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Wiener–Hopf Type Operators and Their Generalized Determinants

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Abstract. We recall some results on generalized determinants which support a theory of operator τ-functions in the context of their predeterminants which are operators valued in a Banach–Lie group that are derived from the transition maps of certain Banach bundles. Related to this study is a class of Banach–Lie algebras known as L*-algebras from which several results are obtained in relationship to tau functions. We survey the applicability of this theory to that of Schlesinger systems associated with (operator) equations of Fuschsian type and discuss how meromorphic connections may play a role here.

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1. Introduction

This contribution is in part based on my talk at IWOTA 2011 in Seville (Spain). At first glances it has the look of a survey bringing together some known results under one roof, but then it unfolds to a more general perspective, and eventually suggests some new directions via applications.

After delving into the background to generalized determinants and operator-valued meromorphic functions, it seemed fitting for these Proceedings to acknowledge the important work in this area that was accomplished by Professor Gohberg along with several of his coworkers towards the development of some foundational concepts which enter into part of the survey here (as realized in [28, 29, 30, 31], for instance). We expect that the fruits of his profound mathematical insight will continue to influence many research projects in the years to come.

The scene is set by recalling some earlier work regarding the existence of determinants in the Banach algebra category, along with a class of operators belonging to a Banach–Lie group for which the concept of determinant can be defined. This is developed in the context of transition maps of a bundle theory over a class of infinite-dimensional manifolds, as was the case in [23]. The resulting operators, in a certain sense, can be viewed as *generalized Wiener–Hopf operators*, and these turn out to be a shade more general than the meaning that can be found in the current literature (see, e.g., [16, 35]). It is mainly this class of operators that encapsulates several of the 'predeterminants' that are probed into.

One principal theme deals with a particular class of generalized determinant operators giving rise to an assortment of τ-*functions*, the background to which is discussed in §4.1. This subject is motivated from several sources such as [36, 37, 59, 63] in the Grassmannian setting which includes flows on invariant subspaces [35], and the close relationship with the Painlevé equations (see Appendix B). Familiar examples arise from Toeplitz and Fredholm determinants in the case where the algebra is $\mathcal{L}(H)$ (H a Hilbert space), in which case, studying the various classes of integral operators and their corresponding determinants seems to be relevant here. In this respect it is worth mentioning several ideas that were previously introduced in [22, 23, 24, 25] connecting to the theory of integrable systems (for instance, involving Lax Pairs and the KP-Hierarchy) with operator theory. Further, we will recall, from, e.g., [5], the class of Banach–Lie algebras known as (simple) L*-algebras, which along with Kac–Moody algebras can be interwoven into this study. Some new observations in this direction are obtained in the form of Propositions 6.1, 6.2, and 6.3.

There is already a significant amount of work that links the τ-function theory to *Schlesinger systems* in the framework of the Riemann–Hilbert problem and isomonodromic deformations (see [3, 9, 11, 36, 37] and references therein). The present approach, as taken in an infinite-dimensional vector bundle (with connection) setting, suggests something more general since we introduce and apply certain operator-valued mechanisms. Of interest are (closed) 1-forms of the type $d \ln \tau$. Partial motivation for doing this is suggested by the work of Katsnelson and Volok [39] who considered this problem from the point of view of matrix-operator differential equations of Fuchsian type along with their associated Schlesinger systems. The instrumentation of generalized determinants and meromorphic operator-valued functions is one such example, and here some attention is paid to the idea of an *operator meromorphic connection* besides suggesting several examples where this can be realized (§7.2).

2. Background to the geometry

2.1. A principal bundle and its transition map

We will start by outlining a general construction from which a large class of interesting and well-studied operators can be obtained directly from the transition functions of an infinite-dimensional bundle theory. A class of these operators will in fact produce the 'predeterminants' for several types of operator-valued functions of a determinant type that we keep in mind.

Let A be a (complex, associative) unital Banach algebra with group of units $G(A)$ and space of idempotents $P(A)$ (in some cases A may be semisimple, and this is assumed if needs be). For a given $p \in P(A)$, we denote by $\Lambda = \mathrm{Sim}(p, A)$ the similarity orbit of p under the inner automorphic action of $G(A)$. There exists a natural map [23, §5]

$$\pi_\Lambda : \Lambda \longrightarrow \mathrm{Gr}(p, A), \tag{2.1}$$

where $\mathrm{Gr}(p, A)$ is an associated Grassmannian of closed subspaces $W = \mathrm{Im}(p)$ (see [21, §6]). In the following we shall be considering certain Banach–Lie subgroups (subalgebras) of $G(A)$ (respectively, of $\mathfrak{g}(A)$).

Remark 2.1. For the standard theory of Banach algebras and associated classes of linear operators we refer to [18, 29]. For the general theory of Banach–Lie groups (algebras) and the infinite-dimensional manifolds modeled on these (such as $\mathrm{Gr}(p, A)$ above) reference [5](cf. [20]) provides a comprehensive account from the operator algebra perspective including many references to the related work of other authors, while much of the development of the relationships between the Banach manifolds $\mathrm{Gr}(p, A)$ and Λ appeared in [21, 23]. As far as parts of this preliminary section is concerned there is a significant amount of related work that has been progressively developed in [5, 6, 7] pertaining to the theory of holomorphic vector bundles over infinite-dimensional flag manifolds.

The detailed framework outlined in [23, §5 and §6] produces a principal $G(pAp)$-bundle with connection

$$(Q', \omega_{Q'}) \longrightarrow \Lambda, \tag{2.2}$$

and an associated vector bundle (with Koszul connection) $(\gamma'_\Lambda, \nabla'_\Lambda) \longrightarrow \Lambda$, whose structure group is $G(pAp)$. This associated vector bundle is constructed via the usual means (cf. [43, Chap. 37]). There is the transition map

$$t_\Lambda : Q' \times_\Lambda Q' \longrightarrow G(pAp), \tag{2.3}$$

which for a pair of sections α, β, is given by $\alpha t_\Lambda(\alpha, \beta) = \beta$. For now we take $A = \mathcal{L}(E)$ where E is a (complex) Banach space, and then observe that the operator $\mathsf{T}_{(\alpha,\beta)} = t_\Lambda(\alpha, \beta) \in G(pAp)$ belongs to some class (to be made more precise later).

Remark 2.2. It will be instructive to point out that $\omega_{Q'}$ in (2.2) is constructed via the properties of a $\mathrm{Lie}(G(pAp))$-valued connection map $\mathcal{V} : TQ' \longrightarrow TQ'$ (see [23, §5.2]) to be used within the meromorphic context later in §7.2.

2.2. Reduction of the structure group

Suppose $B \subseteq A$ is a Banach subalgebra and $G \subseteq G(pBp) \subseteq G(pAp)$ is a Banach–Lie subgroup. Then in the standard way, granting the existence of a cross section $\Lambda \longrightarrow Q'/G$, we obtain from (2.2), a (reduced) principal G-bundle $Q \longrightarrow \Lambda$. The means of doing this is formally the same as seen in, e.g., [41, Propositions 5.5, 5.6] and [43, p. 381]. We also assume, that under appropriate conditions (cf. [43, p. 381] and [41, Theorem 7.1]), that the connection $\omega_{Q'}$ has been reduced accordingly. This can be achieved by commencing from a Banach–Lie group homomorphism $\psi : G \longrightarrow G(pAp)$, and a principal bundle homomorphism

$$\Psi : (Q, G, \Lambda) \longrightarrow (Q', G(pAp), \Lambda), \tag{2.4}$$

such that $\Psi : Q \longrightarrow Q'$ is smooth, and $\Psi(u \cdot g) = \Psi(u) \cdot \psi(g)$ (see [43, p. 381]). Hence we obtain a commutative diagram

$$\begin{array}{ccc} Q & \xrightarrow{\Psi} & Q' \\ \downarrow & & \downarrow \\ \Lambda & \xrightarrow{\widetilde{\Psi}} & \Lambda \end{array} \tag{2.5}$$

A pull-back connection ω_Q on Q is then obtained as $\omega_Q = \Psi^* \omega_{Q'}$, and so leads to a principal G-bundle with connection $(Q, \omega_Q) \longrightarrow \Lambda$, along with its related objects. In particular, these likewise include a G-valued transition map

$$t_\Lambda : Q \times_\Lambda Q \longrightarrow G, \tag{2.6}$$

and an associated vector bundle (with Koszul connection) $(\gamma, \nabla_\gamma) \longrightarrow \Lambda$, whose structure group is G.

3. On generalized determinants in the Banach algebra setting

3.1. Two approaches for generalized determinants

Proceeding with $A = \mathcal{L}(E)$, let us recall the notion of the *socle* of A, denoted $\mathrm{soc}(A)$. This consists of the sum of all minimal left ideals (or right ideals) if they exist, or else it is zero. For the situation in question we follow [2, §2] and take $\mathrm{soc}(A)$ to be generated by the minimal projections of A, that is, elements $p \in P(A)$ such that $pAp = \mathbb{C}p$ (meaning that the restriction of elements of A to $\mathrm{Im}(p)$ is the identity on $\mathrm{Im}(p)$). Also, given $\mathcal{L}(E)$ contains finite rank operators, we have $\mathrm{soc}(A) \neq 0$. For $a \in A$, the spectral rank of a, is given by $\sup_{x \in A} \#(\mathrm{spec}(xa) - \{0\})$.

The *maximal finite rank elements* are those elements of A such that we have rank $a = \#(\mathrm{spec}(a) - \{0\})$, and these elements admit spectral representations of the form $a = \lambda_1 p_1 + \cdots + \lambda_n p_n$ (for some n, and where the p_i are minimal projections).

For arbitrary $a \in \mathrm{soc}(A)$, we have rank $a = \sum m(\lambda_i, a)$ (sum over non-zero elements) where the multiplicity $m(\lambda_i, a)$ is the rank of the Riesz projection for λ_i (see, e.g., [18]). Taking λ (below) as a sum over $\mathrm{spec}(A)$, we have the trace and determinant well-defined for $a \in \mathrm{soc}(A)$, as given by [2, §2]:

$$\begin{aligned} \mathrm{Tr}\ a &= \sum\nolimits_\lambda \lambda\ m(\lambda, a), \\ \mathrm{Det}(1 + a) &= \Pi_\lambda (1 + \lambda)^{m(\lambda, a)}. \end{aligned} \tag{3.1}$$

We refer to [2] (cf. [55]) for further consequences of these constructions.

Another approach for the case $A = \mathcal{L}(E)$ [28] involves taking the subalgebra $\mathcal{F}(E) \subset \mathcal{L}(E)$ of *operators of finite rank* and then define $\mathrm{Det}(1 + F)$ on certain normed subalgebras of E. This starts by considering certain (Banach) subalgebras $B \subset \mathcal{L}(E)$ which are embedded continuously in $\mathcal{L}(E)$, meaning that there is a norm $\|\cdot\|_B$ on B such that:

i) $\|F\|_{\mathcal{L}(E)} \leq C\|F\|_B$, for all $F \in B$, where $C = \text{const.}$, and also assume that,
ii) $\|SF\|_B \leq \|S\|_B\|F\|_B$, for all $S, F \in B$.

If i) and ii) hold then B is called *an embedded subalgebra of* $\mathcal{L}(E)$. If also we have $\mathcal{F}_B = \mathcal{F}(E) \cap B$ dense in B with respect to $\|\cdot\|_B$, then B is said to have *the approximation property.* This property assists the continuous extension of trace and determinant from $\mathcal{F}_B$ to B. If then $B \subset \mathcal{L}(E)$ is an embedded subalgebra with the approximation property, then $\mathrm{Det}(1 + F) : \mathcal{F}_B \longrightarrow \mathbb{C}$ admits a continuous extension in the B-norm from $\mathcal{F}_B$ to B (see [28, Theorem 2.1] which includes related results), and for $F \in \mathcal{F}(E)$ with $|z|$ sufficiently small [28, Theorem 3.3]:

$$\mathrm{Det}(1 + zF) = \exp\Bigg(\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \mathrm{Tr}(F^n) z^n\Bigg). \tag{3.2}$$

If $E = H$ is a Hilbert space and F is a trace class operator, then there is the usual Fredholm determinant given by

$$\mathrm{Det}(1 + F) = \sum_{n=0}^{\infty} \mathrm{Tr}\ \Lambda^n(F), \tag{3.3}$$

(see, e.g., [28, 64, 67]).

3.2. Admissible elements

From [24, §6] we have the principal $G(pAp)$-bundle (*the Stiefel bundle*) denoted $V(p, A) \longrightarrow \mathrm{Gr}(p, A)$ for which an element $v \in V(p, A)$ is manifestly a framing for the Banach algebra A, or simply a basis for its underlying (Banach) vector space. Let us then say that $v \in V(p, A)$ is *admissible* when $\mathrm{Det}(v)$ is defined in the context of a suitable generalized determinant. A particular instance concerns that of 'admissible bases' relative to polarized Hilbert spaces (modules) that can be used to produce an important class of determinant line bundles [57, §7.7] and [63, §3] (cf. [22, 23] and see §5.1 below). Accordingly, when we speak of 'determinants' we will take for granted the existence of the corresponding admissible elements as they were defined in this general criteria.

3.3. Determinants of generalized Wiener–Hopf operators

Again taking $A = \mathcal{L}(E)$ and $B \subset A$ a Banach subalgebra, let $E_1 \subset E$ be a closed subspace, and let $p : E \longrightarrow E_1$ be the projection onto E_1. For some $L \in B$, one could in principle define a *generalized Wiener–Hopf operator* T simply in terms of the relationship $\mathsf{T}_p(L) = pLp$ (see Example 3.1 below). Thus for an appropriate choice of sections, the operator $\mathsf{T}_{(\alpha,\beta)} = t_\Lambda(\alpha, \beta) \in G$ seen above, would then be such an example. Accordingly, if $\mathsf{T} \in \mathrm{soc}(A)$, or if $\mathsf{T} \in B$ where B has the approximation property, then $\mathrm{Tr}(\mathsf{T})$ and $\mathrm{Det}(1 \pm \mathsf{T})$ are well defined as we have seen in §3.1.

Along with applications, much of the work that had been available concerns taking $A = \mathcal{L}(H)$, and so let us recall some particular examples.

Example 3.1. Let us take $B \subset \mathcal{L}(H)$ to be a maximal abelian von Neumann algebra of operators and $H_1 \subset H$ a proper closed subspace of H such that any non-zero vector in H_1 or $H_1^{\perp}$ is separating for B. In [19, §2] an operator $L \in B$ is decreed to be a *generalized Laurent operator*, in which case the triple (H, B, H_1) is called a *Riesz system* for which $\mathsf{T}_p = pLp$ is a *generalized Toeplitz operator* (cf. [16, 51]). This generalizes the well-known case where $H = L^2(S^1, \mathbb{C})$ and $H_1 = H^2(S^1, \mathbb{C})$ is the Hardy space consisting of those Fourier coefficients that vanish on $\mathbb{Z}^-$. Here $p : L^2(S^1, \mathbb{C}) \longrightarrow H^2(S^1, \mathbb{C})$ is the Riesz projection and $\mathsf{T}_p = pm(f)p$ is then the classical Toeplitz operator, where $m(f) \in B$ is multiplication by an essentially bounded function. The term generalized Laurent operator is fitted to such a Riesz system. For the more usual notion of this class of operators see, e.g., [29, §3.1,§16.1]. A further study of the theory of Toeplitz operators relative to bounded domains in $\mathbb{C}^n$ is treated in [69, Chap. 4].

4. Generalized determinants and the τ-function

4.1. Background to the τ-function

For the benefit of readers we provide a short background to the nature of the τ-function besides motivating its introduction into the operator theory context. Originally the function seemed to have played a significant role in the theory of *the Painlevé transcendents* and Hamiltonian systems (see, e.g., [1, Chap. 7] and [14, 54]), whereas in classical Sturm–Liouville theory, the logarithmic derivative of the τ-function differentiates to a 'potential'. Let us exemplify this latter case now, and postpone a short discussion of the Painlevé equations to Appendix B.

Example 4.1. For the Sturm–Liouville (SL) theory, the classical setting considers the Hilbert space $H = L^2([a, b], r(x)dx)$ endowed with the usual inner product $\langle f, g\rangle = \int_a^b f(x)\overline{g(x)}r(x)\, dx$. As seen in, e.g., [29, §6.5], for the appropriate differentiable (real-valued) functions p, q and $r(\neq 0)$, we have the SL-operator L as given by

$$Ly(x) = \frac{1}{r(x)} \left(-\frac{d}{dx} \left[p(x)y'(x)\right] + q(x)y(x) \right). \tag{4.1}$$

The SL-equation $Ly = \lambda y$, along with boundary conditions, is a well-mined eigenvalue problem. Typically, L is a self-adjoint (unbounded) operator with ordered real eigenvalues $\lambda_1 < \lambda_2 < \cdots$, with associated orthonormal eigenfunctions y_1, $y_2, \ldots$ Just as any good student knows, there are many classes of second-order (linear) ODEs that can be expressed in SL-form and numerous applications of the SL-equation to mathematical physics. It is here that we discover 'potentials' $q(x)$

as given by

$$q(x) = -\left(\frac{d}{dx}\right)^2 \left[\ln \tau\right] = -\frac{d}{dx}\left[\frac{\tau'}{\tau}\right]. \tag{4.2}$$

Of interest is the type of functions such as τ and how they may arise. For instance, on commencing from the SL-equation, the work in [49] leads to aspects of such a τ-function relative to differential rings and 'vessels' in a fashion applicable to systems theory.

The main thrust of the τ-function theory has arisen in the theory of integrable systems, particulary in the theory of nonlinear waves (such as in the KP-hierarchy), the inverse scattering transform method, statistical physics and in a number of related areas enjoying deep mathematical connections, as seen in [1, 3, 17, 36, 37, 59, 60, 63, 66]. For instance, there are certain τ-functions for which the expression $\omega = \partial_x \ln \tau$ (where ∂_x denotes a spatial derivative) provides a 'Kähler potential' in the theory of self-dual Einstein gravity [56]. The point being that it is mainly through this work that the τ-function is realizable as a (generalized) determinant, though often in a seemingly formal sense.

Example 4.2. The formal way of defining a τ-function commences with symmetric functions in n-variables indexed by partitions (cf. characters of irreducible representations of $\mathrm{GL}(n,\mathbb{R})$). Consider a partition $\nu = \nu_1 + \nu_2 + \cdots + \nu_n$, with ν_k non-negative integers, $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_n$. Alternating polynomials are given by

$$\begin{aligned} A_{(\nu_1,\ldots,\nu_n)}(x_1,\ldots,x_n) &= \det\left[x_i^{\nu_j}\right] \\ &= \sum_{\sigma \in S_n} \epsilon(\sigma)\, x_{\sigma(1)}^{\nu_1} \cdots x_{\sigma(n)}^{\nu_n}, \end{aligned} \tag{4.3}$$

leading to the following expression for the *Schur function*

$$\mathsf{s}_\nu(x) := \frac{A_{(\nu_1+n-1,\nu_2+n-2,\ldots,\nu_n)}(x_1,x_2,\ldots,x_n)}{\Delta_{\mathrm{vm}}}, \tag{4.4}$$

where $\Delta_{\mathrm{vm}} = \prod_{1\leq j<k\leq n} (x_j - x_k)$ is a Vandermonde determinant.

For a large class of wave functions the formal τ-function is given by a linear combination of Schur functions relative to a partition

$$\tau(\mathbf{t}) = \sum_\nu c_\nu\, \mathsf{s}_\nu(\mathbf{t}), \tag{4.5}$$

where the constants c_ν depend on 'embedding coordinates' [63, §8].

Many significant developments, somewhat in the context of Example 4.2, blossomed forth in the 1980's (see for instance [36, 37, 59, 60]). The essential ideas were later studied in a more unified geometric-analytic context in the groundbreaking paper of Segal and Wilson [63]. In this latter setting the τ-functions, such as those to be defined in (5.5) below (see also Example 4.3), are associated to points W in a Grassmannian of the type $\mathrm{Gr}(p, A)$ (here A denotes the 'restricted'

Banach *-algebra to be outlined in 5.2) and are denoted τ_W-functions. These τ_W-functions are shown in [63, §8] to be closely related to the formal τ-function of Example 4.2 and the basic construction of these will be seen in §5.3.

We have in mind a short survey showing how the τ-function arises from a generalized (e.g., Fredholm or Toeplitz) determinant along with several algebraic, and analytic ramifications (see, e.g., [9, 11]). Also significant is the construction of the associated predeterminant operators in §5.1. A further aspect of the τ-function is its role in the theory of Schlesinger systems as associated with differential equations of Fuchsian type [39, 40, 42, 46], where the operator-theoretic setting is briefly discussed in §7. The classical theta functions are in fact closely tied to τ-functions via exponential multiplication [63, Theorem 9.11] (see also [58]). Hence the acclaimed *Fay trisecant identity* [26], which is ubiquitous in the theory of integrable systems, is expressible in terms of τ-functions as seen in, e.g., [58, §10](cf. [50, §2] in the setting of the KP-hierarchy). Perhaps more in line with this present work is a matrix-operator account of this subject as treated in [4, §4] which touches on 'moduli' questions, as does [8, §5] which also establishes such a trisecant identity in terms of nonabelian theta functions.

Example 4.3. This involves briefly introducing loop groups 'LU' [57, Chap. 6]. Take $H = L^2(S^1, \mathbb{C})$ and a polarization $H = H_+ \oplus H_-$ (with $H_+ \cap H_- = \{0\}$). Following [48, §2.3], for $g \in \Gamma^+ \subset \mathrm{LU}(1)$, we have a Toeplitz operator $\mathsf{T}_g : H_- \longrightarrow H_+$ given by $v \mapsto (gv)_-$. Note that $\mathsf{T}_{g_1 g_2} \neq \mathsf{T}_{g_2 g_1}$, in general, but $\mathsf{T}_{g_+ g g_-} = \mathsf{T}_{g_- g g_+}$ when $g_\pm \in \mathrm{GL}(H_\pm)$. Taking $K_+ = f^{-1}H_+$, for some $f \in \mathrm{GL}(H)$, leads to another polarization (K_+, K_-), and a Toeplitz operator given by $\mathsf{T}'_g = \mathsf{T}_f^{-1}\mathsf{T}_{fg}$. Taking $k \in \mathrm{GL}_+(H)$ and $h \in \mathrm{GL}_-(H)$, then there exists a *Toeplitz-τ-function* given by

$$\tau(h^{-1}H_+, H_+, kH_-, H_-) = \mathrm{Det}(\mathsf{T}_h^{-1}\mathsf{T}_{hk}\mathsf{T}_k^{-1}). \tag{4.6}$$

Example 4.4. References [11, 12](cf. [67]) also reveal a large class of Fredholm determinants, within the representation theory of the infinite-dimensional unitary group, actually to be τ-functions of various integrable systems associated to a particular Painlevé type [1]. Typical of this approach is to commence with T taken to be an integral operator, and on restricting its kernel $K_{\mathsf{T}}(x, y)$ to an interval $\mathcal{J} = \bigcup_{j=1}^m (a_{2j-1}, a_{2j}) \subset \mathbb{R}$, to take the Fredholm determinant $\mathrm{Det}(1 - \lambda\mathsf{T}|_{\mathcal{J}})$, for a suitable $\lambda \in \mathbb{C}$ [11, 67]. In all cases, the kernel $K_{\mathsf{T}}(x, y)$ is explicit, though in [11, §2] the authors implement a continuous, so-called ${}_2F_1$-kernel based on the Gauss hypergeometric function. This ${}_2F_1$-kernel, $K_{\mathsf{T}}(x, y)|_{\mathcal{J}}$ is seen to be of trace class and, in particular, $\mathrm{Det}(1 - \lambda\mathsf{T}|_{\mathcal{J}})$ is a τ-function for the Schlesinger equation (see, e.g., [11, §11] and [36, §5]; also see the assortment of examples in §7).

Example 4.5. In [9] a moment functional $\mathfrak{M} : \mathbb{C}[z, z^{-1}] \longrightarrow \mathbb{C}$, is considered along with a corresponding Lax operator $Q_{\mathcal{I}}(\mathfrak{M})$ depending on an index set $\mathcal{I}$ (here $Q_{ij} = \mathfrak{M}(r_i x p_j)$). The shifted Toeplitz determinant yields a τ-function such that

$$d \ln \tau = \sum_{\rho=1}^{n} \frac{1}{\rho}\ \mathrm{Tr}_n(Q_{\mathcal{I}})\ dt_\rho. \tag{4.7}$$

The Hankel determinant of a semiclassical moment functional on the space of polynomials can be identified with the isomonodromic τ-functions of [36] and this together with the above Toeplitz determinant (of equivalent order) with respect to $\mathfrak{M}$ above can be related by sequences of Schlesinger transformations (cf. §7).

5. The predeterminant $\mathcal{T}$-function and its τ-function

5.1. The predeterminant $\mathcal{T}$-function

We return to the general setting of §2.1, with A a unital Banach algebra and then specialize. Firstly, we recall from §3.2 the principal $G(pAp)$-Stiefel bundle $V(p, A) \longrightarrow \mathrm{Gr}(p, A)$, endowed with a *canonical section* denoted S_p, and proceed to extract part of a construction in [23, §8] to which we refer for complete details. Recalling the map π_Λ in (2.1), and for $p \in P(A)$, we set

$$W_p = \pi_\Lambda^{-1}(p + pA\hat{p}), \tag{5.1}$$

(where $\hat{p} = 1 - p$), and then consider the subset

$$W_p^0 = \{r \in W_p : \phi_p(r) := rp + \hat{r}\hat{p} \in G(A)\}. \tag{5.2}$$

Also recalling $\Lambda = \mathrm{Sim}(p, A)$, the development of [23, §8.1] entailed defining two sections α, β of the principal $G(pAp)$-bundle $Q' \longrightarrow \Lambda$ having the following properties:

i) With respect to π_Λ in (2.1), $\alpha_p = \pi_\Lambda^*(S_p)$ is defined over $\pi_\Lambda(W_p) \subset \mathrm{Gr}(p, A)$.
ii) For β_p with $g = \phi_p(r)$ and $r \in W_p^0$, we have $g \in G(A)$. The assignment $rp : p \longrightarrow r$, yields a proper partial isomorphism which projects along $\mathrm{Ker}(r)$. Then we set $\beta_p(r) = (r, rp)$.

Next, on recalling the transition map t_Λ in (2.3), we set

$$\mathcal{T}(r) = t_\Lambda(\alpha_p(r), \beta_p(r)) \in G(pAp). \tag{5.3}$$

In [23, §8] we called the left operator of (5.3) a *$\mathcal{T}$-function* which can be viewed as a type of generalized Wiener–Hopf operator T as described above. Note that this essentially algebraic construction would work equally well for any of the reduced subgroups G in §2.1, and hence for the principal bundles $Q \longrightarrow \Lambda$ described in §2.2.

5.2. The restricted Banach algebra A

To see how this was used in [23], we first of all take a (separable) Hilbert module $H_{\mathcal{A}}$ over a unital C*-algebra $\mathcal{A}$, and a polarization $H_{\mathcal{A}} = \mathsf{H}_+ \oplus \mathsf{H}_-$ (with $\mathsf{H}_+ \cap \mathsf{H}_- = \{0\}$). Let J be an $\mathcal{A}$-module map satisfying $J^2 = 1$. The 'restricted' Banach algebra $A = \mathcal{L}_J(H_{\mathcal{A}})$ is a Banach *-algebra under the Hilbert–Schmidt modified norm $\|T\|_J = \|T\| + \|[J, T]\|_2$ (here we have used the generalization of the Schatten p-classes to Hilbert modules following [65] and cf. [28]).

If $\mathfrak{P}$ denotes the space of polarizations on $H_{\mathcal{A}}$, then by [23, Theorem 4.1], there exists an analytic diffeomorphism $\varphi : \mathfrak{P} \longrightarrow \Lambda$. Instrumental in [23, §8] and [70, §3] was to take the above sections $\alpha_p(r)$ and $\beta_p(r)$, when viewed as sections of the (universal) vector bundle $\gamma_\Lambda \longrightarrow \Lambda$ in §2.2, to be *covariantly constant* with respect to the connection ∇_Λ. The $\mathcal{T}$-function (operator) is more general than the Zelikin $\mathfrak{T}$-function which for the case $\boldsymbol{\mathcal{A}} \cong \mathbb{C}$ utilizes a cross-ratio coordinate system on $\mathfrak{P}$ [70, §3]. Also, Det $\mathcal{T}$ is definable in terms of 'admissible elements (bases)' for a sufficiently large class of Banach algebras in the setting of §3.

5.3. The τ_W-function

Suppose we take a pair of polarizations $(\mathsf{H}_+, \mathsf{H}_-), (\mathsf{K}_+, \mathsf{K}_-) \in \mathfrak{P}$ to be such that H_+ is the graph of a linear map $S : \mathsf{K}_+ \longrightarrow \mathsf{K}_-$, and H_- is the graph of a linear map $T : \mathsf{K}_- \longrightarrow \mathsf{K}_+$. Then on $H_{\boldsymbol{\mathcal{A}}}$ consider the identity map $\mathsf{H}_+ \oplus \mathsf{H}_- \longrightarrow \mathsf{K}_+ \oplus \mathsf{K}_-$, as represented by the block form

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{5.4}$$

where $a : \mathsf{H}_+ \longrightarrow \mathsf{K}_+$, $d : \mathsf{H}_- \longrightarrow \mathsf{K}_-$ are zero-index Fredholm operators, and $b : \mathsf{H}_- \longrightarrow \mathsf{K}_+$, $c : \mathsf{H}_+ \longrightarrow \mathsf{K}_-$, belong to $\mathcal{K}(H_{\boldsymbol{\mathcal{A}}})$ (the compact operators), such that $S = ca^{-1}$ and $T = bd^{-1}$. When b, c are taken to be Hilbert–Schmidt operators, then ST is of trace-class, and the operator $(1 - ST)$ can be expressed as a $\mathfrak{T}$-function as above. We denote this by $\mathfrak{T}(\mathsf{H}_+, \mathsf{H}_-, \mathsf{K}_+, \mathsf{K}_-)$ (it is essentially a transition map in the $\mathcal{T}$-function setting).

Following [63, §2] and [57, §7.1] (cf. [24, A2]) points $W \in \mathrm{Gr}(p, A)$ represent closed subspaces W of $H_{\boldsymbol{\mathcal{A}}}$ such that i) the orthogonal projection $p_+ : W \longrightarrow \mathsf{H}_+$ is in $\mathrm{Fred}(H_{\boldsymbol{\mathcal{A}}})$, and ii) the orthogonal projection $p_- : W \longrightarrow \mathsf{H}_-$ is in $\mathcal{L}_2(\mathsf{H}_+, \mathsf{H}_-)$ (Hilbert–Schmidt operators). Relative to such points $W \in \mathrm{Gr}(p, A)$, the corresponding τ_W-function is constructed in [63, §3] and is seen to be of the form

$$\begin{aligned} \tau_W(\mathsf{H}_+, \mathsf{H}_-, \mathsf{K}_+, \mathsf{K}_-) &= \mathrm{Det}\ \mathfrak{T}(\mathsf{H}_+, \mathsf{H}_-, \mathsf{K}_+, \mathsf{K}_-) \\ &= \mathrm{Det}(1 - ca^{-1}bd^{-1}) \in \mathbb{C} \otimes 1_{\boldsymbol{\mathcal{A}}}, \end{aligned} \tag{5.5}$$

(cf. [23, 48, 70]). This particular τ-function frequently appears in the Lax equation method in nonlinear wave theory (for instance, *solitons*) and the general setting for the latter within the KP-hierarchy (see, e.g., [1, 3, 17, 63, 60]).

Remark 5.1. In several examples above and in those in the sequel, the main class of operators surveyed will often turn out to be integral operators. One way of seeing this proceeds as follows. Suppose (Ω, μ) is some σ-finite measure space. Let $H = L^2(\Omega, \mu)$ (be separable), $A = \mathcal{L}(H)$ and let $\mathcal{L}_2(H)$ denote the Hilbert–Schmidt operators on H. Then projections $p \in P(A) \cap \mathcal{L}_2(H)$ may be realizable as integral operators, and likewise for elements $q \in U(p, A)$ in the unitary orbit of p (see, e.g., [32, §15] and related results therein). For instance, let $\Omega \subset\subset \mathbb{C}^n$ be a bounded domain. A familiar example is the Bergman projection $p_B : L^2(\Omega) \longrightarrow \mathrm{Hol}(\Omega) \cap L^2(\Omega)$, orthogonally projecting $L^2(\Omega)$ onto its holomorphic subspace. In terms of its associated kernel function $K_\Omega(z, w)$, it is given by $p_B f(z) = \int_\Omega K_\Omega(z, w) f(w)\ dw$. This projection p_B may be viewed as a generalized Calderón–Zygmund operator with respect to certain local pseudometrics [45]. Other familiar and partially related examples are presented in [69, Chap. 4].

6. Simple L*-algebras and Kac–Moody algebras

6.1. Simple L*-algebras

Henceforth, unless otherwise stated, we now restrict in this section to $A = \mathcal{L}(H)$, where H is a separable Hilbert space, and proceed to consider a class of involutive Banach–Lie algebras called *L*-algebras* (see [5, Chap. 7] and [34, Chap. II, III] following the earlier work of [61, 62]).

An *L*-algebra* is a Lie algebra $\mathfrak{g}$ (over $\mathbb{R}$ or $\mathbb{C}$) whose underlying vector space is a Hilbert space, together with a map $x \mapsto x^*$ that satisfies $\langle [x, y], z\rangle = \langle y, [x^*, z]\rangle$, for all $x, y, z \in \mathfrak{g}$. In particular, the Hilbert–Schmidt operators $\mathcal{L}_2(H)$ form a complex simple L*-algebra (see [5, Theorem 7.18] and [34, II.5]) in which the $*$-map specifies an adjoint representation, while noting that by Remark 5.1 above, this latter observation subsequently provides a potential supply of integral operators on H. We denote this L*-algebra by $\mathfrak{g}_{\mathbf{A}}$, which is a simple Lie algebra of type $\mathbf{A}$ in the Cartan classification. In fact, all simple Lie algebras of type $\mathbf{A}$ are isomorphic, up to some multiple of the inner product, to an L*-subalgebra of $\mathfrak{g}_{\mathbf{A}}$.

There are also the simple L*-algebras of Cartan type $\mathbf{B}, \mathbf{C}$ and $\mathbf{D}$ (see [5, 34, 52, 61, 62] for a comprehensive treatment of this infinite-dimensional structure theory). Because of the frequent instrumentation of the Hilbert–Schmidt operators, we will restrict matters here to $\mathfrak{g}_{\mathbf{A}}$, and note that $\mathfrak{g}_{\mathbf{A}}$ as a simple L*-algebra can be approximated by taking the limit of a strictly increasing sequence $\{\mathfrak{g}_n\}$ of simple finite-dimensional Lie algebras of Cartan type $\mathbf{A}$ [61, §3.2]. Specifically, we shall take $G \subseteq G(pBp) \subseteq G(pAp)$ to be a Banach–Lie group whose Banach–Lie algebra (over $\mathbb{C}$) is the simple L*-algebra $\mathfrak{g} = \mathfrak{g}_{\mathbf{A}}$.

A *Cartan subalgebra* $\mathfrak{h}$ of $\mathfrak{g}$ is defined as a maximal self-adjoint abelian (closed) subalgebra. With respect to $\mathfrak{h}$, a Cartan decomposition of $\mathfrak{g}$ can be formed (see below). Let Δ denote the set of non-zero roots of $\mathfrak{g}$ relative to $\mathfrak{h}$. In the form of a lemma we collect together several basic results about the structural theory of the L*-algebra $\mathfrak{g}$ to be used later.

Lemma 6.1. *Let $\mathfrak{g} = \mathfrak{g}_{\mathbf{A}}$ be the (complex) simple L*-algebra as above. Then we have the properties:*

(1) *There exist simple closed $\mathfrak{g}$-ideals $\mathfrak{g}_k$, indexed by some set $\mathcal{J}$, leading to a Hilbert space direct sum*
$$\mathfrak{g} = \bigoplus_{k\in\mathcal{J}} \mathfrak{g}_k. \tag{6.1}$$

(2) *Given a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, there exists a Cartan (Hilbert space) decomposition*
$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\nu\in\Delta} V_\nu, \tag{6.2}$$
where the root spaces V_ν are one-dimensional.

(3) *We have $\mathfrak{g} = \mathrm{cl}(\bigcup \mathfrak{g}_n)$, where each $\mathfrak{g}_n$ is a finite-dimensional simple Lie algebra (of Cartan classification $\mathbf{A}$) appearing in a strictly increasing sequence $\cdots \subset \mathfrak{g}_n \subset \mathfrak{g}_{n+1} \subset \cdots$*

Proof. Part (1) is stated in [61, §1.2, Theorem 1]. The existence of the Cartan decomposition in Part (2) follows from [61, §2.2, Theorem 2] (cf. [62]) and the one-dimensionality of the root spaces V_ν follows from [61, §2.3]. As for Part (3), this follows from [61, §3.2, (iv)]. □

Remark 6.1. The corresponding results for the other types in the Cartan classification are essentially the same. A treatment of these results and the Cartan decomposition also appear in [34, Proposition 11, II.22] (cf. [5, Chap. 7] for certain extensions of the theory) and more generally in [13, Theorem 1] for semisimple Banach–Lie algebras of compact operators.

Having introduced this class of L*-algebras, we recall the setting of the principal G-bundles $Q \longrightarrow \Lambda$ in §2.2, where G is the corresponding Lie group of the L*-algebra $\mathfrak{g}$ and the operator $\mathcal{T}$ arises via the transition map t_Λ in (2.6). This leads to some further observations concerning generalized determinants and τ-functions in particular. The first follows by a straightforward restriction argument.

Proposition 6.1. *For each operator $\mathcal{T} \in G$ of the (predetermiant) type in (5.3) there exists a family of generalized determinants defined relative to the closed ideals of $\mathfrak{g}$.*

Proof. For each $\mathcal{T} \in G$ in (5.3) we can assign a corresponding Hilbert–Schmidt operator $\mathsf{T} \in \mathfrak{g}$. Recall from Lemma 6.1(1) we have a Hilbert space direct sum

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_k \oplus \cdots \tag{6.3}$$

in terms of closed $\mathfrak{g}$-ideals. We then consider restrictions $\mathsf{T}|\mathfrak{g}_k$, which for each k, define a family of generalized determinants by taking the Fredholm determinant $\mathrm{Det}(1 - \lambda \mathsf{T}|\mathfrak{g}_k)$ (for some suitable λ). □

It is clear that the means behind Proposition 6.1 would apply by restricting to a Hilbert space summand in any decomposition of $\mathfrak{g}$. The next observation is in the context of Example 4.4 and follows from the development of ideas in [11].

Proposition 6.2. *Let $G \subseteq U(pAp)$ be a simple Banach–Lie subgroup with associated (simple) L*-algebra which is a real form of $\mathfrak{g} = \mathfrak{g}_{\mathbf{A}}$ as above. Then for each $\mathcal{T} \in G$ in (5.3) we obtain a family of τ-functions via a smooth integrable kernel K defined on a subset $\mathfrak{J} \subset \mathbb{R}$.*

Proof. Without loss of generality we assume that $G \subseteq U(pAp)$ is isomorphic to a unitary (Banach–Lie) subgroup of the infinite-dimensional unitary group $U(\infty)$. The idea is that starting from the operator $\mathcal{T} \in G \subseteq U(\infty)$, we arrive at a integrable kernel defined with respect to $\mathfrak{J}$. Firstly, the results of [12, §1-§3] show that G can be decomposed into irreducible parts via probability measures on the space of all irreducible representations. Specifically, if $\chi : G \longrightarrow \mathbb{C}$ denotes a character of G, then there exists a spectral measure μ_χ on parameter set Ω so that

$$\chi = \int_\Omega \chi_w \mu_\chi \, dw, \tag{6.4}$$

as established in [11, Theorem 1.1]. Following from this, the construction of the kernel K is given in [11, §5]. Taking a projection $p(\mu_\chi)$ onto $\mathfrak{J} = (s, \infty)$ to give a restriction, (see Example 4.4) the Fredholm determinant $\mathcal{D}(s) = \mathrm{Det}(1 - \lambda \mathsf{K}|_{(s,\infty)})$ leads to a family of τ-functions (see [11, §7]). □

Remark 6.2. In [11, 12] the distribution $\mathcal{D}(s)$ and the corresponding τ-function above lead to classes of differential equations of the various Painlevé type (see Appendix B. In [12, §10] the τ-functions arising there are interpreted as correlation functions that are manifestly determinants of certain kernels. As we pointed out in §4.1, there is a considerable amount of background material to this topic in the framework of integrable systems and statistical physics, and so for now we are obliged to refer the reader to works such as [1, 11, 12, 15, 36, 37, 67] in order to get a more complete picture.

6.2. A link with affine Kac–Moody algebras

In Appendix §A.1 we recall how the Cartan matrix of any semisimple Lie algebra induces an associated (*affine*) *Kac–Moody algebra* [38, Chap. 7]. This concept we will adopt in view of the simple L*-algebras discussed previously. Firstly, a lemma that uses the construction and which will be put to use in §6.3 below.

Lemma 6.2. *Consider the* (*complex*) *simple* L^**-algebra* $\mathfrak{g} = \mathfrak{g}_{\mathbf{A}}$. *Then there is an associated* (*strictly*) *increasing sequence of affine Kac–Moody algebras* $\{\widehat{\mathfrak{g}}_{\kappa_n}\}$ *where* κ_n *is an invariant inner product on each simple finite-dimensional Lie algebra* $\mathfrak{g}_n$ *as in Lemma* 6.1(3) *above.*

Proof. Recall from Lemma 6.1(3) we have $\mathfrak{g} = \mathrm{cl}(\bigcup \mathfrak{g}_n)$ with each $\mathfrak{g}_n$ a simple finite-dimensional Lie algebra. Let $\mathbb{C}((t)) = \mathbb{C}[t, t^{-1}]$ denote the algebra of Laurent polynomials in t, and let κ_n denote an invariant inner product on $\mathfrak{g}_n$. Following [38, §7.1] (see also [27, §1]), the affine Kac–Moody (Lie) algebra $\widehat{\mathfrak{g}}_{\kappa_n}$ is the one-dimensional central extension of $\mathfrak{g}_n \otimes \mathbb{C}((t))$ with

$$[\phi_1 \otimes f(t), \phi_2 \otimes g(t)] = [\phi_1, \phi_2] \otimes f(t)g(t) - (\kappa_n(\phi_1, \phi_2)\mathrm{Res}\, f dg) K_n, \qquad (6.5)$$

where $\phi_1, \phi_2 \in \mathfrak{g}_n,\ f, g \in \mathbb{C}((t))$, and K_n denotes the central element. The fact that the sequence $\{\widehat{\mathfrak{g}}_{\kappa_n}\}$ is increasing, is clear, since because $\mathfrak{g}_n \subset \mathfrak{g}_{n+1}$, we have a naturally induced sequence $\mathfrak{g}_n \otimes \mathbb{C}((t)) \subset \mathfrak{g}_{n+1} \otimes \mathbb{C}((t))$. □

6.3. A representation of the Weyl group and τ-functions

Here we use the root lattice (for roots ν) and refer to [53, 68](see Appendix §A.2). Following [53, 68], we give a short outline of a representation-theoretic definition of the τ-function. Firstly, for each generalized Cartan matrix $C = [c_{ij}]$ of affine type, there exists a representation of the Weyl group $\mathcal{W} = \mathcal{W}(C)$ on a field $\mathbb{C}(\nu; \theta; \tau)$ of rational functions with respect to infinitely many variables ν_i, θ_i, τ_i $(i \in \mathcal{I})$,

$$\varrho : \mathcal{W} \longrightarrow \mathbb{C}(\nu; \theta; \tau). \qquad (6.6)$$

This representation is characterized by the action of the generators s_i (see §A.2), such that whenever defined,

$$\begin{aligned} &s_i(\nu_j) = \nu_j - \nu_i c_{ij} \text{ (reflections)},\ s_i(\theta_j) = \theta_j + \frac{\nu_i}{\theta_i} u_{ij}, \\ &s_i(\tau_j) = \tau_j(\theta_j\ \Pi_{k\in\mathcal{I}}\ \tau_k^{-c_{kj}})^{\delta_{ij}}, \end{aligned} \tag{6.7}$$

where the $u_{ij}(i, j \in \mathcal{I})$ satisfy certain conditions as seen in [53, §2] and [68, §1]. The τ_j are realizable as τ-functions for the variables θ_j, and in this representation the τ_j are seen to correspond to the *fundamental weights* Ξ_j, while the θ-variables correspond to the simple roots ν_i [53, §2]. The motivation for regarding the τ_j as τ-functions is commented upon in Remark 6.3 below.

Proposition 6.3. *Consider the (complex) simple L^*-algebra $\mathfrak{g} = \mathfrak{g}_{\mathbf{A}}$. Then there exists a representation $\mathfrak{g} \longrightarrow \mathbb{C}(\nu;\theta;\tau)$ relative to which a family of Fredholm determinants of an (integral) operator $\mathsf{T} \in \mathfrak{g}$ is assigned to a family of τ-functions of the above type.*

Proof. Firstly, we recall from Lemma 6.1(3) the sequence of finite-dimensional simple Lie algebras $\mathfrak{g}_j$. Commencing from a Hilbert–Schmidt (integral) operator $\mathsf{T} \in \mathfrak{g}$, we proceed to the Fredholm determinant $\mathrm{Det}(1 - \lambda\mathsf{T}|\mathfrak{g}_j)$. The latter as a generalized determinant is assigned to each τ_j, and hence to the corresponding weight Ξ_j.

For each $\mathfrak{g}_j$, we have by Lemma 6.2 an associated increasing sequence of affine Kac–Moody algebras $\{\widehat{\mathfrak{g}}_{\kappa_j}\}$. The affine Weyl group $\mathcal{W}(C_n)$ acts upon the root system of each $\widehat{\mathfrak{g}}_{\kappa_j}$ (for each j) as well as on the rational function field $\mathbb{C}(\nu;\theta;\tau)$ [53, §2]. Now since $\mathbb{C}(\nu;\theta;\tau)$ depends on the sum $\sum_j \Xi_j$ of fundamental weights, we thus arrive at a representation

$$\begin{aligned} \mathfrak{g} &\longrightarrow \mathbb{C}(\nu;\theta;\tau), \\ \mathrm{Det}(1 - \lambda\mathsf{T}|\mathfrak{g}_j) &\mapsto (\Xi_j \leftrightarrow \tau_j), \end{aligned} \tag{6.8}$$

which yields the desired result. □

On referring to Appendix §A.2, we introduce the dual $\mathbb{Z}$-module of the coroot lattice $R^\vee$ denoted in (A.4) by $R^\star = \mathrm{Hom}_{\mathbb{Z}}(R^\vee, \mathbb{Z})$. As shown in [53, §2], any family $\{\phi_w(\lambda)\}$ $(w \in \mathcal{W}, \lambda \in R^\star)$ can be identified with a mapping

$$\begin{aligned} &\phi : \mathcal{W} \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(R^\star, \mathbb{C}(\nu;\theta)^*), \\ &w \mapsto \phi_w, \end{aligned} \tag{6.9}$$

where $\mathbb{C}(\nu;\theta)^*$ denotes the multiplicative group of $\mathbb{C}(\nu;\theta)$ regarded as a $\mathbb{Z}$-module, and for which

$$\phi_{w_1 w_2}(\lambda) = w_1(\phi_{w_2}(\lambda))\phi_{w_1}(w_2 \cdot \lambda),\ \forall w_1, w_2, \in \mathcal{W},\ \text{and } \lambda \in R^\star. \tag{6.10}$$

The map ϕ in (6.9) now viewed as a (linear) map $\phi : \mathbb{C}[\mathcal{W}] \longrightarrow M$ on the group algebra, where $M = \mathrm{Hom}_{\mathbb{Z}}(R^\star, \mathbb{C}(\nu;\theta)^*)$, is shown in [53, §2] to define a Hochschild 1-cocycle of $\mathbb{C}[\mathcal{W}]$ with respect to the natural $\mathcal{W}$-bimodule structure of M (we refer to [44, §1-§4] for the basic theory of Hochschild complexes). Since it can be shown that $w(\tau^\lambda) = \phi_w(\lambda)\tau^{w\cdot\lambda}$ ($w \in \mathcal{W}, \lambda \in R^\star$), then the cocycle induced by ϕ in (6.10) becomes the coboundary of the 0-cochain

$$\begin{aligned} &\tau \in \mathrm{Hom}_{\mathbb{Z}}(R^\star, \mathbb{C}(\nu;\theta;\tau)^*), \\ &\lambda \mapsto \tau^\lambda, \end{aligned} \tag{6.11}$$

on extending the $\mathcal{W}$-module $\mathbb{C}(\nu;\theta)$ to $\mathbb{C}(\nu;\theta;\tau)$. It is in this way that the τ-functions are seen to trivialize the Hochschild 1-cocycle as defined by these variables.

Remark 6.3. As outlined in [53, 68], triples $(\nu_j, \theta_j, \tau_j)$ can be seen to lead to a family of discrete dynamical systems classified by certain types denoted $\mathbf{A}_\ell^{(1)}, \ldots, \mathbf{D}_\ell^{(1)}$, for various ℓ, whose corresponding affine Weyl groups are realized as Bäcklund transformations of the Painlevé equations of class $P_{\mathrm{II}}, \ldots, P_{\mathrm{VI}}$ (see Appendix B). Here the pairs (ν_j, θ_j) play the role of discrete time dependent variables and the τ_j are τ-functions associated to the corresponding Painlevé types.

7. Applications: Schlesinger systems, isomonodromic transformations and meromorphic connections

7.1. Holomorphic maps to $\Lambda = \mathrm{Sim}(p, A)$

We return now to the case where $A = \mathcal{L}(H_{\mathcal{A}})$ in §5.2, and commence by considering holomorphic maps $\hat{f} : X \longrightarrow \mathrm{Gr}(p, A)$, where X in this section denotes a compact Riemann surface of genus g_X.

Proposition 7.1. *Given a (non-constant) holomorphic map $\hat{f} : X \longrightarrow \mathrm{Gr}(p, A)$, then $\hat{f}$ can be extended to a holomorphic map $f : X \longrightarrow \Lambda$.*

Proof. For each $x \in X$, let us set $\hat{f}(x) = (\mathsf{K}_+)_x$, and then let $(\mathsf{K}_-)_x$ be a closed complemented subspace for $(\mathsf{K}_+)_x$ in $H_{\mathcal{A}}$, so that $(\mathsf{K}_+)_x \cap (\mathsf{K}_-)_x = \{0\}$. Thus we have produced a polarizing pair $((\mathsf{K}_+)_x, (\mathsf{K}_-)_x)$ that depends on $x \in X$, and hence f extends to the space $\mathfrak{P}$ of polarizations in §5.2. Following from [23, Theorem 4.1 (3)], we next make use of the analytic diffeomorphism $\varphi : \mathfrak{P} \xrightarrow{\cong} \Lambda$, and then finally, the desired holomorphic map $f : X \longrightarrow \Lambda$ is taken to be the composition $f = \varphi \circ \hat{f}$. □

The Krichever correspondence which is based on certain holomorphic data as described in [63, §6] provides a prototypical example, namely, a holomorphic embedding $\hat{f} : X \longrightarrow \mathrm{Gr}(p, A)$ (as applied in [24, 25]). The point of extending such a map to Λ is that the subsequent calculations as carried out in [23] tend to be relatively straightforward, and the relevant analytic objects defined over Λ can be pulled-back via f. For instance, recall the setting of §2.1 and consider the possible pull-back vector bundles with connection under $f : X \longrightarrow \Lambda$:

(i) $(f^*\gamma, {}^*\nabla_\gamma) \longrightarrow X$
(ii) $(f^*\mathrm{Det}_\gamma, f^*\nabla_{\mathrm{Det}_\gamma}) \longrightarrow X$

(iii) More generally, consider $(V, \nabla_V) \longrightarrow X$, with structure group $G \subseteq G(pAp)$ and corresponding connection form ω_V as pulled back under f from a vector bundle with connection $(\mathbf{V}, \nabla_{\mathbf{V}}) \longrightarrow \Lambda$, with corresponding connection form denoted $\omega_{\mathbf{V}}$ (typically associated to a principal G-bundle $Q \longrightarrow \Lambda$ as in (2.5) with connection 1-form ω_Q).

Next we take an integral operator T that corresponds to the $G \subseteq G(pAp)$-valued $\mathcal{T}$-function for Q as described in §5.1 for which, via a holomorphic embedding $f : X \longrightarrow \Lambda$, the Fredholm determinant

$$\mathrm{Det}(1 - \lambda \mathsf{T}|_{\mathfrak{J}}), \tag{7.1}$$

is suitably supported on a countable number of points, or on a union of curve segments $\mathfrak{J}$ in the (complex) ζ-plane supporting T in X. In the context of Example 4.4, the determinant (7.1) provides the τ-function of an isomonodromic family of meromorphic covariant derivative operators D_ζ.

Example 7.1. Specifically from [33] (for $g_X = 1$), the τ-functions in question are of the form $\tau = \tau(a_1, \ldots, a_r; b_1, \ldots, b_n)$, where the $a_1, \ldots, a_r$ are 'asymptotic elements' and the $b_1, \ldots, b_n$ are 'pole locations' collectively parametrizing τ for the given pair (r, n). Moreover, as shown in [33, Theorem 2.6], τ is effectively the Segal–Wilson τ_W-function in [63, §3]. In this case there is the 1-form

$$\omega = d \ln \tau = \sum_{i=1}^{r} K_i \; da_i + \sum_{j=1}^{n} H_j \; db_j, \tag{7.2}$$

where the pairs (H_i, K_j) are Poisson-commuting Hamiltonians. In [25, §4.2] it was shown that a class of τ-functions, denoted τ_Λ, are essentially the same as the τ_W, and are linked via pullback. Each of the τ_Λ serves as a logarithmic potential for the curvature of the connection $\nabla_{\mathrm{Det}_\gamma}$ seen above, and so (7.2) can likewise be interpreted for $\omega = d \ln \tau_\Lambda$.

The further significance of relations of the type (7.2) has been pointed out in [46], where ω on the one hand represents the poles of solutions to the Schlesinger equation, and on the other hand, it is the Hamiltonian of this equation with respect to a natural Poisson structure. The task undertaken in [46] was to give a detailed explanation of the equivalence of these two approaches.

7.2. Operator meromorphic connection

Next we consider how a *meromorphic connection* denoted $\omega_{\mathbf{V}}$ can be constructed on a complex vector bundle $\mathbf{V} \longrightarrow \Lambda$, by commencing from a *meromorphic operator* ($\mathfrak{g}$-*valued*) *function*. Operator-valued meromorphic functions as providing suitable potentials, denoted say $B(\zeta)$ with respect to a local coordinate ζ on some domain in X, have been studied to a significant extent in, e.g., [30, 31, 47]. Typically $B(\zeta)$ is of the form

$$B(\zeta) = \sum_{j=-n}^{\infty} (\zeta - \zeta_0)^j B_j, \tag{7.3}$$

satisfying some mild technical condition (such as 'normality' [30, §3]).

Recall from Remark 2.2 the connection map $\mathcal{V} : TQ \longrightarrow TQ$ used in constructing the connection 1-form ω_Q. To say that $\mathcal{V}$ is *meromorphic* means that $\mathcal{V}$ is pointwise an analytic map with a countable number of poles. The resulting connection ω_Q is then said to be *a meromorphic connection* if, in its local representation, it contains the data of the Laurent coefficients as exhibited in (7.3). This property passes over to the induced connection 1-form $\omega_{\mathbf{V}}$ on the associated complex vector bundle $\mathbf{V} \longrightarrow \Lambda$ (see below).

Granted a supply of holomorphic maps from X to $\mathrm{Gr}(p, A)$, and thus to Λ by Proposition 7.1, we give an application in the following. To do this, let us first recall that $\gamma \longrightarrow \Lambda$ is the vector bundle associated to a principal G-bundle $Q \longrightarrow \Lambda$ as in §2.1 (where $G \subseteq G(pAp)$). Let $\mathcal{V} : TQ \longrightarrow TQ$ be a meromorphic connection map on the principal bundle $Q \longrightarrow \Lambda$. In the usual way, this induces a connection map, denoted the same, $\mathcal{V} : T\gamma \longrightarrow T\gamma$ on the associated vector bundle (the formal details for doing this can be seen in, e.g., [43, p. 381]), and hence we obtain a meromorphic connection on the holomorphic vector bundle $\gamma \longrightarrow \nabla_{\gamma}$.

When 'Det' is well defined under the criteria we had discussed earlier, then we obtain an induced connection map $\mathcal{V}_{Det} : T\mathrm{Det}_{\gamma} \longrightarrow T\mathrm{Det}_{\gamma}$ yielding a local (operator) connection meromorphic 1-form $\widehat{\omega}$. Thus as an immediate consequence of the usual pull-back construction, if $f : X \longrightarrow \Lambda$ is a non-constant holomorphic map, and $\widehat{\omega}$ is a meromorphic connection 1-form on Det_{γ}, then on X we obtain a holomorphic vector bundle $(f^*\mathrm{Det}_{\gamma}, f^*\widehat{\omega}) \longrightarrow X$ with a meromorphic connection.

Remark 7.1. Following, e.g., [66], one may look at 'free energy pre-potentials' in the context of matrix models and topological conformal field theories where the relevant object to consider is $\ln \tau$. Indeed, as we have pointed out in §4.1, it has been known that $\ln \tau$ is a 'prepotential' (for a local connection 1-form, say) as apparent in [10, 36, 37, 42, 46, 56, 66]. Thus somewhat in the spirit of how one applies monodromy preserving transformations [33, 36, 66], we are led from the matrix case to operator-valued meromorphic functions comprised of Laurent coefficients (such as $B(\zeta)$ above) and hence towards a class of distinguished meromorphic (connection) 1-forms

$$\omega = d \ln \tau = \sum_{n=-\infty}^{\infty} \mathrm{Tr}\ B_n(\zeta - \zeta_0)^n\ dt_n. \tag{7.4}$$

7.3. Examples

Given that a meromorphic 1-form in the operator setting of Λ can be pulled back via a holomorphic map from X to $\mathrm{Gr}(p, A)$, or to Λ using Proposition 7.1, let us see some motivation for doing this in the operator context as provided by the following examples of Schlesinger systems relative to X (cf. [39] for matrix operator coefficients). One should keep in mind the role of the τ-function, and the setting of the determinant line bundles with meromorphic connection $(f^*\mathrm{Det}_{\gamma}, f^*\widehat{\omega}) \longrightarrow X$, as we have described above.

Example 7.2. Inspired by the setting of the Riemann–Hilbert problem, let us consider an equation of the form

$$\frac{d\Psi}{d\zeta}(\zeta) = B(\zeta)\Psi(\zeta), \tag{7.5}$$

where $\Psi(\zeta)$ is a meromorphic G-valued function, and $B(\zeta)$ in (7.3) has poles $\{b_1, b_2, \ldots, b_n\}$ that are viewed as variables, and $\{b_1, b_2, \ldots, b_n, \infty\}$ branch points for $B(\zeta)$ when viewed as points of X. Granted Ψ can be continued along a closed path γ, away from the branch points within this embedding, we consider a transformation of the type

$$\Psi(\zeta) \longrightarrow \Psi(\zeta)Z_\gamma, \tag{7.6}$$

where $Z_\gamma \in G \subseteq G(pAp)$ is a (constant) invertible operator depending only on the homotopy class $[\gamma]$ of γ. In this way Z_γ induces a monodromy representation of the fundamental group

$$\pi_1(X/\{b_1, b_2, \ldots, b_n\}) \longrightarrow G, \text{ where } [\gamma] \mapsto Z_\gamma. \tag{7.7}$$

We recall that *a Schlesinger transformation* can be regarded as a discrete monodromy preserving transformation of a meromorphic connection matrix that shifts by elements of $\mathbb{Z}$, the eigenvalues of its residues [11, 42]. A modification of this theory to matrix operator coefficients of differential equations of Fuchsian type is studied in [39], and as was mentioned in the Introduction, motivates a broader scope of applications using operator-valued functions.

Example 7.3. For $g_X = 0$ and $G \cong \mathrm{SL}(2, \mathbb{C})$, the corresponding Schlesinger system satisfies [42, §II]

$$\begin{aligned} \frac{\partial B_i}{\partial \zeta_j} &= \frac{[B_i, B_j]}{\zeta_i - \zeta_j}, \text{ for } i \neq j, \\ \frac{\partial B_i}{\partial \zeta_i} &= -\sum_{j \neq i} \frac{[B_i, B_j]}{\zeta_i - \zeta_j}, \end{aligned} \tag{7.8}$$

where the $B_j = B_j(b_1, b_2, \ldots, b_n)$ are taken to be certain meromorphic functions (cf. [36, 37, 39, 46]). Here the role of the meromorphic connection is prominent in [46]. One starts with a holomorphically trivial vector bundle $(\mathbf{V}_0, \nabla_{\mathbf{V}_0}) \longrightarrow \mathbb{C}P^1$, with meromorphic connection having logarithmic poles at points $(b_1^0, \ldots, b_n^0, \infty)$. This can be deformed isomonodromically to a holomorphic vector bundle with meromorphic connection $(\mathbf{V}, \nabla_{\mathbf{V}}) \longrightarrow \mathbb{C}P^1 \times \mathcal{D}$, where $\mathcal{D}$ is certain deformation space, and for which the restriction $(\mathbf{V}, \nabla_{\mathbf{V}})|(z, b_1^0, \ldots, b_n^0, \infty) \cong (\mathbf{V}_0, \nabla_{\mathbf{V}_0})$. The integrability of this isomonodromic extension leads to the Schlesinger equation:

$$dB_i = -\sum_{i \neq j} \frac{[B_i, B_j]\ d(b_i - b_j)}{(b_i - b_j)}. \tag{7.9}$$

A class of meromorphic connections also appears in [10] which considers a similar deformation problem.

Example 7.4. There is a class of wave functions $\Psi = \Psi(\zeta)$, defined relative to a spectral parameter ζ, whose monodromy data is independent of the deformation parameter if and only if Ψ satisfies the deformation equation (cf. (7.5))

$$\partial_\zeta \Psi = dB(\zeta)\Psi, \tag{7.10}$$

where $dB(\zeta) = \sum_i B_i d\zeta_i$ is a 1-form with rational coefficients $B_i = B_i(\zeta)$. Now to any solution of (7.10) there is an associated 1-form

$$\omega = -\sum_k \mathrm{Res}_{\zeta=\zeta_k} \mathrm{Tr}[g^{(k)-1}\partial_\zeta g^{(k)} d\xi^{(k)}]d\zeta, \tag{7.11}$$

for certain analytic functions $g = g(\zeta)$ and diagonal matrices ξ [3, §8.4], where the deformation equation (7.10) implies that $d\omega = 0$. In fact, we have $\omega = d\ln\tau$, and hence for each solution of (7.10) a τ-function can be associated and one that is transformable under an elementary Schlesinger transformation [3, §8.6].

Example 7.5. Following, e.g., [11, §6] and [36, 42], for the case $g_X = 1$, we have an $\mathfrak{sl}(2,\mathbb{C})$-valued 1-form ω given by

$$\omega = \sum_{j=1}^{k} \sum_{\substack{1\le\ell\le n\\ \ell\neq j}} \frac{\mathrm{Tr}(B_j B_k)}{b_j - b_\ell}\, db_\ell. \tag{7.12}$$

In this case $\tau = \tau(b_1, b_2, \dots, b_k)$ is a τ-function for the system such as appears in (7.8) if $d\ln\tau = \omega$ (see [11, 36]), so τ is defined at least locally. With regards to the Painlevé property, any solution $\{B_j\}_{j=1}^n$ of the Schlesinger equations are analytic functions in $(b_1, b_2, \dots, b_n)$ that have at most poles in addition to the fixed singularities $b_j = b_\ell$, for some $j \neq \ell$ [11] (see also [40]).

Appendix A. Briefly Kac–Moody–Lie algebras

A.1. The basic definitions

The $\ell \times \ell$ Cartan matrix $C = [c_{ij}]$ $(i, j \in \mathcal{I})$ of any semisimple Lie algebra $\mathfrak{g}_\mathbb{C}$ satisfies the conditions

(1) $c_{ij} \in \mathbb{Z}$, for all i, j;
(2) $c_{ii} = 2$, for all i;
(3) $c_{ij} \le 0$ if $i \neq j$;
(4) $c_{ij} = 0$ whenever $c_{ji} = 0$;
(5) The matrix C is positive definite in the sense that all of the principal minors of C are positive.

Conversely, if we have an $\ell \times \ell$ matrix C satisfying (1)–(4), the Cartan structural relations of $\mathfrak{g}_\mathbb{C}$ define an abstract complex Lie algebra $\mathfrak{g}'$ called *the Kac–Moody–Lie algebra defined by* C. If in addition C satisfies (5), then $\mathfrak{g}'$ will be finite-dimensional and also semisimple. But if (5) does not hold, then $\mathfrak{g}'$ will be infinite-dimensional. There is a way of modifying this latter case so that much of the finite-dimensional theory can apply directly. We refer to [38, Chap. 1] and [57, §5.3] for details and retain the notation $\mathfrak{g}'$ once this modification has been done.

Let $\mathfrak{L} = \mathbb{C}((t))(= \mathbb{C}[t, t^{-1}])$ denote the algebra of Laurent polynomials in t. Following [38, §7.1], we have the loop algebra $\mathfrak{L}(\mathfrak{g}') = \mathfrak{L} \otimes_{\mathbb{C}} \mathfrak{g}'$. The central extension $\widetilde{\mathfrak{L}\mathfrak{g}}'$ of $\mathfrak{L}\mathfrak{g}'$ satisfies (1)–(4), and (5)$'$: det $C = 0$, and all the proper principal minors of C are positive. Conversely, the Kac–Moody–Lie algebras corresponding to Cartan matrices following (1)–(4) and (5)$'$ are called *affine Kac–Moody–Lie algebras* (see in particular [38, Theorem 7.4]).

Remark A.1. We could assume that C is *locally finite*, meaning that for each $j \in \mathbb{Z}$, we have $c_{ij} = 0$, except for a finite number of the i's.

A.2. The Weyl–Coxeter group

With respect to a collection of roots $\{\nu_j\}$, the *root lattice* $R = R(C)$ and the *co-root lattice* $R^\vee = R^\vee(C)$ are defined by

$$R = \bigoplus_{j \in \mathcal{I}} \mathbb{Z}\nu_j, \text{ and } R^\vee = \bigoplus_{j \in \mathcal{I}} \mathbb{Z}\nu_j^\vee, \tag{A.1}$$

respectively, together with the pairing $\langle , \rangle : R^\vee \times R \longrightarrow \mathbb{Z}$. Then let $\mathcal{W} = \mathcal{W}(C)$ be the *Weyl–Coxeter group* as defined by generators $s_i (i \in \mathcal{I})$ satisfying the relations

$$s_i^2 = 1, \ (s_i s_j)^{m_{ij}} = 1, \ (i, j \in \mathcal{I}, i \neq j), \tag{A.2}$$

where $m_{ij} = 2, 3, 4, 6$ or ∞, according to whether $c_{ij} c_{ji} = 0, 1, 2, 3$ or ≥ 4, respectively. As is well known in the theory of Coxeter groups, the generators s_i act naturally on the root lattice R by reflections

$$s_i(\nu_j) = \nu_j - \nu_i < \nu_i^\vee, \nu_j > = \nu_j - \nu_i c_{ij}, \tag{A.3}$$

(see, e.g., [38, Chap. 6], and in particular [53, §2] in relationship to §6.3). Let

$$R^\star = \mathrm{Hom}_{\mathbb{Z}}(R^\vee, \mathbb{Z}), \tag{A.4}$$

denote the dual $\mathbb{Z}$-module of the coroot lattice $R^\vee$. Taking the dual basis $\{\Lambda_j\}_{j \in \mathcal{I}}$ of $\{\nu_j^\vee\}$ so that $R^\star = \oplus_{j \in \mathcal{I}} \mathbb{Z}\Lambda_j$, there exists a natural $\mathcal{W}$-homomorphism $R \longrightarrow R^\star$, such that $\nu_j \longrightarrow \sum_{i \in \mathcal{I}} \Lambda_i c_{ij}$ [53, §2].

Appendix B. The Painlevé equations

This subject has a rich and illustrious history as finely surveyed in the monograph [15](see also [14]) to which we refer the reader who wishes pursue further details as well as the interesting historical background.

Painlevé studied second-order nonlinear ODEs of the form

$$y'' = F(y, y', z), \ y = y(z), \tag{B.1}$$

where $F(y', y, z)$ is rational in y and y', and analytic in z. The problem (originally posed by E. Picard) was to identify all equations of the form (B.1) for which the solutions have "no movable critical points", i.e., the locations of any branch points or essential singularities do not depend on the constants of integration of

(B.1). Painlevé studied an assortment of 50 'canonical equations', up to Möbius transformations, whose solutions have no movable critical points and then reduced the study to six particular types (P_{I}–P_{VI}, the solutions to which are often called *the Painlevé transcendents*) [15] and [14, §1]:

P_{I} $y'' = 6y^2 + z$

P_{II} $y'' = 2y^3 + zy + \alpha$

P_{III} $y'' = \frac{(y')^2}{y} - \frac{y'}{z} + \frac{\alpha y^2+\beta}{z} + \gamma y^3 + \frac{\delta}{y}$

P_{IV} $y'' = \frac{(y')^2}{2y} + \frac{3}{2}y^3 + 4zy^2 + 2(z^2-\alpha)y + \frac{\beta}{y}$

P_{V} $y'' = \left(\frac{1}{2y} + \frac{1}{y-1}\right)(y')^2 - \frac{y'}{z} + \frac{(y-1)^2}{z^2}\left(\alpha y + \frac{\beta}{y}\right) + \frac{\gamma y}{z} + \frac{\delta y(y+1)}{y-1}$

P_{VI} $y'' = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-z}\right)(y')^2 - \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{y-z}\right)y'$
$\quad + \frac{y(y-1)(y-z)}{z^2(z-1)^2}\left\{\alpha + \frac{\beta z}{y^2} + \frac{\gamma(z-1)}{(y-1)^2} + \frac{\delta z(z-1)}{(y-z)^2}\right\}$

where the $\alpha, \ldots, \delta$ are constants. The solutions are aptly named 'transcendents' because they cannot be expressed in terms of traditional special functions.

Example B.1. Following [54], we exemplify properties of P_{I}. Let $y = y(z)$ be a solution of

$$y'' = 6y^2 + z. \tag{B.2}$$

The function y is meromorphic on $\mathbb{C}$, and there is a function $\tau = \tau(z)$ holomorphic on $\mathbb{C}$ such that

$$y(z) = -\left(\frac{d}{dz}\right)^2 \ln\tau = \frac{(\tau')^2 - \tau\tau''}{\tau^2}. \tag{B.3}$$

Setting $\eta = \frac{\tau'}{\tau}$, then η is a solution of a third-order ODE in τ:

$$\eta'' - 4(\eta')^3 - 2z - 2\eta = 0. \tag{B.4}$$

Consider the polynomial in y and μ:

$$H_{I}(z; y, \mu) = \frac{1}{2}\mu^2 - 2y^3 - zy. \tag{B.5}$$

Equation (B.2) is equivalent to a Hamiltonian system:

$$\begin{cases} y' &= \frac{\partial H}{\partial \mu} \\ \mu' &= -\frac{\partial H}{\partial y} \end{cases} \tag{B.6}$$

In fact, each type of Painlevé equation can be written as a system such as (B.6) [54]. If $(y(z), \mu(z))$ is a solution to (B.6), then

$$\tau(z) = \exp\left[\int H(z)\; dz\right], \tag{B.7}$$

where $H(z) = H_{I}(z; y(z), \mu(z))$.

Example B.2. A modified version of the Korteweg–de Vries equation is

$$u_t - 6u^2u_x + u_{xxx} = 0. \tag{B.8}$$

As pointed out in [14], (B.8) can be re-scaled by setting $u(x,t) = (3t)^{-\frac{1}{3}} y(z)$ where $z = x(3t)^{-\frac{1}{3}}$, and this is solvable via an inverse scattering transform, where $y(z)$ satisfies the equation $\mathrm{P_{II}}$: $y'' = 2y^3 + zy + \alpha$.

In applications of the Painlevé equations to integrable systems it is usually the case that for each type PI-PVI there is a companion τ-function playing a significant role, as Example B.1 reveals. Likewise, this special relationship shows up in other areas such as plasma physics, quantum optics, general relativity and quantum gravity (cf. §4.1).

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