kernel of $\mathbb{A}$ is A and therefore, A is t-self-adjoint bi-extension of $\dot{A}$. Further we use the following equality (see [6])

$$A[\varphi,u]=(\varphi,\dot{A}^*u),\ \varphi\in\mathcal{D}[\dot{A}],\ u\in\mathcal{D}[A]\cap\mathcal{H}_+.$$

Using (2.7) we get for all $\varphi\in\mathrm{Dom}(A_F)$ and all $u\in\mathcal{D}[A]\cap\mathcal{H}_+$:

$$A[\varphi,u]=(\varphi,\dot{A}^*u)=((\dot{A}^*)^*\varphi,u)=((\dot{A}^*-\mathcal{R}^{-1}\dot{A}^*P^+_{\mathfrak{M}})\varphi,u).$$

Hence, $\mathrm{Dom}(A_F)\subset\mathrm{Dom}(\mathbb{A})$ and

$$\mathbb{A}\varphi=(\dot{A}^*-\mathcal{R}^{-1}\dot{A}^*P^+_{\mathfrak{M}})\varphi,\ \varphi\in\mathrm{Dom}(A_F).$$

Since $\mathrm{Dom}(A)\subset\mathrm{Dom}(\mathbb{A})$ and $\mathbb{A}$ is a t-self-adjoint bi-extension of $\dot{A}$ with quasi-kernel A, we get

$$\mathrm{Dom}(\mathbb{A})=\mathrm{Dom}(A)+\mathrm{Dom}(A_F).$$

Taking into account (2.3), we conclude that $\mathbb{A}$ is generated by A_F. □

The following statement is an immediate consequence of Theorems 4.4 and 4.5.

Corollary 4.6 ([7]). *The operator $\dot{A}$ admits non-negative self-adjoint bi-extensions in $[\mathcal{H}_+,\mathcal{H}_-]$ if and only if A_K and A_F are transversal.*

It was announced in [26] that the transversality condition in Corollary 4.6 is necessary (and sufficient for the case of finite deficiency indices) for the existence of non-negative self-adjoint bi-extensions in $[\mathcal{H}_+,\mathcal{H}_-]$.

Denote by $\mathcal{P}(\dot{A})$ the set of all non-negative self-adjoint bi-extensions of $\dot{A}$. As has been proved in Theorem 4.4 the set $\mathcal{P}(\dot{A})$ is nonempty if and only if A_F and A_K are disjoint in which case the set $\mathcal{P}(\dot{A})$ contains the operator $\mathbb{A}_K$ with the following properties:

1. the operator $\mathbb{A}_K$ is associated with the closed form $A_K[u,v]$, $u,v\in\mathcal{D}[A_K]\cap\mathcal{H}_+$, i.e.,
$$\mathcal{D}[\mathbb{A}_K]=\mathcal{D}[A_K]\cap\mathcal{H}_+,$$
$$\mathbb{A}_K[u,v]=A_K[u,v],\quad u\in\mathrm{Dom}(\mathbb{A}_K),\ v\in\mathcal{D}[A_K]\cap\mathcal{H}_+,$$
2. the operator $\mathbb{A}_K$ is a t-self-adjoint bi-extension of $\dot{A}$ with quasi-kernel A_K and generated by A_F,
3. $\mathcal{P}(\dot{A})\ni\mathbb{A}\Rightarrow\mathcal{D}[\mathbb{A}]\subseteq\mathcal{D}[\mathbb{A}_K]$, $\mathbb{A}[u]\ge\mathbb{A}_K[u]=A_K[u]$, $u\in\mathcal{D}[\mathbb{A}]$.

Thus, $\mathbb{A}_K$ is the *minimal element* of $\mathcal{P}(\dot{A})$ and is an analog of Kreĭn–von Neumann extension. The minimality property is a consequence of Theorem 3.2. Notice that if A_K and A_F are transversal and the deficiency number of $\dot{A}$ is infinite, then the set $\mathcal{P}(\dot{A})$ contains $+\to-$ bounded and unbounded operators.

Let A_1 be a non-negative self-adjoint extension of $\dot{A}$. Let $\mathcal{P}(A_1)$ be the set of all non-negative t-self-adjoint bi-extension of $\dot{A}$ with quasi-kernel A_1. According to Theorem 4.5 the set $\mathcal{P}(A_1) \neq \emptyset$ if and only if A_1 is disjoint with A_F. Using Theorem 3.1, and the equalities

$$\mathcal{D}[A_1] = \left\{ f \in \mathcal{H} : \sup_{h \in \mathrm{Dom}(A_1)} \frac{|(A_1 h, f)|^2}{(A_1 h, h)} < \infty \right\},$$

$$\frac{|(\mathcal{R}A_1 h, f)_+|^2}{(\mathcal{R}A_1 h, h)_+} = \frac{|(A_1 h, f)|^2}{(A_1 h, h)}, \ f \in \mathcal{H}_+,$$

we get: if A_1 and A_F are disjoint, then the operator $\mathbb{A}_{1K} : \mathcal{H}_+ \supseteq \mathrm{Dom}(\mathbb{A}_{1K}) \to \mathcal{H}_-$, associated with closed in $\mathcal{H}_+$ non-negative form $A_1[u,v]$, $u, v \in \mathcal{D}[A_1] \cap \mathcal{H}_+$, is the minimal element of the set $\mathcal{P}(A_1)$ in the sense of quadratic forms. According to Theorem 4.5 this operator is generated by A_F. It is an analog of the Kreĭn–von Neumann type extension of A_1 in the rigged Hilbert space $\mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_-$. The operator $\mathbb{A}_K$ is the minimal element of the sets $\mathcal{P}(A_K)$ and $\mathcal{P}(\dot{A})$. The next theorem parameterizes the set $\mathcal{P}(A_1)$.

Theorem 4.7. *Let $\dot{A}$ be a non-negative closed symmetric operator with disjoint non-negative self-adjoint extensions A_F and A_K. Suppose $\mathbb{A}$ is a t-self-adjoint bi-extension of $\dot{A}$ with quasi-kernel A_1 and generated by A_0. Then $\mathbb{A}$ is non-negative if and only if*

$$A_0 \geq A_1 \geq 0.$$

Proof. We will use (2.5)

$$(\mathbb{A}f, f) = (A_1 f_1, f_1) + (A_0 f_0, f_0) + 2\mathrm{Re}\,(A_1 f_1, f_0),$$

for $f = f_1 + f_0$, $f_1 \in \mathrm{Dom}(A_1)$, $f_0 \in \mathrm{Dom}(A_0)$. It follows that $A_1 \geq 0$ and $A_0 \geq 0$. Replacing f_1 by λf_1 and f_0 by μf_0 we have

$$|\lambda|^2 (A_1 f_1, f_1) + |\mu|^2 (A_0 f_0, f_0) + \lambda\bar{\mu}(A_1 f_1, f_0) + \mu\bar{\lambda}(f_0, A_1 f_1) \geq 0$$

for all $\lambda, \mu \in \mathbb{C}$. Thus, the 2×2 matrix

$$\begin{pmatrix} (A_1 f_1, f_1) & (A_1 f_1, f_0) \\ (f_0, A_1 f_1) & (A_0 f_0, f_0) \end{pmatrix}$$

is non-negative. Hence

$$|(A_1 f_1, f_0)|^2 \leq (A_1 f_1, f_1)(A_0 f_0, f_0)$$

and

$$\sup_{f_1 \in \mathrm{Dom}(A_1)} \frac{|(A_1 f_1, f_0)|^2}{(A_1 f_1, f_1)} \leq (A_0 f_0, f_0). \tag{4.6}$$

This means that

$$f_0 \in \mathcal{D}[A_1] \quad \text{and} \quad A_1[f_1] \leq (A_0 f_0, f_0) \ (= A_0[f_0]).$$

If $\{f_2^{(n)}\}_{n=1}^{\infty} \subset \mathrm{Dom}(A_0)$ and A_0-converges to $\varphi_0 \in \mathcal{D}[A_0]$, then (4.6) yields

$$\sup_{f_1 \in \mathrm{Dom}(A_1)} \frac{|(A_1 f_1, \varphi_0)|^2}{(A_1 f_1, f_1)} \leq A_0[\varphi_0].$$

Thus

$$\mathcal{D}[A_0] \subset \mathcal{D}[A_1] \quad \text{and} \quad A_1[\varphi_0] \leq A_0[\varphi_0] \quad \text{for all} \quad \varphi_0 \in \mathcal{D}[A_0],$$

i.e., $A_1 \leq A_0$.

Conversely. Suppose $0 \leq A_1 \leq A_0$. Then for an arbitrary $f_1 \in \mathrm{Dom}(A_1)$, $f_0 \in \mathrm{Dom}(A_0)$ we get

$$\begin{aligned}
(\mathbb{A}(f_1+f_0), f_1+f_0) &= (A_1 f_1, f_1) + (A_0 f_0, f_0) + 2\mathrm{Re}\,(A_1 f_1, f_0) \\
&= ||A_1^{1/2} f_1||^2 + ||A_0^{1/2} f_0||^2 + 2\mathrm{Re}\,(A_1^{1/2} f_1, A_1^{1/2} f_0) \\
&= ||A_1^{1/2}(f_1+f_0)||^2 + ||A_0^{1/2} f_0||^2 - ||A_1^{1/2} f_0||^2 \geq 0.
\end{aligned}$$

This proves the theorem. □

Let A_1 and A_0 be two non-negative self-adjoint extensions of $\dot{A}$. Consider a form defined on $\mathrm{Dom}(A_1) \times \mathrm{Dom}(A_0)$ as follows

$$\mathcal{B}(f_1, f_0) = (A_1 f_1, f_1) + (A_0 f_0, f_0) + 2\mathrm{Re}\,(A_1 f_1, f_0), \tag{4.7}$$

where $f_l \in \mathrm{Dom}(A_l)$, $(l = 0, 1)$. Let

$$\phi_l = \frac{1}{2}(I + A_l) f_l, \qquad S_l \phi_l = \frac{1}{2}(I - A_l) f_l,$$

be the Cayley transform of A_l for $l = 0, 1$. Then

$$f_l = (I + S_l)\phi_l, \qquad A_l f_l = (I - S_l)\phi_l, \quad (l = 0, 1). \tag{4.8}$$

Substituting (4.8) into (4.7) we obtain a form defined on $\mathcal{H} \times \mathcal{H}$

$$\tilde{\mathcal{B}}(\phi_1, \phi_0) = \|\phi_1 + \phi_0\|^2 - \|S_1\phi_1 + S_0\phi_0\|^2 - 2\,\mathrm{Re}\,((S_1 - S_0)\phi_1, \phi_0)\,. \tag{4.9}$$

Let us set

$$F = \frac{1}{2}(S_1 - S_0), \quad G = \frac{1}{2}(S_1 + S_0), \quad u = \frac{1}{2}(\phi_1 + \phi_2), \quad v = \frac{1}{2}(\phi_1 - \phi_0). \tag{4.10}$$

Then

$$\tilde{\mathcal{B}}(\phi_1, \phi_0) = 4H(u, v) := \|u\|^2 + (Fv, v) - (Fu, u) - \|Fv + Gu\|^2. \tag{4.11}$$

Moreover, $F \pm G$ are contractive operators. From the above reasoning we conclude that non-negativity of the form $\mathcal{B}(f_1, f_0)$ on $\mathrm{Dom}(A_1) \times \mathrm{Dom}(A_0)$ is equivalent to non-negativity of the form $H(u, v)$ on $\mathcal{H} \times \mathcal{H}$. The next statement is established in [3], see also [7].

Proposition 4.8. *The form $H(u, v)$ in* (4.11) *is non-negative for all $u, v \in \mathcal{H}$ if and only if operator F defined in* (4.10) *is non-negative.*

Proposition 4.8 can be used for another proof of Theorem 4.7 (see [3]).

Let A_1 and A_0 be two disjoint non-negative self-adjoint extensions of $\dot{A}$. We say that A_1 and A_0 form an *admissible pair* $\langle A_1, A_0 \rangle$ if

$$A_0 \geq A_1 \iff (A_1 + I)^{-1} \geq (A_0 + I)^{-1}.$$

If $S_j = (I - A_j))(I + A_j)^{-1}$, $j = 1, 2$, then the pair $\langle A_1, A_0\rangle$ is admissible if and only if $\ker(S_1 - S_0) = \mathrm{Dom}(\dot{S})$ and $S_1 \geq S_0$. Let X_j, $j = 0, 1$ be self-adjoint contractions in $\mathfrak{N}$ such that

$$S_j = \frac{1}{2}(S_M + S_\mu) + \frac{1}{2}(S_M - S_\mu)^{1/2} X_j (S_M - S_\mu)^{1/2}.$$

Then it follows from (3.5) that the pair of non-negative self-adjoint extensions $A_j = (I - S_j)(I + S_j)^{-1}$, $j = 0, 1$ is admissible if and only if

$$\ker(X_1 - X_0) \cap \mathrm{Ran}((S_M - S_\mu)^{1/2}) = \{0\} \quad \text{and} \quad X_1 - X_0 \geq 0.$$

Associated closed forms. The next statement describes $\mathbb{A}[u, v]$ (the closure of the form $(\mathbb{A}f, f)$), where $\mathbb{A}$ is a non-negative t-self-adjoint bi-extension of $\dot{A}$ with quasi-kernel A_1 and generated by A_0 (compare with Theorem 3.3).

Theorem 4.9. *Let $\langle A_1, A_0\rangle$ be an admissible pair and let $\mathbb{A}$ be a non-negative t-self-adjoint bi-extension of $\dot{A}$ with quasi-kernel A_1 and generated by A_0. Let $\mathbb{A}[\cdot, \cdot]$ be the closure of the form $(\mathbb{A}f, g)$, $f, g \in \mathrm{Dom}(\dot{A})$. Then*

$$\begin{aligned}
\mathcal{D}[\mathbb{A}] &= \mathrm{Dom}(A_1) \dot{+} \mathrm{Ran}\left((S_1 - S_0)^{1/2}\right) = \mathrm{Dom}(A_0) \dot{+} \mathrm{Ran}\left((S_1 - S_0)^{1/2}\right), \\
\mathbb{A}[u] &= ||h||^2 - ||S_1 h + w||^2 + 2\mathrm{Re}\,(h, w) + 2||(S_1 - S_0)^{-1/2} w||^2 \\
&= A_1[u] + ||(S_1 - S_0)^{-1/2} w||^2 - ||(S_1 - S_\mu)^{-1/2} w||^2, \\
u &= (I + S_1)h + w,
\end{aligned} \tag{4.12}$$

where $S_l = (I - A_l)(I + A_l)$, $l = 0, 1$, $h \in \mathcal{H}$, $w \in \mathrm{Ran}\left((S_1 - S_0)^{1/2}\right)$.

Proof. Let $f = f_1 + f_0$, $f_1 \in \mathrm{Dom}(A_1)$, $f_0 \in \mathrm{Dom}(A_0)$. Then

$$(\mathbb{A}f, f) = (A_1 f_1, f_1) + (A_0 f_0, f_0) + 2\mathrm{Re}\,(A_1 f_1, f_0).$$

Due to (4.9)

$$(\mathbb{A}f, f) = \tilde{\mathcal{B}}(\phi_1, \phi_0) = \|\phi_1 + \phi_0\|^2 - \|S_1\phi_1 + S_0\phi_0\|^2 - 2\,\mathrm{Re}\,\left((S_1 - S_0)\phi_1, \phi_0\right),$$

where

$$\phi_l = \frac{1}{2}(I + A_l) f_l,\ S_l\phi_l = \frac{1}{2}(I - A_l) f_l,\ f_l = (I + S_l)\phi_l,\ A_l f_l = (I - S_l)\phi_l,\ l = 0, 1.$$

Represent $f = f_1 + f_0 = (I + S_1)\phi_1 + (I + S_0)\phi_0$ in the form

$$f = (I + S_1)(\phi_1 + \phi_0) - (S_1 - S_0)\phi_0.$$

Then

$$\begin{aligned}
(\mathbb{A}f, f) &= \|\phi_1 + \phi_0\|^2 - \|S_1(\phi_1 + \phi_0) - (S_1 - S_0)\phi_0\|^2 \\
&\qquad - 2\,\mathrm{Re}\,(\phi_1 + \phi_0, (S_1 - S_0)\phi_0) + 2||(S_1 - S_0)^{1/2}\phi_0||^2.
\end{aligned} \tag{4.13}$$

Suppose that

$$\lim_{n\to\infty} f^{(n)} = u \text{ in } \mathcal{H}_+, \text{ and } \lim_{n\to\infty} (\mathbb{A}(f^{(n)} - f^{(m)}), f^{(n)} - f^{(m)}) = 0.$$

We have

$$f^{(n)}=(I+S_1)(\phi_1^{(n)}+\phi_0^{(n)})-(S_1-S_0)\phi_0^{(n)}.$$

Due to the direct decomposition

$$\mathcal{H}_+=\mathrm{Dom}(A_1)\dot{+}\mathfrak{N}_{-1}$$

and inclusions $\{(I+S_1)(\phi_1^{(n)}+\phi_0^{(n)})\}\subset\mathrm{Dom}(A_1)$, $\{(S_1-S_0)\phi_0^{(n)}\}\subset\mathfrak{N}_{-1}$, we get that the sequences $\{(I+S_1)(\phi_1^{(n)}+\phi_0^{(n)})\}$ and $\{(S_1-S_0)\phi_0^{(n)}\}$ converge in $\mathcal{H}_+$. By definition $||w||_+^2=2||w||^2$, $\forall w\in\mathfrak{N}_{-1}$. Hence $\{(S_1-S_0)\phi_0^{(n)}\}$ converges in $\mathcal{H}$. On the other hand convergence of $\{(I+S_1)(\phi_1^{(n)}+\phi_0^{(n)})\}$ in $\mathcal{H}_+$ yields convergence of $\{\phi_1^{(n)}+\phi_0^{(n)}\}$ in $\mathcal{H}$. Let

$$h=\lim_{n\to\infty}(\phi_1^{(n)}+\phi_0^{(n)})\text{ in }\mathcal{H},$$

$$\mathrm{Dom}(\mathcal{A}_1)\ni y=\lim_{n\to\infty}(I+S_1)(\phi_1^{(n)}+\phi_0^{(n)})\text{ in }\mathcal{H}_+,$$

$$w'=\lim_{n\to\infty}(S_1-S_0)\phi_0^{(n)}.$$

From $\lim\limits_{n\to\infty}(\mathbb{A}(f^{(n)}-f^{(m)}),f^{(n)}-f^{(m)})=0$ and (4.13) we obtain that the sequence $\{(S_1-S_0)^{1/2}\phi_0^{(n)}\}$ converges in $\mathcal{H}$. Let

$$g=\lim_{n\to\infty}(S_1-S_0)^{1/2}\phi_0^{(n)}.$$

Then $w'=(S_1-S_0)^{1/2}g$. Set $w=-w'$. Thus

$$u=y+w,$$

where $y=(I+S_1)h\in\mathrm{Dom}(A_1)$, $w\in\mathrm{Ran}\left((S_1-S_0)^{1/2}\right)$. We get that

$$\begin{aligned}\lim_{n\to\infty}(\mathbb{A}f^{(n)},f^{(n)})&=||h||^2-||S_1h-(S_1-S_0)^{1/2}g||^2\\&\quad-2\mathrm{Re}\,(h,(S_1-S_0)^{1/2}g)+2||g||^2\\&=||h||^2-||S_1h+w||^2+2\mathrm{Re}\,(h,w)+2||(S_1-S_0)^{-1/2}w||^2.\end{aligned}$$

Now let us prove that the quadratic form

$$\eta(u)=||h||^2-||S_1h+w||^2+2\mathrm{Re}\,(h,w)+2||(S_1-S_0)^{-1/2}w||^2,$$
$$u=(I+S_1)h+w,\ h\in\mathcal{H},w\in\mathrm{Ran}(S_1-S_0)^{1/2}$$

is non-negative and closed in $\mathcal{H}_+$ as defined on

$$\mathrm{Dom}(\eta)=\mathrm{Dom}(A_1)\dot{+}\mathrm{Ran}\left((S_1-S_0)^{1/2}\right).$$

Notice that the equality $S_1-S_0=2(A_1+I)^{-1}-2(A_0+I)^{-1}$ yields

$$\mathrm{Dom}(A_1)\dot{+}\mathrm{Ran}\left((S_1-S_0)^{1/2}\right)=\mathrm{Dom}(A_0)\dot{+}\mathrm{Ran}\left((S_1-S_0)^{1/2}\right).$$

First we calculate $A_1[u]$ for $u\in\mathrm{Dom}(A_1)\dot{+}(\mathcal{D}[A_1]\cap\mathfrak{N}_{-1})$. Let us represent u as

$$u=(I+S_1)h+(I+S_1)^{1/2}\omega,$$

where $h \in \mathcal{H}$, $\omega \in \Omega = \{g \in \mathcal{H} : (I+S_1)^{1/2}\omega \in \mathfrak{N}_{-1}\}$. Recall that by (3.7) and (3.10) we have

$$\mathrm{Ran}((S_1 - S_\mu)^{1/2}) = (I+S_1)^{1/2}\Omega = \mathcal{D}[A_1] \cap \mathfrak{N}_{-1}.$$

Using (3.5) we obtain

$$\begin{aligned} A_1[u] =& -||u||^2 + 2||I+S_1)^{-1/2}u||^2 \\ =& -||(I+S_1)h + (I+S_1)^{1/2}\omega||^2 + 2||(I+S_1)^{1/2}h + \omega||^2 \\ =& -||I+S_1)h||^2 - ||(I+S_1)^{1/2}\omega||^2 - 2\mathrm{Re}\,((I+S_1)h, (I+S_1)^{1/2}\omega) \\ & + 2||(I+S_1)^{1/2}h||^2 + 2||\omega||^2 + 4\mathrm{Re}\,((I+S_1)^{1/2}h, \omega) \\ =& \,||h||^2 - ||S_1 h||^2 - ||(I+S_1)^{1/2}\omega||^2 - 2\mathrm{Re}\,(S_1 h, (I+S_1)^{1/2}\omega) \\ & + 2\mathrm{Re}\,(h, (I+S_1)^{1/2}\omega) + 2||\omega||^2 \\ =& \,||h||^2 - ||S_1 h + (I+S_1)^{1/2}\omega||^2 + 2||\omega||^2 + 2\mathrm{Re}\,(h, (I+S_1)^{1/2}\omega). \end{aligned}$$

Denoting $w = (I+S_1)^{1/2}\omega$ and using the equality (see (3.7)) $||(S_1 - S_\mu)^{-1/2}w|| = ||(I+S_1)^{-1/2}w||^2$, we arrive at the equality

$$A_1[u] = ||h||^2 - ||S_1 h + w||^2 + 2\mathrm{Re}\,(h, w) + 2||(S_1 - S_\mu)^{-1/2}w||^2 \geq 0.$$

Furthermore, since $S_1 - S_\mu \geq S_1 - S_0$, we get that

$$\mathrm{Ran}((S_1 - S_\mu)^{1/2}) \supset \mathrm{Ran}((S_1 - S_0)^{1/2})$$

and $||(S_1 - S_0)^{-1/2}w||^2 \geq ||(S_1 - S_\mu)^{-1/2}w||^2$ for all $w \in \mathrm{Ran}((S_1 - S_0)^{1/2})$. So,

$$\begin{aligned} \eta(u) = A_1[u] + ||(S_1 - S_0)^{-1/2}w||^2 - ||(S_1 - S_\mu)^{-1/2}w||^2 \geq 0, \\ u \in \mathrm{Dom}(A_1)\dot{+}\mathrm{Ran}((S_1 - S_0)^{1/2}) \geq 0. \end{aligned}$$

In addition, one can easily see that the right-hand side of (4.12) is closed on $\mathrm{Dom}(A_1)\dot{+}\mathrm{Ran}((S_1 - S_0)^{1/2})$ in $\mathcal{H}_+$. Now we can conclude that (4.12) is valid. □

Define for $\mathbb{A} \in \mathcal{P}(\dot{A})$ the "*dual*" quadratic form

$$\mathbb{A}'[u] = 2\mathrm{Re}\,(\dot{A}^* u, u) - \mathbb{A}[u],\; u \in \mathcal{D}[\mathbb{A}]$$

and let

$$A'_K[u] = 2\mathrm{Re}\,(\dot{A}^* u, u) - A_K[u],\; u \in \mathcal{D}[A_K] \cap \mathcal{H}_+. \tag{4.14}$$

Recall that a linear operator T in a Hilbert space $\mathfrak{H}$ is called *accretive* [20] if $\mathrm{Re}\,(Tf, f) \geq 0$ for all $f \in \mathrm{Dom}(T)$ and *maximal accretive* (m-accretive) if it is accretive and has no accretive extensions in $\mathfrak{H}$. The following statements are equivalent [27]:

(i) the operator T is m-accretive;
(ii) the operator T is accretive and its resolvent set contains points from the left half-plane;
(iii) the operators T and T^* are accretive.

Theorem 4.10. *If A_F and A_K are disjoint, then each non-negative self-adjoint bi-extension $\mathbb{A}$ of $\dot{A}$ possess the properties*

$$\mathcal{D}[\mathbb{A}] \subseteq \mathcal{D}[A_K], \quad \mathbb{A}[u] \geq A_K[u], \quad \mathbb{A}'[u] \leq A'_K[u], \quad u \in \mathcal{D}[\mathbb{A}]. \tag{4.15}$$

In addition, if T is quasi-selfadjoint accretive extension of $\dot{A}$ ($\dot{A} \subset T \subset \dot{A}^*$), then

$$A_K[u] \leq \mathrm{Re}\,(Tu, u) \leq A'_K[u], \quad u \in \mathrm{Dom}(T). \tag{4.16}$$

Proof. As it follows from the proofs of Theorems 4.4 and 4.5 in $\mathcal{H}_+$ the Kreĭn–von Neumann extension of the operator $\dot{B} = \mathcal{R}\dot{A}$ coincides with the Kreĭn–von Neumann extension of the operator $\dot{B}' = \mathcal{R}A_K$. Therefore, using the minimality of A_K among all non-negative self-adjoint extensions of $\dot{A}$ we arrive at (4.15).

It is established in [4] that for each quasi-self-adjoint accretive extension T of $\dot{A}$ one has

$$\mathrm{Dom}(T) \subset \mathcal{D}[A_K], \quad A_K[u] \leq \mathrm{Re}\,(Tu, u), \quad u \in \mathrm{Dom}(T).$$

Using the above and (4.14) we get (4.16). □

Explicit expressions for non-negative t-self-adjoint bi-extensions. Evidently, the linear manifold $\mathrm{Dom}(A_F)$ is a subspace in $\mathcal{H}_+$. Let $\mathfrak{N}_F$ be the orthogonal complement to $\mathrm{Dom}(\dot{A})$ in $\mathrm{Dom}(A_F)$ with respect to the inner product $(\cdot,\cdot)_+$ and let $\mathfrak{M}_F = \mathcal{H}_+ \ominus \mathrm{Dom}(A_F)$. Then $\mathfrak{M}_F = A_F\mathfrak{N}_F$. Thus we have the $(+)$-orthogonal decomposition

$$\mathcal{H}_+ = \mathrm{Dom}(\dot{A}) \oplus \mathfrak{N}_F \oplus \mathfrak{M}_F.$$

Let

$$\mathfrak{N}_0 = \mathrm{Ran}(A_F^{1/2}) \cap \mathfrak{N}_F.$$

Clearly, $A_F^{-\frac{1}{2}}(\mathfrak{N}_0) \subset \mathrm{Dom}(A_F)$. The following equalities take place

$$\dot{A}^* A_F e = -e, \; e \in \mathfrak{N}_F,$$
$$A_F \dot{A}^* g = -g, \; g \in \mathfrak{M}_F.$$

Theorem 4.11 ([11]). *The condition $\mathfrak{N}_0 = \{0\}$ is necessary and sufficient for the uniqueness of non-negative self-adjoint extension of $\dot{A}$. Suppose $\mathfrak{N}_0 \neq \{0\}$. Then the formulas*

$$\begin{aligned} \mathrm{Dom}(\widetilde{A}) &= \mathrm{Dom}(\dot{A}) \oplus (I + A_F\widetilde{U})\mathrm{Dom}(\widetilde{U}), \\ \widetilde{A}(x + h + A_F\widetilde{U}h) &= A_F(x+h) - \widetilde{U}h, \quad x \in \mathrm{Dom}(\dot{A}), \; h \in \mathrm{Dom}(\widetilde{U}) \end{aligned} \tag{4.17}$$

give a one-to-one correspondence between all non-negative self-adjoint extensions $\widetilde{A}$ of $\dot{A}$ and all $(+)$-self-adjoint operators $\widetilde{U}$ in $\mathfrak{N}_F$ satisfying the condition

$$0 \leq \widetilde{U} \leq W_0^{-1}$$

where W_0^{-1} determines the operator inverse with respect to the $(+)$-non-negative self-adjoint relation W_0 in $\mathfrak{N}_F$ associated with the $(+)$-closed in $\mathfrak{N}_F$ non-negative form

$$\omega_0[x, y] = (A_F^{[-1/2]}x, A_F^{[-1/2]}y)_+ = (A_F^{1/2}x, A_F^{1/2}y) + (A_F^{[-1/2]}x, A_F^{[-1/2]}y), \; x, y \in \mathfrak{N}_0.$$

Here $A_F^{[-1/2]}$ *is the Moore–Penrose pseudo-inverse. Operator* $\widetilde{A}$ *coincides with the Kreĭn–von Neumann non-negative self-adjoint extension* A_K *if and only if* $\widetilde{U} = W_0^{-1}$.

Moreover,

- *the extensions* A_F *and* A_K *are disjoint* $\Longleftrightarrow$ $\mathfrak{N}_0$ *is dense in* $\mathfrak{N}_F$,
- *the extensions* A_F *and* A_K *are transversal* $\Longleftrightarrow$ $\mathfrak{N}_0 = \mathfrak{N}_F$.

The associated with $\widetilde{A}$ *closed form is given by the following equalities:*

$$\mathcal{D}[\widetilde{A}] = \mathcal{D}[\dot{A}]\dot{+}A_F\mathcal{R}(\widetilde{U}^{1/2}), \tag{4.18}$$

$$\widetilde{A}[\varphi + A_F h] = ||A_F^{1/2}\varphi - A_F^{[-1/2]}h||^2 + \widetilde{U}^{-1}[h] - w_0[h], \quad \varphi \in \mathcal{D}[A],\, h \in \mathcal{R}(\widetilde{U}^{1/2}).$$

Let A_1 and A_0 be two non-negative self-adjoint extensions. From (4.17) and (4.18) it follows that A_1 and A_0, determined by parameters U_1 and U_0, respectively, then

- A_1 and A_0 are disjoint if and only if $\mathfrak{N}_0$ is dense in $\mathfrak{N}_F$ and $\ker(U_1 - U_0) = \{0\}$,
- $A_1 \le A_0$ if and only if $U_1 \ge U_0$,
- $A_1 \le A_0$ and A_1 and A_0 are transversal if and only if $\mathfrak{N}_0 = \mathfrak{N}_F$, $\mathrm{Ran}(U_1) = \mathfrak{N}_F$, $U_1 \ge U_0$, and $\mathrm{Ran}(I - U_1^{-1}U_0) = \mathfrak{N}_F$.

Denote by $P^+_{\mathfrak{N}_F}$, $P^+_{\mathfrak{M}_F}$ the orthogonal projection in $\mathcal{H}_+$ onto $\mathfrak{N}_F$ and $\mathfrak{M}_F = A_F\mathfrak{N}_F$. Notice that

$$\mathfrak{M} = \mathfrak{N}_i \oplus \mathfrak{N}_{-i} = \mathfrak{N}_F \oplus \mathfrak{M}_F.$$

Recall that each self-adjoint bi-extensions of $\dot{A}$ is of the form (2.1), where $\mathcal{S}$ is a (+)-self-adjoint operator in $\mathfrak{M}$.

Theorem 4.12. *Suppose* A_K *and* A_F *are disjoint. Then*

1. *the operator* $\mathbb{A}_K$ *is of the form*

$$\mathbb{A}_K = \dot{A}^* - \mathcal{R}^{-1}A_F(P^+_{\mathfrak{N}_F} + W_0\dot{A}^*P^+_{\mathfrak{M}_F}); \tag{4.19}$$

2. *the operator* $\mathbb{A} = A^* + \mathcal{R}^{-1}(\mathcal{S} - \dot{A}^*/2)P^+_{\mathfrak{M}}$ *belongs to* $\mathcal{P}(\dot{A})$ *if and only if*

$$\mathcal{S} \ge \mathcal{S}_K = -A_F W_0 \dot{A}^* P^+_{\mathfrak{M}_F} + \frac{1}{2}\left(\dot{A}^*P^+_{\mathfrak{M}_F} - A_F P^+_{\mathfrak{N}_F}\right)$$

in the sense of quadratic forms;

3. *if* A_1 *is a non-negative self-adjoint extension of* $\dot{A}$ *disjoint with* A_F *and if* $A_0 \ge A_1$, *then the non-negative t-self-adjoint bi-extension of* $\dot{A}$ *with quasi-kernel* A_1 *and generated by* A_0 *is of the form*

$$\mathbb{A} = \dot{A}^* + \mathcal{R}^{-1}\left(\mathcal{S} - \frac{1}{2}\dot{A}^*\right)P^+_{\mathfrak{M}},$$

where $\mathcal{S}$ is a (+)-self-adjoint operator in $\mathfrak{M}$ given by

$$\begin{cases} \mathrm{Dom}(\mathcal{S}) = (I + A_F U_1)\mathrm{Dom}(U_1)\dot{+}(I + A_F U_0)\mathrm{Dom}(U_0) \\ \mathcal{S}(I + A_F U_1)e = \frac{1}{2}(A_F - U_1)e,\ e \in \mathrm{Dom}(U_1) \\ \mathcal{S}(I + A_F U_0)g = \frac{1}{2}(-A_F + U_0)g,\ g \in \mathrm{Dom}(U_0) \end{cases}, \tag{4.20}$$

and U_1, U_0 determine A_1 and A_0 in formulas (4.17)*. In particular, if $A_0 = A_F$, then*

$$\mathcal{S} = -A_F U_1^{-1} \dot{A}^* P^+_{\mathfrak{M}_F} + \frac{1}{2}\left(\dot{A}^* P^+_{\mathfrak{M}_F} - A_F P^+_{\mathfrak{N}_F}\right). \tag{4.21}$$

Proof. From (4.17) we get the equality

$$\mathrm{Dom}(\widetilde{A}) \ominus \mathrm{Dom}(\dot{A}) = (I + A_F \widetilde{U})\mathrm{Dom}(\widetilde{U})$$

for an arbitrary non-negative self-adjoint extension $\widetilde{A}$ of $\dot{A}$. Then equalities (2.4) yield (4.20). When $A_0 = A_F$, we have $U_0 = 0$. This gives the equality

$$f = (I + A_F U_1)(-U_1^{-1}\dot{A}^*)P^+_{\mathfrak{M}_F} f + (P^+_{\mathfrak{N}_F} + U_1^{-1}\dot{A}^* P^+_{\mathfrak{M}_F})f.$$

Then by virtue of (4.20) we obtain (4.21). The case $A_1 = A_K$ holds true if and only if $U_1 = W_0^{-1}$ and leads to

$$\mathcal{S}_K = -A_F W_0 \dot{A}^* P^+_{\mathfrak{M}_F} + \frac{1}{2}\left(\dot{A}^* P^+_{\mathfrak{M}_F} - A_F P^+_{\mathfrak{N}_F}\right).$$

Then applying (2.1) we get (4.19). Statement (2.) follows from the fact that $\mathbb{A}_K$ is the minimal element of $\mathcal{P}(\dot{A})$. □

5. Extremal non-negative self-adjoint bi-extensions

Let $\dot{S}$ be a symmetric contraction defined in subspace $\mathrm{Dom}(S)$. We call a sc-extension S of $\dot{S}$ *extremal* if

$$\inf_{g_S \in \mathrm{Dom}(\dot{S})} \|(I - S^2)^{1/2}(g - g_S)\| = 0, \quad \forall g \in \mathcal{H}.$$

We can also offer an equivalent definition of an extremal sc-extension. Let $\mathfrak{N} = \mathcal{H} \ominus \mathrm{Dom}(S)$. We call a sc-extension S of $\dot{S}$ *extremal* if $(I - S^2)_{\mathfrak{N}} = 0$, where $(I - S^2)_{\mathfrak{N}}$ is the Kreĭn shorted operator (see (3.8), (3.9)). The following equality was proved in [10]

$$(I - S^2)_{\mathfrak{N}} = (S_M - S_\mu)^{1/2}(I - X^2)(S_M - S_\mu)^{1/2}, \tag{5.1}$$

where X is corresponding to S (via formula (3.5)) contraction in $\overline{\mathrm{Ran}(S_M - S_\mu)}$. Formula (5.1) implies that S is extremal if and only if X is self-adjoint and unitary, i.e., $X = X^*$ and $X^2 = I$.

Now let $\dot{A}$ be a non-negative closed densely defined symmetric operator. Recall (see Section 3) that a non-negative self-adjoint extension A of $\dot{A}$ is *extremal* [3] if

$$\inf_{\varphi\in\mathrm{Dom}(\dot{A})}(A(h-\varphi),h-\varphi)=0,\quad \forall h\in\mathrm{Dom}(A).$$

If

$$\dot{S}=(I-\dot{A})(I+\dot{A})^{-1},\qquad S=(I-A)(I+A)^{-1}, \tag{5.2}$$

then $(Ah,h)=((I-S^2)g,g)$ where $g=(I+S)^{-1}h$. This yields

$$\inf_{\varphi\in\mathrm{Dom}(\dot{A})}(A(h-\varphi),h-\varphi)=\inf_{g_S\in\mathrm{Dom}(\dot{S})}\|(I-S^2)^{1/2}(g-g_S)\|^2,$$

where $\mathrm{Dom}(\dot{S})=(I+\dot{A})\mathrm{Dom}(\dot{A})$. Therefore, A is extremal non-negative self-adjoint extension of $\dot{A}$ if and only if S is extremal sc-extension of symmetric contraction $\dot{S}$. The Friedrichs and Kreĭn–von Neumann extensions are extremal.

Let $\mathbb{A}$ be a non-negative self-adjoint bi-extension of the symmetric operator $\dot{A}$. We call the operator $\mathbb{A}$ an *extremal bi-extension* if

$$\inf_{\varphi\in\mathrm{Dom}(\dot{A})}(\mathbb{A}(f-\varphi),f-\varphi)=0,\quad \forall f\in\mathrm{Dom}(\mathbb{A}).$$

In what follows we assume that the operators A_F and A_K are disjoint.

Theorem 5.1. *A t-self-adjoint bi-extension $\mathbb{A}$ is extremal if and only if it is generated by an admissible pair $\langle A_1,A_0\rangle$ of extremal non-negative self-adjoint extensions of $\dot{A}$.*

Proof. Let A_1 and A_0 be the quasi-kernels of $\mathbb{A}$ and $\mathbb{A}'$, respectively. Let also $\mathbb{A}$ be an extremal self-adjoint bi-extension. It follows from (2.5) then that

$$(\mathbb{A}f_k,f_k)=(A_kf_k,f_k),\quad \forall f_k\in\mathrm{Dom}(A_k),\ k=0,1.$$

Since $\mathbb{A}$ extends A_1 and is generated by A_0, it follows from (2.5) that

$$(\mathbb{A}f,f)=(A_1f_1,f_1)+(A_0f_0,f_0)+2\mathrm{Re}\,(A_1f_1,f_0)=\mathcal{B}(f_1,f_0),$$

where $f\in\mathrm{Dom}(\mathbb{A})$, $f=f_1+f_0$, $f_k\in\mathrm{Dom}(A_k)$, $k=0,1$. Applying (4.10) and (4.11) we get

$$\begin{aligned}\inf_{\varphi\in\mathrm{Dom}(\dot{A})}(\mathbb{A}(f-\varphi),f-\varphi)&=\inf_{h_S\in\mathrm{Dom}(\dot{S})}H(x-h_S,y)\\ &=\inf_{h_S\in\mathrm{Dom}(\dot{S})}\left(\|x-h_S\|^2-(x,Fx)+(y,Fy)-\|Fy+G(x-h_s)\|^2\right).\end{aligned} \tag{5.3}$$

Since $\inf_{f_A\in\mathrm{Dom}(\dot{A})}(\mathbb{A}(f_k-f_A),f_k-f_A)=0$ for all $f_k\in\mathrm{Dom}(A_k)$, $k=0,1$, the operators A_1 and A_0 are extremal non-negative self-adjoint extensions of $\dot{A}$.

Hence, the extremality of $\mathbb{A}$ implies that the non-negative self-adjoint extensions A_1 and A_0 are also extremal. Since $\mathbb{A}$ is a non-negative self-adjoint bi-extension, then the pair $\langle A_1,A_0\rangle$ is an admissible extremal pair.

Conversely, let us assume that $\langle A_1,A_0\rangle$ is an admissible pair of extremal non-negative self-adjoint extensions of $\dot{A}$. We are going to prove that the corresponding non-negative self-adjoint bi-extension $\mathbb{A}$ with quasi-kernel A_1 and generated by A_0 is extremal. The corresponding (via (5.2)) to A_1 and A_0 sc-extensions S_1 and S_0 are extremal. Also, the fact that $\langle A_1, A_0\rangle$ is an admissible pair, implies that $S_1 - S_0 \geq 0$.

Let

$$S_k = \frac{1}{2}(S_M + S_\mu) + \frac{1}{2}(S_M - S_\mu)^{1/2} X_k (S_M - S_\mu)^{1/2}, \quad k = 0, 1,$$

where X_k, $k = 0, 1$ are self-adjoint contractions in $\mathfrak{N}$. Since S_k, $k = 0, 1$ are extremal sc-extensions, then X_k, $k = 0, 1$, are self-adjoint unitary operators and hence $P_k = (I + X_k)/2$, $k = 0, 1$, are orthogonal projections. Also, $X_1 - X_0 \geq 0$ implies that $P_1 - P_0 \geq 0$ and $\operatorname{Ran}(P_1) \supset \operatorname{Ran}(P_0)$. Since $X_k = 2P_k - I$, $k = 0, 1$, then

$$G = \frac{1}{2}(S_M + S_\mu) + \frac{1}{2}(S_M - S_\mu)^{1/2}(P_1 + P_0 - I)(S_M - S_\mu)^{1/2},$$

and

$$F = (S_M - S_\mu)^{1/2}(P_1 - P_0)(S_M - S_\mu)^{1/2}.$$

Since $I - (P_1 + P_0 - I)^2 = P_1 - P_0$, then (5.1) implies that $(I - G^2)\restriction\mathfrak{N} = F$. Consequently, applying the definition of the operator $(I - G^2)\restriction\mathfrak{N}$ we obtain

$$F = (I - G^2)^{1/2} P_G (I - G^2)^{1/2},$$

where P_G is an orthoprojection onto the subspace

$$\mathcal{H}_G = ((I - G^2)^{1/2})^{-1}\{\mathfrak{N}\} \cap \overline{\operatorname{Ran}}\,((I - G^2)^{1/2}).$$

Therefore,

$$\begin{aligned}
H(x-h_s,y) &= \|x-h_S\|^2 - (x,Fx) + (y,Fy) - \|Fy + G(x-h_s)\|^2\\
&= \|(I-G^2)^{1/2}(x-h_s)\|^2 - \|P_G(I-G^2)^{1/2}x\|^2 + \|P_G(I-G^2)^{1/2}y\|^2\\
&\quad - \|(I-G^2)^{1/2}P_G(I-G^2)^{1/2}y\|^2 - 2\operatorname{Re}((I-G^2)^{1/2}P_G(I-G^2)^{1/2}y, G(x-h_s))\\
&= \|(I-G^2)^{1/2}(x-h_s)\|^2 - \|P_G(I-G^2)^{1/2}x\|^2 + \|GP_G(I-G^2)^{1/2}y\|^2\\
&\quad - 2\operatorname{Re}\Big(GP_G(I-G^2)^{1/2}y, (I-G^2)^{1/2}(x-h_s)\Big)\\
&= \|(I-G^2)^{1/2}(x-h_s) - GP_G(I-G^2)^{1/2}y\|^2 - \|P_G(I-G^2)^{1/2}x\|^2.
\end{aligned}$$

Thus, since $(I - G^2)^{1/2}\operatorname{Dom}(\dot S) \perp \mathcal{H}_G$, then

$$\begin{aligned}
\inf_{h_S \in \operatorname{Dom}(\dot S)} H(x - h_S, y) &= \|P_G(I - G^2)^{1/2}x - P_G G P_G (I - G^2)^{1/2} y\|^2\\
&\qquad - \|P_G(I - G^2)^{1/2}x\|^2, \quad \forall x, y \in \mathcal{H}.
\end{aligned} \tag{5.4}$$

Since A_1 and A_0 are extremal non-negative self-adjoint extensions, then the definition of the functional H and (4.11) imply

$$\inf_{h_S \in \operatorname{Dom}(\dot S)} H(x - h_S, x) = 0, \qquad \inf_{h_S \in \operatorname{Dom}(\dot S)} H(x - h_S, -x) = 0,$$

for all $x \in \mathcal{H}$. Relation (5.4) yields

$$\|P_G(I-G^2)^{1/2}x - P_G G P_G (I-G^2)^{1/2}x\|^2 - \|P_G(I-G^2)^{1/2}x\|^2 = 0,$$

and

$$\|P_G(I-G^2)^{1/2}x + P_G G P_G (I-G^2)^{1/2}x\|^2 - \|P_G(I-G^2)^{1/2}x\|^2 = 0,$$

for all $x \in \mathcal{H}$. Thus, $P_G G P_G (I-G^2)^{1/2}x = 0$ for all $x \in \mathcal{H}$. Applying (5.4) again we get

$$\inf_{h_S \in \operatorname{Dom}(\dot{S})} H(x-h_S, y) = 0, \qquad \forall x, y \in \mathcal{H}.$$

Now we can use (5.3) to confirm that

$$\inf_{f_A \in \operatorname{Dom}(\dot{A})} (\mathbb{A}(f-f_A), f-f_A) = 0,$$

which means that $\mathbb{A}$ is an extremal non-negative self-adjoint bi-extension. □

Recall that the non-negative self-adjoint bi-extension $\mathbb{A}_K$ is associated with the closed in $\mathcal{H}_+$ form $A_K[u,v]$, $u, v \in \mathcal{D}[A_K] \cap \mathcal{H}_+$. The quasi-kernel of $\mathbb{A}_K$ is the Kreĭn–von Neumann extension A_K and $\mathbb{A}_K$ is generated by A_F. Clearly, $\mathbb{A}_K$ is extremal non-negative self-adjoint bi-extension of $\dot{A}$.

Theorem 5.2.

(1) *Let A_F and A_K be transversal. Then the operator $\mathbb{A}_K$ is the unique extremal non-negative t-self-adjoint bi-extension.*

(2) *Let A_F and A_K be disjoint but not transversal. Then except $\mathbb{A}_K$ there exist infinitely many extremal non-negative t-self-adjoint bi-extensions.*

Proof. (1) Suppose that A_F and A_K are transversal. Let also $\mathbb{A}$ be an extremal t-self-adjoint bi-extension with the quasi-kernel A_1 and A_0 be the quasi-kernel of $\mathbb{A}'$. According to Theorem 5.1 for $S_k = (I - A_k)(I + A_k)^{-1}$, $k = 0, 1$ the following relations hold

$$S_k = S_\mu + (S_M - S_\mu)^{1/2} P_k (S_M - S_\mu)^{1/2}, \qquad k = 0, 1, \tag{5.5}$$

where P_k, $k = 0, 1$, are orthoprojections in $\mathfrak{N}$. Since A_1 and A_0 are disjoint, we have $\ker((S_1 - S_0)\restriction \mathfrak{N} = \{0\}$. But

$$\ker((S_1 - S_0)\restriction \mathfrak{N} = \ker((S_M - S_\mu)^{1/2}(P_1 - P_0)(S_M - S_\mu)^{1/2}\restriction \mathfrak{N}).$$

Since $P_1 - P_0 \geq 0$, then $Q = P_1 - P_0$ is an orthoprojection. Also, $\operatorname{Ran}(S_M - S_\mu) = \mathfrak{N}$ implies $\ker(P_1 - P_0) = \{0\}$ or equivalently $P_1 - P_0 = I$. The latter yields $P_1 = I$ and $P_0 = 0$. Consequently, $S_1 = S_M$, $S_0 = S_\mu$ and the quasi-kernels of $\mathbb{A}$ and $\mathbb{A}'$ coincide with A_F and A_K.

(2) Let A_F and A_K be disjoint but not transversal. Then $\operatorname{Ran}((S_M - S_\mu)^{1/2}) \neq \mathfrak{N}$ and $\ker((S_M - S_\mu)^{1/2}) = \{0\}$. We chose a subspace $\mathfrak{L} \subset \mathfrak{N}$ in a way that $\mathfrak{L} \cap \operatorname{Ran}(S_M - S_\mu)^{1/2} = \{0\}$. Let $\mathfrak{N}_1$ be such that $\{0\} \subseteq \mathfrak{N}_1 \subseteq \mathfrak{L}$. Let also P_1 be an orthogonal projection operator on $\mathfrak{N} \ominus \mathfrak{N}_1$, Q an orthoprojection on $\mathfrak{N} \ominus \mathfrak{L}$, and $P_0 = P_1 - Q$. Then $P_1 - P_0 = Q \geq 0$ and $\ker(P_1) \cap \operatorname{Ran}(S_M - S_\mu)^{1/2} = \{0\}$. Let S_k, $k = 0, 1$, be defined by (5.5). Hence, S_1 and S_0 are extremal sc-extensions and

A_k, $k = 0, 1$ are extremal non-negative self-adjoint extensions of $\dot{A}$ and $\langle A_1, A_0\rangle$ is an admissible pair. Therefore, according to Theorem 5.1, if $\mathbb{A} \supset A_1$ and $\mathbb{A}$ is generated by A_0, then $\mathbb{A}$ is extremal t-self-adjoint bi-extension of $\dot{A}$. It follows from the construction of $\mathbb{A}$ that there is infinite number of these bi-extensions. □

6. Boundary triplets and self-adjoint bi-extensions

Let $\dot{A}$ be a closed densely defined symmetric operator in $\mathcal{H}$ with equal deficiency numbers.

Definition 6.1 ([21]). The triplet $\Pi = \{\mathcal{N}, \Gamma_1, \Gamma_0\}$ is called a *boundary triplet* for $\dot{A}^*$ if $\mathcal{N}$ is a Hilbert space and Γ_0, Γ_1 are bounded linear operators from the Hilbert space $\mathcal{H}_+ = \mathrm{Dom}(\dot{A}^*)$ (with the inner product (1.1)) into $\mathcal{N}$ such that the mapping

$$\Gamma := \left\langle \Gamma_0, \Gamma_1 \right\rangle : \mathcal{H}_+ \to \mathcal{N} \oplus \mathcal{N},$$

is surjective and the abstract Green identity

$$\left(\dot{A}^* f, g\right) - \left(f, \dot{A}^* g\right) = (\Gamma_1 f, \Gamma_0 g)_{\mathcal{N}} - (\Gamma_0 f, \Gamma_1 g)_{\mathcal{N}},$$

holds for all f, $g \in \mathcal{H}_+$.

It follows from Definition 6.1 (see [17], [18]) that the operators

$$\mathrm{Dom}(A_k) := \ker \Gamma_k, \quad A_k := \dot{A}^* \upharpoonright \mathrm{Dom}(A_k), \quad (k = 0, 1),$$

are self-adjoint extensions of $\dot{A}$. Moreover, they are transversal, i.e.,

$$\mathrm{Dom}(\dot{A}^*) = \mathrm{Dom}(A_0) + \mathrm{Dom}(A_1).$$

Notice that if $\Pi = \{\mathcal{N}, \Gamma_1, \Gamma_0\}$ is a boundary triplet for $\dot{A}^*$, then $\Pi' = \{\mathcal{N}, -\Gamma_0, \Gamma_1\}$ is the boundary triplet for $\dot{A}^*$ too.

We are going to provide connections between self-adjoint bi-extensions and boundary triplets [7]. The proposition below follows from Definition 6.1.

Theorem 6.2. *Let $\dot{A}$ be a closed densely defined symmetric operator with equal deficiency indices in the Hilbert space $\mathcal{H}$. Suppose $\mathcal{N}$ is a Hilbert space, $\Gamma_0, \Gamma_1 \in [\mathcal{H}_+, \mathcal{N}]$, and the operator $\left\langle \Gamma_0, \Gamma_1 \right\rangle \in [\mathcal{H}_+, \mathcal{N} \oplus \mathcal{N}]$ is surjective. Then the following statements are equivalent.*

(i) $\Pi = \{\mathcal{N}, \Gamma_1, \Gamma_0\}$ *is the boundary triplet for* $\dot{A}^*$;

(ii) *the sesquilinear form*

$$w(f, g) := (\dot{A}^* f, g) - (\Gamma_1 f, \Gamma_0 g)_{\mathcal{N}}, \quad f, g \in \mathcal{H}_+ = \mathrm{Dom}(\dot{A}^*) \tag{6.1}$$

is Hermitian, i.e., $w(f, g) = \overline{w(g, f)}$;

(iii) *the sesquilinear form*

$$w'(f, g) := (\dot{A}^* f, g) + (\Gamma_0 f, \Gamma_1 g)_{\mathcal{N}}, \quad f, g \in \mathcal{H}_+ = \mathrm{Dom}(\dot{A}^*) \tag{6.2}$$

is Hermitian,

If $\mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_-$ is a rigged Hilbert space, $\mathcal{N}$ is a Hilbert space, and $\Gamma \in [\mathcal{H}_+, \mathcal{N}]$, then by $\Gamma^\times$ we will denote the adjoint operator from $[\mathcal{N}, \mathcal{H}_-]$, i.e., $(\Gamma h, g)_\mathcal{N} = (h, \Gamma^\times g)$ for all $h \in \mathcal{H}_+$ and all $g \in \mathcal{N}$.

The following theorem [7] sets up the connection between boundary triplets and t-self-adjoint bi-extensions.

Theorem 6.3. *Let $\dot{A}$ be a closed densely defined symmetric operator with equal deficiency numbers in the Hilbert space $\mathcal{H}$. Consider the rigged Hilbert space $\mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_-$ generated by $\dot{A}$.*

1. *Let $\Pi = \{\mathcal{N}, \Gamma_1, \Gamma_0\}$ for $\dot{A}^*$ be a boundary triplet for $\dot{A}^*$. Define operators $\mathbb{A}$ and $\mathbb{A}'$*
$$\mathbb{A} := \dot{A}^* - \Gamma_0^\times \Gamma_1, \quad \mathbb{A}' := \dot{A}^* + \Gamma_1^\times \Gamma_0,$$
where $\Gamma_0^\times$ and $\Gamma_1^\times \in [\mathcal{N}, \mathcal{H}_-]$ are the adjoint operators to Γ_0 and Γ_1, respectively. Then $\mathbb{A}$ and $\mathbb{A}'$ belong to $[\mathcal{H}_+, \mathcal{H}_-]$ and are t-self-adjoint bi-extensions of $\dot{A}$. Moreover,
$$\mathbb{A} \supset A_1, \quad \mathbb{A}' \supset A_0.$$
2. *If $\mathbb{A}$ is a t-self-adjoint bi-extension of $\dot{A}$ with quasi-kernel A_1 and generated by A_0, then there exists a boundary triplet $\Pi = \{\mathcal{N}, \Gamma_1, \Gamma_0\}$ for $\dot{A}^*$ such that $\dot{A}^* \restriction \ker \Gamma_1 = A_1$ and $\mathbb{A} = \dot{A}^* - \Gamma_0^\times \Gamma_1$.*

It is shown in the proof of Theorem 6.3 that the form $w(f,g)$ in (6.1) corresponds to $\mathbb{A}$, the boundary triplet $\Pi = \{\mathcal{N}, \Gamma_1, \Gamma_0\}$, and $w(f,g) = (\mathbb{A}f, g)$. Similarly, $w'(f,g) = (\mathbb{A}'f, g)$, where $w'(f,g)$ is defined in (6.2), and the boundary triplet is $\Pi' = \{\mathcal{N}, -\Gamma_0, \Gamma_1\}$.

Definition 6.4 ([3]). Suppose that $\dot{A}$ is a non-negative symmetric operator. A boundary triplet $\Pi = \{\mathcal{N}, \Gamma_1, \Gamma_0\}$ is called *non-negative* if
$$w(f,f) = (\dot{A}^* f, f) - (\Gamma_1 f, \Gamma_0 f)_\mathcal{N} \geq 0 \text{ for all } f \in \mathcal{H}_+.$$

The operator $\mathbb{A} = \dot{A}^* - \Gamma_0^\times \Gamma_1$ corresponding to the boundary triplet $\Pi = \{\mathcal{N}, \Gamma_1, \Gamma_0\}$ is [3] a t-self-adjoint non-negative bi-extension of $\dot{A}$ and belongs to $[\mathcal{H}_+, \mathcal{H}_+]$. If $\dot{A}$ is a positive-definite operator, then for the positive-definite self-adjoint extension A we have $\mathcal{H}_+ = \mathrm{Dom}(\dot{A}^*) = \mathrm{Dom}(A) \dot{+} \ker(\dot{A}^*)$. Consequently, A_F and A_K are transversal. Let P be a projection in $\mathcal{H}_+$ onto $\mathrm{Dom}(A)$ parallel to $\ker(\dot{A}^*)$, $\Pi = \{\mathcal{N}, \Gamma_K, \Gamma\}$ be a boundary triplet such that $\ker(\Gamma_K) = \mathrm{Dom}(A_K)$, Then
$$(\dot{A}^* f, f) - (\Gamma_K f, \Gamma f)_\mathcal{N} = (APf, Pf), \; f \in \mathcal{H}_+,$$
i.e., $\{\mathcal{N}, \Gamma_K, \Gamma\}$ is a positive boundary triplet. The latter equality has been assumed as the definition of a positive boundary triplet (the space of boundary values) in the case of a positive-definite operator $\dot{A}$ in [22].

It was shown in [3] that a positive boundary triplet exists if and only if A_F and A_K are transversal. The following theorem naturally follows from the preceding discussion.

Theorem 6.5. *Let $\dot{A}$ be a closed densely defined non-negative symmetric operator such that A_F and A_K are transversal. Then*

1. *to every non-negative boundary triplet $\Pi = \{\mathcal{N}, \Gamma_1, \Gamma_0\}$ there corresponds a non-negative t-self-adjoint bi-extension $\mathbb{A} = \dot{A}^* - \Gamma_0^\times \Gamma_1$;*
2. *to every non-negative t-self-adjoint bi-extension $\mathbb{A}$ there corresponds (up to equivalence[1]) a non-negative boundary triplet.*

Let $\Pi = \{\mathcal{N}, \Gamma_F, \Gamma_K\}$ be a non-negative boundary triplet such that $\mathrm{Dom}(A_K) = \ker \Gamma_K$, and $\mathrm{Dom}(A_F) = \ker \Gamma_F$. In [3] this boundary triplet is called *basic*. It is not hard to see that the corresponding to the basic boundary triplet non-negative t-self-adjoint bi-extension

$$\mathbb{A}_0 = \dot{A}^* - \Gamma_F^\times \Gamma_K \tag{6.3}$$

is such that the quasi-kernel of $\mathbb{A}_0$ is A_K. At the same time, A_F is the quasi-kernel of the bi-extension $\mathbb{A}'_0 = \dot{A}^* + \Gamma_K^\times \Gamma_F$. It follows that $\mathbb{A}_0 = \mathbb{A}_K$ is the minimal element of $\mathcal{P}(\dot{A})$. The following theorem is established in [3].

Theorem 6.6. *Let $\Pi = \{\mathcal{N}, \Gamma_F, \Gamma_K\}$ be a basic boundary triplet. Then a boundary triplet $\widetilde{\Pi} = \left\{\widetilde{\mathcal{N}}, \widetilde{\Gamma}_1, \widetilde{\Gamma}_0\right\}$ is non-negative if and only if*

$$\widetilde{\Gamma}_1 = X(\Gamma_K - B_1\Gamma_F), \qquad \widetilde{\Gamma}_0 = X^{*-1}[(I + B_2B_1)\Gamma_F - B_2\Gamma_K],$$

where B_1, B_2 are non-negative bounded operators in $\mathcal{H}$ and X is a linear homeomorphism from $\mathcal{H}$ onto $\widetilde{\mathcal{H}}$.

Theorem 6.6 essentially provides us with another way to describe all non-negative t-self-adjoint bi-extensions in $[\mathcal{H}_+, \mathcal{H}_-]$. Namely, if $\Pi = \{\mathcal{N}, \Gamma_F, \Gamma_K\}$ is a basic non-negative boundary triplet, then the formula

$$\mathbb{A} = \dot{A}^* - [\Gamma_F^\times (I + B_1B_2) - \Gamma_K^\times B_2](\Gamma_K - B_1\Gamma_F), \tag{6.4}$$

where B_1, B_2 are non-negative bounded operators in $\mathcal{H}$, gives that description. Formulas (6.3) and (6.4) yield the following expression for quadratic forms

$$(\mathbb{A}f, f) = (\mathbb{A}_0 f, f) + b(f, f), \quad f \in \mathcal{H}_+,$$

where

$$\begin{aligned} b(f,f) &= (B_1\Gamma_F f, \Gamma_F f) + (B_2\Gamma_K f, \Gamma_K f) + (B_1\Gamma_F f, B_2B_1\Gamma_F f) \\ &\quad - 2\mathrm{Re}\,(B_1\Gamma_K f, B_2\Gamma_K f) \\ &= ||B_1^{1/2}\Gamma_F f||^2_{\mathcal{N}} + ||B_2^{1/2}(B_1\Gamma_F - \Gamma_K)f||^2_{\mathcal{N}}. \end{aligned}$$

For the corresponding dual self-adjoint bi-extension

$$\mathbb{A}' = \dot{A}^* + (\Gamma_K^\times - \Gamma_F^\times B_1)((I + B_2B_1)\Gamma_F - B_2\Gamma_K),$$

[1] Two boundary triplets $\{\mathcal{N}, \Gamma_1, \Gamma_0\}$ and $\left\{\widetilde{\mathcal{N}}, \widetilde{\Gamma}_1, \widetilde{\Gamma}_0\right\}$ are called *equivalent* [3] if $\ker \Gamma_k = \ker \widetilde{\Gamma}_k$, $k = 0, 1$.

we have

$$(\mathbb{A}'f, f) = (\mathbb{A}'_0 f, f) - b(f, f), \qquad \forall f \in \mathcal{H}_+.$$

Set

$$\mathcal{N} = \mathfrak{N}_F,\ \Gamma_0 = -\dot{A}^* P^+_{\mathfrak{M}_F},\ \Gamma_1 = P^+_{\mathfrak{N}_F}.$$

One can easily check that $\{\mathcal{N}, \Gamma_1, \Gamma_0\}$ is a boundary triplet for $\dot{A}^*$. Clearly

$$\ker(\Gamma_0) = \mathrm{Dom}(A_F).$$

Calculating $\Gamma_0^\times$ and $\Gamma_1^\times$ one obtains

$$\Gamma_0^\times = \mathcal{R}^{-1} A_F P^+_{\mathfrak{N}_F},\ \Gamma_1^\times = \mathcal{R}^{-1} P^+_{\mathfrak{N}_F}.$$

Using Theorem 4.11 we get that the domains of all non-negative self-adjoint extensions $\widetilde{A}$ of $\dot{A}$ takes the form

$$\mathrm{Dom}(\widetilde{A}) = \{v \in \mathrm{Dom}(\dot{A}^*) : \Gamma_0 v = \widetilde{U}\Gamma_1 v\},$$

where $\widetilde{U}$ is an arbitrary (+)-self-adjoint and non-negative operator in $\mathfrak{N}_F$, satisfying $0 \le \widetilde{U} \le W_0^{-1}$, and

$$\mathrm{Dom}(A_K) = \{v \in \mathrm{Dom}(\dot{A}^*) : \Gamma_0 v = W_0^{-1}\Gamma_1 v\}.$$

Now suppose that A_F and A_K are disjoint (transversal). Then W_0 is a densely defined (everywhere defined) in $\mathfrak{N}_F$ and (+)-self-adjoint and we can rewrite $\mathrm{Dom}(A_K)$ as

$$\mathrm{Dom}(A_K) = \ker(\Gamma_1 - W_0\Gamma_0).$$

The operator

$$\mathbb{A}_K = \dot{A}^* - \Gamma_0^\times(\Gamma_1 - W_0\Gamma_0)$$

is t-self-adjoint bi-extension with quasi-kernel A_K and generated by A_F. This is the minimal element of the set $\mathcal{P}(\dot{A})$. Then we get the explicit expressions for $\mathbb{A}_K$ and $\mathbb{A}'_K$ (cf. (4.19)):

$$\mathbb{A}_K = \dot{A}^* - \mathcal{R}^{-1} A_F (P^+_{\mathfrak{N}_F} + W_0 \dot{A}^* P^+_{\mathfrak{M}_F}),$$
$$\mathbb{A}'_K = \dot{A}^* - \mathcal{R}^{-1} (P^+_{\mathfrak{N}_F} - A_F W_0 P^+_{\mathfrak{N}_F}) \dot{A}^* P^+_{\mathfrak{M}_F}.$$

If A_F and A_K are transversal, then we set

$$\Gamma_F = \Gamma_0 = -\dot{A}^* P^+_{\mathfrak{M}_F},\ \Gamma_K = \Gamma_1 - W_0\Gamma_0 = P^+_{\mathfrak{N}_F} + W_0 \dot{A}^* P^+_{\mathfrak{M}_F}.$$

Consequently, we obtain that $\{\mathfrak{N}_F, \Gamma_K, \Gamma_F\}$ is a basic boundary triplet for $\dot{A}^*$. Applying (6.4) we get a complete description of the set of all t-self-adjoint non-negative bi-extensions of $\dot{A}$ in $[\mathcal{H}_+, \mathcal{H}_-]$ given by the following formula

$$\mathbb{A} = \dot{A}^* - \mathcal{R}^{-1} \left[A_F(I + (W_0 + B_1)B_2) - B_2\right] \left[P^+_{\mathfrak{N}_F} + (W_0 + B_1) A^* P^+_{\mathfrak{M}_F}\right],$$

where B_1 and B_2 are an arbitrary (+)-bounded and non-negative self-adjoint operators in $\mathfrak{N}_F$.

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