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Some New Refined Hardy Type Inequalities with Breaking Points $p = 2$ or $p = 3$

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Abstract. For usual Hardy type inequalities the natural "breaking point" (the parameter value where the inequality reverses) is $p = 1$. Recently, J. Oguntuase and L.-E. Persson proved a refined Hardy type inequality with breaking point at $p = 2$. In this paper we show that this refinement is not unique and can be replaced by another refined Hardy type inequality with breaking point at $p = 2$. Moreover, a new refined Hardy type inequality with breaking point at $p = 3$ is obtained. One key idea is to prove some new Jensen type inequalities related to convex or superquadratic funcions, which are also of independent interest.

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1. Introduction

First we consider the following well-known Hardy-type inequality: If the function f is non-negative and measurable on $(0, \infty)$, then

$$\int_0^b \left(\frac{1}{x} \int_0^x f(y)\, dy \right)^p x^\alpha dx \leq \left(\frac{p}{p-1-\alpha} \right)^p \int_0^b f^p(x)\, x^\alpha \left(1 - \left(\frac{x}{b} \right)^{\frac{p-\alpha-1}{p}} \right) dx, \tag{1.1}$$

where $p \geq 1$ and $\alpha < p - 1$. The constant on the right-hand side is sharp. A simple proof of this inequality was recently presented in [12], where also some historical remarks can be found. In particular, for $\alpha = 0$ and for $b = \infty$ (1.1) is the classical Hardy inequality, stated by G.H. Hardy in 1920 (see [2]) and where it was finally proved in 1925 (see [3]). Moreover, for $b = \infty$ (1.1) coincides with the first weighted version also proved by Hardy himself in 1928 (see [4]). Further development of Hardy-type inequalities can be found in the books [6], [7] and [8]. Even if the inequality (1.1) is sharp, it is possible to refine it by inserting a second positive term in the left-hand side of (1.1). Such a result was first proved by C.O. Imoru in 1977 (see [5], and also its generalization in [10]). In all cases the "breaking point", i.e., the point where the inequality reverses is $p = 1$. In 2008 J.A. Oguntuase and L.-E. Persson proved the following refined Hardy inequality with "breaking point" $p = 2$ (see [9] and cf. also [11]): Let $p \geq 1$, $\alpha < p - 1$ and $0 < b \leq \infty$. If $p \geq 2$, and the function f is non-negative and locally integrable on $(0, b)$ and $\int_0^b x^\alpha f^p(x)\, dx < \infty$, then

$$\begin{aligned}\int_0^b \left(\frac{1}{x}\int_0^x f(y)\,dy\right)^p x^\alpha dx + \frac{p-1-\alpha}{p}\int_0^b \int_t^b \left|\frac{p}{p-\alpha-1}\left(\frac{t}{x}\right)^{1-\frac{p-\alpha-1}{p}} f(t)\right. \\ \left. -\frac{1}{x}\int_0^x f(\tau)\,d\tau\right|^p x^{\alpha-\frac{p-\alpha-1}{p}} dx \cdot t^{\frac{p-\alpha-1}{p}-1} dt \\ \leq \left(\frac{p}{p-\alpha-1}\right)^p \int_0^b f^p(x)\, x^\alpha \left(1-\left(\frac{x}{b}\right)^{\frac{p-\alpha-1}{p}}\right) dx. \qquad (1.2)\end{aligned}$$

If $1 < p \leq 2$, then (1.2) holds in the reversed direction. In particular, for $p = 2$ we have equality in (1.2).

In this paper we derive a new refined Hardy type inequality different than (1.2) but again, with a natural breaking point at $p = 2$ (see Theorem 3.2). We also present and prove another new refined Hardy type inequality with breaking point at $p = 3$ (see Theorem 3.4).

One key idea is to prove some new Jensen type inequalities connected to functions of the type $F(x) = x\varphi(x)$, where φ is convex/concave or supequadratic/subquadratic. These results are of independent interest. Next we present and prove some new inequalities for such functions (see Propositions 3.1 and 3.3), which, in their turn, are crucial for the proofs of the new Hardy type inequalities.

The paper is organized as follows: In order not to disturb our discussions later on, the new Jensen type inequalities and other preliminaries are collected in Section 2. The main results concerning refined Hardy type inequalities are stated and discussed in Section 3 and the proofs are given in Section 4.

2. Preliminaries

Our first important Lemma reads:

Lemma 2.1. *Let $K(x) = x\varphi(x)$, where $\varphi(x)$ is convex on $[0, b)$. Then*

$$K(y) - K(x) \geq \varphi(x)(y-x) + C_\varphi(x)\, y\,(y-x), \qquad (2.1)$$

holds for $x \in [0, b)$, $y \in [0, b)$, C_φ is the constant in the definition of convexity and the inequality

$$\begin{aligned}\int_\Omega K(f(s))\,d\mu(s) - K\left(\int_\Omega f(s)\,d\mu(s)\right) \qquad (2.2) \\ \geq \int_\Omega C_\varphi(x) f(s)(f(s)-x)\,d\mu(s) = \int_\Omega C_\varphi(x)(f(s)-x)^2\,d\mu(s)\end{aligned}$$

holds, where f is any non-negative μ-integrable function on the probability measure space (Ω, μ) and $x = \int_\Omega f d\mu$. The inequalities (2.1) and (2.2) hold in the reverse direction when φ is concave.

Example. The inequalities (2.1) and (2.2) are satisfied in particular by $K(x) = x^p$, $p \geq 2$. For $1 < p \leq 2$ the reverse inequalities hold. They reduce to equalities for $p = 2$.

Proof of Lemma 2.1. Multiplying by y the inequality satisfied by any convex function

$$\varphi(y) - \varphi(x) \geq C_\varphi(x)(y - x), \tag{2.3}$$

by simple manipulations we get that $K(x) = x\varphi(x)$ satisfies (2.1) when φ is convex.

By putting $y = f(s)$ and $x = \int_\Omega f d\mu$ in (2.1) and integrating with respect to the probability measure μ we arrive at (2.2). Moreover, if φ is concave, then (2.3) holds in the reverse direction so the same proof as above shows that in fact (2.1) and (2.2) both hold in the reverse direction when $K(x) = x\varphi(x)$, where $\varphi(x)$ is concave. □

Remark 2.2. Inequality (2.2) may be regarded as a new type of Jensen inequality, for the functions $K(x) = x\varphi(x)$, where φ is convex/concave.

Next we define the crucial concept of superquadratic and subquadratic functions (see [1]).

Definition 2.3. Let $\varphi : [0, b) \to \mathbb{R}$. The function φ is superquadratic if for all $x \in [0, b)$ there exists $C_\varphi \in \mathbb{R}$ such that

$$\varphi(y) - \varphi(x) \geq C_\varphi(x)(y - x) + \varphi(|y - x|) \tag{2.4}$$

for all $y \in [0, b)$.

The function φ is subquadratic if $-\varphi$ is superquadratic and the reverse inequality of (2.4) holds.

Remark 2.4. Inequality (2.4) holds for all $\varphi(x) = x^p$, $x \geq 0$, $p \geq 2$. It holds in the reverse direction if $0 < p < 2$ and it reduces to equality for $\varphi(x) = x^2$.

The following result is useful (see [1, Lemma 2.1]):

Lemma 2.5. *Let φ be a superquadratic function with $C_\varphi(x)$ as in (2.4).*

(i) *Then $\varphi(0) \leq 0$.*
(ii) *If $\varphi(0) = \varphi'(0) = 0$, then $C_\varphi(x) = \varphi'(x)$ whenever φ is differentiable at $x > 0$.*
(iii) *If $\varphi \geq 0$, then φ is convex and $\varphi(0) = \varphi'(0) = 0$.*

We are now ready to formulate the similar result as in Lemma 2.1 but when φ is superquadratic instead of convex.

Lemma 2.6. *Let $K(x) = x\varphi(x)$, where $\varphi(x)$ is superquadratic on $[0, b)$. Then*

$$K(y) - K(x) \geq \varphi(x)(y - x) + C_{\varphi}(x)\, y\,(y - x) + y\varphi(|y - x|) \tag{2.5}$$

holds for $x \in [0, b)$, $y \in [0, b)$, $C_{\varphi}(x)$ is defined by (2.4) and

$$\begin{aligned} &\int_{\Omega} K(f(s))\, d\mu(s) \quad - K\left(\int_{\Omega} f(s)\, d\mu(s)\right) \\ &\geq \int_{\Omega} \left[C_{\varphi}(x) f(s)(f(s) - x) + f(s)\varphi(|f(s) - x|)\right] d\mu(s) \end{aligned} \tag{2.6}$$

holds, where f is any non-negative μ-integrable function on the probability measure space (Ω, μ) and $x = \int_{\Omega} f d\mu$. The inequalities (2.5) and (2.6) hold in the reverse direction when φ is subquadratic.

Example. The inequalities (2.5) and (2.6) are satisfied in particular by $K(x) = x^p$, $p \geq 3$. For $1 < p \leq 3$ the reverse inequalities hold. They reduce to equalities for $p = 3$.

Proof of Lemma 2.6*.* Multiplying (2.4) by y, by simple manipulations we get that $K(x) = x\varphi(x)$ satisfies (2.5) when φ is superquadratic.

Next we consider (2.5) with $x = \int_{\Omega} f d\mu$ and $y = f(s)$, and integrate with respect to the probability measure μ and obtain (2.6). Furthermore, if φ is subquadratic, then (2.4) holds in the reverse direction and the same proof as above shows that (2.5) and (2.6) hold in the reverse direction in this case. □

Remark 2.7. Inequality (2.6) may be regarded as a new Jensen type inequality for the functions $K(x) = x\varphi(x)$ where φ is superquadratic/subquadratic.

3. The main results

In this section we state the main results concerning Hardy type inequalities related to $K(x) = x\varphi(x)$, where $\varphi(x)$ is convex/concave and then to $K(x) = x\varphi(x)$, where $\varphi(x)$ is superquadratic/subquadratic. The proofs are given in the next section.

The following result is crucial for the proof of Theorem 3.2 and of independent interest.

Proposition 3.1. *Let $0 < b \leq \infty$, $u : (0, \infty) \to \mathbb{R}$ be a non-negative weight function such that $\frac{u(x)}{x^2}$ is locally integrable on $(0, \infty)$ and let the weight function v be defined by*

$$v(t) = t \int_t^b \frac{u(x)}{x^2} dx, \quad t \in (0, b). \tag{3.1}$$

If the function φ *is integrable and convex on* $[0,b)$ *and* $K(x) = x\varphi(x)$, *then the inequality*

$$\begin{aligned}
&\int_0^b K(f(x))\frac{v(x)}{x}dx - \int_0^b K\left(\frac{1}{x}\int_0^x (f(t)dt)\right)\frac{u(x)}{x}dx \qquad (3.2)\\
&\geq \int_0^b \int_t^b f(t)\left(f(t) - \frac{1}{x}\int_0^x f(\tau)\,d\tau\right) C_\varphi\left(\frac{1}{x}\int_0^x f(\tau)\,d\tau\right)\frac{u(x)}{x^2}dxdt\\
&= \int_0^b \int_t^b \left(f(t) - \frac{1}{x}\int_0^x f(\tau)\,d\tau\right)^2 C_\varphi\left(\frac{1}{x}\int_0^x f(\tau)\,d\tau\right)\frac{u(x)}{x^2}dxdt
\end{aligned}$$

holds for all non-negative locally integrable functions f *and with* C_φ *defined by* (2.3). *If* φ *is concave, then* (3.2) *holds in the reverse direction.*

Example. From Proposition 3.1 for $\varphi(x) = x^{p-1}$, $p \geq 2$ (therefore $C_\varphi(x) = \varphi'(x) = (p-1)x^{p-2}$), choosing $u(x) = 1$, we find that

$$\begin{aligned}
&\int_0^b \left(1 - \frac{x}{b}\right) f^p(x)\frac{dx}{x} - \int_0^b \left(\frac{1}{x}\int_0^x f(t)\,dt\right)^p \frac{dx}{x} \qquad (3.3)\\
&\geq (p-1)\int_0^b \int_t^b \left(f(t) - \frac{1}{x}\int_0^x f(\tau)\,d\tau\right)^2 \left(\frac{1}{x}\int_0^x f(\tau)\,d\tau\right)^{p-2}\frac{dx}{x^2}dt.
\end{aligned}$$

The inequality (3.3) holds in the reverse direction for $1 < p < 2$.

By using (3.3) we are now ready to derive our new refined Hardy type inequality with breaking point $p = 2$.

Theorem 3.2. *Let* $p \geq 2$, $k > 1$, $0 < b \leq \infty$, *and let the function* f *be non-negative and locally integrable on* $(0,b)$. *Then*

$$\begin{aligned}
&\left(\frac{p}{k-1}\right)^p \int_0^b \left(1 - \left(\frac{x}{b}\right)^{\frac{k-1}{p}}\right) x^{p-k} f^p(x)\,dx - \int_0^b x^{-k}\left(\int_0^x f(t)\,dt\right)^p dx\\
&\geq \frac{(p-1)}{p}(k-1)\int_0^b \int_t^b \left(f(t)\frac{p}{k-1}\left(\frac{t}{x}\right)^{1-\frac{k-1}{p}} - \frac{1}{x}\int_0^x f(\sigma)\,d\sigma\right)^2\\
&\cdot \left(\frac{1}{x}\int_0^x f(\sigma)\,d\sigma\right)^{p-2} x^{\left(1-\frac{k-1}{p}\right)(p+1)} t^{\frac{k-1}{p}-1}\frac{dx}{x^2}dt. \qquad (3.4)
\end{aligned}$$

Moreover, the double integral of the right-hand side of (3.4) *is non-negative. If* $1 < p \leq 2$, *then the inequality* (3.4) *holds in reverse direction. Equality holds when* $p = 2$.

Next we formulate a result which is similar to Proposition 3.1 but where $\varphi(x)$ is superquadratic instead of convex.

Proposition 3.3. *Let* $0 < b \leq \infty$, $u : (0,\infty) \to \mathbb{R}$ *be a non-negative weight function such that* $\frac{u(x)}{x^2}$ *is locally integrable on* $(0,\infty)$ *and let the weight function* v *be* *defined by*

$$v(t)=t\int_t^b \frac{u(x)}{x^2}dx, \quad t\in(0,b)\,. \tag{3.5}$$

If the function φ *is integrable and superquadratic on* $[0,b)$ *and* $K(x)=x\varphi(x)$, *then the inequality*

$$\begin{aligned}
&\int_0^b K(f(x))\frac{v(x)}{x}dx-\int_0^b K\left(\frac{1}{x}\int_0^x (f(t)dt)\right)\frac{u(x)}{x}dx \\
&\geq \int_0^b\int_t^b f(t)\left(f(t)-\frac{1}{x}\int_0^x f(\tau)\,d\tau\right)C_\varphi\left(\frac{1}{x}\int_0^x f(\tau)\,d\tau\right)\frac{u(x)}{x^2}dxdt \\
&\quad+\int_0^b\int_t^b f(t)\varphi\left(\left|f(t)-\frac{1}{x}\int_0^x f(\tau)\,d\tau\right|\right)\frac{u(x)}{x^2}dxdt \\
&=\int_0^b\int_t^b \left(f(t)-\frac{1}{x}\int_0^x f(\tau)\,d\tau\right)^2 C_\varphi\left(\frac{1}{x}\int_0^x f(\tau)\,d\tau\right)\frac{u(x)}{x^2}dxdt \\
&\quad+\int_0^b\int_t^b f(t)\varphi\left(\left|f(t)-\frac{1}{x}\int_0^x f(\tau)\,d\tau\right|\right)\frac{u(x)}{x^2}dxdt,
\end{aligned} \tag{3.6}$$

holds for all non-negative locally integrable functions f *and with* C_φ *defined by* (2.4). *If the function* φ *is subquadratic, then* (3.6) *holds in the reverse direction.*

By using (3.6) we are now ready to state our next new refined Hardy type inequality with breaking point $p=3$.

Theorem 3.4. *Let* $p\geq 3$, $k>1$, $0<b\leq\infty$, *and let the function* f *be non-negative and locally integrable on* $(0,b)$. *Then*

$$\begin{aligned}
&\left(\frac{p}{k-1}\right)^p\int_0^b\left(1-\left(\frac{x}{b}\right)^{\frac{k-1}{p}}\right)x^{p-k}f^p(x)\,dx-\int_0^b x^{-k}\left(\int_0^x f(t)\,dt\right)^p dx \\
&\geq \frac{p-1}{p}(k-1)\int_0^b\int_t^b\left(f(t)\frac{p}{k-1}\left(\frac{t}{x}\right)^{1-\frac{k-1}{p}}-\frac{1}{x}\int_0^x f(\sigma)\,d\sigma\right)^2 \\
&\quad\cdot\left(\frac{1}{x}\int_0^x f(\sigma)\,d\sigma\right)^{p-2}x^{\left(1-\frac{k-1}{p}\right)(p+1)}t^{\frac{k-1}{p}-1}\frac{dx}{x^2}dt \\
&\quad+\int_0^b\int_t^b\left(f(t)\,t^{1-\frac{k-1}{p}}\right)\left(\left|f(t)\frac{p}{k-1}\left(\frac{t}{x}\right)^{1-\frac{k-1}{p}}-\frac{1}{x}\int_0^x f(\sigma)\,d\sigma\right|\right)^{p-1} \\
&\quad\cdot x^{\left(1-\frac{k-1}{p}\right)p}t^{\frac{k-1}{p}-1}\frac{dx}{x^2}dt.
\end{aligned} \tag{3.7}$$

Moreover, each double integral of the right-hand side of (3.7) *is non-negative. If* $1<p\leq 3$, *then the inequality* (3.7) *holds in the reverse direction. Equality holds when* $p=3$.

4. Proofs

Proof of Proposition 3.1. Let us choose in (2.2) the probability measure $d\mu(t) = \frac{1}{x}dt$, $0 \le t \le x$. Then

$$\begin{aligned} &\frac{1}{x}\int_0^x K(f(t))\,dt - K\left(\frac{1}{x}\int_0^x f(t)\,dt\right) \\ &\quad \ge C_\varphi\left(\frac{1}{x}\int_0^x f(t)\,dt\right)\frac{1}{x}\int_0^x f(t)\left(f(t) - \frac{1}{x}\int_0^x f(\tau)\,d\tau\right)dt \\ &\quad = C_\varphi\left(\frac{1}{x}\int_0^x f(t)\,dt\right)\frac{1}{x}\int_0^x \left(f(t) - \frac{1}{x}\int_0^x f(\tau)\,d\tau\right)^2 dt. \end{aligned} \tag{4.1}$$

Multiplying (4.1) by $\frac{u(x)}{x}$ and integrating over $0 \le x \le b$, we get that

$$\begin{aligned} &\int_0^b \int_0^x K(f(t))\,dt\frac{u(x)}{x^2}dx - \int_0^b K\left(\frac{1}{x}\int_0^x f(t)\,dt\right)\frac{u(x)}{x}dx \\ &\quad \ge \int_0^b \int_0^x \left(f(t) - \frac{1}{x}\int_0^x f(\tau)\,d\tau\right)^2 C_\varphi\left(\frac{1}{x}\int_0^x f(\tau)\,d\tau\right)\frac{u(x)}{x^2}dtdx. \end{aligned} \tag{4.2}$$

Now, by using (3.1) and Fubini's theorem, we find that

$$\begin{aligned} \int_0^b \int_0^x K(f(t))\frac{u(x)}{x^2}dtdx &= \int_0^b \frac{K(f(t))}{t}\left(t\int_t^b \frac{u(x)}{x^2}dx\right)dt \\ &= \int_0^b K(f(t))\frac{v(t)}{t}dt = \int_0^b K(f(x))\frac{v(x)}{x}dx. \end{aligned} \tag{4.3}$$

(4.2) and (4.3) lead to (3.2). In the case that $K(x) = x\varphi(x)$ and φ is concave then (4.1), (4.2) and (4.3) hold in the reverse direction, which leads to the reverse of (3.2). The proof is complete. □

Proof of Theorem 3.2. We denote the right-hand side of (3.3) by R and replace the parameter b by $b^{\frac{k-1}{p}}$ and $f(x)$ by $f\left(x^{\frac{p}{k-1}}\right)x^{\frac{p}{k-1}-1}$. Then

$$\begin{aligned} R = \int_0^{b^{\frac{k-1}{p}}} \int_t^{b^{\frac{k-1}{p}}} &\left(f\left(t^{\frac{p}{k-1}}\right)t^{\frac{p}{k-1}-1} - \frac{1}{x}\int_0^x f\left(\tau^{\frac{p}{k-1}}\right)\tau^{\frac{p}{k-1}-1}d\tau\right)^2 \\ &\cdot (p-1)\left(\frac{1}{x}\int_0^x f\left(\tau^{\frac{p}{k-1}}\right)\tau^{\frac{p}{k-1}-1}d\tau\right)^{p-2}\frac{dx}{x^2}dt\,. \end{aligned} \tag{4.4}$$

We now make the substitutions

$$y = x^{\frac{p}{k-1}} \qquad \text{and} \qquad s = t^{\frac{p}{k-1}} \Leftrightarrow x = y^{\frac{k-1}{p}} \quad t = s^{\frac{k-1}{p}}$$

from which it follows that

$$t = b^{\frac{k-1}{p}} \Rightarrow s = b, \quad x = b^{\frac{k-1}{p}} \Rightarrow y = b, \quad dt = \tfrac{k-1}{p}s^{\frac{k-1}{p}-1}ds, \quad \tfrac{k-1}{p}ds = t^{\frac{p}{k-1}-1}dt,$$

$$dx = y^{\frac{k-1}{p}-1}\tfrac{k-1}{p}dy, \quad dy = \tfrac{p}{k-1}x^{\frac{p}{k-1}-1}dx, \quad \text{and} \quad t^{\frac{p}{k-1}-1} = s^{1-\frac{k-1}{p}}.$$

By using these substitutions we get from (4.4) that

$$R=(p-1)\left(\frac{k-1}{p}\right)^{p+2}\int_0^b\int_s^b\left(\frac{p}{k-1}f(s)\left(\frac{s}{y}\right)^{\left(1-\frac{k-1}{p}\right)}-\frac{1}{y}\int_0^y f(\sigma)\,d\sigma\right)^2 \cdot\left(\frac{1}{y}\int_0^y f(\sigma)\,d\sigma\right)^{p-2} y^{\left(1-\frac{k-1}{p}\right)(p+1)}s^{\frac{k-1}{p}-1}\frac{dy}{y^2}ds\,. \tag{4.5}$$

Now we make the same changes on the left-hand side of (3.3), denoted by L, that is, we replace b by $b^{\frac{k-1}{p}}$ and $f(x)$ by $f\left(x^{\frac{p}{k-1}}\right)x^{\frac{p}{k-1}-1}$ and by the substitution $y=x^{\frac{p}{k-1}}$ we get that

$$L=\int_0^b\frac{k-1}{p}\left(1-\left(\frac{y}{b}\right)^{\frac{k-1}{p}}\right)y^{p-k}\left(f(y)\right)^p dy -\left(\frac{k-1}{p}\right)^{p+1}\int_0^b y^{-k}\left(\int_0^y f(s)\,ds\right)^p dy. \tag{4.6}$$

Therefore from (4.4)–(4.6), after dividing L and R by $\left((k-1)/p\right)^{p+1}$, we get (3.4).

The reverse of the crucial inequality (3.4) holds for $1<p\le 2$ because in this case the function $\varphi(x)=x^{p-1}$, $x>0$, is concave. Hence the proof follows in the same way also in this case.

This completes the proof of the theorem. □

Proof of Proposition 3.3. Let us choose in (2.6) the probability measure $d\mu(t)=\frac{1}{x}dt$, $0\le t\le x$. Then

$$\begin{aligned}\frac{1}{x}\int_0^x K(f(t))\,dt-K\left(\frac{1}{x}\int_0^x f(t)\,dt\right) &\geq C_\varphi\left(\frac{1}{x}\int_0^x f(t)\,dt\right)\frac{1}{x}\int_0^x f(t)\left(f(t)-\frac{1}{x}\int_0^x f(\tau)\,d\tau\right)dt\\ &\quad+\frac{1}{x}\int_0^x f(t)\,\varphi\left(\left|f(t)-\frac{1}{x}\int_0^x f(\tau)\,d\tau\right|\right)dt\\ &=C_\varphi\left(\frac{1}{x}\int_0^x f(t)\,dt\right)\frac{1}{x}\int_0^x\left(f(t)-\frac{1}{x}\int_0^x f(\tau)\,d\tau\right)^2 dt\\ &\quad+\frac{1}{x}\int_0^x f(t)\,\varphi\left(\left|f(t)-\frac{1}{x}\int_0^x f(\tau)\,d\tau\right|\right)dt.\end{aligned} \tag{4.7}$$

Multiplying (4.7) by $\frac{u(x)}{x}$ and integrating over $0\le x\le b$, we find that

$$\int_0^b\int_0^x K(f(t))\,dt\frac{u(x)}{x^2}dx-\int_0^b K\left(\frac{1}{x}\int_0^x f(t)\,dt\right)\frac{u(x)}{x}dx \geq\int_0^b\int_0^x f(t)\left(f(t)-\frac{1}{x}\int_0^x f(\tau)\,d\tau\right)C_\varphi\left(\frac{1}{x}\int_0^x f(\tau)\,d\tau\right)\frac{u(x)}{x^2}dtdx \tag{4.8}$$

$$+\int_0^b\int_0^x f(t)\,\varphi\left(\left|f(t)-\frac{1}{x}\int_0^x f(\tau)\,d\tau\right|\right)\frac{u(x)}{x^2}dtdx$$
$$=\int_0^b\int_0^x\left(f(t)-\frac{1}{x}\int_0^x f(\tau)\,d\tau\right)^2 C_\varphi\left(\frac{1}{x}\int_0^x f(\tau)\,d\tau\right)\frac{u(x)}{x^2}dtdx$$
$$+\int_0^b\int_0^x f(t)\,\varphi\left(\left|f(t)-\frac{1}{x}\int_0^x f(\tau)\,d\tau\right|\right)\frac{u(x)}{x^2}dtdx.$$

Now, by using (3.5) and Fubini's theorem, we obtain that

$$\int_0^b\int_0^x K(f(t))\frac{u(x)}{x^2}dtdx=\int_0^b\frac{K(f(t))}{t}\left(t\int_t^b\frac{u(x)}{x}dx\right)dt \tag{4.9}$$
$$=\int_0^b K(f(t))\frac{v(t)}{t}dt=\int_0^b K(f(x))\frac{v(x)}{x}dt.$$

By combining (4.8) and (4.9) we obtain (3.6).

If φ is subquadratic, then the crucial inequality (2.6) holds in the reverse direction. Hence, the proof follows in the same way also in this case. □

Proof of Theorem 3.4. First we note that by using Proposition 3.3 with $\varphi(x)=x^{p-1}$, $p\geq 3$ (therefore $C_\varphi(x)=\varphi'(x)=(p-1)\,x^{p-2}$), choosing $u(x)=1$, we obtain that

$$\int_0^b\left(1-\frac{x}{b}\right)f^p(x)\frac{dx}{x}-\int_0^b\left(\frac{1}{x}\int_0^x f(t)\,dt\right)^p\frac{dx}{x} \tag{4.10}$$
$$\geq\int_0^b\int_t^b f(t)\left(f(t)-\frac{1}{x}\int_0^x f(\tau)\,d\tau\right)(p-1)\left(\frac{1}{x}\int_0^x f(\tau)\,d\tau\right)^{p-2}\frac{dx}{x^2}dt$$
$$+\int_0^b\int_t^b f(t)\left(\left|f(t)-\frac{1}{x}\int_0^x f(\tau)\,d\tau\right|\right)^{p-1}\frac{dx}{x^2}dt$$
$$=\int_0^b\int_t^b\left(f(t)-\frac{1}{x}\int_0^x f(\tau)\,d\tau\right)^2(p-1)\left(\frac{1}{x}\int_0^x f(\tau)\,d\tau\right)^{p-2}\frac{dx}{x^2}dt$$
$$+\int_0^b\int_t^b f(t)\left(\left|f(t)-\frac{1}{x}\int_0^x f(\tau)\,d\tau\right|\right)^{p-1}\frac{dx}{x^2}dt.$$

By now making the same steps and variable substitutions as in the proof of Theorem 3.2 but now with (4.10) as the crucial inequality instead of (3.3) we obtain the proof of Theorem 3.4. We leave out the details. □

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