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Products of Toeplitz and Hankel Operators on the Hardy Space of the Unit Sphere

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Abstract. The aim of this note is to discuss boundedness and compactness of Hankel products and mixed Toeplitz–Hankel products on the Hardy space of the unit sphere in several complex variables. The main adopted tool is an auxiliary pioneering operator involved in an earlier investigation of dual Toeplitz operators on the orthogonal complement of the Hardy space on the unit sphere.

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1. Introduction

Dual Toeplitz operators on the orthogonal complement of the Bergman space have been introduced and well investigated by Stroethoff and Zheng [11]. The higher-dimensional case of dual Toeplitz operators in both Hardy and Bergman space settings has been studied in [2, 5]; see also the relevant references therein. On the other hand, Toeplitz and Hankel operators in the latter setting have been extensively studied; in this respect we refer to [3, 4, 7, 8, 9, 12, 13].

For our purpose, let $\mathbb{B}_n$, $n > 1$, be the unit ball of $\mathbb{C}^n$ and $\mathbb{S}_n$ be its boundary (the unit sphere). Denote by $L^2(\mathbb{S}_n)$ the Lebesgue space of square integrable functions and by $\mathcal{H}^2(\mathbb{S}_n)$ its Hardy subspace, (for more details see [1, 3, 10, 15]). While on the circle the orthogonal complement of the Hardy space can be characterized by $(H^2)^\perp = \overline{zH^2}$, the matter is much more involved in higher dimensions because

$$L^2(\mathbb{S}_n) \ominus \left\{ \mathcal{H}^2(\mathbb{S}_n) + \overline{\mathcal{H}^2(\mathbb{S}_n)} \right\}$$

is large enough to cause capital differences from the one-dimensional case.

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Therefore, in contrast to the case of the circle, (where dual Toeplitz operators are anti-unitarily equivalent to Toeplitz operators in view of the above symmetry between $(H^2)^\perp$ and H^2, [11] or [5]), dual Toeplitz operators on the orthogonal complement of the Hardy space on the unit sphere cannot be analogously reduced to Toeplitz operators. Accordingly, they may constitute a worth studying new class of Haplitz-type operators. In [5], such dual Toeplitz operators have been introduced and studied from various points of view.

An interesting auxiliary operator, namely $\mathscr{S}_w$, has been introduced and used in studying products of these dual Toeplitz operators. In particular, commuting dual Toeplitz operators have then been characterized through certain necessary and sufficient conditions on the symbols. Besides, a Brown–Halmos type theorem has been proved; it tells us when exactly the product of two dual Toeplitz operators is again a dual Toeplitz operator. Several consequences of the two latter issues, such as the characterization of zero divisors, have been also inferred. For the sake of completeness, we summarize a few of these results here and refer to [5] for details.

Sarason's problem related to the boundedness of Toeplitz products has been extensively investigated by many authors, we refer to [6, 14] for details. The higher-dimensional case of Toeplitz products and Hankel products has been considered by many authors [2, 7, 8, 9, 12, 13]; see also the relevant references therein.

In the present paper, a more prominent role of the operator $\mathscr{S}_w$ is emphasized. More precisely, making use of this transformation, we discuss necessary conditions ensuring boundedness and compactness of products of Hankel operators $H_f H_g^*$, (equivalently the "dual" semicommutators $\mathcal{S}_{fg} - \mathcal{S}_f \mathcal{S}_g$), and mixed Hankel–Toeplitz products $T_f H_g^*$ and $H_g T_{\overline{f}}$; as well as the commutators $\mathcal{S}_f \mathcal{S}_g - \mathcal{S}_g \mathcal{S}_f$. These represent the main results of this communication.

2. Preliminaries

For $f \in L^\infty(\mathbb{S}_n)$, define the dual Toeplitz operator $\mathcal{S}_f$ as the operator on $\left(\mathcal{H}^2(\mathbb{S}_n)\right)^\perp$ defined to be a multiplication followed by a projection as follows:

$$\begin{aligned} \mathcal{S}_f : \left(\mathcal{H}^2(\mathbb{S}_n)\right)^\perp &\longrightarrow \left(\mathcal{H}^2(\mathbb{S}_n)\right)^\perp \\ u &\longrightarrow \mathcal{S}_f(u) := \mathcal{Q}(fu). \end{aligned}$$

Here, $\mathcal{Q}$ is the orthogonal projection from $L^2(\mathcal{S}_n)$ onto $\left(\mathcal{H}^2(\mathbb{S}_n)\right)^\perp$ defined by

$$\begin{aligned} \mathcal{Q} : L^2(\mathbb{S}_n) &\longrightarrow \left(\mathcal{H}^2(\mathbb{S}_n)\right)^\perp \\ g &\longrightarrow \mathcal{Q}(g) := (I - \mathcal{P})(g), \end{aligned}$$

with $\mathcal{P}$ being the customary (Hardy) orthogonal projection from $L^2(\mathbb{S}_n)$ onto the Hardy space $\mathcal{H}^2(\mathbb{S}_n)$. Since the projection $\mathcal{Q}$ has norm 1, then for any $h \in \left(\mathcal{H}^2(\mathbb{S}_n)\right)^\perp$, we have

$$||\mathcal{S}_f(h)||_2 = ||\mathcal{Q}(fh)||_2 \leq ||fh||_2 \leq ||f||_\infty ||h||_2.$$

Immediate algebraic properties of dual Toeplitz operators can be easily observed; for instance for $f, g \in L^\infty(\mathbb{S}_n)$, $\alpha, \beta \in \mathbb{C}$, we have

$$\mathcal{S}_f^* = \mathcal{S}_{\overline{f}} \ \text{ and } \ \mathcal{S}_{\alpha f + \beta g} = \alpha \mathcal{S}_f + \beta \mathcal{S}_g.$$

Dual Toeplitz operators appear naturally if one observes that under the orthogonal decomposition:

$$L^2(\mathcal{S}_n) = \mathcal{H}^2(\mathbb{S}_n) \oplus \left(\mathcal{H}^2(\mathbb{S}_n)\right)^\perp,$$

the multiplication operator $\mathcal{M}_f$, $f \in L^\infty(\mathbb{S}_n)$, can be represented as follows:

$$\mathcal{M}_f = \begin{pmatrix} T_f & H^*_{\overline{f}} \\ H_f & \mathcal{S}_f \end{pmatrix},$$

where the Toeplitz and Hankel operators are defined respectively by

$$\begin{aligned} T_f : \ \mathcal{H}^2(\mathbb{S}_n) &\longrightarrow \mathcal{H}^2(\mathbb{S}_n) \\ g &\longrightarrow T_f(g) := \mathcal{P}(fg), \end{aligned}$$

and

$$\begin{aligned} H_f : \ \mathcal{H}^2(\mathbb{S}_n) &\longrightarrow \left(\mathcal{H}^2(\mathbb{S}_n)\right)^\perp \\ g &\longrightarrow H_f(g) := \mathcal{Q}(fg). \end{aligned}$$

This representation gives rise to *dual Toeplitz operators* on $\left(\mathcal{H}^2(\mathbb{S}_n)\right)^\perp$. At once, we observe the following algebraic relationships connecting them with Haplitz operators, namely: for $f, g \in L^\infty(\mathbb{S}_n)$, the product identity $\mathcal{M}_f \mathcal{M}_g = \mathcal{M}_{fg}$ implies that

$$\begin{pmatrix} T_f & H^*_{\overline{f}} \\ H_f & \mathcal{S}_f \end{pmatrix} \begin{pmatrix} T_g & H^*_{\overline{g}} \\ H_g & \mathcal{S}_g \end{pmatrix} = \begin{pmatrix} T_{fg} & H^*_{\overline{fg}} \\ H_{fg} & \mathcal{S}_{fg} \end{pmatrix}.$$

Hence, we infer that

$$\begin{aligned} T_{fg} &= T_f T_g + H^*_{\overline{f}} H_g. \\ \mathcal{S}_{fg} &= H_f H^*_{\overline{g}} + \mathcal{S}_f \mathcal{S}_g. \\ H_{fg} &= H_f T_g + \mathcal{S}_f H_g. \end{aligned} \tag{2.1}$$

It follows that the commutator $[\ \mathcal{S}_f,\ \mathcal{S}_g] = \ \mathcal{S}_f \mathcal{S}_g - \mathcal{S}_g \mathcal{S}_f$ is given by

$$[\ \mathcal{S}_f,\ \mathcal{S}_g] = H_g H^*_{\overline{f}} - H_f H^*_{\overline{g}}. \tag{2.2}$$

In particular, such identities reduce to the following ones, since the Hankel operator is trivial if the symbol is analytic:

Lemma 2.1. *Let $f \in \mathcal{H}^\infty(\mathbb{S}_n)$, then we have*

i) $H_g T_f = \mathcal{S}_f H_g$.
ii) $T_{\overline{f}} H^*_g = H^*_g \mathcal{S}_{\overline{f}}$.
iii) $\mathcal{S}_{fg} = \mathcal{S}_f \mathcal{S}_g$.
iv) $\mathcal{S}_{g\overline{f}} = \mathcal{S}_g \mathcal{S}_{\overline{f}}$.

A key property which usually proves very useful in establishing more fundamental properties of Toeplitz type operators is the so-called the *spectral inclusion theorem*. It turns out [5] that our dual Toeplitz operators do satisfy such property.

Let us denote by $\mathscr{R}(f)$ the essential range of the essentially bounded function f, and by $\sigma(T)$ the spectrum of an operator T. Then, we have

Proposition 2.2. [5]

1. *If f is in $L^\infty(\mathbb{S}_n)$, then* $\mathscr{R}(f) = \sigma(M_f) \subseteq \sigma(\mathcal{S}_f)$.
2. *Let f be in $L^\infty(\mathbb{S}_n)$. Then, we have* $||\mathcal{S}_f|| = ||f||_\infty$.
3. *If f is in $L^\infty(\mathbb{S}_n)$, then* $\mathcal{S}_f = 0$ *if and only if* $f = 0$.

3. The auxiliary operator $\mathscr{S}_w$

Let z, w be in $\mathbb{B}_n$, and recall that the Hardy space $\mathcal{H}^2(\mathbb{B}_n)$ is a reproducing kernel Hilbert space with kernel function given by

$$K_w(z) = \frac{1}{(1 - \langle z, w\rangle)^n},$$

while the normalized reproducing kernel is denoted by k_w.

For f and g in $L^2(\mathbb{S}_n)$, consider the rank one operator defined by $(f \otimes g)h = \langle h, g\rangle\, f, \forall f \in L^2(\mathbb{S}_n)$; and note that $\|f \otimes g\| = \|f\|\ \|g\|$.

The unitary operator $\mathbb{U}_w$ is defined by

$$\mathbb{U}_w f = (f \circ \varphi_w) k_w. \tag{3.1}$$

Observe that $\mathbb{U}_w 1 = k_w$. Also, for a Toeplitz operator, we have

$$\mathbb{U}_w T_f \mathbb{U}_w = T_{f\circ\varphi_w}. \tag{3.2}$$

Further, we know that

$$\langle z, w\rangle^j = \sum_{|m|=j} \frac{j!}{m!} z^m\ \overline{w}^m.$$

Thus, by the binomial rule, we obtain

$$\begin{aligned} K_w^{-1}(z) = (1 - \langle z, w\rangle)^n &= \sum_{j=0}^{n} \frac{n!}{j!(n-j)!}\ (-1)^j\ \langle z, w\rangle^j \\ &= \sum_{j=0}^{n} \sum_{|m|=j} \frac{(-1)^j n!}{j!(n-j)!} \frac{j!}{m!}\ z^m \overline{w}^m \\ &= \sum_{j=0}^{n} \sum_{|m|=j} \lambda_{j,m}\ z^m \overline{w}^m, \quad \text{with} \quad \lambda_{j,m} = \frac{(-1)^j n!}{(n-j)! m!}. \end{aligned} \tag{3.3}$$

On the other hand, the customary ball automorphism $\varphi_w \in \mathbb{A}ut(\mathbb{B}_n)$ is defined by

$$\varphi_w(z) = \frac{w - P_w(z) - \sqrt{1 - |w|^2} Q_w(z)}{1 - \langle z, w\rangle},\ \ z, w \in \mathbb{B}_n, \tag{3.4}$$

where P_w denotes the orthogonal projection onto the subspace generated by w defined by $P_0 = 0$ and $P_w(z) = \frac{\langle z,w\rangle}{\langle w,w\rangle},\ w \neq 0$, and Q_w denotes the orthogonal projection onto its orthogonal complement given by $Q_w(z) = z - P_w(z)$. In particular, it satisfies the universal identity:

$$1-|\varphi_w(z)|^2=\frac{\left(1-|w|^2\right)\left(1-|z|^2\right)}{|1-\langle z,w\rangle|^2},\ z,w\in\mathbb{B}_n. \tag{3.5}$$

Finally, for operators $\mathbf{T}$ and $\mathbf{S}$, we can easily verify that:

$$\mathbf{T}(f\otimes g)\mathbf{S}^*=\mathbf{T}f\otimes\mathbf{S}g. \tag{3.6}$$

Matching all that together, we obtain the following key assertion:

Proposition 3.1. *On the Hardy space of the unit sphere $\mathcal{H}^2(\mathbb{S}_n)$, we have*

$$k_w\otimes k_w=\sum_{j=0}^{n}\sum_{|m|=j}\lambda_{j,m}T_{\varphi_w^m}T_{\overline{\varphi_w}^m},\ \forall w\in B_n, \tag{3.7}$$

with $\lambda_{j,m}$ as in (3.3) *above.*

Proof. Consider a Hardy function $f\in\mathcal{H}^2(\mathbb{B}_n)$. The invariant volume mean value property tells us that

$$f(\psi(0))=\int_{\mathbb{B}_n}(f\circ\psi)(w)d\nu(w),\ \forall f\in L^\infty(\mathbb{B}_n),\forall\psi\in\mathbb{A}ut(\mathbb{B}_n).$$

In particular, for the identity map which is in $\mathbb{A}ut(\mathbb{B}_n)$, we get

$$f(0)=\int_{\mathbb{B}_n}f(w)dA(w),\forall f\in\mathcal{H}^\infty(\mathbb{B}_n).$$

Inserting $K_w(z)K_w^{-1}(z)$ and noticing that $(1\otimes 1)f=f(0)$, we obtain

$$(1\otimes 1)f=\int_{\mathbb{B}_n}K_w^{-1}(z)K_w(z)f(w)dA(w),\forall f\in\mathcal{H}^\infty(\mathbb{B}_n). \tag{3.8}$$

Owing to Formula (3.3), we infer that

$$\begin{aligned}(1\otimes 1)f&=\sum_{j=0}^{n}\sum_{|m|=j}\lambda_{j,m}z^m\int_{\mathbb{B}_n}\overline{w}^m f(w)\overline{K_z(w)}dA(w)\\&=\sum_{j=0}^{n}\sum_{|m|=j}\lambda_{j,m}z^m\left(T_{\overline{w}^m}f\right)(z),\ \forall f\in\mathcal{H}^\infty(\mathbb{B}_n).\end{aligned}$$

Therefore, we obtain the following operator identity

$$(1\otimes 1)=\sum_{j=0}^{n}\sum_{|m|=j}\lambda_{j,m}T_{z^m}T_{\overline{w}^m}.$$

Introducing the unitary operator $\mathbb{U}_w$, we get

$$\mathbb{U}_w(1\otimes 1)\mathbb{U}_w=\sum_{j=0}^{n}\sum_{|m|=j}\lambda_{j,m}\left(\mathbb{U}_wT_{z^m}\mathbb{U}_w\right)\left(\mathbb{U}_wT_{\overline{w}^m}\mathbb{U}_w\right).$$

Notice also that by Formulas (3.1) and (3.6), we have

$$\mathbb{U}_w(1 \otimes 1)\mathbb{U}_w = (\mathbb{U}_w 1) \otimes (\mathbb{U}_w 1) = k_w \otimes k_w.$$

Using the latter two equations along with Identity (3.2), we infer that on $\mathcal{H}^2(\mathbb{B}_n)$ we have

$$k_w \otimes k_w = \sum_{j=0}^{n} \sum_{|m|=j} \lambda_{j,m} T_{\varphi_w^m} T_{\overline{\varphi_w}^m}, \ \forall w \in B_n,$$

which is valid on the sphere as well. □

Proposition 3.1 suggests the introduction of the following transformation of operators: for a bounded linear operator T on $\left(\mathcal{H}^2(\mathbb{S}_n)\right)^{\perp}$ and $w \in \mathbb{B}_n$, define the linear operator $\mathscr{S}_w(T)$ by

$$\mathscr{S}_w(T) = \sum_{j=0}^{n} \sum_{|m|=j} \lambda_{j,m} \ \mathcal{S}_{\varphi_w^m} T \mathcal{S}_{\overline{\varphi_w}^m}. \tag{3.9}$$

This pioneering operator $\mathscr{S}_w$ has an amazing story. It has been discovered first by K. Stroethoff and D. Zheng [11] in the Bergman space setting, where it looks like a two-term perturbation of the identity. In the setting of the Hardy space on the circle, it was adopted by H. Guediri and took the form of a one-term perturbation of the identity. In Lu & Chang [8], in H. Guediri [5] and in the present paper it takes the form of a multi-term perturbation of the identity as in the latter formula (3.9). This phenomenon seems to be very connected to the degree of the denominator, (equivalently to the dimension of the manifold), in the reproducing kernel expression of the underlying space.

The operator $\mathscr{S}_w$ reveals on a characterization of Hardy space dual Toeplitz operators:

Proposition 3.2. *If $\mathcal{S}_f$ is a dual Toeplitz operator on $\left(\mathcal{H}^2(\mathbb{S}_n)\right)^{\perp}$, then*

$$\mathscr{S}_w(\mathcal{S}_f) = 0, \quad \textit{for all } w \in \mathbb{B}_n.$$

Proof. Fix a $w \in \mathbb{B}_n$ and consider a dual Toeplitz operator $\mathcal{S}_f$ on $\left(\mathcal{H}^2(\mathbb{S}_n)\right)^{\perp}$, with symbol $f \in L^\infty(\mathbb{S}_n)$. By (3.9), we have

$$\mathscr{S}_w(\mathcal{S}_f) = \sum_{j=0}^{n} \sum_{|m|=j} \lambda_{j,m} \ \mathcal{S}_{\varphi_w^m} \mathcal{S}_f \mathcal{S}_{\overline{\varphi_w}^m} = \sum_{j=0}^{n} \sum_{|m|=j} \lambda_{j,m} \ \mathcal{S}_{|\varphi_w^m|^2 f} = \mathcal{S}_\Psi,$$

with $\Psi = f \sum_{j=0}^{n} \sum_{|m|=j} \lambda_{j,m} |\varphi_w^m|^2$.

Now, applying Formula (3.3) to $\varphi_w(z)$ with $z \in \mathbb{S}_n$, and invoking Identity (3.5), we see that

$$\sum_{j=0}^{n} \sum_{|m|=j} \lambda_{j,m} |\varphi_w^m(z)|^2 = (1 - \langle \varphi_w(z), \varphi_w(z) \rangle)^n = 0.$$

Therefore, we see that $\mathscr{S}_w(\mathcal{S}_f) = 0$. □

Remark 3.3. *Theorem* 3.2 *can be restated as follows: if T is a dual Toeplitz operator, then* $T \in \bigcap_{w\in\mathbb{D}} \ker \mathscr{S}_w$.

It seems to be interesting to see whether the following complete characterization is valid: given $w \in \mathbb{D}$ (fixed), T is a dual Toeplitz operator if and only if $\mathscr{S}_w(T) = 0$?

The following novel assertion plays a central role in the sequel:

Theorem 3.4. *Let T be a compact operator on $\big(\mathcal{H}^2(\mathbb{S}_n)\big)^\perp$, then* $\|\mathscr{S}_w(T)\| \longrightarrow 0$ *as* $|w| \to 1^-$.

Proof. First, we claim that the operator $\mathscr{S}_w$ admits the following representation:

$$\mathscr{S}_w(T) = \sum_{j=0}^{n-1} \sum_{|m|=j} \lambda_{j,m}^{(n-1)} \mathcal{S}_{\varphi_w^m} \left(T - \sum_{i=1}^{n} \mathcal{S}_{\varphi_{w,i}} T \mathcal{S}_{\overline{\varphi_{w,i}}} \right) \mathcal{S}_{\overline{\varphi_w}^m}, \tag{3.10}$$

where $\varphi_{w,i}$ denotes the ith component of φ_w and $\lambda_{j,m}^{(n-1)} = \dfrac{(-1)^j (n-1)!}{(n-1-j)!m!}$.

Indeed, setting $\alpha_i = (\underbrace{0, 0, \ldots, 1}_{i\text{th component}}, 0, \ldots, 0)$ for $1 \le i \le n$, we have

$$\begin{aligned}
&\sum_{j=0}^{n-1} \sum_{|m|=j} \lambda_{j,m}^{(n-1)} \mathcal{S}_{\varphi_w^m} \left(T - \sum_{i=1}^{n} \mathcal{S}_{\varphi_{w,i}} T \mathcal{S}_{\overline{\varphi_{w,i}}} \right) \mathcal{S}_{\overline{\varphi_w}^m} \\
&= \sum_{j=0}^{n-1} \sum_{|m|=j} \lambda_{j,m}^{(n-1)} \mathcal{S}_{\varphi_w^m} T \mathcal{S}_{\overline{\varphi_w}^m} - \sum_{j=0}^{n-1} \sum_{|m|=j} \lambda_{j,m}^{(n-1)} \sum_{i=1}^{n} \mathcal{S}_{\varphi_w^{m+\alpha_i}} T \mathcal{S}_{\overline{\varphi_w}^{m+\alpha_i}} \\
&= \sum_{j=0}^{n-1} \sum_{|m|=j} \lambda_{j,m}^{(n-1)} \mathcal{S}_{\varphi_w^m} T \mathcal{S}_{\overline{\varphi_w}^m} - \sum_{k=1}^{n} \sum_{|p|=k} |p| \lambda_{k-1,p}^{(n-1)} \mathcal{S}_{\varphi_w^p} T \mathcal{S}_{\overline{\varphi_w}^p} \\
&= T + \sum_{j=1}^{n-1} \sum_{|m|=j} \left(\lambda_{j,m}^{(n-1)} - |m| \lambda_{j-1,m}^{(n-1)} \right) \mathcal{S}_{\varphi_w^m} T \mathcal{S}_{\overline{\varphi_w}^m} - \sum_{|m|=n} n \lambda_{n-1,m}^{(n-1)} \mathcal{S}_{\varphi_w^m} T \mathcal{S}_{\overline{\varphi_w}^m} \\
&= T + \sum_{j=1}^{n-1} \sum_{|m|=j} \lambda_{j,m} \mathcal{S}_{\varphi_w^m} T \mathcal{S}_{\overline{\varphi_w}^m} + \sum_{|m|=n} \lambda_{n,m} \mathcal{S}_{\varphi_w^m} T \mathcal{S}_{\overline{\varphi_w}^m} \\
&= \sum_{j=0}^{n} \sum_{|m|=j} \lambda_{j,m} \mathcal{S}_{\varphi_w^m} T \mathcal{S}_{\overline{\varphi_w}^m} = \mathscr{S}_w(T).
\end{aligned}$$

Next, using identity (3.10), we only need to verify that

$$\left\| T - \sum_{i=1}^{n} \mathcal{S}_{\varphi_{w,i}} T \mathcal{S}_{\overline{\varphi_{w,i}}} \right\| \longrightarrow 0 \quad \text{as} \quad |w| \longrightarrow 1^-. \tag{3.11}$$

Owing to the density of finite rank operators in the set of compact operators, we only need to verify the latter for rank one operators. For let $f, g \in \big(\mathcal{H}^2(\mathbb{S}_n)\big)^\perp$;

then one has

$$\left\|f\otimes g-\sum_{i=1}^{n}\mathcal{S}_{\varphi_{w,i}}(f\otimes g)\mathcal{S}_{\overline{\varphi_{w,i}}}\right\|=\left\|\sum_{i=1}^{n}\left\{(\zeta_i f)\otimes(\zeta_i g)-\left(\mathcal{S}_{\varphi_{w,i}}f\right)\otimes\left(\mathcal{S}_{\varphi_{w,i}}g\right)\right\}\right\|$$
$$\leq\sum_{i=1}^{n}\left\{\left\|\left(\zeta_i f-\mathcal{S}_{\varphi_{w,i}}f\right)\otimes(\zeta_i g)\right\|+\left\|\left(\mathcal{S}_{\varphi_{w,i}}f\right)\otimes\left(\zeta_i g-\mathcal{S}_{\varphi_{w,i}}g\right)\right\|\right\}. \qquad (3.12)$$

Now, for $z\in\mathbb{S}_n$ and $w\in\mathbb{B}_n$, observe that $w-\varphi_w(z)\longrightarrow 0$ *a.e.* as $|w|\to 1^-$; and thus componentwise, for $i=1,2,3,\ldots,n$, we have $w_i-\varphi_{w,i}(z)\longrightarrow 0$ *a.e.* as $|w|\to 1^-$. Making appeal to the dominated convergence theorem, we infer that for $f\in\left(\mathcal{H}^2(\mathbb{S}_n)\right)^{\perp}$ one has

$$\|w_i f-\varphi_{w,i}f\|_2^2=\int_{\mathbb{S}_n}|w_i f(z)-\varphi_{w,i}(z)f(z)|^2\,d\sigma(z)\longrightarrow 0 \text{ as } |w|\longrightarrow 1^-.$$

Hence, for $i=1,2,3,\ldots,n$, we see that $\|\zeta_i f-\varphi_{w,i}f\|_2\longrightarrow 0$ as $\mathbb{B}_n\ni w\longrightarrow\zeta\in\mathbb{S}_n$. Because of the identity $(I-\mathcal{P})(\zeta_i f(z))=\zeta_i f(z)$, we see that

$$\left\|\zeta_i f-\mathcal{S}_{\varphi_{w,i}}f\right\|_2=\|(I-\mathcal{P})\left(\zeta_i f-\varphi_{w,i}f\right)\|_2\longrightarrow 0 \text{ as } \mathbb{B}_n\ni w\longrightarrow\zeta\in\mathbb{S}_n.$$

The latter together with Inequality (3.12) yield:

$$\left\|f\otimes g-\sum_{i=1}^{n}\mathcal{S}_{\varphi_{w,i}}(f\otimes g)\mathcal{S}_{\overline{\varphi_{w,i}}}\right\|\longrightarrow 0 \text{ as } \mathbb{B}_n\ni w\longrightarrow\zeta\in\mathbb{S}_n.$$ □

4. Products of dual Toeplitz operators

Lemma 2.1 suggests that $\mathcal{S}_f$ and $\mathcal{S}_g$ commute if f and g are both analytic or conjugate analytic. If a non-trivial linear combination of the symbols f and g is constant, they do commute as well. In this section, we are interested to see whether these are the only cases where commutativity holds. The same question in related settings has been considered for instance in [2, 11]. An answer to this question [5] is reported again in the following:

Theorem 4.1. *Suppose that φ,ψ are bounded functions on the unit sphere $\mathbb{S}^n$. Then, the corresponding dual Toeplitz operators commute on $\left(\mathcal{H}^2(\mathbb{S}_n)\right)^{\perp}$, (i.e., $\mathcal{S}_\varphi\mathcal{S}_\psi=\mathcal{S}_\psi\mathcal{S}_\varphi$), if and only if φ and ψ satisfy one of the following conditions:*

1. *They are both analytic on $\mathbb{S}_n$.*
2. *They are both co-analytic on $\mathbb{S}_n$.*
3. *One of them is constant on $\mathbb{S}_n$.*
4. *A non-trivial linear combination of them is constant on $\mathbb{S}_n$.*

Proof. The if part is trivial due to Lemma 2.1. Regarding the only if part, observe that by Proposition 3.1 and parts (i) and (ii) of Lemma 2.1 one has

$$H_f(k_w\otimes k_w)H_{\bar{g}}^*=\sum_{j=0}^{n}\sum_{|m|=j}\lambda_{j,m}\left(\mathcal{S}_{\varphi_w^m}H_f\right)\left(H_{\bar{g}}^*\mathcal{S}_{\overline{\varphi_w}^m}\right)=\mathscr{S}_w(H_f H_{\bar{g}}^*). \qquad (4.1)$$

Similarly, we have

$$H_g(k_w \otimes k_w)H^*_{\overline{f}} = \mathscr{S}_w(H_g H^*_{\overline{f}}). \tag{4.2}$$

Combining the two latter identities and owing again to Identity (3.6) as well as to Equation (2.2), we see that

$$(H_g k_w) \otimes (H_{\overline{f}} k_w) - (H_f k_w) \otimes (H_{\overline{g}} k_w) = \mathscr{S}_w([\mathcal{S}_f, \mathcal{S}_g]).$$

By assumption, we get

$$(H_g k_w) \otimes (H_{\overline{f}} k_w) = (H_f k_w) \otimes (H_{\overline{g}} k_w), \forall w \in \mathbb{B}_n.$$

In particular, for $w = 0$ one has $k_0 = 1$; whence $H_g 1 \otimes H_{\overline{f}} 1 = H_f 1 \otimes H_{\overline{g}} 1$, which can be rewritten as

$$\left\langle h, H_{\overline{f}} 1 \right\rangle H_g 1 = \langle h, H_{\overline{g}} 1\rangle H_f 1, \forall h \in \left(\mathcal{H}^2(\mathbb{B}_n)\right)^{\perp}.$$

At this stage, we distinguish several cases:

1) If $H_g 1 = 0$, then g is analytic. Also we must have either $H_f 1 = 0$ or $H_{\overline{g}} 1 = 0$, which means that either f is analytic, (which corresponds to condition (1)), or g is co-analytic, (in this case g must be constant, which corresponds to condition (3)).

2) If $H_{\overline{g}} 1 = 0$, then g is co-analytic. Also we see that either $H_g 1 = 0$ or $H_{\overline{f}} 1 = 0$. This means that either g is analytic, (which implies that g is constant and corresponds to condition (3)), or f is co-analytic, (which agrees with condition (2)).

3) If both $H_g 1 \neq 0$ and $H_{\overline{g}} 1 \neq 0$, then there exists a complex number $\lambda \neq 0$ such that $H_f 1 = \lambda H_g 1$ and $H_{\overline{f}} 1 = \overline{\lambda} H_{\overline{g}} 1$. That is to say $\mathcal{Q}(f - \lambda g) = \mathcal{Q}(\overline{f} - \overline{\lambda}\overline{g}) = 0$; whence $f - \lambda g$ and $\overline{f} - \overline{\lambda}\overline{g}$ are both analytic. Thus $f - \lambda g$ is constant, which corresponds to condition (4). □

Products of bounded dual Toeplitz operators can be bounded operators in numerous cases. But the crucial question is when does the product of two dual Toeplitz operators produce a dual Toeplitz operator? The answer of this question [5] is given in the following Brown-Halmos type theorem:

Theorem 4.2. *Let f and g be in $L^\infty(\mathbb{S}_n)$. Then, the dual Toeplitz product $\mathcal{S}_f \mathcal{S}_g$ is again a dual Toeplitz operator if and only if one of the following conditions holds:*

1. *f is analytic.*
2. *g is co-analytic.*

In either cases $\mathcal{S}_f \mathcal{S}_g = \mathcal{S}_{fg}$.

Proof. From the elementary properties of dual Toeplitz operators, namely Lemma 2.1, the "if part" is obvious, whereas the "only if part" is less trivial. For, suppose that $\mathcal{S}_f \mathcal{S}_g = \mathcal{S}_h$ for some $h \in L^\infty(\mathbb{S}_n)$. From Identity (2.1), we have

$$0 = \mathcal{S}_h - \mathcal{S}_f \mathcal{S}_g = \mathcal{S}_{h-fg} + H_f H^*_{\overline{g}}.$$

Introducing the operator $\mathscr{S}_w$, we see from Relation (4.1) that

$$\mathscr{S}_w(\mathcal{S}_{fg-h}) = \mathscr{S}_w\left(H_f H_{\overline{g}}^*\right) = H_f(k_w \otimes k_w)H_{\overline{g}}^*. \tag{4.3}$$

Since S_{fg-h} is a dual Toeplitz operator, Proposition 3.2 reduces Equation (4.3) to

$$(H_f(k_w)) \otimes (H_{\overline{g}}(k_w)) = 0.$$

In particular, if $w = 0$ one gets $k_0 = 1$; whence we obtain

$$(H_f 1) \otimes (H_{\overline{g}}^* 1) = 0.$$

Since $\|H_f 1 \otimes H_{\overline{g}} 1\| = \|H_f 1\| \, \|H_{\overline{g}} 1\|$, we see that at least one of the two factors vanishes. Therefore, we have two possibilities

- If $H_f 1 = 0$, we see that f is analytic, (which corresponds to condition (1)).
- If $H_{\overline{g}} 1 = 0$, then $\overline{g}$ is analytic, whence g is co-analytic, (which corresponds to (2)).

The additional conclusion of the theorem, then, follows from Lemma 2.1. □

The so-called zero product problem is then a simple corollary of the latter:

Corollary 4.3. $\mathcal{S}_f \mathcal{S}_g = 0$ *if and only if either* $f = 0$ *or* $g = 0$*; i.e., among the class of dual Toeplitz operators on* $\left(\mathcal{H}^2(\mathbb{B}_n)\right)^{\perp}$ *there are no zero divisors.*

5. Haplitz products

Based on the above concepts, (namely the operator $\mathscr{S}_w$ defined by (3.9), Proposition 3.1 and Theorem 3.4), we now discuss certain characterizations of boundedness and compactness of Hankel products $H_f H_g^*$ as well as of mixed Haplitz products $T_f H_g^*$ and $H_g T_f$ on the sphere. Notice that Toeplitz products on the circle have been studied by Zheng [14], whereas Hankel and mixed Haplitz products have been discussed by Hamada [6]. In case of several complex variables, analog investigations have been done by Zheng [13], Nie [9], Xia [12], Le [7] and Lu & Shang [8].

The following theorem gives a necessary condition for the boundedness of a Hankel product $H_f H_g^*$:

Theorem 5.1. *Let* f *and* g *be in* $L^2(\mathbb{S}_n)$*. If the Hankel product* $H_f H_g^*$ *is bounded, then*

$$\sup_{w\in\mathbb{B}_n} \|f\circ\varphi_w - \mathcal{P}(f\circ\varphi_w)\|_2 \, \|g\circ\varphi_w - \mathcal{P}(g\circ\varphi_w)\|_2 < \infty. \tag{5.1}$$

Proof. According to Zheng [13], we have

$$\|H_f k_w\|_2 \, \|H_g k_w\|_2 = \|f\circ\varphi_w - \mathcal{P}(f\circ\varphi_w)\|_2 \, \|g\circ\varphi_w - \mathcal{P}(g\circ\varphi_w)\|_2 \,. \tag{5.2}$$

On the other hand, by the above norm formula of rank one operators and Equation (3.6), we have

$$\|H_f k_w\|_2 \, \|H_g k_w\|_2 = \|(H_f k_w) \otimes (H_g k_w)\| = \left\|H_f(k_w \otimes k_w)H_g^*\right\|. \tag{5.3}$$

So, it suffices that the R.H.S. of the latter is bounded.

Since $\varphi_w \in \mathcal{H}^\infty(\mathbb{S}_n)$, we see by Lemma 2.1 that $H_f T_{\varphi_w} = \mathcal{S}_{\varphi_w} H_f$ and $T_{\overline{\varphi_w}} H_g^* = H_g^* \mathcal{S}_{\overline{\varphi_w}}$. Thus, inserting H_f and H_g^* into Formula (3.7), we see that

$$\begin{aligned} H_f(k_w \otimes k_w)H_g^* &= \sum_{j=0}^{n} \sum_{|m|=j} \lambda_{j,m} H_f T_{\varphi_w^m} T_{\overline{\varphi_w}^m} H_g^* \\ &= \sum_{j=0}^{n} \sum_{|m|=j} \lambda_{j,m} \mathcal{S}_{\varphi_w^m} \left(H_f H_g^*\right) \mathcal{S}_{\overline{\varphi_w}^m} = \mathscr{S}_w \left(H_f H_g^*\right). \end{aligned} \tag{5.4}$$

By Proposition 2.2, we have $\left\|\mathcal{S}_{\varphi_w^m}\right\| = \|\mathcal{S}_{\overline{\varphi_w}^m}\| = \|\varphi_w^m\|_\infty \le 1$. Thus, we infer that

$$\begin{aligned} \left\|H_f(k_w \otimes k_w)H_g^*\right\| &\le \sum_{j=0}^{n} \sum_{|m|=j} |\lambda_{j,m}| \left\|\mathcal{S}_{\varphi_w^m}\right\| \left\|H_f H_g^*\right\| \|\mathcal{S}_{\overline{\varphi_w}^m}\| \\ &\le \left\|H_f H_g^*\right\| \sum_{j=0}^{n} \sum_{|m|=j} |\lambda_{j,m}| < \infty; \end{aligned} \tag{5.5}$$

whence, the theorem is proved. □

The following result gives a necessary condition for the compactness of a Hankel product $H_f H_g^*$. Notice that the compactness matter in the "dual case" of $H_f^* H_g$ has been considered by J. Xia [12]. In that paper, J. Xia proves that Condition (5.6) fails to be necessary for the compactness of $H_f^* H_g$. Later on, T. Le [7] provided a certain progress in this direction.

Theorem 5.2. *Let f and g be in $L^2(\mathbb{S}_n)$. If the Hankel product $H_f H_g^*$ is compact, then*

$$\lim_{w\to\mathbb{S}_n} \|f\circ\varphi_w - \mathcal{P}(f\circ\varphi_w)\|_2 \, \|g\circ\varphi_w - \mathcal{P}(g\circ\varphi_w)\|_2 = 0. \tag{5.6}$$

Proof. By Equations (5.2), (5.3) and (5.4), we see that

$$\|f\circ\varphi_w - \mathcal{P}(f\circ\varphi_w)\|_2 \, \|g\circ\varphi_w - \mathcal{P}(g\circ\varphi_w)\|_2 = \left\|\mathscr{S}_w\left(H_f H_g^*\right)\right\|. \tag{5.7}$$

Consequently, if $H_f H_g^*$ is compact, we see by Theorem 3.4 that

$$\lim_{w\to\mathbb{S}_n} \left\|\mathscr{S}_w\left(H_f H_g^*\right)\right\| = 0,$$

and the claimed assertion follows. □

Similar characterizations of bounded and compact mixed Haplitz products $T_f H_g^*$ and $H_g T_f$ are given as follows:

Theorem 5.3. *Let f be in $\mathcal{H}^2(\mathbb{S}_n)$ and g be in $L^2(\mathbb{S}_n)$. If one of the mixed Haplitz products $T_f H_g^*$ or $H_g T_{\overline{f}}$ is bounded, then*

$$\sup_{w\in\mathbb{B}_n} \|f\circ\varphi_w\|_2 \, \|g\circ\varphi_w - \mathcal{P}(g\circ\varphi_w)\|_2 < \infty.$$

Proof. Relying on the fact that $\varphi_w \in \mathcal{H}^\infty(\mathbb{S}_n)$ and owing to the analyticity of f, we see by Lemma 2.1 that $T_f T_{\varphi_w} = T_{\varphi_w} T_f$ and $T_{\overline{\varphi_w}} H_g^* = H_g^* \mathcal{S}_{\overline{\varphi_w}}$. Thus, as in the proof of Theorem 5.1, inserting T_f and H_g^* into Formula (3.7), we see that

$$T_f(k_w \otimes k_w)H_g^* = \sum_{j=0}^{n} \sum_{|m|=j} \lambda_{j,m} T_f T_{\varphi_w^m} T_{\overline{\varphi_w}^m} H_g^* = \sum_{j=0}^{n} \sum_{|m|=j} \lambda_{j,m} T_{\varphi_w^m} \left(T_f H_g^*\right) \mathcal{S}_{\overline{\varphi_w}^m}. \tag{5.8}$$

Estimating the norms of Toeplitz and dual Toeplitz operators with automorphic symbols, we get $\left\|T_{\varphi_w^m}\right\| \leq 1$ and $\|\mathcal{S}_{\overline{\varphi_w}^m}\| \leq 1$. Thus, if $T_f H_g^*$ is bounded, we infer that

$$\left\|T_f(k_w \otimes k_w)H_g^*\right\| \leq \left\|T_f H_g^*\right\| \sum_{j=0}^{n} \sum_{|m|=j} |\lambda_{j,m}| < \infty. \tag{5.9}$$

Whence, as in Equations (5.2) and (5.3), we obtain the claimed estimate. Similar argument can be used to handel the remaining case. □

Compact mixed Haplitz products can also be characterized similarly:

Theorem 5.4. *Let $f \in \mathcal{H}^\infty(\mathbb{S}_n)$ and $g \in L^2(\mathbb{S}_n)$. If one of the mixed Haplitz products $T_f H_g^*$ or $H_g T_{\overline{f}}$ is compact, then*

$$\lim_{w \to \mathbb{S}_n} \|f \circ \varphi_w\|_2 \, \|g \circ \varphi_w - \mathcal{P}(g \circ \varphi_w)\|_2 = 0.$$

Proof. As in the proof of Theorem 3.4, for any operator $T : \left(\mathcal{H}^2(\mathbb{S}_n)\right)^\perp \longrightarrow \mathcal{H}^2(\mathbb{S}_n)$, we have

$$\sum_{j=0}^{n} \sum_{|m|=j} \lambda_{j,m} T_{\varphi_w^m} T \mathcal{S}_{\overline{\varphi_w}^m} = \sum_{j=0}^{n-1} \sum_{|m|=j} \lambda_{j,m}^{(n-1)} T_{\varphi_w^m} \left(T - \sum_{i=1}^{n} T_{\varphi_{w,i}} T \mathcal{S}_{\overline{\varphi_{w,i}}} \right) \mathcal{S}_{\overline{\varphi_w}^m}. \tag{5.10}$$

We claim that if such a T is compact, then

$$\lim_{|w| \longrightarrow 1^-} \sum_{j=0}^{n} \sum_{|m|=j} \lambda_{j,m} T_{\varphi_w^m} T \mathcal{S}_{\overline{\varphi_w}^m} = 0. \tag{5.11}$$

By Identity (5.10), we only need to verify that

$$\left\| T - \sum_{i=1}^{n} T_{\varphi_{w,i}} T \mathcal{S}_{\overline{\varphi_{w,i}}} \right\| \longrightarrow 0 \text{ as } |w| \longrightarrow 1^-. \tag{5.12}$$

Using the density of finite rank operators in the set of compact operators, we only need to verify the latter for rank one operators acting from $\left(\mathcal{H}^2(\mathbb{S}_n)\right)^\perp$ into $\mathcal{H}^2(\mathbb{S}_n)$. For let $f \in \mathcal{H}^2(\mathbb{S}_n)$ and $g \in \left(\mathcal{H}^2(\mathbb{S}_n)\right)^\perp$. Then, one has

$$\begin{aligned} &\left\| f \otimes g - \sum_{i=1}^{n} T_{\varphi_{w,i}} (f \otimes g) \mathcal{S}_{\overline{\varphi_{w,i}}} \right\| \\ &\leq \sum_{i=1}^{n} \left\{ \left\| \left(\zeta_i f - T_{\varphi_{w,i}} f\right) \otimes (\zeta_i g) \right\| + \left\| \left(T_{\varphi_{w,i}} f\right) \otimes \left(\zeta_i g - \mathcal{S}_{\varphi_{w,i}} g\right) \right\| \right\}. \end{aligned} \tag{5.13}$$

Now, for $z \in \mathbb{S}_n$ and $w \in \mathbb{B}_n$, observe that $w - \varphi_w(z) \longrightarrow 0$ *a.e.* as $|w| \to 1^-$; and thus componentwise, for $i = 1, 2, 3, \ldots, n$, we have $w_i - \varphi_{w,i}(z) \longrightarrow 0$ *a.e.* as $|w| \to 1^-$. Making use of the dominated convergence theorem, we infer that for

$f \in \mathcal{H}^2(\mathbb{S}_n)$ and $g \in \left(\mathcal{H}^2(\mathbb{S}_n)\right)^{\perp}$ one has

$$\|w_i f - \varphi_{w,i} f\|_2^2 = \int\limits_{\mathbb{S}_n} |w_i f(z) - \varphi_{w,i}(z) f(z)|^2 \, d\sigma(z) \longrightarrow 0 \;\text{ as }\; |w| \longrightarrow 1^-,$$

and

$$\|w_i g - \varphi_{w,i} g\|_2^2 = \int\limits_{\mathbb{S}_n} |w_i g(z) - \varphi_{w,i}(z) g(z)|^2 \, d\sigma(z) \longrightarrow 0 \;\text{ as }\; |w| \longrightarrow 1^-.$$

Hence, for $i = 1, 2, 3, \ldots, n$, we see that

$$\|\zeta_i f - \varphi_{w,i} f\|_2 \longrightarrow 0 \quad \text{and} \quad \|\zeta_i g - \varphi_{w,i} g\|_2 \longrightarrow 0$$

as $\mathbb{B}_n \ni w \longrightarrow \zeta \in \mathbb{S}_n$. Because of the identities $\mathcal{P}(\zeta_i f(z)) = \zeta_i f(z)$ and $(I - \mathcal{P})(\zeta_i g(z)) = \zeta_i g(z)$, we see that

$$\left\|\zeta_i f - T_{\varphi_{w,i}} f\right\|_2 = \|\mathcal{P}\left(\zeta_i f - \varphi_{w,i} f\right)\|_2 \longrightarrow 0 \;\text{ as }\; \mathbb{B}_n \ni w \longrightarrow \zeta \in \mathbb{S}_n,$$

and

$$\left\|\zeta_i g - \mathcal{S}_{\varphi_{w,i}} g\right\|_2 = \|(I - \mathcal{P})\left(\zeta_i g - \varphi_{w,i} g\right)\|_2 \longrightarrow 0 \;\text{ as }\; \mathbb{B}_n \ni w \longrightarrow \zeta \in \mathbb{S}_n.$$

Combining the latter two limits together with Inequality (5.13), we infer that

$$\left\| f \otimes g - \sum_{i=1}^{n} T_{\varphi_{w,i}} (f \otimes g) \mathcal{S}_{\overline{\varphi_{w,i}}} \right\| \longrightarrow 0 \;\text{ as }\; \mathbb{B}_n \ni w \longrightarrow \zeta \in \mathbb{S}_n;$$

which proves (5.11).

Next, suppose for instance that $T_f H_g^*$ is compact, (the other case related to $H_g T_{\overline{f}}$, can be handled similarly), then by (5.8) and (5.11), we see that

$$\left\|T_f (k_w \otimes k_w) H_g^*\right\| \longrightarrow 0 \;\text{ as }\; \mathbb{B}_n \ni w \longrightarrow \zeta \in \mathbb{S}_n.$$

Thus, as in Equations (5.2) and (5.3), we obtain the claimed condition. □

Owing to the alternative representation (2.2) of the commutator of two dual Toeplitz operators, we can characterize its compactness:

Theorem 5.5. *Let f and g be bounded measurable on $\mathbb{S}_n$. If the commutator $[\, \mathcal{S}_f, \; \mathcal{S}_g]$ is compact, then*

$$\left\|(H_g k_w) \otimes \left(H_{\overline{f}} k_w\right) - (H_f k_w) \otimes (H_{\overline{g}} k_w)\right\| \longrightarrow 0 \quad \textit{as} \quad |w| \to 1^-.$$

Proof. Making use of Formulas (2.2) and (5.4), we obtain:

$$\mathscr{S}_w\left([\,\mathcal{S}_f,\ \mathcal{S}_g]\right) = \left(H_g k_w\right) \otimes \left(H_{\overline{f}} k_w\right) - \left(H_f k_w\right) \otimes \left(H_{\overline{g}} k_w\right).$$

So, if the commutator is compact, then the result follows from Theorem 3.4. □

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