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1 Quaternion Fourier Transform: Re-tooling Image and Signal Processing Analysis

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'Did you ask a good question today?' – Janet Teig

Abstract. Quaternion Fourier transforms (QFT's) provide expressive power and elegance in the analysis of higher-dimensional linear invariant systems. But, this power comes at a cost – an overwhelming number of choices in the QFT definition, each with consequences. This chapter explores the evolution of QFT definitions as a framework from which to solve specific problems in vector-image and vector-signal processing.

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1. Introduction

In recent years there has been an increasing recognition on the part of engineers and investigators in image and signal processing of holistic vector approaches to spectral analysis. Generally speaking, this type of spectral analysis treats the vector components of a system not in an iterated, channel-wise fashion but instead in a holistic, gestalt fashion. The Quaternion Fourier transform (QFT) is one such analysis tool.

One of the earliest documented attempts (1987) at describing this type of spectral analysis was in the area of two-dimensional nuclear magnet resonance. Ernst, *et al.* [6, pp. 307–308] briefly discusses using a hypercomplex Fourier transform as a method to independently adjust phase angles with respect to two frequency variables in two-dimensional spectroscopy. After introducing the concept they immediately fall back to an iterated approach leaving the idea unexplored. For similar reasons, Ell [2] in 1992 independently explored the use of QFTs as a tool in the analysis of linear time-invariant systems of partial differential equations (PDEs). Ell specifically 'designed' a quaternion Fourier transform whose spectral operators allowed him to disambiguate partial derivatives with respect to two different independent variables. Ell's original QFT was given by

$$H\left[\boldsymbol{j}\omega,\boldsymbol{k}\nu\right]=\int_{\mathbb{R}^2} e^{-\boldsymbol{j}\omega t}h\left(t,\tau\right)e^{-\boldsymbol{k}\nu\tau}dtd\tau\,, \tag{1.1}$$

where $H\left[\boldsymbol{j}\omega,\boldsymbol{k}\nu\right]\in\mathbb{H}$ (the set of quaternions), $\boldsymbol{j}$ and $\boldsymbol{k}$ are Hamilton's hypercomplex operators, and $h\left(t,\tau\right):\mathbb{R}\times\mathbb{R}\rightarrow\mathbb{R}$ (the set of reals). The partial-differential equivalent spectral operators for this transform are given by

$$\frac{\partial}{\partial t}h\left(t,\tau\right)\Leftrightarrow\boldsymbol{j}\omega\,H\left[\boldsymbol{j}\omega,\boldsymbol{k}\nu\right],\quad \frac{\partial}{\partial\tau}h\left(t,\tau\right)\Leftrightarrow H\left[\boldsymbol{j}\omega,\boldsymbol{k}\nu\right]\boldsymbol{k}\nu. \tag{1.2}$$

These two differentials have clearly different spectral signatures in contrast to the two-dimensional iterated complex Fourier transform where

$$\frac{\partial}{\partial t}h\left(t,\tau\right)\Leftrightarrow\boldsymbol{j}\omega\,H\left[\boldsymbol{j}\omega,\boldsymbol{j}\nu\right],\quad \frac{\partial}{\partial\tau}h\left(t,\tau\right)\Leftrightarrow\boldsymbol{j}\nu H\left[\boldsymbol{j}\omega,\boldsymbol{j}\nu\right], \tag{1.3}$$

especially when $\omega=\nu$, at which point the complex spectral domain responses are indistinguishable. This was the first step towards stability analysis in designing controllers for systems described by PDEs.

The slow adoption of QFTs at the present time by the engineering community is due in part to their lack of practical understanding of its properties. This slow adoption is further exacerbated by the variety of transform definitions available. But, in the middle of difficulty lies opportunity. Instead of attempting to find *the* single best QFT (which cannot meet every design engineer's needs) we provide instead the means to allow the designer to select the definition most appropriate to his specific problem. That means, allow him to *re-tool* for the analysis problem at hand.

For example, when QFTs were later applied to colour-image processing [4], where each colour pixel in an image is treated as a 3-vector with basis $\{\boldsymbol{i},\boldsymbol{j},\boldsymbol{k}\}\in\mathbb{H}$, it became apparent that there was no preferential association of colour-space axes with either the basis or the QFT's exponential-kernel axis. This lead to the next generalization of the QFT defined as

$$\mathcal{F}^{+}\left[\omega,\nu\right]=\int_{\mathbb{R}^2} e^{-\boldsymbol{\mu}(\omega t+\nu\tau)}f\left(t,\tau\right)dtd\tau\,, \tag{1.4}$$

where the transform kernel axis $\boldsymbol{\mu}$ is *any* pure unit quaternion, *i.e.*,

$$\boldsymbol{\mu}\in\left\{\boldsymbol{i}x+\boldsymbol{j}y+\boldsymbol{k}z\in\mathbb{H}\mid x^2+y^2+z^2=1\right\}$$

so that $\boldsymbol{\mu}^2=-1$. Still later it was realized [9] that since there is no preferred direction of indexing the image's pixels then the sign on the transform kernel is also arbitrary, so that a *forward* QFT could also be defined as

$$\mathcal{F}^{-}\left[\omega,\nu\right]=\int_{\mathbb{R}^2} e^{+\boldsymbol{\mu}(\omega t+\nu\tau)}f\left(t,\tau\right)dtd\tau\,, \tag{1.5}$$

and the two definitions could be intermixed without concern of creating a non-causal set of image processing filters. This led to several simplifications of a spectral form of the vector correlation operation on two images [8].

Bearing in mind such diverse application of various QFTs, the focus of this work is to detail as broad a set of QFT definitions as possible, and where known, some of the issues associated with applying them to problems in signal and image processing. It also includes a review of approaches taken to define the interrelations between the various QFT definitions.

2. Preliminaries

To provide a basis for discussion this section gives nomenclature, basic facts on quaternions, and some useful subsets and algebraic equations.

2.1. Just the Facts

The quaternion algebra over the reals $\mathbb{R}$, denoted by

$$\mathbb{H} = \{q = r_0 + \boldsymbol{i}r_1 + \boldsymbol{j}r_2 + \boldsymbol{k}r_3 \mid r_0, r_1, r_2, r_3 \in \mathbb{R}\}, \tag{2.1}$$

is an associative non-commutative four-dimensional algebra, which obeys Hamilton's multiplication rules

$$\boldsymbol{ij} = \boldsymbol{k} = -\boldsymbol{jk},\ \boldsymbol{jk} = \boldsymbol{i} = -\boldsymbol{kj},\ \boldsymbol{ki} = \boldsymbol{j} = -\boldsymbol{ik}, \tag{2.2}$$

$$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1. \tag{2.3}$$

The quaternion conjugate is defined by

$$\bar{q} = r_0 - \boldsymbol{i}r_1 - \boldsymbol{j}r_2 - \boldsymbol{k}r_3, \tag{2.4}$$

which is an anti-involution, *i.e.*, $\bar{\bar{q}} = q$, $\overline{p+q} = \overline{p} + \overline{q}$, and $\overline{qp} = \bar{p}\,\bar{q}$. The norm of a quaternion is defined as

$$|q| = \sqrt{q\bar{q}} = \sqrt{r_0^2 + r_1^2 + r_2^2 + r_3^2}. \tag{2.5}$$

Using the conjugate and norm of q, one can define the inverse of $q \in \mathbb{H} \setminus \{0\}$ as

$$q^{-1} = \bar{q}\Big/|q|^2. \tag{2.6}$$

Two classical operators on quaternions are the vector- and scalar-part, $V[.]$ and $S[.]$, respectively; these are defined as

$$V[q] = \boldsymbol{i}r_1 + \boldsymbol{j}r_2 + \boldsymbol{k}r_3, \qquad S[q] = r_0. \tag{2.7}$$

2.2. Useful Subsets

Various subsets of the quaternions are of interest and used repeatedly throughout this work. The 3-vector subset of $\mathbb{H}$ is the set of pure quaternions defined as

$$V[\mathbb{H}] = \{q = \boldsymbol{i}r_1 + \boldsymbol{j}r_2 + \boldsymbol{k}r_3 \in \mathbb{H}\}. \tag{2.8}$$

The set of pure, unit length quaternions is denoted $\mathbb{S}^3_{\mathbb{H}}$, *i.e.*,

$$\mathbb{S}^3_{\mathbb{H}} = \left\{\boldsymbol{\mu} = \boldsymbol{i}r_1 + \boldsymbol{j}r_2 + \boldsymbol{k}r_3 \in \mathbb{H} \mid r_1^2 + r_2^2 + r_3^2 = 1\right\}. \tag{2.9}$$

Each element of $\mathbb{S}^3_{\mathbb{H}}$ creates a distinct copy of the complex numbers because $\boldsymbol{\mu}^2 = -1$, that is, each creates an injective ring homomorphism from $\mathbb{C}$ to $\mathbb{H}$. So, for each $\boldsymbol{\mu} \in \mathbb{S}^3_{\mathbb{H}}$, we associate a complex sub-field of $\mathbb{H}$ denoted

$$\mathbb{C}_{\boldsymbol{\mu}} = \left\{\alpha + \beta\boldsymbol{\mu}; \mid \alpha, \beta \in \mathbb{R}, \boldsymbol{\mu} \in \mathbb{S}^3_{\mathbb{H}}\right\}. \tag{2.10}$$

2.3. Useful Algebraic Equations

In various quaternion equations the non-commutativity of the multiplication causes difficulty, however, there are algebraic forms which assist in making simplifications. The following three defined forms appear to be the most useful.

Definition 2.1 (Even-Odd Form). Every $f : \mathbb{R}^2 \to \mathbb{H}$ can be split into even and odd parts along the x- and y-axis as

$$f(x,y) = f_{\text{ee}}(x,y) + f_{\text{eo}}(x,y) + f_{\text{oe}}(x,y) + f_{\text{oo}}(x,y) \tag{2.11}$$

where f_{eo} denotes the part of f that is even with respect to x and odd with respect to y, *etc.*, given as

$$\begin{aligned}
f_{\text{ee}}(x,y) &= \tfrac{1}{4}\left(f(x,y) + f(-x,y) + f(x,-y) + f(-x,-y)\right), \\
f_{\text{eo}}(x,y) &= \tfrac{1}{4}\left(f(x,y) + f(-x,y) - f(x,-y) - f(-x,-y)\right), \\
f_{\text{oe}}(x,y) &= \tfrac{1}{4}\left(f(x,y) - f(-x,y) + f(x,-y) - f(-x,-y)\right), \\
f_{\text{oo}}(x,y) &= \tfrac{1}{4}\left(f(x,y) - f(-x,y) - f(x,-y) + f(-x,-y)\right).
\end{aligned} \tag{2.12}$$

Definition 2.2 (Symplectic Form [3]). Every $q = r_0 + \boldsymbol{i}r_1 + \boldsymbol{j}r_2 + \boldsymbol{k}r_3 \in \mathbb{H}$ can be rewritten in terms of a new basis of operators $\{\boldsymbol{\mu}_1, \boldsymbol{\mu}_2, \boldsymbol{\mu}_3\}$ as

$$q = r'_0 + \boldsymbol{\mu}_1 r'_1 + \boldsymbol{\mu}_2 r'_2 + \boldsymbol{\mu}_3 r'_3 = \left(r'_0 + \boldsymbol{\mu}_1 r'_1\right) + \left(r'_2 + \boldsymbol{\mu}_1 r'_3\right)\boldsymbol{\mu}_2, \tag{2.13}$$

where $\boldsymbol{\mu}_1\boldsymbol{\mu}_2 = \boldsymbol{\mu}_3$, $\boldsymbol{\mu}_{1,2,3} \in \mathbb{S}^3_{\mathbb{H}}$, hence they form an orthogonal triad.

Remark 2.3. The mapping $\{r_1, r_2, r_3\} \to \{r'_1, r'_2, r'_3\}$ is a change in basis from $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ to $\{\boldsymbol{\mu}_1, \boldsymbol{\mu}_2, \boldsymbol{\mu}_3\}$ *via*

$$r'_0 = r_0, \quad r'_n = -\tfrac{1}{2}\left(V[q]\,\boldsymbol{\mu}_n + \boldsymbol{\mu}_n V[q]\right), \quad n = \{1,2,3\}. \tag{2.14}$$

Remark 2.4. The symplectic form essentially decomposes a quaternion with respect to a specific complex sub-field. That is

$$q = \left(r'_0 + \boldsymbol{\mu}_1 r'_1\right) + \left(r'_2 + \boldsymbol{\mu}_1 r'_3\right)\boldsymbol{\mu}_2 = c_1 + c_2\boldsymbol{\mu}_2, \tag{2.15}$$

where $c_{1,2} \in \mathbb{C}_{\boldsymbol{\mu}_1}$. The author coined the terms *simplex* and *perplex* parts of q, for c_1 and c_2, respectively.

Remark 2.5. The symplectic form works for any permutation of the basis $\{\boldsymbol{\mu}_1, \boldsymbol{\mu}_2, \boldsymbol{\mu}_3\}$ so that the simplex and complex parts can be taken from any complex sub-field $\mathbb{C}_{\boldsymbol{\mu}_n}$ Further, the *swap rule* applies to the last term, *i.e.*, $c_2\boldsymbol{\mu}_2 = \boldsymbol{\mu}_2\overline{c_2}$, where the over bar denotes both quaternion and complex sub-field conjugation.

Definition 2.6 (Split Form [7]). Every $q \in \mathbb{H}$ can be split as

$$q = q_+ + q_-, \quad q_\pm = \tfrac{1}{2}\left(q \pm \boldsymbol{\mu}_1 q \boldsymbol{\mu}_2\right), \tag{2.16}$$

where $\boldsymbol{\mu}_1\boldsymbol{\mu}_2 = \boldsymbol{\mu}_3$, and $\boldsymbol{\mu}_{1,2,3} \in \mathbb{S}^3_{\mathbb{H}}$.

Remark 2.7. The split form allows for the explicit ordering of factors with respect to the operators. So, for example, $q = r_0 + \boldsymbol{\mu}_1 r_1 + \boldsymbol{\mu}_2 r_2 + \boldsymbol{\mu}_3 r_3$ becomes

$$\begin{aligned} q_\pm &= \left\{(r_0 \pm r_3) + \boldsymbol{\mu}_1(r_1 \pm r_2)\right\}\frac{1 \pm \boldsymbol{\mu}_3}{2} \\ &= \frac{1 \pm \boldsymbol{\mu}_3}{2}\left\{(r_0 \pm r_3) - \boldsymbol{\mu}_2(r_1 \pm r_2)\right\}. \end{aligned} \tag{2.17}$$

Euler's formula holds for quaternions, so any unit length quaternion can be written as $\cos a + \boldsymbol{\mu}\sin a = e^{\boldsymbol{\mu}a}$, for $a \in \mathbb{R}$ and $\boldsymbol{\mu} \in \mathbb{S}^3_{\mathbb{H}}$. Here $\boldsymbol{\mu}$ is referred to as the (Eigen-) axis and a as the (Eigen-) phase angle. Although in general $e^q\, e^p \neq e^{q+p}$ for $p, q \in \mathbb{H}$, their exponential product is a linear combination of exponentials of the sum and difference of their phase angles. This can be written in two ways as shown in the following two propositions.

Proposition 2.8 (Exponential Split). *Let $\boldsymbol{\mu}_{1,2} \in \mathbb{S}^3_{\mathbb{H}}$ and $a, b \in \mathbb{R}$, then*

$$e^{\boldsymbol{\mu}_1 a} e^{\boldsymbol{\mu}_2 b} = e^{\boldsymbol{\mu}_1(a-b)}\frac{1+\boldsymbol{\mu}_3}{2} + e^{\boldsymbol{\mu}_1(a+b)}\frac{1-\boldsymbol{\mu}_3}{2} \tag{2.18}$$

and

$$e^{\boldsymbol{\mu}_1 a} e^{\boldsymbol{\mu}_2 b} = \frac{1+\boldsymbol{\mu}_3}{2}e^{\boldsymbol{\mu}_2(b-a)} + \frac{1-\boldsymbol{\mu}_3}{2}e^{\boldsymbol{\mu}_2(b+a)}, \tag{2.19}$$

where $\boldsymbol{\mu}_1\boldsymbol{\mu}_2 = \boldsymbol{\mu}_3$ and $\boldsymbol{\mu}_3 \in \mathbb{S}^3_{\mathbb{H}}$.

Proof. Application of split form to the exponential product. □

Proposition 2.9 (Exponential Modulation). *Let $\boldsymbol{\mu}_{1,2} \in \mathbb{S}^3_{\mathbb{H}}$ and $a, b \in \mathbb{R}$, then*

$$e^{\boldsymbol{\mu}_1 a} e^{\boldsymbol{\mu}_2 b} = \tfrac{1}{2}\left(e^{\boldsymbol{\mu}_1(a+b)} + e^{\boldsymbol{\mu}_1(a-b)}\right) - \tfrac{1}{2}\boldsymbol{\mu}_1\left(e^{\boldsymbol{\mu}_1(a+b)} - e^{\boldsymbol{\mu}_1(a-b)}\right)\boldsymbol{\mu}_2 \tag{2.20}$$

and

$$e^{\boldsymbol{\mu}_1 a} e^{\boldsymbol{\mu}_2 b} = \tfrac{1}{2}\left(e^{\boldsymbol{\mu}_2(b+a)} + e^{\boldsymbol{\mu}_2(b-a)}\right) - \tfrac{1}{2}\boldsymbol{\mu}_1\left(e^{\boldsymbol{\mu}_2(b+a)} - e^{\boldsymbol{\mu}_2(b-a)}\right)\boldsymbol{\mu}_2. \tag{2.21}$$

Proof. Direct application of Euler's formula and trigonometric identities. □

Remark 2.10. The sandwich terms (*i.e.*, $\boldsymbol{\mu}_1(.)\boldsymbol{\mu}_2$) in the exponential equations introduce 4-space rotations into the interpretation of the product [1]. For if $p = \boldsymbol{\mu}_1 q \boldsymbol{\mu}_2$, then p is a rotated version of q about the $(\boldsymbol{\mu}_1, \boldsymbol{\mu}_2)$-plane by $\frac{\pi}{2}$.

3. Quaternion Fourier Transforms

The purpose of this section is to enumerate a list of *possible* definitions for a quaternion Fourier transform. This is followed by a discussion regarding various operator properties used in the engineering fields that require simple Fourier transform pairs between the non-transformed operation and the equivalent Fourier domain operation, *i.e.*, the so-called *operator pairs* as seen in most engineering textbooks on Fourier analysis. Finally, a discussion on how the inter-relationship between QFT definitions are explored, not so as to reduce them to a single canonical form, but to provide the investigator a tool to cross between definitions when necessary so as to gain insight into operator properties.

3.1. Transform Definitions

Although there has been much use of the QFT forms currently in circulation, there are however more available. Not all 'degrees-of-freedom' have been exploited. The non-commutativity of the quaternion multiplication gave rise to the left- and right-handed QFT kernels. The infinite number of square-roots of -1 (the cardinality of $\mathbb{S}_{\mathbb{H}}$) gave rise to the two-sided, or sandwiched kernel. One concept left unexplored is the implication of the exponential product of two quaternions, *i.e.*, $e^p e^q \neq e^{p+q}$. When this is also taken into account, the list expands to *eight* distinct QFTs as enumerated in Table 1.

TABLE 1. QFT kernel definitions for $f : \mathbb{R}^2 \to \mathbb{H}$.

	Left	Right	Sandwich
Single-axis	$e^{-\boldsymbol{\mu}_1(\omega x+\boldsymbol{\nu} y)}\ f(.)$	$f(.)\ e^{-\boldsymbol{\mu}_1(\omega x+\boldsymbol{\nu} y)}$	$e^{-\boldsymbol{\mu}_1\omega x}\ f(.)\ e^{-\boldsymbol{\mu}_1\boldsymbol{\nu} y}$
Dual-axis	$e^{-(\boldsymbol{\mu}_1\omega x+\boldsymbol{\mu}_2\boldsymbol{\nu} y)}\ f(.)$	$f(.)\ e^{-(\boldsymbol{\mu}_1\omega x+\boldsymbol{\mu}_2\boldsymbol{\nu} y)}$	–
Factored	$e^{-\boldsymbol{\mu}_1\omega x}e^{-\boldsymbol{\mu}_2\boldsymbol{\nu} y}\ f(.)$	$f(.)\ e^{-\boldsymbol{\mu}_1\omega x}e^{-\boldsymbol{\mu}_2\boldsymbol{\nu} y}$	$e^{-\boldsymbol{\mu}_1\omega x}\ f(.)\ e^{-\boldsymbol{\mu}_2\boldsymbol{\nu} y}$

Depending on the value space of $f(x, y)$, the available number of distinct QFT forms changes. Table 1 shows the options when $f : \mathbb{R}^2 \to \mathbb{H}$. However, if $f : \mathbb{R}^2 \to \mathbb{R}$, then all chirality options (left, right, and sandwiched) collapse to the same form, leaving three distinct choices: single-axis and factored and un-factored dual-axis forms.

If neither x nor y are time-like, so that causality of the solution is not a factor, then the number of given QFTs doubles. Variations created by conjugating the quaternion-exponential kernel of both the forward and inverse transform are usually a matter of convention – the signs must be opposites. For non-causal systems, however, the sign on the kernel can be taken both ways; each defining its own *forward* transform from the spatial to spatial-frequency domain. To distinguish the two versions of the forward transform, one is called *forward* the other *reverse*. Of course, the *inverse* transform is still obtained by conjugating the corresponding forward (or reverse) kernel. Hence, one may define the single-axis, left- and right-sided, forward and reverse transforms as follows[1].

Definition 3.1 (Single-axis, Left-sided QFT). The single-axis, left-sided, forward ($\mathcal{F}^{+L}$) and reverse ($\mathcal{F}^{-L}$) QFTs are defined as

$$\mathcal{F}^{\pm L}\left[f\left(x,y\right)\right] = \iint_{\mathbb{R}^2} e^{\mp \boldsymbol{\mu}_1 (\omega x + \nu y)} f\left(x,y\right) dxdy = F^{\pm L}\left[\omega,\nu\right]. \tag{3.1}$$

Definition 3.2 (Single-axis, Right-sided QFT). The single-axis, right-sided, forward ($\mathcal{F}^{+R}$) and reverse ($\mathcal{F}^{-R}$) QFTs are defined as

$$\mathcal{F}^{\pm R}\left[f\left(x,y\right)\right] = \iint_{\mathbb{R}^2} f\left(x,y\right) e^{\mp \boldsymbol{\mu}_1 (\omega x + \nu y)} dxdy = F^{\pm R}\left[\omega,\nu\right]. \tag{3.2}$$

All of the entries in Table 1 exploit the fact that unit length complex numbers act as rotation operators within the complex plane. There is, however, another rotation operator – the 3-space rotation operator for which quaternions are famous. Table 2 lists additional definitions under the provision that f takes on values restricted to $V[\mathbb{H}]$. Note the factor of $\frac{1}{2}$ in the kernel exponent, this is included so that the frequency scales between the various definitions align.

TABLE 2. QFT kernel definitions exclusively for $f : \mathbb{R}^2 \to V[\mathbb{H}]$.

	3-Space Rotator
Single-axis	$e^{-\boldsymbol{\mu}_1 \omega x/2}\ f(.)\ e^{+\boldsymbol{\mu}_1 \nu y/2}$
Dual-axis	$e^{-(\boldsymbol{\mu}_1 \omega x + \boldsymbol{\mu}_2 \nu y)/2}\ f(.)\ e^{+(\boldsymbol{\mu}_1 \omega x + \boldsymbol{\mu}_2 \nu y)/2}$
Dual-axis, factored	$e^{-\boldsymbol{\mu}_1 \omega x/2} e^{-\boldsymbol{\mu}_2 \nu y/2}\ f(.)\ e^{+\boldsymbol{\mu}_2 \nu y/2} e^{+\boldsymbol{\mu}_1 \omega x/2}$

Taking all these permutations in mind, one arrives at 22 unique QFT definitions.

3.2. Functional Relationships

There are several properties used in *complex* Fourier transform (CFT) analysis that one hopes will carry over to the QFT in some fashion. These are listed in Table 3 from which we will discuss the challenges which arise in QFT analysis. In what follows let $f\left(x,y\right) \Leftrightarrow F\left[\omega,\nu\right]$ denote transform pairs, *i.e.*, $\mathcal{F}\left[f\left(x,y\right)\right] = F\left[\omega,\nu\right]$ is the forward (or reverse) transform and $\mathcal{F}^{-1}\left[F\left(\omega,\nu\right)\right] = f\left(x,y\right)$ is its inversion.

Inversion. Every transform should be invertible. Although this seems obvious, there are instances where a given transform is not. For example, if the restriction of $f : \mathbb{R}^2 \to V[\mathbb{H}]$ were not imposed on the inputs of Table 2, then

[1] Note, that in (3.1) and (3.2) the arguments x and y of f in $\mathcal{F}^{\pm L,R}\left[f\left(x,y\right)\right]$ are shown for clarity, but are actually dummy arguments, which are integrated out. A more mathematical notation would be $\mathcal{F}^{\pm L,R}\{f\}(\omega,\nu) = F^{\pm L,R}(\omega,\nu)$.

TABLE 3. Fourier transform $\mathcal{F}$ properties. $[\alpha, \beta, \gamma, \delta \in \mathbb{R}]$

Property	Definition
Inversion	$\mathcal{F}^{-1}\left[\mathcal{F}\left[f\left(x,y\right)\right]\right] = f\left(x,y\right)$
Linearity	$\alpha f(x,y) + \beta g(x,y)$
Complex Degenerate	$(\boldsymbol{\mu}_1 = \boldsymbol{i}$ and $f : \mathbb{R}^2 \to \mathbb{C}_{\boldsymbol{i}}) \to (\text{QFT}\cong\text{CFT})$
Convolution	$f \circ g(x,y) = (?)$
Correlation	$f \star g(x,y) = (?)$
Modulation	$e^{\boldsymbol{\mu}_1 \omega_0 x} f(.),\ e^{\boldsymbol{\mu}_2 \omega_0 x} f(.),\ f(.) e^{\boldsymbol{\mu}_1 \nu_0 y},\ f(.) e^{\boldsymbol{\mu}_2 \nu_0 y}$, *etc.*
Scaling	$f(x/\alpha, y/\beta)$
Translation	$f(x - x_0, y - y_0)$
Rotation	$f(x\cos\alpha - y\sin\alpha, x\sin\alpha + y\cos\alpha)$
Axis-reversal	$f(-x,y), f(x,-y), f(-x,-y)$
Re-coordinate	$f(\alpha x + \beta y, \gamma x + \delta y)$
Conjugation	$\overline{f(x,y)}$
Differentials	$\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial^2}{\partial x \partial y}$, *etc.*

every transform of that table would cease to be invertible. This is because any real-valued function, or the scalar part of full quaternion valued functions, commute with the kernel factors which then vanish from under the integral.

Linearity. $\alpha f(x,y) + \beta g(x,y) \Leftrightarrow \alpha F[\omega,\nu] + \beta G[\omega,\nu]$ where $\alpha, \beta \in \mathbb{R}$. A quick check verifies that this property holds for all proposed QFT definitions.

Complex Degenerate. For the single-axis transforms, if $\boldsymbol{\mu}_1 = \boldsymbol{i}$ and $f : \mathbb{R}^2 \to \mathbb{C}_{\boldsymbol{i}}$, then the QFT should ideally degenerate to the twice iterated complex Fourier transform. This degenerate property cannot apply to the dual axis, factored forms of Tables 1 and 2 since if $\boldsymbol{\mu}_1 = \boldsymbol{\mu}_2 = \boldsymbol{i}$, these forms reduce to their single-axis versions.

Convolution (Faltung) theorem. Rarely does the standard, complex transform type pair $f \circ g\,(x,y) \Leftrightarrow F\,[\omega,\nu]\,G\,[\omega,\nu]$ exist in such a simple form for the QFT. Even the very *definition* of convolution needs an update since $f \circ g \neq g \circ f$ when f and g are $\mathbb{H}$-valued. The definition is altered again based on which of the two functions is translated, *i.e.*, is the integrand $f(x - x', y - y')g(x,y)$ or $f(x,y)g(x - x', y - y')$. This gives rise to at least four distinct convolution definitions. This will also alter the spectral operator pair.

Further, if f is an input function, then g is typically related to the *impulse response* of a system. But, if $g : \mathbb{R}^2 \to V[\mathbb{H}]$, is a *single* impulse response sufficient to describe such a system? Or does it take at least two orthogonally oriented impulses, say $\boldsymbol{\mu}_1\delta(x,y)$ and $\boldsymbol{\mu}_2\delta(x,y)$, where $\boldsymbol{\mu}_1 \perp \boldsymbol{\mu}_2$ and $\delta(x,y)$ is the Dirac delta function?

Correlation. Consider the correlation definition of two $\mathbb{R}$-valued functions f and g (let $f, g : \mathbb{R} \to \mathbb{R}$ for simplicity of discussion)

$$f \star g\,(t) = \int_{\mathbb{R}} f\,(\tau)\, g\,(\tau - t)\, d\tau = \int_{\mathbb{R}} f\,(\tau + t)\, g\,(\tau)\, d\tau, \tag{3.3}$$

where substituting $\tau - t = \tau'$ yields the second form. For the correlation of real-valued functions this is entirely sufficient.

However, for $\mathbb{C}$-valued functions a conjugation operation is required to ensure the relation of the autocorrelation functions $(f \star f)$ to the power spectrum as required by the Wiener-Khintchine theorem. This effectively ensures that the power spectrum of a complex auto-correlation is $\mathbb{R}$-valued. The complex extension to the cross-correlation function can then be given as

$$f \star g\,(t) = \int_{\mathbb{R}} f\,(\tau)\, \overline{g\,(\tau - t)}\, d\tau, \tag{3.4}$$

or alternatively as

$$f \star g\,(t) = \int_{\mathbb{R}} \overline{f\,(\tau)}\, g\,(\tau + t)\, d\tau. \tag{3.5}$$

In general, the literature does not give significance to the direction of the shifted signal $(\tau \pm t)$. However, in the case of vector correlation matching problems, such as colour image registration, direction is fundamental.

Taking this into consideration, for $\mathbb{H}$-valued functions the equivalent correlation could be either

$$f \star g\,(x, y) = \int_{\mathbb{R}^2} f\,(x', y')\, \overline{g\,(x' - x, y' - y)} dx' dy', \tag{3.6}$$

or

$$f \star g\,(x, y) = \int_{\mathbb{R}^2} f\,(x' + x, y' + y)\, \overline{g\,(x', y')} dx' dy', \tag{3.7}$$

depending on which shift direction is required. For more details see [9, 4]. Note that the correlation result is not necessarily $\mathbb{R}$-valued.

Modulation. There are multiple types of frequency modulation that need to be addressed. The modulating exponential can be applied from the left or right, can be driven as a function of either input parameter (*i.e.*, x or y), and be pointing along one of the kernel axes (*i.e.*, $\boldsymbol{\mu}_1$ or $\boldsymbol{\mu}_2$). Some options are detailed in the Karnaugh map of Table 4.

In summary, when addressing the QFT operator properties one often needs to regress back to the basic operator definitions and their underlying assumptions, then verify they are still valid for generalization to quaternion forms. Either the operator definition itself needs to be modified (as in the case of correlation) or the number of permutations on the definition increases (as in convolution and modulation).

Table 4. Frequency Modulations

	Left	Right	
x	$e^{\boldsymbol{\mu}_2 \omega_0 x} f(.)$	$f(.) e^{\boldsymbol{\mu}_2 \omega_0 x}$	$\boldsymbol{\mu}_2$
x	$e^{\boldsymbol{\mu}_1 \omega_0 x} f(.)$	$f(.) e^{\boldsymbol{\mu}_1 \omega_0 x}$	$\boldsymbol{\mu}_1$
y	$e^{\boldsymbol{\mu}_1 \boldsymbol{\nu}_0 y} f(.)$	$f(.) e^{\boldsymbol{\mu}_1 \boldsymbol{\nu}_0 y}$	$\boldsymbol{\mu}_1$
y	$e^{\boldsymbol{\mu}_2 \boldsymbol{\nu}_0 y} f(.)$	$f(.) e^{\boldsymbol{\mu}_2 \boldsymbol{\nu}_0 y}$	$\boldsymbol{\mu}_2$

3.3. Relationships between Transforms

At the heart of all methods for determining inter-relationships between various QFTs is a decomposing process, of either the input function $f(.)$ or the exponential-kernel, so that their parts can be commuted into an alternate QFT form. Ell and Sangwine [5] used the *symplectic form* to link the single-axis, left and right, forward and reverse forms of the QFT *via* simplex and perplex complex sub-fields. Yeh [10] reworked these relationships and made further connections to the dual-axis, factored form QFT, but instead used *even-odd decomposition* of the input function. This approach essentially split each QFT into cosine and sine QFTs. Hitzer's [7] approach was to use the *split form* to factor the input function and kernel into factors with respect to the hypercomplex operators, so as to manipulate the result to an alternate QFT.

The inter-relationships between the various transform definitions not only give insight into the subsequent Fourier analysis, they are also used to simplify operator pairs. For example, the inter-relationships between the single-axis forms as given in Definitions 3.2 and 3.1 were used with the symplectic form (Def. 2.2) by Ell and Sangwine [5] to arrive at operator pairs for the convolution operator. Let the single-sided convolutions be defined as follows.

Definition 3.3 (Convolution [5]). The left- and right-sided convolution are defined, respectively, as

$$
\begin{aligned}
h_L \circ f(x, y) &= \iint_{\mathbb{R}^2} h_L(x', y') f(x - x', y - y')\, dx' dy', \\
f \circ h_R(x, y) &= \iint_{\mathbb{R}^2} f(x - x', y - y')\, h_R(x', y')\, dx' dy'.
\end{aligned}
\tag{3.8}
$$

Now, let the QFT of the input function f be symplectically decomposed with respect to $\boldsymbol{\mu}_1$ as

$$\mathcal{F}^{\pm L}[f(x, y)] = F_1^{\pm L}[\omega, \nu] + F_2^{\pm L}[\omega, \nu]\, \boldsymbol{\mu}_2$$

and

$$\mathcal{F}^{\pm(L,R)}[h_R(x, y)] = H_R^{\pm(L,R)}[\omega, \nu],$$

then the right-convolution operator can be written as

$$\mathcal{F}^{\pm L}\left[f \circ h_R\right](\omega,\nu) = F_1^{\pm L}\left[\omega,\nu\right] H_R^{\pm L}\left[\omega,\nu\right] + F_2^{\pm L}\left[\omega,\nu\right] \boldsymbol{\mu}_2 H_R^{\mp L}\left[\omega,\nu\right].$$

Note the use of both forward and reverse QFT transforms. Such a compact operator formula would not be possible without the intermixing of QFT definitions.

4. Conclusions

The three currently defined quaternion Fourier transforms have been shown to be incomplete. By careful consideration of the underlying reasons for those three forms, this list has been extended to no less than twenty-two unique definitions. Future work may show that some of these definitions hold little of practical value or, without loss of generality, they may be reduced to but a few. The shift from iterated, channel-wise vector analysis to gestalt vector-image and vector-signal analysis shows promise. This promise raises several challenges:

1. Are there other, more suitable quaternion Fourier transform definitions?
2. Can these transforms be reduced to a salient few?
3. Are there additional decomposition methods, like the even-odd, split, and symplectic discussed herein, which can be used?
4. All the decomposition methods used to simplify the operator formulas are at odds with the very gestalt, holistic approach espoused, can this be done otherwise?

These questions will be the focus of future efforts.

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