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9 Clifford–Fourier Transform and Spinor Representation of Images

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Abstract. We propose in this chapter to introduce a spinor representation for images based on the work of T. Friedrich. This spinor representation generalizes the usual Weierstrass representation of minimal surfaces (*i.e.*, surfaces with constant mean curvature equal to zero) to arbitrary surfaces (immersed in $\mathbb{R}^3$). We investigate applications to image processing focusing on segmentation and Clifford–Fourier analysis. All these applications involve sections of the spinor bundle of image graphs, that is spinor fields, satisfying the so-called Dirac equation.

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1. Introduction

The idea of this chapter is to perform grey-level image processing using the geometric information given by the Gauss map variations of image graphs. While it is well known that one can parameterize the Gauss map of a minimal surface by a meromorphic function (see below), it is a much more recent result (see [5]) that such a parametrization can be extended to arbitrary surfaces of $\mathbb{R}^3$ when dealing with spin geometry.

Let us first recall that a minimal surface Σ immersed in $\mathbb{R}^3$, that is a surface with constant mean curvature equal to zero, can be described with one holomorphic function φ and one meromorphic function ψ such that the product $\varphi\psi^2$ is holomorphic. This is the so-called Weierstrass representation of Σ (see [6] or [8] for details). The function ψ is nothing else but the composition of the Gauss map of Σ with the stereographic projection from the unit sphere to the complex plane.

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The main result of T. Friedrich in [5] states that there is a one-to-one correspondance between spinor fields φ^* of constant length on a Riemannian surface (Σ, g) and satisfying

$$D\varphi^* = H\varphi^* \tag{1.1}$$

where D is a Dirac operator in one hand, and isometric immersions of Σ in $\mathbb{R}^3$ with mean curvature equal to H, on the other hand. The Weierstrass representation appears to be the particular case corresponding to $H \equiv 0$.

Let us describe now the method introduced in the following. Let

$$\begin{aligned} \chi : \Omega \subset \mathbb{R}^2 &\longrightarrow \mathbb{R}^3 \\ (x, y) &\longmapsto (x, y, I(x, y)) \end{aligned} \tag{1.2}$$

be the immersion in the three-dimensional Euclidean space of a grey-level image I defined on a domain Ω of $\mathbb{R}^2$. The first step (see § 2) consists in computing the spinor field φ^* that describes the image surface Σ. We follow here the paper of T. Friedrich [5]: φ^* is obtained from the restriction to the surface Σ of a parallel spinor ϕ on $\mathbb{R}^3$. The computation of φ^* requires us to deal with irreducible representations of the complex Clifford algebra $C\ell_{3,0} \otimes \mathbb{C}$ and with the generalized Weierstrass representation of Σ based on period forms. In practice, φ^* is given by a field of elements of $\mathbb{C}^2$.

As said before, the spinor field φ^* characterizes the geometry of the surface Σ immersed in $\mathbb{R}^3$ by the parametrization (1.2). In the same way that the normal of a minimal surface is parameterized by the meromorphic function ψ, the normal of the surface Σ is parameterized by the spinor field φ^*. The latter explains how the tangent plane to Σ varies in the ambient space.

There are many reasons to believe that such a generalized Weierstrass parametrization may reveal itself to be an efficient tool in the context of image processing:

1. The field φ^* of elements of $\mathbb{C}^2$ (see (2.26)) encodes the Riemannian structure of the surface Σ in a very tractable way (although the definition of φ^* may appear quite complicated).
2. The geometrical methods based on the study of the so-called structure tensor involve only the eigenvalues of the structure tensor, that means in some sense the values of the first fundamental form of the surface. The spinor field φ^* contains both intrinsic and extrinsic information. Studying the variations of φ^* allows us to get not only information about the variations (derivative) of the first fundamental form, but also about the geometric embedding of the surface Σ and in particular about the mean curvature.
3. We are dealing here with first-order instead of zero-order geometric variations of Σ. As shown later, this appears to be more relevant by taking into account both edges and textures.
4. As will be detailed in the sequel, the spinor field φ^* can be decomposed as a series of basic spinor fields using a suitable Clifford–Fourier transform. This series corresponds to a harmonic decomposition of the surface Σ adapted to

the Riemannian geometry. This is in fact the main novelty of this chapter since the usual techniques of Fourier analysis do not involve geometric data.

5. One can envisage the possibility of performing diffusion in this context. The usual Laplace Beltrami operator can be replaced by the squared Atiyah Singer Dirac operator [7] (the Atiyah Singer Dirac operator acting as an elliptic operator of order one on spinor fields).

To illustrate some of these ideas, we investigate rapidly in § 3 applications to segmentation and more precisely to edge and texture detection. As stated before, the basic idea is to replace the usual order-one structure tensor by an order-two structure tensor called the spinor tensor obtained from the derivative of the spinor field φ^*. This spinor tensor measures the variations of the unit normal of the image surface. Experiments show that this approach is particularly well adapted to texture detection.

We define in § 4 the Clifford–Fourier transform of a spinor field. For this, we follow the approach of [3] that relies on a spin generalization of the usual notion of group character. We are led to compute the group morphisms from $\mathbb{Z}/M\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$ to $\mathrm{Spin}(3)$. Since this last group acts on the sections of the spinor bundle, a Clifford–Fourier transform can be defined by averaging this action. One of the key ideas here is to split the spinor bundle of the surface according to the Clifford multiplication by the bivector coding the tangent plane to the surface. This has two advantages: the first one is to involve the geometry in the process, the second one is to reduce the computation of the Clifford–Fourier transform to two usual complex Fourier transforms. It is important to notice that although the Fourier transform we propose is, as usual, a global transformation on the image, the way it is computed takes into account local geometric data. We finally introduce the harmonic decomposition mentioned above and show some results of filtering on standard images.

The reader will find in Appendix A the mathematical definitions and results used throughout the text.

2. Spinor Representation of Images

This section is devoted to the explicit computation of the spinor field φ^* of a given surface immersed in Euclidean space. It is obtained as the restriction of a constant spinor field of $\mathbb{R}^3$ the components of which are determined using period forms.

2.1. Spinors and Graphs

Let $I : \Omega \longrightarrow \mathbb{R}$ be a differentiable function defined on a domain Ω of $\mathbb{R}^2$. We consider the surface Σ immersed in $\mathbb{R}^3$ by the parametrization:

$$\chi(x, y) = (x, y, I(x, y)). \tag{2.1}$$

Also, let g be the metric on Σ induced by the Euclidean metric of $\mathbb{R}^3$. The couple (Σ, g) is a Riemannian surface of global chart (Ω, χ). We denote by M the Riemannian manifold $(\mathbb{R}^3, \| \, \|_2)$ and by (z_1, z_2, ν) an orthonormal frame field of

M with (z_1, z_2) an orthonormal frame field on Σ, and by ν the global unit field normal to Σ. One can choose (z_1, z_2, ν) with the following matrix representation

$$\begin{pmatrix} \dfrac{I_x}{\sqrt{(I_x^2+I_y^2)(I_x^2+I_y^2+1)}} & \dfrac{-I_y}{\sqrt{I_x^2+I_y^2}} & \dfrac{-I_x}{\sqrt{I_x^2+I_y^2+1}} \\ \dfrac{I_y}{\sqrt{(I_x^2+I_y^2)(I_x^2+I_y^2+1)}} & \dfrac{I_x}{\sqrt{I_x^2+I_y^2}} & \dfrac{-I_y}{\sqrt{I_x^2+I_y^2+1}} \\ \dfrac{I_x^2+I_y^2}{\sqrt{(I_x^2+I_y^2)(I_x^2+I_y^2+1)}} & 0 & \dfrac{1}{\sqrt{I_x^2+I_y^2+1}} \end{pmatrix}. \tag{2.2}$$

Note that z_1 and z_2 are not defined when $I_x = I_y = 0$. This has no consequence in the sequel since we deal only with the normal ν.

Following [5] the surface Σ can be represented by a spinor field φ^* with constant length satisfying the Dirac equation:

$$D\varphi^* = H\varphi^* \tag{2.3}$$

where H denotes the mean curvature of Σ. We recall here the basic idea (see Appendix A for notations and definitions). Let ϕ be a parallel spinor field of M, *i.e.*, satisfying

$$\nabla_X^M \phi = 0 \tag{2.4}$$

for all vector fields X on M. Let also φ be the restriction $\phi_{|\Sigma}$ of ϕ to Σ. The spinor field φ decomposes into

$$\varphi = \varphi^+ + \varphi^- \tag{2.5}$$

with

$$\varphi^+ = \frac{1}{2}(\varphi + i\nu \cdot \varphi) \qquad \varphi^- = \frac{1}{2}(\varphi - i\nu \cdot \varphi) \tag{2.6}$$

and satisfies

$$D\varphi = -H \cdot \nu \cdot \varphi. \tag{2.7}$$

This last equation reads

$$D(\varphi^+ + \varphi^-) = -H \cdot \nu \cdot (\varphi^+ + \varphi^-) \tag{2.8}$$

and implies

$$D\varphi^+ = -iH\varphi^- \qquad D\varphi^- = iH\varphi^+. \tag{2.9}$$

If we set $\varphi^* = \varphi^+ - i\varphi^-$ then $D\varphi^* = H\varphi^*$ and φ^* is of constant length.

Proposition 2.1. *The spinor fields φ^+, φ^- and φ^* are given by*

$$\varphi^+ = \frac{1}{2}\left(\begin{array}{c} \left(1 - \frac{I_y}{\sqrt{1+I_x^2+I_y^2}}\right) u + \left(\frac{I_x - i}{\sqrt{1+I_x^2+I_y^2}}\right) v \\ \left(1 + \frac{I_y}{\sqrt{1+I_x^2+I_y^2}}\right) v + \left(\frac{I_x + i}{\sqrt{1+I_x^2+I_y^2}}\right) u \end{array}\right) \tag{2.10}$$

$$\varphi^- = \frac{1}{2}\left(\begin{array}{c} \left(1 + \frac{I_y}{\sqrt{1+I_x^2+I_y^2}}\right) u - \left(\frac{I_x - i}{\sqrt{1+I_x^2+I_y^2}}\right) v \\ \left(1 - \frac{I_y}{\sqrt{1+I_x^2+I_y^2}}\right) v - \left(\frac{I_x + i}{\sqrt{1+I_x^2+I_y^2}}\right) u \end{array}\right) \tag{2.11}$$

and

$$\varphi^* = \frac{1}{2}(1-i)\left(\begin{array}{c} \left(1 - \frac{iI_y}{\sqrt{1+I_x^2+I_y^2}}\right) u + \left(\frac{1 + iI_x}{\sqrt{1+I_x^2+I_y^2}}\right) v \\ \left(1 + \frac{iI_y}{\sqrt{1+I_x^2+I_y^2}}\right) v + \left(\frac{iI_x - 1}{\sqrt{1+I_x^2+I_y^2}}\right) u \end{array}\right) \tag{2.12}$$

where u and v are (constant) complex numbers.

Proof. Since ϕ is a parallel spinor field on M, $\phi = (u, v)$ where u and v are two (constant) complex numbers. Let ρ_2 be the irreducible complex representation of $\mathbb{C}l(3)$ described in Appendix A.1. Recall that

$$\nu = \frac{1}{\Delta}(-I_x e_1 - I_y e_2 + e_3) \tag{2.13}$$

where $\Delta = \sqrt{I_x^2 + I_y^2 + 1}$, so that

$$\rho_2(\nu) = -\frac{I_x}{\Delta}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} - \frac{I_y}{\Delta}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} + \frac{1}{\Delta}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{2.14}$$

By definition:

$$\nu \cdot \varphi = \rho_2(\nu)\begin{pmatrix} u \\ v \end{pmatrix}. \tag{2.15}$$

Simple computations lead now to the result. $\square$

The next step consists in computing the components (u, v) of the constant field ϕ. This is done by considering a quaternionic structure on the spinor bundle $S(\Sigma)$ of the surface Σ and period forms.

2.2. Quaternionic Structure and Period Forms

Let I be the complex structure on $S(\Sigma)$ given by the multiplication by i. A quaternionic structure on $S(\Sigma)$ is a linear map J that satisfies $J^2 = -Id$ and $IJ = -JI$. In the sequel J is given by

$$J\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} -\overline{\varphi_2} \\ \overline{\varphi_1} \end{pmatrix}. \tag{2.16}$$

If we write $\varphi_1 = \alpha_1 + i\beta_1$ and $\varphi_2 = \alpha_2 + i\beta_2$, the corresponding quaternion is given by

$$\varphi_1 + \varphi_2 j = (\alpha_1 + i\beta_1) + (\alpha_2 + i\beta_2)j = \alpha_1 + i\beta_1 + \alpha_2 j + \beta_2 k \tag{2.17}$$

and

$$j(\varphi_1 + \varphi_2 j) = -\overline{\varphi_2} + \overline{\varphi_1} j, \tag{2.18}$$

i.e., J is the left multiplication by j. Since

$$S^+(\Sigma) = \left\{ \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix},\ \varphi_1 = \frac{I_x - i}{I_y + \Delta}\varphi_2 \right\} \tag{2.19}$$

and

$$S^+(\Sigma) = \left\{ \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix},\ \varphi_1 = \frac{I_x - i}{I_y - \Delta}\varphi_2 \right\} \tag{2.20}$$

then $JS^+(\Sigma) \subset S^-(\Sigma)$ and $JS^-(\Sigma) \subset S^+(\Sigma)$. We also denote by J the quaternionic structure (obtained in the same way) on $S(M)$.

Let us consider $\phi = (u, v)$ a constant spinor field on M and φ^* its restriction on Σ. Let also $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ and $g : \mathbb{R}^3 \longrightarrow \mathbb{C}$ be the functions defined by

$$f(m) = -\Im(m \cdot \phi, \phi) \tag{2.21}$$

and

$$g(m) = i(m \cdot \phi, J(\phi)) \tag{2.22}$$

where $(\ ,\ )$ denotes the Hermitian product. Using the representation ρ_2, one can check that

$$m \cdot \phi = \begin{pmatrix} -im_2 u + (im_1 - m_3)v \\ (im_1 + m_3)u + im_2 v \end{pmatrix} \tag{2.23}$$

for $m = (m_1, m_2, m_3)$. The equations $f(m) = m_1$ and $g(m) = m_2 + im_3$ are equivalent to:

$$|u|^2 = |v|^2, \quad u\overline{v} = -\frac{1}{2} \tag{2.24}$$

and

$$uv = -\frac{1}{2}, \quad u^2 + v^2 = 1, \quad u^2 = v^2. \tag{2.25}$$

This implies $u = \pm 1/\sqrt{2}$ and $v = -u$.

Definition 2.2. The spinor representation of the image given by the parametrization (2.1) is defined by

$$\varphi^* = \frac{1}{2\sqrt{2}}(1-i)\begin{pmatrix} \left(1 - \dfrac{1+i(\ \ I_x + I_y)}{\sqrt{1+I_x^2+I_y^2}}\right) \\ -\left(1 + \dfrac{1+i(-I_x + I_y)}{\sqrt{1+I_x^2+I_y^2}}\right) \end{pmatrix}. \tag{2.26}$$

This means that $u = 1/\sqrt{2}$ and $v = -1/\sqrt{2}$ in the expression (2.12).

The two 1-forms

$$\eta_f(X) = 2\,\Re(X \cdot (\varphi^*)^+, (\varphi^*)^-) = -\Im(X \cdot \varphi, \varphi) \tag{2.27}$$

$$\begin{aligned}\eta_g(X) &= i(X \cdot (\varphi^*)^+, J((\varphi^*)^+)) + i(X \cdot (\varphi^*)^-, J((\varphi^*)^-)) \\ &= i(X \cdot \varphi, J(\varphi))\end{aligned} \tag{2.28}$$

are exact and verify $d(f_{|\Sigma}) = \eta_f$, $d(g_{|\Sigma}) = \eta_g$. The generalized Weierstrass parametrization is actually given by the isometric immersion:

$$\int(\eta_f, \eta_g) : \Sigma \longrightarrow M. \tag{2.29}$$

2.3. Dirac Equation and Mean Curvature

We only mention here some results that can be used when dealing with diffusion. We do not go into further details since we will not treat this problem in the present chapter. Let (Σ, g) be an oriented two-dimensional Riemannian manifold and φ a spinor field without zeros solution of the Dirac equation $D\varphi = \lambda\varphi$. Then φ defines an isometric immersion

$$(\widetilde{\Sigma}, |\varphi|^4 g) \longrightarrow \mathbb{R}^3 \tag{2.30}$$

with mean curvature $H = \lambda/|\varphi|^2$ (see [5]).

3. Spinors and Segmentation

The aim of this section is to introduce the spinor tensor corresponding to the variations of the unit normal and to show its capability to detect both edges and textures.

3.1. The Spinor Tensor

We propose here to deal with a second-order version of the classical approach of edge detection based on the so-called structure tensor (see [10]). Instead of measuring edges from eigenvalues of the Riemannian metric, we focus here on the eigenvalues of the tensor obtained from the derivative of the spinor field φ^*. More precisely let

$$\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \tag{3.1}$$

be a section of the spinor bundle $S(\Sigma)$ given in an orthonormal frame, *i.e.*, $|\varphi|^2 = |\varphi_1|^2 + |\varphi_2|^2$ and let $X = (X_1, X_2)$ be a section of the tangent bundle $T(\Sigma)$. We consider the connection ∇ on $S(\Sigma)$ given by the connection 1-form $\omega = 0$. Thus

$$\nabla_X \varphi = \begin{pmatrix} X_1 \dfrac{\partial \varphi_1}{\partial x} + X_2 \dfrac{\partial \varphi_1}{\partial y} \\ X_1 \dfrac{\partial \varphi_2}{\partial x} + X_2 \dfrac{\partial \varphi_2}{\partial y} \end{pmatrix} \tag{3.2}$$

and

$$\begin{aligned} |\nabla_X \varphi|^2 = {} & X_1^2 \left| \frac{\partial \varphi_1}{\partial x} \right|^2 + 2X_1X_2 \Re \left(\frac{\partial \varphi_1}{\partial x} \overline{\frac{\partial \varphi_1}{\partial y}} \right) + X_2^2 \left| \frac{\partial \varphi_1}{\partial y} \right|^2 \\ & + X_1^2 \left| \frac{\partial \varphi_2}{\partial x} \right|^2 + 2X_1X_2 \Re \left(\frac{\partial \varphi_2}{\partial x} \overline{\frac{\partial \varphi_2}{\partial y}} \right) + X_2^2 \left| \frac{\partial \varphi_2}{\partial y} \right|^2 . \end{aligned} \tag{3.3}$$

If we denote

$$G_\varphi = \begin{pmatrix} \left| \dfrac{\partial \varphi_1}{\partial x} \right|^2 + \left| \dfrac{\partial \varphi_2}{\partial x} \right|^2 & \Re \left(\dfrac{\partial \varphi_1}{\partial x} \overline{\dfrac{\partial \varphi_1}{\partial y}} + \dfrac{\partial \varphi_2}{\partial x} \overline{\dfrac{\partial \varphi_2}{\partial y}} \right) \\ \Re \left(\dfrac{\partial \varphi_1}{\partial x} \overline{\dfrac{\partial \varphi_1}{\partial y}} + \dfrac{\partial \varphi_2}{\partial x} \overline{\dfrac{\partial \varphi_2}{\partial y}} \right) & \left| \dfrac{\partial \varphi_1}{\partial y} \right|^2 + \left| \dfrac{\partial \varphi_2}{\partial y} \right|^2 \end{pmatrix} \tag{3.4}$$

then

$$(X_1 \ X_2) G_\varphi (X_1 \ X_2)^T = |\nabla_X \varphi|^2 . \tag{3.5}$$

G_φ is a field of real symmetric matrices.

As in the case of the usual structure tensor (*i.e.*, Di Zenzo tensor, see [10]) the optima of $|\nabla_X \varphi|^2$ under the constraint $\|X\| = 1$ (for the Euclidean norm) are given by the field of eigenvalues of G_φ. Applying the above formula to the spinor φ^* of Definition 2.2 leads to

$$G_{\varphi^*} = \frac{1}{2(1 + I_x^2 + I_y^2)^2} \begin{pmatrix} G_{\varphi^*}^{11} & G_{\varphi^*}^{12} \\ G_{\varphi^*}^{21} & G_{\varphi^*}^{22} \end{pmatrix} \tag{3.6}$$

with

$$\begin{aligned} G_{\varphi^*}^{11} &= I_{xx}^2 + I_{xy}^2 + I_{xx}^2 I_y^2 + I_{xy}^2 I_x^2 - 2I_{xx}I_{xy}I_xI_y \\ G_{\varphi^*}^{22} &= I_{yy}^2 + I_{xy}^2 + I_{yy}^2 I_x^2 + I_{xy}^2 I_y^2 - 2I_{yy}I_{xy}I_xI_y \\ G_{\varphi^*}^{12} &= I_{xx}I_{xy} + I_{xy}I_{yy} + I_{xx}I_{xy}I_y^2 + I_{xy}I_{yy}I_x^2 - I_{xy}^2 I_xI_y - I_{xx}I_{yy}I_xI_y \\ G_{\varphi^*}^{21} &= G_{\varphi^*}^{12} . \end{aligned} \tag{3.7}$$

Definition 3.1. The tensor G_{φ^*} is called the spinor tensor of the surface Σ.

Note that as already mentioned this last tensor corresponds to the tensor involved in the measure of the variations of the unit normal ν introduced in § 2.1. Indeed, we have

$$(X_1\ X_2)G_{\varphi^*}(X_1\ X_2)^T = \|d_X\nu\|^2. \tag{3.8}$$

3.2. Experiments

We compare in Figure 1 the edge and texture detection methods based on the usual structure tensor (Figure 1(b) and 1(d)) and on the spinor tensor (Figure 1(e) and 1(f)).

The structure tensor only takes into account the first-order derivatives of the function I. The subsequent segmentation method detects the strongest grey-level variations of the image. As a consequence, this method provides thick edges, as can be observed.

The spinor tensor takes into account the second-order derivatives of the function I too. By definition, it measures the strongest variations of the unit normal to the surface parametrized by the graph of I. We observe that this new approach provides thinner edges than the first one. It appears also to be more relevant to detect textures.

4. Spinors and Clifford–Fourier Transform

We first define a Clifford–Fourier transform using spin characters that is group morphisms from $\mathbb{R}^2$ to Spin(3). Then, we introduce a harmonic decomposition of spinor fields and show some results of filtering applied to images.

4.1. Clifford–Fourier Transform with Spin Characters

Let us recall the idea of the construction of the Clifford–Fourier transform for colour image processing introduced in [3]. From the mathematical viewpoint, a Fourier transform is defined through group actions and more precisely through irreducible and unitary representations of the involved group. This is closely related to the well-known shift theorem stating that:

$$\mathcal{F}f_\alpha(u) = e^{i\alpha u}\mathcal{F}f(u) \tag{4.1}$$

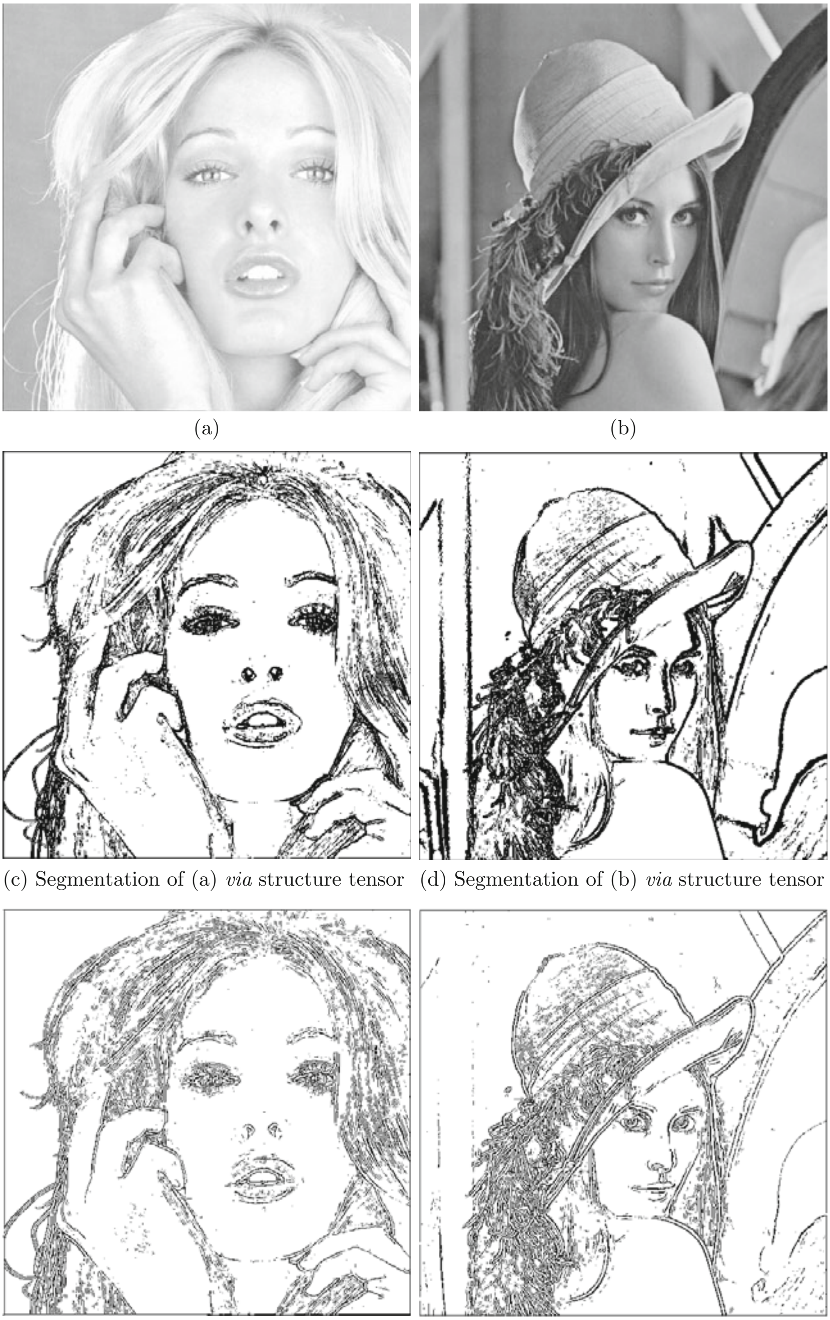
where $f_\alpha(u) = f(\alpha + u)$. The group morphism

$$\alpha \longmapsto e^{i\alpha u} \tag{4.2}$$

is a so-called *character* of the additive group $(\mathbb{R}, +)$, that is an irreducible unitary representation of dimension 1.

The definition proposed in [3] relies on a Clifford generalization of this notion by introducing spin characters. It can be shown that the group morphisms from $\mathbb{Z}/M\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$ to Spin(3) are given by

$$\rho_{u,v,B} : (m, n) \longmapsto e^{2\pi(um/M+vn/N)B} \tag{4.3}$$



(a)

(b)

(c) Segmentation of (a) *via* structure tensor

(d) Segmentation of (b) *via* structure tensor

(e) Segmentation of (a) *via* spinor tensor

(e) Segmentation of (b) *via* spinor tensor

FIGURE 1. Segmentation: Structure tensor vs. spinor tensor

where

$$e^{2\pi\left(\frac{um}{M}+\frac{vn}{N}\right)B} = \cos 2\pi\left(\frac{um}{M}+\frac{vn}{N}\right) + \sin 2\pi\left(\frac{um}{M}+\frac{vn}{N}\right)B \tag{4.4}$$

$(u,v) \in \mathbb{Z}/M\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$, and

$$B = \gamma_1 e_1 e_2 + \gamma_2 e_1 e_3 + \gamma_3 e_2 e_3 \tag{4.5}$$

is a unit bivector, *i.e.*, $\gamma_1^2+\gamma_2^2+\gamma_3^2 = 1$. The map $\rho_{u,v,B}$ is called a spin character of the group $\mathbb{Z}/M\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$. Recalling that Spin(3) acts on the sections of the spinor bundle, we are led to propose the following definition.

Definition 4.1. The Clifford–Fourier transform of a spinor φ of $S(\Sigma)$ is given by

$$\mathcal{F}(\varphi)(u,v) = \sum_{\substack{n\in\mathbb{Z}/N\mathbb{Z}\\ m\in\mathbb{Z}/M\mathbb{Z}}} \rho_{u,v,z_1\wedge z_2\,(m,n)}(-m,-n)\cdot\varphi(m,n) \tag{4.6}$$

where (z_1,z_2) is an orthonormal frame of $T(\Sigma)$.

Since the spinor bundle of Σ splits into

$$S(\Sigma) = S^+_{z_1\wedge z_2}(\Sigma) \oplus S^-_{z_1\wedge z_2}(\Sigma) \tag{4.7}$$

we have

$$\begin{aligned}\rho_{u,v,z_1\wedge z_2(m,n)}(-m,-n)\cdot\varphi(m,n) &= e^{2\pi i\left(\frac{um}{M}+\frac{vn}{N}\right)}\varphi^+(m,n)\,v_{-i}(m,n)\\ &\quad + e^{-2\pi i\left(\frac{um}{M}+\frac{vn}{N}\right)}\varphi^-(m,n)\,v_i(m,n)\end{aligned} \tag{4.8}$$

where v_{-i}, respectively v_i, is the unit eigenspinor field of eigenvalue $-i$, respectively i, relatively to the operator $z_1\wedge z_2\,\cdot$ (here $\cdot$ denotes the Clifford multiplication). Consequently

$$\mathcal{F}(\varphi)(u,v) = \left(\widehat{\varphi^+}^{-1}(u,v),\ \widehat{\varphi^-}(u,v)\right) \tag{4.9}$$

in the frame (v_{-i}, v_i), where $\widehat{\ }$ and $\widehat{\ }^{-1}$ denote the Fourier transform on

$$L^2(\mathbb{Z}/M\mathbb{Z}\times\mathbb{Z}/N\mathbb{Z},\mathbb{C}),$$

also called discrete Fourier transform, and its inverse.

4.2. Spinor Field Decomposition

The inverse Clifford–Fourier transform of φ is

$$\mathcal{F}^{-1}(\varphi)(u,v) = \sum_{\substack{n\in\mathbb{Z}/N\mathbb{Z}\\ m\in\mathbb{Z}/M\mathbb{Z}}} \rho_{u,v,z_1\wedge z_2(m,n)}(m,n)\,\cdot\,\varphi(m,n) \tag{4.10}$$

This means that every spinor field φ may be written as a superposition of basic spinor fields, *i.e.*,

$$\varphi = \sum \varphi_{m,n} \tag{4.11}$$

where

$$\varphi_{m,n} : (u,v) \longmapsto \rho_{u,v,z_1\wedge z_2(m,n)}(m,n)\cdot\mathcal{F}(\varphi)(m,n) \tag{4.12}$$

Following the splitting $S(\Sigma) = S^+_{z_1\wedge z_2}(\Sigma) \oplus S^-_{z_1\wedge z_2}(\Sigma)$, we have

$$\varphi_{m,n} = \left(\varphi^+_{m,n}, \varphi^-_{m,n}\right)$$

in the frame (v_{-i}, v_i), with

$$\varphi^+_{m,n} \colon (u,v) \longmapsto e^{-2\pi i(um/M+vn/N)}\,\widehat{\varphi^+}^{-1}(m,n)$$

and

$$\varphi^-_{m,n} \colon (u,v) \longmapsto e^{2\pi i(um/M+vn/N)}\,\widehat{\varphi^-}(m,n)$$

Moreover,

$$|\varphi_{m,n}|^2 = |\varphi^+_{m,n}|^2 + |\varphi^-_{m,n}|^2$$

since $S^+_{z_1\wedge z_2}(\Sigma)$ and $S^-_{z_1\wedge z_2}(\Sigma)$ are orthogonal.

4.3. Experiments

Let us now give an example of applications of the Clifford–Fourier transform on spinor fields to image processing. In order to perform filtering with the decomposition (4.11), we proceed as follows. Let I be a grey-level image, and φ^* be the corresponding spinor representation given in Definition 2.2. We apply a Gaussian mask T_σ of variance σ in the spectrum $\mathcal{F}\varphi^*$ of φ^*. Then, we consider the norm of its inverse Fourier transform, *i.e.*, $|\mathcal{F}^{-1}T_\sigma\mathcal{F}\varphi^*|$ and the function $|\mathcal{F}^{-1}T_\sigma\mathcal{F}\varphi^*|\,I$.

Figures 2 and 3 show results of this process for different values of σ (left column $|\mathcal{F}^{-1}T_\sigma\mathcal{F}\varphi^*|$ and right column $|\mathcal{F}^{-1}T_\sigma\mathcal{F}\varphi^*|\,I$). It is clear that for σ sufficiently high, we have $|\mathcal{F}^{-1}T_\sigma\mathcal{F}\varphi^*|\,I \simeq I$ and $|\mathcal{F}^{-1}T_\sigma\mathcal{F}\varphi^*| \simeq 1$ since $|\varphi^*| = 1$. This explains why the two left lower images are almost white and the two right lower images are almost the same as the originals.

We can see in the left columns of Figures 2 and 3 that the filtering acts through φ^* as a smoothing of the geometry of the image. More precisely, when σ is small, the modulus $|\mathcal{F}^{-1}T_\sigma\mathcal{F}\varphi^*|$ is small at points corresponding to nearly all the geometric variations of the image. When σ increases the modulus is affected only at points corresponding to the strongest geometric variations, *i.e.*, to both edges and textures (and also where the noise is high).

The right columns of Figures 2 and 3 show that the filtering acts through $|\mathcal{F}^{-1}T_\sigma\mathcal{F}\varphi^*|\,I$ as a diffusion that leaves the geometric data untouched (the higher the value of σ, the more important is the diffusion). This appears clearly in Figure 4 (compare the plumes of the hat) or in Figure 5 (compare the hair).

These experiments show that our approach is relevant to deal with harmonic analysis together with Riemannian geometry.

Conclusion

Spin geometry is a powerful mathematical tool to deal with many theoretical and applied geometric problems. In this chapter we have shown how to take advantage of the generalized Weierstrass representation to perform grey-level image processing, in particular edge and texture detection. Our main contribution is the

FIGURE 2. Left: $|\mathcal{F}^{-1}(T_\sigma\,\mathcal{F}\varphi^*)|$ for $\sigma = 100, 1000, 10000, 100000$ (from top to bottom). Right: $|\mathcal{F}^{-1}(T_\sigma\,\mathcal{F}\varphi^*)|I$

definition of a Clifford–Fourier transform for spinor fields that relies on a generalization of the usual notion of character (the spin character). One important fact is that this new transform takes into account the Riemannian geometry of the

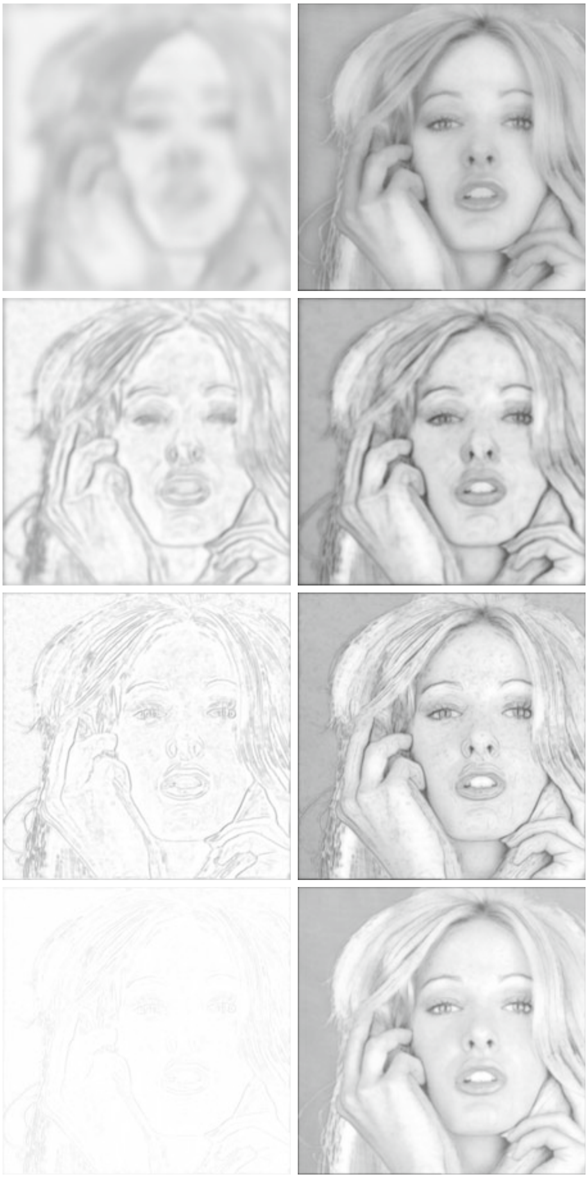


FIGURE 3. Left: $|\mathcal{F}^{-1}(T_\sigma\, \mathcal{F}\varphi^*)|$ for $\sigma = 100, 1000, 10000, 100000$ (from top to bottom). Right: $|\mathcal{F}^{-1}(T_\sigma\, \mathcal{F}\varphi^*)|I$

image surface by involving the spinor field that parameterizes the normal and the bivector field coding the tangent plane. We have also introduced what appears to

FIGURE 4. Left: original. Right: $|\mathcal{F}^{-1}T_\sigma\mathcal{F}\varphi^*|\,I$ with $\sigma = 100$

FIGURE 5. Left: original. Right: $|\mathcal{F}^{-1}T_\sigma\mathcal{F}\varphi^*|\,I$ with $\sigma = 100$

be a harmonic decomposition of the parametrization and investigated applications to filtering.

Note that there are only two cases where the Grassmannian $G_{n,2}$ of 2-planes in $\mathbb{R}^n$ admits a rational parametrization. In fact, one can show that $G_{3,2} \simeq \mathbb{C}P^1$ and $G_{4,2} \simeq \mathbb{C}P^1 \times \mathbb{C}P^1$ (see [9]). The case treated here corresponds to $G_{3,2}$. As a consequence the generalization to colour images is not straightforward. Nevertheless, a quite different approach is possible to tackle this problem and will be the subject of a forthcoming paper.

Let us also mention that one may envisage performing diffusion on grey-level images through the heat equation given by the Dirac operator. The latter is well known be a square root of the Laplacian. Preliminary results are discussed in [2] that show that this diffusion better preserves edges and textures than the usual Riemannian approaches.

Appendix A. Mathematical Background

We recall here some definitions and results concerning spin geometry. The reader may refer to [7] for details and conventions. We focus on the particular case of an oriented surface immersed in $\mathbb{R}^3$.

A.1. Complex Representations of $\boldsymbol{C\ell_{3,0}} \otimes \mathbb{C}$

Let (e_1, e_2, e_3) be an orthonormal basis of $\mathbb{R}^3$. The Clifford algebra $C\ell_{3,0}$ is the quotient of the tensor algebra of the vectorial space $\mathbb{R}^3$ by the ideal generated by the elements $u \otimes u + Q(u)$ where Q is the Euclidean quadratic form. It can be shown that $C\ell_{3,0}$ is isomorphic to the product $\mathbb{H}\times\mathbb{H}$ of two copies of the quaternion algebra. The complex Clifford algebra $C\ell_{3,0}\otimes\mathbb{C}$ is isomorphic to $\mathbb{C}(2)\oplus\mathbb{C}(2)$ where $\mathbb{C}(2)$ denotes the algebra of 2×2-matrices with complex entries. This decomposition is given by

$$C\ell_{3,0} \otimes \mathbb{C} \simeq (C\ell_{3,0} \otimes \mathbb{C})^+ \oplus (C\ell_{3,0} \otimes \mathbb{C})^- \tag{A.1}$$

where

$$(C\ell_{3,0} \otimes \mathbb{C})^\pm = (1 \pm \omega_3) C\ell_{3,0} \otimes \mathbb{C} \tag{A.2}$$

and ω_3 is the pseudoscalar $\boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3$. More precisely, the subalgebra $(C\ell_{3,0} \otimes \mathbb{C})^+$ is generated by the elements

$$\alpha_1 = \frac{1 + \boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3}{2},\ \alpha_2 = \frac{\boldsymbol{e}_2\boldsymbol{e}_3 - \boldsymbol{e}_1}{2},\ \alpha_3 = \frac{\boldsymbol{e}_2 + \boldsymbol{e}_1\boldsymbol{e}_3}{2},\ \alpha_4 = \frac{\boldsymbol{e}_3 - \boldsymbol{e}_1\boldsymbol{e}_2}{2} \tag{A.3}$$

and an isomorphism with $\mathbb{C}(2)$ is given by sending these elements to the matrices

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, A_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, A_4 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{A.4}$$

In the same way, $(C\ell_{3,0} \otimes \mathbb{C})^-$ is generated by

$$\beta_1 = \frac{1 - \boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3}{2}, \beta_2 = \frac{\boldsymbol{e}_2\boldsymbol{e}_3 + \boldsymbol{e}_1}{2}, \beta_3 = \frac{\boldsymbol{e}_1\boldsymbol{e}_3 - \boldsymbol{e}_2}{2}, \beta_4 = \frac{-\boldsymbol{e}_3 - \boldsymbol{e}_1\boldsymbol{e}_2}{2} \tag{A.5}$$

and an isomorphism is given by sending these elements to the above matrices A_1, A_2, A_3 and A_4.

Let us denote by ρ the natural representation of $\mathbb{C}(2)$ on $\mathbb{C}^2$. The two equivalent classes ρ_1 and ρ_2 of irreducible complex representations of $C\ell_{3,0} \otimes \mathbb{C}$ are given by

$$\rho_1(\varphi_1 + \varphi_2) = \rho(\varphi_1) \qquad \rho_2(\varphi_1 + \varphi_2) = \rho(\varphi_2). \tag{A.6}$$

They are characterized by

$$\rho_1(\omega_3) = Id \text{ and } \rho_2(\omega_3) = -Id \tag{A.7}$$

For the sake of completeness, let us list these representations explicitly:

$$\begin{aligned}
\rho_1(1) = \rho(\alpha_1) &= \ \ A_1, & \rho_1(\boldsymbol{e}_1) = \rho(-\alpha_2) &= -A_2 \\
\rho_1(\boldsymbol{e}_2) = \rho(\alpha_3) &= \ \ A_3, & \rho_1(\boldsymbol{e}_3) = \rho(\alpha_4) &= \ \ A_4 \\
\rho_1(\boldsymbol{e}_1\boldsymbol{e}_2) = \rho(-\alpha_4) &= -A_4, & \rho_1(\boldsymbol{e}_1\boldsymbol{e}_3) = \rho(\alpha_3) &= \ \ A_3 \\
\rho_1(\boldsymbol{e}_2\boldsymbol{e}_3) = \rho(\alpha_2) &= \ \ A_2, & \rho_1(\omega_3) = \rho(\alpha_1) &= \ \ A_1
\end{aligned} \tag{A.8}$$

and

$$\begin{aligned} \rho_2(1) = \rho(\beta_1) &= A_1, & \rho_2(\boldsymbol{e}_1) = \rho(\beta_2) &= A_2 \\ \rho_2(\boldsymbol{e}_2) = \rho(-\beta_3) &= -A_3, & \rho_2(\boldsymbol{e}_3) = \rho(-\beta_4) &= -A_4 \\ \rho_2(\boldsymbol{e}_1\boldsymbol{e}_2) = \rho(-\beta_4) &= -A_4, & \rho_2(\boldsymbol{e}_1\boldsymbol{e}_3) = \rho(\beta_3) &= A_3 \\ \rho_2(\boldsymbol{e}_2\boldsymbol{e}_3) = \rho(\beta_2) &= A_2, & \rho_2(\omega_3) = \rho(-\beta_1) &= -A_1. \end{aligned} \tag{A.9}$$

The complex spin representation of Spin(3) is the homomorphism

$$\Delta_3 : \mathrm{Spin}(3) \longrightarrow \mathbb{C}(2) \tag{A.10}$$

given by restricting an irreducible complex representation of $C\ell_{3,0} \otimes \mathbb{C}$ to the spinor group $\mathrm{Spin}(3) \subset (C\ell_{3,0} \otimes \mathbb{C})^0$ (see for example [4] for the definition of the Spin group). Note that Δ_3 is independant of the chosen representation.

A.2. Spin Structures and Spinor Bundles

Let us denote by M the Riemannian manifold $\mathbb{R}^3$ and $P_{SO}(M)$ the principal $SO(3)$-bundle of oriented orthonormal frames of M. A spin structure on M is a principal Spin(3)-bundle $P_{\mathrm{Spin}}(M)$ together with a 2-sheeted covering

$$P_{\mathrm{Spin}}(M) \longrightarrow P_{SO}(M) \tag{A.11}$$

that is compatible with $SO(3)$ and Spin(3) actions. The Spinor bundle $S(M)$ is the bundle associated to the spin structure $P_{\mathrm{Spin}}(M)$ and the complex spin representation Δ_3. More precisely, it is the quotient of the product $P_{\mathrm{Spin}}(M) \times \mathbb{C}^2$ by the action

$$\mathrm{Spin}(3) \times P_{\mathrm{Spin}}(M) \times \mathbb{C}^2 \longrightarrow P_{\mathrm{Spin}}(M) \times \mathbb{C}^2 \tag{A.12}$$

that sends (τ, p, z) to $(p\tau^{-1}, \Delta_3(\tau)z)$. We will write

$$S(M) = P_{\mathrm{Spin}}(M) \times_{\Delta_3} \mathbb{C}^2. \tag{A.13}$$

It appears that the fiber bundle $S(M)$ is a bundle of complex left modules over the Clifford bundle $Cl(M) = P_{\mathrm{Spin}}(M) \times_{Ad} \mathbb{C}l(3)$ of M. In the sequel

$$(u, \phi) \longmapsto u \cdot \phi \tag{A.14}$$

denotes the corresponding multiplication for $u \in T(M)$ and ϕ a section of $S(M)$.

We consider now an oriented surface Σ embedded in M. Let us denote by (z_1, z_2) an orthonormal frame of $T(\Sigma)$ and ν the global unit field normal to Σ. Using the map

$$(z_1, z_2) \longmapsto (z_1, z_2, \nu) \tag{A.15}$$

it is possible to pull back the bundle $P_{\mathrm{Spin}}(M)_{|\Sigma}$ to obtain a spin structure $P_{\mathrm{Spin}}(\Sigma)$ on Σ. Since $C\ell_{2,0} \otimes \mathbb{C}$ is isomorphic to $(C\ell_{3,0} \otimes \mathbb{C})^0$ under the map α defined by

$$\alpha(\eta^0 + \eta^1) = \eta^0 + \eta^1\nu \tag{A.16}$$

the algebra $C\ell_{2,0} \otimes \mathbb{C}$ acts on $\mathbb{C}^2$ *via* ρ_2. This representation leads to the complex spinor representation Δ_2 of Spin(2). It can be shown that the induced bundle

$$S(\Sigma) = P_{\mathrm{Spin}}(\Sigma) \times_{\Delta_3 \circ \alpha} \mathbb{C}^2 \tag{A.17}$$

coincides with the spinor bundle of the induced spin structure on Σ. Once again $S(\Sigma)$ is a bundle of complex left modules over the Clifford bundle $Cl(\Sigma)$ of Σ: the Clifford multiplication is given by the map

$$(v, \varphi) \longmapsto v \cdot \nu \cdot \varphi \tag{A.18}$$

for $v \in T(\Sigma)$ and φ a section of $T(\Sigma)$.

The Spinor bundle $S(\Sigma)$ decomposes into

$$S(\Sigma) = S^{+}(\Sigma) \oplus S^{-}(\Sigma) \tag{A.19}$$

where

$$S^{\pm}(\Sigma) = \{\varphi \in S(\Sigma),\ i \cdot z_1 \cdot z_2 \cdot \varphi = \pm\varphi\} \tag{A.20}$$

(compare [5]). Since $\rho_2(z_1 z_2 \nu)$ is minus the identity, this is equivalent to

$$S^{\pm}(\Sigma) = \{\varphi \in S(\Sigma),\ i\nu \cdot \varphi = \pm\varphi\}. \tag{A.21}$$

A.3. Spinor Connections and Dirac Operators

Let ∇^M and ∇^Σ be the Levi–Civita connections on the tangent bundles $T(M)$ and $T(\Sigma)$ respectively. The classical Gauss formula asserts that

$$\nabla^M_X Y = \nabla^\Sigma_X Y - \langle \nabla^M_X \nu, Y\rangle \nu \tag{A.22}$$

where X and Y are vector fields on Σ. A similar formula exists when dealing with spinor fields. Let us first recall that one may construct on $S(M)$ and $S(\Sigma)$ some spinor Levi–Civita connections compatible with the Clifford multiplication, that is connections which we continue to denote by ∇^M and ∇^Σ verifying

$$\nabla^M_X (Y \cdot \varphi) = (\nabla^M_X Y) \cdot \varphi + Y \cdot \nabla^M_X \varphi \tag{A.23}$$

when X and Y are vector fields on M and φ is a section of $S(M)$ and a similar formula for ∇^Σ. The analog of the Gauss formula reads

$$\nabla^M_X \varphi = \nabla^\Sigma_X \varphi - \frac{1}{2}(\nabla^M_X \nu) \cdot \nu \cdot \varphi \tag{A.24}$$

for φ a section of $S(\Sigma)$ and X a vector field on Σ (see [1] for a proof). If (z_1, z_2) is an orthonormal frame of $T(\Sigma)$, following [5], the Dirac operator on $S(\Sigma)$ is defined by

$$D = z_1 \cdot \nabla^\Sigma_{z_1} + z_2 \cdot \nabla^\Sigma_{z_2} \tag{A.25}$$

and it can be verified that $DS^{\pm}(\Sigma) \subset S^{\mp}(\Sigma)$.

Let now ϕ and φ be respectively a section of $S(M)$ and the section of $S(\Sigma)$ given by the restriction $\phi_{|\Sigma}$. We obtain from the Gauss spinor formula

$$z_1 \cdot \nabla^M_{z_1} \phi + z_2 \cdot \nabla^M_{z_2} \phi = D\varphi - \frac{1}{2}(z_1 \cdot (\nabla^M_{z_1} \nu) \cdot \nu \cdot \varphi + z_2 \cdot (\nabla^M_{z_2} \nu) \cdot \nu \cdot \varphi). \tag{A.26}$$

Since

$$z_1 \cdot (\nabla^M_{z_1} \nu) + z_2 \cdot (\nabla^M_{z_2} \nu) = -2H \tag{A.27}$$

where H is the mean curvature of Σ, it follows that

$$D\varphi = z_1 \cdot \nabla^M_{z_1} \phi + z_2 \cdot \nabla^M_{z_2} \phi - H \cdot \nu \cdot \varphi. \tag{A.28}$$

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