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Rational Matrix Solutions of a Bezout Type Equation on the Half-plane

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Dedicated to Leonia Lerer on the occasion of his 70th birthday, in friendship

Abstract. A state space description is given of all stable rational matrix solutions of a general rational Bezout type equation on the right half-plane. Included are a state space formula for a particular solution satisfying a certain H^2 minimality condition, a state space formula for the inner function describing the null space of the multiplication operator corresponding to the Bezout equation, and a parameterization of all solutions using the particular solution and this inner function. A state space version of the related Tolokonnikov lemma is also presented.

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1. Introduction

In this paper G is a stable rational $m \times p$ matrix function. Here *stable* means that G is proper, that is, the limit of $G(s)$ as $s \to \infty$ exists, and G has all its poles in the open left half-plane $\{s \in \mathbb{C} \mid \Re(s) < 0\}$. In other words, G is a rational matrix-valued H^∞ function, where the latter means that G is analytic and bounded on the open right half-plane. In this paper p will be larger than m, and thus G will be a "fat" non-square matrix function. We shall be interested in stable rational $p \times m$ matrix-valued solutions X of the Bezout type equation

$$G(s)X(s) = I_m, \quad \Re s \geq 0. \tag{1.1}$$

The symbol I_m on the right-hand side denotes the $m \times m$ identity matrix.

Throughout we shall assume that G admits a state space realization of the form

$$G(s) = C(sI_n - A)^{-1}B + D. \tag{1.2}$$

Here A is an $n \times n$ matrix which is assumed to be stable, that is, all the eigenvalues of A are contained in the open left half-plane. Moreover, B, C and D are matrices of appropriate sizes. Our aim is to give necessary and sufficient conditions for the solvability of (1.1), and to give a full description of all stable rational matrix-valued solutions, in terms of the matrices appearing in the realization (1.2). The results we present are the continuous analogs of the main theorems in [6] and [8].

To state the main results we need some additional notation. By P we denote the controllability Gramian associated with the realization (1.2), that is, P is the (unique) solution of the Lyapunov equation

$$AP + PA^* + BB^* = 0. \tag{1.3}$$

Consider the algebraic Riccati equation

$$A^*Q + QA + (C - \Gamma^*Q)^*(DD^*)^{-1}(C - \Gamma^*Q) = 0, \tag{1.4}$$

where Γ is defined by

$$\Gamma = BD^* + PC^*. \tag{1.5}$$

Here it is assumed that D is right invertible, which is a natural condition. Indeed, if (1.1) has a stable rational matrix solution X, then using (1.2) and the fact that X is proper, we see that $DX(\infty) = \lim_{s\to\infty} G(s)X(s) = I_m$. Hence $X(\infty)$ is a right inverse of D, and thus D is right invertible. A solution Q of (1.4) is called the *stabilizing solution* of the algebraic Riccati equation (1.4) if Q is Hermitian and the $n \times n$ matrix A_0 given by

$$A_0 = A - \Gamma(DD^*)^{-1}(C - \Gamma^*Q) \tag{1.6}$$

is stable. If it exists, a stabilizing solution is unique (cf. formula (2.11)). The following is the first main result of this paper.

Theorem 1.1. *There is a stable rational $p \times m$ matrix function X satisfying the equation $G(s)X(s) = I_m$ if and only if the following three conditions hold*

1. *The matrix D is right invertible,*
2. *there exists a stabilizing solution Q of the Riccati equation* (1.4), *and*
3. *the matrix $I_n - PQ$ is invertible.*

In that case a particular solution of (1.1) *is given by*

$$\Xi(s) = \Big(I_p - C_1(sI_n - A_0)^{-1}(I_n - PQ)^{-1}B\Big)D^*(DD^*)^{-1}, \tag{1.7}$$

where A_0 is the stable $n \times n$ matrix given by (1.6) *and*

$$C_1 = D^*(DD^*)^{-1}(C - \Gamma^*Q) + B^*Q. \tag{1.8}$$

The matrix $D^*(DD^*)^{-1}$ appearing in (1.7) is *Moore–Penrose right inverse* of D. In what follows we shall often denote $D^*(DD^*)^{-1}$ by D^+. Note that $\dim \operatorname{Ker} D = p - m$.

The rational $p \times p$ matrix function appearing in the right-hand side of (1.7) between the brackets will be denoted by Y, that is,

$$Y(s) = I_p - C_1(sI_n - A_0)^{-1}(I_n - PQ)^{-1}B. \tag{1.9}$$

Note that the value of Y at infinity is invertible. Hence $Y(s)^{-1}$ is a well-defined rational matrix function. We shall see that $Y(s)^{-1}$ is again stable. Thus both $Y(s)$ and $Y(s)^{-1}$ are stable rational matrix functions. In other words the entries of both $Y(s)$ and $Y(s)^{-1}$ are H^∞ functions. In this case we say that Y is *invertible outer.*

Among other things the following theorem describes the set of all stable rational solutions to $G(s)X(s) = I_m$.

Theorem 1.2. *There exists a stable rational $p \times m$ matrix function X satisfying $G(s)X(s) = I_m$ if and only if D is right invertible and there exists a stable rational $p \times p$ matrix function Y which is invertible outer and satisfies the equation $G(s)Y(s) = D$. In this case one such Y is given by* (1.9) *and the inverse of this Y is given by*

$$Y(s)^{-1} = I_p + C_1(I_n - PQ)^{-1}(sI_n - A)^{-1}B. \tag{1.10}$$

Moreover, using this function Y the following holds.

(i) *Let E be any isometry mapping $\mathbb{C}^{p-m}$ into $\mathbb{C}^p$ such that* $\operatorname{Im} E = \operatorname{Ker} D$. *Then the function*

$$\Theta(s) = Y(s)E = \Big(I_p - C_1(sI_n - A_0)^{-1}(I_n - PQ)^{-1}B\Big)E \tag{1.11}$$

is a stable rational $p \times (p-m)$ matrix function satisfying $G(s)\Theta(s) = 0$, and Θ is inner, that is, $\Theta(-\bar{s})^*\Theta(s) = I_{p-m}$.

(ii) *If h is any $\mathbb{C}^p$-valued H^2 function satisfying $G(s)h(s) = 0$, then there exists a unique $\mathbb{C}^{(p-m)}$-valued H^2 function ω such that $h(s) = \Theta(s)\omega(s)$. In fact, $\omega(s) = \Theta(-\bar{s})^*h(s)$.*

(iii) *The set of all stable rational $p \times m$ matrix functions X satisfying $G(s)X(s) = I_m$ is given by*

$$\begin{aligned} X(s) = \Big(I_p - C_1(sI_n - A_0)^{-1}(I_n - PQ)^{-1}B\Big) &\times \\ \times \Big(D^*(DD^*)^{-1} + EZ(s)\Big)&, \end{aligned} \tag{1.12}$$

where Z is an arbitrary stable rational $(p-m) \times m$ matrix function. Moreover, if X satisfies $G(s)X(s) = I_m$, then Z in (1.12) *is given by $Z(s) = E^*Y(s)^{-1}X(s)$.*

(iv) *The rational $p \times p$ matrix function*

$$G_{ext}(s) = \begin{bmatrix} G(s) \\ E^*Y(s)^{-1} \end{bmatrix}, \quad \Re s \geq 0, \tag{1.13}$$

is invertible outer and its inverse is given by

$$G_{ext}(s)^{-1} = \begin{bmatrix} \Xi(s) & \Theta(s) \end{bmatrix}, \quad \Re s \geq 0. \tag{1.14}$$

Note that item (ii) tells us that the null space $\operatorname{Ker} M_G$ of the multiplication operator M_G defined by G, mapping H^2_p into H^2_m, is given by $\operatorname{Ker} M_G = \Theta H^2_{p-m}$. Thus Θ plays the role of the inner function in the Beurling–Lax theorem specified for $\operatorname{Ker} M_G$. Furthermore, (1.12) in item (iii) can be rewritten in the following equivalent form $X(s) = \Xi(s) + \Theta(s)Z(s)$. Using this form of (1.12) we expect our state space formulas also to be useful in deriving rational H^∞ solutions of (1.1) that satisfy an additional H^∞ norm constraint, by reducing the norm constraint problem to a generalized Sarason problem (cf. Section I.7 in [7]). Finally, item (iv) is inspired by Tolokonnikov's lemma (see [18] and [16, Appendix 3]).

The formulas in Theorems 1.1 and 1.2 can be easily converted into a Matlab program to compute Ξ in (1.7), the function Y in (1.9), and Θ in (1.11).

We see Theorems 1.1 and 1.2 as the closed right half-plane analogues of Theorem 1.1 in [6] and Theorem 1.1 in [8], which deal with equation (1.1) in the setting of rational matrix functions analytic in the closed unit disc. Obviously, a way to obtain the set of all stable rational matrix solutions to equation (1.1) is to use the Cayley transform to derive the right half-plane solutions from their analogues in the disc case as given in [8]. However, note that in the present half-plane case there is an additional difficulty: The constant functions are not in L^2, whereas in the disc case they are in L^2. Furthermore, the particular solution Ξ in Theorem 1.1 is not the analogue of the least squares solution in [6]. On the other hand, as we shall show in Section 4, the function Ξ has an interpretation in terms of solutions to a somewhat different minimization problem (see Theorem 4.1).

We take this occasion to mention that Theorem 1.1 *in* [6] *and Theorem* 1.1 *in* [8] *have predecessors in the papers* [14] *and* [13]. *In particular, see Lemma* 4.1 *and Theorem* 4.2 *in* [13]. *We are grateful to Dr. Sander Wahls for mentioning to us these and several related other references. It is interesting to see the role the Bezout equation plays in solving the engineering problems considered in* [14] *and* [13]. *The proofs in* [6] *and* [8] *are quite different from those in* [14] *and* [13]*; also different Riccati equations are used and different state space formulas are obtained.*

There is an extensive literature on the Bezout equation and the related corona equation, see, e.g., the classical papers [4], [9], [18], the books [16], [15], [1], the more recent papers [19], [20], [21], [22], and the references therein. Also, finding rational matrix solutions in state space form for Bezout equations is a classical topic in mathematical system theory; see, e.g., the book [23], and the papers [11], [10]. However, as far as we know the formulas we present here are new and cannot easily be obtained using the methods presented in the classical sources. The interpretation of the special solution (1.7) as a limit of solutions of minimization problems also seems to be new. Moreover, the approach we follow in the present paper and the earlier papers [6, 8] can be extended to a Wiener space setting. In fact, in a Wiener space setting the function Y given by (1.9) appears in a very natural way; see also the comment at the end of Section 2. We plan to return to this in a future paper, also for the discrete case.

The paper consists of four sections, including this introduction. In the second section we present the preliminaries from operator theory used in the proofs, and we explain the role of the Riccati equation (1.4), and prove the necessity of conditions 1, 2, 3 in Theorem 1.1. The third section contains the proofs of Theorem 1.1 and 1.2. In the final section we consider an optimization problem, which helps in identifying Ξ in as a solution with a special minimality property.

2. Operator theory and Riccati equation

In this section we prove the necessity of conditions 1, 2, 3 in Theorem 1.1. Our proof requires some preliminaries from operator theory and uses the Riccati equation (1.4).

Let Ω be any proper rational $k \times r$ matrix function with no pole on the imaginary axis $i\mathbb{R}$. With Ω we associate the Wiener–Hopf operator T_Ω and the Hankel operator H_Ω, both mapping $L^2_r(\mathbb{R}^+)$ into $L^2_k(\mathbb{R}^+)$. These operators are the integral operators defined by

$$(T_\Omega f)(t) = \Omega(\infty)f(t) + \int_0^t \omega(t-\tau)f(\tau)d\tau, \quad t \geq 0, \quad f \in L^2_r(\mathbb{R}^+), \tag{2.1}$$

$$(H_\Omega f)(t) = \int_0^\infty \omega(t+\tau)f(\tau)d\tau, \qquad t \geq 0 \quad f \in L^2_r(\mathbb{R}^+). \tag{2.2}$$

Here ω is the Lebesque integrable (continuous) matrix function on the imaginary axis determined by Ω via the Fourier transform:

$$\Omega(s) = \Omega(\infty) + \int_{-\infty}^\infty e^{-s\tau}\omega(\tau)\, d\tau, \quad s \in i\mathbb{R}.$$

In the sequel we shall freely use the basic theory of Wiener–Hopf and Hankel operators which can be found in Chapters XII and XIII of [12]. Note that in [12] the Fourier transform is taken with respect to the real line instead of the imaginary axis as is done here.

Now let G be the stable rational $p \times m$ function given by (1.2). Then

$$G(s) = D + \int_0^\infty e^{-s\tau}Ce^{\tau A}B\, d\tau, \quad s \in i\mathbb{R}.$$

Hence the Wiener–Hopf operator T_G and the Hankel operator H_G are given by

$$(T_G f)(t) = Df(t) + \int_0^t Ce^{(t-\tau)A}Bf(\tau)d\tau, \quad t \geq 0, \tag{2.3}$$

$$(H_G f)(t) = \int_0^\infty Ce^{(t+\tau)A}Bf(\tau)d\tau, \qquad t \geq 0. \tag{2.4}$$

With G we also associate the rational $m \times m$ matrix function R given by $R(s) = G(s)G(-\overline{s})^*$. Note that R is a proper rational $m \times m$ matrix function with no pole on the imaginary axis. By T_R we denote the corresponding Wiener–Hopf operator acting on $L^2_m(\mathbb{R}^+)$. It is well known (see, e.g., formula (24) in Section XII.2 of [12]) that

$$T_R = T_G T_G^* + H_G H_G^*. \tag{2.5}$$

Next assume that the equation $G(s)X(s) = I_m$ has a stable rational matrix solution X. The fact that X is stable implies that X is proper and has no poles on the imaginary axis, and thus T_X is well defined. Furthermore, $T_G T_X = T_{GX}$; see [12, Proposition XIII.1.2]. Since GX is identically equal to the $m \times m$ identity matrix T_{GX} is the identity operator on $L^2_m(\mathbb{R}^+)$, and hence T_X is a right inverse of T_G. The fact T_G that is right invertible, implies that $T_G T_G^*$ is invertible and hence strictly positive. The identity (2.5) then shows that T_R is strictly positive too, and hence is invertible.

In the following proposition we use the algebraic Riccati equation (1.4) to obtain necessary and sufficient conditions for T_R to be invertible in terms of the matrices A, B, and C appearing in the realization (1.2). As in Section 1, we denote by P the controllability Gramian associated with the realization (1.2), that is, P is the solution of the Lyapunov equation (1.3). Finally, Γ is the $n \times m$ matrix defined by (1.5).

Proposition 2.1. *Let $R(s) = G(s)G(-\overline{s})^*$. Then the operator T_R is invertible if and only if the algebraic Riccati equation*

$$A^*Q + QA + (C - \Gamma^*Q)^* (DD^*)^{-1} (C - \Gamma^*Q) = 0 \tag{2.6}$$

has a stabilizing solution Q, that is, Q is a Hermitian solution of (2.6) *and the operator A_0, defined by*

$$A_0 = A - \Gamma C_0, \ \textit{where } C_0 = (DD^*)^{-1} (C - \Gamma^*Q), \tag{2.7}$$

is stable.

Proof. The proposition is an immediate consequence of Theorem 14.8 in [3]. To see this, we first show that

$$R(s) = DD^* + C(sI_n - A)^{-1}\Gamma - \Gamma^*(sI_n + A^*)^{-1}C^*. \tag{2.8}$$

This partial fraction expansion for R follows from the Lyapunov equation (1.3), and its immediate consequence

$$-(sI_n - A)^{-1}BB^*(sI_n + A^*)^{-1} = (sI_n - A)^{-1}P - P(sI_n + A^*)^{-1}.$$

By employing $G(s) = D + C(sI_n - A)^{-1}B$ the identity (2.8) then follows from

$$\begin{aligned} R(s) = G(s)G(-\overline{s})^* &= (D + C(sI_n - A)^{-1}B)(D^* - B^*(sI_n + A^*)C^*) \\ &= DD^* + C(sI_n - A)^{-1}BD^* - DB^*(sI_n + A^*)C^* \\ &\quad + C(sI_n - A)^{-1}PC^* - CP(sI_n + A^*)^{-1}C^*. \end{aligned}$$

Using $\Gamma = BD^* + PC^*$, this yields (2.8). Given (2.8) we can apply Theorem 14.8 in [3], replacing J by DD^* and B by Γ, and rewriting the corresponding algebraic Riccati equation in the form (2.6). $\square$

From the partial fraction expansion (2.8) it follows that the action of the Wiener–Hopf operator T_R on $L^2_m(\mathbb{R}^+)$ is given by

$$\begin{aligned}(T_R f)(t) = DD^* f(t) + \int_0^t Ce^{(t-\tau)A}\Gamma f(\tau)\,d\tau \\ + \int_t^\infty \Gamma^* e^{-(t-\tau)A^*} C^* f(\tau)\,d\tau. \quad t \geq 0.\end{aligned} \tag{2.9}$$

By $W_{\rm obs}$ and $W_{0,\,\rm obs}$ we denote the observability operators mapping the state space $\mathbb{C}^n$ into $L^2_m(\mathbb{R}^+)$ defined by

$$(W_{\rm obs}x)(t) = Ce^{tA}x \quad \text{and} \quad (W_{0,\,\rm obs}x)(t) = C_0 e^{tA_0}x, \ \text{where } x \in \mathbb{C}^n. \tag{2.10}$$

Proposition 2.2. *Assume that T_R is invertible, or equivalently, there exists a stabilizing solution Q to the algebraic Riccati equation* (2.6)*. Then this stabilizing solution is uniquely determined by*

$$Q = W^*_{\rm obs} T_R^{-1} W_{\rm obs}. \tag{2.11}$$

Proof. To establish this, let us first show that Q satisfies the following Lyapunov equation

$$A^*Q + QA_0 + C^*C_0 = 0. \tag{2.12}$$

Recall that $A_0 = A - \Gamma C_0$. Then (2.12) follows from the Riccati equation

$$\begin{aligned}0 &= A^*Q + QA + (C - \Gamma^*Q)^* (DD^*)^{-1} (C - \Gamma^*Q) \\ &= A^*Q + Q(A_0 + \Gamma C_0) + (C - \Gamma^*Q)^* C_0 \\ &= A^*Q + QA_0 + C^*C_0.\end{aligned}$$

Thus (2.12) holds. Because A and A_0 are both stable, the stabilizing solution Q can also be written as

$$Q = \int_0^\infty e^{tA^*} C^* C_0 e^{tA_0}\,dt = W^*_{\rm obs} W_{0,\,\rm obs}. \tag{2.13}$$

Next we prove that

$$T_R^{-1} W_{\rm obs} = W_{0,\,\rm obs}. \tag{2.14}$$

This essentially follows from [2], Corollary 6.3. For completeness we provide a proof. It suffices to compute $T_R W_{0,\,\rm obs}$. To do this, we use (2.9). Fix $x \in \mathbb{C}^n$. From the second identity in (2.10) and the first identiy in (2.7) it follows that

$$\begin{aligned}&\int_0^t Ce^{(t-\tau)A}\Gamma (W_{0,\,\rm obs}x)(\tau)\,d\tau = \int_0^t Ce^{(t-\tau)A}\Gamma C_0 e^{\tau A_0}x\,d\tau \\ &\quad = \int_0^t Ce^{(t-\tau)A}(A - A_0)e^{\tau A_0}x\,d\tau = Ce^{tA}\Big(\int_0^t Ce^{-\tau A}(A - A_0)e^{\tau A_0}\,d\tau\Big)x \\ &\quad = -Ce^{tA}\Big(\int_0^t \frac{d}{d\tau}(e^{-\tau A}e^{\tau A_0})\,d\tau\Big)x = -Ce^{tA_0}x + Ce^{tA}x.\end{aligned}$$

Furthermore, using the Lyapunov identity (2.12) we obtain

$$\begin{aligned}
&\int_t^\infty \Gamma^* e^{-(t-\tau)A^*} C^* (W_{0,\,\mathrm{obs}}x)(\tau)\,d\tau = \int_t^\infty \Gamma^* e^{-(t-\tau)A^*} C^* C_0 e^{\tau A_0} x\,d\tau \\
&\quad = -\int_t^\infty \Gamma^* e^{-(t-\tau)A^*} (A^*Q + QA_0) e^{\tau A_0} x\,d\tau \\
&\quad = -\Gamma^* e^{-tA^*} \Big(\int_t^\infty e^{\tau A^*} (A^*Q + QA_0) e^{\tau A_0}\,d\tau \Big) x \\
&\quad = -\Gamma^* e^{-tA^*} \Big(\int_t^\infty \frac{d}{d\tau} (e^{\tau A^*} Q e^{\tau A_0})\,d\tau \Big) x = -\Gamma^* e^{-tA^*} \Big(- e^{tA^*} Q e^{tA_0} \Big) x \\
&\quad = \Gamma^* Q e^{tA_0} x.
\end{aligned}$$

Using (2.9) and the second identity in (2.7) we conclude that

$$\begin{aligned}
(T_R W_{0,\,\mathrm{obs}})(t) &= DD^* C_0 e^{tA_0} + (-C e^{tA_0} + C e^{tA}) + \Gamma^* Q e^{tA_0} \\
&= (DD^* C_0 + \Gamma^* Q) e^{tA_0} - C e^{tA_0} + C e^{tA} = C e^{tA}.
\end{aligned}$$

This proves $T_R W_{0,\,\mathrm{obs}} = W_{\mathrm{obs}}$, and hence (2.14) holds. Together (2.13) and (2.14) show that $W_{\mathrm{obs}}^* T_R^{-1} W_{\mathrm{obs}} = W_{\mathrm{obs}}^* W_{0,\,\mathrm{obs}} = Q$. In particular, the stabilizing solution is uniquely determined by (2.11). $\square$

Lemma 2.3. *Assume T_R is invertible. Then $I - H_G^* T_R^{-1} H_G$ is positive. Furthermore, the following are equivalent:*

(i) *T_G is right invertible,*
(ii) *$I - H_G^* T_R^{-1} H_G$ is strictly positive,*
(iii) *$I - H_G^* T_R^{-1} H_G$ is invertible.*

Proof. Rewriting (2.5) as $T_G T_G^* = T_R - H_G H_G^*$, and multiplying the latter identity from the left and from the right by $T_R^{-1/2}$ shows that

$$T_R^{-1/2} T_G T_G^* T_R^{-1/2} = I - T_R^{-1/2} H_G H_G^* T_R^{-1/2}. \tag{2.15}$$

Hence $I - T_R^{-1/2} H_G H_G^* T_R^{-1/2}$ is positive which shows that $H_G^* T_R^{-1/2}$ is a contraction. But then $H_G^* T_R^{-1} H_G = \big(H_G^* T_R^{-1/2}\big)\big(H_G^* T_R^{-1/2}\big)^*$ is also a contraction, and thus the operator $I - H_G^* T_R^{-1} H_G$ is positive.

Since $I - H_G^* T_R^{-1} H_G$ is positive, the equivalence of items (ii) and (iii) is trivial. Assume (ii) holds. Then $T_R^{-1/2} H_G$ is a strict contraction, and hence the same holds true for $T_R^{-1/2} H_G H_G^* T_R^{-1/2}$. But then $I - T_R^{-1/2} H_G H_G^* T_R^{-1/2}$ is strictly positive, and (2.15) shows that T_G is right invertible. The converse implication is proved in a similar way. $\square$

Corollary 2.4. *Assume that T_R is invertible, or equivalently, there exists a stabilizing solution Q to the algebraic Riccati equation* (2.6). *Then the spectral radius of QP is at most one.*

Furthermore, the following are equivalent:

(i) T_G *is right invertible,*
(ii) $r_{\rm spec}(QP) < 1$,
(iii) $I_n - QP$ *is invertible.*

Proof. Let $W_{\rm con}$ be the controllability operator mapping $L_2^p(\mathbb{R}^+)$ into $\mathbb{C}^n$ defined by

$$W_{\rm con}h = \int_0^\infty e^{tA}Bh(t)dt, \qquad h \in L_p^2(\mathbb{R}^+).$$

Then $P = W_{\rm con}W_{\rm con}^*$ and $H_G = W_{\rm obs}W_{\rm con}$. Using these two identities and (2.11), we obtain for the spectral radius of $H_G^*T_R^{-1}H_G$ that

$$\begin{aligned} r_{\rm spec}(H_G^*T_R^{-1}H_G) &= r_{\rm spec}(W_{\rm con}^*W_{\rm obs}^*T_R^{-1}W_{\rm obs}W_{\rm con}) \\ &= r_{\rm spec}(W_{\rm con}^*QW_{\rm con}) \\ &= r_{\rm spec}(QW_{\rm con}W_{\rm con}^*) = r_{\rm spec}(QP). \end{aligned} \tag{2.16}$$

By Lemma 2.3 the operator $I - H_G^*T_R^{-1}H_G$ is positive. Hence the spectral radius of $H_G^*T_R^{-1}H_G$ is at most one, and the preceding calculation shows that $r_{\rm spec}(QP) \leq 1$

Since $r_{\rm spec}(QP) \leq 1$, the equivalence of items (ii) and (iii) is trivial. Assume $r_{\rm spec}(QP) < 1$. Then (2.16) shows that $I - H_G^*T_R^{-1}H_G$ is invertible, and Lemma 2.3 tells us that T_G is right invertible. To prove the converse implication, assume that T_G is right invertible. Then, by Lemma 2.3, the operator $I - H_G^*T_R^{-1}H_G$ is strictly positive. Hence $r_{\rm spec}(I - H_G^*T_R^{-1}H_G) < 1$, and (2.16) shows that $r_{\rm spec}(QP) < 1$. □

Necessity of the conditions 1, 2, 3 in Theorem 1.1. Assume that the equation $G(s)X(s) = I_m$ has a stable rational matrix solution X. As was shown in the paragraph preceding Theorem 1.1, this implies that D is right invertible. Thus condition 1 is necessary. Furthermore, in the paragraph directly after (2.5) it was shown that $G(s)X(s) = I_m$ has a stable rational matrix solution also implies that T_G is right invertible and T_R is invertible. Given the latter we can apply Proposition 2.1 to show that condition 2 is necessary. Finally, using Corollary 2.4, we see that T_G is right invertible and T_R is invertible imply that $I_n - PQ$ is invertible, which shows that condition 3 is necessary. □

Comment. The identities appearing in this section can also be used to give an alternative formula for the function Y in (1.9), namely

$$Y(s) = I_p - \int_0^\infty e^{-st}y(t)dt, \ \Re s \geq 0, \quad \text{where} \quad y = T_G^*(T_GT_G^*)^{-1}g. \tag{2.17}$$

This formula also makes sense in a Wiener space setting. From formula (2.17) for Y it follows that $T_Gy = g$, which immediately implies that $G(s)Y(s) = D$. The latter identity will be derived in the next section (see the second paragraph of the proof of Theorem 1.1) using state space computations. We plan to prove (2.17) in a future paper.

3. Proof of the two main theorems

It will be convenient first to prove the two identities given in the following lemma.

Lemma 3.1. *Assume conditions* 1, 2, 3 *in Theorem* 1.1 *are satisfied. Then*

$$BC_1 = A(I_n - PQ) - (I_n - PQ)A_0, \tag{3.1}$$

$$DC_1 = C(I_n - PQ). \tag{3.2}$$

Proof. Recall that C_1 and C_0 are respectively defined in (1.8) and (2.7). This implies that

$$C_1 = D^*C_0 + B^*Q. \tag{3.3}$$

To prove the first identity, we use the Lyapunov equation (2.12) with Γ defined in (1.5) to compute

$$\begin{aligned} BC_1 &= BD^*C_0 + BB^*Q = (\Gamma - PC^*)C_0 - (AP + PA^*)Q \\ &= \Gamma C_0 + PA^*Q + PQA_0 - APQ - PA^*Q \\ &= \Gamma C_0 + PQA_0 - APQ = A - A_0 + PQA_0 - APQ \\ &= A(I_n - PQ) - (I_n - PQ)A_0. \end{aligned}$$

The second identity follows from

$$\begin{aligned} DC_1 &= C - \Gamma^*Q + DB^*Q = C - DB^*Q - CPQ + DB^*Q \\ &= C(I_n - PQ). \end{aligned}$$

Thus both identities are proved. □

Proof of Theorem 1.1. In the previous section we have seen that the conditions 1, 2, 3 in Theorem 1.1 are necessary. Therefore in what follows we assume these three conditions are fullfilled. The latter allows us to introduce the $p \times p$ rational matrix function

$$Y(s) = I_p - C_1(sI_n - A_0)^{-1}(I_n - PQ)^{-1}B. \tag{3.4}$$

Note that Y is stable, because the matrix A_0 which is given by (1.6) is stable. The latter follows from the fact that condition 2 is satisfied. We claim that

$$Y(s)^{-1} = I_p + C_1(I_n - PQ)^{-1}(sI_n - A)^{-1}B. \tag{3.5}$$

Since A is stable, we see that Y is invertible outer. To prove (3.5), we use (3.1). Indeed, using (3.1), we obtain

$$\begin{aligned} A_0 + (I_n - PQ)^{-1}BC_1 &= A_0 + (I_n - PQ)^{-1}A(I_n - PQ) - A_0 \\ &= (I_n - PQ)^{-1}A(I_n - PQ). \end{aligned} \tag{3.6}$$

Recall that the inverse of $I_p - \gamma(sI_n - \alpha)^{-1}\beta$ is the state space realization given by $I_p + \gamma\big(sI_n - (\alpha + \beta\gamma)\big)^{-1}\beta$. Using this for the state space realization for Y in (3.4) with (3.6), we obtain

$$\begin{aligned} Y(s)^{-1} &= I_p + C_1(sI_n - (I_n - PQ)^{-1}A(I_n - PQ))^{-1}(I_n - PQ)^{-1}B \\ &= I_p + C_1(I_n - PQ)^{-1}(sI_n - A)^{-1}B. \end{aligned}$$

Hence the inverse of $Y(s)$ is given by (3.5). In particular, Y is an invertible outer function.

Next we show that $G(s)Y(s) = D$. To do this we use (3.2) together with the state space formula for $Y(s)^{-1}$ in (3.5), to obtain

$$\begin{aligned} DY(s)^{-1} &= D + DC_1(I_n - PQ)^{-1}(sI_n - A)^{-1}B \\ &= D + C(sI_n - A)^{-1}B = G(s). \end{aligned}$$

In other words, $G(s) = DY(s)^{-1}$. By multiplying the latter identity from the left by $Y(s)$ we obtain $G(s)Y(s) = D$.

Finally, by comparing (1.7) and (3.4), we see that $\Xi(s) = Y(s)D^*(DD^*)^{-1}$. It follows that

$$G(s)\Xi(s) = G(s)Y(s)D^*(DD^*)^{-1} = DD^*(DD^*)^{-1} = I_m.$$

This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. Given the above proof of Theorem 1.1 it remains to prove items (i)–(iv) in Theorem 1.2. We do this in four steps.

STEP 1. First we show that Θ is inner. To do this, recall that

$$\Theta(s) = E + C_1(sI_n - A_0)^{-1}B_i, \text{ where } B_i = -(I_n - PQ)^{-1}BE. \tag{3.7}$$

We shall make use of the following Lyapunov equation

$$A_0^*(Q - QPQ) + (Q - QPQ)A_0 + C_1^*C_1 = 0. \tag{3.8}$$

To see this, notice, that (3.1), (3.2) and (3.3) with (2.7) and (2.12) yield

$$\begin{aligned} C_1^*C_1 &= (C_0^*D + QB)C_1 \\ &= C_0^*C(I_n - QP) + QA(I_n - PQ) - Q(I_n - PQ)A_0 \\ &= -(QA + A_0^*Q)(I_n - QP) + QA(I_n - PQ) - Q(I_n - PQ)A_0 \\ &= -A_0^*(Q - QPQ) - (Q - QPQ)A_0. \end{aligned}$$

Therefore (3.8) holds. The Lyapunov equation in (3.8) also yields

$$\begin{aligned} &-(sI_n + A_0^*)^{-1}C_1^*C_1(sI_n - A_0)^{-1} \\ &\qquad = (Q - QPQ)(sI_n - A_0)^{-1} - (sI_n + A_0^*)^{-1}(Q - QPQ). \end{aligned} \tag{3.9}$$

To see this, simply multiply the previous equation by $sI_n + A_0^*$ on the left and $sI_n - A_0$ on the right.

To show that Θ is an inner function, notice that (3.9) gives

$$\begin{aligned}\Theta(-\overline{s})^*\Theta(s) &= \left(E^* - B_i^*(sI_n + A_0^*)^{-1}C_1^*\right)\left(E + C_1(sI_n - A_0)^{-1}B_i\right)\\ &= I_{p-m} + E^*C_1(sI_n - A_0)^{-1}B_i - B_i^*(sI_n + A_0^*)^{-1}C_1^*E\\ &\quad - B_i^*(sI_n + A_0^*)^{-1}C_1^*C_1(sI_n - A_0)^{-1}B_i\\ &= I_{p-m} + E^*C_1(sI_n - A_0)^{-1}B_i - B_i^*(sI_n + A_0^*)^{-1}C_1^*E\\ &\quad + B_i^*(Q - QPQ)(sI_n - A_0)^{-1}B_i\\ &\quad - B_i^*(sI_n + A_0^*)^{-1}(Q - QPQ)B_i\\ &= I_{p-m} + \left(B_i^*(Q - QPQ) + E^*C_1\right)(sI_n - A_0)^{-1}B_i\\ &\quad - B_i^*(sI_n + A_0^*)^{-1}\left(C_1^*E + (Q - QPQ)B_i\right) = I_{p-m}.\end{aligned}$$

The last equality follows from the fact that

$$B_i^*(Q - QPQ) + E^*C_1 = 0. \tag{3.10}$$

To verify this, observe that

$$\begin{aligned}B_i^*(Q - QPQ) &= -E^*B^*(I_n - QP)^{-1}(Q - QPQ) = -E^*B^*Q\\ E^*C_1 &= E^*(B^*Q + D^*C_0) = E^*B^*Q.\end{aligned}$$

Hence $B_i^*(Q - QPQ) + E^*C_1 = 0$. Therefore $\Theta(s)$ is an inner function.

Step 2. It will be convenient first to prove item (iv). Take s in the right half-plane, i.e., $\Re s \geq 0$. Using the definition of Y in (1.9), and the identities (1.7) and (1.11), we see that $\Xi(s) = Y(s)D^+$ and $\Theta(s) = Y(s)E$, and hence

$$\begin{bmatrix}\Xi(s) & \Theta(s)\end{bmatrix} = Y(s)\begin{bmatrix}D^+ & E\end{bmatrix}.$$

Next observe that the $p \times p$ matrix $\begin{bmatrix}D^+ & E\end{bmatrix}$ is invertible, and

$$\begin{bmatrix}D^+ & E\end{bmatrix}\begin{bmatrix}D\\ E^*\end{bmatrix} = I_p.$$

Thus $\begin{bmatrix}\Xi(s) & \Theta(s)\end{bmatrix}$ is invertible, and

$$\begin{bmatrix}\Xi(s) & \Theta(s)\end{bmatrix}^{-1} = \begin{bmatrix}D\\ E^*\end{bmatrix}Y(s)^{-1} = \begin{bmatrix}G(s)\\ E^*Y(s)^{-1}\end{bmatrix}.$$

This proves (1.14). Since the function $\begin{bmatrix}\Xi & \Theta\end{bmatrix}$ is a stable rational $p \times p$ matrix function, we see that the function defined by (1.13) is invertible outer.

Step 3. In this part we prove item (iii). Let X be given by (1.12). In other words $X(s) = Y(s)(D^*(DD^*)^{-1} + EZ(s))$, where Z is an arbitrary stable rational matrix function of size $(p - m) \times m$. Since $G(s)Y(s) = D$, we see $G(s)X(s) = DD^*(DD^*)^{-1} + DEZ(s)$. But $DD^*(DD^*)^{-1} = I_m$ and $DE = 0$. We conclude that $G(s)X(s) = I_m$, as desired.

Next we deal with the reverse implication. Let X be any stable rational $p \times m$ matrix function satsfying the equation $G(s)X(s) = I_m$. Put $H = X - \Xi$. Then H is a rational matrix-valued function, and $G(s)H(s) = 0$. Notice that

$E^*Y(s)^{-1}\Xi(s) = 0$. Using item (iv) we obtain

$$\begin{aligned} H(s) &= \begin{bmatrix}\Xi(s) & \Theta(s)\end{bmatrix}\begin{bmatrix} G(s) \\ E^*Y(s)^{-1}\end{bmatrix} H(s) \\ &= \begin{bmatrix}\Xi(s) & \Theta(s)\end{bmatrix}\begin{bmatrix} 0 \\ E^*Y(s)^{-1}H(s)\end{bmatrix} \\ &= \Theta(s)E^*Y(s)^{-1}H(s) = \Theta(s)E^*Y(s)^{-1}X(s). \end{aligned} \tag{3.11}$$

Thus $H(s) = \Theta(s)Z(s)$, where $Z(s) = E^*Y(s)^{-1}X(s)$. Since Y is invertible outer, the inverse $Y(\cdot)^{-1}$ is a rational $p \times p$ matrix function. Thus Z is a rational matrix function of size $(p-m) \times m$, and X has the desired representation (1.12).

STEP 4. We prove item (ii). Let h be any $\mathbb{C}^p$-valued H^2 function such that $G(s)h(s) = 0$ for $\Re s > 0$. Repeating the first three identities in (3.11) with h in place of H, we see that $h(s) = \Theta(s)\omega(s)$, where $\omega(s) = E^*Y(s)^{-1}h(s)$. Since Y is invertible outer, the entries of $Y(\cdot)^{-1}$ are H^∞ functions. Hence the entries of ω are H^2 functions. Furthermore, using the fact that Θ is inner, we see that $\omega(s) = \Theta(-\overline{s})^*h(s)$ for $\Re s > 0$. $\square$

To complete this section, let us establish the following useful (see the next section) identity

$$\Theta(-\overline{s})^*\Xi(s) = -B_i^*(sI_n + A_0^*)^{-1}C_0^*. \tag{3.12}$$

For convenience, let us set $B_1 = -(I_n - PQ)^{-1}BD^+$. Then (3.12) follows from (3.3), (3.9) and (3.10), that is,

$$\begin{aligned} \Theta(-\overline{s})^*\Xi(s) &= \big(E^* - B_i^*(sI_n + A_0^*)^{-1}C_1^*\big)\big(D^+ + C_1(sI_n - A_0)^{-1}B_1\big) \\ &= E^*C_1(sI_n - A_0)^{-1}B_1 - B_i^*(sI_n + A_0^*)^{-1}C_1^*D^+ \\ &\quad - B_i^*(sI_n + A_0^*)^{-1}C_1^*C_1(sI_n - A_0)^{-1}B_1 \\ &= \big((E^*C_1 + B_i^*(Q - QPQ)\big)(sI_n - A_0)^{-1}B_1 \\ &\quad - B_i^*(sI_n + A_0^*)^{-1}\big(C_1^*D^+ + (Q - QPQ)B_1\big) \\ &= -B_i^*(sI_n + A_0^*)^{-1}\big(C_1^*D^+ + (Q - QPQ)B_1\big) \\ &= -B_i^*(sI_n + A_0^*)^{-1}\left(C_1^* - QB\right)D^*(DD^*)^{-1} \\ &= -B_i^*(sI_n + A_0^*)^{-1}C_0^*. \end{aligned}$$

This establishes (3.12).

4. The minimization problem

Throughout this section G is a stable rational $m \times p$ matrix function, and we assume that G is given by the stable state space representation (1.2). We also assume that T_G is right invertible.

For each $\gamma > 0$ let w_γ be the scalar weight function given by $w_\gamma(s) = (s+\gamma)^{-1}$. Note that for each $X \in H^\infty_{p\times m}$ the function $w_\gamma X$ belongs to $H^2_{p\times m}$. With G and the weight function w_γ we associate the following minimization problem:

$$\inf\left\{\|w_\gamma X\|_2 \mid G(s)X(s) = I_m \ (\Re s > 0) \text{ and } X \in H^\infty_{p\times m}\right\}. \tag{4.1}$$

The problem is to check whether or not the infimum is a minimum, and if so, to find a minimizing function. We shall show in this section that such a minimizing X exists and is unique. It what follows this minimizing function will be denoted by Ξ_γ. The next theorem shows that Ξ_γ is a stable rational matrix function and provides an explicit formula for Ξ_γ.

Theorem 4.1. *For each $\gamma > 0$ there is a unique solution to the optimization problem* (4.1)*, and this solution is given by*

$$\Xi_\gamma(s) = \Xi(s) - \Theta(s)B_i^*(\gamma I - A_0^*)^{-1}C_0^*, \quad \Re s > 0. \tag{4.2}$$

Here Ξ and Θ are the rational matrix functions given by (1.7) *and* (1.11)*, respectively, the matrix A_0 is defined by* (1.6) *and C_0 by* (2.7)*. In particular, we have $\Xi(s) = \lim_{\gamma\to\infty} \Xi_\gamma(s)$.*

Proof. Fix $\gamma > 0$. Since for each $X \in H^\infty_{p\times m}$ the function $w_\gamma X$ belongs to $H^2_{p\times m}$, we have

$$\begin{aligned}\|w_\gamma \Xi_\gamma\|_2 &= \inf\left\{\|w_\gamma X\|_2 \mid w_\gamma G X = w_\gamma I_m \text{ and } X \in H^\infty_{p\times m}\right\} &(4.3)\\ &\geq \inf\left\{\|Z\|_2 \mid GZ = w_\gamma I_m \text{ and } Z \in H^2_{p\times m}\right\} = \|Z_\gamma\|_2. &(4.4)\end{aligned}$$

The last optimization problem is a least squares optimization problem. So the optimal solution Z_γ for the problem (4.4) is unique. We first derive a formula for Z_γ.

From item (ii) in Theorem 1.2 we know that $\operatorname{Ker} T_G = \operatorname{Im} T_\Theta$. By taking the Fourier transform, we see that Z_γ is the unique matrix function in $H^2_{p\times m}$ such that $GZ_\gamma = w_\gamma I_m$ and Z_γ is orthogonal to $\Theta H^2_{(p-m)\times m}$. Using $G\Xi = I_m$, we obtain that all H^2 solutions to $GZ = w_\gamma I_m$ are given by

$$Z = w_\gamma \Xi + \Theta H^2_{(p-m)\times m}.$$

So we are looking for a H^2 function Z_γ such that

$$Z_\gamma = w_\gamma \Xi + \Theta F \quad \text{and} \quad Z_\gamma \perp \Theta H^2_{(p-m)\times m},$$

where F is a matrix function in $H^2_{(p-m)\times m}$. By exploiting that Θ is inner, we obtain

$$w_\gamma \Theta^* \Xi + F \perp H^2_{(p-m)\times m}.$$

But then (3.12) tells us that the latter is equivalent to

$$-w_\gamma B_i^*(sI_n + A_0^*)^{-1}C_0^* + F \perp H^2_{(p-m)\times m}. \tag{4.5}$$

However, $-w_\gamma(s)B_i^*(sI_n + A_0^*)^{-1}C_0^*$ admits a partial fraction expansion of the form

$$-w_\gamma(s)B_i^*(sI_n + A_0^*)^{-1}C_0^* = (s+\gamma)^{-1}B_i^*(\gamma I_n - A_0^*)^{-1}C_0^* + \Omega^*(s),$$

where Ω is in $H^2_{(p-m)\times m}$. Using this in the orthogonality relation (4.5), we see that $F(s) = -w_\gamma(s)B_i^*(\gamma I_n - A_0^*)^{-1}C_0^*$. In other words,

$$Z_\gamma(s) = w_\gamma(s)\Xi(s) - w_\gamma(s)\Theta(s)B_i^*(\gamma I_n - A_0^*)^{-1}C_0^*. \tag{4.6}$$

Next put $\Xi_\gamma(s) = (s+\gamma)Z_\gamma(s)$. Then (4.6) implies that Ξ_γ is given by (4.2). Hence Ξ_γ is a stable rational matrix function. In particular, Ξ_γ belongs to $H^\infty_{p\times m}$. Furthermore, it follows that the inequality on the left-hand side of (4.4) is an equality. We conclude that Ξ_γ given by (4.2) is the unique solution to the minimization problem (4.1). □

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